

Topics In Analysis

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Contents

I	Review Of Advanced Calculus	17
1	Set Theory	19
1.1	Basic Definitions	19
1.2	The Schroder Bernstein Theorem	22
1.3	Equivalence Relations	25
1.4	Partially Ordered Sets	26
2	Continuous Functions Of One Variable	27
2.1	Exercises	28
2.2	Theorems About Continuous Functions	29
3	The Riemann Stieltjes Integral	35
3.1	Upper And Lower Riemann Stieltjes Sums	35
3.2	Exercises	39
3.3	Functions Of Riemann Integrable Functions	40
3.4	Properties Of The Integral	43
3.5	Fundamental Theorem Of Calculus	47
3.6	Exercises	51
4	Some Important Linear Algebra	53
4.1	Algebra in $\mathbb{F}^n$	55
4.2	Exercises	56
4.3	Subspaces Spans And Bases	57
4.4	An Application To Matrices	61
4.5	The Mathematical Theory Of Determinants	63
4.6	Exercises	76
4.7	The Cayley Hamilton Theorem	76
4.8	An Identity Of Cauchy	78
4.9	Block Multiplication Of Matrices	79
4.10	Exercises	83
4.11	Shur's Theorem	85
4.12	The Right Polar Decomposition	91

5	Multi-variable Calculus	97
5.1	Continuous Functions	97
5.1.1	Distance In $\mathbb{F}^n$	97
5.2	Open And Closed Sets	100
5.3	Continuous Functions	102
5.3.1	Sufficient Conditions For Continuity	102
5.4	Exercises	103
5.5	Limits Of A Function	104
5.6	Exercises	108
5.7	The Limit Of A Sequence	108
5.7.1	Sequences And Completeness	110
5.7.2	Continuity And The Limit Of A Sequence	111
5.8	Properties Of Continuous Functions	112
5.9	Exercises	113
5.10	Proofs Of Theorems	113
5.11	The Space $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$	118
5.11.1	The Operator Norm	118
5.12	The Frechet Derivative	120
5.13	C^1 Functions	124
5.14	C^k Functions	129
5.15	Mixed Partial Derivatives	130
5.16	Implicit Function Theorem	132
5.16.1	More Continuous Partial Derivatives	136
5.17	The Method Of Lagrange Multipliers	137
6	Metric Spaces And General Topological Spaces	139
6.1	Metric Space	139
6.2	Compactness In Metric Space	141
6.3	Some Applications Of Compactness	145
6.4	Ascoli Arzela Theorem	147
6.5	The Tietze Extension Theorem	150
6.6	General Topological Spaces	153
6.7	Connected Sets	158
7	Weierstrass Approximation Theorem	163
7.1	The Bernstein Polynomials	163
7.2	Stone Weierstrass Theorem	167
7.2.1	The Case Of Compact Sets	167
7.2.2	The Case Of Locally Compact Sets	170
7.2.3	The Case Of Complex Valued Functions	171
7.3	Exercises	172

II	Real And Abstract Analysis	175
8	Abstract Measure And Integration	177
8.1	σ Algebras	177
8.2	Exercises	188
8.3	The Abstract Lebesgue Integral	190
8.3.1	Preliminary Observations	190
8.3.2	Definition Of The Lebesgue Integral For Nonnegative Measurable Functions	191
8.3.3	The Lebesgue Integral For Nonnegative Simple Functions	193
8.3.4	Simple Functions And Measurable Functions	196
8.3.5	The Monotone Convergence Theorem	200
8.3.6	Other Definitions	202
8.3.7	Fatou's Lemma	202
8.3.8	The Righteous Algebraic Desires Of The Lebesgue Integral	205
8.4	The Space L^1	205
8.5	Vitali Convergence Theorem	212
8.6	Exercises	214
9	The Construction Of Measures	219
9.1	Outer Measures	219
9.2	Regular Measures	224
9.3	Urysohn's lemma	227
9.4	Positive Linear Functionals	234
9.5	One Dimensional Lebesgue Measure	244
9.6	One Dimensional Lebesgue Stieltjes Measure	245
9.7	The Distribution Function	247
9.8	Product Measures	251
9.9	Alternative Treatment Of Product Measure	263
9.9.1	Monotone Classes And Algebras	263
9.9.2	Product Measure	266
9.10	Completion Of Measures	271
9.11	Another Version Of Product Measures	275
9.11.1	General Theory	275
9.11.2	Completion Of Product Measure Spaces	279
9.12	Disturbing Examples	281
9.13	Exercises	283
10	Lebesgue Measure	285
10.1	Basic Properties	285
10.2	The Vitali Covering Theorem	289
10.3	The Vitali Covering Theorem (Elementary Version)	291
10.4	Vitali Coverings	294
10.5	Change Of Variables For Linear Maps	298
10.6	Change Of Variables For C^1 Functions	301

10.7	Mappings Which Are Not One To One	308
10.8	Lebesgue Measure And Iterated Integrals	309
10.9	Spherical Coordinates In Many Dimensions	311
10.10	The Brouwer Fixed Point Theorem	313
10.11	The Brouwer Fixed Point Theorem Another Proof	317
11	Some Extension Theorems	321
11.1	Caratheodory Extension Theorem	321
11.2	The Tychonoff Theorem	323
11.3	Kolmogorov Extension Theorem	326
11.4	Exercises	333
12	The L^p Spaces	337
12.1	Basic Inequalities And Properties	337
12.2	Density Considerations	345
12.3	Separability	347
12.4	Continuity Of Translation	349
12.5	Mollifiers And Density Of Smooth Functions	350
12.6	Exercises	355
13	Banach Spaces	361
13.1	Theorems Based On Baire Category	361
13.1.1	Baire Category Theorem	361
13.1.2	Uniform Boundedness Theorem	365
13.1.3	Open Mapping Theorem	366
13.1.4	Closed Graph Theorem	368
13.2	Hahn Banach Theorem	370
13.2.1	Partially Ordered Sets	370
13.2.2	Gauge Functions And Hahn Banach Theorem	371
13.2.3	The Complex Version Of The Hahn Banach Theorem	373
13.2.4	The Dual Space And Adjoint Operators	375
13.3	Approximation With Sums	378
13.4	Weak And Weak $*$ Topologies	382
13.4.1	Basic Definitions	382
13.4.2	Banach Alaoglu Theorem	383
13.4.3	Eberlein Smulian Theorem	385
13.5	Exercises	388
14	Locally Convex Topological Vector Spaces	393
14.1	Fundamental Considerations	393
14.2	Separation Theorems	397
14.2.1	Convex Functionals	400
14.2.2	More Separation Theorems	403
14.3	The Weak And Weak $*$ Topologies	406
14.4	The Tychonoff Fixed Point Theorem	407
14.5	Set-Valued Maps	412

15 Hilbert Spaces	421
15.1 Basic Theory	421
15.2 The Hilbert Space $L(U)$	427
15.3 Approximations In Hilbert Space	432
15.4 The Müntz Theorem	436
15.5 Orthonormal Sets	440
15.6 Fourier Series, An Example	441
15.7 Compact Operators	443
15.7.1 Compact Operators In Hilbert Space	443
15.7.2 Nuclear Operators	448
15.7.3 Hilbert Schmidt Operators	451
15.8 Compact Operators In Banach Space	456
15.9 The Fredholm Alternative	457
15.10 Square Roots	459
15.11 Quotient Spaces	464
15.12 General Theory Of Continuous Semigroups	468
15.12.1 An Evolution Equation	477
15.12.2 Adjoints, Hilbert Space	480
15.12.3 Adjoints, Reflexive Banach Space	484
16 Representation Theorems	489
16.1 Radon Nikodym Theorem	489
16.2 Vector Measures	495
16.3 Representation Theorems For The Dual Space Of L^p	503
16.4 The Dual Space Of $C(X)$	510
16.5 The Dual Space Of $C_0(X)$	513
16.6 More Attractive Formulations	514
16.7 Sequential Compactness In L^1	516
16.8 Exercises	521
17 Integrals And Derivatives	525
17.1 The Fundamental Theorem Of Calculus	525
17.2 Absolutely Continuous Functions	530
17.3 Differentiation Of Measures With Respect To Lebesgue Measure	535
17.4 Exercises	541
18 Orlicz Spaces	545
18.1 Basic Theory	545
18.2 Dual Spaces In Orlicz Space	560
19 Hausdorff Measure	565
19.1 Definition Of Hausdorff Measures	565
19.1.1 Properties Of Hausdorff Measure	566
19.2 $\mathcal{H}^n$ And m_n	569
19.3 Technical Considerations	571
19.3.1 Steiner Symmetrization	573

19.3.2	The Isodiametric Inequality	575
19.4	The Proper Value Of $\beta(n)$	576
19.4.1	A Formula For $\alpha(n)$	577
19.4.2	Hausdorff Measure And Linear Transformations	579
19.5	The Area Formula	581
19.5.1	Preliminary Results	581
19.5.2	The Area Formula	589
19.6	The Area Formula Alternate Version	592
19.6.1	Preliminary Results	592
19.6.2	The Area Formula	599
19.7	The Divergence Theorem	601
20	Differentiation With Respect To General Radon Measures	615
20.1	Besicovitch Covering Theorem	615
20.2	Fundamental Theorem Of Calculus For Radon Measures	620
20.3	Slicing Measures	624
20.4	Vitali Coverings	630
20.5	Differentiation Of Radon Measures	632
20.6	The Radon Nikodym Theorem For Radon Measures	635
21	Fourier Transforms	637
21.1	An Algebra Of Special Functions	637
21.2	Fourier Transforms Of Functions In $\mathcal{G}$	638
21.3	Fourier Transforms Of Just About Anything	641
21.3.1	Fourier Transforms Of $\mathcal{G}^*$	641
21.3.2	Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$	645
21.3.3	Fourier Transforms Of Functions In $L^2(\mathbb{R}^n)$	648
21.3.4	The Schwartz Class	652
21.3.5	Convolution	654
21.4	Exercises	656
22	Fourier Analysis In $\mathbb{R}^n$ An Introduction	661
22.1	The Marcinkiewicz Interpolation Theorem	661
22.2	The Calderon Zygmund Decomposition	664
22.3	Mihlin's Theorem	666
22.4	Singular Integrals	679
22.5	Helmholtz Decompositions	689
23	The Bochner Integral	697
23.1	Strong And Weak Measurability	697
23.2	The Bochner Integral	705
23.2.1	Definition And Basic Properties	705
23.2.2	Taking A Closed Operator Out Of The Integral	710
23.3	Operator Valued Functions	715
23.3.1	Review Of Hilbert Schmidt Theorem	717
23.3.2	Measurable Compact Operators	721

23.4	Fubini's Theorem For Bochner Integrals	721
23.5	The Spaces $L^p(\Omega; X)$	724
23.6	Measurable Representatives	732
23.7	Vector Measures	734
23.8	The Riesz Representation Theorem	739
23.9	Maximal Monotone Operators	745
23.9.1	Basic Theory	745
23.9.2	Evolution Inclusions	752
23.10	Exercises	757
24	The Yankov von Neumann Aumann theorem	759
24.1	Multifunctions	770
III	Complex Analysis	773
25	The Complex Numbers	775
25.1	The Extended Complex Plane	777
25.2	Exercises	778
26	Riemann Stieltjes Integrals	779
26.1	Exercises	789
27	Fundamentals Of Complex Analysis	791
27.1	Analytic Functions	791
27.1.1	Cauchy Riemann Equations	793
27.1.2	An Important Example	795
27.2	Exercises	796
27.3	Cauchy's Formula For A Disk	797
27.4	Exercises	804
27.5	Zeros Of An Analytic Function	807
27.6	Liouville's Theorem	809
27.7	The General Cauchy Integral Formula	810
27.7.1	The Cauchy Goursat Theorem	810
27.7.2	A Redundant Assumption	813
27.7.3	Classification Of Isolated Singularities	814
27.7.4	The Cauchy Integral Formula	817
27.7.5	An Example Of A Cycle	824
27.8	Exercises	828
28	The Open Mapping Theorem	831
28.1	A Local Representation	831
28.2	Branches Of The Logarithm	833
28.3	Maximum Modulus Theorem	835
28.4	Extensions Of Maximum Modulus Theorem	837
28.4.1	Phragmén Lindelöf Theorem	837

28.4.2	Hadamard Three Circles Theorem	839
28.4.3	Schwarz’s Lemma	840
28.4.4	One To One Analytic Maps On The Unit Ball	841
28.5	Exercises	842
28.6	Counting Zeros	844
28.7	An Application To Linear Algebra	848
28.8	Exercises	852
29	Residues	855
29.1	Rouche’s Theorem And The Argument Principle	858
29.1.1	Argument Principle	858
29.1.2	Rouche’s Theorem	861
29.1.3	A Different Formulation	862
29.2	Singularities And The Laurent Series	863
29.2.1	What Is An Annulus?	863
29.2.2	The Laurent Series	866
29.2.3	Contour Integrals And Evaluation Of Integrals	870
29.3	Exercises	879
30	Some Important Functional Analysis Applications	881
30.1	The Spectral Radius Of A Bounded Linear Transformation	881
30.2	Analytic Semigroups	884
30.2.1	Sectorial Operators And Analytic Semigroups	884
30.2.2	The Numerical Range	895
30.2.3	An Interesting Example	897
30.2.4	Fractional Powers Of Sectorial Operators	900
30.2.5	A Scale Of Banach Spaces	915
31	Complex Mappings	919
31.1	Conformal Maps	919
31.2	Fractional Linear Transformations	920
31.2.1	Circles And Lines	920
31.2.2	Three Points To Three Points	922
31.3	Riemann Mapping Theorem	923
31.3.1	Montel’s Theorem	924
31.3.2	Regions With Square Root Property	926
31.4	Analytic Continuation	930
31.4.1	Regular And Singular Points	930
31.4.2	Continuation Along A Curve	932
31.5	The Picard Theorems	933
31.5.1	Two Competing Lemmas	935
31.5.2	The Little Picard Theorem	938
31.5.3	Schottky’s Theorem	939
31.5.4	A Brief Review	943
31.5.5	Montel’s Theorem	945

31.5.6 The Great Big Picard Theorem	946
31.6 Exercises	948
32 Approximation By Rational Functions	951
32.1 Runge's Theorem	951
32.1.1 Approximation With Rational Functions	951
32.1.2 Moving The Poles And Keeping The Approximation	953
32.1.3 Merten's Theorem.	953
32.1.4 Runge's Theorem	958
32.2 The Mittag-Leffler Theorem	960
32.2.1 A Proof From Runge's Theorem	960
32.2.2 A Direct Proof Without Runge's Theorem	962
32.2.3 Functions Meromorphic On $\mathbb{C}$	964
32.2.4 A Great And Glorious Theorem About Simply Connected Regions	964
32.3 Exercises	968
33 Infinite Products	969
33.1 Analytic Function With Prescribed Zeros	973
33.2 Factoring A Given Analytic Function	978
33.2.1 Factoring Some Special Analytic Functions	980
33.3 The Existence Of An Analytic Function With Given Values	982
33.4 Jensen's Formula	986
33.5 Blaschke Products	989
33.5.1 The Müntz-Szasz Theorem Again	992
33.6 Exercises	994
34 Elliptic Functions	1003
34.1 Periodic Functions	1004
34.1.1 The Unimodular Transformations	1008
34.1.2 The Search For An Elliptic Function	1011
34.1.3 The Differential Equation Satisfied By $\wp$	1014
34.1.4 A Modular Function	1016
34.1.5 A Formula For λ	1022
34.1.6 Mapping Properties Of λ	1024
34.1.7 A Short Review And Summary	1032
34.2 The Picard Theorem Again	1036
34.3 Exercises	1037
IV Sobolev Spaces	1039
35 Weak Derivatives	1041
35.1 Weak * Convergence	1041
35.2 Test Functions And Weak Derivatives	1042
35.3 Weak Derivatives In L^p_{loc}	1046

35.4	Morrey's Inequality	1049
35.5	Rademacher's Theorem	1052
35.6	Change Of Variables Formula Lipschitz Maps	1055
36	The Area And Coarea Formulas	1065
36.1	The Area Formula Again	1065
36.2	Mappings That Are Not One To One	1068
36.3	The Coarea Formula	1072
36.4	A Nonlinear Fubini's Theorem	1083
37	Integration On Manifolds	1085
37.1	Partitions Of Unity	1085
37.2	Integration On Manifolds	1089
37.3	Comparison With $\mathcal{H}^n$	1095
38	Basic Theory Of Sobolev Spaces	1097
38.1	Embedding Theorems For $W^{m,p}(\mathbb{R}^n)$	1106
38.2	An Extension Theorem	1120
38.3	General Embedding Theorems	1127
38.4	More Extension Theorems	1130
39	Sobolev Spaces Based On L^2	1137
39.1	Fourier Transform Techniques	1137
39.2	Fractional Order Spaces	1142
39.3	An Intrinsic Norm	1144
39.4	Embedding Theorems	1151
39.5	The Trace On The Boundary Of A Half Space	1152
39.6	Sobolev Spaces On Manifolds	1160
39.6.1	General Theory	1160
39.6.2	The Trace On The Boundary	1165
40	Weak Solutions	1171
40.1	The Lax Milgram Theorem	1171
41	Korn's Inequality	1177
41.1	A Fundamental Inequality	1177
41.2	Korn's Inequality	1183
42	Elliptic Regularity And Nirenberg Differences	1185
42.1	The Case Of A Half Space	1185
42.2	The Case Of Bounded Open Sets	1195
43	Interpolation In Banach Space	1205
43.1	An Assortment Of Important Theorems	1205
43.1.1	Weak Vector Valued Derivatives	1205
43.1.2	Some Imbedding Theorems	1215

43.2 The K Method	1220
43.3 The J Method	1225
43.4 Duality And Interpolation	1231
44 Trace Spaces	1243
44.1 Definition And Basic Theory Of Trace Spaces	1243
44.2 Trace And Interpolation Spaces	1250
45 Traces Of Sobolev Spaces And Fractional Order Spaces	1257
45.1 Traces Of Sobolev Spaces On The Boundary Of A Half Space	1257
45.2 A Right Inverse For The Trace For A Half Space	1260
45.3 Intrinsic Norms	1262
45.4 Fractional Order Sobolev Spaces	1283
46 Sobolev Spaces On Manifolds	1289
46.1 Basic Definitions	1289
46.2 The Trace On The Boundary Of An Open Set	1291
V Topics In Probability	1295
47 Basic Probability	1297
47.1 Random Variables And Independence	1297
47.1.1 Independence	1301
47.1.2 Independence For Banach Space Valued Random Variables . .	1304
47.1.3 0,1 Laws	1309
47.1.4 Kolmogorov's Inequality, Strong Law Of Large Numbers . . .	1312
47.2 The Characteristic Function	1318
47.3 Conditional Probability	1319
47.4 Characteristic Functions And Independence	1323
47.5 Characteristic Functions For Measures	1326
47.6 Characteristic Functions And Independence In Banach Space	1329
47.7 Convolution And Sums	1331
47.8 The Convergence Of Sums Of Symmetric Random Variables	1337
47.9 The Multivariate Normal Distribution	1343
47.10 Use Of Characteristic Functions To Find Moments	1350
47.11 The Central Limit Theorem	1351
47.12 Characteristic Functions Of Probability Measures	1359
47.13 Positive Definite Functions, Bochner's Theorem	1363
48 Conditional Expectation And Martingales	1371
48.1 Conditional Expectation	1371
48.2 Discrete Martingales	1374
48.2.1 Upcrossings	1376
48.2.2 The Submartingale Convergence Theorem	1378
48.3 Optional Sampling And Stopping Times	1379

48.3.1	Stopping Times And Their Properties	1379
48.4	Optional Stopping Times And Martingales	1383
48.4.1	Stopping Times And Their Properties	1383
48.4.2	Maximal Inequalities	1389
48.4.3	The Upcrossing Estimate	1391
48.5	The Submartingale Convergence Theorem	1393
48.6	A Reverse Submartingale Convergence Theorem	1397
48.7	Strong Law Of Large Numbers	1399
49	Probability In Infinite Dimensions	1403
49.1	Conditional Expectation In Banach Spaces	1403
49.2	Probability Measures And Tightness	1407
49.3	A Major Existence And Convergence Theorem	1414
49.4	Bochner's Theorem In Infinite Dimensions	1420
49.5	The Multivariate Normal Distribution	1425
49.6	Gaussian Measures	1430
49.6.1	Definitions And Basic Properties	1430
49.6.2	Fernique's Theorem	1434
49.7	Gaussian Measures For A Separable Hilbert Space	1439
49.8	Abstract Wiener Spaces	1449
49.9	White Noise	1463
49.10	Existence Of Abstract Wiener Spaces	1463
50	Stochastic Processes	1469
50.1	Fundamental Definitions And Properties	1469
50.2	Filtrations	1475
50.3	Martingales	1479
50.3.1	Some Maximal Estimates	1480
50.4	Optional Sampling Theorems	1484
50.4.1	Stopping Times And Their Properties	1484
50.4.2	Doob Optional Sampling Theorem	1487
50.5	Doob Optional Sampling Continuous Case	1492
50.5.1	Stopping Times	1492
50.5.2	The Optional Sampling Theorem Continuous Case	1495
50.6	Right Continuity Of Submartingales	1503
50.7	Some Maximal Inequalities	1510
50.8	Continuous Submartingale Convergence Theorem	1513
50.9	Hitting This Before That	1517
50.10	The Space $\mathcal{M}_T^p(E)$	1520
51	The Quadratic Variation Of A Martingale	1525
51.1	How To Recognize A Martingale	1525
51.2	The Burkholder Davis Gundy Inequality	1536
51.3	The Case Of $\mathcal{M}_T^2(H)$	1539
51.4	The Case Of Local Martingales	1542

51.5	The Quadratic Variation And Stochastic Integration	1550
51.6	Doob Meyer Decomposition	1558
51.7	Levy's Theorem	1578
52	Wiener Processes	1587
52.1	Real Wiener Processes	1587
52.2	Nowhere Differentiability Of Wiener Processes	1591
52.3	Wiener Processes In Separable Banach Space	1592
52.4	An Example Of Martingales, Independent Increments	1598
52.5	Hilbert Space Valued Wiener Processes	1602
52.6	Levy's Theorem In Hilbert Space	1616
53	Stochastic Integration	1621
53.1	Integrals Of Elementary Processes	1621
53.2	Integrals Of More General Processes	1634
53.3	Progressively Measurable	1641
53.4	Localization	1645
53.5	The Integral For A More General Case	1651
53.6	Taking Out A Linear Transformation	1655
53.7	The Case Where Q Is Not Trace Class	1657
54	The Ito Formula	1663
54.1	The Case Of A Q Wiener Process	1663
54.2	The Case Where Q Is Not Trace Class	1675
55	Stochastic Differential Equations	1679
55.1	Theorems Based On Contraction Mapping	1679
55.1.1	Predictable And Stochastic Continuity	1679
55.1.2	Existence And Uniqueness For A Stochastic Equation	1683
55.2	Gelfand Triples	1687
55.3	An Ito Formula	1689
55.4	Finite To Infinite Dimensions	1707
55.4.1	Some Nonlinear Operators.	1707
55.4.2	The Situation	1710
55.4.3	The Finite Dimensional Problem, Estimates	1712
55.4.4	Passing To The Limit	1714
55.5	Finite Dimensional Problems	1722
55.5.1	The Ito Formula	1722
55.5.2	An Unbelievably Technical Lemma	1723
55.5.3	An Existence Theorem	1737
55.6	Infinite Dimensional Problems	1746

56 Another Approach, Semigroups	1751
56.1 Stochastic Fubini Theorem	1751
56.2 Correlation Operators	1756
56.3 Weak Solutions To Stochastic Differential Equations	1761
56.3.1 Convolution With A Semigroup	1762
56.3.2 Continuity Of $W_A(t)$	1772
56.3.3 Temporal Regularity Analytic Semigroups	1787
56.3.4 Spacial Regularity	1790
56.3.5 Weak Solutions	1797
A The Hausdorff Maximal Theorem	1807
A.1 The Hamel Basis	1811
A.2 Exercises	1812
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Part I

Review Of Advanced
Calculus

Set Theory

1.1 Basic Definitions

A set is a collection of things called elements of the set. For example, the set of integers, the collection of signed whole numbers such as 1,2,-4, etc. This set whose existence will be assumed is denoted by $\mathbb{Z}$. Other sets could be the set of people in a family or the set of donuts in a display case at the store. Sometimes parentheses, $\{ \}$ specify a set by listing the things which are in the set between the parentheses. For example the set of integers between -1 and 2, including these numbers could be denoted as $\{-1, 0, 1, 2\}$. The notation signifying x is an element of a set S , is written as $x \in S$. Thus, $1 \in \{-1, 0, 1, 2, 3\}$. Here are some axioms about sets. Axioms are statements which are accepted, not proved.

1. Two sets are equal if and only if they have the same elements.
2. To every set, A , and to every condition $S(x)$ there corresponds a set, B , whose elements are exactly those elements x of A for which $S(x)$ holds.
3. For every collection of sets there exists a set that contains all the elements that belong to at least one set of the given collection.
4. The Cartesian product of a nonempty family of nonempty sets is nonempty.
5. If A is a set there exists a set, $\mathcal{P}(A)$ such that $\mathcal{P}(A)$ is the set of all subsets of A . This is called the power set.

These axioms are referred to as the axiom of extension, axiom of specification, axiom of unions, axiom of choice, and axiom of powers respectively.

It seems fairly clear you should want to believe in the axiom of extension. It is merely saying, for example, that $\{1, 2, 3\} = \{2, 3, 1\}$ since these two sets have the same elements in them. Similarly, it would seem you should be able to specify a new set from a given set using some “condition” which can be used as a test to determine whether the element in question is in the set. For example, the set of all integers which are multiples of 2. This set could be specified as follows.

$$\{x \in \mathbb{Z} : x = 2y \text{ for some } y \in \mathbb{Z}\}.$$

In this notation, the colon is read as “such that” and in this case the condition is being a multiple of 2.

Another example of political interest, could be the set of all judges who are not judicial activists. I think you can see this last is not a very precise condition since there is no way to determine to everyone’s satisfaction whether a given judge is an activist. Also, just because something is grammatically correct does not mean it makes any sense. For example consider the following nonsense.

$$S = \{x \in \text{set of dogs} : \text{it is colder in the mountains than in the winter}\}.$$

So what is a condition?

We will leave these sorts of considerations and assume our conditions make sense. The axiom of unions states that for any collection of sets, there is a set consisting of all the elements in each of the sets in the collection. Of course this is also open to further consideration. What is a collection? Maybe it would be better to say “set of sets” or, given a set whose elements are sets there exists a set whose elements consist of exactly those things which are elements of at least one of these sets. If $\mathcal{S}$ is such a set whose elements are sets,

$$\cup \{A : A \in \mathcal{S}\} \text{ or } \cup \mathcal{S}$$

signify this union.

Something is in the Cartesian product of a set or “family” of sets if it consists of a single thing taken from each set in the family. Thus $(1, 2, 3) \in \{1, 4, 2\} \times \{1, 2, 7\} \times \{4, 3, 7, 9\}$ because it consists of exactly one element from each of the sets which are separated by $\times$. Also, this is the notation for the Cartesian product of finitely many sets. If $\mathcal{S}$ is a set whose elements are sets,

$$\prod_{A \in \mathcal{S}} A$$

signifies the Cartesian product.

The Cartesian product is the set of choice functions, a choice function being a function which selects exactly one element of each set of $\mathcal{S}$. You may think the axiom of choice, stating that the Cartesian product of a nonempty family of nonempty sets is nonempty, is innocuous but there was a time when many mathematicians were ready to throw it out because it implies things which are very hard to believe, things which never happen without the axiom of choice.

A is a subset of B , written $A \subseteq B$, if every element of A is also an element of B . This can also be written as $B \supseteq A$. A is a proper subset of B , written $A \subset B$ or $B \supset A$ if A is a subset of B but A is not equal to B , $A \neq B$. $A \cap B$ denotes the intersection of the two sets, A and B and it means the set of elements of A which are also elements of B . The axiom of specification shows this is a set. The empty set is the set which has no elements in it, denoted as $\emptyset$. $A \cup B$ denotes the union of the two sets, A and B and it means the set of all elements which are in either of the sets. It is a set because of the axiom of unions.

The complement of a set, (the set of things which are not in the given set) must be taken with respect to a given set called the universal set which is a set which contains the one whose complement is being taken. Thus, the complement of A , denoted as A^C (or more precisely as $X \setminus A$) is a set obtained from using the axiom of specification to write

$$A^C \equiv \{x \in X : x \notin A\}$$

The symbol $\notin$ means: “is not an element of”. Note the axiom of specification takes place relative to a given set. Without this universal set it makes no sense to use the axiom of specification to obtain the complement.

Words such as “all” or “there exists” are called quantifiers and they must be understood relative to some given set. For example, the set of all integers larger than 3. Or there exists an integer larger than 7. Such statements have to do with a given set, in this case the integers. Failure to have a reference set when quantifiers are used turns out to be illogical even though such usage may be grammatically correct. Quantifiers are used often enough that there are symbols for them. The symbol $\forall$ is read as “for all” or “for every” and the symbol $\exists$ is read as “there exists”. Thus $\forall \exists \exists$ could mean for every upside down A there exists a backwards E .

DeMorgan’s laws are very useful in mathematics. Let $\mathcal{S}$ be a set of sets each of which is contained in some universal set, U . Then

$$\cup \{A^C : A \in \mathcal{S}\} = (\cap \{A : A \in \mathcal{S}\})^C$$

and

$$\cap \{A^C : A \in \mathcal{S}\} = (\cup \{A : A \in \mathcal{S}\})^C.$$

These laws follow directly from the definitions. Also following directly from the definitions are:

Let $\mathcal{S}$ be a set of sets then

$$B \cup \cup \{A : A \in \mathcal{S}\} = \cup \{B \cup A : A \in \mathcal{S}\}.$$

and: Let $\mathcal{S}$ be a set of sets show

$$B \cap \cup \{A : A \in \mathcal{S}\} = \cup \{B \cap A : A \in \mathcal{S}\}.$$

Unfortunately, there is no single universal set which can be used for all sets. Here is why: Suppose there were. Call it S . Then you could consider A the set of all elements of S which are not elements of themselves, this from the axiom of specification. If A is an element of itself, then it fails to qualify for inclusion in A . Therefore, it must not be an element of itself. However, if this is so, it qualifies for inclusion in A so it is an element of itself and so this can’t be true either. Thus the most basic of conditions you could imagine, that of being an element of, is meaningless and so allowing such a set causes the whole theory to be meaningless. The solution is to not allow a universal set. As mentioned by Halmos in Naive set theory, “Nothing contains everything”. Always beware of statements involving quantifiers wherever they occur, even this one.

1.2 The Schroder Bernstein Theorem

It is very important to be able to compare the size of sets in a rational way. The most useful theorem in this context is the Schroder Bernstein theorem which is the main result to be presented in this section. The Cartesian product is discussed above. The next definition reviews this and defines the concept of a function.

Definition 1.2.1 *Let X and Y be sets.*

$$X \times Y \equiv \{(x, y) : x \in X \text{ and } y \in Y\}$$

A relation is defined to be a subset of $X \times Y$. A function, f , also called a mapping, is a relation which has the property that if (x, y) and (x, y_1) are both elements of the f , then $y = y_1$. The domain of f is defined as

$$D(f) \equiv \{x : (x, y) \in f\},$$

written as $f : D(f) \rightarrow Y$.

It is probably safe to say that most people do not think of functions as a type of relation which is a subset of the Cartesian product of two sets. A function is like a machine which takes inputs, x and makes them into a unique output, $f(x)$. Of course, that is what the above definition says with more precision. An ordered pair, (x, y) which is an element of the function or mapping has an input, x and a unique output, y , denoted as $f(x)$ while the name of the function is f . “mapping” is often a noun meaning function. However, it also is a verb as in “ f is mapping A to B ”. That which a function is thought of as doing is also referred to using the word “maps” as in: f maps X to Y . However, a set of functions may be called a set of maps so this word might also be used as the plural of a noun. There is no help for it. You just have to suffer with this nonsense.

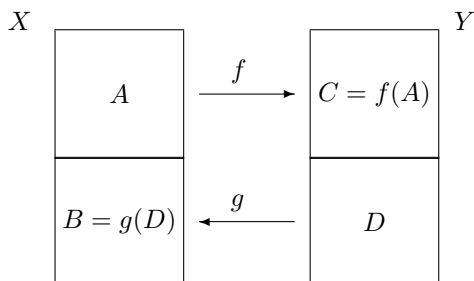
The following theorem which is interesting for its own sake will be used to prove the Schroder Bernstein theorem.

Theorem 1.2.2 *Let $f : X \rightarrow Y$ and $g : Y \rightarrow X$ be two functions. Then there exist sets A, B, C, D , such that*

$$A \cup B = X, C \cup D = Y, A \cap B = \emptyset, C \cap D = \emptyset,$$

$$f(A) = C, g(D) = B.$$

The following picture illustrates the conclusion of this theorem.



Proof: Consider the empty set, $\emptyset \subseteq X$. If $y \in Y \setminus f(\emptyset)$, then $g(y) \notin \emptyset$ because $\emptyset$ has no elements. Also, if A, B, C , and D are as described above, A also would have this same property that the empty set has. However, A is probably larger. Therefore, say $A_0 \subseteq X$ satisfies $\mathcal{P}$ if whenever $y \in Y \setminus f(A_0)$, $g(y) \notin A_0$.

$$\mathcal{A} \equiv \{A_0 \subseteq X : A_0 \text{ satisfies } \mathcal{P}\}.$$

Let $A = \cup \mathcal{A}$. If $y \in Y \setminus f(A)$, then for each $A_0 \in \mathcal{A}$, $y \in Y \setminus f(A_0)$ and so $g(y) \notin A_0$. Since $g(y) \notin A_0$ for all $A_0 \in \mathcal{A}$, it follows $g(y) \notin A$. Hence A satisfies $\mathcal{P}$ and is the largest subset of X which does so. Now define

$$C \equiv f(A), D \equiv Y \setminus C, B \equiv X \setminus A.$$

It only remains to verify that $g(D) = B$.

Suppose $x \in B = X \setminus A$. Then $A \cup \{x\}$ does not satisfy $\mathcal{P}$ and so there exists $y \in Y \setminus f(A \cup \{x\}) \subseteq D$ such that $g(y) \in A \cup \{x\}$. But $y \notin f(A)$ and so since A satisfies $\mathcal{P}$, it follows $g(y) \notin A$. Hence $g(y) = x$ and so $x \in g(D)$ and this proves the theorem.

Theorem 1.2.3 (Schroder Bernstein) *If $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are one to one, then there exists $h : X \rightarrow Y$ which is one to one and onto.*

Proof: Let A, B, C, D be the sets of Theorem 1.2.2 and define

$$h(x) \equiv \begin{cases} f(x) & \text{if } x \in A \\ g^{-1}(x) & \text{if } x \in B \end{cases}$$

Then h is the desired one to one and onto mapping.

Recall that the Cartesian product may be considered as the collection of choice functions.

Definition 1.2.4 *Let I be a set and let X_i be a set for each $i \in I$. f is a choice function written as*

$$f \in \prod_{i \in I} X_i$$

if $f(i) \in X_i$ for each $i \in I$.

The axiom of choice says that if $X_i \neq \emptyset$ for each $i \in I$, for I a set, then

$$\prod_{i \in I} X_i \neq \emptyset.$$

Sometimes the two functions, f and g are onto but not one to one. It turns out that with the axiom of choice, a similar conclusion to the above may be obtained.

Corollary 1.2.5 *If $f : X \rightarrow Y$ is onto and $g : Y \rightarrow X$ is onto, then there exists $h : X \rightarrow Y$ which is one to one and onto.*

Proof: For each $y \in Y$, $f^{-1}(y) \equiv \{x \in X : f(x) = y\} \neq \emptyset$. Therefore, by the axiom of choice, there exists $f_0^{-1} \in \prod_{y \in Y} f^{-1}(y)$ which is the same as saying that for each $y \in Y$, $f_0^{-1}(y) \in f^{-1}(y)$. Similarly, there exists $g_0^{-1}(x) \in g^{-1}(x)$ for all $x \in X$. Then f_0^{-1} is one to one because if $f_0^{-1}(y_1) = f_0^{-1}(y_2)$, then

$$y_1 = f(f_0^{-1}(y_1)) = f(f_0^{-1}(y_2)) = y_2.$$

Similarly g_0^{-1} is one to one. Therefore, by the Schroder Bernstein theorem, there exists $h : X \rightarrow Y$ which is one to one and onto.

Definition 1.2.6 A set S , is finite if there exists a natural number n and a map θ which maps $\{1, \dots, n\}$ one to one and onto S . S is infinite if it is not finite. A set S , is called countable if there exists a map θ mapping $\mathbb{N}$ one to one and onto S . (When θ maps a set A to a set B , this will be written as $\theta : A \rightarrow B$ in the future.) Here $\mathbb{N} \equiv \{1, 2, \dots$ the natural numbers. S is at most countable if there exists a map $\theta : \mathbb{N} \rightarrow S$ which is onto.

The property of being at most countable is often referred to as being countable because the question of interest is normally whether one can list all elements of the set, designating a first, second, third etc. in such a way as to give each element of the set a natural number. The possibility that a single element of the set may be counted more than once is often not important.

Theorem 1.2.7 If X and Y are both at most countable, then $X \times Y$ is also at most countable. If either X or Y is countable, then $X \times Y$ is also countable.

Proof: It is given that there exists a mapping $\eta : \mathbb{N} \rightarrow X$ which is onto. Define $\eta(i) \equiv x_i$ and consider X as the set $\{x_1, x_2, x_3, \dots\}$. Similarly, consider Y as the set $\{y_1, y_2, y_3, \dots\}$. It follows the elements of $X \times Y$ are included in the following rectangular array.

$$\begin{array}{ccccccc} (x_1, y_1) & (x_1, y_2) & (x_1, y_3) & \cdots & \leftarrow & \text{Those which have } x_1 \text{ in first slot.} \\ (x_2, y_1) & (x_2, y_2) & (x_2, y_3) & \cdots & \leftarrow & \text{Those which have } x_2 \text{ in first slot.} \\ (x_3, y_1) & (x_3, y_2) & (x_3, y_3) & \cdots & \leftarrow & \text{Those which have } x_3 \text{ in first slot.} \\ \vdots & \vdots & \vdots & & & \vdots \end{array}$$

Follow a path through this array as follows.

$$\begin{array}{ccccc} (x_1, y_1) & \rightarrow & (x_1, y_2) & & (x_1, y_3) \rightarrow \\ & \searrow & & \nearrow & \\ (x_2, y_1) & & (x_2, y_2) & & \\ \downarrow & \nearrow & & & \\ (x_3, y_1) & & & & \end{array}$$

Thus the first element of $X \times Y$ is (x_1, y_1) , the second element of $X \times Y$ is (x_1, y_2) , the third element of $X \times Y$ is (x_2, y_1) etc. This assigns a number from $\mathbb{N}$ to each element of $X \times Y$. Thus $X \times Y$ is at most countable.

It remains to show the last claim. Suppose without loss of generality that X is countable. Then there exists $\alpha : \mathbb{N} \rightarrow X$ which is one to one and onto. Let $\beta : X \times Y \rightarrow \mathbb{N}$ be defined by $\beta((x, y)) \equiv \alpha^{-1}(x)$. Thus β is onto $\mathbb{N}$. By the first part there exists a function from $\mathbb{N}$ onto $X \times Y$. Therefore, by Corollary 1.2.5, there exists a one to one and onto mapping from $X \times Y$ to $\mathbb{N}$. This proves the theorem.

Theorem 1.2.8 *If X and Y are at most countable, then $X \cup Y$ is at most countable. If either X or Y are countable, then $X \cup Y$ is countable.*

Proof: As in the preceding theorem,

$$X = \{x_1, x_2, x_3, \dots\}$$

and

$$Y = \{y_1, y_2, y_3, \dots\}.$$

Consider the following array consisting of $X \cup Y$ and path through it.

$$\begin{array}{ccccc} x_1 & \rightarrow & x_2 & & x_3 & \rightarrow \\ & \swarrow & & \nearrow & & \\ y_1 & \rightarrow & y_2 & & & \end{array}$$

Thus the first element of $X \cup Y$ is x_1 , the second is x_2 the third is y_1 the fourth is y_2 etc.

Consider the second claim. By the first part, there is a map from $\mathbb{N}$ onto $X \times Y$. Suppose without loss of generality that X is countable and $\alpha : \mathbb{N} \rightarrow X$ is one to one and onto. Then define $\beta(y) \equiv 1$, for all $y \in Y$, and $\beta(x) \equiv \alpha^{-1}(x)$. Thus, β maps $X \times Y$ onto $\mathbb{N}$ and this shows there exist two onto maps, one mapping $X \cup Y$ onto $\mathbb{N}$ and the other mapping $\mathbb{N}$ onto $X \cup Y$. Then Corollary 1.2.5 yields the conclusion. This proves the theorem.

1.3 Equivalence Relations

There are many ways to compare elements of a set other than to say two elements are equal or the same. For example, in the set of people let two people be equivalent if they have the same weight. This would not be saying they were the same person, just that they weighed the same. Often such relations involve considering one characteristic of the elements of a set and then saying the two elements are equivalent if they are the same as far as the given characteristic is concerned.

Definition 1.3.1 *Let S be a set. $\sim$ is an equivalence relation on S if it satisfies the following axioms.*

1. $x \sim x$ for all $x \in S$. (Reflexive)
2. If $x \sim y$ then $y \sim x$. (Symmetric)

3. If $x \sim y$ and $y \sim z$, then $x \sim z$. (Transitive)

Definition 1.3.2 $[x]$ denotes the set of all elements of S which are equivalent to x and $[x]$ is called the equivalence class determined by x or just the equivalence class of x .

With the above definition one can prove the following simple theorem.

Theorem 1.3.3 Let $\sim$ be an equivalence class defined on a set, S and let $\mathcal{H}$ denote the set of equivalence classes. Then if $[x]$ and $[y]$ are two of these equivalence classes, either $x \sim y$ and $[x] = [y]$ or it is not true that $x \sim y$ and $[x] \cap [y] = \emptyset$.

1.4 Partially Ordered Sets

Definition 1.4.1 Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that

$$x \leq x \text{ for all } x \in \mathcal{F}.$$

$$\text{If } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 1.4.2 (Hausdorff Maximal Principle) Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.

Continuous Functions Of One Variable

There is a theorem about the integral of a continuous function which requires the notion of uniform continuity. This is discussed in this section. Consider the function $f(x) = \frac{1}{x}$ for $x \in (0, 1)$. This is a continuous function because, it is continuous at every point of $(0, 1)$. However, for a given $\varepsilon > 0$, the δ needed in the ε, δ definition of continuity becomes very small as x gets close to 0. The notion of uniform continuity involves being able to choose a single δ which works on the whole domain of f . Here is the definition.

Definition 2.0.3 *Let $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then f is uniformly continuous if for every $\varepsilon > 0$, there exists a δ **depending only on** ε such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$.*

It is an amazing fact that under certain conditions continuity implies uniform continuity.

Definition 2.0.4 *A set, $K \subseteq \mathbb{R}$ is sequentially compact if whenever $\{a_n\} \subseteq K$ is a sequence, there exists a subsequence, $\{a_{n_k}\}$ such that this subsequence converges to a point of K .*

The following theorem is part of the Heine Borel theorem.

Theorem 2.0.5 *Every closed interval, $[a, b]$ is sequentially compact.*

Proof: Let $\{x_n\} \subseteq [a, b] \equiv I_0$. Consider the two intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ each of which has length $(b - a)/2$. At least one of these intervals contains x_n for infinitely many values of n . Call this interval I_1 . Now do for I_1 what was done for I_0 . Split it in half and let I_2 be the interval which contains x_n for infinitely many values of n . Continue this way obtaining a sequence of nested intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots$ where the length of I_n is $(b - a)/2^n$. Now pick n_1 such that $x_{n_1} \in I_1$, n_2 such that $n_2 > n_1$ and $x_{n_2} \in I_2$, n_3 such that $n_3 > n_2$ and $x_{n_3} \in I_3$, etc. (This can be done because in each case the intervals contained x_n for infinitely many values of n .) By

the nested interval lemma there exists a point, c contained in all these intervals. Furthermore,

$$|x_{n_k} - c| < (b - a) 2^{-k}$$

and so $\lim_{k \rightarrow \infty} x_{n_k} = c \in [a, b]$. This proves the theorem.

Theorem 2.0.6 *Let $f : K \rightarrow \mathbb{R}$ be continuous where K is a sequentially compact set in $\mathbb{R}$. Then f is uniformly continuous on K .*

Proof: If this is not true, there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a pair of points, x_δ and y_δ such that even though $|x_\delta - y_\delta| < \delta$, $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. Taking a succession of values for δ equal to $1, 1/2, 1/3, \dots$, and letting the exceptional pair of points for $\delta = 1/n$ be denoted by x_n and y_n ,

$$|x_n - y_n| < \frac{1}{n}, |f(x_n) - f(y_n)| \geq \varepsilon.$$

Now since K is sequentially compact, there exists a subsequence, $\{x_{n_k}\}$ such that $x_{n_k} \rightarrow z \in K$. Now $n_k \geq k$ and so

$$|x_{n_k} - y_{n_k}| < \frac{1}{k}.$$

Consequently, $y_{n_k} \rightarrow z$ also. (x_{n_k} is like a person walking toward a certain point and y_{n_k} is like a dog on a leash which is constantly getting shorter. Obviously y_{n_k} must also move toward the point also. You should give a precise proof of what is needed here.) By continuity of f

$$0 = |f(z) - f(z)| = \lim_{k \rightarrow \infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon,$$

an obvious contradiction. Therefore, the theorem must be true.

The following corollary follows from this theorem and Theorem 2.0.5.

Corollary 2.0.7 *Suppose I is a closed interval, $I = [a, b]$ and $f : I \rightarrow \mathbb{R}$ is continuous. Then f is uniformly continuous.*

2.1 Exercises

1. A function, $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, K such that

$$|f(x) - f(y)| \leq K|x - y|$$

for all $x, y \in D$. Show every Lipschitz function is uniformly continuous.

2. If $|x_n - y_n| \rightarrow 0$ and $x_n \rightarrow z$, show that $y_n \rightarrow z$ also.
3. Consider $f : (1, \infty) \rightarrow \mathbb{R}$ given by $f(x) = \frac{1}{x}$. Show f is uniformly continuous even though the set on which f is defined is not sequentially compact.

4. If f is uniformly continuous, does it follow that $|f|$ is also uniformly continuous? If $|f|$ is uniformly continuous does it follow that f is uniformly continuous? Answer the same questions with “uniformly continuous” replaced with “continuous”. Explain why.

2.2 Theorems About Continuous Functions

In this section, proofs of some theorems which have not been proved yet are given.

Theorem 2.2.1 *The following assertions are valid*

1. *The function, $af + bg$ is continuous at x when f, g are continuous at $x \in D(f) \cap D(g)$ and $a, b \in \mathbb{R}$.*
2. *If f and g are each real valued functions continuous at x , then fg is continuous at x . If, in addition to this, $g(x) \neq 0$, then f/g is continuous at x .*
3. *If f is continuous at x , $f(x) \in D(g) \subseteq \mathbb{R}$, and g is continuous at $f(x)$, then $g \circ f$ is continuous at x .*
4. *The function $f : \mathbb{R} \rightarrow \mathbb{R}$, given by $f(x) = |x|$ is continuous.*

Proof: First consider 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|x - y| < \delta_1$, it follows $|f(x) - f(y)| < \frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\delta_2 > 0$ such that whenever $|x - y| < \delta_2$, it follows that $|g(x) - g(y)| < \frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|x - y| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$\begin{aligned} & |af(x) + bf(x) - (ag(y) + bg(y))| \\ & \leq |a||f(x) - f(y)| + |b||g(x) - g(y)| \\ & < |a|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right) + |b|\left(\frac{\varepsilon}{2(|a|+|b|+1)}\right) < \varepsilon. \end{aligned}$$

Now consider 2.) There exists $\delta_1 > 0$ such that if $|y - x| < \delta_1$, then

$$|f(x) - f(y)| < 1.$$

Therefore, for such y ,

$$|f(y)| < 1 + |f(x)|.$$

It follows that for such y ,

$$|fg(x) - fg(y)| \leq |f(x)g(x) - g(x)f(y)| + |g(x)f(y) - f(y)g(y)|$$

$$\begin{aligned} &\leq |g(x)| |f(x) - f(y)| + |f(y)| |g(x) - g(y)| \\ &\leq (1 + |g(x)| + |f(y)|) [|g(x) - g(y)| + |f(x) - f(y)|]. \end{aligned}$$

Now let $\varepsilon > 0$ be given. There exists δ_2 such that if $|x - y| < \delta_2$, then

$$|g(x) - g(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)},$$

and there exists δ_3 such that if $|x - y| < \delta_3$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)}$$

Now let $0 < \delta \leq \min(\delta_1, \delta_2, \delta_3)$. Then if $|x - y| < \delta$, all the above hold at once and so

$$\begin{aligned} &|fg(x) - fg(y)| \leq \\ &(1 + |g(x)| + |f(y)|) [|g(x) - g(y)| + |f(x) - f(y)|] \\ &< (1 + |g(x)| + |f(y)|) \left(\frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} + \frac{\varepsilon}{2(1 + |g(x)| + |f(y)|)} \right) = \varepsilon. \end{aligned}$$

This proves the first part of 2.) To obtain the second part, let δ_1 be as described above and let $\delta_0 > 0$ be such that for $|x - y| < \delta_0$,

$$|g(x) - g(y)| < |g(x)|/2$$

and so by the triangle inequality,

$$-|g(x)|/2 \leq |g(y)| - |g(x)| \leq |g(x)|/2$$

which implies $|g(y)| \geq |g(x)|/2$, and $|g(y)| < 3|g(x)|/2$.

Then if $|x - y| < \min(\delta_0, \delta_1)$,

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| &= \left| \frac{f(x)g(y) - f(y)g(x)}{g(x)g(y)} \right| \\ &\leq \frac{|f(x)g(y) - f(y)g(x)|}{\left(\frac{|g(x)|^2}{2}\right)} \\ &= \frac{2|f(x)g(y) - f(y)g(x)|}{|g(x)|^2} \\ &\leq \frac{2}{|g(x)|^2} [|f(x)g(y) - f(y)g(y) + f(y)g(y) - f(y)g(x)|] \\ &\leq \frac{2}{|g(x)|^2} [|g(y)| |f(x) - f(y)| + |f(y)| |g(y) - g(x)|] \\ &\leq \frac{2}{|g(x)|^2} \left[\frac{3}{2} |g(x)| |f(x) - f(y)| + (1 + |f(x)|) |g(y) - g(x)| \right] \\ &\leq \frac{2}{|g(x)|^2} (1 + 2|f(x)| + 2|g(x)|) [|f(x) - f(y)| + |g(y) - g(x)|] \\ &\equiv M [|f(x) - f(y)| + |g(y) - g(x)|] \end{aligned}$$

where M is defined by

$$M \equiv \frac{2}{|g(x)|^2} (1 + 2|f(x)| + 2|g(x)|)$$

Now let δ_2 be such that if $|x-y| < \delta_2$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2} M^{-1}$$

and let δ_3 be such that if $|x-y| < \delta_3$, then

$$|g(y) - g(x)| < \frac{\varepsilon}{2} M^{-1}.$$

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|x-y| < \delta$, everything holds and

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| &\leq M [|f(x) - f(y)| + |g(y) - g(x)|] \\ &< M \left[\frac{\varepsilon}{2} M^{-1} + \frac{\varepsilon}{2} M^{-1} \right] = \varepsilon. \end{aligned}$$

This completes the proof of the second part of 2.)

Note that in these proofs no effort is made to find some sort of “best” δ . The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now consider 3.). If f is continuous at x , $f(x) \in D(g) \subseteq \mathbb{R}^p$, and g is continuous at $f(x)$, then $g \circ f$ is continuous at x . Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$ such that if $|y - f(x)| < \eta$ and $y \in D(g)$, it follows that $|g(y) - g(f(x))| < \varepsilon$. From continuity of f at x , there exists $\delta > 0$ such that if $|x - z| < \delta$ and $z \in D(f)$, then $|f(z) - f(x)| < \eta$. Then if $|x - z| < \delta$ and $z \in D(g \circ f) \subseteq D(f)$, all the above hold and so

$$|g(f(z)) - g(f(x))| < \varepsilon.$$

This proves part 3.)

To verify part 4.), let $\varepsilon > 0$ be given and let $\delta = \varepsilon$. Then if $|x-y| < \delta$, the triangle inequality implies

$$\begin{aligned} |f(x) - f(y)| &= ||x| - |y|| \\ &\leq |x-y| < \delta = \varepsilon. \end{aligned}$$

This proves part 4.) and completes the proof of the theorem.

Next here is a proof of the intermediate value theorem.

Theorem 2.2.2 Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and suppose $f(a) < c < f(b)$. Then there exists $x \in (a, b)$ such that $f(x) = c$.

Proof: Let $d = \frac{a+b}{2}$ and consider the intervals $[a, d]$ and $[d, b]$. If $f(d) \geq c$, then on $[a, d]$, the function is $\leq c$ at one end point and $\geq c$ at the other. On the other hand, if $f(d) \leq c$, then on $[d, b]$ $f \geq 0$ at one end point and ≤ 0 at the

other. Pick the interval on which f has values which are at least as large as c and values no larger than c . Now consider that interval, divide it in half as was done for the original interval and argue that on one of these smaller intervals, the function has values at least as large as c and values no larger than c . Continue in this way. Next apply the nested interval lemma to get x in all these intervals. In the n^{th} interval, let x_n, y_n be elements of this interval such that $f(x_n) \leq c, f(y_n) \geq c$. Now $|x_n - x| \leq (b - a) 2^{-n}$ and $|y_n - x| \leq (b - a) 2^{-n}$ and so $x_n \rightarrow x$ and $y_n \rightarrow x$. Therefore,

$$f(x) - c = \lim_{n \rightarrow \infty} (f(x_n) - c) \leq 0$$

while

$$f(x) - c = \lim_{n \rightarrow \infty} (f(y_n) - c) \geq 0.$$

Consequently $f(x) = c$ and this proves the theorem.

Lemma 2.2.3 *Let $\phi : [a, b] \rightarrow \mathbb{R}$ be a continuous function and suppose ϕ is 1 - 1 on (a, b) . Then ϕ is either strictly increasing or strictly decreasing on $[a, b]$.*

Proof: First it is shown that ϕ is either strictly increasing or strictly decreasing on (a, b) .

If ϕ is not strictly decreasing on (a, b) , then there exists $x_1 < y_1, x_1, y_1 \in (a, b)$ such that

$$(\phi(y_1) - \phi(x_1))(y_1 - x_1) > 0.$$

If for some other pair of points, $x_2 < y_2$ with $x_2, y_2 \in (a, b)$, the above inequality does not hold, then since ϕ is 1 - 1,

$$(\phi(y_2) - \phi(x_2))(y_2 - x_2) < 0.$$

Let $x_t \equiv tx_1 + (1 - t)x_2$ and $y_t \equiv ty_1 + (1 - t)y_2$. Then $x_t < y_t$ for all $t \in [0, 1]$ because

$$tx_1 \leq ty_1 \text{ and } (1 - t)x_2 \leq (1 - t)y_2$$

with strict inequality holding for at least one of these inequalities since not both t and $(1 - t)$ can equal zero. Now define

$$h(t) \equiv (\phi(y_t) - \phi(x_t))(y_t - x_t).$$

Since h is continuous and $h(0) < 0$, while $h(1) > 0$, there exists $t \in (0, 1)$ such that $h(t) = 0$. Therefore, both x_t and y_t are points of (a, b) and $\phi(y_t) - \phi(x_t) = 0$ contradicting the assumption that ϕ is one to one. It follows ϕ is either strictly increasing or strictly decreasing on (a, b) .

This property of being either strictly increasing or strictly decreasing on (a, b) carries over to $[a, b]$ by the continuity of ϕ . Suppose ϕ is strictly increasing on (a, b) , a similar argument holding for ϕ strictly decreasing on (a, b) . If $x > a$, then pick $y \in (a, x)$ and from the above, $\phi(y) < \phi(x)$. Now by continuity of ϕ at a ,

$$\phi(a) = \lim_{x \rightarrow a+} \phi(z) \leq \phi(y) < \phi(x).$$

Therefore, $\phi(a) < \phi(x)$ whenever $x \in (a, b)$. Similarly $\phi(b) > \phi(x)$ for all $x \in (a, b)$. This proves the lemma.

Corollary 2.2.4 *Let $f : (a, b) \rightarrow \mathbb{R}$ be one to one and continuous. Then $f(a, b)$ is an open interval, (c, d) and $f^{-1} : (c, d) \rightarrow (a, b)$ is continuous.*

Proof: Since f is either strictly increasing or strictly decreasing, it follows that $f(a, b)$ is an open interval, (c, d) . Assume f is decreasing. Now let $x \in (a, b)$. Why is f^{-1} continuous at $f(x)$? Since f is decreasing, if $f(x) < f(y)$, then $y \equiv f^{-1}(f(y)) < x \equiv f^{-1}(f(x))$ and so f^{-1} is also decreasing. Let $\varepsilon > 0$ be given. Let $\varepsilon > \eta > 0$ and $(x - \eta, x + \eta) \subseteq (a, b)$. Then $f(x) \in (f(x + \eta), f(x - \eta))$. Let $\delta = \min(f(x) - f(x + \eta), f(x - \eta) - f(x))$. Then if

$$|f(z) - f(x)| < \delta,$$

it follows

$$z \equiv f^{-1}(f(z)) \in (x - \eta, x + \eta) \subseteq (x - \varepsilon, x + \varepsilon)$$

so

$$|f^{-1}(f(z)) - x| = |f^{-1}(f(z)) - f^{-1}(f(x))| < \varepsilon.$$

This proves the theorem in the case where f is strictly decreasing. The case where f is increasing is similar.

The Riemann Stieltjes Integral

The integral originated in attempts to find areas of various shapes and the ideas involved in finding integrals are much older than the ideas related to finding derivatives. In fact, Archimedes¹ was finding areas of various curved shapes about 250 B.C. using the main ideas of the integral. What is presented here is a generalization of these ideas. The main interest is in the Riemann integral but if it is easy to generalize to the so called Stieltjes integral in which the length of an interval, $[x, y]$ is replaced with an expression of the form $F(y) - F(x)$ where F is an increasing function, then the generalization is given. However, there is much more that can be written about Stieltjes integrals than what is presented here. A good source for this is the book by Apostol, [3].

3.1 Upper And Lower Riemann Stieltjes Sums

The Riemann integral pertains to bounded functions which are defined on a bounded interval. Let $[a, b]$ be a closed interval. A set of points in $[a, b]$, $\{x_0, \dots, x_n\}$ is a partition if

$$a = x_0 < x_1 < \dots < x_n = b.$$

Such partitions are denoted by P or Q . For f a bounded function defined on $[a, b]$, let

$$\begin{aligned} M_i(f) &\equiv \sup\{f(x) : x \in [x_{i-1}, x_i]\}, \\ m_i(f) &\equiv \inf\{f(x) : x \in [x_{i-1}, x_i]\}. \end{aligned}$$

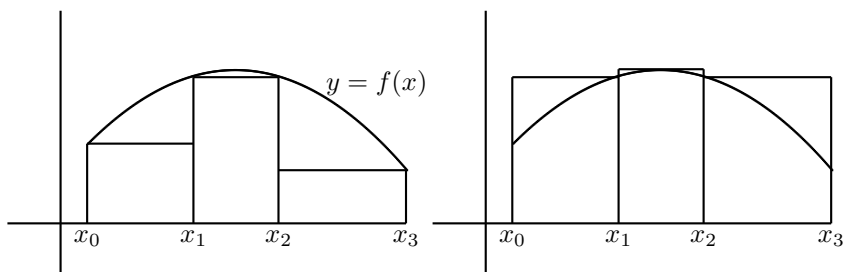
¹Archimedes 287-212 B.C. found areas of curved regions by stuffing them with simple shapes which he knew the area of and taking a limit. He also made fundamental contributions to physics. The story is told about how he determined that a gold smith had cheated the king by giving him a crown which was not solid gold as had been claimed. He did this by finding the amount of water displaced by the crown and comparing with the amount of water it should have displaced if it had been solid gold.

Definition 3.1.1 Let F be an increasing function defined on $[a, b]$ and let $\Delta F_i \equiv F(x_i) - F(x_{i-1})$. Then define upper and lower sums as

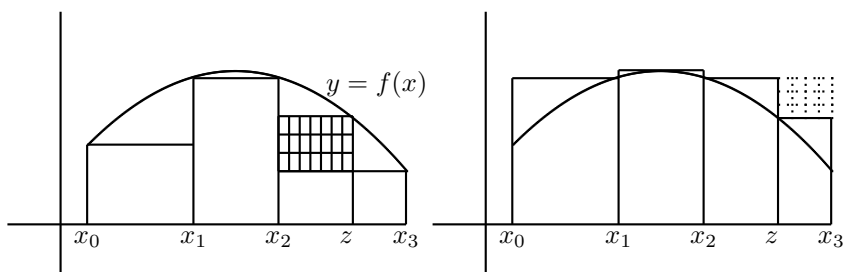
$$U(f, P) \equiv \sum_{i=1}^n M_i(f) \Delta F_i \text{ and } L(f, P) \equiv \sum_{i=1}^n m_i(f) \Delta F_i$$

respectively. The numbers, $M_i(f)$ and $m_i(f)$, are well defined real numbers because f is assumed to be bounded and $\mathbb{R}$ is complete. Thus the set $S = \{f(x) : x \in [x_{i-1}, x_i]\}$ is bounded above and below.

In the following picture, the sum of the areas of the rectangles in the picture on the left is a lower sum for the function in the picture and the sum of the areas of the rectangles in the picture on the right is an upper sum for the same function which uses the same partition. In these pictures the function, F is given by $F(x) = x$ and these are the ordinary upper and lower sums from calculus.



What happens when you add in more points in a partition? The following pictures illustrate in the context of the above example. In this example a single additional point, labeled z has been added in.



Note how the lower sum got larger by the amount of the area in the shaded rectangle and the upper sum got smaller by the amount in the rectangle shaded by dots. In general this is the way it works and this is shown in the following lemma.

Lemma 3.1.2 If $P \subseteq Q$ then

$$U(f, Q) \leq U(f, P), \text{ and } L(f, P) \leq L(f, Q).$$

Proof: This is verified by adding in one point at a time. Thus let

$$P = \{x_0, \dots, x_n\}$$

and let

$$Q = \{x_0, \dots, x_k, y, x_{k+1}, \dots, x_n\}.$$

Thus exactly one point, y , is added between x_k and x_{k+1} . Now the term in the upper sum which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f, P)$ is

$$\sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(x_k)) \quad (3.1.1)$$

and the term which corresponds to the interval $[x_k, x_{k+1}]$ in $U(f, Q)$ is

$$\begin{aligned} & \sup \{f(x) : x \in [x_k, y]\} (F(y) - F(x_k)) \\ & + \sup \{f(x) : x \in [y, x_{k+1}]\} (F(x_{k+1}) - F(y)) \\ & \equiv M_1 (F(y) - F(x_k)) + M_2 (F(x_{k+1}) - F(y)) \end{aligned} \quad (3.1.2)$$

All the other terms in the two sums coincide. Now $\sup \{f(x) : x \in [x_k, x_{k+1}]\} \geq \max(M_1, M_2)$ and so the expression in 3.1.2 is no larger than

$$\begin{aligned} & \sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(y)) \\ & + \sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(y) - F(x_k)) \\ & = \sup \{f(x) : x \in [x_k, x_{k+1}]\} (F(x_{k+1}) - F(x_k)), \end{aligned}$$

the term corresponding to the interval, $[x_k, x_{k+1}]$ and $U(f, P)$. This proves the first part of the lemma pertaining to upper sums because if $Q \supseteq P$, one can obtain Q from P by adding in one point at a time and each time a point is added, the corresponding upper sum either gets smaller or stays the same. The second part about lower sums is similar and is left as an exercise.

Lemma 3.1.3 *If P and Q are two partitions, then*

$$L(f, P) \leq U(f, Q).$$

Proof: By Lemma 3.1.2,

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

Definition 3.1.4

$$\bar{I} \equiv \inf \{U(f, Q) \text{ where } Q \text{ is a partition}\}$$

$$\underline{I} \equiv \sup \{L(f, P) \text{ where } P \text{ is a partition}\}.$$

Note that $\underline{I}$ and $\bar{I}$ are well defined real numbers.

Theorem 3.1.5 $\underline{I} \leq \bar{I}$.

Proof: From Lemma 3.1.3,

$$\underline{I} = \sup\{L(f, P) \text{ where } P \text{ is a partition}\} \leq U(f, Q)$$

because $U(f, Q)$ is an upper bound to the set of all lower sums and so it is no smaller than the least upper bound. Therefore, since Q is arbitrary,

$$\begin{aligned} \underline{I} &= \sup\{L(f, P) \text{ where } P \text{ is a partition}\} \\ &\leq \inf\{U(f, Q) \text{ where } Q \text{ is a partition}\} \equiv \bar{I} \end{aligned}$$

where the inequality holds because it was just shown that $\underline{I}$ is a lower bound to the set of all upper sums and so it is no larger than the greatest lower bound of this set. This proves the theorem.

Definition 3.1.6 A bounded function f is Riemann Stieltjes integrable, written as

$$f \in R([a, b])$$

if

$$\underline{I} = \bar{I}$$

and in this case,

$$\int_a^b f(x) dF \equiv \underline{I} = \bar{I}.$$

When $F(x) = x$, the integral is called the Riemann integral and is written as

$$\int_a^b f(x) dx.$$

Thus, in words, the Riemann integral is the unique number which lies between all upper sums and all lower sums if there is such a unique number.

Recall the following Proposition which comes from the definitions.

Proposition 3.1.7 Let S be a nonempty set and suppose $\sup(S)$ exists. Then for every $\delta > 0$,

$$S \cap (\sup(S) - \delta, \sup(S)] \neq \emptyset.$$

If $\inf(S)$ exists, then for every $\delta > 0$,

$$S \cap [\inf(S), \inf(S) + \delta) \neq \emptyset.$$

This proposition implies the following theorem which is used to determine the question of Riemann Stieltjes integrability.

Theorem 3.1.8 A bounded function f is Riemann integrable if and only if for all $\varepsilon > 0$, there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon. \quad (3.1.3)$$

Proof: First assume f is Riemann integrable. Then let P and Q be two partitions such that

$$U(f, Q) < \bar{I} + \varepsilon/2, \quad L(f, P) > \underline{I} - \varepsilon/2.$$

Then since $\underline{I} = \bar{I}$,

$$U(f, Q \cup P) - L(f, P \cup Q) \leq U(f, Q) - L(f, P) < \bar{I} + \varepsilon/2 - (\underline{I} - \varepsilon/2) = \varepsilon.$$

Now suppose that for all $\varepsilon > 0$ there exists a partition such that 3.1.3 holds. Then for given ε and partition P corresponding to ε

$$\bar{I} - \underline{I} \leq U(f, P) - L(f, P) \leq \varepsilon.$$

Since ε is arbitrary, this shows $\underline{I} = \bar{I}$ and this proves the theorem.

The condition described in the theorem is called the Riemann criterion.

Not all bounded functions are Riemann integrable. For example, let $F(x) = x$ and

$$f(x) \equiv \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad (3.1.4)$$

Then if $[a, b] = [0, 1]$ all upper sums for f equal 1 while all lower sums for f equal 0. Therefore the Riemann criterion is violated for $\varepsilon = 1/2$.

3.2 Exercises

1. Prove the second half of Lemma 3.1.2 about lower sums.
2. Verify that for f given in 3.1.4, the lower sums on the interval $[0, 1]$ are all equal to zero while the upper sums are all equal to one.
3. Let $f(x) = 1 + x^2$ for $x \in [-1, 3]$ and let $P = \{-1, -\frac{1}{3}, 0, \frac{1}{2}, 1, 2\}$. Find $U(f, P)$ and $L(f, P)$ for $F(x) = x$ and for $F(x) = x^3$.
4. Show that if $f \in R([a, b])$ for $F(x) = x$, there exists a partition, $\{x_0, \dots, x_n\}$ such that for any $z_k \in [x_k, x_{k+1}]$,

$$\left| \int_a^b f(x) dx - \sum_{k=1}^n f(z_k)(x_k - x_{k-1}) \right| < \varepsilon$$

This sum, $\sum_{k=1}^n f(z_k)(x_k - x_{k-1})$, is called a Riemann sum and this exercise shows that the Riemann integral can always be approximated by a Riemann sum. For the general Riemann Stieltjes case, does anything change?

5. Let $P = \{1, 1\frac{1}{4}, 1\frac{1}{2}, 1\frac{3}{4}, 2\}$ and $F(x) = x$. Find upper and lower sums for the function, $f(x) = \frac{1}{x}$ using this partition. What does this tell you about $\ln(2)$?
6. If $f \in R([a, b])$ with $F(x) = x$ and f is changed at finitely many points, show the new function is also in $R([a, b])$. Is this still true for the general case where F is only assumed to be an increasing function? Explain.

7. In the case where $F(x) = x$, define a “left sum” as

$$\sum_{k=1}^n f(x_{k-1})(x_k - x_{k-1})$$

and a “right sum”,

$$\sum_{k=1}^n f(x_k)(x_k - x_{k-1}).$$

Also suppose that all partitions have the property that $x_k - x_{k-1}$ equals a constant, $(b-a)/n$ so the points in the partition are equally spaced, and define the integral to be the number these right and left sums get close to as n gets larger and larger. Show that for f given in 3.1.4, $\int_0^x f(t) dt = 1$ if x is rational and $\int_0^x f(t) dt = 0$ if x is irrational. It turns out that the correct answer should always equal zero for that function, regardless of whether x is rational. This is shown when the Lebesgue integral is studied. This illustrates why this method of defining the integral in terms of left and right sums is total nonsense. Show that even though this is the case, it makes no difference if f is continuous.

3.3 Functions Of Riemann Integrable Functions

It is often necessary to consider functions of Riemann integrable functions and a natural question is whether these are Riemann integrable. The following theorem gives a partial answer to this question. This is not the most general theorem which will relate to this question but it will be enough for the needs of this book.

Theorem 3.3.1 *Let f, g be bounded functions and let*

$$f([a, b]) \subseteq [c_1, d_1], \quad g([a, b]) \subseteq [c_2, d_2].$$

Let $H : [c_1, d_1] \times [c_2, d_2] \rightarrow \mathbb{R}$ satisfy,

$$|H(a_1, b_1) - H(a_2, b_2)| \leq K[|a_1 - a_2| + |b_1 - b_2|]$$

for some constant K . Then if $f, g \in R([a, b])$ it follows that $H \circ (f, g) \in R([a, b])$.

Proof: In the following claim, $M_i(h)$ and $m_i(h)$ have the meanings assigned above with respect to some partition of $[a, b]$ for the function, h .

Claim: The following inequality holds.

$$\begin{aligned} & |M_i(H \circ (f, g)) - m_i(H \circ (f, g))| \leq \\ & K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|]. \end{aligned}$$

Proof of the claim: By the above proposition, there exist $x_1, x_2 \in [x_{i-1}, x_i]$ be such that

$$H(f(x_1), g(x_1)) + \eta > M_i(H \circ (f, g)),$$

and

$$H(f(x_2), g(x_2)) - \eta < m_i(H \circ (f, g)).$$

Then

$$\begin{aligned} & |M_i(H \circ (f, g)) - m_i(H \circ (f, g))| \\ & < 2\eta + |H(f(x_1), g(x_1)) - H(f(x_2), g(x_2))| \\ & < 2\eta + K[|f(x_1) - f(x_2)| + |g(x_1) - g(x_2)|] \\ & \leq 2\eta + K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|]. \end{aligned}$$

Since $\eta > 0$ is arbitrary, this proves the claim.

Now continuing with the proof of the theorem, let P be such that

$$\sum_{i=1}^n (M_i(f) - m_i(f)) \Delta F_i < \frac{\varepsilon}{2K}, \quad \sum_{i=1}^n (M_i(g) - m_i(g)) \Delta F_i < \frac{\varepsilon}{2K}.$$

Then from the claim,

$$\begin{aligned} & \sum_{i=1}^n (M_i(H \circ (f, g)) - m_i(H \circ (f, g))) \Delta F_i \\ & < \sum_{i=1}^n K[|M_i(f) - m_i(f)| + |M_i(g) - m_i(g)|] \Delta F_i < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $H \circ (f, g)$ satisfies the Riemann criterion and hence $H \circ (f, g)$ is Riemann integrable as claimed. This proves the theorem.

This theorem implies that if f, g are Riemann Stieltjes integrable, then so is $af + bg, |f|, f^2$, along with infinitely many other such continuous combinations of Riemann Stieltjes integrable functions. For example, to see that $|f|$ is Riemann integrable, let $H(a, b) = |a|$. Clearly this function satisfies the conditions of the above theorem and so $|f| = H(f, f) \in R([a, b])$ as claimed. The following theorem gives an example of many functions which are Riemann integrable.

Theorem 3.3.2 *Let $f : [a, b] \rightarrow \mathbb{R}$ be either increasing or decreasing on $[a, b]$ and suppose F is continuous. Then $f \in R([a, b])$.*

Proof: Let $\varepsilon > 0$ be given and let

$$x_i = a + i \left(\frac{b-a}{n} \right), \quad i = 0, \dots, n.$$

Since F is continuous, it follows from Corollary 2.0.7 on Page 28 that it is uniformly continuous. Therefore, if n is large enough, then for all i ,

$$F(x_i) - F(x_{i-1}) < \frac{\varepsilon}{f(b) - f(a) + 1}$$

Then since f is increasing,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n (f(x_i) - f(x_{i-1})) (F(x_i) - F(x_{i-1})) \\ &\leq \frac{\varepsilon}{f(b) - f(a) + 1} \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \\ &= \frac{\varepsilon}{f(b) - f(a) + 1} (f(b) - f(a)) < \varepsilon. \end{aligned}$$

Thus the Riemann criterion is satisfied and so the function is Riemann Stieltjes integrable. The proof for decreasing f is similar.

Corollary 3.3.3 *Let $[a, b]$ be a bounded closed interval and let $\phi : [a, b] \rightarrow \mathbb{R}$ be Lipschitz continuous and suppose F is continuous. Then $\phi \in R([a, b])$. Recall that a function, ϕ , is Lipschitz continuous if there is a constant, K , such that for all x, y ,*

$$|\phi(x) - \phi(y)| < K|x - y|.$$

Proof: Let $f(x) = x$. Then by Theorem 3.3.2, f is Riemann Stieltjes integrable. Let $H(a, b) \equiv \phi(a)$. Then by Theorem 3.3.1 $H \circ (f, f) = \phi \circ f = \phi$ is also Riemann Stieltjes integrable. This proves the corollary. In fact, it is enough to assume ϕ is continuous, although this is harder. This is the content of the next theorem which is where the difficult theorems about continuity and uniform continuity are used. This is the main result on the existence of the Riemann Stieltjes integral for this book.

Theorem 3.3.4 *Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and F is just an increasing function defined on $[a, b]$. Then $f \in R([a, b])$.*

Proof: By Corollary 2.0.7 on Page 28, f is uniformly continuous on $[a, b]$. Therefore, if $\varepsilon > 0$ is given, there exists a $\delta > 0$ such that if $|x_i - x_{i-1}| < \delta$, then $M_i - m_i < \frac{\varepsilon}{F(b) - F(a) + 1}$. Let

$$P \equiv \{x_0, \dots, x_n\}$$

be a partition with $|x_i - x_{i-1}| < \delta$. Then

$$\begin{aligned} U(f, P) - L(f, P) &< \sum_{i=1}^n (M_i - m_i) (F(x_i) - F(x_{i-1})) \\ &< \frac{\varepsilon}{F(b) - F(a) + 1} (F(b) - F(a)) < \varepsilon. \end{aligned}$$

By the Riemann criterion, $f \in R([a, b])$. This proves the theorem.

3.4 Properties Of The Integral

The integral has many important algebraic properties. First here is a simple lemma.

Lemma 3.4.1 *Let S be a nonempty set which is bounded above and below. Then if $-S \equiv \{-x : x \in S\}$,*

$$\sup(-S) = -\inf(S) \quad (3.4.5)$$

and

$$\inf(-S) = -\sup(S). \quad (3.4.6)$$

Proof: Consider 3.4.5. Let $x \in S$. Then $-x \leq \sup(-S)$ and so $x \geq -\sup(-S)$. It follows that $-\sup(-S)$ is a lower bound for S and therefore, $-\sup(-S) \leq \inf(S)$. This implies $\sup(-S) \geq -\inf(S)$. Now let $-x \in -S$. Then $x \in S$ and so $x \geq \inf(S)$ which implies $-x \leq -\inf(S)$. Therefore, $-\inf(S)$ is an upper bound for $-S$ and so $-\inf(S) \geq \sup(-S)$. This shows 3.4.5. Formula 3.4.6 is similar and is left as an exercise.

In particular, the above lemma implies that for $M_i(f)$ and $m_i(f)$ defined above $M_i(-f) = -m_i(f)$, and $m_i(-f) = -M_i(f)$.

Lemma 3.4.2 *If $f \in R([a, b])$ then $-f \in R([a, b])$ and*

$$-\int_a^b f(x) dF = \int_a^b -f(x) dF.$$

Proof: The first part of the conclusion of this lemma follows from Theorem 3.3.2 since the function $\phi(y) \equiv -y$ is Lipschitz continuous. Now choose P such that

$$\int_a^b -f(x) dF - L(-f, P) < \varepsilon.$$

Then since $m_i(-f) = -M_i(f)$,

$$\varepsilon > \int_a^b -f(x) dF - \sum_{i=1}^n m_i(-f) \Delta F_i = \int_a^b -f(x) dF + \sum_{i=1}^n M_i(f) \Delta F_i$$

which implies

$$\varepsilon > \int_a^b -f(x) dF + \sum_{i=1}^n M_i(f) \Delta F_i \geq \int_a^b -f(x) dF + \int_a^b f(x) dF.$$

Thus, since ε is arbitrary,

$$\int_a^b -f(x) dF \leq -\int_a^b f(x) dF$$

whenever $f \in R([a, b])$. It follows

$$\int_a^b -f(x) dF \leq -\int_a^b f(x) dF = -\int_a^b -(-f(x)) dF \leq \int_a^b -f(x) dF$$

and this proves the lemma.

Theorem 3.4.3 *The integral is linear,*

$$\int_a^b (\alpha f + \beta g)(x) dF = \alpha \int_a^b f(x) dF + \beta \int_a^b g(x) dF.$$

whenever $f, g \in R([a, b])$ and $\alpha, \beta \in \mathbb{R}$.

Proof: First note that by Theorem 3.3.1, $\alpha f + \beta g \in R([a, b])$. To begin with, consider the claim that if $f, g \in R([a, b])$ then

$$\int_a^b (f + g)(x) dF = \int_a^b f(x) dF + \int_a^b g(x) dF. \quad (3.4.7)$$

Let P_1, Q_1 be such that

$$U(f, Q_1) - L(f, Q_1) < \varepsilon/2, \quad U(g, P_1) - L(g, P_1) < \varepsilon/2.$$

Then letting $P \equiv P_1 \cup Q_1$, Lemma 3.1.2 implies

$$U(f, P) - L(f, P) < \varepsilon/2, \quad \text{and} \quad U(g, P) - L(g, P) < \varepsilon/2.$$

Next note that

$$m_i(f + g) \geq m_i(f) + m_i(g), \quad M_i(f + g) \leq M_i(f) + M_i(g).$$

Therefore,

$$L(g + f, P) \geq L(f, P) + L(g, P), \quad U(g + f, P) \leq U(f, P) + U(g, P).$$

For this partition,

$$\begin{aligned} \int_a^b (f + g)(x) dF &\in [L(f + g, P), U(f + g, P)] \\ &\subseteq [L(f, P) + L(g, P), U(f, P) + U(g, P)] \end{aligned}$$

and

$$\int_a^b f(x) dF + \int_a^b g(x) dF \in [L(f, P) + L(g, P), U(f, P) + U(g, P)].$$

Therefore,

$$\begin{aligned} \left| \int_a^b (f + g)(x) dF - \left(\int_a^b f(x) dF + \int_a^b g(x) dF \right) \right| &\leq \\ U(f, P) + U(g, P) - (L(f, P) + L(g, P)) &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves 3.4.7 since ε is arbitrary.

It remains to show that

$$\alpha \int_a^b f(x) dF = \int_a^b \alpha f(x) dF.$$

Suppose first that $\alpha \geq 0$. Then

$$\begin{aligned} \int_a^b \alpha f(x) dF &\equiv \sup\{L(\alpha f, P) : P \text{ is a partition}\} = \\ &\alpha \sup\{L(f, P) : P \text{ is a partition}\} \equiv \alpha \int_a^b f(x) dF. \end{aligned}$$

If $\alpha < 0$, then this and Lemma 3.4.2 imply

$$\begin{aligned} \int_a^b \alpha f(x) dF &= \int_a^b (-\alpha)(-f(x)) dF \\ &= (-\alpha) \int_a^b (-f(x)) dF = \alpha \int_a^b f(x) dF. \end{aligned}$$

This proves the theorem.

In the next theorem, suppose F is defined on $[a, b] \cup [b, c]$.

Theorem 3.4.4 *If $f \in R([a, b])$ and $f \in R([b, c])$, then $f \in R([a, c])$ and*

$$\int_a^c f(x) dF = \int_a^b f(x) dF + \int_b^c f(x) dF. \quad (3.4.8)$$

Proof: Let P_1 be a partition of $[a, b]$ and P_2 be a partition of $[b, c]$ such that

$$U(f, P_i) - L(f, P_i) < \varepsilon/2, \quad i = 1, 2.$$

Let $P \equiv P_1 \cup P_2$. Then P is a partition of $[a, c]$ and

$$\begin{aligned} &U(f, P) - L(f, P) \\ &= U(f, P_1) - L(f, P_1) + U(f, P_2) - L(f, P_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned} \quad (3.4.9)$$

Thus, $f \in R([a, c])$ by the Riemann criterion and also for this partition,

$$\begin{aligned} \int_a^b f(x) dF + \int_b^c f(x) dF &\in [L(f, P_1) + L(f, P_2), U(f, P_1) + U(f, P_2)] \\ &= [L(f, P), U(f, P)] \end{aligned}$$

and

$$\int_a^c f(x) dF \in [L(f, P), U(f, P)].$$

Hence by 3.4.9,

$$\left| \int_a^c f(x) dF - \left(\int_a^b f(x) dF + \int_b^c f(x) dF \right) \right| < U(f, P) - L(f, P) < \varepsilon$$

which shows that since ε is arbitrary, 3.4.8 holds. This proves the theorem.

Corollary 3.4.5 *Let F be continuous and let $[a, b]$ be a closed and bounded interval and suppose that*

$$a = y_1 < y_2 < \cdots < y_l = b$$

and that f is a bounded function defined on $[a, b]$ which has the property that f is either increasing on $[y_j, y_{j+1}]$ or decreasing on $[y_j, y_{j+1}]$ for $j = 1, \dots, l-1$. Then $f \in R([a, b])$.

Proof: This follows from Theorem 3.4.4 and Theorem 3.3.2.

The symbol, $\int_a^b f(x) dF$ when $a > b$ has not yet been defined.

Definition 3.4.6 *Let $[a, b]$ be an interval and let $f \in R([a, b])$. Then*

$$\int_b^a f(x) dF \equiv - \int_a^b f(x) dF.$$

Note that with this definition,

$$\int_a^a f(x) dF = - \int_a^a f(x) dF$$

and so

$$\int_a^a f(x) dF = 0.$$

Theorem 3.4.7 *Assuming all the integrals make sense,*

$$\int_a^b f(x) dF + \int_b^c f(x) dF = \int_a^c f(x) dF.$$

Proof: This follows from Theorem 3.4.4 and Definition 3.4.6. For example, assume

$$c \in (a, b).$$

Then from Theorem 3.4.4,

$$\int_a^c f(x) dF + \int_c^b f(x) dF = \int_a^b f(x) dF$$

and so by Definition 3.4.6,

$$\begin{aligned} \int_a^c f(x) dF &= \int_a^b f(x) dF - \int_c^b f(x) dF \\ &= \int_a^b f(x) dF + \int_b^c f(x) dF. \end{aligned}$$

The other cases are similar.

The following properties of the integral have either been established or they follow quickly from what has been shown so far.

$$\text{If } f \in R([a, b]) \text{ then if } c \in [a, b], f \in R([a, c]), \quad (3.4.10)$$

$$\int_a^b \alpha dF = \alpha (F(b) - F(a)), \quad (3.4.11)$$

$$\int_a^b (\alpha f + \beta g)(x) dF = \alpha \int_a^b f(x) dF + \beta \int_a^b g(x) dF, \quad (3.4.12)$$

$$\int_a^b f(x) dF + \int_b^c f(x) dF = \int_a^c f(x) dF, \quad (3.4.13)$$

$$\int_a^b f(x) dF \geq 0 \text{ if } f(x) \geq 0 \text{ and } a < b, \quad (3.4.14)$$

$$\left| \int_a^b f(x) dF \right| \leq \int_a^b |f(x)| dF. \quad (3.4.15)$$

The only one of these claims which may not be completely obvious is the last one. To show this one, note that

$$|f(x)| - f(x) \geq 0, \quad |f(x)| + f(x) \geq 0.$$

Therefore, by 3.4.14 and 3.4.12, if $a < b$,

$$\int_a^b |f(x)| dF \geq \int_a^b f(x) dF$$

and

$$\int_a^b |f(x)| dF \geq - \int_a^b f(x) dF.$$

Therefore,

$$\int_a^b |f(x)| dF \geq \left| \int_a^b f(x) dF \right|.$$

If $b < a$ then the above inequality holds with a and b switched. This implies 3.4.15.

3.5 Fundamental Theorem Of Calculus

In this section $F(x) = x$ so things are specialized to the ordinary Riemann integral. With these properties, it is easy to prove the fundamental theorem of calculus².

²This theorem is why Newton and Leibnitz are credited with inventing calculus. The integral had been around for thousands of years and the derivative was by their time well known. However the connection between these two ideas had not been fully made although Newton's predecessor, Isaac Barrow had made some progress in this direction.

Let $f \in R([a, b])$. Then by 3.4.10 $f \in R([a, x])$ for each $x \in [a, b]$. The first version of the fundamental theorem of calculus is a statement about the derivative of the function

$$x \rightarrow \int_a^x f(t) \, dt.$$

Theorem 3.5.1 *Let $f \in R([a, b])$ and let*

$$F(x) \equiv \int_a^x f(t) \, dt.$$

Then if f is continuous at $x \in (a, b)$,

$$F'(x) = f(x).$$

Proof: Let $x \in (a, b)$ be a point of continuity of f and let h be small enough that $x + h \in [a, b]$. Then by using 3.4.13,

$$h^{-1}(F(x+h) - F(x)) = h^{-1} \int_x^{x+h} f(t) \, dt.$$

Also, using 3.4.11,

$$f(x) = h^{-1} \int_x^{x+h} f(x) \, dt.$$

Therefore, by 3.4.15,

$$\begin{aligned} |h^{-1}(F(x+h) - F(x)) - f(x)| &= \left| h^{-1} \int_x^{x+h} (f(t) - f(x)) \, dt \right| \\ &\leq \left| h^{-1} \int_x^{x+h} |f(t) - f(x)| \, dt \right|. \end{aligned}$$

Let $\varepsilon > 0$ and let $\delta > 0$ be small enough that if $|t - x| < \delta$, then

$$|f(t) - f(x)| < \varepsilon.$$

Therefore, if $|h| < \delta$, the above inequality and 3.4.11 shows that

$$|h^{-1}(F(x+h) - F(x)) - f(x)| \leq |h|^{-1} \varepsilon |h| = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\lim_{h \rightarrow 0} h^{-1}(F(x+h) - F(x)) = f(x)$$

and this proves the theorem.

Note this gives existence for the initial value problem,

$$F'(x) = f(x), \quad F(a) = 0$$

whenever f is Riemann integrable and continuous.³

The next theorem is also called the fundamental theorem of calculus.

Theorem 3.5.2 *Let $f \in R([a, b])$ and suppose there exists an antiderivative for f, G , such that*

$$G'(x) = f(x)$$

for every point of (a, b) and G is continuous on $[a, b]$. Then

$$\int_a^b f(x) dx = G(b) - G(a). \quad (3.5.16)$$

Proof: Let $P = \{x_0, \dots, x_n\}$ be a partition satisfying

$$U(f, P) - L(f, P) < \varepsilon.$$

Then

$$\begin{aligned} G(b) - G(a) &= G(x_n) - G(x_0) \\ &= \sum_{i=1}^n G(x_i) - G(x_{i-1}). \end{aligned}$$

By the mean value theorem,

$$\begin{aligned} G(b) - G(a) &= \sum_{i=1}^n G'(z_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^n f(z_i) \Delta x_i \end{aligned}$$

where z_i is some point in $[x_{i-1}, x_i]$. It follows, since the above sum lies between the upper and lower sums, that

$$G(b) - G(a) \in [L(f, P), U(f, P)],$$

and also

$$\int_a^b f(x) dx \in [L(f, P), U(f, P)].$$

Therefore,

$$\left| G(b) - G(a) - \int_a^b f(x) dx \right| < U(f, P) - L(f, P) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, 3.5.16 holds. This proves the theorem.

³Of course it was proved that if f is continuous on a closed interval, $[a, b]$, then $f \in R([a, b])$ but this is a hard theorem using the difficult result about uniform continuity.

The following notation is often used in this context. Suppose F is an antiderivative of f as just described with F continuous on $[a, b]$ and $F' = f$ on (a, b) . Then

$$\int_a^b f(x) \, dx = F(b) - F(a) \equiv F(x) \Big|_a^b.$$

Definition 3.5.3 Let f be a bounded function defined on a closed interval $[a, b]$ and let $P \equiv \{x_0, \dots, x_n\}$ be a partition of the interval. Suppose $z_i \in [x_{i-1}, x_i]$ is chosen. Then the sum

$$\sum_{i=1}^n f(z_i)(x_i - x_{i-1})$$

is known as a Riemann sum. Also,

$$\|P\| \equiv \max\{|x_i - x_{i-1}| : i = 1, \dots, n\}.$$

Proposition 3.5.4 Suppose $f \in R([a, b])$. Then there exists a partition, $P \equiv \{x_0, \dots, x_n\}$ with the property that for any choice of $z_k \in [x_{k-1}, x_k]$,

$$\left| \int_a^b f(x) \, dx - \sum_{k=1}^n f(z_k)(x_k - x_{k-1}) \right| < \varepsilon.$$

Proof: Choose P such that $U(f, P) - L(f, P) < \varepsilon$ and then both $\int_a^b f(x) \, dx$ and $\sum_{k=1}^n f(z_k)(x_k - x_{k-1})$ are contained in $[L(f, P), U(f, P)]$ and so the claimed inequality must hold. This proves the proposition.

It is significant because it gives a way of approximating the integral.

The definition of Riemann integrability given in this chapter is also called Darboux integrability and the integral defined as the unique number which lies between all upper sums and all lower sums which is given in this chapter is called the Darboux integral. The definition of the Riemann integral in terms of Riemann sums is given next.

Definition 3.5.5 A bounded function, f defined on $[a, b]$ is said to be Riemann integrable if there exists a number, I with the property that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$P \equiv \{x_0, x_1, \dots, x_n\}$$

is any partition having $\|P\| < \delta$, and $z_i \in [x_{i-1}, x_i]$,

$$\left| I - \sum_{i=1}^n f(z_i)(x_i - x_{i-1}) \right| < \varepsilon.$$

The number $\int_a^b f(x) \, dx$ is defined as I .

Thus, there are two definitions of the Riemann integral. It turns out they are equivalent which is the following theorem of Darboux.

Theorem 3.5.6 *A bounded function defined on $[a, b]$ is Riemann integrable in the sense of Definition 3.5.5 if and only if it is integrable in the sense of Darboux. Furthermore the two integrals coincide.*

The proof of this theorem is left for the exercises in Problems 10 - 12. It isn't essential that you understand this theorem so if it does not interest you, leave it out. Note that it implies that given a Riemann integrable function f in either sense, it can be approximated by Riemann sums whenever $\|P\|$ is sufficiently small. Both versions of the integral are obsolete but entirely adequate for most applications and as a point of departure for a more up to date and satisfactory integral. The reason for using the Darboux approach to the integral is that all the existence theorems are easier to prove in this context.

3.6 Exercises

1. Let $F(x) = \int_{x^2}^{x^3} \frac{t^5 + 7}{t^7 + 87t^6 + 1} dt$. Find $F'(x)$.
2. Let $F(x) = \int_2^x \frac{1}{1+t^4} dt$. Sketch a graph of F and explain why it looks the way it does.
3. Let a and b be positive numbers and consider the function,

$$F(x) = \int_0^{ax} \frac{1}{a^2 + t^2} dt + \int_b^{a/x} \frac{1}{a^2 + t^2} dt.$$

Show that F is a constant.

4. Solve the following initial value problem from ordinary differential equations which is to find a function y such that

$$y'(x) = \frac{x^7 + 1}{x^6 + 97x^5 + 7}, \quad y(10) = 5.$$

5. If $F, G \in \int f(x) dx$ for all $x \in \mathbb{R}$, show $F(x) = G(x) + C$ for some constant, C . Use this to give a different proof of the fundamental theorem of calculus which has for its conclusion $\int_a^b f(t) dt = G(b) - G(a)$ where $G'(x) = f(x)$.
6. Suppose f is Riemann integrable on $[a, b]$ and continuous. (In fact continuous implies Riemann integrable.) Show there exists $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx.$$

Hint: You might consider the function $F(x) \equiv \int_a^x f(t) dt$ and use the mean value theorem for derivatives and the fundamental theorem of calculus.

7. Suppose f and g are continuous functions on $[a, b]$ and that $g(x) \neq 0$ on (a, b) . Show there exists $c \in (a, b)$ such that

$$f(c) \int_a^b g(x) dx = \int_a^b f(x) g(x) dx.$$

Hint: Define $F(x) \equiv \int_a^x f(t) g(t) dt$ and let $G(x) \equiv \int_a^x g(t) dt$. Then use the Cauchy mean value theorem on these two functions.

8. Consider the function

$$f(x) \equiv \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Is f Riemann integrable? Explain why or why not.

9. Prove the second part of Theorem 3.3.2 about decreasing functions.
10. Suppose f is a bounded function defined on $[a, b]$ and $|f(x)| < M$ for all $x \in [a, b]$. Now let Q be a partition having n points, $\{x_0^*, \dots, x_n^*\}$ and let P be any other partition. Show that

$$|U(f, P) - L(f, P)| \leq 2Mn \|P\| + |U(f, Q) - L(f, Q)|.$$

Hint: Write the sum for $U(f, P) - L(f, P)$ and split this sum into two sums, the sum of terms for which $[x_{i-1}, x_i]$ contains at least one point of Q , and terms for which $[x_{i-1}, x_i]$ does not contain any points of Q . In the latter case, $[x_{i-1}, x_i]$ must be contained in some interval, $[x_{k-1}^*, x_k^*]$. Therefore, the sum of these terms should be no larger than $|U(f, Q) - L(f, Q)|$.

11. $\uparrow$ If $\varepsilon > 0$ is given and f is a Darboux integrable function defined on $[a, b]$, show there exists $\delta > 0$ such that whenever $\|P\| < \delta$, then

$$|U(f, P) - L(f, P)| < \varepsilon.$$

12. $\uparrow$ Prove Theorem 3.5.6.

Some Important Linear Algebra

This chapter contains some important linear algebra as distinguished from that which is normally presented in undergraduate courses consisting mainly of uninteresting things you can do with row operations.

The notation, $\mathbb{C}^n$ refers to the collection of ordered lists of n complex numbers. Since every real number is also a complex number, this simply generalizes the usual notion of $\mathbb{R}^n$, the collection of all ordered lists of n real numbers. In order to avoid worrying about whether it is real or complex numbers which are being referred to, the symbol $\mathbb{F}$ will be used. If it is not clear, always pick $\mathbb{C}$.

Definition 4.0.1 *Define*

$$\mathbb{F}^n \equiv \{(x_1, \dots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \dots, n\}.$$

$(x_1, \dots, x_n) = (y_1, \dots, y_n)$ if and only if for all $j = 1, \dots, n$, $x_j = y_j$. When

$$(x_1, \dots, x_n) \in \mathbb{F}^n,$$

it is conventional to denote $(x_1, \dots, x_n)$ by the single bold face letter, $\mathbf{x}$. The numbers, x_j are called the coordinates. The set

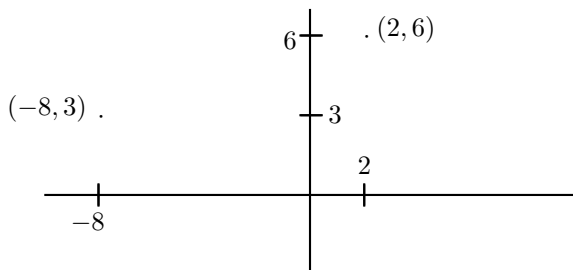
$$\{(0, \dots, 0, t, 0, \dots, 0) : t \in \mathbb{F}\}$$

for t in the i^{th} slot is called the i^{th} coordinate axis. The point $\mathbf{0} \equiv (0, \dots, 0)$ is called the origin.

Thus $(1, 2, 4i) \in \mathbb{F}^3$ and $(2, 1, 4i) \in \mathbb{F}^3$ but $(1, 2, 4i) \neq (2, 1, 4i)$ because, even though the same numbers are involved, they don't match up. In particular, the first entries are not equal.

The geometric significance of $\mathbb{R}^n$ for $n \leq 3$ has been encountered already in calculus or in precalculus. Here is a short review. First consider the case when $n = 1$. Then from the definition, $\mathbb{R}^1 = \mathbb{R}$. Recall that $\mathbb{R}$ is identified with the points of a line. Look at the number line again. Observe that this amounts to

identifying a point on this line with a real number. In other words a real number determines where you are on this line. Now suppose $n = 2$ and consider two lines which intersect each other at right angles as shown in the following picture.



Notice how you can identify a point shown in the plane with the ordered pair, $(2, 6)$. You go to the right a distance of 2 and then up a distance of 6. Similarly, you can identify another point in the plane with the ordered pair $(-8, 3)$. Go to the left a distance of 8 and then up a distance of 3. The reason you go to the left is that there is a $-$ sign on the eight. From this reasoning, every ordered pair determines a unique point in the plane. Conversely, taking a point in the plane, you could draw two lines through the point, one vertical and the other horizontal and determine unique points, x_1 on the horizontal line in the above picture and x_2 on the vertical line in the above picture, such that the point of interest is identified with the ordered pair, (x_1, x_2) . In short, points in the plane can be identified with ordered pairs similar to the way that points on the real line are identified with real numbers. Now suppose $n = 3$. As just explained, the first two coordinates determine a point in a plane. Letting the third component determine how far up or down you go, depending on whether this number is positive or negative, this determines a point in space. Thus, $(1, 4, -5)$ would mean to determine the point in the plane that goes with $(1, 4)$ and then to go below this plane a distance of 5 to obtain a unique point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

You can't stop here and say that you are only interested in $n \leq 3$. What if you were interested in the motion of two objects? You would need three coordinates to describe where the first object is and you would need another three coordinates to describe where the other object is located. Therefore, you would need to be considering $\mathbb{R}^6$. If the two objects moved around, you would need a time coordinate as well. As another example, consider a hot object which is cooling and suppose you want the temperature of this object. How many coordinates would be needed? You would need one for the temperature, three for the position of the point in the object and one more for the time. Thus you would need to be considering $\mathbb{R}^5$. Many other examples can be given. Sometimes n is very large. This is often the case in applications to business when they are trying to maximize profit subject to constraints. It also occurs in numerical analysis when people try to solve hard problems on a computer.

There are other ways to identify points in space with three numbers but the one presented is the most basic. In this case, the coordinates are known as Cartesian coordinates after Descartes¹ who invented this idea in the first half of the seventeenth century. I will often not bother to draw a distinction between the point in n dimensional space and its Cartesian coordinates.

The geometric significance of $\mathbb{C}^n$ for $n > 1$ is not available because each copy of $\mathbb{C}$ corresponds to the plane or $\mathbb{R}^2$.

4.1 Algebra in $\mathbb{F}^n$

There are two algebraic operations done with elements of $\mathbb{F}^n$. One is addition and the other is multiplication by numbers, called scalars. In the case of $\mathbb{C}^n$ the scalars are complex numbers while in the case of $\mathbb{R}^n$ the only allowed scalars are real numbers. Thus, the scalars always come from $\mathbb{F}$ in either case.

Definition 4.1.1 *If $\mathbf{x} \in \mathbb{F}^n$ and $a \in \mathbb{F}$, also called a scalar, then $a\mathbf{x} \in \mathbb{F}^n$ is defined by*

$$a\mathbf{x} = a(x_1, \dots, x_n) \equiv (ax_1, \dots, ax_n). \quad (4.1.1)$$

This is known as scalar multiplication. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ then $\mathbf{x} + \mathbf{y} \in \mathbb{F}^n$ and is defined by

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1, \dots, x_n) + (y_1, \dots, y_n) \\ &\equiv (x_1 + y_1, \dots, x_n + y_n) \end{aligned} \quad (4.1.2)$$

With this definition, the algebraic properties satisfy the conclusions of the following theorem.

Theorem 4.1.2 *For $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$ and α, β scalars, (real numbers), the following hold.*

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}, \quad (4.1.3)$$

the commutative law of addition,

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}), \quad (4.1.4)$$

the associative law for addition,

$$\mathbf{v} + \mathbf{0} = \mathbf{v}, \quad (4.1.5)$$

the existence of an additive identity,

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0}, \quad (4.1.6)$$

¹René Descartes 1596-1650 is often credited with inventing analytic geometry although it seems the ideas were actually known much earlier. He was interested in many different subjects, physiology, chemistry, and physics being some of them. He also wrote a large book in which he tried to explain the book of Genesis scientifically. Descartes ended up dying in Sweden.

the existence of an additive inverse, Also

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w}, \quad (4.1.7)$$

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}, \quad (4.1.8)$$

$$\alpha(\beta\mathbf{v}) = \alpha\beta(\mathbf{v}), \quad (4.1.9)$$

$$1\mathbf{v} = \mathbf{v}. \quad (4.1.10)$$

In the above $\mathbf{0} = (0, \dots, 0)$.

You should verify these properties all hold. For example, consider 4.1.7

$$\begin{aligned} \alpha(\mathbf{v} + \mathbf{w}) &= \alpha(v_1 + w_1, \dots, v_n + w_n) \\ &= (\alpha(v_1 + w_1), \dots, \alpha(v_n + w_n)) \\ &= (\alpha v_1 + \alpha w_1, \dots, \alpha v_n + \alpha w_n) \\ &= (\alpha v_1, \dots, \alpha v_n) + (\alpha w_1, \dots, \alpha w_n) \\ &= \alpha\mathbf{v} + \alpha\mathbf{w}. \end{aligned}$$

As usual subtraction is defined as $\mathbf{x} - \mathbf{y} \equiv \mathbf{x} + (-\mathbf{y})$.

4.2 Exercises

1. Verify all the properties 4.1.3-4.1.10.
2. Compute $5(1, 2 + 3i, 3, -2) + 6(2 - i, 1, -2, 7)$.
3. Draw a picture of the points in $\mathbb{R}^2$ which are determined by the following ordered pairs.
 - (a) $(1, 2)$
 - (b) $(-2, -2)$
 - (c) $(-2, 3)$
 - (d) $(2, -5)$
4. Does it make sense to write $(1, 2) + (2, 3, 1)$? Explain.
5. Draw a picture of the points in $\mathbb{R}^3$ which are determined by the following ordered triples.
 - (a) $(1, 2, 0)$
 - (b) $(-2, -2, 1)$
 - (c) $(-2, 3, -2)$

4.3 Subspaces Spans And Bases

Definition 4.3.1 Let $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ be vectors in $\mathbb{F}^n$. A linear combination is any expression of the form

$$\sum_{i=1}^p c_i \mathbf{x}_i$$

where the c_i are scalars. The set of all linear combinations of these vectors is called $\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_n)$. If $V \subseteq \mathbb{F}^n$, then V is called a subspace if whenever α, β are scalars and $\mathbf{u}$ and $\mathbf{v}$ are vectors of V , it follows $\alpha\mathbf{u} + \beta\mathbf{v} \in V$. That is, it is “closed under the algebraic operations of vector addition and scalar multiplication”. A linear combination of vectors is said to be trivial if all the scalars in the linear combination equal zero. A set of vectors is said to be linearly independent if the only linear combination of these vectors which equals the zero vector is the trivial linear combination. Thus $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is called linearly independent if whenever

$$\sum_{k=1}^p c_k \mathbf{x}_k = \mathbf{0}$$

it follows that all the scalars, c_k equal zero. A set of vectors, $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$, is called linearly dependent if it is not linearly independent. Thus the set of vectors is linearly dependent if there exist scalars, $c_i, i = 1, \dots, n$, not all zero such that $\sum_{k=1}^p c_k \mathbf{x}_k = \mathbf{0}$.

Lemma 4.3.2 A set of vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ is linearly independent if and only if none of the vectors can be obtained as a linear combination of the others.

Proof: Suppose first that $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ is linearly independent. If

$$\mathbf{x}_k = \sum_{j \neq k} c_j \mathbf{x}_j,$$

then

$$\mathbf{0} = 1\mathbf{x}_k + \sum_{j \neq k} (-c_j) \mathbf{x}_j,$$

a nontrivial linear combination, contrary to assumption. This shows that if the set is linearly independent, then none of the vectors is a linear combination of the others.

Now suppose no vector is a linear combination of the others. Is $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ linearly independent? If it is not there exist scalars, c_i , not all zero such that

$$\sum_{i=1}^p c_i \mathbf{x}_i = \mathbf{0}.$$

Say $c_k \neq 0$. Then you can solve for $\mathbf{x}_k$ as

$$\mathbf{x}_k = \sum_{j \neq k} (-c_j) / c_k \mathbf{x}_j$$

contrary to assumption. This proves the lemma.

The following is called the exchange theorem.

Theorem 4.3.3 (*Exchange Theorem*) *Let $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ be a linearly independent set of vectors such that each $\mathbf{x}_i$ is in $\text{span}(\mathbf{y}_1, \dots, \mathbf{y}_s)$. Then $r \leq s$.*

Proof: Define $\text{span}\{\mathbf{y}_1, \dots, \mathbf{y}_s\} \equiv V$, it follows there exist scalars, $c_1, \dots, c_s$ such that

$$\mathbf{x}_1 = \sum_{i=1}^s c_i \mathbf{y}_i. \quad (4.3.11)$$

Not all of these scalars can equal zero because if this were the case, it would follow that $\mathbf{x}_1 = \mathbf{0}$ and so $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ would not be linearly independent. Indeed, if $\mathbf{x}_1 = \mathbf{0}$, $1\mathbf{x}_1 + \sum_{i=2}^r 0\mathbf{x}_i = \mathbf{x}_1 = \mathbf{0}$ and so there would exist a nontrivial linear combination of the vectors $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ which equals zero.

Say $c_k \neq 0$. Then solve (4.3.11) for $\mathbf{y}_k$ and obtain

$$\mathbf{y}_k \in \text{span} \left(\mathbf{x}_1, \overbrace{\mathbf{y}_1, \dots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \dots, \mathbf{y}_s}^{\text{s-1 vectors here}} \right).$$

Define $\{\mathbf{z}_1, \dots, \mathbf{z}_{s-1}\}$ by

$$\{\mathbf{z}_1, \dots, \mathbf{z}_{s-1}\} \equiv \{\mathbf{y}_1, \dots, \mathbf{y}_{k-1}, \mathbf{y}_{k+1}, \dots, \mathbf{y}_s\}$$

Therefore, $\text{span}\{\mathbf{x}_1, \mathbf{z}_1, \dots, \mathbf{z}_{s-1}\} = V$ because if $\mathbf{v} \in V$, there exist constants $c_1, \dots, c_s$ such that

$$\mathbf{v} = \sum_{i=1}^{s-1} c_i \mathbf{z}_i + c_s \mathbf{y}_k.$$

Now replace the $\mathbf{y}_k$ in the above with a linear combination of the vectors,

$$\{\mathbf{x}_1, \mathbf{z}_1, \dots, \mathbf{z}_{s-1}\}$$

to obtain

$$\mathbf{v} \in \text{span}\{\mathbf{x}_1, \mathbf{z}_1, \dots, \mathbf{z}_{s-1}\}.$$

The vector $\mathbf{y}_k$, in the list $\{\mathbf{y}_1, \dots, \mathbf{y}_s\}$, has now been replaced with the vector $\mathbf{x}_1$ and the resulting modified list of vectors has the same span as the original list of vectors, $\{\mathbf{y}_1, \dots, \mathbf{y}_s\}$.

Now suppose that $r > s$ and that

$$\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_l, \mathbf{z}_1, \dots, \mathbf{z}_p) = V$$

where the vectors, $\mathbf{z}_1, \dots, \mathbf{z}_p$ are each taken from the set, $\{\mathbf{y}_1, \dots, \mathbf{y}_s\}$ and $l+p = s$. This has now been done for $l = 1$ above. Then since $r > s$, it follows that $l \leq s < r$

and so $l + 1 \leq r$. Therefore, $\mathbf{x}_{l+1}$ is a vector not in the list, $\{\mathbf{x}_1, \dots, \mathbf{x}_l\}$ and since $\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_l, \mathbf{z}_1, \dots, \mathbf{z}_p\} = V$, there exist scalars, c_i and d_j such that

$$\mathbf{x}_{l+1} = \sum_{i=1}^l c_i \mathbf{x}_i + \sum_{j=1}^p d_j \mathbf{z}_j. \quad (4.3.12)$$

Now not all the d_j can equal zero because if this were so, it would follow that $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ would be a linearly dependent set because one of the vectors would equal a linear combination of the others. Therefore, (4.3.12) can be solved for one of the $\mathbf{z}_i$, say $\mathbf{z}_k$, in terms of $\mathbf{x}_{l+1}$ and the other $\mathbf{z}_i$ and just as in the above argument, replace that $\mathbf{z}_i$ with $\mathbf{x}_{l+1}$ to obtain

$$\text{span} \left(\mathbf{x}_1, \dots, \mathbf{x}_l, \mathbf{x}_{l+1}, \overbrace{\mathbf{z}_1, \dots, \mathbf{z}_{k-1}, \mathbf{z}_{k+1}, \dots, \mathbf{z}_p}^{\text{p-1 vectors here}} \right) = V.$$

Continue this way, eventually obtaining

$$\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_s) = V.$$

But then $\mathbf{x}_r \in \text{span}(\mathbf{x}_1, \dots, \mathbf{x}_s)$ contrary to the assumption that $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is linearly independent. Therefore, $r \leq s$ as claimed.

Definition 4.3.4 A finite set of vectors, $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is a basis for $\mathbb{F}^n$ if

$$\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_r) = \mathbb{F}^n$$

and $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is linearly independent.

Corollary 4.3.5 Let $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_s\}$ be two bases² of $\mathbb{F}^n$. Then $r = s = n$.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Now note the vectors,

$$\mathbf{e}_i = \overbrace{(0, \dots, 0, 1, 0 \dots, 0)}^{1 \text{ is in the } i^{\text{th}} \text{ slot}}$$

for $i = 1, 2, \dots, n$ are a basis for $\mathbb{F}^n$. This proves the corollary.

Lemma 4.3.6 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ be a set of vectors. Then $V \equiv \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_r)$ is a subspace.

²This is the plural form of basis. We could say basiss but it would involve an inordinate amount of hissing as in "The sixth shiek's sixth sheep is sick". This is the reason that bases is used instead of basiss.

Proof: Suppose α, β are two scalars and let $\sum_{k=1}^r c_k \mathbf{v}_k$ and $\sum_{k=1}^r d_k \mathbf{v}_k$ are two elements of V . What about

$$\alpha \sum_{k=1}^r c_k \mathbf{v}_k + \beta \sum_{k=1}^r d_k \mathbf{v}_k?$$

Is it also in V ?

$$\alpha \sum_{k=1}^r c_k \mathbf{v}_k + \beta \sum_{k=1}^r d_k \mathbf{v}_k = \sum_{k=1}^r (\alpha c_k + \beta d_k) \mathbf{v}_k \in V$$

so the answer is yes. This proves the lemma.

Definition 4.3.7 A finite set of vectors, $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is a basis for a subspace, V of $\mathbb{F}^n$ if $\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_r) = V$ and $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is linearly independent.

Corollary 4.3.8 Let $\{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_s\}$ be two bases for V . Then $r = s$.

Proof: From the exchange theorem, $r \leq s$ and $s \leq r$. Therefore, this proves the corollary.

Definition 4.3.9 Let V be a subspace of $\mathbb{F}^n$. Then $\dim(V)$ read as the dimension of V is the number of vectors in a basis.

Of course you should wonder right now whether an arbitrary subspace even has a basis. In fact it does and this is in the next theorem. First, here is an interesting lemma.

Lemma 4.3.10 Suppose $\mathbf{v} \notin \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is linearly independent. Then $\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}\}$ is also linearly independent.

Proof: Suppose $\sum_{i=1}^k c_i \mathbf{u}_i + d\mathbf{v} = \mathbf{0}$. It is required to verify that each $c_i = 0$ and that $d = 0$. But if $d \neq 0$, then you can solve for $\mathbf{v}$ as a linear combination of the vectors, $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$,

$$\mathbf{v} = -\sum_{i=1}^k \left(\frac{c_i}{d}\right) \mathbf{u}_i$$

contrary to assumption. Therefore, $d = 0$. But then $\sum_{i=1}^k c_i \mathbf{u}_i = \mathbf{0}$ and the linear independence of $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ implies each $c_i = 0$ also. This proves the lemma.

Theorem 4.3.11 Let V be a nonzero subspace of $\mathbb{F}^n$. Then V has a basis.

Proof: Let $\mathbf{v}_1 \in V$ where $\mathbf{v}_1 \neq \mathbf{0}$. If $\text{span}\{\mathbf{v}_1\} = V$, stop. $\{\mathbf{v}_1\}$ is a basis for V . Otherwise, there exists $\mathbf{v}_2 \in V$ which is not in $\text{span}\{\mathbf{v}_1\}$. By Lemma 4.3.10 $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set of vectors. If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = V$ stop, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for V . If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} \neq V$, then there exists $\mathbf{v}_3 \notin \text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a larger linearly independent set of vectors. Continuing this way, the process must stop before $n + 1$ steps because if not, it would be possible to obtain $n + 1$ linearly independent vectors contrary to the exchange theorem. This proves the theorem.

In words the following corollary states that any linearly independent set of vectors can be enlarged to form a basis.

Corollary 4.3.12 *Let V be a subspace of $\mathbb{F}^n$ and let $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ be a linearly independent set of vectors in V . Then either it is a basis for V or there exist vectors, $\mathbf{v}_{r+1}, \dots, \mathbf{v}_s$ such that $\{\mathbf{v}_1, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_s\}$ is a basis for V .*

Proof: This follows immediately from the proof of Theorem 47.9.4. You do exactly the same argument except you start with $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ rather than $\{\mathbf{v}_1\}$.

It is also true that any spanning set of vectors can be restricted to obtain a basis.

Theorem 4.3.13 *Let V be a subspace of $\mathbb{F}^n$ and suppose $\text{span}(\mathbf{u}_1 \cdots, \mathbf{u}_p) = V$ where the $\mathbf{u}_i$ are nonzero vectors. Then there exist vectors, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ such that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$ and $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is a basis for V .*

Proof: Let r be the smallest positive integer with the property that for some set, $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\} \subseteq \{\mathbf{u}_1 \cdots, \mathbf{u}_p\}$,

$$\text{span}(\mathbf{v}_1 \cdots, \mathbf{v}_r) = V.$$

Then $r \leq p$ and it must be the case that $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ is linearly independent because if it were not so, one of the vectors, say $\mathbf{v}_k$ would be a linear combination of the others. But then you could delete this vector from $\{\mathbf{v}_1 \cdots, \mathbf{v}_r\}$ and the resulting list of $r - 1$ vectors would still span V contrary to the definition of r . This proves the theorem.

4.4 An Application To Matrices

The following is a theorem of major significance.

Theorem 4.4.1 *Suppose A is an $n \times n$ matrix. Then A is one to one if and only if A is onto. Also, if B is an $n \times n$ matrix and $AB = I$, then it follows $BA = I$.*

Proof: First suppose A is one to one. Consider the vectors, $\{A\mathbf{e}_1, \dots, A\mathbf{e}_n\}$ where $\mathbf{e}_k$ is the column vector which is all zeros except for a 1 in the k^{th} position. This set of vectors is linearly independent because if

$$\sum_{k=1}^n c_k A\mathbf{e}_k = \mathbf{0},$$

then since A is linear,

$$A \left(\sum_{k=1}^n c_k \mathbf{e}_k \right) = \mathbf{0}$$

and since A is one to one, it follows

$$\sum_{k=1}^n c_k \mathbf{e}_k = \mathbf{0}^3$$

which implies each $c_k = 0$. Therefore, $\{A\mathbf{e}_1, \dots, A\mathbf{e}_n\}$ must be a basis for $\mathbb{F}^n$ because if not there would exist a vector, $\mathbf{y} \notin \text{span}(A\mathbf{e}_1, \dots, A\mathbf{e}_n)$ and then by Lemma 4.3.10, $\{A\mathbf{e}_1, \dots, A\mathbf{e}_n, \mathbf{y}\}$ would be an independent set of vectors having $n + 1$ vectors in it, contrary to the exchange theorem. It follows that for $\mathbf{y} \in \mathbb{F}^n$ there exist constants, c_i such that

$$\mathbf{y} = \sum_{k=1}^n c_k A\mathbf{e}_k = A \left(\sum_{k=1}^n c_k \mathbf{e}_k \right)$$

showing that, since $\mathbf{y}$ was arbitrary, A is onto.

Next suppose A is onto. This means the span of the columns of A equals $\mathbb{F}^n$. If these columns are not linearly independent, then by Lemma 4.3.2 on Page 57, one of the columns is a linear combination of the others and so the span of the columns of A equals the span of the $n - 1$ other columns. This violates the exchange theorem because $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ would be a linearly independent set of vectors contained in the span of only $n - 1$ vectors. Therefore, the columns of A must be independent and this equivalent to saying that $A\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. This implies A is one to one because if $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x} - \mathbf{y}) = \mathbf{0}$ and so $\mathbf{x} - \mathbf{y} = \mathbf{0}$.

Now suppose $AB = I$. Why is $BA = I$? Since $AB = I$ it follows B is one to one since otherwise, there would exist, $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$ and then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0} \neq I\mathbf{x}$. Therefore, from what was just shown, B is also onto. In addition to this, A must be one to one because if $A\mathbf{y} = \mathbf{0}$, then $\mathbf{y} = B\mathbf{x}$ for some $\mathbf{x}$ and then $\mathbf{x} = AB\mathbf{x} = A\mathbf{y} = \mathbf{0}$ showing $\mathbf{y} = \mathbf{0}$. Now from what is given to be so, it follows $(AB)A = A$ and so using the associative law for matrix multiplication,

$$A(BA) - A = A(BA - I) = \mathbf{0}.$$

But this means $(BA - I)\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$ since otherwise, A would not be one to one. Hence $BA = I$ as claimed. This proves the theorem.

This theorem shows that if an $n \times n$ matrix, B acts like an inverse when multiplied on one side of A it follows that $B = A^{-1}$ and it will act like an inverse on both sides of A .

The conclusion of this theorem pertains to square matrices only. For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix} \quad (4.4.13)$$

Then

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

but

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

4.5 The Mathematical Theory Of Determinants

It is assumed the reader is familiar with matrices. However, the topic of determinants is often neglected in linear algebra books these days. Therefore, I will give a fairly quick and grubby treatment of this topic which includes all the main results. Two books which give a good introduction to determinants are Apostol [3] and Rudin [57]. A recent book which also has a good introduction is Baker [8].

Let $(i_1, \dots, i_n)$ be an ordered list of numbers from $\{1, \dots, n\}$. This means the order is important so $(1, 2, 3)$ and $(2, 1, 3)$ are different.

The following Lemma will be essential in the definition of the determinant.

Lemma 4.5.1 *There exists a unique function, sgn_n which maps each list of n numbers from $\{1, \dots, n\}$ to one of the three numbers, 0, 1, or -1 which also has the following properties.*

$$\text{sgn}_n(1, \dots, n) = 1 \quad (4.5.14)$$

$$\text{sgn}_n(i_1, \dots, p, \dots, q, \dots, i_n) = -\text{sgn}_n(i_1, \dots, q, \dots, p, \dots, i_n) \quad (4.5.15)$$

In words, the second property states that if two of the numbers are switched, the value of the function is multiplied by -1 . Also, in the case where $n > 1$ and $\{i_1, \dots, i_n\} = \{1, \dots, n\}$ so that every number from $\{1, \dots, n\}$ appears in the ordered list, $(i_1, \dots, i_n)$,

$$\begin{aligned} \text{sgn}_n(i_1, \dots, i_{\theta-1}, n, i_{\theta+1}, \dots, i_n) &\equiv \\ (-1)^{n-\theta} \text{sgn}_{n-1}(i_1, \dots, i_{\theta-1}, i_{\theta+1}, \dots, i_n) \end{aligned} \quad (4.5.16)$$

where $n = i_\theta$ in the ordered list, $(i_1, \dots, i_n)$.

Proof: To begin with, it is necessary to show the existence of such a function. This is clearly true if $n = 1$. Define $\text{sgn}_1(1) \equiv 1$ and observe that it works. No switching is possible. In the case where $n = 2$, it is also clearly true. Let $\text{sgn}_2(1, 2) = 1$ and $\text{sgn}_2(2, 1) = -1$ while $\text{sgn}_2(2, 2) = \text{sgn}_2(1, 1) = 0$ and verify it works. Assuming such a function exists for n , sgn_{n+1} will be defined in terms of sgn_n . If there are any repeated numbers in $(i_1, \dots, i_{n+1})$, $\text{sgn}_{n+1}(i_1, \dots, i_{n+1}) \equiv 0$. If there are no repeats, then $n+1$ appears somewhere in the ordered list. Let θ be the position of the number $n+1$ in the list. Thus, the list is of the form $(i_1, \dots, i_{\theta-1}, n+1, i_{\theta+1}, \dots, i_{n+1})$. From 4.5.16 it must be that

$$\begin{aligned} \text{sgn}_{n+1}(i_1, \dots, i_{\theta-1}, n+1, i_{\theta+1}, \dots, i_{n+1}) &\equiv \\ (-1)^{n+1-\theta} \text{sgn}_n(i_1, \dots, i_{\theta-1}, i_{\theta+1}, \dots, i_{n+1}). \end{aligned}$$

It is necessary to verify this satisfies 4.5.14 and 4.5.15 with n replaced with $n + 1$. The first of these is obviously true because

$$\text{sgn}_{n+1}(1, \dots, n, n+1) \equiv (-1)^{n+1-(n+1)} \text{sgn}_n(1, \dots, n) = 1.$$

If there are repeated numbers in $(i_1, \dots, i_{n+1})$, then it is obvious 4.5.15 holds because both sides would equal zero from the above definition. It remains to verify 4.5.15 in the case where there are no numbers repeated in $(i_1, \dots, i_{n+1})$. Consider

$$\text{sgn}_{n+1}(i_1, \dots, \overset{r}{p}, \dots, \overset{s}{q}, \dots, i_{n+1}),$$

where the r above the p indicates the number, p is in the r^{th} position and the s above the q indicates that the number, q is in the s^{th} position. Suppose first that $r < \theta < s$. Then

$$\begin{aligned} \text{sgn}_{n+1}(i_1, \dots, \overset{r}{p}, \dots, \overset{\theta}{n+1}, \dots, \overset{s}{q}, \dots, i_{n+1}) &\equiv \\ (-1)^{n+1-\theta} \text{sgn}_n(i_1, \dots, \overset{r}{p}, \dots, \overset{s-1}{q}, \dots, i_{n+1}) \end{aligned}$$

while

$$\begin{aligned} \text{sgn}_{n+1}(i_1, \dots, \overset{r}{q}, \dots, \overset{\theta}{n+1}, \dots, \overset{s}{p}, \dots, i_{n+1}) &= \\ (-1)^{n+1-\theta} \text{sgn}_n(i_1, \dots, \overset{r}{q}, \dots, \overset{s-1}{p}, \dots, i_{n+1}) \end{aligned}$$

and so, by induction, a switch of p and q introduces a minus sign in the result. Similarly, if $\theta > s$ or if $\theta < r$ it also follows that 4.5.15 holds. The interesting case is when $\theta = r$ or $\theta = s$. Consider the case where $\theta = r$ and note the other case is entirely similar.

$$\begin{aligned} \text{sgn}_{n+1}(i_1, \dots, \overset{r}{n+1}, \dots, \overset{s}{q}, \dots, i_{n+1}) &= \\ (-1)^{n+1-r} \text{sgn}_n(i_1, \dots, \overset{s-1}{q}, \dots, i_{n+1}) \end{aligned} \tag{4.5.17}$$

while

$$\begin{aligned} \text{sgn}_{n+1}(i_1, \dots, \overset{r}{q}, \dots, \overset{s}{n+1}, \dots, i_{n+1}) &= \\ (-1)^{n+1-s} \text{sgn}_n(i_1, \dots, \overset{r}{q}, \dots, i_{n+1}). \end{aligned} \tag{4.5.18}$$

By making $s-1-r$ switches, move the q which is in the $s-1^{\text{th}}$ position in 4.5.17 to the r^{th} position in 4.5.18. By induction, each of these switches introduces a factor of -1 and so

$$\text{sgn}_n(i_1, \dots, \overset{s-1}{q}, \dots, i_{n+1}) = (-1)^{s-1-r} \text{sgn}_n(i_1, \dots, \overset{r}{q}, \dots, i_{n+1}).$$

Therefore,

$$\begin{aligned}
 & \operatorname{sgn}_{n+1} \left(i_1, \dots, n+1, \dots, \overset{r}{q}, \dots, \overset{s}{q}, \dots, i_{n+1} \right) \\
 &= (-1)^{n+1-r} \operatorname{sgn}_n \left(i_1, \dots, \overset{s-1}{q}, \dots, i_{n+1} \right) \\
 &= (-1)^{n+1-r} (-1)^{s-1-r} \operatorname{sgn}_n \left(i_1, \dots, \overset{r}{q}, \dots, i_{n+1} \right) \\
 &= (-1)^{n+s} \operatorname{sgn}_n \left(i_1, \dots, \overset{r}{q}, \dots, i_{n+1} \right) \\
 &= (-1)^{2s-1} (-1)^{n+1-s} \operatorname{sgn}_n \left(i_1, \dots, \overset{r}{q}, \dots, i_{n+1} \right) \\
 &= -\operatorname{sgn}_{n+1} \left(i_1, \dots, \overset{r}{q}, \dots, n+1, \dots, \overset{s}{q}, \dots, i_{n+1} \right).
 \end{aligned}$$

This proves the existence of the desired function.

To see this function is unique, note that you can obtain any ordered list of distinct numbers from a sequence of switches. If there exist two functions, f and g both satisfying 4.5.14 and 4.5.15, you could start with $f(1, \dots, n) = g(1, \dots, n)$ and applying the same sequence of switches, eventually arrive at $f(i_1, \dots, i_n) = g(i_1, \dots, i_n)$. If any numbers are repeated, then 4.5.15 gives both functions are equal to zero for that ordered list. This proves the lemma.

In what follows sgn will often be used rather than sgn_n because the context supplies the appropriate n .

Definition 4.5.2 *Let f be a real valued function which has the set of ordered lists of numbers from $\{1, \dots, n\}$ as its domain. Define*

$$\sum_{(k_1, \dots, k_n)} f(k_1 \cdots k_n)$$

to be the sum of all the

$$f(k_1 \cdots k_n)$$

for all possible choices of ordered lists

$$(k_1, \dots, k_n)$$

of numbers of

$$\{1, \dots, n\}.$$

For example,

$$\sum_{(k_1, k_2)} f(k_1, k_2) = f(1, 2) + f(2, 1) + f(1, 1) + f(2, 2).$$

Definition 4.5.3 Let $(a_{ij}) = A$ denote an $n \times n$ matrix. The determinant of A , denoted by $\det(A)$ is defined by

$$\det(A) \equiv \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{1k_1} \cdots a_{nk_n}$$

where the sum is taken over all ordered lists of numbers from $\{1, \dots, n\}$. Note it suffices to take the sum over only those ordered lists in which there are no repeats because if there are, $\operatorname{sgn}(k_1, \dots, k_n) = 0$ and so that term contributes 0 to the sum.

Let A be an $n \times n$ matrix, $A = (a_{ij})$ and let $(r_1, \dots, r_n)$ denote an ordered list of n numbers from $\{1, \dots, n\}$. Let $A(r_1, \dots, r_n)$ denote the matrix whose k^{th} row is the r_k row of the matrix, A . Thus

$$\det(A(r_1, \dots, r_n)) = \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n} \quad (4.5.19)$$

and

$$A(1, \dots, n) = A.$$

Proposition 4.5.4 Let

$$(r_1, \dots, r_n)$$

be an ordered list of numbers from $\{1, \dots, n\}$. Then

$$\operatorname{sgn}(r_1, \dots, r_n) \det(A)$$

$$= \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n} \quad (4.5.20)$$

$$= \det(A(r_1, \dots, r_n)). \quad (4.5.21)$$

Proof: Let $(1, \dots, n) = (1, \dots, r, \dots, s, \dots, n)$ so $r < s$.

$$\det(A(1, \dots, r, \dots, s, \dots, n)) = \quad (4.5.22)$$

$$\sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_r, \dots, k_s, \dots, k_n) a_{1k_1} \cdots a_{rk_r} \cdots a_{sk_s} \cdots a_{nk_n},$$

and renaming the variables, calling k_s, k_r and k_r, k_s , this equals

$$= \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_s, \dots, k_r, \dots, k_n) a_{1k_1} \cdots a_{rk_s} \cdots a_{sk_r} \cdots a_{nk_n}$$

$$= \sum_{(k_1, \dots, k_n)} -\operatorname{sgn}\left(k_1, \dots, \overbrace{k_r, \dots, k_s}^{\text{These got switched}}, \dots, k_n\right) a_{1k_1} \cdots a_{sk_r} \cdots a_{rk_s} \cdots a_{nk_n}$$

$$= -\det(A(1, \dots, s, \dots, r, \dots, n)). \quad (4.5.23)$$

Consequently,

$$\begin{aligned} & \det(A(1, \dots, s, \dots, r, \dots, n)) = \\ & -\det(A(1, \dots, r, \dots, s, \dots, n)) = -\det(A) \end{aligned}$$

Now letting $A(1, \dots, s, \dots, r, \dots, n)$ play the role of A , and continuing in this way, switching pairs of numbers,

$$\det(A(r_1, \dots, r_n)) = (-1)^p \det(A)$$

where it took p switches to obtain $(r_1, \dots, r_n)$ from $(1, \dots, n)$. By Lemma 4.5.1, this implies

$$\det(A(r_1, \dots, r_n)) = (-1)^p \det(A) = \operatorname{sgn}(r_1, \dots, r_n) \det(A)$$

and proves the proposition in the case when there are no repeated numbers in the ordered list, $(r_1, \dots, r_n)$. However, if there is a repeat, say the r^{th} row equals the s^{th} row, then the reasoning of 4.5.22–4.5.23 shows that $A(r_1, \dots, r_n) = 0$ and also $\operatorname{sgn}(r_1, \dots, r_n) = 0$ so the formula holds in this case also.

Observation 4.5.5 *There are $n!$ ordered lists of distinct numbers from $\{1, \dots, n\}$.*

To see this, consider n slots placed in order. There are n choices for the first slot. For each of these choices, there are $n - 1$ choices for the second. Thus there are $n(n - 1)$ ways to fill the first two slots. Then for each of these ways there are $n - 2$ choices left for the third slot. Continuing this way, there are $n!$ ordered lists of distinct numbers from $\{1, \dots, n\}$ as stated in the observation.

With the above, it is possible to give a more symmetric description of the determinant from which it will follow that $\det(A) = \det(A^T)$.

Corollary 4.5.6 *The following formula for $\det(A)$ is valid.*

$$\det(A) = \frac{1}{n!}.$$

$$\sum_{(r_1, \dots, r_n)} \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(r_1, \dots, r_n) \operatorname{sgn}(k_1, \dots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}. \quad (4.5.24)$$

And also $\det(A^T) = \det(A)$ where A^T is the transpose of A . (Recall that for $A^T = (a_{ij}^T)$, $a_{ij}^T = a_{ji}$.)

Proof: From Proposition 4.5.4, if the r_i are distinct,

$$\det(A) = \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(r_1, \dots, r_n) \operatorname{sgn}(k_1, \dots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}.$$

Summing over all ordered lists, $(r_1, \dots, r_n)$ where the r_i are distinct, (If the r_i are not distinct, $\operatorname{sgn}(r_1, \dots, r_n) = 0$ and so there is no contribution to the sum.)

$$n! \det(A) =$$

$$\sum_{(r_1, \dots, r_n)} \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(r_1, \dots, r_n) \operatorname{sgn}(k_1, \dots, k_n) a_{r_1 k_1} \cdots a_{r_n k_n}.$$

This proves the corollary since the formula gives the same number for A as it does for A^T .

Corollary 4.5.7 *If two rows or two columns in an $n \times n$ matrix, A , are switched, the determinant of the resulting matrix equals (-1) times the determinant of the original matrix. If A is an $n \times n$ matrix in which two rows are equal or two columns are equal then $\det(A) = 0$. Suppose the i^{th} row of A equals $(xa_1 + yb_1, \dots, xa_n + yb_n)$. Then*

$$\det(A) = x \det(A_1) + y \det(A_2)$$

where the i^{th} row of A_1 is $(a_1, \dots, a_n)$ and the i^{th} row of A_2 is $(b_1, \dots, b_n)$, all other rows of A_1 and A_2 coinciding with those of A . In other words, $\det$ is a linear function of each row A . The same is true with the word “row” replaced with the word “column”.

Proof: By Proposition 4.5.4 when two rows are switched, the determinant of the resulting matrix is (-1) times the determinant of the original matrix. By Corollary 4.5.6 the same holds for columns because the columns of the matrix equal the rows of the transposed matrix. Thus if A_1 is the matrix obtained from A by switching two columns,

$$\det(A) = \det(A^T) = -\det(A_1^T) = -\det(A_1).$$

If A has two equal columns or two equal rows, then switching them results in the same matrix. Therefore, $\det(A) = -\det(A)$ and so $\det(A) = 0$.

It remains to verify the last assertion.

$$\begin{aligned} \det(A) &\equiv \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{1k_1} \cdots (xa_{ik_i} + yb_{ik_i}) \cdots a_{nk_n} \\ &= x \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{1k_1} \cdots a_{ik_i} \cdots a_{nk_n} \\ &\quad + y \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{1k_1} \cdots b_{ik_i} \cdots a_{nk_n} \\ &\equiv x \det(A_1) + y \det(A_2). \end{aligned}$$

The same is true of columns because $\det(A^T) = \det(A)$ and the rows of A^T are the columns of A .

Definition 4.5.8 *A vector, $\mathbf{w}$, is a linear combination of the vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ if there exists scalars, $c_1, \dots, c_r$ such that $\mathbf{w} = \sum_{k=1}^r c_k \mathbf{v}_k$. This is the same as saying $\mathbf{w} \in \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$.*

The following corollary is also of great use.

Corollary 4.5.9 Suppose A is an $n \times n$ matrix and some column (row) is a linear combination of r other columns (rows). Then $\det(A) = 0$.

Proof: Let $A = (\mathbf{a}_1 \cdots \mathbf{a}_n)$ be the columns of A and suppose the condition that one column is a linear combination of r of the others is satisfied. Then by using Corollary 4.5.7 you may rearrange the columns to have the n^{th} column a linear combination of the first r columns. Thus $\mathbf{a}_n = \sum_{k=1}^r c_k \mathbf{a}_k$ and so

$$\det(A) = \det(\mathbf{a}_1 \cdots \mathbf{a}_r \cdots \mathbf{a}_{n-1} \sum_{k=1}^r c_k \mathbf{a}_k).$$

By Corollary 4.5.7

$$\det(A) = \sum_{k=1}^r c_k \det(\mathbf{a}_1 \cdots \mathbf{a}_r \cdots \mathbf{a}_{n-1} \mathbf{a}_k) = 0.$$

The case for rows follows from the fact that $\det(A) = \det(A^T)$. This proves the corollary.

Recall the following definition of matrix multiplication.

Definition 4.5.10 If A and B are $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$, $AB = (c_{ij})$ where

$$c_{ij} \equiv \sum_{k=1}^n a_{ik} b_{kj}.$$

One of the most important rules about determinants is that the determinant of a product equals the product of the determinants.

Theorem 4.5.11 Let A and B be $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B).$$

Proof: Let c_{ij} be the ij^{th} entry of AB . Then by Proposition 4.5.4,

$$\begin{aligned} \det(AB) &= \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) c_{1k_1} \cdots c_{nk_n} \\ &= \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) \left(\sum_{r_1} a_{1r_1} b_{r_1 k_1} \right) \cdots \left(\sum_{r_n} a_{nr_n} b_{r_n k_n} \right) \\ &= \sum_{(r_1, \dots, r_n)} \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) b_{r_1 k_1} \cdots b_{r_n k_n} (a_{1r_1} \cdots a_{nr_n}) \\ &= \sum_{(r_1, \dots, r_n)} \operatorname{sgn}(r_1 \cdots r_n) a_{1r_1} \cdots a_{nr_n} \det(B) = \det(A) \det(B). \end{aligned}$$

This proves the theorem.

Lemma 4.5.12 Suppose a matrix is of the form

$$M = \begin{pmatrix} A & * \\ \mathbf{0} & a \end{pmatrix} \quad (4.5.25)$$

or

$$M = \begin{pmatrix} A & \mathbf{0} \\ * & a \end{pmatrix} \quad (4.5.26)$$

where a is a number and A is an $(n-1) \times (n-1)$ matrix and $*$ denotes either a column or a row having length $n-1$ and the $\mathbf{0}$ denotes either a column or a row of length $n-1$ consisting entirely of zeros. Then

$$\det(M) = a \det(A).$$

Proof: Denote M by (m_{ij}) . Thus in the first case, $m_{nn} = a$ and $m_{ni} = 0$ if $i \neq n$ while in the second case, $m_{nn} = a$ and $m_{in} = 0$ if $i \neq n$. From the definition of the determinant,

$$\det(M) \equiv \sum_{(k_1, \dots, k_n)} \text{sgn}_n(k_1, \dots, k_n) m_{1k_1} \cdots m_{nk_n}$$

Letting θ denote the position of n in the ordered list, $(k_1, \dots, k_n)$ then using the earlier conventions used to prove Lemma 4.5.1, $\det(M)$ equals

$$\sum_{(k_1, \dots, k_n)} (-1)^{n-\theta} \text{sgn}_{n-1} \left(k_1, \dots, k_{\theta-1}, k_{\theta+1}, \dots, k_n \right) m_{1k_1} \cdots m_{nk_n}$$

Now suppose 4.5.26. Then if $k_n \neq n$, the term involving m_{nk_n} in the above expression equals zero. Therefore, the only terms which survive are those for which $\theta = n$ or in other words, those for which $k_n = n$. Therefore, the above expression reduces to

$$a \sum_{(k_1, \dots, k_{n-1})} \text{sgn}_{n-1}(k_1, \dots, k_{n-1}) m_{1k_1} \cdots m_{(n-1)k_{n-1}} = a \det(A).$$

To get the assertion in the situation of 4.5.25 use Corollary 4.5.6 and 4.5.26 to write

$$\det(M) = \det(M^T) = \det \left(\begin{pmatrix} A^T & \mathbf{0} \\ * & a \end{pmatrix} \right) = a \det(A^T) = a \det(A).$$

This proves the lemma.

In terms of the theory of determinants, arguably the most important idea is that of Laplace expansion along a row or a column. This will follow from the above definition of a determinant.

Definition 4.5.13 Let $A = (a_{ij})$ be an $n \times n$ matrix. Then a new matrix called the cofactor matrix, $\text{cof}(A)$ is defined by $\text{cof}(A) = (c_{ij})$ where to obtain c_{ij} delete the i^{th} row and the j^{th} column of A , take the determinant of the $(n-1) \times (n-1)$ matrix which results, (This is called the ij^{th} minor of A .) and then multiply this number by $(-1)^{i+j}$. To make the formulas easier to remember, $\text{cof}(A)_{ij}$ will denote the ij^{th} entry of the cofactor matrix.

The following is the main result. Earlier this was given as a definition and the outrageous totally unjustified assertion was made that the same number would be obtained by expanding the determinant along any row or column. The following theorem proves this assertion.

Theorem 4.5.14 *Let A be an $n \times n$ matrix where $n \geq 2$. Then*

$$\det(A) = \sum_{j=1}^n a_{ij} \operatorname{cof}(A)_{ij} = \sum_{i=1}^n a_{ij} \operatorname{cof}(A)_{ij}. \quad (4.5.27)$$

The first formula consists of expanding the determinant along the i^{th} row and the second expands the determinant along the j^{th} column.

Proof: Let $(a_{i1}, \dots, a_{in})$ be the i^{th} row of A . Let B_j be the matrix obtained from A by leaving every row the same except the i^{th} row which in B_j equals $(0, \dots, 0, a_{ij}, 0, \dots, 0)$. Then by Corollary 4.5.7,

$$\det(A) = \sum_{j=1}^n \det(B_j)$$

Denote by A^{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and the j^{th} column of A . Thus $\operatorname{cof}(A)_{ij} \equiv (-1)^{i+j} \det(A^{ij})$. At this point, recall that from Proposition 4.5.4, when two rows or two columns in a matrix, M , are switched, this results in multiplying the determinant of the old matrix by -1 to get the determinant of the new matrix. Therefore, by Lemma 4.5.12,

$$\begin{aligned} \det(B_j) &= (-1)^{n-j} (-1)^{n-i} \det \left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix} \right) \\ &= (-1)^{i+j} \det \left(\begin{pmatrix} A^{ij} & * \\ \mathbf{0} & a_{ij} \end{pmatrix} \right) = a_{ij} \operatorname{cof}(A)_{ij}. \end{aligned}$$

Therefore,

$$\det(A) = \sum_{j=1}^n a_{ij} \operatorname{cof}(A)_{ij}$$

which is the formula for expanding $\det(A)$ along the i^{th} row. Also,

$$\begin{aligned} \det(A) &= \det(A^T) = \sum_{j=1}^n a_{ij}^T \operatorname{cof}(A^T)_{ij} \\ &= \sum_{j=1}^n a_{ji} \operatorname{cof}(A)_{ji} \end{aligned}$$

which is the formula for expanding $\det(A)$ along the i^{th} column. This proves the theorem.

Note that this gives an easy way to write a formula for the inverse of an $n \times n$ matrix.

Theorem 4.5.15 A^{-1} exists if and only if $\det(A) \neq 0$. If $\det(A) \neq 0$, then $A^{-1} = (a_{ij}^{-1})$ where

$$a_{ij}^{-1} = \det(A)^{-1} \operatorname{cof}(A)_{ji}$$

for $\operatorname{cof}(A)_{ij}$ the ij^{th} cofactor of A .

Proof: By Theorem 4.5.14 and letting $(a_{ir}) = A$, if $\det(A) \neq 0$,

$$\sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ir} \det(A)^{-1} = \det(A) \det(A)^{-1} = 1.$$

Now consider

$$\sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

when $k \neq r$. Replace the k^{th} column with the r^{th} column to obtain a matrix, B_k whose determinant equals zero by Corollary 4.5.7. However, expanding this matrix along the k^{th} column yields

$$0 = \det(B_k) \det(A)^{-1} = \sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1}$$

Summarizing,

$$\sum_{i=1}^n a_{ir} \operatorname{cof}(A)_{ik} \det(A)^{-1} = \delta_{rk}.$$

Using the other formula in Theorem 4.5.14, and similar reasoning,

$$\sum_{j=1}^n a_{rj} \operatorname{cof}(A)_{kj} \det(A)^{-1} = \delta_{rk}$$

This proves that if $\det(A) \neq 0$, then A^{-1} exists with $A^{-1} = (a_{ij}^{-1})$, where

$$a_{ij}^{-1} = \operatorname{cof}(A)_{ji} \det(A)^{-1}.$$

Now suppose A^{-1} exists. Then by Theorem 4.5.11,

$$1 = \det(I) = \det(AA^{-1}) = \det(A) \det(A^{-1})$$

so $\det(A) \neq 0$. This proves the theorem.

The next corollary points out that if an $n \times n$ matrix, A has a right or a left inverse, then it has an inverse.

Corollary 4.5.16 Let A be an $n \times n$ matrix and suppose there exists an $n \times n$ matrix, B such that $BA = I$. Then A^{-1} exists and $A^{-1} = B$. Also, if there exists C an $n \times n$ matrix such that $AC = I$, then A^{-1} exists and $A^{-1} = C$.

Proof: Since $BA = I$, Theorem 4.5.11 implies

$$\det B \det A = 1$$

and so $\det A \neq 0$. Therefore from Theorem 4.5.15, A^{-1} exists. Therefore,

$$A^{-1} = (BA) A^{-1} = B (AA^{-1}) = BI = B.$$

The case where $CA = I$ is handled similarly.

The conclusion of this corollary is that left inverses, right inverses and inverses are all the same in the context of $n \times n$ matrices.

Theorem 4.5.15 says that to find the inverse, take the transpose of the cofactor matrix and divide by the determinant. The transpose of the cofactor matrix is called the adjugate or sometimes the classical adjoint of the matrix A . It is an abomination to call it the adjoint although you do sometimes see it referred to in this way. In words, A^{-1} is equal to one over the determinant of A times the adjugate matrix of A .

In case you are solving a system of equations, $A\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, it follows that if A^{-1} exists,

$$\mathbf{x} = (A^{-1}A) \mathbf{x} = A^{-1} (A\mathbf{x}) = A^{-1}\mathbf{y}$$

thus solving the system. Now in the case that A^{-1} exists, there is a formula for A^{-1} given above. Using this formula,

$$x_i = \sum_{j=1}^n a_{ij}^{-1} y_j = \sum_{j=1}^n \frac{1}{\det(A)} \operatorname{cof}(A)_{ji} y_j.$$

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{pmatrix} * & \cdots & y_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & y_n & \cdots & * \end{pmatrix},$$

where here the i^{th} column of A is replaced with the column vector, $(y_1 \cdots y_n)^T$, and the determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

Definition 4.5.17 A matrix M , is upper triangular if $M_{ij} = 0$ whenever $i > j$. Thus such a matrix equals zero below the main diagonal, the entries of the form M_{ii} as shown.

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & * \end{pmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

With this definition, here is a simple corollary of Theorem 4.5.14.

Corollary 4.5.18 *Let M be an upper (lower) triangular matrix. Then $\det(M)$ is obtained by taking the product of the entries on the main diagonal.*

Definition 4.5.19 *A submatrix of a matrix A is the rectangular array of numbers obtained by deleting some rows and columns of A . Let A be an $m \times n$ matrix. The **determinant rank** of the matrix equals r where r is the largest number such that some $r \times r$ submatrix of A has a non zero determinant. The **row rank** is defined to be the dimension of the span of the rows. The **column rank** is defined to be the dimension of the span of the columns.*

Theorem 4.5.20 *If A has determinant rank, r , then there exist r rows of the matrix such that every other row is a linear combination of these r rows.*

Proof: Suppose the determinant rank of $A = (a_{ij})$ equals r . If rows and columns are interchanged, the determinant rank of the modified matrix is unchanged. Thus rows and columns can be interchanged to produce an $r \times r$ matrix in the upper left corner of the matrix which has non zero determinant. Now consider the $(r+1) \times (r+1)$ matrix, M ,

$$\begin{pmatrix} a_{11} & \cdots & a_{1r} & a_{1p} \\ \vdots & & \vdots & \vdots \\ a_{r1} & \cdots & a_{rr} & a_{rp} \\ a_{l1} & \cdots & a_{lr} & a_{lp} \end{pmatrix}$$

where C will denote the $r \times r$ matrix in the upper left corner which has non zero determinant. I claim $\det(M) = 0$.

There are two cases to consider in verifying this claim. First, suppose $p > r$. Then the claim follows from the assumption that A has determinant rank r . On the other hand, if $p < r$, then the determinant is zero because there are two identical columns. Expand the determinant along the last column and divide by $\det(C)$ to obtain

$$a_{lp} = - \sum_{i=1}^r \frac{\text{cof}(M)_{ip}}{\det(C)} a_{ip}.$$

Now note that $\text{cof}(M)_{ip}$ does not depend on p . Therefore the above sum is of the form

$$a_{lp} = \sum_{i=1}^r m_i a_{ip}$$

which shows the l^{th} row is a linear combination of the first r rows of A . Since l is arbitrary, this proves the theorem.

Corollary 4.5.21 *The determinant rank equals the row rank.*

Proof: From Theorem 4.5.20, the row rank is no larger than the determinant rank. Could the row rank be smaller than the determinant rank? If so, there exist p rows for $p < r$ such that the span of these p rows equals the row space. But this implies that the $r \times r$ submatrix whose determinant is nonzero also has row rank no larger than p which is impossible if its determinant is to be nonzero because at least one row is a linear combination of the others.

Corollary 4.5.22 *If A has determinant rank, r , then there exist r columns of the matrix such that every other column is a linear combination of these r columns. Also the column rank equals the determinant rank.*

Proof: This follows from the above by considering A^T . The rows of A^T are the columns of A and the determinant rank of A^T and A are the same. Therefore, from Corollary 4.5.21, column rank of $A = \text{row rank of } A^T = \text{determinant rank of } A^T = \text{determinant rank of } A$.

The following theorem is of fundamental importance and ties together many of the ideas presented above.

Theorem 4.5.23 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. $\det(A) = 0$.
2. A, A^T are not one to one.
3. A is not onto.

Proof: Suppose $\det(A) = 0$. Then the determinant rank of $A = r < n$. Therefore, there exist r columns such that every other column is a linear combination of these columns by Theorem 4.5.20. In particular, it follows that for some m , the m^{th} column is a linear combination of all the others. Thus letting $A = (\mathbf{a}_1 \ \cdots \ \mathbf{a}_m \ \cdots \ \mathbf{a}_n)$ where the columns are denoted by $\mathbf{a}_i$, there exists scalars, α_i such that

$$\mathbf{a}_m = \sum_{k \neq m} \alpha_k \mathbf{a}_k.$$

Now consider the column vector, $\mathbf{x} \equiv (\alpha_1 \ \cdots \ -1 \ \cdots \ \alpha_n)^T$. Then

$$A\mathbf{x} = -\mathbf{a}_m + \sum_{k \neq m} \alpha_k \mathbf{a}_k = \mathbf{0}.$$

Since also $A\mathbf{0} = \mathbf{0}$, it follows A is not one to one. Similarly, A^T is not one to one by the same argument applied to A^T . This verifies that 1.) implies 2.).

Now suppose 2.). Then since A^T is not one to one, it follows there exists $\mathbf{x} \neq \mathbf{0}$ such that

$$A^T \mathbf{x} = \mathbf{0}.$$

Taking the transpose of both sides yields

$$\mathbf{x}^T A = \mathbf{0}$$

where the $\mathbf{0}$ is a $1 \times n$ matrix or row vector. Now if $A\mathbf{y} = \mathbf{x}$, then

$$|\mathbf{x}|^2 = \mathbf{x}^T (A\mathbf{y}) = (\mathbf{x}^T A) \mathbf{y} = \mathbf{0}\mathbf{y} = 0$$

contrary to $\mathbf{x} \neq \mathbf{0}$. Consequently there can be no $\mathbf{y}$ such that $A\mathbf{y} = \mathbf{x}$ and so A is not onto. This shows that 2.) implies 3.).

Finally, suppose 3.). If 1.) does not hold, then $\det(A) \neq 0$ but then from Theorem 4.5.15 A^{-1} exists and so for every $\mathbf{y} \in \mathbb{F}^n$ there exists a unique $\mathbf{x} \in \mathbb{F}^n$ such that $A\mathbf{x} = \mathbf{y}$. In fact $\mathbf{x} = A^{-1}\mathbf{y}$. Thus A would be onto contrary to 3.). This shows 3.) implies 1.) and proves the theorem.

Corollary 4.5.24 *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. $\det(A) \neq 0$.
2. A and A^T are one to one.
3. A is onto.

Proof: This follows immediately from the above theorem.

4.6 Exercises

1. Let $m < n$ and let A be an $m \times n$ matrix. Show that A is **not** one to one.
Hint: Consider the $n \times n$ matrix, A_1 which is of the form

$$A_1 \equiv \begin{pmatrix} A \\ 0 \end{pmatrix}$$

where the 0 denotes an $(n - m) \times n$ matrix of zeros. Thus $\det A_1 = 0$ and so A_1 is not one to one. Now observe that $A_1\mathbf{x}$ is the vector,

$$A_1\mathbf{x} = \begin{pmatrix} A\mathbf{x} \\ \mathbf{0} \end{pmatrix}$$

which equals zero if and only if $A\mathbf{x} = \mathbf{0}$.

4.7 The Cayley Hamilton Theorem

Definition 4.7.1 *Let A be an $n \times n$ matrix. The characteristic polynomial is defined as*

$$p_A(t) \equiv \det(tI - A)$$

and the solutions to $p_A(t) = 0$ are called eigenvalues. For A a matrix and $p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$, denote by $p(A)$ the matrix defined by

$$p(A) \equiv A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I.$$

The explanation for the last term is that A^0 is interpreted as I , the identity matrix.

The Cayley Hamilton theorem states that every matrix satisfies its characteristic equation, that equation defined by $P_A(t) = 0$. It is one of the most important theorems in linear algebra. The following lemma will help with its proof.

Lemma 4.7.2 *Suppose for all $|\lambda|$ large enough,*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = 0,$$

where the A_i are $n \times n$ matrices. Then each $A_i = 0$.

Proof: Multiply by λ^{-m} to obtain

$$A_0\lambda^{-m} + A_1\lambda^{-m+1} + \cdots + A_{m-1}\lambda^{-1} + A_m = 0.$$

Now let $|\lambda| \rightarrow \infty$ to obtain $A_m = 0$. With this, multiply by λ to obtain

$$A_0\lambda^{-m+1} + A_1\lambda^{-m+2} + \cdots + A_{m-1} = 0.$$

Now let $|\lambda| \rightarrow \infty$ to obtain $A_{m-1} = 0$. Continue multiplying by λ and letting $\lambda \rightarrow \infty$ to obtain that all the $A_i = 0$. This proves the lemma.

With the lemma, here is a simple corollary.

Corollary 4.7.3 *Let A_i and B_i be $n \times n$ matrices and suppose*

$$A_0 + A_1\lambda + \cdots + A_m\lambda^m = B_0 + B_1\lambda + \cdots + B_m\lambda^m$$

for all $|\lambda|$ large enough. Then $A_i = B_i$ for all i . Consequently if λ is replaced by any $n \times n$ matrix, the two sides will be equal. That is, for C any $n \times n$ matrix,

$$A_0 + A_1C + \cdots + A_mC^m = B_0 + B_1C + \cdots + B_mC^m.$$

Proof: Subtract and use the result of the lemma.

With this preparation, here is a relatively easy proof of the Cayley Hamilton theorem.

Theorem 4.7.4 *Let A be an $n \times n$ matrix and let $p(\lambda) \equiv \det(\lambda I - A)$ be the characteristic polynomial. Then $p(A) = 0$.*

Proof: Let $C(\lambda)$ equal the transpose of the cofactor matrix of $(\lambda I - A)$ for $|\lambda|$ large. (If $|\lambda|$ is large enough, then λ cannot be in the finite list of eigenvalues of A and so for such λ , $(\lambda I - A)^{-1}$ exists.) Therefore, by Theorem 4.5.15

$$C(\lambda) = p(\lambda)(\lambda I - A)^{-1}.$$

Note that each entry in $C(\lambda)$ is a polynomial in λ having degree no more than $n-1$. Therefore, collecting the terms,

$$C(\lambda) = C_0 + C_1\lambda + \cdots + C_{n-1}\lambda^{n-1}$$

for C_j some $n \times n$ matrix. It follows that for all $|\lambda|$ large enough,

$$(A - \lambda I) (C_0 + C_1 \lambda + \cdots + C_{n-1} \lambda^{n-1}) = p(\lambda) I$$

and so Corollary 4.7.3 may be used. It follows the matrix coefficients corresponding to equal powers of λ are equal on both sides of this equation. Therefore, if λ is replaced with A , the two sides will be equal. Thus

$$0 = (A - A) (C_0 + C_1 A + \cdots + C_{n-1} A^{n-1}) = p(A) I = p(A).$$

This proves the Cayley Hamilton theorem.

4.8 An Identity Of Cauchy

There is a very interesting identity for determinants due to Cauchy.

Theorem 4.8.1 *The following identity holds.*

$$\prod_{i,j} (a_i + b_j) \begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix} = \prod_{j < i} (a_i - a_j) (b_i - b_j). \quad (4.8.28)$$

Proof: What is the exponent of a_2 on the right? It occurs in $(a_2 - a_1)$ and in $(a_m - a_2)$ for $m > 2$. Therefore, there are exactly $n - 1$ factors which contain a_2 . Therefore, a_2 has an exponent of $n - 1$. Similarly, each a_k is raised to the $n - 1$ power and the same holds for the b_k as well. Therefore, the right side of 4.8.28 is of the form

$$c a_1^{n-1} a_2^{n-1} \cdots a_n^{n-1} b_1^{n-1} \cdots b_n^{n-1}$$

where c is some constant. Now consider the left side of 4.8.28.

This is of the form

$$\frac{1}{n!} \prod_{i,j} (a_i + b_j) \sum_{i_1 \cdots i_n, j_1, \cdots, j_n} \text{sgn}(i_1 \cdots i_n) \text{sgn}(j_1 \cdots j_n) \cdot \frac{1}{a_{i_1} + b_{j_1}} \frac{1}{a_{i_2} + b_{j_2}} \cdots \frac{1}{a_{i_n} + b_{j_n}}.$$

For a given $i_1 \cdots i_n, j_1, \cdots, j_n$, let

$$S(i_1 \cdots i_n, j_1, \cdots, j_n) \equiv \{(i_1, j_1), (i_2, j_2) \cdots, (i_n, j_n)\}.$$

This equals

$$\frac{1}{n!} \sum_{i_1 \cdots i_n, j_1, \cdots, j_n} \text{sgn}(i_1 \cdots i_n) \text{sgn}(j_1 \cdots j_n) \prod_{(i,j) \notin \{(i_1, j_1), (i_2, j_2) \cdots, (i_n, j_n)\}} (a_i + b_j)$$

where you can assume the i_k are all distinct and the j_k are also all distinct because otherwise sgn will produce a 0. Therefore, in

$$\prod_{(i,j) \notin \{(i_1,j_1), (i_2,j_2), \dots, (i_n,j_n)\}} (a_i + b_j),$$

there are exactly $n - 1$ factors which contain a_k for each k and similarly, there are exactly $n - 1$ factors which contain b_k for each k . Therefore, the left side of 4.8.28 is of the form

$$da_1^{n-1} a_2^{n-1} \dots a_n^{n-1} b_1^{n-1} \dots b_n^{n-1}$$

and it remains to verify that $c = d$. Using the properties of determinants, the left side of 4.8.28 is of the form

$$\prod_{i \neq j} (a_i + b_j) \begin{vmatrix} 1 & \frac{a_1+b_1}{a_1+b_2} & \dots & \frac{a_1+b_1}{a_1+b_n} \\ \frac{a_2+b_2}{a_2+b_1} & 1 & \dots & \frac{a_2+b_2}{a_2+b_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_n+b_n}{a_n+b_1} & \frac{a_n+b_n}{a_n+b_2} & \dots & 1 \end{vmatrix}$$

Let $a_k \rightarrow -b_k$. Then this converges to $\prod_{i \neq j} (-b_i + b_j)$. The right side of 4.8.28 converges to

$$\prod_{j < i} (-b_i + b_j) (b_i - b_j) = \prod_{i \neq j} (-b_i + b_j).$$

Therefore, $d = c$ and this proves the identity.

4.9 Block Multiplication Of Matrices

Consider the following problem

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

You know how to do this. You get

$$\begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}.$$

Now what if instead of numbers, the entries, A, B, C, D, E, F, G are matrices of a size such that the multiplications and additions needed in the above formula all make sense. Would the formula be true in this case? I will show below that this is true.

Suppose A is a matrix of the form

$$A = \begin{pmatrix} A_{11} & \dots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{r1} & \dots & A_{rm} \end{pmatrix} \tag{4.9.29}$$

where A_{ij} is a $s_i \times p_j$ matrix where s_i is constant for $j = 1, \dots, m$ for each $i = 1, \dots, r$. Such a matrix is called a **block matrix**, also a **partitioned matrix**. How do you get the block A_{ij} ? Here is how for A an $m \times n$ matrix:

$$\left(\begin{array}{ccc} & \overbrace{I_{s_i \times s_i}}^{s_i \times m} & \\ 0 & & 0 \end{array} \right) A \left(\begin{array}{c} \overbrace{0}^{n \times p_j} \\ I_{p_j \times p_j} \\ 0 \end{array} \right). \quad (4.9.30)$$

In the block column matrix on the right, you need to have $c_j - 1$ rows of zeros above the small $p_j \times p_j$ identity matrix where the columns of A involved in A_{ij} are $c_j, \dots, c_j + p_j$ and in the block row matrix on the left, you need to have $r_i - 1$ columns of zeros to the left of the $s_i \times s_i$ identity matrix where the rows of A involved in A_{ij} are $r_i, \dots, r_i + s_i$. An important observation to make is that the matrix on the right specifies columns to use in the block and the one on the left specifies the rows used. There is no overlap between the blocks of A . Thus the identity $n \times n$ identity matrix corresponding to multiplication on the right of A is of the form

$$\left(\begin{array}{ccc} I_{p_1 \times p_1} & & 0 \\ & \ddots & \\ 0 & & I_{p_m \times p_m} \end{array} \right)$$

these little identity matrices don't overlap. A similar conclusion follows from consideration of the matrices $I_{s_i \times s_i}$.

Next consider the question of multiplication of two block matrices. Let B be a block matrix of the form

$$\left(\begin{array}{ccc} B_{11} & \cdots & B_{1p} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rp} \end{array} \right) \quad (4.9.31)$$

and A is a block matrix of the form

$$\left(\begin{array}{ccc} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{p1} & \cdots & A_{pm} \end{array} \right) \quad (4.9.32)$$

and that for all i, j , it makes sense to multiply $B_{is}A_{sj}$ for all $s \in \{1, \dots, p\}$. (That is the two matrices, B_{is} and A_{sj} are conformable.) and that for fixed ij , it follows $B_{is}A_{sj}$ is the same size for each s so that it makes sense to write $\sum_s B_{is}A_{sj}$.

The following theorem says essentially that when you take the product of two matrices, you can do it two ways. One way is to simply multiply them forming BA . The other way is to partition both matrices, formally multiply the blocks to get another block matrix and this one will be BA partitioned. Before presenting this theorem, here is a simple lemma which is really a special case of the theorem.

Lemma 4.9.1 *Consider the following product.*

$$\begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & I & \mathbf{0} \end{pmatrix}$$

where the first is $n \times r$ and the second is $r \times n$. The small identity matrix I is an $r \times r$ matrix and there are l zero rows above I and l zero columns to the left of I in the right matrix. Then the product of these matrices is a block matrix of the form

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Proof: From the definition of the way you multiply matrices, the product is

$$\left(\begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \mathbf{0} \cdots \begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \mathbf{0} \begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \mathbf{e}_1 \cdots \begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \mathbf{e}_r \begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \mathbf{0} \cdots \begin{pmatrix} \mathbf{0} \\ I \\ \mathbf{0} \end{pmatrix} \mathbf{0} \right)$$

which yields the claimed result. In the formula $\mathbf{e}_j$ refers to the column vector of length r which has a 1 in the j^{th} position. This proves the lemma.

Theorem 4.9.2 *Let B be a $q \times p$ block matrix as in 4.9.31 and let A be a $p \times n$ block matrix as in 4.9.32 such that B_{is} is conformable with A_{sj} and each product, $B_{is}A_{sj}$ for $s = 1, \dots, p$ is of the same size so they can be added. Then BA can be obtained as a block matrix such that the ij^{th} block is of the form*

$$\sum_s B_{is}A_{sj}. \quad (4.9.33)$$

Proof: From 4.9.30

$$B_{is}A_{sj} = \begin{pmatrix} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{pmatrix} B \begin{pmatrix} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{pmatrix} A \begin{pmatrix} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{pmatrix}$$

where here it is assumed B_{is} is $r_i \times p_s$ and A_{sj} is $p_s \times q_j$. The product involves the s^{th} block in the i^{th} row of blocks for B and the s^{th} block in the j^{th} column of A . Thus there are the same number of rows above the $I_{p_s \times p_s}$ as there are columns to the left of $I_{p_s \times p_s}$ in those two inside matrices. Then from Lemma 4.9.1

$$\begin{pmatrix} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

Since the blocks of small identity matrices do not overlap,

$$\sum_s \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} I_{p_1 \times p_1} & & 0 \\ & \ddots & \\ 0 & & I_{p_p \times p_p} \end{pmatrix} = I$$

and so

$$\begin{aligned} \sum_s B_{is} A_{sj} &= \\ \sum_s \begin{pmatrix} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{pmatrix} B \begin{pmatrix} \mathbf{0} \\ I_{p_s \times p_s} \\ \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & I_{p_s \times p_s} & \mathbf{0} \end{pmatrix} A \begin{pmatrix} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{pmatrix} B I A \begin{pmatrix} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & I_{r_i \times r_i} & \mathbf{0} \end{pmatrix} B A \begin{pmatrix} \mathbf{0} \\ I_{q_j \times q_j} \\ \mathbf{0} \end{pmatrix} \end{aligned}$$

Hence the ij^{th} block of BA equals the formal multiplication according to matrix multiplication,

$$\sum_s B_{is} A_{sj}.$$

This proves the theorem.

Example 4.9.3 Let an $n \times n$ matrix have the form

$$A = \begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & P \end{pmatrix}$$

where P is $n-1 \times n-1$. Multiply it by

$$B = \begin{pmatrix} p & \mathbf{q} \\ \mathbf{r} & Q \end{pmatrix}$$

where B is also an $n \times n$ matrix and Q is $n-1 \times n-1$.

You use block multiplication

$$\begin{pmatrix} a & \mathbf{b} \\ \mathbf{c} & P \end{pmatrix} \begin{pmatrix} p & \mathbf{q} \\ \mathbf{r} & Q \end{pmatrix} = \begin{pmatrix} ap + \mathbf{br} & a\mathbf{q} + \mathbf{b}Q \\ p\mathbf{c} + P\mathbf{r} & \mathbf{c}Q + PQ \end{pmatrix}$$

Note that this all makes sense. For example, $\mathbf{b} = 1 \times n-1$ and $\mathbf{r} = n-1 \times 1$ so $\mathbf{br}$ is a 1×1 . Similar considerations apply to the other blocks.

Here is an interesting and significant application of block multiplication. In this theorem, $p_M(t)$ denotes the characteristic polynomial, $\det(tI - M)$. Thus the zeros of this polynomial are the eigenvalues of the matrix, M .

Theorem 4.9.4 Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then

$$p_{BA}(t) = t^{n-m} p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n-m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

Since the two matrices above are similar it follows that $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials. Therefore, noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m} \det(tI - AB) = t^{n-m} p_{AB}(t)$. This proves the theorem.

4.10 Exercises

1. Show that matrix multiplication is associative. That is, $(AB)C = A(BC)$.
2. Show the inverse of a matrix, if it exists, is unique. Thus if $AB = BA = I$, then $B = A^{-1}$.
3. In the proof of Theorem 4.5.15 it was claimed that $\det(I) = 1$. Here $I = (\delta_{ij})$. Prove this assertion. Also prove Corollary 4.5.18.
4. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors in $\mathbb{F}^n$ and let $M(\mathbf{v}_1, \dots, \mathbf{v}_n)$ denote the matrix whose i^{th} column equals $\mathbf{v}_i$. Define

$$d(\mathbf{v}_1, \dots, \mathbf{v}_n) \equiv \det(M(\mathbf{v}_1, \dots, \mathbf{v}_n)).$$

Prove that d is linear in each variable, (multilinear), that

$$d(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_j, \dots, \mathbf{v}_n) = -d(\mathbf{v}_1, \dots, \mathbf{v}_j, \dots, \mathbf{v}_i, \dots, \mathbf{v}_n), \quad (4.10.34)$$

and

$$d(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1 \quad (4.10.35)$$

where here $\mathbf{e}_j$ is the vector in $\mathbb{F}^n$ which has a zero in every position except the j^{th} position in which it has a one.

5. Suppose $f : \mathbb{F}^n \times \dots \times \mathbb{F}^n \rightarrow \mathbb{F}$ satisfies 4.10.34 and 4.10.35 and is linear in each variable. Show that $f = d$.

6. Show that if you replace a row (column) of an $n \times n$ matrix A with itself added to some multiple of another row (column) then the new matrix has the same determinant as the original one.
7. If $A = (a_{ij})$, show $\det(A) = \sum_{(k_1, \dots, k_n)} \operatorname{sgn}(k_1, \dots, k_n) a_{k_1 1} \cdots a_{k_n n}$.
8. Use the result of Problem 6 to evaluate by hand the determinant

$$\det \begin{pmatrix} 1 & 2 & 3 & 2 \\ -6 & 3 & 2 & 3 \\ 5 & 2 & 2 & 3 \\ 3 & 4 & 6 & 4 \end{pmatrix}.$$

9. Find the inverse if it exists of the matrix,

$$\begin{pmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{pmatrix}.$$

10. Let $Ly = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$ where the a_i are given continuous functions defined on a closed interval, (a, b) and y is some function which has n derivatives so it makes sense to write Ly . Suppose $Ly_k = 0$ for $k = 1, 2, \dots, n$. The Wronskian of these functions, y_i is defined as

$$W(y_1, \dots, y_n)(x) \equiv \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{pmatrix}$$

Show that for $W(x) = W(y_1, \dots, y_n)(x)$ to save space,

$$W'(x) = \det \begin{pmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & & \vdots \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{pmatrix}.$$

Now use the differential equation, $Ly = 0$ which is satisfied by each of these functions, y_i and properties of determinants presented above to verify that $W' + a_{n-1}(x)W = 0$. Give an explicit solution of this linear differential equation, Abel's formula, and use your answer to verify that the Wronskian of these solutions to the equation, $Ly = 0$ either vanishes identically on (a, b) or never.

11. Two $n \times n$ matrices, A and B , are similar if $B = S^{-1}AS$ for some invertible $n \times n$ matrix, S . Show that if two matrices are similar, they have the same characteristic polynomials.

12. Suppose the characteristic polynomial of an $n \times n$ matrix, A is of the form

$$t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$$

and that $a_0 \neq 0$. Find a formula A^{-1} in terms of powers of the matrix, A . Show that A^{-1} exists if and only if $a_0 \neq 0$.

13. In constitutive modeling of the stress and strain tensors, one sometimes considers sums of the form $\sum_{k=0}^{\infty} a_k A^k$ where A is a 3×3 matrix. Show using the Cayley Hamilton theorem that if such a thing makes any sense, you can always obtain it as a finite sum having no more than n terms.

4.11 Shur's Theorem

Every matrix is related to an upper triangular matrix in a particularly significant way. This is Shur's theorem and it is the most important theorem in the spectral theory of matrices.

Lemma 4.11.1 *Let*

$$\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$$

be a basis for $\mathbb{F}^n$. Then there exists an orthonormal basis for $\mathbb{F}^n$,

$$\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$$

which has the property that for each $k \leq n$,

$$\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k).$$

Proof: Let $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a basis for $\mathbb{F}^n$. Let $\mathbf{u}_1 \equiv \mathbf{x}_1 / |\mathbf{x}_1|$. Thus for $k = 1$, $\text{span}(\mathbf{u}_1) = \text{span}(\mathbf{x}_1)$ and $\{\mathbf{u}_1\}$ is an orthonormal set. Now suppose for some $k < n$, $\mathbf{u}_1, \dots, \mathbf{u}_k$ have been chosen such that $(\mathbf{u}_j \cdot \mathbf{u}_l) = \delta_{jl}$ and $\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$. Then define

$$\mathbf{u}_{k+1} \equiv \frac{\mathbf{x}_{k+1} - \sum_{j=1}^k (\mathbf{x}_{k+1} \cdot \mathbf{u}_j) \mathbf{u}_j}{\left| \mathbf{x}_{k+1} - \sum_{j=1}^k (\mathbf{x}_{k+1} \cdot \mathbf{u}_j) \mathbf{u}_j \right|}, \quad (4.11.36)$$

where the denominator is not equal to zero because the $\mathbf{x}_j$ form a basis and so

$$\mathbf{x}_{k+1} \notin \text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k) = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k)$$

Thus by induction,

$$\mathbf{u}_{k+1} \in \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}).$$

Also, $\mathbf{x}_{k+1} \in \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1})$ which is seen easily by solving 4.11.36 for $\mathbf{x}_{k+1}$ and it follows

$$\text{span}(\mathbf{x}_1, \dots, \mathbf{x}_k, \mathbf{x}_{k+1}) = \text{span}(\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
 (\mathbf{u}_{k+1} \cdot \mathbf{u}_l) &= C \left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^k (\mathbf{x}_{k+1} \cdot \mathbf{u}_j) (\mathbf{u}_j \cdot \mathbf{u}_l) \right) \\
 &= C \left((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - \sum_{j=1}^k (\mathbf{x}_{k+1} \cdot \mathbf{u}_j) \delta_{lj} \right) \\
 &= C ((\mathbf{x}_{k+1} \cdot \mathbf{u}_l) - (\mathbf{x}_{k+1} \cdot \mathbf{u}_l)) = 0.
 \end{aligned}$$

The vectors, $\{\mathbf{u}_j\}_{j=1}^n$, generated in this way are therefore an orthonormal basis because each vector has unit length.

The process by which these vectors were generated is called the Gram Schmidt process. Recall the following definition.

Definition 4.11.2 An $n \times n$ matrix, U , is unitary if $UU^* = I = U^*U$ where U^* is defined to be the transpose of the conjugate of U .

Theorem 4.11.3 Let A be an $n \times n$ matrix. Then there exists a unitary matrix, U such that

$$U^*AU = T, \quad (4.11.37)$$

where T is an upper triangular matrix having the eigenvalues of A on the main diagonal listed according to multiplicity as roots of the characteristic equation.

Proof: Let $\mathbf{v}_1$ be a unit eigenvector for A . Then there exists λ_1 such that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad |\mathbf{v}_1| = 1.$$

Extend $\{\mathbf{v}_1\}$ to a basis and then use Lemma 4.11.1 to obtain $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$, an orthonormal basis in $\mathbb{F}^n$. Let U_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then from the above, it follows U_0 is unitary. Then $U_0^*AU_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is an $n-1 \times n-1$ matrix. Repeat the process for the matrix, A_1 above. There exists a unitary matrix $\tilde{U}_1$ such that $\tilde{U}_1^*A_1\tilde{U}_1$ is of the form

$$\begin{pmatrix} \lambda_2 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_2 & \\ 0 & & & \end{pmatrix}.$$

Now let U_1 be the $n \times n$ matrix of the form

$$\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{U}_1 \end{pmatrix}.$$

This is also a unitary matrix because by block multiplication,

$$\begin{aligned} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{U}_1 \end{pmatrix}^* \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{U}_1 \end{pmatrix} &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{U}_1^* \end{pmatrix} \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{U}_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{U}_1^* \tilde{U}_1 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix} \end{aligned}$$

Then using block multiplication, $U_1^* U_0^* A U_0 U_1$ is of the form

$$\begin{pmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{pmatrix}$$

where A_2 is an $n-2 \times n-2$ matrix. Continuing in this way, there exists a unitary matrix, U given as the product of the U_i in the above construction such that

$$U^* A U = T$$

where T is some upper triangular matrix. Since the matrix is upper triangular, the characteristic equation is $\prod_{i=1}^n (\lambda - \lambda_i)$ where the λ_i are the diagonal entries of T . Therefore, the λ_i are the eigenvalues.

What if A is a real matrix and you only want to consider real unitary matrices?

Theorem 4.11.4 *Let A be a real $n \times n$ matrix. Then there exists a real unitary matrix, Q and a matrix T of the form*

$$T = \begin{pmatrix} P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & P_r \end{pmatrix} \quad (4.11.38)$$

where P_i equals either a real 1×1 matrix or P_i equals a real 2×2 matrix having two complex eigenvalues of A such that $Q^T A Q = T$. The matrix, T is called the real Schur form of the matrix A .

Proof: Suppose

$$A \mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad |\mathbf{v}_1| = 1$$

where λ_1 is real. Then let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an orthonormal basis of vectors in $\mathbb{R}^n$. Let Q_0 be a matrix whose i^{th} column is $\mathbf{v}_i$. Then $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is a real $(n-1) \times (n-1)$ matrix. This is just like the proof of Theorem 4.11.3 up to this point.

Now in case $\lambda_1 = \alpha + i\beta$, it follows since A is real that $\mathbf{v}_1 = \mathbf{z}_1 + i\mathbf{w}_1$ and that $\bar{\mathbf{v}}_1 = \mathbf{z}_1 - i\mathbf{w}_1$ is an eigenvector for the eigenvalue, $\alpha - i\beta$. Here $\mathbf{z}_1$ and $\mathbf{w}_1$ are real vectors. It is clear that $\{\mathbf{z}_1, \mathbf{w}_1\}$ is an independent set of vectors in $\mathbb{R}^n$. Indeed, $\{\mathbf{v}_1, \bar{\mathbf{v}}_1\}$ is an independent set and it follows $\text{span}(\mathbf{v}_1, \bar{\mathbf{v}}_1) = \text{span}(\mathbf{z}_1, \mathbf{w}_1)$. Now using the Gram Schmidt theorem in $\mathbb{R}^n$, there exists $\{\mathbf{u}_1, \mathbf{u}_2\}$, an orthonormal set of real vectors such that $\text{span}(\mathbf{u}_1, \mathbf{u}_2) = \text{span}(\mathbf{v}_1, \bar{\mathbf{v}}_1)$. Now let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be an orthonormal basis in $\mathbb{R}^n$ and let Q_0 be a unitary matrix whose i^{th} column is $\mathbf{u}_i$. Then $A\mathbf{u}_j$ are both in $\text{span}(\mathbf{u}_1, \mathbf{u}_2)$ for $j = 1, 2$ and so $\mathbf{u}_k^T A\mathbf{u}_j = 0$ whenever $k \geq 3$. It follows that $Q_0^* A Q_0$ is of the form

$$\begin{pmatrix} * & * & \cdots & * \\ * & * & & \\ 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{pmatrix}$$

where A_1 is now an $(n-2) \times (n-2)$ matrix. In this case, find $\tilde{Q}_1$ an $(n-2) \times (n-2)$ matrix to put A_1 in an appropriate form as above and come up with A_2 either an $(n-4) \times (n-4)$ matrix or an $(n-3) \times (n-3)$ matrix. Then the only other difference is to let

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & \tilde{Q}_1 & \\ 0 & 0 & & & \end{pmatrix}$$

thus putting a 2×2 identity matrix in the upper left corner rather than a one. Repeating this process with the above modification for the case of a complex eigenvalue leads eventually to 4.11.38 where Q is the product of real unitary matrices Q_i above. Finally,

$$\lambda I - T = \begin{pmatrix} \lambda I_1 - P_1 & \cdots & * \\ & \ddots & \vdots \\ 0 & & \lambda I_r - P_r \end{pmatrix}$$

where I_k is the 2×2 identity matrix in the case that P_k is 2×2 and is the number 1 in the case where P_k is a 1×1 matrix. Now, it follows that $\det(\lambda I - T) = \prod_{k=1}^r \det(\lambda I_k - P_k)$. Therefore, λ is an eigenvalue of T if and only if it is an eigenvalue of some P_k . This proves the theorem since the eigenvalues of T are the same as those of A because they have the same characteristic polynomial due to the similarity of A and T .

Definition 4.11.5 When a linear transformation, A , mapping a linear space, V to V has a basis of eigenvectors, the linear transformation is called non defective.

Otherwise it is called defective. An $n \times n$ matrix, A , is called normal if $AA^* = A^*A$. An important class of normal matrices is that of the Hermitian or self adjoint matrices. An $n \times n$ matrix, A is self adjoint or Hermitian if $A = A^*$.

The next lemma is the basis for concluding that every normal matrix is unitarily similar to a diagonal matrix.

Lemma 4.11.6 *If T is upper triangular and normal, then T is a diagonal matrix.*

Proof: Since T is normal, $T^*T = TT^*$. Writing this in terms of components and using the description of the adjoint as the transpose of the conjugate, yields the following for the ik^{th} entry of $T^*T = TT^*$.

$$\sum_j t_{ij} t_{jk}^* = \sum_j t_{ij} \overline{t_{kj}} = \sum_j t_{ij}^* t_{jk} = \sum_j \overline{t_{ji}} t_{jk}.$$

Now use the fact that T is upper triangular and let $i = k = 1$ to obtain the following from the above.

$$\sum_j |t_{1j}|^2 = \sum_j |t_{j1}|^2 = |t_{11}|^2$$

You see, $t_{j1} = 0$ unless $j = 1$ due to the assumption that T is upper triangular. This shows T is of the form

$$\begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & * \end{pmatrix}.$$

Now do the same thing only this time take $i = k = 2$ and use the result just established. Thus, from the above,

$$\sum_j |t_{2j}|^2 = \sum_j |t_{j2}|^2 = |t_{22}|^2,$$

showing that $t_{2j} = 0$ if $j > 2$ which means T has the form

$$\begin{pmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Next let $i = k = 3$ and obtain that T looks like a diagonal matrix in so far as the first 3 rows and columns are concerned. Continuing in this way it follows T is a diagonal matrix.

Theorem 4.11.7 *Let A be a normal matrix. Then there exists a unitary matrix, U such that U^*AU is a diagonal matrix.*

Proof: From Theorem 4.11.3 there exists a unitary matrix, U such that U^*AU equals an upper triangular matrix. The theorem is now proved if it is shown that the property of being normal is preserved under unitary similarity transformations. That is, verify that if A is normal and if $B = U^*AU$, then B is also normal. But this is easy.

$$\begin{aligned} B^*B &= U^*A^*UU^*AU = U^*A^*AU \\ &= U^*AA^*U = U^*AUU^*A^*U = BB^*. \end{aligned}$$

Therefore, U^*AU is a normal and upper triangular matrix and by Lemma 4.11.6 it must be a diagonal matrix. This proves the theorem.

Corollary 4.11.8 *If A is Hermitian, then all the eigenvalues of A are real and there exists an orthonormal basis of eigenvectors.*

Proof: Since A is normal, there exists unitary, U such that $U^*AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A . Therefore, $D^* = U^*A^*U = U^*AU = D$ showing D is real.

Finally, let

$$U = (\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n)$$

where the $\mathbf{u}_i$ denote the columns of U and

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

The equation, $U^*AU = D$ implies

$$\begin{aligned} AU &= (A\mathbf{u}_1 \quad A\mathbf{u}_2 \quad \cdots \quad A\mathbf{u}_n) \\ &= UD = (\lambda_1\mathbf{u}_1 \quad \lambda_2\mathbf{u}_2 \quad \cdots \quad \lambda_n\mathbf{u}_n) \end{aligned}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\mathbf{u}_i = \lambda_i\mathbf{u}_i$ and since the matrix is unitary, the ij^{th} entry of U^*U equals δ_{ij} and so

$$\delta_{ij} = \overline{\mathbf{u}_i}^T \mathbf{u}_j = \overline{\mathbf{u}_i^T \mathbf{u}_j} = \overline{\mathbf{u}_i \cdot \mathbf{u}_j}.$$

This proves the corollary because it shows the vectors $\{\mathbf{u}_i\}$ form an orthonormal basis.

Corollary 4.11.9 *If A is a real symmetric matrix, then A is Hermitian and there exists a real unitary matrix, U such that $U^T AU = D$ where D is a diagonal matrix.*

Proof: This follows from Theorem 4.11.4 and Corollary 4.11.8.

4.12 The Right Polar Decomposition

The right polar decomposition involves writing a matrix as a product of two other matrices, one which preserves distances and the other which stretches and distorts. First here are some lemmas.

Lemma 4.12.1 *Let A be a Hermitian matrix such that all its eigenvalues are non-negative. Then there exists a Hermitian matrix, $A^{1/2}$ such that $A^{1/2}$ has all non-negative eigenvalues and $(A^{1/2})^2 = A$.*

Proof: Since A is Hermitian, there exists a diagonal matrix D having all real nonnegative entries and a unitary matrix U such that $A = U^*DU$. Then denote by $D^{1/2}$ the matrix which is obtained by replacing each diagonal entry of D with its square root. Thus $D^{1/2}D^{1/2} = D$. Then define

$$A^{1/2} \equiv U^*D^{1/2}U.$$

Then

$$(A^{1/2})^2 = U^*D^{1/2}UU^*D^{1/2}U = U^*DU = A.$$

Since $D^{1/2}$ is real,

$$(U^*D^{1/2}U)^* = U^*(D^{1/2})^*(U^*)^* = U^*D^{1/2}U$$

so $A^{1/2}$ is Hermitian. This proves the lemma.

There is also a useful observation about orthonormal sets of vectors which is stated in the next lemma.

Lemma 4.12.2 *Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_r\}$ is an orthonormal set of vectors. Then if $c_1, \dots, c_r$ are scalars,*

$$\left| \sum_{k=1}^r c_k \mathbf{x}_k \right|^2 = \sum_{k=1}^r |c_k|^2.$$

Proof: This follows from the definition. From the properties of the dot product and using the fact that the given set of vectors is orthonormal,

$$\begin{aligned} \left| \sum_{k=1}^r c_k \mathbf{x}_k \right|^2 &= \left(\sum_{k=1}^r c_k \mathbf{x}_k, \sum_{j=1}^r c_j \mathbf{x}_j \right) \\ &= \sum_{k,j} c_k \overline{c_j} (\mathbf{x}_k, \mathbf{x}_j) = \sum_{k=1}^r |c_k|^2. \end{aligned}$$

This proves the lemma.

Next it is helpful to recall the Gram Schmidt algorithm and observe a certain property stated in the next lemma.

Lemma 4.12.3 Suppose $\{\mathbf{w}_1, \dots, \mathbf{w}_r, \mathbf{v}_{r+1}, \dots, \mathbf{v}_p\}$ is a linearly independent set of vectors such that $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ is an orthonormal set of vectors. Then when the Gram Schmidt process is applied to the vectors in the given order, it will not change any of the $\mathbf{w}_1, \dots, \mathbf{w}_r$.

Proof: Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be the orthonormal set delivered by the Gram Schmidt process. Then $\mathbf{u}_1 = \mathbf{w}_1$ because by definition, $\mathbf{u}_1 \equiv \mathbf{w}_1 / |\mathbf{w}_1| = \mathbf{w}_1$. Now suppose $\mathbf{u}_j = \mathbf{w}_j$ for all $j \leq k \leq r$. Then if $k < r$, consider the definition of $\mathbf{u}_{k+1}$.

$$\mathbf{u}_{k+1} \equiv \frac{\mathbf{w}_{k+1} - \sum_{j=1}^{k+1} (\mathbf{w}_{k+1}, \mathbf{u}_j) \mathbf{u}_j}{\left| \mathbf{w}_{k+1} - \sum_{j=1}^{k+1} (\mathbf{w}_{k+1}, \mathbf{u}_j) \mathbf{u}_j \right|}$$

By induction, $\mathbf{u}_j = \mathbf{w}_j$ and so this reduces to $\mathbf{w}_{k+1} / |\mathbf{w}_{k+1}| = \mathbf{w}_{k+1}$. This proves the lemma.

This lemma immediately implies the following lemma.

Lemma 4.12.4 Let V be a subspace of dimension p and let $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ be an orthonormal set of vectors in V . Then this orthonormal set of vectors may be extended to an orthonormal basis for V ,

$$\{\mathbf{w}_1, \dots, \mathbf{w}_r, \mathbf{y}_{r+1}, \dots, \mathbf{y}_p\}$$

Proof: First extend the given linearly independent set $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ to a basis for V and then apply the Gram Schmidt theorem to the resulting basis. Since $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ is orthonormal it follows from Lemma 4.12.3 the result is of the desired form, an orthonormal basis extending $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$. This proves the lemma.

Here is another lemma about preserving distance.

Lemma 4.12.5 Suppose R is an $m \times n$ matrix with $m > n$ and R preserves distances. Then $R^*R = I$.

Proof: Since R preserves distances, $|R\mathbf{x}| = |\mathbf{x}|$ for every $\mathbf{x}$. Therefore from the axioms of the dot product,

$$\begin{aligned} & |\mathbf{x}|^2 + |\mathbf{y}|^2 + (\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{x}) \\ &= |\mathbf{x} + \mathbf{y}|^2 \\ &= (R(\mathbf{x} + \mathbf{y}), R(\mathbf{x} + \mathbf{y})) \\ &= (R\mathbf{x}, R\mathbf{x}) + (R\mathbf{y}, R\mathbf{y}) + (R\mathbf{x}, R\mathbf{y}) + (R\mathbf{y}, R\mathbf{x}) \\ &= |\mathbf{x}|^2 + |\mathbf{y}|^2 + (R^*R\mathbf{x}, \mathbf{y}) + (\mathbf{y}, R^*R\mathbf{x}) \end{aligned}$$

and so for all $\mathbf{x}, \mathbf{y}$,

$$(R^*R\mathbf{x} - \mathbf{x}, \mathbf{y}) + (\mathbf{y}, R^*R\mathbf{x} - \mathbf{x}) = 0$$

Hence for all $\mathbf{x}, \mathbf{y}$,

$$\operatorname{Re}(R^*R\mathbf{x} - \mathbf{x}, \mathbf{y}) = 0$$

Now for a $\mathbf{x}, \mathbf{y}$ given, choose $\alpha \in \mathbb{C}$ such that

$$\alpha (R^* R \mathbf{x} - \mathbf{x}, \mathbf{y}) = |(R^* R \mathbf{x} - \mathbf{x}, \mathbf{y})|$$

Then

$$\begin{aligned} 0 &= \operatorname{Re} (R^* R \mathbf{x} - \mathbf{x}, \bar{\alpha} \mathbf{y}) = \operatorname{Re} \alpha (R^* R \mathbf{x} - \mathbf{x}, \mathbf{y}) \\ &= |(R^* R \mathbf{x} - \mathbf{x}, \mathbf{y})| \end{aligned}$$

Thus $|(R^* R \mathbf{x} - \mathbf{x}, \mathbf{y})| = 0$ for all $\mathbf{x}, \mathbf{y}$ because the given $\mathbf{x}, \mathbf{y}$ were arbitrary. Let $\mathbf{y} = R^* R \mathbf{x} - \mathbf{x}$ to conclude that for all $\mathbf{x}$,

$$R^* R \mathbf{x} - \mathbf{x} = \mathbf{0}$$

which says $R^* R = I$ since $\mathbf{x}$ is arbitrary. This proves the lemma.

With this preparation, here is the big theorem about the right polar decomposition.

Theorem 4.12.6 *Let F be an $m \times n$ matrix where $m \geq n$. Then there exists a Hermitian $n \times n$ matrix, U which has all nonnegative eigenvalues and an $m \times n$ matrix, R which preserves distances and satisfies $R^* R = I$ such that*

$$F = RU.$$

Proof: Consider $F^* F$. This is a Hermitian matrix because

$$(F^* F)^* = F^* (F^*)^* = F^* F$$

Also the eigenvalues of the $n \times n$ matrix $F^* F$ are all nonnegative. This is because if $\mathbf{x}$ is an eigenvalue,

$$\lambda (\mathbf{x}, \mathbf{x}) = (F^* F \mathbf{x}, \mathbf{x}) = (F \mathbf{x}, F \mathbf{x}) \geq 0.$$

Therefore, by Lemma 4.12.1, there exists an $n \times n$ Hermitian matrix, U having all nonnegative eigenvalues such that

$$U^2 = F^* F.$$

Consider the subspace $U(\mathbb{F}^n)$. Let $\{U \mathbf{x}_1, \dots, U \mathbf{x}_r\}$ be an orthonormal basis for $U(\mathbb{F}^n) \subseteq \mathbb{F}^n$. Note that $U(\mathbb{F}^n)$ might not be all of $\mathbb{F}^n$. Using Lemma 4.12.4, extend to an orthonormal basis for all of $\mathbb{F}^n$,

$$\{U \mathbf{x}_1, \dots, U \mathbf{x}_r, \mathbf{y}_{r+1}, \dots, \mathbf{y}_n\}.$$

Next observe that $\{F \mathbf{x}_1, \dots, F \mathbf{x}_r\}$ is also an orthonormal set of vectors in $\mathbb{F}^m$. This is because

$$\begin{aligned} (F \mathbf{x}_k, F \mathbf{x}_j) &= (F^* F \mathbf{x}_k, \mathbf{x}_j) = (U^2 \mathbf{x}_k, \mathbf{x}_j) \\ &= (U \mathbf{x}_k, U^* \mathbf{x}_j) = (U \mathbf{x}_k, U \mathbf{x}_j) = \delta_{jk} \end{aligned}$$

Therefore, from Lemma 4.12.4 again, this orthonormal set of vectors can be extended to an orthonormal basis for $\mathbb{F}^m$,

$$\{F\mathbf{x}_1, \dots, F\mathbf{x}_r, \mathbf{z}_{r+1}, \dots, \mathbf{z}_m\}$$

Thus there are at least as many $\mathbf{z}_k$ as there are $\mathbf{y}_j$. Now for $\mathbf{x} \in \mathbb{F}^n$, since

$$\{U\mathbf{x}_1, \dots, U\mathbf{x}_r, \mathbf{y}_{r+1}, \dots, \mathbf{y}_n\}$$

is an orthonormal basis for $\mathbb{F}^n$, there exist unique scalars,

$$c_1 \dots, c_r, d_{r+1}, \dots, d_n$$

such that

$$\mathbf{x} = \sum_{k=1}^r c_k U\mathbf{x}_k + \sum_{j=r+1}^n d_j \mathbf{y}_j$$

Define

$$R\mathbf{x} \equiv \sum_{k=1}^r c_k F\mathbf{x}_k + \sum_{j=r+1}^n d_j \mathbf{z}_j \quad (4.12.39)$$

Then also there exist scalars b_k such that

$$U\mathbf{x} = \sum_{k=1}^r b_k U\mathbf{x}_k$$

and so from 4.12.39, applied to $U\mathbf{x}$ in place of x

$$RU\mathbf{x} = \sum_{k=1}^r b_k F\mathbf{x}_k = F\left(\sum_{k=1}^r b_k \mathbf{x}_k\right)$$

Is $F\left(\sum_{k=1}^r b_k \mathbf{x}_k\right) = F(\mathbf{x})$?

$$\begin{aligned} & \left(F\left(\sum_{k=1}^r b_k \mathbf{x}_k\right) - F(\mathbf{x}), F\left(\sum_{k=1}^r b_k \mathbf{x}_k\right) - F(\mathbf{x}) \right) \\ &= \left((F^* F)\left(\sum_{k=1}^r b_k \mathbf{x}_k - \mathbf{x}\right), \left(\sum_{k=1}^r b_k \mathbf{x}_k - \mathbf{x}\right) \right) \\ &= \left(U^2\left(\sum_{k=1}^r b_k \mathbf{x}_k - \mathbf{x}\right), \left(\sum_{k=1}^r b_k \mathbf{x}_k - \mathbf{x}\right) \right) \\ &= \left(U\left(\sum_{k=1}^r b_k \mathbf{x}_k - \mathbf{x}\right), U\left(\sum_{k=1}^r b_k \mathbf{x}_k - \mathbf{x}\right) \right) \\ &= \left(\sum_{k=1}^r b_k U\mathbf{x}_k - U\mathbf{x}, \sum_{k=1}^r b_k U\mathbf{x}_k - U\mathbf{x} \right) = 0 \end{aligned}$$

Therefore, $F(\sum_{k=1}^r b_k \mathbf{x}_k) = F(\mathbf{x})$ and this shows

$$RU\mathbf{x} = F\mathbf{x}.$$

From 4.12.39 and Lemma 4.12.2 R preserves distances. Therefore, by Lemma 4.12.5 $R^*R = I$. This proves the theorem.

Multi-variable Calculus

5.1 Continuous Functions

In what follows, $\mathbb{F}$ will denote either $\mathbb{R}$ or $\mathbb{C}$. It turns out it is more efficient to not make a distinction. However, the main interest is in $\mathbb{R}$ so if you like, you can think $\mathbb{R}$ whenever you see $\mathbb{F}$.

5.1.1 Distance In $\mathbb{F}^n$

It is necessary to give a generalization of the dot product for vectors in $\mathbb{C}^n$. This definition reduces to the usual one in the case the components of the vector are real.

Definition 5.1.1 *Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Thus $\mathbf{x} = (x_1, \dots, x_n)$ where each $x_k \in \mathbb{C}$ and a similar formula holding for $\mathbf{y}$. Then the dot product of these two vectors is defined to be*

$$\mathbf{x} \cdot \mathbf{y} \equiv \sum_j x_j \overline{y_j} \equiv x_1 \overline{y_1} + \dots + x_n \overline{y_n}.$$

Notice how you put the conjugate on the entries of the vector, $\mathbf{y}$. It makes no difference if the vectors happen to be real vectors but with complex vectors you must do it this way. The reason for this is that when you take the dot product of a vector with itself, you want to get the square of the length of the vector, a positive number. Placing the conjugate on the components of $\mathbf{y}$ in the above definition assures this will take place. Thus

$$\mathbf{x} \cdot \mathbf{x} = \sum_j x_j \overline{x_j} = \sum_j |x_j|^2 \geq 0.$$

If you didn't place a conjugate as in the above definition, things wouldn't work out correctly. For example,

$$(1 + i)^2 + 2^2 = 4 + 2i$$

and this is not a positive number.

The following properties of the dot product follow immediately from the definition and you should verify each of them.

Properties of the dot product:

1. $\mathbf{u} \cdot \mathbf{v} = \overline{\mathbf{v} \cdot \mathbf{u}}$.
2. If a, b are numbers and $\mathbf{u}, \mathbf{v}, \mathbf{z}$ are vectors then $(a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{z} = a(\mathbf{u} \cdot \mathbf{z}) + b(\mathbf{v} \cdot \mathbf{z})$.
3. $\mathbf{u} \cdot \mathbf{u} \geq 0$ and it equals 0 if and only if $\mathbf{u} = \mathbf{0}$.

The norm is defined in the usual way.

Definition 5.1.2 For $\mathbf{x} \in \mathbb{C}^n$,

$$|\mathbf{x}| \equiv \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} = (\mathbf{x} \cdot \mathbf{x})^{1/2}$$

Here is a fundamental inequality called the **Cauchy Schwarz inequality** which is stated here in $\mathbb{C}^n$. First here is a simple lemma.

Lemma 5.1.3 If $z \in \mathbb{C}$ there exists $\theta \in \mathbb{C}$ such that $\theta z = |z|$ and $|\theta| = 1$.

Proof: Let $\theta = 1$ if $z = 0$ and otherwise, let $\theta = \frac{\bar{z}}{|z|}$. Recall that for $z = x + iy, \bar{z} = x - iy$ and $\bar{z}z = |z|^2$.

Theorem 5.1.4 (Cauchy Schwarz) The following inequality holds for x_i and $y_i \in \mathbb{C}$.

$$|(\mathbf{x} \cdot \mathbf{y})| = \left| \sum_{i=1}^n x_i \bar{y}_i \right| \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2} = |\mathbf{x}| |\mathbf{y}| \quad (5.1.1)$$

Proof: Let $\theta \in \mathbb{C}$ such that $|\theta| = 1$ and

$$\theta \sum_{i=1}^n x_i \bar{y}_i = \left| \sum_{i=1}^n x_i \bar{y}_i \right|$$

Thus

$$\theta \sum_{i=1}^n x_i \bar{y}_i = \sum_{i=1}^n x_i \overline{(\theta y_i)} = \left| \sum_{i=1}^n x_i \bar{y}_i \right|.$$

Consider $p(t) \equiv \sum_{i=1}^n (x_i + t\bar{\theta}y_i) \overline{(x_i + t\bar{\theta}y_i)}$ where $t \in \mathbb{R}$.

$$\begin{aligned} 0 &\leq p(t) = \sum_{i=1}^n |x_i|^2 + 2t \operatorname{Re} \left(\theta \sum_{i=1}^n x_i \bar{y}_i \right) + t^2 \sum_{i=1}^n |y_i|^2 \\ &= |\mathbf{x}|^2 + 2t \left| \sum_{i=1}^n x_i \bar{y}_i \right| + t^2 |\mathbf{y}|^2 \end{aligned}$$

If $|\mathbf{y}| = 0$ then 5.1.1 is obviously true because both sides equal zero. Therefore, assume $|\mathbf{y}| \neq 0$ and then $p(t)$ is a polynomial of degree two whose graph opens up. Therefore, it either has no zeroes, two zeros or one repeated zero. If it has two zeros, the above inequality must be violated because in this case the graph must dip below the x axis. Therefore, it either has no zeros or exactly one. From the quadratic formula this happens exactly when

$$4 \left| \sum_{i=1}^n x_i \bar{y}_i \right|^2 - 4 |\mathbf{x}|^2 |\mathbf{y}|^2 \leq 0$$

and so

$$\left| \sum_{i=1}^n x_i \bar{y}_i \right| \leq |\mathbf{x}| |\mathbf{y}|$$

as claimed. This proves the inequality.

By analogy to the case of $\mathbb{R}^n$, length or magnitude of vectors in $\mathbb{C}^n$ can be defined.

Definition 5.1.5 Let $\mathbf{z} \in \mathbb{C}^n$. Then $|\mathbf{z}| \equiv (\mathbf{z} \cdot \mathbf{z})^{1/2}$. Also numbers in $\mathbb{F}$ will often be referred to as scalars.

Theorem 5.1.6 For length defined in Definition 5.1.5, the following hold.

$$|\mathbf{z}| \geq 0 \text{ and } |\mathbf{z}| = 0 \text{ if and only if } \mathbf{z} = \mathbf{0} \quad (5.1.2)$$

$$\text{If } \alpha \text{ is a scalar, } |\alpha \mathbf{z}| = |\alpha| |\mathbf{z}| \quad (5.1.3)$$

$$|\mathbf{z} + \mathbf{w}| \leq |\mathbf{z}| + |\mathbf{w}|. \quad (5.1.4)$$

Proof: The first two claims are left as exercises. To establish the third,

$$\begin{aligned} |\mathbf{z} + \mathbf{w}|^2 &= (\mathbf{z} + \mathbf{w}, \mathbf{z} + \mathbf{w}) \\ &= \mathbf{z} \cdot \mathbf{z} + \mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{z} + \mathbf{z} \cdot \mathbf{w} \\ &= |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2 \operatorname{Re} \mathbf{w} \cdot \mathbf{z} \\ &\leq |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2 |\mathbf{w} \cdot \mathbf{z}| \\ &\leq |\mathbf{z}|^2 + |\mathbf{w}|^2 + 2 |\mathbf{w}| |\mathbf{z}| = (|\mathbf{z}| + |\mathbf{w}|)^2. \end{aligned}$$

The main difference between $\mathbb{C}^n$ and $\mathbb{R}^n$ is that the scalars are complex numbers.

Definition 5.1.7 Suppose you have a vector space, V and for $\mathbf{z}, \mathbf{w} \in V$ and α a scalar a norm is a way of measuring distance or magnitude which satisfies the properties 5.1.2 - 5.1.4. Thus a norm is something which does the following.

$$\|\mathbf{z}\| \geq 0 \text{ and } \|\mathbf{z}\| = 0 \text{ if and only if } \mathbf{z} = \mathbf{0} \quad (5.1.5)$$

$$\text{If } \alpha \text{ is a scalar, } \|\alpha \mathbf{z}\| = |\alpha| \|\mathbf{z}\| \quad (5.1.6)$$

$$\|\mathbf{z} + \mathbf{w}\| \leq \|\mathbf{z}\| + \|\mathbf{w}\|. \quad (5.1.7)$$

Here is understood that for all $\mathbf{z} \in V$, $\|\mathbf{z}\| \in [0, \infty)$.

Note that $|\cdot|$ provides a norm on $\mathbb{F}^n$ from the above.

5.2 Open And Closed Sets

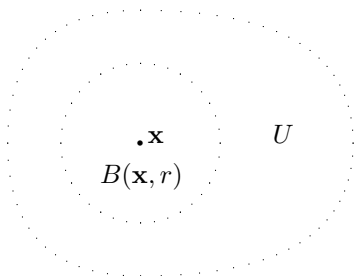
Eventually, one must consider functions which are defined on subsets of $\mathbb{F}^n$ and their properties. The next definition will end up being quite important. It describes a type of subset of $\mathbb{F}^n$ with the property that if $\mathbf{x}$ is in this set, then so is $\mathbf{y}$ whenever $\mathbf{y}$ is close enough to $\mathbf{x}$.

Definition 5.2.1 Let $U \subseteq \mathbb{F}^n$. U is an **open set** if whenever $\mathbf{x} \in U$, there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U$. More generally, if U is any subset of $\mathbb{F}^n$, $\mathbf{x} \in U$ is an **interior point** of U if there exists $r > 0$ such that $B(\mathbf{x}, r) \subseteq U$. In other words U is an open set exactly when every point of U is an interior point of U .

If there is something called an open set, surely there should be something called a closed set and here is the definition of one.

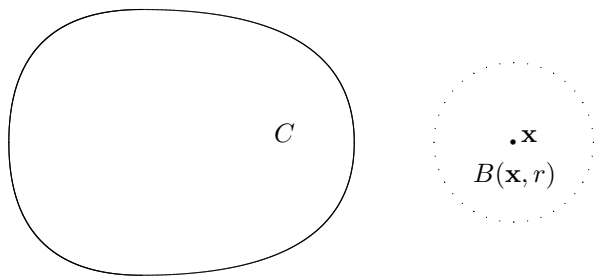
Definition 5.2.2 A subset, C , of $\mathbb{F}^n$ is called a **closed set** if $\mathbb{F}^n \setminus C$ is an open set. The symbol, $\mathbb{F}^n \setminus C$ denotes everything in $\mathbb{F}^n$ which is not in C . It is also called the **complement** of C . The symbol, S^C is a short way of writing $\mathbb{F}^n \setminus S$.

To illustrate this definition, consider the following picture.



You see in this picture how the edges are dotted. This is because an open set, can not include the edges or the set would fail to be open. For example, consider what would happen if you picked a point out on the edge of U in the above picture. Every open ball centered at that point would have in it some points which are outside U . Therefore, such a point would violate the above definition. You also see the edges of $B(\mathbf{x}, r)$ dotted suggesting that $B(\mathbf{x}, r)$ ought to be an open set. This is intuitively clear but does require a proof. This will be done in the next theorem and will give examples of open sets. Also, you can see that if $\mathbf{x}$ is close to the edge of U , you might have to take r to be very small.

It is roughly the case that open sets don't have their skins while closed sets do. Here is a picture of a closed set, C .



Note that $\mathbf{x} \notin C$ and since $\mathbb{F}^n \setminus C$ is open, there exists a ball, $B(\mathbf{x}, r)$ contained entirely in $\mathbb{F}^n \setminus C$. If you look at $\mathbb{F}^n \setminus C$, what would be its skin? It can't be in $\mathbb{F}^n \setminus C$ and so it must be in C . This is a rough heuristic explanation of what is going on with these definitions. Also note that $\mathbb{F}^n$ and $\emptyset$ are both open and closed. Here is why. If $\mathbf{x} \in \emptyset$, then there must be a ball centered at $\mathbf{x}$ which is also contained in $\emptyset$. This must be considered to be true because there is nothing in $\emptyset$ so there can be no example to show it false¹. Therefore, from the definition, it follows $\emptyset$ is open. It is also closed because if $\mathbf{x} \notin \emptyset$, then $B(\mathbf{x}, 1)$ is also contained in $\mathbb{F}^n \setminus \emptyset = \mathbb{F}^n$. Therefore, $\emptyset$ is both open and closed. From this, it follows $\mathbb{F}^n$ is also both open and closed.

Theorem 5.2.3 *Let $\mathbf{x} \in \mathbb{F}^n$ and let $r \geq 0$. Then $B(\mathbf{x}, r)$ is an open set. Also,*

$$D(\mathbf{x}, r) \equiv \{\mathbf{y} \in \mathbb{F}^n : |\mathbf{y} - \mathbf{x}| \leq r\}$$

is a closed set.

Proof: Suppose $\mathbf{y} \in B(\mathbf{x}, r)$. It is necessary to show there exists $r_1 > 0$ such that $B(\mathbf{y}, r_1) \subseteq B(\mathbf{x}, r)$. Define $r_1 \equiv r - |\mathbf{x} - \mathbf{y}|$. Then if $|\mathbf{z} - \mathbf{y}| < r_1$, it follows from the above triangle inequality that

$$\begin{aligned} |\mathbf{z} - \mathbf{x}| &= |\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{x}| \\ &\leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}| \\ &< r_1 + |\mathbf{y} - \mathbf{x}| = r - |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}| = r. \end{aligned}$$

Note that if $r = 0$ then $B(\mathbf{x}, r) = \emptyset$, the empty set. This is because if $\mathbf{y} \in \mathbb{F}^n$, $|\mathbf{x} - \mathbf{y}| \geq 0$ and so $\mathbf{y} \notin B(\mathbf{x}, 0)$. Since $\emptyset$ has no points in it, it must be open because

¹To a mathematician, the statment: Whenever a pig is born with wings it can fly must be taken as true. We do not consider biological or aerodynamic considerations in such statements. There is no such thing as a winged pig and therefore, all winged pigs must be superb flyers since there can be no example of one which is not. On the other hand we would also consider the statement: Whenever a pig is born with wings it can't possibly fly, as equally true. The point is, you can say anything you want about the elements of the empty set and no one can gainsay your statement. Therefore, such statements are considered as true by default. You may say this is a very strange way of thinking about truth and ultimately this is because mathematics is not about truth. It is more about consistency and logic.

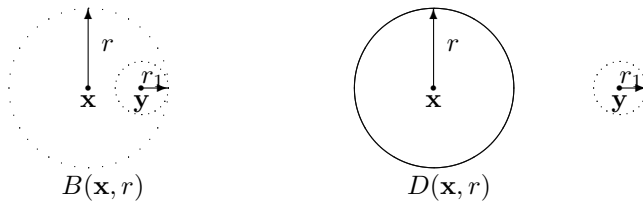
every point in it, (There are none.) satisfies the desired property of being an interior point.

Now suppose $\mathbf{y} \notin D(\mathbf{x}, r)$. Then $|\mathbf{x} - \mathbf{y}| > r$ and defining $\delta \equiv |\mathbf{x} - \mathbf{y}| - r$, it follows that if $\mathbf{z} \in B(\mathbf{y}, \delta)$, then by the triangle inequality,

$$\begin{aligned} |\mathbf{x} - \mathbf{z}| &\geq |\mathbf{x} - \mathbf{y}| - |\mathbf{y} - \mathbf{z}| > |\mathbf{x} - \mathbf{y}| - \delta \\ &= |\mathbf{x} - \mathbf{y}| - (|\mathbf{x} - \mathbf{y}| - r) = r \end{aligned}$$

and this shows that $B(\mathbf{y}, \delta) \subseteq \mathbb{F}^n \setminus D(\mathbf{x}, r)$. Since $\mathbf{y}$ was an arbitrary point in $\mathbb{F}^n \setminus D(\mathbf{x}, r)$, it follows $\mathbb{F}^n \setminus D(\mathbf{x}, r)$ is an open set which shows from the definition that $D(\mathbf{x}, r)$ is a closed set as claimed.

A picture which is descriptive of the conclusion of the above theorem which also implies the manner of proof is the following.



5.3 Continuous Functions

With the above definition of the norm in $\mathbb{F}^p$, it becomes possible to define continuity.

Definition 5.3.1 A function $\mathbf{f} : D(\mathbf{f}) \subseteq \mathbb{F}^p \rightarrow \mathbb{F}^q$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mathbf{y} \in D(\mathbf{f})$ and

$$|\mathbf{y} - \mathbf{x}| < \delta$$

it follows that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon.$$

$\mathbf{f}$ is continuous if it is continuous at every point of $D(\mathbf{f})$.

Note the total similarity to the scalar valued case.

5.3.1 Sufficient Conditions For Continuity

The next theorem is a fundamental result which will allow us to worry less about the $\varepsilon \delta$ definition of continuity.

Theorem 5.3.2 The following assertions are valid.

1. The function, $a\mathbf{f} + b\mathbf{g}$ is continuous at $\mathbf{x}$ whenever $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.

2. If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
3. If $\mathbf{f} = (f_1, \dots, f_q) : D(\mathbf{f}) \rightarrow \mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function.
4. The function $f : \mathbb{F}^p \rightarrow \mathbb{F}$, given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous.

The proof of this theorem is in the last section of this chapter. Its conclusions are not surprising. For example the first claim says that $(a\mathbf{f} + b\mathbf{g})(\mathbf{y})$ is close to $(a\mathbf{f} + b\mathbf{g})(\mathbf{x})$ when $\mathbf{y}$ is close to $\mathbf{x}$ provided the same can be said about $\mathbf{f}$ and $\mathbf{g}$. For the second claim, if $\mathbf{y}$ is close to $\mathbf{x}$, $\mathbf{f}(\mathbf{x})$ is close to $\mathbf{f}(\mathbf{y})$ and so by continuity of $\mathbf{g}$ at $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{f}(\mathbf{y}))$ is close to $\mathbf{g}(\mathbf{f}(\mathbf{x}))$. To see the third claim is likely, note that closeness in $\mathbb{F}^p$ is the same as closeness in each coordinate. The fourth claim is immediate from the triangle inequality.

For functions defined on $\mathbb{F}^n$, there is a notion of polynomial just as there is for functions defined on $\mathbb{R}$.

Definition 5.3.3 Let α be an n dimensional multi-index. This means

$$\alpha = (\alpha_1, \dots, \alpha_n)$$

where each α_i is a natural number or zero. Also, let

$$|\alpha| \equiv \sum_{i=1}^n |\alpha_i|$$

The symbol, $\mathbf{x}^\alpha$, means

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

An n dimensional polynomial of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} d_\alpha \mathbf{x}^\alpha.$$

where the d_α are complex or real numbers.

The above theorem implies that polynomials are all continuous.

5.4 Exercises

1. Let $\mathbf{f}(t) = (t, \sin t)$. Show f is continuous at every point t .
2. Suppose $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$ where K is a constant. Show that $\mathbf{f}$ is everywhere continuous. Functions satisfying such an inequality are called Lipschitz functions.
3. Suppose $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|^\alpha$ where K is a constant and $\alpha \in (0, 1)$. Show that $\mathbf{f}$ is everywhere continuous.

4. Suppose $f : \mathbb{F}^3 \rightarrow \mathbb{F}$ is given by $f(\mathbf{x}) = 3x_1x_2 + 2x_3^2$. Use Theorem 5.3.2 to verify that f is continuous. **Hint:** You should first verify that the function, $\pi_k : \mathbb{F}^3 \rightarrow \mathbb{F}$ given by $\pi_k(\mathbf{x}) = x_k$ is a continuous function.
5. Generalize the previous problem to the case where $f : \mathbb{F}^q \rightarrow \mathbb{F}$ is a polynomial.
6. State and prove a theorem which involves quotients of functions encountered in the previous problem.

5.5 Limits Of A Function

As in the case of scalar valued functions of one variable, a concept closely related to continuity is that of the limit of a function. The notion of limit of a function makes sense at points, $\mathbf{x}$, which are limit points of $D(\mathbf{f})$ and this concept is defined next.

Definition 5.5.1 Let $A \subseteq \mathbb{F}^m$ be a set. A point, $\mathbf{x}$, is a limit point of A if $B(\mathbf{x}, r)$ contains infinitely many points of A for every $r > 0$.

Definition 5.5.2 Let $\mathbf{f} : D(\mathbf{f}) \subseteq \mathbb{F}^p \rightarrow \mathbb{F}^q$ be a function and let $\mathbf{x}$ be a limit point of $D(\mathbf{f})$. Then

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$$

if and only if the following condition holds. For all $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$0 < |\mathbf{y} - \mathbf{x}| < \delta, \text{ and } \mathbf{y} \in D(\mathbf{f})$$

then,

$$|\mathbf{L} - \mathbf{f}(\mathbf{y})| < \varepsilon.$$

Theorem 5.5.3 If $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$ and $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}_1$, then $\mathbf{L} = \mathbf{L}_1$.

Proof: Let $\varepsilon > 0$ be given. There exists $\delta > 0$ such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \varepsilon, \quad |\mathbf{f}(\mathbf{y}) - \mathbf{L}_1| < \varepsilon.$$

Pick such a $\mathbf{y}$. There exists one because $\mathbf{x}$ is a limit point of $D(\mathbf{f})$. Then

$$|\mathbf{L} - \mathbf{L}_1| \leq |\mathbf{L} - \mathbf{f}(\mathbf{y})| + |\mathbf{f}(\mathbf{y}) - \mathbf{L}_1| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\mathbf{L} = \mathbf{L}_1$.

As in the case of functions of one variable, one can define what it means for $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x}) = \pm\infty$.

Definition 5.5.4 If $f(\mathbf{x}) \in \mathbb{F}$, $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{x}) = \infty$ if for every number l , there exists $\delta > 0$ such that whenever $|\mathbf{y} - \mathbf{x}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $f(\mathbf{x}) > l$.

The following theorem is just like the one variable version presented earlier.

Theorem 5.5.5 Suppose $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$ and $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{g}(\mathbf{y}) = \mathbf{K}$ where $\mathbf{K}, \mathbf{L} \in \mathbb{F}^q$. Then if $a, b \in \mathbb{F}$,

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} (a\mathbf{f}(\mathbf{y}) + b\mathbf{g}(\mathbf{y})) = a\mathbf{L} + b\mathbf{K}, \quad (5.5.8)$$

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f} \cdot \mathbf{g}(\mathbf{y}) = \mathbf{L}\mathbf{K} \quad (5.5.9)$$

and if g is scalar valued with $\lim_{\mathbf{y} \rightarrow \mathbf{x}} g(\mathbf{y}) = K \neq 0$,

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) g(\mathbf{y}) = \mathbf{L}K. \quad (5.5.10)$$

Also, if $\mathbf{h}$ is a continuous function defined near $\mathbf{L}$, then

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{h} \circ \mathbf{f}(\mathbf{y}) = \mathbf{h}(\mathbf{L}). \quad (5.5.11)$$

Suppose $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$. If $|\mathbf{f}(\mathbf{y}) - \mathbf{b}| \leq r$ for all $\mathbf{y}$ sufficiently close to $\mathbf{x}$, then $|\mathbf{L} - \mathbf{b}| \leq r$ also.

Proof: The proof of 5.5.8 is left for you. It is like a corresponding theorem for continuous functions. Now 5.5.9 is to be verified. Let $\varepsilon > 0$ be given. Then by the triangle inequality,

$$\begin{aligned} |\mathbf{f} \cdot \mathbf{g}(\mathbf{y}) - \mathbf{L} \cdot \mathbf{K}| &\leq |\mathbf{f}\mathbf{g}(\mathbf{y}) - \mathbf{f}(\mathbf{y}) \cdot \mathbf{K}| + |\mathbf{f}(\mathbf{y}) \cdot \mathbf{K} - \mathbf{L} \cdot \mathbf{K}| \\ &\leq |\mathbf{f}(\mathbf{y})| |\mathbf{g}(\mathbf{y}) - \mathbf{K}| + |\mathbf{K}| |\mathbf{f}(\mathbf{y}) - \mathbf{L}|. \end{aligned}$$

There exists δ_1 such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta_1$ and $\mathbf{y} \in D(\mathbf{f})$, then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < 1,$$

and so for such $\mathbf{y}$, the triangle inequality implies, $|\mathbf{f}(\mathbf{y})| < 1 + |\mathbf{L}|$. Therefore, for $0 < |\mathbf{y} - \mathbf{x}| < \delta_1$,

$$|\mathbf{f} \cdot \mathbf{g}(\mathbf{y}) - \mathbf{L} \cdot \mathbf{K}| \leq (1 + |\mathbf{K}| + |\mathbf{L}|) [|\mathbf{g}(\mathbf{y}) - \mathbf{K}| + |\mathbf{f}(\mathbf{y}) - \mathbf{L}|]. \quad (5.5.12)$$

Now let $0 < \delta_2$ be such that if $\mathbf{y} \in D(\mathbf{f})$ and $0 < |\mathbf{x} - \mathbf{y}| < \delta_2$,

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \frac{\varepsilon}{2(1 + |\mathbf{K}| + |\mathbf{L}|)}, \quad |\mathbf{g}(\mathbf{y}) - \mathbf{K}| < \frac{\varepsilon}{2(1 + |\mathbf{K}| + |\mathbf{L}|)}.$$

Then letting $0 < \delta \leq \min(\delta_1, \delta_2)$, it follows from 5.5.12 that

$$|\mathbf{f} \cdot \mathbf{g}(\mathbf{y}) - \mathbf{L} \cdot \mathbf{K}| < \varepsilon$$

and this proves 5.5.9.

The proof of 5.5.10 is left to you.

Consider 5.5.11. Since $\mathbf{h}$ is continuous near $\mathbf{L}$, it follows that for $\varepsilon > 0$ given, there exists $\eta > 0$ such that if $|\mathbf{y} - \mathbf{L}| < \eta$, then

$$|\mathbf{h}(\mathbf{y}) - \mathbf{h}(\mathbf{L})| < \varepsilon$$

Now since $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L}$, there exists $\delta > 0$ such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta$, then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \eta.$$

Therefore, if $0 < |\mathbf{y} - \mathbf{x}| < \delta$,

$$|\mathbf{h}(\mathbf{f}(\mathbf{y})) - \mathbf{h}(\mathbf{L})| < \varepsilon.$$

It only remains to verify the last assertion. Assume $|\mathbf{f}(\mathbf{y}) - \mathbf{b}| \leq r$ for all $\mathbf{y}$ close enough to $\mathbf{x}$. It is required to show that $|\mathbf{L} - \mathbf{b}| \leq r$. If this is not true, then $|\mathbf{L} - \mathbf{b}| > r$. Consider $B(\mathbf{L}, |\mathbf{L} - \mathbf{b}| - r)$. Since $\mathbf{L}$ is the limit of $\mathbf{f}$, it follows $\mathbf{f}(\mathbf{y}) \in B(\mathbf{L}, |\mathbf{L} - \mathbf{b}| - r)$ whenever $\mathbf{y} \in D(\mathbf{f})$ is close enough to $\mathbf{x}$. Thus, by the triangle inequality,

$$|\mathbf{f}(\mathbf{y}) - \mathbf{L}| < |\mathbf{L} - \mathbf{b}| - r$$

and so

$$\begin{aligned} r &< |\mathbf{L} - \mathbf{b}| - |\mathbf{f}(\mathbf{y}) - \mathbf{L}| \leq ||\mathbf{b} - \mathbf{L}| - |\mathbf{f}(\mathbf{y}) - \mathbf{L}|| \\ &\leq |\mathbf{b} - \mathbf{f}(\mathbf{y})|, \end{aligned}$$

a contradiction to the assumption that $|\mathbf{b} - \mathbf{f}(\mathbf{y})| \leq r$.

Theorem 5.5.6 For $\mathbf{f} : D(\mathbf{f}) \rightarrow \mathbb{F}^q$ and $\mathbf{x} \in D(\mathbf{f})$ a limit point of $D(\mathbf{f})$, $\mathbf{f}$ is continuous at $\mathbf{x}$ if and only if

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x}).$$

Proof: First suppose $\mathbf{f}$ is continuous at $\mathbf{x}$ a limit point of $D(\mathbf{f})$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$. In particular, this holds if $0 < |\mathbf{x} - \mathbf{y}| < \delta$ and this is just the definition of the limit. Hence $\mathbf{f}(\mathbf{x}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y})$.

Next suppose $\mathbf{x}$ is a limit point of $D(\mathbf{f})$ and $\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{x})$. This means that if $\varepsilon > 0$ there exists $\delta > 0$ such that for $0 < |\mathbf{x} - \mathbf{y}| < \delta$ and $\mathbf{y} \in D(\mathbf{f})$, it follows $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| < \varepsilon$. However, if $\mathbf{y} = \mathbf{x}$, then $|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| = |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| = 0$ and so whenever $\mathbf{y} \in D(\mathbf{f})$ and $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$, showing $\mathbf{f}$ is continuous at $\mathbf{x}$.

The following theorem is important.

Theorem 5.5.7 Suppose $\mathbf{f} : D(\mathbf{f}) \rightarrow \mathbb{F}^q$. Then for $\mathbf{x}$ a limit point of $D(\mathbf{f})$,

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \mathbf{f}(\mathbf{y}) = \mathbf{L} \tag{5.5.13}$$

if and only if

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_k(\mathbf{y}) = L_k \tag{5.5.14}$$

where $\mathbf{f}(\mathbf{y}) \equiv (f_1(\mathbf{y}), \dots, f_p(\mathbf{y}))$ and $\mathbf{L} \equiv (L_1, \dots, L_p)$.

Proof: Suppose 5.5.13. Then letting $\varepsilon > 0$ be given there exists $\delta > 0$ such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta$, it follows

$$|f_k(\mathbf{y}) - L_k| \leq |\mathbf{f}(\mathbf{y}) - \mathbf{L}| < \varepsilon$$

which verifies 5.5.14.

Now suppose 5.5.14 holds. Then letting $\varepsilon > 0$ be given, there exists δ_k such that if $0 < |\mathbf{y} - \mathbf{x}| < \delta_k$, then

$$|f_k(\mathbf{y}) - L_k| < \frac{\varepsilon}{\sqrt{p}}.$$

Let $0 < \delta < \min(\delta_1, \dots, \delta_p)$. Then if $0 < |\mathbf{y} - \mathbf{x}| < \delta$, it follows

$$\begin{aligned} |\mathbf{f}(\mathbf{y}) - \mathbf{L}| &= \left(\sum_{k=1}^p |f_k(\mathbf{y}) - L_k|^2 \right)^{1/2} \\ &< \left(\sum_{k=1}^p \frac{\varepsilon^2}{p} \right)^{1/2} = \varepsilon. \end{aligned}$$

This proves the theorem.

This theorem shows it suffices to consider the components of a vector valued function when computing the limit.

Example 5.5.8 Find $\lim_{(x,y) \rightarrow (3,1)} \left(\frac{x^2-9}{x-3}, y \right)$.

It is clear that $\lim_{(x,y) \rightarrow (3,1)} \frac{x^2-9}{x-3} = 6$ and $\lim_{(x,y) \rightarrow (3,1)} y = 1$. Therefore, this limit equals $(6, 1)$.

Example 5.5.9 Find $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$.

First of all observe the domain of the function is $\mathbb{F}^2 \setminus \{(0,0)\}$, every point in $\mathbb{F}^2$ except the origin. Therefore, $(0,0)$ is a limit point of the domain of the function so it might make sense to take a limit. However, just as in the case of a function of one variable, the limit may not exist. In fact, this is the case here. To see this, take points on the line $y = 0$. At these points, the value of the function equals 0. Now consider points on the line $y = x$ where the value of the function equals $1/2$. Since arbitrarily close to $(0,0)$ there are points where the function equals $1/2$ and points where the function has the value 0, it follows there can be no limit. Just take $\varepsilon = 1/10$ for example. You can't be within $1/10$ of $1/2$ and also within $1/10$ of 0 at the same time.

Note it is necessary to rely on the definition of the limit much more than in the case of a function of one variable and it is the case there are no easy ways to do limit problems for functions of more than one variable. It is what it is and you will not deal with these concepts without agony.

5.6 Exercises

1. Find the following limits if possible

(a) $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$

(b) $\lim_{(x,y) \rightarrow (0,0)} \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$

(c) $\lim_{(x,y) \rightarrow (0,0)} \frac{(x^2 - y^4)^2}{(x^2 + y^4)^2}$ **Hint:** Consider along $y = 0$ and along $x = y^2$.

(d) $\lim_{(x,y) \rightarrow (0,0)} x \sin\left(\frac{1}{x^2 + y^2}\right)$

(e) The limit as $(x, y) \rightarrow (1, 2)$ of the expression

$$\frac{-2yx^2 + 8yx + 34y + 3y^3 - 18y^2 + 6x^2 - 13x - 20 - xy^2 - x^3}{-y^2 + 4y - 5 - x^2 + 2x}.$$

Hint: It might help to write this in terms of the variables $(s, t) = (x - 1, y - 2)$.

2. In the definition of limit, why must $\mathbf{x}$ be a limit point of $D(\mathbf{f})$? **Hint:** If $\mathbf{x}$ were not a limit point of $D(\mathbf{f})$, show there exists $\delta > 0$ such that $B(\mathbf{x}, \delta)$ contains no points of $D(\mathbf{f})$ other than possibly $\mathbf{x}$ itself. Argue that 33.3 is a limit and that so is 22 and 7 and 11. In other words the concept is totally worthless.

5.7 The Limit Of A Sequence

As in the case of real numbers, one can consider the limit of a sequence of points in $\mathbb{F}^p$.

Definition 5.7.1 A sequence $\{\mathbf{a}_n\}_{n=1}^{\infty}$ converges to $\mathbf{a}$, and write

$$\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a} \text{ or } \mathbf{a}_n \rightarrow \mathbf{a}$$

if and only if for every $\varepsilon > 0$ there exists n_ε such that whenever $n \geq n_\varepsilon$,

$$|\mathbf{a}_n - \mathbf{a}| < \varepsilon.$$

In words the definition says that given any measure of closeness, ε , the terms of the sequence are eventually all this close to $\mathbf{a}$. There is absolutely no difference between this and the definition for sequences of numbers other than here bold face is used to indicate $\mathbf{a}_n$ and $\mathbf{a}$ are points in $\mathbb{F}^p$.

Theorem 5.7.2 If $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}$ and $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}_1$ then $\mathbf{a}_1 = \mathbf{a}$.

Proof: Suppose $\mathbf{a}_1 \neq \mathbf{a}$. Then let $0 < \varepsilon < |\mathbf{a}_1 - \mathbf{a}|/2$ in the definition of the limit. It follows there exists n_ε such that if $n \geq n_\varepsilon$, then $|\mathbf{a}_n - \mathbf{a}| < \varepsilon$ and $|\mathbf{a}_n - \mathbf{a}_1| < \varepsilon$. Therefore, for such n ,

$$\begin{aligned} |\mathbf{a}_1 - \mathbf{a}| &\leq |\mathbf{a}_1 - \mathbf{a}_n| + |\mathbf{a}_n - \mathbf{a}| \\ &< \varepsilon + \varepsilon < |\mathbf{a}_1 - \mathbf{a}|/2 + |\mathbf{a}_1 - \mathbf{a}|/2 = |\mathbf{a}_1 - \mathbf{a}|, \end{aligned}$$

a contradiction.

As in the case of a vector valued function, it suffices to consider the components. This is the content of the next theorem.

Theorem 5.7.3 *Let $\mathbf{a}_n = (a_1^n, \dots, a_p^n) \in \mathbb{F}^p$. Then $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a} \equiv (a_1, \dots, a_p)$ if and only if for each $k = 1, \dots, p$,*

$$\lim_{n \rightarrow \infty} a_k^n = a_k. \quad (5.7.15)$$

Proof: First suppose $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}$. Then given $\varepsilon > 0$ there exists n_ε such that if $n > n_\varepsilon$, then

$$|a_k^n - a_k| \leq |\mathbf{a}_n - \mathbf{a}| < \varepsilon$$

which establishes 5.7.15.

Now suppose 5.7.15 holds for each k . Then letting $\varepsilon > 0$ be given there exist n_k such that if $n > n_k$,

$$|a_k^n - a_k| < \varepsilon/\sqrt{p}.$$

Therefore, letting $n_\varepsilon > \max(n_1, \dots, n_p)$, it follows that for $n > n_\varepsilon$,

$$|\mathbf{a}_n - \mathbf{a}| = \left(\sum_{k=1}^n |a_k^n - a_k|^2 \right)^{1/2} < \left(\sum_{k=1}^n \frac{\varepsilon^2}{p} \right)^{1/2} = \varepsilon,$$

showing that $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}$. This proves the theorem.

Example 5.7.4 *Let $\mathbf{a}_n = \left(\frac{1}{n^2+1}, \frac{1}{n} \sin(n), \frac{n^2+3}{3n^2+5n} \right)$.*

It suffices to consider the limits of the components according to the following theorem. Thus the limit is $(0, 0, 1/3)$.

Theorem 5.7.5 *Suppose $\{\mathbf{a}_n\}$ and $\{\mathbf{b}_n\}$ are sequences and that*

$$\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a} \text{ and } \lim_{n \rightarrow \infty} \mathbf{b}_n = \mathbf{b}.$$

Also suppose x and y are numbers in $\mathbb{F}$. Then

$$\lim_{n \rightarrow \infty} x\mathbf{a}_n + y\mathbf{b}_n = x\mathbf{a} + y\mathbf{b} \quad (5.7.16)$$

$$\lim_{n \rightarrow \infty} \mathbf{a}_n \cdot \mathbf{b}_n = \mathbf{a} \cdot \mathbf{b} \quad (5.7.17)$$

If $b_n \in \mathbb{F}$, then

$$\mathbf{a}_n b_n \rightarrow \mathbf{a}b.$$

Proof: The first of these claims is left for you to do. To do the second, let $\varepsilon > 0$ be given and choose n_1 such that if $n \geq n_1$ then

$$|\mathbf{a}_n - \mathbf{a}| < 1.$$

Then for such n , the triangle inequality and Cauchy Schwarz inequality imply

$$\begin{aligned} |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a} \cdot \mathbf{b}| &\leq |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a}_n \cdot \mathbf{b}| + |\mathbf{a}_n \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b}| \\ &\leq |\mathbf{a}_n| |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| |\mathbf{a}_n - \mathbf{a}| \\ &\leq (|\mathbf{a}| + 1) |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| |\mathbf{a}_n - \mathbf{a}|. \end{aligned}$$

Now let n_2 be large enough that for $n \geq n_2$,

$$|\mathbf{b}_n - \mathbf{b}| < \frac{\varepsilon}{2(|\mathbf{a}| + 1)}, \text{ and } |\mathbf{a}_n - \mathbf{a}| < \frac{\varepsilon}{2(|\mathbf{b}| + 1)}.$$

Such a number exists because of the definition of limit. Therefore, let

$$n_\varepsilon > \max(n_1, n_2).$$

For $n \geq n_\varepsilon$,

$$\begin{aligned} |\mathbf{a}_n \cdot \mathbf{b}_n - \mathbf{a} \cdot \mathbf{b}| &\leq (|\mathbf{a}| + 1) |\mathbf{b}_n - \mathbf{b}| + |\mathbf{b}| |\mathbf{a}_n - \mathbf{a}| \\ &< (|\mathbf{a}| + 1) \frac{\varepsilon}{2(|\mathbf{a}| + 1)} + |\mathbf{b}| \frac{\varepsilon}{2(|\mathbf{b}| + 1)} \leq \varepsilon. \end{aligned}$$

This proves 5.7.16. The proof of 5.7.17 is entirely similar and is left for you.

5.7.1 Sequences And Completeness

Recall the definition of a Cauchy sequence.

Definition 5.7.6 $\{\mathbf{a}_n\}$ is a Cauchy sequence if for all $\varepsilon > 0$, there exists n_ε such that whenever $n, m \geq n_\varepsilon$,

$$|\mathbf{a}_n - \mathbf{a}_m| < \varepsilon.$$

A sequence is Cauchy means the terms are “bunching up to each other” as m, n get large.

Theorem 5.7.7 Let $\{\mathbf{a}_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{F}^p$. Then there exists a unique $\mathbf{a} \in \mathbb{F}^p$ such that $\mathbf{a}_n \rightarrow \mathbf{a}$.

Proof: Let $\mathbf{a}_n = (a_1^n, \dots, a_p^n)$. Then

$$|a_k^n - a_k^m| \leq |\mathbf{a}_n - \mathbf{a}_m|$$

which shows for each $k = 1, \dots, p$, it follows $\{a_k^n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$. This requires that both the real and imaginary parts of a_k^n are Cauchy sequences in $\mathbb{R}$ which means the real and imaginary parts converge in $\mathbb{R}$. This shows $\{a_k^n\}_{n=1}^\infty$

must converge to some a_k . That is $\lim_{n \rightarrow \infty} a_k^n = a_k$. Letting $\mathbf{a} = (a_1, \dots, a_p)$, it follows from Theorem 5.7.3 that

$$\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{a}.$$

This proves the theorem.

Theorem 5.7.8 *The set of terms in a Cauchy sequence in $\mathbb{F}^p$ is bounded in the sense that for all n , $|\mathbf{a}_n| < M$ for some $M < \infty$.*

Proof: Let $\varepsilon = 1$ in the definition of a Cauchy sequence and let $n > n_1$. Then from the definition,

$$|\mathbf{a}_n - \mathbf{a}_{n_1}| < 1.$$

It follows that for all $n > n_1$,

$$|\mathbf{a}_n| < 1 + |\mathbf{a}_{n_1}|.$$

Therefore, for all n ,

$$|\mathbf{a}_n| \leq 1 + |\mathbf{a}_{n_1}| + \sum_{k=1}^{n_1} |\mathbf{a}_k|.$$

This proves the theorem.

Theorem 5.7.9 *If a sequence $\{\mathbf{a}_n\}$ in $\mathbb{F}^p$ converges, then the sequence is a Cauchy sequence.*

Proof: Let $\varepsilon > 0$ be given and suppose $\mathbf{a}_n \rightarrow \mathbf{a}$. Then from the definition of convergence, there exists n_ε such that if $n > n_\varepsilon$, it follows that

$$|\mathbf{a}_n - \mathbf{a}| < \frac{\varepsilon}{2}$$

Therefore, if $m, n \geq n_\varepsilon + 1$, it follows that

$$|\mathbf{a}_n - \mathbf{a}_m| \leq |\mathbf{a}_n - \mathbf{a}| + |\mathbf{a} - \mathbf{a}_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

showing that, since $\varepsilon > 0$ is arbitrary, $\{\mathbf{a}_n\}$ is a Cauchy sequence.

5.7.2 Continuity And The Limit Of A Sequence

Just as in the case of a function of one variable, there is a very useful way of thinking of continuity in terms of limits of sequences found in the following theorem. In words, it says a function is continuous if it takes convergent sequences to convergent sequences whenever possible.

Theorem 5.7.10 *A function $\mathbf{f} : D(\mathbf{f}) \rightarrow \mathbb{F}^q$ is continuous at $\mathbf{x} \in D(\mathbf{f})$ if and only if, whenever $\mathbf{x}_n \rightarrow \mathbf{x}$ with $\mathbf{x}_n \in D(\mathbf{f})$, it follows $\mathbf{f}(\mathbf{x}_n) \rightarrow \mathbf{f}(\mathbf{x})$.*

Proof: Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$ and let $\mathbf{x}_n \rightarrow \mathbf{x}$. Let $\varepsilon > 0$ be given. By continuity, there exists $\delta > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$. However, there exists n_δ such that if $n \geq n_\delta$, then $|\mathbf{x}_n - \mathbf{x}| < \delta$ and so for all n this large,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_n)| < \varepsilon$$

which shows $\mathbf{f}(\mathbf{x}_n) \rightarrow \mathbf{f}(\mathbf{x})$.

Now suppose the condition about taking convergent sequences to convergent sequences holds at $\mathbf{x}$. Suppose $\mathbf{f}$ fails to be continuous at $\mathbf{x}$. Then there exists $\varepsilon > 0$ and $\mathbf{x}_n \in D(\mathbf{f})$ such that $|\mathbf{x} - \mathbf{x}_n| < \frac{1}{n}$, yet

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_n)| \geq \varepsilon.$$

But this is clearly a contradiction because, although $\mathbf{x}_n \rightarrow \mathbf{x}$, $\mathbf{f}(\mathbf{x}_n)$ fails to converge to $\mathbf{f}(\mathbf{x})$. It follows $\mathbf{f}$ must be continuous after all. This proves the theorem.

5.8 Properties Of Continuous Functions

Functions of p variables have many of the same properties as functions of one variable. First there is a version of the extreme value theorem generalizing the one dimensional case.

Theorem 5.8.1 *Let C be closed and bounded and let $f : C \rightarrow \mathbb{R}$ be continuous. Then f achieves its maximum and its minimum on C . This means there exist, $\mathbf{x}_1, \mathbf{x}_2 \in C$ such that for all $\mathbf{x} \in C$,*

$$f(\mathbf{x}_1) \leq f(\mathbf{x}) \leq f(\mathbf{x}_2).$$

There is also the long technical theorem about sums and products of continuous functions. These theorems are proved in the next section.

Theorem 5.8.2 *The following assertions are valid*

1. *The function, $a\mathbf{f} + b\mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.*
2. *If f and g are each $\mathbb{F}$ valued functions continuous at $\mathbf{x}$, then fg is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then f/g is continuous at $\mathbf{x}$.*
3. *If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.*
4. *If $\mathbf{f} = (f_1, \dots, f_q) : D(\mathbf{f}) \rightarrow \mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function.*
5. *The function $f : \mathbb{F}^p \rightarrow \mathbb{F}$, given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous.*

5.9 Exercises

1. $\mathbf{f} : D \subseteq \mathbb{F}^p \rightarrow \mathbb{F}^q$ is Lipschitz continuous or just Lipschitz for short if there exists a constant, K such that

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in D$. Show every Lipschitz function is uniformly continuous which means that given $\varepsilon > 0$ there exists $\delta > 0$ independent of $\mathbf{x}$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$.

2. If $\mathbf{f}$ is uniformly continuous, does it follow that $|\mathbf{f}|$ is also uniformly continuous? If $|\mathbf{f}|$ is uniformly continuous does it follow that $\mathbf{f}$ is uniformly continuous? Answer the same questions with “uniformly continuous” replaced with “continuous”. Explain why.

5.10 Proofs Of Theorems

This section contains the proofs of the theorems which were just stated without proof.

Theorem 5.10.1 *The following assertions are valid*

1. The function, $a\mathbf{f} + b\mathbf{g}$ is continuous at $\mathbf{x}$ when $\mathbf{f}, \mathbf{g}$ are continuous at $\mathbf{x} \in D(\mathbf{f}) \cap D(\mathbf{g})$ and $a, b \in \mathbb{F}$.
2. If f and g are each $\mathbb{F}$ valued functions continuous at $\mathbf{x}$, then fg is continuous at $\mathbf{x}$. If, in addition to this, $g(\mathbf{x}) \neq 0$, then f/g is continuous at $\mathbf{x}$.
3. If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$.
4. If $\mathbf{f} = (f_1, \dots, f_q) : D(\mathbf{f}) \rightarrow \mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function.
5. The function $f : \mathbb{F}^p \rightarrow \mathbb{F}$, given by $f(\mathbf{x}) = |\mathbf{x}|$ is continuous.

Proof: Begin with 1.) Let $\varepsilon > 0$ be given. By assumption, there exist $\delta_1 > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_1$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \frac{\varepsilon}{2(|a|+|b|+1)}$ and there exists $\delta_2 > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_2$, it follows that $|\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| < \frac{\varepsilon}{2(|a|+|b|+1)}$. Then let $0 < \delta \leq \min(\delta_1, \delta_2)$. If $|\mathbf{x} - \mathbf{y}| < \delta$, then everything happens at once. Therefore, using the triangle inequality

$$|a\mathbf{f}(\mathbf{x}) + b\mathbf{f}(\mathbf{x}) - (a\mathbf{g}(\mathbf{y}) + b\mathbf{g}(\mathbf{y}))|$$

$$\begin{aligned} &\leq |a| |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| + |b| |\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{y})| \\ &< |a| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) + |b| \left(\frac{\varepsilon}{2(|a| + |b| + 1)} \right) < \varepsilon. \end{aligned}$$

Now begin on 2.) There exists $\delta_1 > 0$ such that if $|\mathbf{y} - \mathbf{x}| < \delta_1$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < 1.$$

Therefore, for such $\mathbf{y}$,

$$|f(\mathbf{y})| < 1 + |f(\mathbf{x})|.$$

It follows that for such $\mathbf{y}$,

$$\begin{aligned} |fg(\mathbf{x}) - fg(\mathbf{y})| &\leq |f(\mathbf{x})g(\mathbf{x}) - g(\mathbf{x})f(\mathbf{y})| + |g(\mathbf{x})f(\mathbf{y}) - f(\mathbf{y})g(\mathbf{y})| \\ &\leq |g(\mathbf{x})| |f(\mathbf{x}) - f(\mathbf{y})| + |f(\mathbf{y})| |g(\mathbf{x}) - g(\mathbf{y})| \\ &\leq (1 + |g(\mathbf{x})| + |f(\mathbf{y})|) [|g(\mathbf{x}) - g(\mathbf{y})| + |f(\mathbf{x}) - f(\mathbf{y})|]. \end{aligned}$$

Now let $\varepsilon > 0$ be given. There exists δ_2 such that if $|\mathbf{x} - \mathbf{y}| < \delta_2$, then

$$|g(\mathbf{x}) - g(\mathbf{y})| < \frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)},$$

and there exists δ_3 such that if $|\mathbf{x} - \mathbf{y}| < \delta_3$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)}$$

Now let $0 < \delta \leq \min(\delta_1, \delta_2, \delta_3)$. Then if $|\mathbf{x} - \mathbf{y}| < \delta$, all the above hold at once and

$$\begin{aligned} &|fg(\mathbf{x}) - fg(\mathbf{y})| \leq \\ &(1 + |g(\mathbf{x})| + |f(\mathbf{y})|) [|g(\mathbf{x}) - g(\mathbf{y})| + |f(\mathbf{x}) - f(\mathbf{y})|] \\ &< (1 + |g(\mathbf{x})| + |f(\mathbf{y})|) \left(\frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)} + \frac{\varepsilon}{2(1 + |g(\mathbf{x})| + |f(\mathbf{y})|)} \right) = \varepsilon. \end{aligned}$$

This proves the first part of 2.) To obtain the second part, let δ_1 be as described above and let $\delta_0 > 0$ be such that for $|\mathbf{x} - \mathbf{y}| < \delta_0$,

$$|g(\mathbf{x}) - g(\mathbf{y})| < |g(\mathbf{x})|/2$$

and so by the triangle inequality,

$$-|g(\mathbf{x})|/2 \leq |g(\mathbf{y})| - |g(\mathbf{x})| \leq |g(\mathbf{x})|/2$$

which implies $|g(\mathbf{y})| \geq |g(\mathbf{x})|/2$, and $|g(\mathbf{y})| < 3|g(\mathbf{x})|/2$.

Then if $|\mathbf{x} - \mathbf{y}| < \min(\delta_0, \delta_1)$,

$$\begin{aligned}
 \left| \frac{f(\mathbf{x})}{g(\mathbf{x})} - \frac{f(\mathbf{y})}{g(\mathbf{y})} \right| &= \left| \frac{f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})}{g(\mathbf{x})g(\mathbf{y})} \right| \\
 &\leq \frac{|f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})|}{\left(\frac{|g(\mathbf{x})|^2}{2}\right)} \\
 &= \frac{2|f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})|}{|g(\mathbf{x})|^2} \\
 &\leq \frac{2}{|g(\mathbf{x})|^2} [|f(\mathbf{x})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{y}) + f(\mathbf{y})g(\mathbf{y}) - f(\mathbf{y})g(\mathbf{x})|] \\
 &\leq \frac{2}{|g(\mathbf{x})|^2} [|g(\mathbf{y})||f(\mathbf{x}) - f(\mathbf{y})| + |f(\mathbf{y})||g(\mathbf{y}) - g(\mathbf{x})|] \\
 &\leq \frac{2}{|g(\mathbf{x})|^2} \left[\frac{3}{2} |\mathbf{g}(\mathbf{x})||f(\mathbf{x}) - f(\mathbf{y})| + (1 + |f(\mathbf{x})|)|g(\mathbf{y}) - g(\mathbf{x})| \right] \\
 &\leq \frac{2}{|g(\mathbf{x})|^2} (1 + 2|f(\mathbf{x})| + 2|g(\mathbf{x})|) [|f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|] \\
 &\equiv M [|f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|]
 \end{aligned}$$

where

$$M \equiv \frac{2}{|g(\mathbf{x})|^2} (1 + 2|f(\mathbf{x})| + 2|g(\mathbf{x})|)$$

Now let δ_2 be such that if $|\mathbf{x} - \mathbf{y}| < \delta_2$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\varepsilon}{2} M^{-1}$$

and let δ_3 be such that if $|\mathbf{x} - \mathbf{y}| < \delta_3$, then

$$|g(\mathbf{y}) - g(\mathbf{x})| < \frac{\varepsilon}{2} M^{-1}.$$

Then if $0 < \delta \leq \min(\delta_0, \delta_1, \delta_2, \delta_3)$, and $|\mathbf{x} - \mathbf{y}| < \delta$, everything holds and

$$\begin{aligned}
 \left| \frac{f(\mathbf{x})}{g(\mathbf{x})} - \frac{f(\mathbf{y})}{g(\mathbf{y})} \right| &\leq M [|f(\mathbf{x}) - f(\mathbf{y})| + |g(\mathbf{y}) - g(\mathbf{x})|] \\
 &< M \left[\frac{\varepsilon}{2} M^{-1} + \frac{\varepsilon}{2} M^{-1} \right] = \varepsilon.
 \end{aligned}$$

This completes the proof of the second part of 2.) Note that in these proofs no effort is made to find some sort of “best” δ . The problem is one which has a yes or a no answer. Either it is or it is not continuous.

Now begin on 3.). If $\mathbf{f}$ is continuous at $\mathbf{x}$, $\mathbf{f}(\mathbf{x}) \in D(\mathbf{g}) \subseteq \mathbb{F}^p$, and $\mathbf{g}$ is continuous at $\mathbf{f}(\mathbf{x})$, then $\mathbf{g} \circ \mathbf{f}$ is continuous at $\mathbf{x}$. Let $\varepsilon > 0$ be given. Then there exists $\eta > 0$

such that if $|\mathbf{y} - \mathbf{f}(\mathbf{x})| < \eta$ and $\mathbf{y} \in D(\mathbf{g})$, it follows that $|\mathbf{g}(\mathbf{y}) - \mathbf{g}(\mathbf{f}(\mathbf{x}))| < \varepsilon$. It follows from continuity of $\mathbf{f}$ at $\mathbf{x}$ that there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{z}| < \delta$ and $\mathbf{z} \in D(\mathbf{f})$, then $|\mathbf{f}(\mathbf{z}) - \mathbf{f}(\mathbf{x})| < \eta$. Then if $|\mathbf{x} - \mathbf{z}| < \delta$ and $\mathbf{z} \in D(\mathbf{g} \circ \mathbf{f}) \subseteq D(\mathbf{f})$, all the above hold and so

$$|\mathbf{g}(\mathbf{f}(\mathbf{z})) - \mathbf{g}(\mathbf{f}(\mathbf{x}))| < \varepsilon.$$

This proves part 3.)

Part 4.) says: If $\mathbf{f} = (f_1, \dots, f_q) : D(\mathbf{f}) \rightarrow \mathbb{F}^q$, then $\mathbf{f}$ is continuous if and only if each f_k is a continuous $\mathbb{F}$ valued function. Then

$$\begin{aligned} |f_k(\mathbf{x}) - f_k(\mathbf{y})| &\leq |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \\ &\equiv \left(\sum_{i=1}^q |f_i(\mathbf{x}) - f_i(\mathbf{y})|^2 \right)^{1/2} \\ &\leq \sum_{i=1}^q |f_i(\mathbf{x}) - f_i(\mathbf{y})|. \end{aligned} \quad (5.10.18)$$

Suppose first that $\mathbf{f}$ is continuous at $\mathbf{x}$. Then there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$. The first part of the above inequality then shows that for each $k = 1, \dots, q$, $|f_k(\mathbf{x}) - f_k(\mathbf{y})| < \varepsilon$. This shows the only if part. Now suppose each function, f_k is continuous. Then if $\varepsilon > 0$ is given, there exists $\delta_k > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta_k$

$$|f_k(\mathbf{x}) - f_k(\mathbf{y})| < \varepsilon/q.$$

Now let $0 < \delta \leq \min(\delta_1, \dots, \delta_q)$. For $|\mathbf{x} - \mathbf{y}| < \delta$, the above inequality holds for all k and so the last part of 5.10.18 implies

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| &\leq \sum_{i=1}^q |f_i(\mathbf{x}) - f_i(\mathbf{y})| \\ &< \sum_{i=1}^q \frac{\varepsilon}{q} = \varepsilon. \end{aligned}$$

This proves part 4.)

To verify part 5.), let $\varepsilon > 0$ be given and let $\delta = \varepsilon$. Then if $|\mathbf{x} - \mathbf{y}| < \delta$, the triangle inequality implies

$$\begin{aligned} |f(\mathbf{x}) - f(\mathbf{y})| &= \|\mathbf{x} - \mathbf{y}\| \\ &\leq |\mathbf{x} - \mathbf{y}| < \delta = \varepsilon. \end{aligned}$$

This proves part 5.) and completes the proof of the theorem.

Here is a multidimensional version of the nested interval lemma.

The following definition is similar to that given earlier. It defines what is meant by a sequentially compact set in $\mathbb{F}^p$.

Definition 5.10.2 A set, $K \subseteq \mathbb{F}^p$ is sequentially compact if and only if whenever $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a sequence of points in K , there exists a point, $\mathbf{x} \in K$ and a subsequence, $\{\mathbf{x}_{n_k}\}_{k=1}^{\infty}$ such that $\mathbf{x}_{n_k} \rightarrow \mathbf{x}$.

It turns out the sequentially compact sets in $\mathbb{F}^p$ are exactly those which are closed and bounded. Only half of this result will be needed in this book and this is proved next. First note that $\mathbb{C}$ can be considered as $\mathbb{R}^2$. Therefore, $\mathbb{C}^p$ may be considered as $\mathbb{R}^{2p}$.

Theorem 5.10.3 Let $C \subseteq \mathbb{F}^p$ be closed and bounded. Then C is sequentially compact.

Proof: Let $\{\mathbf{a}_n\} \subseteq C$. Then let $\mathbf{a}_n = (a_1^n, \dots, a_p^n)$. It follows the real and imaginary parts of the terms of the sequence, $\{a_j^n\}_{n=1}^{\infty}$ are each contained in some sufficiently large closed bounded interval. By Theorem 2.0.5 on Page 27, there is a subsequence of the sequence of real parts of $\{a_j^n\}_{n=1}^{\infty}$ which converges. Also there is a further subsequence of the imaginary parts of $\{a_j^n\}_{n=1}^{\infty}$ which converges. Thus there is a subsequence, n_k with the property that $a_j^{n_k}$ converges to a point, $a_j \in \mathbb{F}$. Taking further subsequences, one obtains the existence of a subsequence, still called n_k such that for each $r = 1, \dots, p$, $a_r^{n_k}$ converges to a point, $a_r \in \mathbb{F}$ as $k \rightarrow \infty$. Therefore, letting $\mathbf{a} \equiv (a_1, \dots, a_p)$, $\lim_{k \rightarrow \infty} \mathbf{a}^{n_k} = \mathbf{a}$. Since C is closed, it follows $\mathbf{a} \in C$. This proves the theorem.

Here is a proof of the extreme value theorem.

Theorem 5.10.4 Let C be closed and bounded and let $f : C \rightarrow \mathbb{R}$ be continuous. Then f achieves its maximum and its minimum on C . This means there exist, $\mathbf{x}_1, \mathbf{x}_2 \in C$ such that for all $\mathbf{x} \in C$,

$$f(\mathbf{x}_1) \leq f(\mathbf{x}) \leq f(\mathbf{x}_2).$$

Proof: Let $M = \sup \{f(\mathbf{x}) : \mathbf{x} \in C\}$. Recall this means $+\infty$ if f is not bounded above and it equals the least upper bound of these values of f if f is bounded above. Then there exists a sequence, $\{\mathbf{x}_n\}$ such that $f(\mathbf{x}_n) \rightarrow M$. Since C is sequentially compact, there exists a subsequence, $\mathbf{x}_{n_k}$, and a point, $\mathbf{x} \in C$ such that $\mathbf{x}_{n_k} \rightarrow \mathbf{x}$. But then since f is continuous at $\mathbf{x}$, it follows from Theorem 5.7.10 on Page 111 that $f(\mathbf{x}) = \lim_{k \rightarrow \infty} f(\mathbf{x}_{n_k}) = M$. This proves f achieves its maximum and also shows its maximum is less than ∞ . Let $\mathbf{x}_2 = \mathbf{x}$. The case of a minimum is handled similarly.

Recall that a function is uniformly continuous if the following definition holds.

Definition 5.10.5 Let $\mathbf{f} : D(\mathbf{f}) \rightarrow \mathbb{F}^q$. Then $\mathbf{f}$ is uniformly continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < \varepsilon$.

Theorem 5.10.6 Let $\mathbf{f} : C \rightarrow \mathbb{F}^q$ be continuous where C is a closed and bounded set in $\mathbb{F}^p$. Then $\mathbf{f}$ is uniformly continuous on C .

Proof: If this is not so, there exists $\varepsilon > 0$ and pairs of points, $\mathbf{x}_n$ and $\mathbf{y}_n$ satisfying $|\mathbf{x}_n - \mathbf{y}_n| < 1/n$ but $|\mathbf{f}(\mathbf{x}_n) - \mathbf{f}(\mathbf{y}_n)| \geq \varepsilon$. Since C is sequentially compact, there exists $\mathbf{x} \in C$ and a subsequence, $\{\mathbf{x}_{n_k}\}$ satisfying $\mathbf{x}_{n_k} \rightarrow \mathbf{x}$. But $|\mathbf{x}_{n_k} - \mathbf{y}_{n_k}| < 1/k$ and so $\mathbf{y}_{n_k} \rightarrow \mathbf{x}$ also. Therefore, from Theorem 5.7.10 on Page 111,

$$\varepsilon \leq \lim_{k \rightarrow \infty} |\mathbf{f}(\mathbf{x}_{n_k}) - \mathbf{f}(\mathbf{y}_{n_k})| = |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| = 0,$$

a contradiction. This proves the theorem.

5.11 The Space $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

Definition 5.11.1 The symbol, $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ will denote the set of linear transformations mapping $\mathbb{F}^n$ to $\mathbb{F}^m$. Thus $L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ means that for α, β scalars and $\mathbf{x}, \mathbf{y}$ vectors in $\mathbb{F}^n$,

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

It is convenient to give a norm for the elements of $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. This will allow the consideration of questions such as whether a function having values in this space of linear transformations is continuous.

5.11.1 The Operator Norm

How do you measure the distance between linear transformations defined on $\mathbb{F}^n$? It turns out there are many ways to do this but I will give the most common one here.

Definition 5.11.2 $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ denotes the space of linear transformations mapping $\mathbb{F}^n$ to $\mathbb{F}^m$. For $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, the **operator norm** is defined by

$$\|A\| \equiv \max \{|Ax|_{\mathbb{F}^m} : |x|_{\mathbb{F}^n} \leq 1\} < \infty.$$

Theorem 5.11.3 Denote by $|\cdot|$ the norm on either $\mathbb{F}^n$ or $\mathbb{F}^m$. Then $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ with this operator norm is a **complete normed linear space** of dimension nm with

$$\|A\mathbf{x}\| \leq \|A\| |\mathbf{x}|.$$

Here **Completeness** means that every Cauchy sequence converges.

Proof: It is necessary to show the norm defined on $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ really is a norm. This means it is necessary to verify

$$\|A\| \geq 0 \text{ and equals zero if and only if } A = 0.$$

For α a scalar,

$$\|\alpha A\| = |\alpha| \|A\|,$$

and for $A, B \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$,

$$\|A + B\| \leq \|A\| + \|B\|$$

The first two properties are obvious but you should verify them. It remains to verify the norm is well defined and also to verify the triangle inequality above. First if $|\mathbf{x}| \leq 1$, and (A_{ij}) is the matrix of the linear transformation with respect to the usual basis vectors, then

$$\begin{aligned} \|A\| &= \max \left\{ \left(\sum_i |(A\mathbf{x})_i|^2 \right)^{1/2} : |\mathbf{x}| \leq 1 \right\} \\ &= \max \left\{ \left(\sum_i \left| \sum_j A_{ij} x_j \right|^2 \right)^{1/2} : |\mathbf{x}| \leq 1 \right\} \end{aligned}$$

which is a finite number by the extreme value theorem.

It is clear that a basis for $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ consists of linear transformations whose matrices are of the form E_{ij} where E_{ij} consists of the $m \times n$ matrix having all zeros except for a 1 in the ij^{th} position. In effect, this considers $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ as $\mathbb{F}^{nm}$. Think of the $m \times n$ matrix as a long vector folded up.

If $\mathbf{x} \neq \mathbf{0}$,

$$|A\mathbf{x}| \frac{1}{|\mathbf{x}|} = \left| A \frac{\mathbf{x}}{|\mathbf{x}|} \right| \leq \|A\| \quad (5.11.19)$$

It only remains to verify completeness. Suppose then that $\{A_k\}$ is a Cauchy sequence in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Then from 5.11.19 $\{A_k \mathbf{x}\}$ is a Cauchy sequence for each $\mathbf{x} \in \mathbb{F}^n$. This follows because

$$|A_k \mathbf{x} - A_l \mathbf{x}| \leq \|A_k - A_l\| |\mathbf{x}|$$

which converges to 0 as $k, l \rightarrow \infty$. Therefore, by completeness of $\mathbb{F}^m$, there exists $A\mathbf{x}$, the name of the thing to which the sequence, $\{A_k \mathbf{x}\}$ converges such that

$$\lim_{k \rightarrow \infty} A_k \mathbf{x} = A\mathbf{x}.$$

Then A is linear because

$$\begin{aligned} A(a\mathbf{x} + b\mathbf{y}) &\equiv \lim_{k \rightarrow \infty} A_k(a\mathbf{x} + b\mathbf{y}) \\ &= \lim_{k \rightarrow \infty} (aA_k \mathbf{x} + bA_k \mathbf{y}) \\ &= a \lim_{k \rightarrow \infty} A_k \mathbf{x} + b \lim_{k \rightarrow \infty} A_k \mathbf{y} \\ &= aA\mathbf{x} + bA\mathbf{y}. \end{aligned}$$

By the first part of this argument, $\|A\| < \infty$ and so $A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. This proves the theorem.

Proposition 5.11.4 *Let $A(\mathbf{x}) \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ for each $\mathbf{x} \in U \subseteq \mathbb{F}^p$. Then letting $(A_{ij}(\mathbf{x}))$ denote the matrix of $A(\mathbf{x})$ with respect to the standard basis, it follows A_{ij} is continuous at $\mathbf{x}$ for each i, j if and only if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then $\|A(\mathbf{x}) - A(\mathbf{y})\| < \varepsilon$. That is, A is a continuous function having values in $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ at $\mathbf{x}$.*

Proof: Suppose first the second condition holds. Then from the material on linear transformations,

$$\begin{aligned} |A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})| &= |\mathbf{e}_i \cdot (A(\mathbf{x}) - A(\mathbf{y})) \mathbf{e}_j| \\ &\leq |\mathbf{e}_i| |(A(\mathbf{x}) - A(\mathbf{y})) \mathbf{e}_j| \\ &\leq \|A(\mathbf{x}) - A(\mathbf{y})\|. \end{aligned}$$

Therefore, the second condition implies the first.

Now suppose the first condition holds. That is each A_{ij} is continuous at $\mathbf{x}$. Let $|\mathbf{v}| \leq 1$.

$$\begin{aligned} |(A(\mathbf{x}) - A(\mathbf{y}))(\mathbf{v})| &= \left(\sum_i \left| \sum_j (A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})) v_j \right|^2 \right)^{1/2} \quad (5.11.20) \\ &\leq \left(\sum_i \left(\sum_j |A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})| |v_j| \right)^2 \right)^{1/2}. \end{aligned}$$

By continuity of each A_{ij} , there exists a $\delta > 0$ such that for each i, j

$$|A_{ij}(\mathbf{x}) - A_{ij}(\mathbf{y})| < \frac{\varepsilon}{n\sqrt{m}}$$

whenever $|\mathbf{x} - \mathbf{y}| < \delta$. Then from 5.11.20, if $|\mathbf{x} - \mathbf{y}| < \delta$,

$$\begin{aligned} |(A(\mathbf{x}) - A(\mathbf{y}))(\mathbf{v})| &< \left(\sum_i \left(\sum_j \frac{\varepsilon}{n\sqrt{m}} |v_j| \right)^2 \right)^{1/2} \\ &\leq \left(\sum_i \left(\sum_j \frac{\varepsilon}{n\sqrt{m}} \right)^2 \right)^{1/2} = \varepsilon \end{aligned}$$

This proves the proposition.

5.12 The Frechet Derivative

Let U be an open set in $\mathbb{F}^n$, and let $\mathbf{f} : U \rightarrow \mathbb{F}^m$ be a function.

Definition 5.12.1 A function $\mathbf{g}$ is $o(\mathbf{v})$ if

$$\lim_{|\mathbf{v}| \rightarrow 0} \frac{\mathbf{g}(\mathbf{v})}{|\mathbf{v}|} = \mathbf{0} \quad (5.12.21)$$

A function $\mathbf{f} : U \rightarrow \mathbb{F}^m$ is differentiable at $\mathbf{x} \in U$ if there exists a linear transformation $L \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ such that

$$\mathbf{f}(\mathbf{x} + \mathbf{v}) = \mathbf{f}(\mathbf{x}) + L\mathbf{v} + o(\mathbf{v})$$

This linear transformation L is the definition of $D\mathbf{f}(\mathbf{x})$. This derivative is often called the Frechet derivative. .

Usually no harm is occasioned by thinking of this linear transformation as its matrix taken with respect to the usual basis vectors.

The definition 5.12.21 means that the error,

$$\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}$$

converges to $\mathbf{0}$ faster than $|\mathbf{v}|$. Thus the above definition is equivalent to saying

$$\lim_{|\mathbf{v}| \rightarrow 0} \frac{|\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}) - L\mathbf{v}|}{|\mathbf{v}|} = 0 \quad (5.12.22)$$

or equivalently,

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) - D\mathbf{f}(\mathbf{x})(\mathbf{y} - \mathbf{x})|}{|\mathbf{y} - \mathbf{x}|} = 0. \quad (5.12.23)$$

Now it is clear this is just a generalization of the notion of the derivative of a function of one variable because in this more specialized situation,

$$\lim_{|v| \rightarrow 0} \frac{|f(x+v) - f(x) - f'(x)v|}{|v|} = 0,$$

due to the definition which says

$$f'(x) = \lim_{v \rightarrow 0} \frac{f(x+v) - f(x)}{v}.$$

For functions of n variables, you can't define the derivative as the limit of a difference quotient like you can for a function of one variable because you can't divide by a vector. That is why there is a need for a more general definition.

The term $o(\mathbf{v})$ is notation that is descriptive of the behavior in 5.12.21 and it is only this behavior that is of interest. Thus, if t and k are constants,

$$o(\mathbf{v}) = o(\mathbf{v}) + o(\mathbf{v}), \quad o(t\mathbf{v}) = o(\mathbf{v}), \quad ko(\mathbf{v}) = o(\mathbf{v})$$

and other similar observations hold. The sloppiness built in to this notation is useful because it ignores details which are not important. It may help to think of $o(\mathbf{v})$ as an adjective describing what is left over after approximating $\mathbf{f}(\mathbf{x} + \mathbf{v})$ by $\mathbf{f}(\mathbf{x}) + D\mathbf{f}(\mathbf{x})\mathbf{v}$.

Theorem 5.12.2 *The derivative is well defined.*

Proof: First note that for a fixed vector, $\mathbf{v}$, $o(t\mathbf{v}) = o(t)$. Now suppose both L_1 and L_2 work in the above definition. Then let $\mathbf{v}$ be any vector and let t be a real scalar which is chosen small enough that $t\mathbf{v} + \mathbf{x} \in U$. Then

$$\mathbf{f}(\mathbf{x} + t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_1 t\mathbf{v} + o(t\mathbf{v}), \quad \mathbf{f}(\mathbf{x} + t\mathbf{v}) = \mathbf{f}(\mathbf{x}) + L_2 t\mathbf{v} + o(t\mathbf{v}).$$

Therefore, subtracting these two yields $(L_2 - L_1)(t\mathbf{v}) = o(t\mathbf{v}) = o(t)$. Therefore, dividing by t yields $(L_2 - L_1)(\mathbf{v}) = \frac{o(t)}{t}$. Now let $t \rightarrow 0$ to conclude that $(L_2 - L_1)(\mathbf{v}) = 0$. Since this is true for all $\mathbf{v}$, it follows $L_2 = L_1$. This proves the theorem.

Lemma 5.12.3 *Let $\mathbf{f}$ be differentiable at $\mathbf{x}$. Then $\mathbf{f}$ is continuous at $\mathbf{x}$ and in fact, there exists $K > 0$ such that whenever $|\mathbf{v}|$ is small enough,*

$$|\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})| \leq K |\mathbf{v}|$$

Proof: From the definition of the derivative, $\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$. Let $|\mathbf{v}|$ be small enough that $\frac{o(|\mathbf{v}|)}{|\mathbf{v}|} < 1$ so that $|o(\mathbf{v})| \leq |\mathbf{v}|$. Then for such $\mathbf{v}$,

$$\begin{aligned} |\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})| &\leq |D\mathbf{f}(\mathbf{x})\mathbf{v}| + |\mathbf{v}| \\ &\leq (|D\mathbf{f}(\mathbf{x})| + 1) |\mathbf{v}| \end{aligned}$$

This proves the lemma with $K = |D\mathbf{f}(\mathbf{x})| + 1$.

Theorem 5.12.4 *(The chain rule) Let U and V be open sets, $U \subseteq \mathbb{F}^n$ and $V \subseteq \mathbb{F}^m$. Suppose $\mathbf{f} : U \rightarrow V$ is differentiable at $\mathbf{x} \in U$ and suppose $\mathbf{g} : V \rightarrow \mathbb{F}^q$ is differentiable at $\mathbf{f}(\mathbf{x}) \in V$. Then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}$ and*

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{x}) = D(\mathbf{g}(\mathbf{f}(\mathbf{x}))) D(\mathbf{f}(\mathbf{x})).$$

Proof: This follows from a computation. Let $B(\mathbf{x}, r) \subseteq U$ and let r also be small enough that for $|\mathbf{v}| \leq r$, it follows that $\mathbf{f}(\mathbf{x} + \mathbf{v}) \in V$. Such an r exists because $\mathbf{f}$ is continuous at $\mathbf{x}$. For $|\mathbf{v}| < r$, the definition of differentiability of $\mathbf{g}$ and $\mathbf{f}$ implies

$$\mathbf{g}(\mathbf{f}(\mathbf{x} + \mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) =$$

$$\begin{aligned} &D\mathbf{g}(\mathbf{f}(\mathbf{x}))(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) + o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ &= D\mathbf{g}(\mathbf{f}(\mathbf{x}))[D\mathbf{f}(\mathbf{x})\mathbf{v} + o(\mathbf{v})] + o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) \\ &= D(\mathbf{g}(\mathbf{f}(\mathbf{x}))) D(\mathbf{f}(\mathbf{x}))\mathbf{v} + o(\mathbf{v}) + o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})). \end{aligned} \quad (5.12.24)$$

It remains to show $o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) = o(\mathbf{v})$.

By Lemma 5.12.3, with K given there, letting $\varepsilon > 0$, it follows that for $|\mathbf{v}|$ small enough,

$$|o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}))| \leq (\varepsilon/K) |\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})| \leq (\varepsilon/K) K |\mathbf{v}| = \varepsilon |\mathbf{v}|.$$

Since $\varepsilon > 0$ is arbitrary, this shows $o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x})) = o(\mathbf{v})$ because whenever $|\mathbf{v}|$ is small enough,

$$\frac{|o(\mathbf{f}(\mathbf{x} + \mathbf{v}) - \mathbf{f}(\mathbf{x}))|}{|\mathbf{v}|} \leq \varepsilon.$$

By 5.12.24, this shows

$$\mathbf{g}(\mathbf{f}(\mathbf{x} + \mathbf{v})) - \mathbf{g}(\mathbf{f}(\mathbf{x})) = D(\mathbf{g}(\mathbf{f}(\mathbf{x}))) D(\mathbf{f}(\mathbf{x})) \mathbf{v} + o(\mathbf{v})$$

which proves the theorem.

The derivative is a linear transformation. What is the matrix of this linear transformation taken with respect to the usual basis vectors? Let $\mathbf{e}_i$ denote the vector of $\mathbb{F}^n$ which has a one in the i^{th} entry and zeroes elsewhere. Then the matrix of the linear transformation is the matrix whose i^{th} column is $D\mathbf{f}(\mathbf{x})\mathbf{e}_i$. What is this? Let $t \in \mathbb{R}$ such that $|t|$ is sufficiently small.

$$\begin{aligned} \mathbf{f}(\mathbf{x} + t\mathbf{e}_i) - \mathbf{f}(\mathbf{x}) &= D\mathbf{f}(\mathbf{x})t\mathbf{e}_i + \mathbf{o}(t\mathbf{e}_i) \\ &= D\mathbf{f}(\mathbf{x})t\mathbf{e}_i + \mathbf{o}(t). \end{aligned}$$

Then dividing by t and taking a limit,

$$D\mathbf{f}(\mathbf{x})\mathbf{e}_i = \lim_{t \rightarrow 0} \frac{\mathbf{f}(\mathbf{x} + t\mathbf{e}_i) - \mathbf{f}(\mathbf{x})}{t} \equiv \frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}).$$

Thus the matrix of $D\mathbf{f}(\mathbf{x})$ with respect to the usual basis vectors is the matrix of the form

$$\begin{pmatrix} f_{1,x_1}(\mathbf{x}) & f_{1,x_2}(\mathbf{x}) & \cdots & f_{1,x_n}(\mathbf{x}) \\ \vdots & \vdots & & \vdots \\ f_{m,x_1}(\mathbf{x}) & f_{m,x_2}(\mathbf{x}) & \cdots & f_{m,x_n}(\mathbf{x}) \end{pmatrix}.$$

As mentioned before, there is no harm in referring to this matrix as $D\mathbf{f}(\mathbf{x})$ but it may also be referred to as $J\mathbf{f}(\mathbf{x})$.

This is summarized in the following theorem.

Theorem 5.12.5 *Let $\mathbf{f} : \mathbb{F}^n \rightarrow \mathbb{F}^m$ and suppose $\mathbf{f}$ is differentiable at $\mathbf{x}$. Then all the partial derivatives $\frac{\partial f_i(\mathbf{x})}{\partial x_j}$ exist and if $J\mathbf{f}(\mathbf{x})$ is the matrix of the linear transformation with respect to the standard basis vectors, then the ij^{th} entry is given by $f_{i,j}$ or $\frac{\partial f_i}{\partial x_j}(\mathbf{x})$.*

What if all the partial derivatives of $\mathbf{f}$ exist? Does it follow that $\mathbf{f}$ is differentiable? Consider the following function.

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Then from the definition of partial derivatives,

$$\lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

and

$$\lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{0 - 0}{h} = 0$$

However f is not even continuous at $(0, 0)$ which may be seen by considering the behavior of the function along the line $y = x$ and along the line $x = 0$. By Lemma 5.12.3 this implies f is not differentiable. Therefore, it is necessary to consider the correct definition of the derivative given above if you want to get a notion which generalizes the concept of the derivative of a function of one variable in such a way as to preserve continuity whenever the function is differentiable.

5.13 C^1 Functions

However, there are theorems which can be used to get differentiability of a function based on existence of the partial derivatives.

Definition 5.13.1 *When all the partial derivatives exist and are continuous the function is called a C^1 function.*

Because of Proposition 5.11.4 on Page 119 and Theorem 5.12.5 which identifies the entries of Jf with the partial derivatives, the following definition is equivalent to the above.

Definition 5.13.2 *Let $U \subseteq \mathbb{F}^n$ be an open set. Then $\mathbf{f} : U \rightarrow \mathbb{F}^m$ is $C^1(U)$ if $\mathbf{f}$ is differentiable and the mapping*

$$\mathbf{x} \rightarrow D\mathbf{f}(\mathbf{x}),$$

is continuous as a function from U to $\mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$.

The following is an important abstract generalization of the familiar concept of partial derivative.

Definition 5.13.3 *Let $\mathbf{g} : U \subseteq \mathbb{F}^n \times \mathbb{F}^m \rightarrow \mathbb{F}^q$, where U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Denote an element of $\mathbb{F}^n \times \mathbb{F}^m$ by $(\mathbf{x}, \mathbf{y})$ where $\mathbf{x} \in \mathbb{F}^n$ and $\mathbf{y} \in \mathbb{F}^m$. Then the map $\mathbf{x} \rightarrow \mathbf{g}(\mathbf{x}, \mathbf{y})$ is a function from the open set in $\mathbb{F}^n$,*

$$\{\mathbf{x} : (\mathbf{x}, \mathbf{y}) \in U\}$$

to $\mathbb{F}^q$. When this map is differentiable, its derivative is denoted by

$$D_1\mathbf{g}(\mathbf{x}, \mathbf{y}), \text{ or sometimes by } D_{\mathbf{x}}\mathbf{g}(\mathbf{x}, \mathbf{y}).$$

Thus,

$$\mathbf{g}(\mathbf{x} + \mathbf{v}, \mathbf{y}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) = D_1\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{v} + o(\mathbf{v}).$$

A similar definition holds for the symbol $D_{\mathbf{y}}\mathbf{g}$ or $D_2\mathbf{g}$. The special case seen in beginning calculus courses is where $\mathbf{g} : U \rightarrow \mathbb{F}^q$ and

$$\mathbf{g}_{x_i}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}(\mathbf{x})}{\partial x_i} \equiv \lim_{h \rightarrow 0} \frac{\mathbf{g}(\mathbf{x} + h\mathbf{e}_i) - \mathbf{g}(\mathbf{x})}{h}.$$

The following theorem will be very useful in much of what follows. It is a version of the mean value theorem. You might call it the mean value inequality.

Theorem 5.13.4 *Suppose U is an open subset of $\mathbb{F}^n$ and $\mathbf{f} : U \rightarrow \mathbb{F}^m$ has the property that $D\mathbf{f}(\mathbf{x})$ exists for all $\mathbf{x}$ in U and that, $\mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in U$ for all $t \in [0, 1]$. (The line segment joining the two points lies in U .) Suppose also that for all points on this line segment,*

$$\|D\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))\| \leq M.$$

Then

$$|\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| \leq M |\mathbf{y} - \mathbf{x}|.$$

Proof: Let

$$S \equiv \{t \in [0, 1] : \text{for all } s \in [0, t],$$

$$|\mathbf{f}(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| \leq (M + \varepsilon) s |\mathbf{y} - \mathbf{x}|\}.$$

Then $0 \in S$ and by continuity of $\mathbf{f}$, it follows that if $t \equiv \sup S$, then $t \in S$ and if $t < 1$,

$$|\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| = (M + \varepsilon) t |\mathbf{y} - \mathbf{x}|. \quad (5.13.25)$$

If $t < 1$, then there exists a sequence of positive numbers, $\{h_k\}_{k=1}^{\infty}$ converging to 0 such that

$$|\mathbf{f}(\mathbf{x} + (t + h_k)(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| > (M + \varepsilon)(t + h_k) |\mathbf{y} - \mathbf{x}|$$

which implies that

$$\begin{aligned} & |\mathbf{f}(\mathbf{x} + (t + h_k)(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))| \\ & + |\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| > (M + \varepsilon)(t + h_k) |\mathbf{y} - \mathbf{x}|. \end{aligned}$$

By 5.13.25, this inequality implies

$$|\mathbf{f}(\mathbf{x} + (t + h_k)(\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))| > (M + \varepsilon) h_k |\mathbf{y} - \mathbf{x}|$$

which yields upon dividing by h_k and taking the limit as $h_k \rightarrow 0$,

$$|D\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(\mathbf{y} - \mathbf{x})| \geq (M + \varepsilon) |\mathbf{y} - \mathbf{x}|.$$

Now by the definition of the norm of a linear operator,

$$\begin{aligned} M |\mathbf{y} - \mathbf{x}| & \geq \|D\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))\| |\mathbf{y} - \mathbf{x}| \\ & \geq |D\mathbf{f}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(\mathbf{y} - \mathbf{x})| \geq (M + \varepsilon) |\mathbf{y} - \mathbf{x}|, \end{aligned}$$

a contradiction. Therefore, $t = 1$ and so

$$|\mathbf{f}(\mathbf{x} + (\mathbf{y} - \mathbf{x})) - \mathbf{f}(\mathbf{x})| \leq (M + \varepsilon) |\mathbf{y} - \mathbf{x}|.$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem.

The next theorem proves that if the partial derivatives exist and are continuous, then the function is differentiable.

Theorem 5.13.5 *Let $\mathbf{g} : U \subseteq \mathbb{F}^n \times \mathbb{F}^m \rightarrow \mathbb{F}^q$. Then $\mathbf{g}$ is $C^1(U)$ if and only if $D_1\mathbf{g}$ and $D_2\mathbf{g}$ both exist and are continuous on U . In this case,*

$$D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{v}) = D_1\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{u} + D_2\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{v}.$$

Proof: Suppose first that $\mathbf{g} \in C^1(U)$. Then if $(\mathbf{x}, \mathbf{y}) \in U$,

$$\mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) = D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{0}) + o(\mathbf{u}).$$

Therefore, $D_1\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{u} = D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{0})$. Then

$$\begin{aligned} |(D_1\mathbf{g}(\mathbf{x}, \mathbf{y}) - D_1\mathbf{g}(\mathbf{x}', \mathbf{y}'))(\mathbf{u})| &= \\ |(D\mathbf{g}(\mathbf{x}, \mathbf{y}) - D\mathbf{g}(\mathbf{x}', \mathbf{y}'))(\mathbf{u}, \mathbf{0})| &\leq \\ \|D\mathbf{g}(\mathbf{x}, \mathbf{y}) - D\mathbf{g}(\mathbf{x}', \mathbf{y}')\| |\mathbf{u}| &= \end{aligned}$$

Therefore,

$$\|D_1\mathbf{g}(\mathbf{x}, \mathbf{y}) - D_1\mathbf{g}(\mathbf{x}', \mathbf{y}')\| \leq \|D\mathbf{g}(\mathbf{x}, \mathbf{y}) - D\mathbf{g}(\mathbf{x}', \mathbf{y}')\|.$$

A similar argument applies for $D_2\mathbf{g}$ and this proves the continuity of the function, $(\mathbf{x}, \mathbf{y}) \rightarrow D_i\mathbf{g}(\mathbf{x}, \mathbf{y})$ for $i = 1, 2$. The formula follows from

$$\begin{aligned} D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{v}) &= D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{0}) + D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{0}, \mathbf{v}) \\ &\equiv D_1\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{u} + D_2\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{v}. \end{aligned}$$

Now suppose $D_1\mathbf{g}(\mathbf{x}, \mathbf{y})$ and $D_2\mathbf{g}(\mathbf{x}, \mathbf{y})$ exist and are continuous.

$$\begin{aligned} \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) &= \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y} + \mathbf{v}) \\ &\quad + \mathbf{g}(\mathbf{x}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) \\ &= \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) + \mathbf{g}(\mathbf{x}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y}) + \\ &\quad [\mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y}) - (\mathbf{g}(\mathbf{x}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y}))] \\ &= D_1\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{u} + D_2\mathbf{g}(\mathbf{x}, \mathbf{y})\mathbf{v} + o(\mathbf{v}) + o(\mathbf{u}) + \\ &\quad [\mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y}) - (\mathbf{g}(\mathbf{x}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y}))]. \end{aligned} \quad (5.13.26)$$

Let $\mathbf{h}(\mathbf{x}, \mathbf{u}) \equiv \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y})$. Then the expression in [] is of the form,

$$\mathbf{h}(\mathbf{x}, \mathbf{u}) - \mathbf{h}(\mathbf{x}, \mathbf{0}).$$

Also

$$D_2\mathbf{h}(\mathbf{x}, \mathbf{u}) = D_1\mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - D_1\mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y})$$

and so, by continuity of $(\mathbf{x}, \mathbf{y}) \rightarrow D_1\mathbf{g}(\mathbf{x}, \mathbf{y})$,

$$\|D_2\mathbf{h}(\mathbf{x}, \mathbf{u})\| < \varepsilon$$

whenever $\|(\mathbf{u}, \mathbf{v})\|$ is small enough. By Theorem 5.13.4 on Page 125, there exists $\delta > 0$ such that if $\|(\mathbf{u}, \mathbf{v})\| < \delta$, the norm of the last term in 5.13.26 satisfies the inequality,

$$\|\mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y}) - (\mathbf{g}(\mathbf{x}, \mathbf{y} + \mathbf{v}) - \mathbf{g}(\mathbf{x}, \mathbf{y}))\| < \varepsilon \|\mathbf{u}\|. \quad (5.13.27)$$

Therefore, this term is $o(\|(\mathbf{u}, \mathbf{v})\|)$. It follows from 5.13.27 and 5.13.26 that

$$\begin{aligned} \mathbf{g}(\mathbf{x} + \mathbf{u}, \mathbf{y} + \mathbf{v}) &= \\ \mathbf{g}(\mathbf{x}, \mathbf{y}) + D_1 \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u} + D_2 \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v} &+ o(\mathbf{u}) + o(\mathbf{v}) + o(\|(\mathbf{u}, \mathbf{v})\|) \\ &= \mathbf{g}(\mathbf{x}, \mathbf{y}) + D_1 \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u} + D_2 \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v} &+ o(\|(\mathbf{u}, \mathbf{v})\|) \end{aligned}$$

Showing that $D\mathbf{g}(\mathbf{x}, \mathbf{y})$ exists and is given by

$$D\mathbf{g}(\mathbf{x}, \mathbf{y})(\mathbf{u}, \mathbf{v}) = D_1 \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{u} + D_2 \mathbf{g}(\mathbf{x}, \mathbf{y}) \mathbf{v}.$$

The continuity of $(\mathbf{x}, \mathbf{y}) \rightarrow D\mathbf{g}(\mathbf{x}, \mathbf{y})$ follows from the continuity of $(\mathbf{x}, \mathbf{y}) \rightarrow D_i \mathbf{g}(\mathbf{x}, \mathbf{y})$. This proves the theorem.

Not surprisingly, it can be generalized to many more factors.

Definition 5.13.6 Let $\mathbf{g} : U \subseteq \prod_{i=1}^n \mathbb{F}^{r_i} \rightarrow \mathbb{F}^q$, where U is an open set. Then the map $\mathbf{x}_i \rightarrow \mathbf{g}(\mathbf{x})$ is a function from the open set in $\mathbb{F}^{r_i}$,

$$\{\mathbf{x} : \mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_n) \in U\}$$

to $\mathbb{F}^q$. When this map is differentiable, its derivative is denoted by $D_i \mathbf{g}(\mathbf{x})$. To aid in the notation, for $\mathbf{v} \in \mathbb{F}^{r_i}$, let $\theta_i \mathbf{v} \in \prod_{i=1}^n \mathbb{F}^{r_i}$ be the vector $(\mathbf{0}, \dots, \mathbf{v}, \dots, \mathbf{0})$ where the $\mathbf{v}$ is in the i^{th} slot and for $\mathbf{v} \in \prod_{i=1}^n \mathbb{F}^{r_i}$, let $\mathbf{v}_i$ denote the entry in the i^{th} slot of $\mathbf{v}$. Thus, by saying $\mathbf{x}_i \rightarrow \mathbf{g}(\mathbf{x})$ is differentiable is meant that for $\mathbf{v} \in \mathbb{F}^{r_i}$ sufficiently small,

$$\mathbf{g}(\mathbf{x} + \theta_i \mathbf{v}) - \mathbf{g}(\mathbf{x}) = D_i \mathbf{g}(\mathbf{x}) \mathbf{v} + \mathbf{o}(\mathbf{v}).$$

Note $D_i \mathbf{g}(\mathbf{x}) \in \mathcal{L}(\mathbb{F}^{r_i}, \prod_{i=1}^n \mathbb{F}^{r_i})$.

Here is a generalization of Theorem 5.13.5.

Theorem 5.13.7 Let $\mathbf{g}, U, \prod_{i=1}^n \mathbb{F}^{r_i}$, be given as in Definition 5.13.6. Then $\mathbf{g}$ is $C^1(U)$ if and only if $D_i \mathbf{g}$ exists and is continuous on U for each i . In this case,

$$D\mathbf{g}(\mathbf{x})(\mathbf{v}) = \sum_k D_k \mathbf{g}(\mathbf{x}) \mathbf{v}_k \quad (5.13.28)$$

where $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$.

Proof: Suppose then that $D_i \mathbf{g}$ exists and is continuous for each i . Note that $\sum_{j=1}^k \theta_j \mathbf{v}_j = (\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{0}, \dots, \mathbf{0})$. Thus $\sum_{j=1}^n \theta_j \mathbf{v}_j = \mathbf{v}$ and define $\sum_{j=1}^0 \theta_j \mathbf{v}_j \equiv \mathbf{0}$. Therefore,

$$\mathbf{g}(\mathbf{x} + \mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^n \left[\mathbf{g} \left(\mathbf{x} + \sum_{j=1}^k \theta_j \mathbf{v}_j \right) - \mathbf{g} \left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j \right) \right] \quad (5.13.29)$$

Consider the terms in this sum.

$$\mathbf{g} \left(\mathbf{x} + \sum_{j=1}^k \theta_j \mathbf{v}_j \right) - \mathbf{g} \left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j \right) = \mathbf{g}(\mathbf{x} + \theta_k \mathbf{v}_k) - \mathbf{g}(\mathbf{x}) + \quad (5.13.30)$$

$$\left(\mathbf{g} \left(\mathbf{x} + \sum_{j=1}^k \theta_j \mathbf{v}_j \right) - \mathbf{g}(\mathbf{x} + \theta_k \mathbf{v}_k) \right) - \left(\mathbf{g} \left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j \right) - \mathbf{g}(\mathbf{x}) \right) \quad (5.13.31)$$

and the expression in 5.13.31 is of the form $\mathbf{h}(\mathbf{v}_k) - \mathbf{h}(\mathbf{0})$ where for small $\mathbf{w} \in \mathbb{F}^{r_k}$,

$$\mathbf{h}(\mathbf{w}) \equiv \mathbf{g} \left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j + \theta_k \mathbf{w} \right) - \mathbf{g}(\mathbf{x} + \theta_k \mathbf{w}).$$

Therefore,

$$D\mathbf{h}(\mathbf{w}) = D_k \mathbf{g} \left(\mathbf{x} + \sum_{j=1}^{k-1} \theta_j \mathbf{v}_j + \theta_k \mathbf{w} \right) - D_k \mathbf{g}(\mathbf{x} + \theta_k \mathbf{w})$$

and by continuity, $\|D\mathbf{h}(\mathbf{w})\| < \varepsilon$ provided $|\mathbf{v}|$ is small enough. Therefore, by Theorem 5.13.4, whenever $|\mathbf{v}|$ is small enough, $|\mathbf{h}(\mathbf{v}_k) - \mathbf{h}(\mathbf{0})| \leq \varepsilon |\mathbf{v}_k| \leq \varepsilon |\mathbf{v}|$ which shows that since ε is arbitrary, the expression in 5.13.31 is $\mathbf{o}(\mathbf{v})$. Now in 5.13.30

$$\mathbf{g}(\mathbf{x} + \theta_k \mathbf{v}_k) - \mathbf{g}(\mathbf{x}) = D_k \mathbf{g}(\mathbf{x}) \mathbf{v}_k + \mathbf{o}(\mathbf{v}_k) = D_k \mathbf{g}(\mathbf{x}) \mathbf{v}_k + \mathbf{o}(\mathbf{v}).$$

Therefore, referring to 5.13.29,

$$\mathbf{g}(\mathbf{x} + \mathbf{v}) - \mathbf{g}(\mathbf{x}) = \sum_{k=1}^n D_k \mathbf{g}(\mathbf{x}) \mathbf{v}_k + \mathbf{o}(\mathbf{v})$$

which shows $D\mathbf{g}$ exists and equals the formula given in 5.13.28.

Next suppose $\mathbf{g}$ is C^1 . I need to verify that $D_k \mathbf{g}(\mathbf{x})$ exists and is continuous. Let $\mathbf{v} \in \mathbb{F}^{r_k}$ sufficiently small. Then

$$\begin{aligned} \mathbf{g}(\mathbf{x} + \theta_k \mathbf{v}) - \mathbf{g}(\mathbf{x}) &= D\mathbf{g}(\mathbf{x}) \theta_k \mathbf{v} + \mathbf{o}(\theta_k \mathbf{v}) \\ &= D\mathbf{g}(\mathbf{x}) \theta_k \mathbf{v} + \mathbf{o}(\mathbf{v}) \end{aligned}$$

since $|\theta_k \mathbf{v}| = |\mathbf{v}|$. Then $D_k \mathbf{g}(\mathbf{x})$ exists and equals

$$D\mathbf{g}(\mathbf{x}) \circ \theta_k$$

Since $\mathbf{x} \rightarrow D\mathbf{g}(\mathbf{x})$ is continuous and $\theta_k : \mathbb{R}^{r_k} \rightarrow \prod_{i=1}^n \mathbb{R}^{r_i}$ is also continuous, this proves the theorem

The way this is usually used is in the following corollary, a case of Theorem 5.13.7 obtained by letting $\mathbb{R}^{r_j} = \mathbb{F}$ in the above theorem.

Corollary 5.13.8 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f} : U \rightarrow \mathbb{F}^m$ be C^1 in the sense that all the partial derivatives of $\mathbf{f}$ exist and are continuous. Then $\mathbf{f}$ is differentiable and*

$$\mathbf{f}(\mathbf{x} + \mathbf{v}) = \mathbf{f}(\mathbf{x}) + \sum_{k=1}^n \frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{x}) v_k + \mathbf{o}(\mathbf{v}).$$

5.14 C^k Functions

Recall the notation for partial derivatives in the following definition.

Definition 5.14.1 *Let $\mathbf{g} : U \rightarrow \mathbb{F}^m$. Then*

$$\mathbf{g}_{x_k}(\mathbf{x}) \equiv \frac{\partial \mathbf{g}}{\partial x_k}(\mathbf{x}) \equiv \lim_{h \rightarrow 0} \frac{\mathbf{g}(\mathbf{x} + h\mathbf{e}_k) - \mathbf{g}(\mathbf{x})}{h}$$

Higher order partial derivatives are defined in the usual way.

$$\mathbf{g}_{x_k x_l}(\mathbf{x}) \equiv \frac{\partial^2 \mathbf{g}}{\partial x_l \partial x_k}(\mathbf{x})$$

and so forth.

To deal with higher order partial derivatives in a systematic way, here is a useful definition.

Definition 5.14.2 $\alpha = (\alpha_1, \dots, \alpha_n)$ for $\alpha_1 \dots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \dots + \alpha_n$ and if $\mathbf{x} \in \mathbb{F}^n$,

$$\mathbf{x} = (x_1, \dots, x_n),$$

and $\mathbf{f}$ a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}, \quad D^\alpha \mathbf{f}(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} \mathbf{f}(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

The following is the definition of what is meant by a C^k function.

Definition 5.14.3 *Let U be an open subset of $\mathbb{F}^n$ and let $\mathbf{f} : U \rightarrow \mathbb{F}^m$. Then for k a nonnegative integer, $\mathbf{f}$ is C^k if for every $|\alpha| \leq k$, $D^\alpha \mathbf{f}$ exists and is continuous.*

5.15 Mixed Partial Derivatives

Under certain conditions the **mixed partial derivatives** will always be equal. This astonishing fact is due to Euler in 1734.

Theorem 5.15.1 Suppose $f : U \subseteq \mathbb{F}^2 \rightarrow \mathbb{R}$ where U is an open set on which f_x, f_y, f_{xy} and f_{yx} exist. Then if f_{xy} and f_{yx} are continuous at the point $(x, y) \in U$, it follows

$$f_{xy}(x, y) = f_{yx}(x, y).$$

Proof: Since U is open, there exists $r > 0$ such that $B((x, y), r) \subseteq U$. Now let $|t|, |s| < r/2$, t, s real numbers and consider

$$\Delta(s, t) \equiv \frac{1}{st} \left\{ \overbrace{f(x+t, y+s) - f(x+t, y)}^{h(t)} - \overbrace{(f(x, y+s) - f(x, y))}^{h(0)} \right\}. \quad (5.15.32)$$

Note that $(x+t, y+s) \in U$ because

$$\begin{aligned} |(x+t, y+s) - (x, y)| &= |(t, s)| = (t^2 + s^2)^{1/2} \\ &\leq \left(\frac{r^2}{4} + \frac{r^2}{4} \right)^{1/2} = \frac{r}{\sqrt{2}} < r. \end{aligned}$$

As implied above, $h(t) \equiv f(x+t, y+s) - f(x+t, y)$. Therefore, by the mean value theorem from calculus and the (one variable) chain rule,

$$\begin{aligned} \Delta(s, t) &= \frac{1}{st} (h(t) - h(0)) = \frac{1}{st} h'(\alpha t) t \\ &= \frac{1}{s} (f_x(x + \alpha t, y+s) - f_x(x + \alpha t, y)) \end{aligned}$$

for some $\alpha \in (0, 1)$. Applying the mean value theorem again,

$$\Delta(s, t) = f_{xy}(x + \alpha t, y + \beta s)$$

where $\alpha, \beta \in (0, 1)$.

If the terms $f(x+t, y)$ and $f(x, y+s)$ are interchanged in 5.15.32, $\Delta(s, t)$ is unchanged and the above argument shows there exist $\gamma, \delta \in (0, 1)$ such that

$$\Delta(s, t) = f_{yx}(x + \gamma t, y + \delta s).$$

Letting $(s, t) \rightarrow (0, 0)$ and using the continuity of f_{xy} and f_{yx} at (x, y) ,

$$\lim_{(s, t) \rightarrow (0, 0)} \Delta(s, t) = f_{xy}(x, y) = f_{yx}(x, y).$$

This proves the theorem.

The following is obtained from the above by simply fixing all the variables except for the two of interest.

Corollary 5.15.2 Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \rightarrow \mathbb{R}$ has the property that for two indices, k, l , f_{x_k} , f_{x_l} , $f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.

By considering the real and imaginary parts of f in the case where f has values in $\mathbb{F}$ you obtain the following corollary.

Corollary 5.15.3 Suppose U is an open subset of $\mathbb{F}^n$ and $f : U \rightarrow \mathbb{F}$ has the property that for two indices, k, l , f_{x_k} , f_{x_l} , $f_{x_l x_k}$, and $f_{x_k x_l}$ exist on U and $f_{x_k x_l}$ and $f_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $f_{x_k x_l}(\mathbf{x}) = f_{x_l x_k}(\mathbf{x})$.

Finally, by considering the components of $\mathbf{f}$ you get the following generalization.

Corollary 5.15.4 Suppose U is an open subset of $\mathbb{F}^n$ and $\mathbf{f} : U \rightarrow \mathbb{F}^m$ has the property that for two indices, k, l , $\mathbf{f}_{x_k}$, $\mathbf{f}_{x_l}$, $\mathbf{f}_{x_l x_k}$, and $\mathbf{f}_{x_k x_l}$ exist on U and $\mathbf{f}_{x_k x_l}$ and $\mathbf{f}_{x_l x_k}$ are both continuous at $\mathbf{x} \in U$. Then $\mathbf{f}_{x_k x_l}(\mathbf{x}) = \mathbf{f}_{x_l x_k}(\mathbf{x})$.

It is necessary to assume the mixed partial derivatives are continuous in order to assert they are equal. The following is a well known example [5].

Example 5.15.5 Let

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

From the definition of partial derivatives it follows immediately that $f_x(0, 0) = f_y(0, 0) = 0$. Using the standard rules of differentiation, for $(x, y) \neq (0, 0)$,

$$f_x = y \frac{x^4 - y^4 + 4x^2 y^2}{(x^2 + y^2)^2}, \quad f_y = x \frac{x^4 - y^4 - 4x^2 y^2}{(x^2 + y^2)^2}$$

Now

$$\begin{aligned} f_{xy}(0, 0) &\equiv \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y} \\ &= \lim_{y \rightarrow 0} \frac{-y^4}{(y^2)^2} = -1 \end{aligned}$$

while

$$\begin{aligned} f_{yx}(0, 0) &\equiv \lim_{x \rightarrow 0} \frac{f_y(x, 0) - f_y(0, 0)}{x} \\ &= \lim_{x \rightarrow 0} \frac{x^4}{(x^2)^2} = 1 \end{aligned}$$

showing that although the mixed partial derivatives do exist at $(0, 0)$, they are not equal there.

5.16 Implicit Function Theorem

The implicit function theorem is one of the greatest theorems in mathematics. There are many versions of this theorem. However, I will give a very simple proof valid in finite dimensional spaces.

Theorem 5.16.1 (*implicit function theorem*) Suppose U is an open set in $\mathbb{R}^n \times \mathbb{R}^m$. Let $\mathbf{f} : U \rightarrow \mathbb{R}^n$ be in $C^1(U)$ and suppose

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}, \quad D_1 \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n). \quad (5.16.33)$$

Then there exist positive constants, δ, η , such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \quad (5.16.34)$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0, \eta))$.

Proof: Let

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} f_1(\mathbf{x}, \mathbf{y}) \\ f_2(\mathbf{x}, \mathbf{y}) \\ \vdots \\ f_n(\mathbf{x}, \mathbf{y}) \end{pmatrix}.$$

Define for $(\mathbf{x}^1, \dots, \mathbf{x}^n) \in \overline{B(\mathbf{x}_0, \delta)}^n$ and $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ the following matrix.

$$J(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y}) \equiv \begin{pmatrix} f_{1,x_1}(\mathbf{x}^1, \mathbf{y}) & \cdots & f_{1,x_n}(\mathbf{x}^1, \mathbf{y}) \\ \vdots & & \vdots \\ f_{n,x_1}(\mathbf{x}^n, \mathbf{y}) & \cdots & f_{n,x_n}(\mathbf{x}^n, \mathbf{y}) \end{pmatrix}.$$

Then by the assumption of continuity of all the partial derivatives, there exists $\delta_0 > 0$ and $\eta_0 > 0$ such that if $\delta < \delta_0$ and $\eta < \eta_0$, it follows that for all $(\mathbf{x}^1, \dots, \mathbf{x}^n) \in \overline{B(\mathbf{x}_0, \delta)}^n$ and $\mathbf{y} \in B(\mathbf{y}_0, \eta)$,

$$\det(J(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y})) > r > 0. \quad (5.16.35)$$

and $\overline{B(\mathbf{x}_0, \delta_0)} \times \overline{B(\mathbf{y}_0, \eta_0)} \subseteq U$. Pick $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ and suppose there exist $\mathbf{x}, \mathbf{z} \in \overline{B(\mathbf{x}_0, \delta)}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{z}, \mathbf{y}) = \mathbf{0}$. Consider f_i and let

$$h(t) \equiv f_i(\mathbf{x} + t(\mathbf{z} - \mathbf{x}), \mathbf{y}).$$

Then $h(1) = h(0)$ and so by the mean value theorem, $h'(t_i) = 0$ for some $t_i \in (0, 1)$. Therefore, from the chain rule and for this value of t_i ,

$$h'(t_i) = Df_i(\mathbf{x} + t_i(\mathbf{z} - \mathbf{x}), \mathbf{y})(\mathbf{z} - \mathbf{x}) = 0. \quad (5.16.36)$$

Then denote by $\mathbf{x}^i$ the vector, $\mathbf{x} + t_i(\mathbf{z} - \mathbf{x})$. It follows from 5.16.36 that

$$J(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y})(\mathbf{z} - \mathbf{x}) = \mathbf{0}$$

and so from 5.16.35 $\mathbf{z} - \mathbf{x} = \mathbf{0}$. Now it will be shown that if η is chosen sufficiently small, then for all $\mathbf{y} \in B(\mathbf{y}_0, \eta)$, there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$.

Claim: If η is small enough, then the function, $h_{\mathbf{y}}(\mathbf{x}) \equiv |\mathbf{f}(\mathbf{x}, \mathbf{y})|^2$ achieves its minimum value on $\overline{B(\mathbf{x}_0, \delta)}$ at a point of $B(\mathbf{x}_0, \delta)$.

Proof of claim: Suppose this is not the case. Then there exists a sequence $\eta_k \rightarrow 0$ and for some $\mathbf{y}_k$ having $|\mathbf{y}_k - \mathbf{y}_0| < \eta_k$, the minimum of $h_{\mathbf{y}_k}$ occurs on a point of the boundary of $\overline{B(\mathbf{x}_0, \delta)}$, $\mathbf{x}_k$ such that $|\mathbf{x}_0 - \mathbf{x}_k| = \delta$. Now taking a subsequence, still denoted by k , it can be assumed that $\mathbf{x}_k \rightarrow \mathbf{x}$ with $|\mathbf{x} - \mathbf{x}_0| = \delta$ and $\mathbf{y}_k \rightarrow \mathbf{y}_0$. Let $\varepsilon > 0$. Then for k large enough, $h_{\mathbf{y}_k}(\mathbf{x}_0) < \varepsilon$ because $\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$. Therefore, from the definition of $\mathbf{x}_k$, $h_{\mathbf{y}_k}(\mathbf{x}_k) < \varepsilon$. Passing to the limit yields $h_{\mathbf{y}_0}(\mathbf{x}) \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $h_{\mathbf{y}_0}(\mathbf{x}) = 0$ which contradicts the first part of the argument in which it was shown that for $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there is at most one point, $\mathbf{x}$ of $\overline{B(\mathbf{x}_0, \delta)}$ where $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Here two have been obtained, $\mathbf{x}_0$ and $\mathbf{x}$. This proves the claim.

Choose $\eta < \eta_0$ and also small enough that the above claim holds and let $\mathbf{x}(\mathbf{y})$ denote a point of $B(\mathbf{x}_0, \delta)$ at which the minimum of $h_{\mathbf{y}}$ on $\overline{B(\mathbf{x}_0, \delta)}$ is achieved. Since $\mathbf{x}(\mathbf{y})$ is an interior point, you can consider $h_{\mathbf{y}}(\mathbf{x}(\mathbf{y}) + t\mathbf{v})$ for $|t|$ small and conclude this function of t has a zero derivative at $t = 0$. Thus

$$Dh_{\mathbf{y}}(\mathbf{x}(\mathbf{y}))\mathbf{v} = 0 = 2\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^T D_1\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})\mathbf{v}$$

for every vector $\mathbf{v}$. But from 5.16.35 and the fact that $\mathbf{v}$ is arbitrary, it follows $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$. This proves the existence of the function $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ such that $\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$ for all $\mathbf{y} \in B(\mathbf{y}_0, \eta)$.

It remains to verify this function is a C^1 function. To do this, let $\mathbf{y}_1$ and $\mathbf{y}_2$ be points of $B(\mathbf{y}_0, \eta)$. Then as before, consider the i^{th} component of $\mathbf{f}$ and consider the same argument using the mean value theorem to write

$$\begin{aligned} 0 &= f_i(\mathbf{x}(\mathbf{y}_1), \mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_2) \\ &= f_i(\mathbf{x}(\mathbf{y}_1), \mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_1) + f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_1) - f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}_2) \\ &= D_1f_i(\mathbf{x}^i, \mathbf{y}_1)(\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)) + D_2f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}^i)(\mathbf{y}_1 - \mathbf{y}_2). \end{aligned}$$

Therefore,

$$J(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y}_1)(\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)) = -M(\mathbf{y}_1 - \mathbf{y}_2) \quad (5.16.37)$$

where M is the matrix whose i^{th} row is $D_2f_i(\mathbf{x}(\mathbf{y}_2), \mathbf{y}^i)$. Then from 5.16.35 there exists a constant, C independent of the choice of $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ such that

$$\left\| J(\mathbf{x}^1, \dots, \mathbf{x}^n, \mathbf{y})^{-1} \right\| < C$$

whenever $(\mathbf{x}^1, \dots, \mathbf{x}^n) \in \overline{B(\mathbf{x}_0, \delta)}^n$. By continuity of the partial derivatives of $\mathbf{f}$ it also follows there exists a constant, C_1 such that $\|D_2f_i(\mathbf{x}, \mathbf{y})\| < C_1$ whenever, $(\mathbf{x}, \mathbf{y}) \in \overline{B(\mathbf{x}_0, \delta)} \times B(\mathbf{y}_0, \eta)$. Hence $\|M\|$ must also be bounded independent of the

choice of $\mathbf{y}_1$ and $\mathbf{y}_2$ in $B(\mathbf{y}_0, \eta)$. From 5.16.37, it follows there exists a constant, C such that for all $\mathbf{y}_1, \mathbf{y}_2$ in $B(\mathbf{y}_0, \eta)$,

$$|\mathbf{x}(\mathbf{y}_1) - \mathbf{x}(\mathbf{y}_2)| \leq C |\mathbf{y}_1 - \mathbf{y}_2|. \quad (5.16.38)$$

It follows as in the proof of the chain rule that

$$\mathbf{o}(\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})) = \mathbf{o}(\mathbf{v}). \quad (5.16.39)$$

Now let $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ and let $|\mathbf{v}|$ be sufficiently small that $\mathbf{y} + \mathbf{v} \in B(\mathbf{y}_0, \eta)$. Then

$$\begin{aligned} \mathbf{0} &= \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y} + \mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \\ &= \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y} + \mathbf{v}) - \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y}) + \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y}) - \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \\ &= D_2 \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y}) \mathbf{v} + D_1 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) (\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})) + \mathbf{o}(|\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})|) \\ &= D_2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v} + D_1 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) (\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})) + \\ &\quad \mathbf{o}(|\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})|) + (D_2 \mathbf{f}(\mathbf{x}(\mathbf{y} + \mathbf{v}), \mathbf{y}) \mathbf{v} - D_2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v}) \\ &= D_2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v} + D_1 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) (\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y})) + \mathbf{o}(\mathbf{v}). \end{aligned}$$

Therefore,

$$\mathbf{x}(\mathbf{y} + \mathbf{v}) - \mathbf{x}(\mathbf{y}) = -D_1 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) \mathbf{v} + \mathbf{o}(\mathbf{v})$$

which shows that $D\mathbf{x}(\mathbf{y}) = -D_1 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})^{-1} D_2 \mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y})$ and $\mathbf{y} \rightarrow D\mathbf{x}(\mathbf{y})$ is continuous. This proves the theorem.

In practice, how do you verify the condition, $D_1 \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$?

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} f_1(x_1, \dots, x_n, y_1, \dots, y_n) \\ \vdots \\ f_n(x_1, \dots, x_n, y_1, \dots, y_n) \end{pmatrix}.$$

The matrix of the linear transformation, $D_1 \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)$ is then

$$\begin{pmatrix} \frac{\partial f_1(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_1} & \dots & \frac{\partial f_1(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_1} & \dots & \frac{\partial f_n(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_n} \end{pmatrix}$$

and from linear algebra, $D_1 \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n)$ exactly when the above matrix has an inverse. In other words when

$$\det \begin{pmatrix} \frac{\partial f_1(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_1} & \dots & \frac{\partial f_1(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_1} & \dots & \frac{\partial f_n(x_1, \dots, x_n, y_1, \dots, y_n)}{\partial x_n} \end{pmatrix} \neq 0$$

at $(\mathbf{x}_0, \mathbf{y}_0)$. The above determinant is important enough that it is given special notation. Letting $\mathbf{z} = \mathbf{f}(\mathbf{x}, \mathbf{y})$, the above determinant is often written as

$$\frac{\partial(z_1, \dots, z_n)}{\partial(x_1, \dots, x_n)}.$$

Of course you can replace $\mathbb{R}$ with $\mathbb{F}$ in the above by applying the above to the situation in which each $\mathbb{F}$ is replaced with $\mathbb{R}^2$.

Corollary 5.16.2 (*implicit function theorem*) Suppose U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Let $\mathbf{f} : U \rightarrow \mathbb{F}^n$ be in $C^1(U)$ and suppose

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}, D_1\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \quad (5.16.40)$$

Then there exist positive constants, δ, η , such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \quad (5.16.41)$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^1(B(\mathbf{y}_0, \eta))$.

The next theorem is a very important special case of the implicit function theorem known as the inverse function theorem. Actually one can also obtain the implicit function theorem from the inverse function theorem. It is done this way in [46] and in [3].

Theorem 5.16.3 (*inverse function theorem*) Let $\mathbf{x}_0 \in U \subseteq \mathbb{F}^n$ and let $\mathbf{f} : U \rightarrow \mathbb{F}^n$. Suppose

$$\mathbf{f} \text{ is } C^1(U), \text{ and } D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \quad (5.16.42)$$

Then there exist open sets, W , and V such that

$$\mathbf{x}_0 \in W \subseteq U, \quad (5.16.43)$$

$$\mathbf{f} : W \rightarrow V \text{ is one to one and onto,} \quad (5.16.44)$$

$$\mathbf{f}^{-1} \text{ is } C^1. \quad (5.16.45)$$

Proof: Apply the implicit function theorem to the function

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{f}(\mathbf{x}) - \mathbf{y}$$

where $\mathbf{y}_0 \equiv \mathbf{f}(\mathbf{x}_0)$. Thus the function $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ defined in that theorem is $\mathbf{f}^{-1}$. Now let

$$W \equiv B(\mathbf{x}_0, \delta) \cap \mathbf{f}^{-1}(B(\mathbf{y}_0, \eta))$$

and

$$V \equiv B(\mathbf{y}_0, \eta).$$

This proves the theorem.

5.16.1 More Continuous Partial Derivatives

Corollary 5.16.2 will now be improved slightly. If $\mathbf{f}$ is C^k , it follows that the function which is implicitly defined is also in C^k , not just C^1 . Since the inverse function theorem comes as a case of the implicit function theorem, this shows that the inverse function also inherits the property of being C^k .

Theorem 5.16.4 (*implicit function theorem*) Suppose U is an open set in $\mathbb{F}^n \times \mathbb{F}^m$. Let $\mathbf{f} : U \rightarrow \mathbb{F}^n$ be in $C^k(U)$ and suppose

$$\mathbf{f}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}, \quad D_1 \mathbf{f}(\mathbf{x}_0, \mathbf{y}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \quad (5.16.46)$$

Then there exist positive constants, δ, η , such that for every $\mathbf{y} \in B(\mathbf{y}_0, \eta)$ there exists a unique $\mathbf{x}(\mathbf{y}) \in B(\mathbf{x}_0, \delta)$ such that

$$\mathbf{f}(\mathbf{x}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \quad (5.16.47)$$

Furthermore, the mapping, $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is in $C^k(B(\mathbf{y}_0, \eta))$.

Proof: From Corollary 5.16.2 $\mathbf{y} \rightarrow \mathbf{x}(\mathbf{y})$ is C^1 . It remains to show it is C^k for $k > 1$ assuming that $\mathbf{f}$ is C^k . From 5.16.47

$$\frac{\partial \mathbf{x}}{\partial y^l} = -D_1(\mathbf{x}, \mathbf{y})^{-1} \frac{\partial \mathbf{f}}{\partial y^l}.$$

Thus the following formula holds for $q = 1$ and $|\alpha| = q$.

$$D^\alpha \mathbf{x}(\mathbf{y}) = \sum_{|\beta| \leq q} M_\beta(\mathbf{x}, \mathbf{y}) D^\beta \mathbf{f}(\mathbf{x}, \mathbf{y}) \quad (5.16.48)$$

where M_β is a matrix whose entries are differentiable functions of $D^\gamma(\mathbf{x})$ for $|\gamma| < q$ and $D^\tau \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q$. This follows easily from the description of $D_1(\mathbf{x}, \mathbf{y})^{-1}$ in terms of the cofactor matrix and the determinant of $D_1(\mathbf{x}, \mathbf{y})$. Suppose 5.16.48 holds for $|\alpha| = q < k$. Then by induction, this yields $\mathbf{x}$ is C^q . Then

$$\frac{\partial D^\alpha \mathbf{x}(\mathbf{y})}{\partial y^p} = \sum_{|\beta| \leq |\alpha|} \frac{\partial M_\beta(\mathbf{x}, \mathbf{y})}{\partial y^p} D^\beta \mathbf{f}(\mathbf{x}, \mathbf{y}) + M_\beta(\mathbf{x}, \mathbf{y}) \frac{\partial D^\beta \mathbf{f}(\mathbf{x}, \mathbf{y})}{\partial y^p}.$$

By the chain rule $\frac{\partial M_\beta(\mathbf{x}, \mathbf{y})}{\partial y^p}$ is a matrix whose entries are differentiable functions of $D^\tau \mathbf{f}(\mathbf{x}, \mathbf{y})$ for $|\tau| \leq q+1$ and $D^\gamma(\mathbf{x})$ for $|\gamma| < q+1$. It follows since y^p was arbitrary that for any $|\alpha| = q+1$, a formula like 5.16.48 holds with q being replaced by $q+1$. By induction, $\mathbf{x}$ is C^k . This proves the theorem.

As a simple corollary this yields an improved version of the inverse function theorem.

Theorem 5.16.5 (*inverse function theorem*) Let $\mathbf{x}_0 \in U \subseteq \mathbb{F}^n$ and let $\mathbf{f} : U \rightarrow \mathbb{F}^n$. Suppose for k a positive integer,

$$\mathbf{f} \text{ is } C^k(U), \quad \text{and} \quad D\mathbf{f}(\mathbf{x}_0)^{-1} \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n). \quad (5.16.49)$$

Then there exist open sets, W , and V such that

$$\mathbf{x}_0 \in W \subseteq U, \quad (5.16.50)$$

$$\mathbf{f} : W \rightarrow V \text{ is one to one and onto,} \quad (5.16.51)$$

$$\mathbf{f}^{-1} \text{ is } C^k. \quad (5.16.52)$$

5.17 The Method Of Lagrange Multipliers

As an application of the implicit function theorem, consider the method of Lagrange multipliers from calculus. Recall the problem is to maximize or minimize a function subject to equality constraints. Let $f : U \rightarrow \mathbb{R}$ be a C^1 function where $U \subseteq \mathbb{R}^n$ and let

$$g_i(\mathbf{x}) = 0, \quad i = 1, \dots, m \quad (5.17.53)$$

be a collection of equality constraints with $m < n$. Now consider the system of nonlinear equations

$$\begin{aligned} f(\mathbf{x}) &= a \\ g_i(\mathbf{x}) &= 0, \quad i = 1, \dots, m. \end{aligned}$$

$\mathbf{x}_0$ is a local maximum if $f(\mathbf{x}_0) \geq f(\mathbf{x})$ for all $\mathbf{x}$ near $\mathbf{x}_0$ which also satisfies the constraints 5.17.53. A local minimum is defined similarly. Let $\mathbf{F} : U \times \mathbb{R} \rightarrow \mathbb{R}^{m+1}$ be defined by

$$\mathbf{F}(\mathbf{x}, a) \equiv \begin{pmatrix} f(\mathbf{x}) - a \\ g_1(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{pmatrix}. \quad (5.17.54)$$

Now consider the $m+1 \times n$ Jacobian matrix,

$$\begin{pmatrix} f_{x_1}(\mathbf{x}_0) & \cdots & f_{x_n}(\mathbf{x}_0) \\ g_{1x_1}(\mathbf{x}_0) & \cdots & g_{1x_n}(\mathbf{x}_0) \\ \vdots & & \vdots \\ g_{mx_1}(\mathbf{x}_0) & \cdots & g_{mx_n}(\mathbf{x}_0) \end{pmatrix}.$$

If this matrix has rank $m+1$ then some $m+1 \times m+1$ submatrix has nonzero determinant. It follows from the implicit function theorem that there exist $m+1$ variables, $x_{i_1}, \dots, x_{i_{m+1}}$ such that the system

$$\mathbf{F}(\mathbf{x}, a) = \mathbf{0} \quad (5.17.55)$$

specifies these $m+1$ variables as a function of the remaining $n - (m+1)$ variables and a in an open set of $\mathbb{R}^{n-m}$. Thus there is a solution $(\mathbf{x}, a)$ to 5.17.55 for some $\mathbf{x}$ close to $\mathbf{x}_0$ whenever a is in some open interval. Therefore, $\mathbf{x}_0$ cannot be either a local minimum or a local maximum. It follows that if $\mathbf{x}_0$ is either a local maximum

or a local minimum, then the above matrix must have rank less than $m + 1$ which requires the rows to be linearly dependent. Thus, there exist m scalars,

$$\lambda_1, \dots, \lambda_m,$$

and a scalar μ , not all zero such that

$$\mu \begin{pmatrix} f_{x_1}(\mathbf{x}_0) \\ \vdots \\ f_{x_n}(\mathbf{x}_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix} + \dots + \lambda_m \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix}. \quad (5.17.56)$$

If the column vectors

$$\begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix}, \dots, \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix} \quad (5.17.57)$$

are linearly independent, then, $\mu \neq 0$ and dividing by μ yields an expression of the form

$$\begin{pmatrix} f_{x_1}(\mathbf{x}_0) \\ \vdots \\ f_{x_n}(\mathbf{x}_0) \end{pmatrix} = \lambda_1 \begin{pmatrix} g_{1x_1}(\mathbf{x}_0) \\ \vdots \\ g_{1x_n}(\mathbf{x}_0) \end{pmatrix} + \dots + \lambda_m \begin{pmatrix} g_{mx_1}(\mathbf{x}_0) \\ \vdots \\ g_{mx_n}(\mathbf{x}_0) \end{pmatrix} \quad (5.17.58)$$

at every point $\mathbf{x}_0$ which is either a local maximum or a local minimum. This proves the following theorem.

Theorem 5.17.1 *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \rightarrow \mathbb{R}$ be a C^1 function. Then if $\mathbf{x}_0 \in U$ is either a local maximum or local minimum of f subject to the constraints 5.17.53, then 5.17.56 must hold for some scalars $\mu, \lambda_1, \dots, \lambda_m$ not all equal to zero. If the vectors in 5.17.57 are linearly independent, it follows that an equation of the form 5.17.58 holds.*

Metric Spaces And General Topological Spaces

6.1 Metric Space

Definition 6.1.1 A metric space is a set, X and a function $d : X \times X \rightarrow [0, \infty)$ which satisfies the following properties.

$$\begin{aligned} d(x, y) &= d(y, x) \\ d(x, y) &\geq 0 \text{ and } d(x, y) = 0 \text{ if and only if } x = y \\ d(x, y) &\leq d(x, z) + d(z, y). \end{aligned}$$

You can check that $\mathbb{R}^n$ and $\mathbb{C}^n$ are metric spaces with $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. However, there are many others. The definitions of open and closed sets are the same for a metric space as they are for $\mathbb{R}^n$.

Definition 6.1.2 A set, U in a metric space is open if whenever $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. As before, $B(x, r) \equiv \{y : d(x, y) < r\}$. Closed sets are those whose complements are open. A point p is a limit point of a set, S if for every $r > 0$, $B(p, r)$ contains infinitely many points of S . A sequence, $\{x_n\}$ converges to a point x if for every $\varepsilon > 0$ there exists N such that if $n \geq N$, then $d(x, x_n) < \varepsilon$. $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that if $m, n \geq N$, then $d(x_n, x_m) < \varepsilon$.

Lemma 6.1.3 In a metric space, X every ball, $B(x, r)$ is open. A set is closed if and only if it contains all its limit points. If p is a limit point of S , then there exists a sequence of distinct points of S , $\{x_n\}$ such that $\lim_{n \rightarrow \infty} x_n = p$.

Proof: Let $z \in B(x, r)$. Let $\delta = r - d(x, z)$. Then if $w \in B(z, \delta)$,

$$d(w, x) \leq d(x, z) + d(z, w) < d(x, z) + r - d(x, z) = r.$$

Therefore, $B(z, \delta) \subseteq B(x, r)$ and this shows $B(x, r)$ is open.

The properties of balls are presented in the following theorem.

Theorem 6.1.4 Suppose (X, d) is a metric space. Then the sets $\{B(x, r) : r > 0, x \in X\}$ satisfy

$$\cup \{B(x, r) : r > 0, x \in X\} = X \quad (6.1.1)$$

If $p \in B(x, r_1) \cap B(z, r_2)$, there exists $r > 0$ such that

$$B(p, r) \subseteq B(x, r_1) \cap B(z, r_2). \quad (6.1.2)$$

Proof: Observe that the union of these balls includes the whole space, X so 6.1.1 is obvious. Consider 6.1.2. Let $p \in B(x, r_1) \cap B(z, r_2)$. Consider

$$r \equiv \min(r_1 - d(x, p), r_2 - d(z, p))$$

and suppose $y \in B(p, r)$. Then

$$d(y, x) \leq d(y, p) + d(p, x) < r_1 - d(x, p) + d(x, p) = r_1$$

and so $B(p, r) \subseteq B(x, r_1)$. By similar reasoning, $B(p, r) \subseteq B(z, r_2)$. This proves the theorem.

Let K be a closed set. This means $K^C \equiv X \setminus K$ is an open set. Let p be a limit point of K . If $p \in K^C$, then since K^C is open, there exists $B(p, r) \subseteq K^C$. But this contradicts p being a limit point because there are no points of K in this ball. Hence all limit points of K must be in K .

Suppose next that K contains its limit points. Is K^C open? Let $p \in K^C$. Then p is not a limit point of K . Therefore, there exists $B(p, r)$ which contains at most finitely many points of K . Since $p \notin K$, it follows that by making r smaller if necessary, $B(p, r)$ contains no points of K . That is $B(p, r) \subseteq K^C$ showing K^C is open. Therefore, K is closed.

Suppose now that p is a limit point of S . Let $x_1 \in (S \setminus \{p\}) \cap B(p, 1)$. If $x_1, \dots, x_k$ have been chosen, let

$$r_{k+1} \equiv \min \left\{ d(p, x_i), i = 1, \dots, k, \frac{1}{k+1} \right\}.$$

Let $x_{k+1} \in (S \setminus \{p\}) \cap B(p, r_{k+1})$. This proves the lemma.

Lemma 6.1.5 If $\{x_n\}$ is a Cauchy sequence in a metric space, X and if some subsequence, $\{x_{n_k}\}$ converges to x , then $\{x_n\}$ converges to x . Also if a sequence converges, then it is a Cauchy sequence.

Proof: Note first that $n_k \geq k$ because in a subsequence, the indices, $n_1, n_2, \dots$ are strictly increasing. Let $\varepsilon > 0$ be given and let N be such that for $k > N$, $d(x, x_{n_k}) < \varepsilon/2$ and for $m, n \geq N$, $d(x_m, x_n) < \varepsilon/2$. Pick $k > n$. Then if $n > N$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, suppose $\lim_{n \rightarrow \infty} x_n = x$. Then there exists N such that if $n > N$, then $d(x_n, x) < \varepsilon/2$. it follows that for $m, n > N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the lemma.

A useful idea is the idea of distance from a point to a set.

Definition 6.1.6 *Let (X, d) be a metric space and let S be a nonempty set in X . Then*

$$\text{dist}(x, S) \equiv \inf \{d(x, y) : y \in S\}.$$

The following lemma is the fundamental result.

Lemma 6.1.7 *The function, $x \rightarrow \text{dist}(x, S)$ is continuous and in fact satisfies*

$$|\text{dist}(x, S) - \text{dist}(y, S)| \leq d(x, y).$$

Proof: Suppose $\text{dist}(x, y)$ is as least as large as $\text{dist}(y, S)$. Then pick $z \in S$ such that $d(y, z) \leq \text{dist}(y, S) + \varepsilon$. Then

$$\begin{aligned} |\text{dist}(x, S) - \text{dist}(y, S)| &= \text{dist}(x, S) - \text{dist}(y, S) \\ &\leq d(x, z) - (d(y, z) - \varepsilon) \\ &= d(x, z) - d(y, z) + \varepsilon \\ &\leq d(x, y) + d(y, z) - d(y, z) + \varepsilon \\ &= d(x, y) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

6.2 Compactness In Metric Space

Many existence theorems in analysis depend on some set being compact. Therefore, it is important to be able to identify compact sets. The purpose of this section is to describe compact sets in a metric space.

Definition 6.2.1 *Let A be a subset of X . A is compact if whenever A is contained in the union of a set of open sets, there exists finitely many of these open sets whose union contains A . (Every open cover admits a finite subcover.) A is “sequentially compact” means every sequence has a convergent subsequence converging to an element of A .*

In a metric space compact is not the same as closed and bounded!

Example 6.2.2 *Let X be any infinite set and define $d(x, y) = 1$ if $x \neq y$ while $d(x, y) = 0$ if $x = y$.*

You should verify the details that this is a metric space because it satisfies the axioms of a metric. The set X is closed and bounded because its complement is $\emptyset$ which is clearly open because every point of $\emptyset$ is an interior point. (There are none.) Also X is bounded because $X = B(x, 2)$. However, X is clearly not compact because $\{B(x, \frac{1}{2}) : x \in X\}$ is a collection of open sets whose union contains X but

since they are all disjoint and nonempty, there is no finite subset of these whose union contains X . In fact $B(x, \frac{1}{2}) = \{x\}$.

From this example it is clear something more than closed and bounded is needed. If you are not familiar with the issues just discussed, ignore them and continue.

Definition 6.2.3 *In any metric space, a set E is totally bounded if for every $\varepsilon > 0$ there exists a finite set of points $\{x_1, \dots, x_n\}$ such that*

$$E \subseteq \cup_{i=1}^n B(x_i, \varepsilon).$$

This finite set of points is called an ε net.

The following proposition tells which sets in a metric space are compact. First here is an interesting lemma.

Lemma 6.2.4 *Let X be a metric space and suppose D is a countable dense subset of X . In other words, it is being assumed X is a separable metric space. Consider the open sets of the form $B(d, r)$ where r is a positive rational number and $d \in D$. Denote this countable collection of open sets by $\mathcal{B}$. Then every open set is the union of sets of $\mathcal{B}$. Furthermore, if $\mathcal{C}$ is any collection of open sets, there exists a countable subset, $\{U_n\} \subseteq \mathcal{C}$ such that $\cup_n U_n = \cup \mathcal{C}$.*

Proof: Let U be an open set and let $x \in U$. Let $B(x, \delta) \subseteq U$. Then by density of D , there exists $d \in D \cap B(x, \delta/4)$. Now pick $r \in \mathbb{Q} \cap (\delta/4, 3\delta/4)$ and consider $B(d, r)$. Clearly, $B(d, r)$ contains the point x because $r > \delta/4$. Is $B(d, r) \subseteq B(x, \delta)$? if so, this proves the lemma because x was an arbitrary point of U . Suppose $z \in B(d, r)$. Then

$$d(z, x) \leq d(z, d) + d(d, x) < r + \frac{\delta}{4} < \frac{3\delta}{4} + \frac{\delta}{4} = \delta$$

Now let $\mathcal{C}$ be any collection of open sets. Each set in this collection is the union of countably many sets of $\mathcal{B}$. Let $\mathcal{B}'$ denote the sets of $\mathcal{B}$ which are contained in some set of $\mathcal{C}$. Thus $\cup \mathcal{B}' = \cup \mathcal{C}$. Then for each $B \in \mathcal{B}'$, pick $U_B \in \mathcal{C}$ such that $B \subseteq U_B$. Then $\{U_B : B \in \mathcal{B}'\}$ is a countable collection of sets of $\mathcal{C}$ whose union equals $\cup \mathcal{C}$. Therefore, this proves the lemma.

Proposition 6.2.5 *Let (X, d) be a metric space. Then the following are equivalent.*

$$(X, d) \text{ is compact,} \tag{6.2.3}$$

$$(X, d) \text{ is sequentially compact,} \tag{6.2.4}$$

$$(X, d) \text{ is complete and totally bounded.} \tag{6.2.5}$$

Proof: Suppose 6.2.3 and let $\{x_k\}$ be a sequence. Suppose $\{x_k\}$ has no convergent subsequence. If this is so, then no value of the sequence is repeated more than finitely many times. Also $\{x_k\}$ has no limit point because if it did, there would exist a subsequence which converges. To see this, suppose p is a limit point of $\{x_k\}$. Then in $B(p, 1)$ there are infinitely many points of $\{x_k\}$. Pick one called x_{k_1} . Now if

$x_{k_1}, x_{k_2}, \dots, x_{k_n}$ have been picked with $x_{k_i} \in B(p, 1/i)$, consider $B(p, 1/(n+1))$. There are infinitely many points of $\{x_k\}$ in this ball also. Pick $x_{k_{n+1}}$ such that $k_{n+1} > k_n$. Then $\{x_{k_n}\}_{n=1}^\infty$ is a subsequence which converges to p and it is assumed this does not happen. Thus $\{x_k\}$ has no limit points. It follows the set

$$C_n = \cup \{x_k : k \geq n\}$$

is a closed set because it has no limit points and if

$$U_n = C_n^C,$$

then

$$X = \cup_{n=1}^\infty U_n$$

but there is no finite subcovering, because no value of the sequence is repeated more than finitely many times. This contradicts compactness of (X, d) . Note x_k is not in U_n whenever $k > n$. Thus 6.2.3 implies 6.2.4.

Now suppose 6.2.4 and let $\{x_n\}$ be a Cauchy sequence. Is $\{x_n\}$ convergent? By sequential compactness $x_{n_k} \rightarrow x$ for some subsequence. By Lemma 6.1.5 it follows that $\{x_n\}$ also converges to x showing that (X, d) is complete. If (X, d) is not totally bounded, then there exists $\varepsilon > 0$ for which there is no ε net. Hence there exists a sequence $\{x_k\}$ with $d(x_k, x_l) \geq \varepsilon$ for all $l \neq k$. By Lemma 6.1.5 again, this contradicts 6.2.4 because no subsequence can be a Cauchy sequence and so no subsequence can converge. This shows 6.2.4 implies 6.2.5.

Now suppose 6.2.5. What about 6.2.4? Let $\{p_n\}$ be a sequence and let $\{x_i^n\}_{i=1}^{m_n}$ be a 2^{-n} net for $n = 1, 2, \dots$. Let

$$B_n \equiv B(x_{i_n}^n, 2^{-n})$$

be such that B_n contains p_k for infinitely many values of k and $B_n \cap B_{n+1} \neq \emptyset$. To do this, suppose B_n contains p_k for infinitely many values of k . Then one of the sets which intersect $B_n, B(x_i^{n+1}, 2^{-(n+1)})$ must contain p_k for infinitely many values of k because all these indices of points from $\{p_n\}$ contained in B_n must be accounted for in one of finitely many sets, $B(x_i^{n+1}, 2^{-(n+1)})$. Thus there exists a strictly increasing sequence of integers, n_k such that

$$p_{n_k} \in B_k.$$

Then if $k \geq l$,

$$\begin{aligned} d(p_{n_k}, p_{n_l}) &\leq \sum_{i=l}^{k-1} d(p_{n_{i+1}}, p_{n_i}) \\ &< \sum_{i=l}^{k-1} 2^{-(i-1)} < 2^{-(l-2)}. \end{aligned}$$

Consequently $\{p_{n_k}\}$ is a Cauchy sequence. Hence it converges because the metric space is complete. This proves 6.2.4.

Now suppose 6.2.4 and 6.2.5 which have now been shown to be equivalent. Let D_n be a n^{-1} net for $n = 1, 2, \dots$ and let

$$D = \cup_{n=1}^{\infty} D_n.$$

Thus D is a countable dense subset of (X, d) .

Now let $\mathcal{C}$ be any set of open sets such that $\cup \mathcal{C} \supseteq X$. By Lemma 6.2.4, there exists a countable subset of $\mathcal{C}$,

$$\tilde{\mathcal{C}} = \{U_n\}_{n=1}^{\infty}$$

such that $\cup \tilde{\mathcal{C}} = \cup \mathcal{C}$. If $\mathcal{C}$ admits no finite subcover, then neither does $\tilde{\mathcal{C}}$ and there exists $p_n \in X \setminus \cup_{k=1}^n U_k$. Then since X is sequentially compact, there is a subsequence $\{p_{n_k}\}$ such that $\{p_{n_k}\}$ converges. Say

$$p = \lim_{k \rightarrow \infty} p_{n_k}.$$

All but finitely many points of $\{p_{n_k}\}$ are in $X \setminus \cup_{k=1}^n U_k$. Therefore $p \in X \setminus \cup_{k=1}^n U_k$ for each n . Hence

$$p \notin \cup_{k=1}^{\infty} U_k$$

contradicting the construction of $\{U_n\}_{n=1}^{\infty}$ which required that $\cup_{n=1}^{\infty} U_n \supseteq X$. Hence X is compact. This proves the proposition.

Consider $\mathbb{R}^n$. In this setting totally bounded and bounded are the same. This will yield a proof of the Heine Borel theorem from advanced calculus.

Lemma 6.2.6 *A subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded.*

Proof: Let A be totally bounded. Is it bounded? Let $\mathbf{x}_1, \dots, \mathbf{x}_p$ be a 1 net for A . Now consider the ball $B(\mathbf{0}, r+1)$ where $r > \max(|\mathbf{x}_i| : i = 1, \dots, p)$. If $\mathbf{z} \in A$, then $\mathbf{z} \in B(\mathbf{x}_j, 1)$ for some j and so by the triangle inequality,

$$|\mathbf{z} - \mathbf{0}| \leq |\mathbf{z} - \mathbf{x}_j| + |\mathbf{x}_j| < 1 + r.$$

Thus $A \subseteq B(\mathbf{0}, r+1)$ and so A is bounded.

Now suppose A is bounded and suppose A is not totally bounded. Then there exists $\varepsilon > 0$ such that there is no ε net for A . Therefore, there exists a sequence of points $\{a_i\}$ with $|a_i - a_j| \geq \varepsilon$ if $i \neq j$. Since A is bounded, there exists $r > 0$ such that

$$A \subseteq [-r, r]^n.$$

($\mathbf{x} \in [-r, r]^n$ means $x_i \in [-r, r]$ for each i .) Now define $\mathcal{S}$ to be all cubes of the form

$$\prod_{k=1}^n [a_k, b_k)$$

where

$$a_k = -r + i2^{-p}r, \quad b_k = -r + (i+1)2^{-p}r,$$

for $i \in \{0, 1, \dots, 2^{p+1} - 1\}$. Thus $\mathcal{S}$ is a collection of $(2^{p+1})^n$ non overlapping cubes whose union equals $[-r, r]^n$ and whose diameters are all equal to $2^{-p}r\sqrt{n}$. Now choose p large enough that the diameter of these cubes is less than ε . This yields a contradiction because one of the cubes must contain infinitely many points of $\{a_i\}$. This proves the lemma.

The next theorem is called the Heine Borel theorem and it characterizes the compact sets in $\mathbb{R}^n$.

Theorem 6.2.7 *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof: Since a set in $\mathbb{R}^n$ is totally bounded if and only if it is bounded, this theorem follows from Proposition 6.2.5 and the observation that a subset of $\mathbb{R}^n$ is closed if and only if it is complete. This proves the theorem.

6.3 Some Applications Of Compactness

The following corollary is an important existence theorem which depends on compactness.

Corollary 6.3.1 *Let X be a compact metric space and let $f : X \rightarrow \mathbb{R}$ be continuous. Then $\max \{f(x) : x \in X\}$ and $\min \{f(x) : x \in X\}$ both exist.*

Proof: First it is shown $f(X)$ is compact. Suppose $\mathcal{C}$ is a set of open sets whose union contains $f(X)$. Then since f is continuous $f^{-1}(U)$ is open for all $U \in \mathcal{C}$. Therefore, $\{f^{-1}(U) : U \in \mathcal{C}\}$ is a collection of open sets whose union contains X . Since X is compact, it follows finitely many of these, $\{f^{-1}(U_1), \dots, f^{-1}(U_p)\}$ contains X in their union. Therefore, $f(X) \subseteq \cup_{k=1}^p U_k$ showing $f(X)$ is compact as claimed.

Now since $f(X)$ is compact, Theorem 6.2.7 implies $f(X)$ is closed and bounded. Therefore, it contains its inf and its sup. Thus f achieves both a maximum and a minimum.

Definition 6.3.2 *Let X, Y be metric spaces and $f : X \rightarrow Y$ a function. f is uniformly continuous if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever x_1 and x_2 are two points of X satisfying $d(x_1, x_2) < \delta$, it follows that $d(f(x_1), f(x_2)) < \varepsilon$.*

A very important theorem is the following.

Theorem 6.3.3 *Suppose $f : X \rightarrow Y$ is continuous and X is compact. Then f is uniformly continuous.*

Proof: Suppose this is not true and that f is continuous but not uniformly continuous. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist points, p_δ and q_δ such that $d(p_\delta, q_\delta) < \delta$ and yet $d(f(p_\delta), f(q_\delta)) \geq \varepsilon$. Let p_n and q_n be the points which go with $\delta = 1/n$. By Proposition 6.2.5 $\{p_n\}$ has a convergent

subsequence, $\{p_{n_k}\}$ converging to a point, $x \in X$. Since $d(p_n, q_n) < \frac{1}{n}$, it follows that $q_{n_k} \rightarrow x$ also. Therefore,

$$\varepsilon \leq d(f(p_{n_k}), f(q_{n_k})) \leq d(f(p_{n_k}), f(x)) + d(f(x), f(q_{n_k}))$$

but by continuity of f , both $d(f(p_{n_k}), f(x))$ and $d(f(x), f(q_{n_k}))$ converge to 0 as $k \rightarrow \infty$ contradicting the above inequality. This proves the theorem.

Another important property of compact sets in a metric space concerns the finite intersection property.

Definition 6.3.4 *If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.*

Theorem 6.3.5 *Suppose $\mathcal{F}$ is a collection of compact sets in a metric space, X which has the finite intersection property. Then there exists a point in their intersection. ($\cap \mathcal{F} \neq \emptyset$).*

Proof: First I show each compact set is closed. Let K be a nonempty compact set and suppose $p \notin K$. Then for each $x \in K$, let $V_x = B(x, d(p, x)/3)$ and $U_x = B(p, d(p, x)/3)$ so that U_x and V_x have empty intersection. Then since V is compact, there are finitely many V_x which cover K say $V_{x_1}, \dots, V_{x_n}$. Then let $U = \cap_{i=1}^n U_{x_i}$. It follows $p \in U$ and U has empty intersection with K . In fact U has empty intersection with $\cup_{i=1}^n V_{x_i}$. Since U is an open set and $p \in K^C$ is arbitrary, it follows K^C is an open set.

Consider now the claim about the intersection. If this were not so, $\cup \{F^C : F \in \mathcal{F}\} = X$ and so, in particular, picking some $F_0 \in \mathcal{F}$, $\{F^C : F \in \mathcal{F}\}$ would be an open cover of F_0 . Since F_0 is compact, some finite subcover, $F_1^C, \dots, F_m^C$ exists. But then $F_0 \subseteq \cup_{k=1}^m F_k^C$ which means $\cap_{k=0}^m F_k = \emptyset$, contrary to the finite intersection property. To see this, note that if $x \in F_0$, then it must fail to be in some F_k and so it is not in $\cap_{k=0}^m F_k$. Since this is true for every x it follows $\cap_{k=0}^m F_k = \emptyset$.

Theorem 6.3.6 *Let X_i be a compact metric space with metric d_i . Then $\prod_{i=1}^m X_i$ is also a compact metric space with respect to the metric, $d(\mathbf{x}, \mathbf{y}) \equiv \max_i (d_i(x_i, y_i))$.*

Proof: This is most easily seen from sequential compactness. Let $\{\mathbf{x}^k\}_{k=1}^\infty$ be a sequence of points in $\prod_{i=1}^m X_i$. Consider the i^{th} component of $\mathbf{x}^k$, x_i^k . It follows $\{x_i^k\}$ is a sequence of points in X_i and so it has a convergent subsequence. Compactness of X_1 implies there exists a subsequence of $\mathbf{x}^k$, denoted by $\{\mathbf{x}^{k_1}\}$ such that

$$\lim_{k_1 \rightarrow \infty} x_1^{k_1} \rightarrow x_1 \in X_1.$$

Now there exists a further subsequence, denoted by $\{\mathbf{x}^{k_2}\}$ such that in addition to this, $x_2^{k_2} \rightarrow x_2 \in X_2$. After taking m such subsequences, there exists a subsequence, $\{\mathbf{x}^l\}$ such that $\lim_{l \rightarrow \infty} x_i^l = x_i \in X_i$ for each i . Therefore, letting $\mathbf{x} \equiv (x_1, \dots, x_m)$, $\mathbf{x}^l \rightarrow \mathbf{x}$ in $\prod_{i=1}^m X_i$. This proves the theorem.

6.4 Ascoli Arzela Theorem

Definition 6.4.1 Let (X, d) be a complete metric space. Then it is said to be locally compact if $\overline{B(x, r)}$ is compact for each $r > 0$.

Thus if you have a locally compact metric space, then if $\{a_n\}$ is a bounded sequence, it must have a convergent subsequence.

Let K be a compact subset of $\mathbb{R}^n$ and consider the continuous functions which have values in a locally compact metric space, (X, d) where d denotes the metric on X . Denote this space as $C(K, X)$.

Definition 6.4.2 For $f, g \in C(K, X)$, where K is a compact subset of $\mathbb{R}^n$ and X is a locally compact complete metric space define

$$\rho_K(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})) : \mathbf{x} \in K\}.$$

Then ρ_K provides a distance which makes $C(K, X)$ into a metric space.

The Ascoli Arzela theorem is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 6.4.3 Let $A \subseteq C(K, X)$ for K a compact subset of $\mathbb{R}^n$. Then A is said to be uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x} - \mathbf{y}| < \delta$ and $f \in A$,

$$d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon.$$

The set, A is said to be uniformly bounded if for some $M < \infty$, and $a \in X$,

$$f(\mathbf{x}) \in B(a, M)$$

for all $f \in A$ and $\mathbf{x} \in K$.

Uniform equicontinuity is like saying that the whole set of functions, A , is uniformly continuous on K uniformly for $f \in A$. The version of the Ascoli Arzela theorem I will present here is the following.

Theorem 6.4.4 Suppose K is a nonempty compact subset of $\mathbb{R}^n$ and $A \subseteq C(K, X)$ is uniformly bounded and uniformly equicontinuous. Then if $\{f_k\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, f_{k_i} such that

$$\lim_{i \rightarrow \infty} \rho_K(f_{k_i}, f) = 0.$$

To give a proof of this theorem, I will first prove some lemmas.

Lemma 6.4.5 If K is a compact subset of $\mathbb{R}^n$, then there exists $D \equiv \{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq K$ such that D is dense in K . Also, for every $\varepsilon > 0$ there exists a finite set of points, $\{\mathbf{x}_1, \dots, \mathbf{x}_m\} \subseteq K$, called an ε net such that

$$\cup_{i=1}^m B(\mathbf{x}_i, \varepsilon) \supseteq K.$$

Proof: For $m \in \mathbb{N}$, pick $x_1^m \in K$. If every point of K is within $1/m$ of x_1^m , stop. Otherwise, pick

$$x_2^m \in K \setminus B(x_1^m, 1/m).$$

If every point of K contained in $B(x_1^m, 1/m) \cup B(x_2^m, 1/m)$, stop. Otherwise, pick

$$x_3^m \in K \setminus (B(x_1^m, 1/m) \cup B(x_2^m, 1/m)).$$

If every point of K is contained in $B(x_1^m, 1/m) \cup B(x_2^m, 1/m) \cup B(x_3^m, 1/m)$, stop. Otherwise, pick

$$x_4^m \in K \setminus (B(x_1^m, 1/m) \cup B(x_2^m, 1/m) \cup B(x_3^m, 1/m))$$

Continue this way until the process stops, say at $N(m)$. It must stop because if it didn't, there would be a convergent subsequence due to the compactness of K . Ultimately all terms of this convergent subsequence would be closer than $1/m$, violating the manner in which they are chosen. Then $D = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{N(m)} \{x_k^m\}$. This is countable because it is a countable union of countable sets. If $\mathbf{y} \in K$ and $\varepsilon > 0$, then for some m , $2/m < \varepsilon$ and so $B(\mathbf{y}, \varepsilon)$ must contain some point of $\{x_k^m\}$ since otherwise, the process stopped too soon. You could have picked $\mathbf{y}$. This proves the lemma.

Lemma 6.4.6 Suppose D is defined above and $\{g_m\}$ is a sequence of functions of A having the property that for every $\mathbf{x}_k \in D$,

$$\lim_{m \rightarrow \infty} g_m(\mathbf{x}_k) \text{ exists.}$$

Then there exists $g \in C(K, X)$ such that

$$\lim_{m \rightarrow \infty} \rho(g_m, g) = 0.$$

Proof: Define g first on D .

$$g(\mathbf{x}_k) \equiv \lim_{m \rightarrow \infty} g_m(\mathbf{x}_k).$$

Next I show that $\{g_m\}$ converges at every point of K . Let $\mathbf{x} \in K$ and let $\varepsilon > 0$ be given. Choose $\mathbf{x}_k$ such that for all $f \in A$,

$$d(f(\mathbf{x}_k), f(\mathbf{x})) < \frac{\varepsilon}{3}.$$

I can do this by the equicontinuity. Now if p, q are large enough, say $p, q \geq M$,

$$d(g_p(\mathbf{x}_k), g_q(\mathbf{x}_k)) < \frac{\varepsilon}{3}.$$

Therefore, for $p, q \geq M$,

$$\begin{aligned} d(g_p(\mathbf{x}), g_q(\mathbf{x})) &\leq d(g_p(\mathbf{x}), g_p(\mathbf{x}_k)) + d(g_p(\mathbf{x}_k), g_q(\mathbf{x}_k)) + d(g_q(\mathbf{x}_k), g_q(\mathbf{x})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

It follows that $\{g_m(\mathbf{x})\}$ is a Cauchy sequence having values X . Therefore, it converges. Let $g(\mathbf{x})$ be the name of the thing it converges to.

Let $\varepsilon > 0$ be given and pick $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ and $|\mathbf{x} - \mathbf{y}| < \delta$, it follows $d(f(\mathbf{x}), f(\mathbf{y})) < \frac{\varepsilon}{3}$ for all $f \in A$. Now let $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ be a δ net for K as in Lemma 6.4.5. Since there are only finitely many points in this δ net, it follows that there exists N such that for all $p, q \geq N$,

$$d(g_q(\mathbf{x}_i), g_p(\mathbf{x}_i)) < \frac{\varepsilon}{3}$$

for all $\{\mathbf{x}_1, \dots, \mathbf{x}_m\}$. Therefore, for arbitrary $\mathbf{x} \in K$, pick $\mathbf{x}_i \in \{\mathbf{x}_1, \dots, \mathbf{x}_m\}$ such that $|\mathbf{x}_i - \mathbf{x}| < \delta$. Then

$$\begin{aligned} d(g_q(\mathbf{x}), g_p(\mathbf{x})) &\leq d(g_q(\mathbf{x}), g_q(\mathbf{x}_i)) + d(g_q(\mathbf{x}_i), g_p(\mathbf{x}_i)) + d(g_p(\mathbf{x}_i), g_p(\mathbf{x})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since N does not depend on the choice of $\mathbf{x}$, it follows this sequence $\{g_m\}$ is uniformly Cauchy. That is, for every $\varepsilon > 0$, there exists N such that if $p, q \geq N$, then

$$\rho(g_p, g_q) < \varepsilon.$$

Next, I need to verify that the function, g is a continuous function. Let N be large enough that whenever $p, q \geq N$, the above holds. Then for all $\mathbf{x} \in K$,

$$d(g(\mathbf{x}), g_p(\mathbf{x})) \leq \frac{\varepsilon}{3} \quad (6.4.6)$$

whenever $p \geq N$. This follows from observing that for $p, q \geq N$,

$$d(g_q(\mathbf{x}), g_p(\mathbf{x})) < \frac{\varepsilon}{3}$$

and then taking the limit as $q \rightarrow \infty$ to obtain 6.4.6. In passing to the limit, you can use the following simple claim.

Claim: In a metric space, if $a_n \rightarrow a$, then $d(a_n, b) \rightarrow d(a, b)$.

Proof of the claim: You note that by the triangle inequality, $d(a_n, b) - d(a, b) \leq d(a_n, a)$ and $d(a, b) - d(a_n, b) \leq d(a_n, a)$ and so

$$|d(a_n, b) - d(a, b)| \leq d(a_n, a).$$

Now let p satisfy 6.4.6 for all $\mathbf{x}$ whenever $p > N$. Also pick $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then

$$d(g_p(\mathbf{x}), g_p(\mathbf{y})) < \frac{\varepsilon}{3}.$$

Then if $|\mathbf{x} - \mathbf{y}| < \delta$,

$$\begin{aligned} d(g(\mathbf{x}), g(\mathbf{y})) &\leq d(g(\mathbf{x}), g_p(\mathbf{x})) + d(g_p(\mathbf{x}), g_p(\mathbf{y})) + d(g_p(\mathbf{y}), g(\mathbf{y})) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε was arbitrary, this shows that g is continuous.

It only remains to verify that $\rho(g, g_k) \rightarrow 0$. But this follows from 6.4.6. This proves the lemma.

With these lemmas, it is time to prove Theorem 6.4.4.

Proof of Theorem 6.4.4: Let $D = \{\mathbf{x}_k\}$ be the countable dense set of K guaranteed by Lemma 6.4.5 and let $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), \dots\}$ be a subsequence of $\mathbb{N}$ such that

$$\lim_{k \rightarrow \infty} f_{(1,k)}(\mathbf{x}_1) \text{ exists.}$$

This is where the local compactness of X is being used. Now let

$$\{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), \dots\}$$

be a subsequence of $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), \dots\}$ which has the property that

$$\lim_{k \rightarrow \infty} f_{(2,k)}(\mathbf{x}_2) \text{ exists.}$$

Thus it is also the case that

$$f_{(2,k)}(\mathbf{x}_1) \text{ converges to } \lim_{k \rightarrow \infty} f_{(1,k)}(\mathbf{x}_1).$$

because every subsequence of a convergent sequence converges to the same thing as the convergent sequence. Continue this way and consider the array

$$\begin{array}{l} f_{(1,1)}, f_{(1,2)}, f_{(1,3)}, f_{(1,4)}, \dots \text{ converges at } \mathbf{x}_1 \\ f_{(2,1)}, f_{(2,2)}, f_{(2,3)}, f_{(2,4)}, \dots \text{ converges at } \mathbf{x}_1 \text{ and } \mathbf{x}_2 \\ f_{(3,1)}, f_{(3,2)}, f_{(3,3)}, f_{(3,4)}, \dots \text{ converges at } \mathbf{x}_1, \mathbf{x}_2, \text{ and } \mathbf{x}_3 \\ \vdots \end{array}$$

Now let $g_k \equiv f_{(k,k)}$. Thus g_k is ultimately a subsequence of $\{f_{(m,k)}\}$ whenever $k > m$ and therefore, $\{g_k\}$ converges at each point of D . By Lemma 6.4.6 it follows there exists $g \in C(K; X)$ such that

$$\lim_{k \rightarrow \infty} \rho(g, g_k) = 0.$$

This proves the Ascoli Arzela theorem.

Actually there is an if and only if version of it but the most useful case is what is presented here. The process used to get the subsequence in the proof is called the Cantor diagonalization procedure.

6.5 The Tietze Extension Theorem

It turns out that if H is a closed subset of a metric space, (X, d) and if $f : H \rightarrow [a, b]$ is continuous, then there exists g defined on all of X such that $g = f$ on H and g is continuous. This is called the Tietze extension theorem. First it is well to recall continuity in the context of metric space.

Definition 6.5.1 Let (X, d) be a metric space and suppose $f : X \rightarrow Y$ is a function where (Y, ρ) is also a metric space. For example, $Y = \mathbb{R}$. Then f is continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(f(x), f(z)) < \varepsilon$ whenever $d(x, z) < \delta$. As is usual in such definitions, f is said to be continuous if it is continuous at every point of X .

The following lemma gives an important example of a continuous real valued function defined on a metric space, (X, d) .

Lemma 6.5.2 Let (X, d) be a metric space and let $S \subseteq X$ be a nonempty subset. Define

$$\text{dist}(x, S) \equiv \inf \{d(x, y) : y \in S\}.$$

Then $x \rightarrow \text{dist}(x, S)$ is a continuous function satisfying the inequality,

$$|\text{dist}(x, S) - \text{dist}(y, S)| \leq d(x, y). \quad (6.5.7)$$

Proof: The continuity of $x \rightarrow \text{dist}(x, S)$ is obvious if the inequality 6.5.7 is established. So let $x, y \in X$. Without loss of generality, assume $\text{dist}(x, S) \geq \text{dist}(y, S)$ and pick $z \in S$ such that $d(y, z) - \varepsilon < \text{dist}(y, S)$. Then

$$\begin{aligned} |\text{dist}(x, S) - \text{dist}(y, S)| &= \text{dist}(x, S) - \text{dist}(y, S) \leq d(x, z) - (d(y, z) - \varepsilon) \\ &\leq d(x, z) + d(y, z) - d(y, z) + \varepsilon = d(x, y) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves 6.5.7.

Lemma 6.5.3 Let H, K be two nonempty disjoint closed subsets of a metric space, (X, d) . Then there exists a continuous function, $g : X \rightarrow [-1, 1]$ such that $g(H) = -1/3$, $g(K) = 1/3$, $g(X) \subseteq [-1/3, 1/3]$.

Proof: Let

$$f(x) \equiv \frac{\text{dist}(x, H)}{\text{dist}(x, H) + \text{dist}(x, K)}.$$

The denominator is never equal to zero because if $\text{dist}(x, H) = 0$, then $x \in H$ because H is closed. (To see this, pick $h_k \in B(x, 1/k) \cap H$. Then $h_k \rightarrow x$ and since H is closed, $x \in H$.) Similarly, if $\text{dist}(x, K) = 0$, then $x \in K$ and so the denominator is never zero as claimed. Hence, by Lemma 6.5.2, f is continuous and from its definition, $f = 0$ on H and $f = 1$ on K . Now let $g(x) \equiv \frac{2}{3}(f(x) - \frac{1}{2})$. Then g has the desired properties.

Definition 6.5.4 For f a real or complex valued bounded continuous function defined on a metric space, M

$$\|f\|_M \equiv \sup \{|f(x)| : x \in M\}.$$

Lemma 6.5.5 Suppose M is a closed set in X where (X, d) is a metric space and suppose $f : M \rightarrow [-1, 1]$ is continuous at every point of M . Then there exists a function, g which is defined and continuous on all of X such that $\|f - g\|_M < \frac{2}{3}$.

Proof: Let $H = f^{-1}([-1, -1/3])$, $K = f^{-1}([1/3, 1])$. Thus H and K are disjoint closed subsets of M . Suppose first H, K are both nonempty. Then by Lemma 6.5.3 there exists g such that g is a continuous function defined on all of X and $g(H) = -1/3$, $g(K) = 1/3$, and $g(X) \subseteq [-1/3, 1/3]$. It follows $\|f - g\|_M < 2/3$. If $H = \emptyset$, then f has all its values in $[-1/3, 1]$ and so letting $g \equiv 1/3$, the desired condition is obtained. If $K = \emptyset$, let $g \equiv -1/3$. This proves the lemma.

Lemma 6.5.6 *Suppose M is a closed set in X where (X, d) is a metric space and suppose $f : M \rightarrow [-1, 1]$ is continuous at every point of M . Then there exists a function, g which is defined and continuous on all of X such that $g = f$ on M and g has its values in $[-1, 1]$.*

Proof: Let g_1 be such that $g_1(X) \subseteq [-1/3, 1/3]$ and $\|f - g_1\|_M \leq \frac{2}{3}$. Suppose $g_1, \dots, g_m$ have been chosen such that $g_j(X) \subseteq [-1/3, 1/3]$ and

$$\left\| f - \sum_{i=1}^m \left(\frac{2}{3}\right)^{i-1} g_i \right\|_M < \left(\frac{2}{3}\right)^m. \quad (6.5.8)$$

Then

$$\left\| \left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^m \left(\frac{2}{3}\right)^{i-1} g_i \right) \right\|_M \leq 1$$

and so $\left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^m \left(\frac{2}{3}\right)^{i-1} g_i \right)$ can play the role of f in the first step of the proof. Therefore, there exists g_{m+1} defined and continuous on all of X such that its values are in $[-1/3, 1/3]$ and

$$\left\| \left(\frac{3}{2}\right)^m \left(f - \sum_{i=1}^m \left(\frac{2}{3}\right)^{i-1} g_i \right) - g_{m+1} \right\|_M \leq \frac{2}{3}.$$

Hence

$$\left\| \left(f - \sum_{i=1}^m \left(\frac{2}{3}\right)^{i-1} g_i \right) - \left(\frac{2}{3}\right)^m g_{m+1} \right\|_M \leq \left(\frac{2}{3}\right)^{m+1}.$$

It follows there exists a sequence, $\{g_i\}$ such that each has its values in $[-1/3, 1/3]$ and for every m 6.5.8 holds. Then let

$$g(x) \equiv \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(x).$$

It follows

$$|g(x)| \leq \left| \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} g_i(x) \right| \leq \sum_{i=1}^m \left(\frac{2}{3}\right)^{i-1} \frac{1}{3} \leq 1$$

and since convergence is uniform, g must be continuous. The estimate 6.5.8 implies $f = g$ on M .

The following is the Tietze extension theorem.

Theorem 6.5.7 *Let M be a closed nonempty subset of a metric space (X, d) and let $f : M \rightarrow [a, b]$ be continuous at every point of M . Then there exists a function, g continuous on all of X which coincides with f on M such that $g(X) \subseteq [a, b]$.*

Proof: Let $f_1(x) = 1 + \frac{2}{b-a}(f(x) - b)$. Then f_1 satisfies the conditions of Lemma 6.5.6 and so there exists $g_1 : X \rightarrow [-1, 1]$ such that g is continuous on X and equals f_1 on M . Let $g(x) = (g_1(x) - 1)\left(\frac{b-a}{2}\right) + b$. This works.

6.6 General Topological Spaces

It turns out that metric spaces are not sufficiently general for some applications. This section is a brief introduction to general topology. In making this generalization, the properties of balls which are the conclusion of Theorem 6.1.4 on Page 140 are stated as axioms for a subset of the power set of a given set which will be known as a basis for the topology. More can be found in [45] and the references listed there.

Definition 6.6.1 *Let X be a nonempty set and suppose $\mathcal{B} \subseteq \mathcal{P}(X)$. Then $\mathcal{B}$ is a basis for a topology if it satisfies the following axioms.*

1.) *Whenever $p \in A \cap B$ for $A, B \in \mathcal{B}$, it follows there exists $C \in \mathcal{B}$ such that $p \in C \subseteq A \cap B$.*

2.) $\cup \mathcal{B} = X$.

Then a subset, U , of X is an open set if for every point, $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Thus the open sets are exactly those which can be obtained as a union of sets of $\mathcal{B}$. Denote these subsets of X by the symbol τ and refer to τ as the topology or the set of open sets.

Note that this is simply the analog of saying a set is open exactly when every point is an interior point.

Proposition 6.6.2 *Let X be a set and let $\mathcal{B}$ be a basis for a topology as defined above and let τ be the set of open sets determined by $\mathcal{B}$. Then*

$$\emptyset \in \tau, X \in \tau, \quad (6.6.9)$$

$$\text{If } \mathcal{C} \subseteq \tau, \text{ then } \cup \mathcal{C} \in \tau \quad (6.6.10)$$

$$\text{If } A, B \in \tau, \text{ then } A \cap B \in \tau. \quad (6.6.11)$$

Proof: If $p \in \emptyset$ then there exists $B \in \mathcal{B}$ such that $p \in B \subseteq \emptyset$ because there are no points in $\emptyset$. Therefore, $\emptyset \in \tau$. Now if $p \in X$, then by part 2.) of Definition 6.6.1 $p \in B \subseteq X$ for some $B \in \mathcal{B}$ and so $X \in \tau$.

If $\mathcal{C} \subseteq \tau$, and if $p \in \cup \mathcal{C}$, then there exists a set, $B \in \mathcal{C}$ such that $p \in B$. However, B is itself a union of sets from $\mathcal{B}$ and so there exists $C \in \mathcal{B}$ such that $p \in C \subseteq B \subseteq \cup \mathcal{C}$. This verifies 6.6.10.

Finally, if $A, B \in \tau$ and $p \in A \cap B$, then since A and B are themselves unions of sets of $\mathcal{B}$, it follows there exists $A_1, B_1 \in \mathcal{B}$ such that $A_1 \subseteq A, B_1 \subseteq B$, and

$p \in A_1 \cap B_1$. Therefore, by 1.) of Definition 6.6.1 there exists $C \in \mathcal{B}$ such that $p \in C \subseteq A_1 \cap B_1 \subseteq A \cap B$, showing that $A \cap B \in \tau$ as claimed. Of course if $A \cap B = \emptyset$, then $A \cap B \in \tau$. This proves the proposition.

Definition 6.6.3 A set X together with such a collection of its subsets satisfying 6.6.9-6.6.11 is called a topological space. τ is called the topology or set of open sets of X .

Definition 6.6.4 A topological space is said to be Hausdorff if whenever p and q are distinct points of X , there exist disjoint open sets U, V such that $p \in U$, $q \in V$. In other words points can be separated with open sets.



Definition 6.6.5 A subset of a topological space is said to be closed if its complement is open. Let p be a point of X and let $E \subseteq X$. Then p is said to be a limit point of E if every open set containing p contains a point of E distinct from p .

Note that if the topological space is Hausdorff, then this definition is equivalent to requiring that every open set containing p contains infinitely many points from E . Why?

Theorem 6.6.6 A subset, E , of X is closed if and only if it contains all its limit points.

Proof: Suppose first that E is closed and let x be a limit point of E . Is $x \in E$? If $x \notin E$, then E^C is an open set containing x which contains no points of E , a contradiction. Thus $x \in E$.

Now suppose E contains all its limit points. Is the complement of E open? If $x \in E^C$, then x is not a limit point of E because E has all its limit points and so there exists an open set, U containing x such that U contains no point of E other than x . Since $x \notin E$, it follows that $x \in U \subseteq E^C$ which implies E^C is an open set because this shows E^C is the union of open sets.

Theorem 6.6.7 If (X, τ) is a Hausdorff space and if $p \in X$, then $\{p\}$ is a closed set.

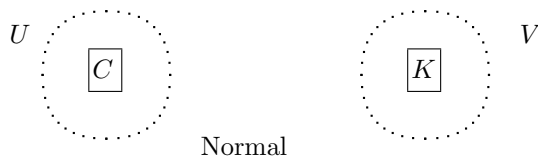
Proof: If $x \neq p$, there exist open sets U and V such that $x \in U$, $p \in V$ and $U \cap V = \emptyset$. Therefore, $\{p\}^C$ is an open set so $\{p\}$ is closed.

Note that the Hausdorff axiom was stronger than needed in order to draw the conclusion of the last theorem. In fact it would have been enough to assume that if $x \neq y$, then there exists an open set containing x which does not intersect y .

Definition 6.6.8 A topological space (X, τ) is said to be regular if whenever C is a closed set and p is a point not in C , there exist disjoint open sets U and V such that $p \in U$, $C \subseteq V$. Thus a closed set can be separated from a point not in the closed set by two disjoint open sets.



Definition 6.6.9 The topological space, (X, τ) is said to be normal if whenever C and K are disjoint closed sets, there exist disjoint open sets U and V such that $C \subseteq U$, $K \subseteq V$. Thus any two disjoint closed sets can be separated with open sets.



Definition 6.6.10 Let E be a subset of X . $\overline{E}$ is defined to be the smallest closed set containing E .

Lemma 6.6.11 The above definition is well defined.

Proof: Let $\mathcal{C}$ denote all the closed sets which contain E . Then $\mathcal{C}$ is nonempty because $X \in \mathcal{C}$.

$$(\cap \{A : A \in \mathcal{C}\})^C = \cup \{A^C : A \in \mathcal{C}\},$$

an open set which shows that $\cap \mathcal{C}$ is a closed set and is the smallest closed set which contains E .

Theorem 6.6.12 $\overline{E} = E \cup \{\text{limit points of } E\}$.

Proof: Let $x \in \overline{E}$ and suppose that $x \notin E$. If x is not a limit point either, then there exists an open set, U , containing x which does not intersect E . But then U^C is a closed set which contains E which does not contain x , contrary to the definition that $\overline{E}$ is the intersection of all closed sets containing E . Therefore, x must be a limit point of E after all.

Now $E \subseteq \overline{E}$ so suppose x is a limit point of E . Is $x \in \overline{E}$? If H is a closed set containing E , which does not contain x , then H^C is an open set containing x which contains no points of E other than x negating the assumption that x is a limit point of E .

The following is the definition of continuity in terms of general topological spaces. It is really just a generalization of the ε - δ definition of continuity given in calculus.

Definition 6.6.13 Let (X, τ) and (Y, η) be two topological spaces and let $f : X \rightarrow Y$. f is continuous at $x \in X$ if whenever V is an open set of Y containing $f(x)$, there exists an open set $U \in \tau$ such that $x \in U$ and $f(U) \subseteq V$. f is continuous if $f^{-1}(V) \in \tau$ whenever $V \in \eta$.

You should prove the following.

Proposition 6.6.14 In the situation of Definition 6.6.13 f is continuous if and only if f is continuous at every point of X .

Definition 6.6.15 Let (X_i, τ_i) be topological spaces. $\prod_{i=1}^n X_i$ is the Cartesian product. Define a product topology as follows. Let $\mathcal{B} = \prod_{i=1}^n A_i$ where $A_i \in \tau_i$. Then $\mathcal{B}$ is a basis for the product topology.

Theorem 6.6.16 The set $\mathcal{B}$ of Definition 6.6.15 is a basis for a topology.

Proof: Suppose $\mathbf{x} \in \prod_{i=1}^n A_i \cap \prod_{i=1}^n B_i$ where A_i and B_i are open sets. Say

$$\mathbf{x} = (x_1, \dots, x_n).$$

Then $x_i \in A_i \cap B_i$ for each i . Therefore, $\mathbf{x} \in \prod_{i=1}^n A_i \cap B_i \in \mathcal{B}$ and $\prod_{i=1}^n A_i \cap B_i \subseteq \prod_{i=1}^n A_i$.

The definition of compactness is also considered for a general topological space. This is given next.

Definition 6.6.17 A subset, E , of a topological space (X, τ) is said to be compact if whenever $\mathcal{C} \subseteq \tau$ and $E \subseteq \cup \mathcal{C}$, there exists a finite subset of $\mathcal{C}$, $\{U_1 \cdots U_n\}$, such that $E \subseteq \cup_{i=1}^n U_i$. (Every open covering admits a finite subcovering.) E is precompact if $\overline{E}$ is compact. A topological space is called locally compact if it has a basis $\mathcal{B}$, with the property that $\overline{B}$ is compact for each $B \in \mathcal{B}$.

In general topological spaces there may be no concept of “bounded”. Even if there is, closed and bounded is not necessarily the same as compactness. However, in any Hausdorff space every compact set must be a closed set.

Theorem 6.6.18 If (X, τ) is a Hausdorff space, then every compact subset must also be a closed set.

Proof: Suppose $p \notin K$. For each $x \in K$, there exist open sets, U_x and V_x such that

$$x \in U_x, \quad p \in V_x,$$

and

$$U_x \cap V_x = \emptyset.$$

If K is assumed to be compact, there are finitely many of these sets, $U_{x_1}, \dots, U_{x_m}$ which cover K . Then let $V \equiv \cap_{i=1}^m V_{x_i}$. It follows that V is an open set containing p which has empty intersection with each of the U_{x_i} . Consequently, V contains no points of K and is therefore not a limit point of K . This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space.

Definition 6.6.19 Suppose (X, τ) is a locally compact Hausdorff space. Then let $\tilde{X} \equiv X \cup \{\infty\}$ where ∞ is just the name of some point which is not in X which is called the point at infinity. A basis for the topology $\tilde{\tau}$ for $\tilde{X}$ is

$$\tau \cup \{K^C \text{ where } K \text{ is a compact subset of } X\}.$$

The complement is taken with respect to $\tilde{X}$ and so the open sets, K^C are basic open sets which contain ∞ .

The reason this is called a compactification is contained in the next lemma.

Lemma 6.6.20 If (X, τ) is a locally compact Hausdorff space, then $(\tilde{X}, \tilde{\tau})$ is a compact Hausdorff space. Also if U is an open set of $\tilde{\tau}$, then $U \setminus \{\infty\}$ is an open set of τ .

Proof: Since (X, τ) is a locally compact Hausdorff space, it follows $(\tilde{X}, \tilde{\tau})$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and ∞ . Since (X, τ) is locally compact, there exists an open set of τ , U having compact closure which contains p . Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, p and ∞ respectively. Now let $\mathcal{C}$ be an open cover of $\tilde{X}$ with sets from $\tilde{\tau}$. Then ∞ must be in some set, U_∞ from $\mathcal{C}$, which must contain a set of the form K^C where K is a compact subset of X . Then there exist sets from $\mathcal{C}$, $U_1, \dots, U_r$ which cover K . Therefore, a finite subcover of $\tilde{X}$ is $U_1, \dots, U_r, U_\infty$.

To see the last claim, suppose U contains ∞ since otherwise there is nothing to show. Notice that if C is a compact set, then $X \setminus C$ is an open set. Therefore, if $x \in U \setminus \{\infty\}$, and if $\tilde{X} \setminus C$ is a basic open set contained in U containing ∞ , then if x is in this basic open set of $\tilde{X}$, it is also in the open set $X \setminus C \subseteq U \setminus \{\infty\}$. If x is not in any basic open set of the form $\tilde{X} \setminus C$ then x is contained in an open set of τ which is contained in $U \setminus \{\infty\}$. Thus $U \setminus \{\infty\}$ is indeed open in τ .

Definition 6.6.21 If every finite subset of a collection of sets has nonempty intersection, the collection has the finite intersection property.

Theorem 6.6.22 Let $\mathcal{K}$ be a set whose elements are compact subsets of a Hausdorff topological space, (X, τ) . Suppose $\mathcal{K}$ has the finite intersection property. Then $\emptyset \neq \cap \mathcal{K}$.

Proof: Suppose to the contrary that $\emptyset = \cap \mathcal{K}$. Then consider

$$\mathcal{C} \equiv \{K^C : K \in \mathcal{K}\}.$$

It follows $\mathcal{C}$ is an open cover of K_0 where K_0 is any particular element of $\mathcal{K}$. But then there are finitely many $K \in \mathcal{K}$, $K_1, \dots, K_r$ such that $K_0 \subseteq \cup_{i=1}^r K_i^C$ implying that $\cap_{i=0}^r K_i = \emptyset$, contradicting the finite intersection property.

Lemma 6.6.23 *Let (X, τ) be a topological space and let $\mathcal{B}$ be a basis for τ . Then X is compact if and only if every open cover of basic open sets admits a finite subcover.*

Proof: Suppose first that X is compact. Then if $\mathcal{C}$ is an open cover consisting of basic open sets, it follows it admits a finite subcover because these are open sets in $\mathcal{C}$.

Next suppose that every basic open cover admits a finite subcover and let $\mathcal{C}$ be an open cover of X . Then define $\tilde{\mathcal{C}}$ to be the collection of basic open sets which are contained in some set of $\mathcal{C}$. It follows $\tilde{\mathcal{C}}$ is a basic open cover of X and so it admits a finite subcover, $\{U_1, \dots, U_p\}$. Now each U_i is contained in an open set of $\mathcal{C}$. Let O_i be a set of $\mathcal{C}$ which contains U_i . Then $\{O_1, \dots, O_p\}$ is an open cover of X . This proves the lemma.

In fact, much more can be said than Lemma 6.6.23. However, this is all which I will present here.

6.7 Connected Sets

Stated informally, connected sets are those which are in one piece. More precisely,

Definition 6.7.1 *A set, S in a general topological space is separated if there exist sets, A, B such that*

$$S = A \cup B, \quad A, B \neq \emptyset, \quad \text{and} \quad \bar{A} \cap B = \bar{B} \cap A = \emptyset.$$

In this case, the sets A and B are said to separate S . A set is connected if it is not separated.

One of the most important theorems about connected sets is the following.

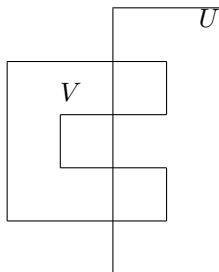
Theorem 6.7.2 *Suppose U and V are connected sets having nonempty intersection. Then $U \cup V$ is also connected.*

Proof: Suppose $U \cup V = A \cup B$ where $\bar{A} \cap B = \bar{B} \cap A = \emptyset$. Consider the sets, $A \cap U$ and $B \cap U$. Since

$$\overline{(A \cap U)} \cap (B \cap U) = (A \cap U) \cap \overline{(B \cap U)} = \emptyset,$$

It follows one of these sets must be empty since otherwise, U would be separated. It follows that U is contained in either A or B . Similarly, V must be contained in either A or B . Since U and V have nonempty intersection, it follows that both V and U are contained in one of the sets, A, B . Therefore, the other must be empty and this shows $U \cup V$ cannot be separated and is therefore, connected.

The intersection of connected sets is not necessarily connected as is shown by the following picture.



Theorem 6.7.3 *Let $f : X \rightarrow Y$ be continuous where X and Y are topological spaces and X is connected. Then $f(X)$ is also connected.*

Proof: To do this you show $f(X)$ is not separated. Suppose to the contrary that $f(X) = A \cup B$ where A and B separate $f(X)$. Then consider the sets, $f^{-1}(A)$ and $f^{-1}(B)$. If $z \in f^{-1}(B)$, then $f(z) \in B$ and so $f(z)$ is not a limit point of A . Therefore, there exists an open set, U containing $f(z)$ such that $U \cap A = \emptyset$. But then, the continuity of f implies that $f^{-1}(U)$ is an open set containing z such that $f^{-1}(U) \cap f^{-1}(A) = \emptyset$. Therefore, $f^{-1}(B)$ contains no limit points of $f^{-1}(A)$. Similar reasoning implies $f^{-1}(A)$ contains no limit points of $f^{-1}(B)$. It follows that X is separated by $f^{-1}(A)$ and $f^{-1}(B)$, contradicting the assumption that X was connected.

An arbitrary set can be written as a union of maximal connected sets called connected components. This is the concept of the next definition.

Definition 6.7.4 *Let S be a set and let $p \in S$. Denote by C_p the union of all connected subsets of S which contain p . This is called the connected component determined by p .*

Theorem 6.7.5 *Let C_p be a connected component of a set S in a general topological space. Then C_p is a connected set and if $C_p \cap C_q \neq \emptyset$, then $C_p = C_q$.*

Proof: Let $\mathcal{C}$ denote the connected subsets of S which contain p . If $C_p = A \cup B$ where

$$\overline{A} \cap B = \overline{B} \cap A = \emptyset,$$

then p is in one of A or B . Suppose without loss of generality $p \in A$. Then every set of $\mathcal{C}$ must also be contained in A also since otherwise, as in Theorem 6.7.2, the set would be separated. But this implies B is empty. Therefore, C_p is connected. From this, and Theorem 6.7.2, the second assertion of the theorem is proved.

This shows the connected components of a set are equivalence classes and partition the set.

A set, I is an interval in $\mathbb{R}$ if and only if whenever $x, y \in I$ then $(x, y) \subseteq I$. The following theorem is about the connected sets in $\mathbb{R}$.

Theorem 6.7.6 *A set, C in $\mathbb{R}$ is connected if and only if C is an interval.*

Proof: Let C be connected. If C consists of a single point, p , there is nothing to prove. The interval is just $[p, p]$. Suppose $p < q$ and $p, q \in C$. You need to show $(p, q) \subseteq C$. If

$$x \in (p, q) \setminus C$$

let $C \cap (-\infty, x) \equiv A$, and $C \cap (x, \infty) \equiv B$. Then $C = A \cup B$ and the sets, A and B separate C contrary to the assumption that C is connected.

Conversely, let I be an interval. Suppose I is separated by A and B . Pick $x \in A$ and $y \in B$. Suppose without loss of generality that $x < y$. Now define the set,

$$S \equiv \{t \in [x, y] : [x, t] \subseteq A\}$$

and let l be the least upper bound of S . Then $l \in \overline{A}$ so $l \notin B$ which implies $l \in A$. But if $l \notin \overline{B}$, then for some $\delta > 0$,

$$(l, l + \delta) \cap B = \emptyset$$

contradicting the definition of l as an upper bound for S . Therefore, $l \in \overline{B}$ which implies $l \notin A$ after all, a contradiction. It follows I must be connected.

The following theorem is a very useful description of the open sets in $\mathbb{R}$.

Theorem 6.7.7 *Let U be an open set in $\mathbb{R}$. Then there exist countably many disjoint open sets, $\{(a_i, b_i)\}_{i=1}^{\infty}$ such that $U = \cup_{i=1}^{\infty} (a_i, b_i)$.*

Proof: Let $p \in U$ and let $z \in C_p$, the connected component determined by p . Since U is open, there exists, $\delta > 0$ such that $(z - \delta, z + \delta) \subseteq U$. It follows from Theorem 6.7.2 that

$$(z - \delta, z + \delta) \subseteq C_p.$$

This shows C_p is open. By Theorem 6.7.6, this shows C_p is an open interval, (a, b) where $a, b \in [-\infty, \infty]$. There are therefore at most countably many of these connected components because each must contain a rational number and the rational numbers are countable. Denote by $\{(a_i, b_i)\}_{i=1}^{\infty}$ the set of these connected components. This proves the theorem.

Definition 6.7.8 *A topological space, E is arcwise connected if for any two points, $p, q \in E$, there exists a closed interval, $[a, b]$ and a continuous function, $\gamma : [a, b] \rightarrow E$ such that $\gamma(a) = p$ and $\gamma(b) = q$. E is locally connected if it has a basis of connected open sets. E is locally arcwise connected if it has a basis of arcwise connected open sets.*

An example of an arcwise connected topological space would be the any subset of $\mathbb{R}^n$ which is the continuous image of an interval. Locally connected is not the same as connected. A well known example is the following.

$$\left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0, 1] \right\} \cup \{ (0, y) : y \in [-1, 1] \} \quad (6.7.12)$$

You can verify that this set of points considered as a metric space with the metric from $\mathbb{R}^2$ is not locally connected or arcwise connected but is connected.

Proposition 6.7.9 *If a topological space is arcwise connected, then it is connected.*

Proof: Let X be an arcwise connected space and suppose it is separated. Then $X = A \cup B$ where A, B are two separated sets. Pick $p \in A$ and $q \in B$. Since X is given to be arcwise connected, there must exist a continuous function $\gamma : [a, b] \rightarrow X$ such that $\gamma(a) = p$ and $\gamma(b) = q$. But then we would have $\gamma([a, b]) = (\gamma([a, b]) \cap A) \cup (\gamma([a, b]) \cap B)$ and the two sets, $\gamma([a, b]) \cap A$ and $\gamma([a, b]) \cap B$ are separated thus showing that $\gamma([a, b])$ is separated and contradicting Theorem 6.7.6 and Theorem 6.7.3. It follows that X must be connected as claimed.

Theorem 6.7.10 *Let U be an open subset of a locally arcwise connected topological space, X . Then U is arcwise connected if and only if U is connected. Also the connected components of an open set in such a space are open sets, hence arcwise connected.*

Proof: By Proposition 6.7.9 it is only necessary to verify that if U is connected and open in the context of this theorem, then U is arcwise connected. Pick $p \in U$. Say $x \in U$ satisfies $\mathcal{P}$ if there exists a continuous function, $\gamma : [a, b] \rightarrow U$ such that $\gamma(a) = p$ and $\gamma(b) = x$.

$$A \equiv \{x \in U \text{ such that } x \text{ satisfies } \mathcal{P}\}$$

If $x \in A$, there exists, according to the assumption that X is locally arcwise connected, an open set, V , containing x and contained in U which is arcwise connected. Thus letting $y \in V$, there exist intervals, $[a, b]$ and $[c, d]$ and continuous functions having values in U , γ, η such that $\gamma(a) = p, \gamma(b) = x, \eta(c) = x$, and $\eta(d) = y$. Then let $\gamma_1 : [a, b + d - c] \rightarrow U$ be defined as

$$\gamma_1(t) \equiv \begin{cases} \gamma(t) & \text{if } t \in [a, b] \\ \eta(t + c - b) & \text{if } t \in [b, b + d - c] \end{cases}$$

Then it is clear that γ_1 is a continuous function mapping p to y and showing that $V \subseteq A$. Therefore, A is open. $A \neq \emptyset$ because there is an open set, V containing p which is contained in U and is arcwise connected.

Now consider $B \equiv U \setminus A$. This is also open. If B is not open, there exists a point $z \in B$ such that every open set containing z is not contained in B . Therefore, letting V be one of the basic open sets chosen such that $z \in V \subseteq U$, there exist

points of A contained in V . But then, a repeat of the above argument shows $z \in A$ also. Hence B is open and so if $B \neq \emptyset$, then $U = B \cup A$ and so U is separated by the two sets, B and A contradicting the assumption that U is connected.

It remains to verify the connected components are open. Let $z \in C_p$ where C_p is the connected component determined by p . Then picking V an arcwise connected open set which contains z and is contained in U , $C_p \cup V$ is connected and contained in U and so it must also be contained in C_p . This proves the theorem.

As an application, consider the following corollary.

Corollary 6.7.11 *Let $f : \Omega \rightarrow \mathbb{Z}$ be continuous where Ω is a connected open set. Then f must be a constant.*

Proof: Suppose not. Then it achieves two different values, k and $l \neq k$. Then $\Omega = f^{-1}(l) \cup f^{-1}(\{m \in \mathbb{Z} : m \neq l\})$ and these are disjoint nonempty open sets which separate Ω . To see they are open, note

$$f^{-1}(\{m \in \mathbb{Z} : m \neq l\}) = f^{-1}\left(\bigcup_{m \neq l} \left(m - \frac{1}{6}, m + \frac{1}{6}\right)\right)$$

which is the inverse image of an open set.

Weierstrass Approximation Theorem

7.1 The Bernstein Polynomials

This short chapter is on the important Weierstrass approximation theorem. It is about approximating an arbitrary continuous function uniformly by a polynomial. It will be assumed only that f has values in $\mathbb{C}$ and that all scalars are in $\mathbb{C}$. First here is some notation.

Definition 7.1.1 $\alpha = (\alpha_1, \dots, \alpha_n)$ for $\alpha_1 \dots \alpha_n$ positive integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \dots + \alpha_n$ and if $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = (x_1, \dots, x_n),$$

and f a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

A polynomial in n variables of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} a_\alpha \mathbf{x}^\alpha.$$

Here α is a multi-index as just described.

The following estimate will be the basis for the Weierstrass approximation theorem. It is actually a statement about the variance of a binomial random variable.

Lemma 7.1.2 The following estimate holds for $x \in [0, 1]$.

$$\sum_{k=0}^m \binom{m}{k} (k - mx)^2 x^k (1 - x)^{m-k} \leq \frac{1}{4}m$$

Proof: By the Binomial theorem,

$$\sum_{k=0}^m \binom{m}{k} (e^t x)^k (1-x)^{m-k} = (1-x+e^t x)^m. \quad (7.1.1)$$

Differentiating both sides with respect to t and then evaluating at $t = 0$ yields

$$\sum_{k=0}^m \binom{m}{k} k x^k (1-x)^{m-k} = m x.$$

Now doing two derivatives of 7.1.1 with respect to t yields

$$\begin{aligned} \sum_{k=0}^m \binom{m}{k} k^2 (e^t x)^k (1-x)^{m-k} &= m(m-1)(1-x+e^t x)^{m-2} e^{2t} x^2 \\ &\quad + m(1-x+e^t x)^{m-1} x e^t. \end{aligned}$$

Evaluating this at $t = 0$,

$$\sum_{k=0}^m \binom{m}{k} k^2 x^k (1-x)^{m-k} = m(m-1)x^2 + m x.$$

Therefore,

$$\begin{aligned} \sum_{k=0}^m \binom{m}{k} (k-mx)^2 x^k (1-x)^{m-k} &= m(m-1)x^2 + m x - 2m^2 x^2 + m^2 x^2 \\ &= m(x-x^2) \leq \frac{1}{4}m. \end{aligned}$$

This proves the lemma.

Now for $\mathbf{x} = (x_1, \dots, x_n) \in [0, 1]^n$ consider the polynomial,

$$\begin{aligned} p_m(\mathbf{x}) &\equiv \sum_{k_1=1}^m \cdots \sum_{k_n=1}^m \binom{m}{k_1} \binom{m}{k_2} \cdots \binom{m}{k_n} x_1^{k_1} (1-x_1)^{m-k_1} x_2^{k_2} (1-x_2)^{m-k_2} \\ &\quad \cdots x_n^{k_n} (1-x_n)^{m-k_n} f\left(\frac{k_1}{m}, \dots, \frac{k_n}{m}\right). \end{aligned} \quad (7.1.2)$$

Also define if I is a set in $\mathbb{R}^n$

$$\|h\|_I \equiv \sup \{|h(\mathbf{x})| : \mathbf{x} \in I\}.$$

Thus p_m converges uniformly to f on a set, I if

$$\lim_{m \rightarrow \infty} \|p_m - f\|_I = 0.$$

Also to simplify the notation, let $\mathbf{k} = (k_1, \dots, k_n)$ where each $k_i \in [0, m]$, $\frac{\mathbf{k}}{m} \equiv \left(\frac{k_1}{m}, \dots, \frac{k_n}{m}\right)$, and let

$$\binom{\mathbf{m}}{\mathbf{k}} \equiv \binom{m}{k_1} \binom{m}{k_2} \cdots \binom{m}{k_n}.$$

Also define

$$\|\mathbf{k}\|_\infty \equiv \max \{k_i, i = 1, 2, \dots, n\}$$

$$\mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} \equiv x_1^{k_1} (1 - x_1)^{m - k_1} x_2^{k_2} (1 - x_2)^{m - k_2} \dots x_n^{k_n} (1 - x_n)^{m - k_n}.$$

Thus in terms of this notation,

$$p_m(\mathbf{x}) = \sum_{\|\mathbf{k}\|_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} f\left(\frac{\mathbf{k}}{\mathbf{m}}\right)$$

Lemma 7.1.3 For $\mathbf{x} \in [0, 1]^n$, f a continuous function defined on $[0, 1]^n$, and p_m given in 7.1.2, p_m converges uniformly to f on $[0, 1]^n$ as $m \rightarrow \infty$.

Proof: The function, f is uniformly continuous because it is continuous on a compact set. Therefore, there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, then

$$|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon.$$

Denote by G the set of $\mathbf{k}$ such that $(k_i - mx_i)^2 < \eta^2 m^2$ for each i where $\eta = \delta/\sqrt{n}$. Note this condition is equivalent to saying that for each i , $|\frac{k_i}{m} - x_i| < \eta$. By the binomial theorem,

$$\sum_{\|\mathbf{k}\|_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} = 1$$

and so for $\mathbf{x} \in [0, 1]^n$,

$$\begin{aligned} |p_m(\mathbf{x}) - f(\mathbf{x})| &\leq \sum_{\|\mathbf{k}\|_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x}) \right| \\ &\leq \sum_{\mathbf{k} \in G} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x}) \right| \\ &\quad + \sum_{\mathbf{k} \in G^c} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} \left| f\left(\frac{\mathbf{k}}{\mathbf{m}}\right) - f(\mathbf{x}) \right| \end{aligned} \quad (7.1.3)$$

Now for $\mathbf{k} \in G$ it follows that for each i

$$\left| \frac{k_i}{m} - x_i \right| < \frac{\delta}{\sqrt{n}} \quad (7.1.4)$$

and so $|f(\frac{\mathbf{k}}{\mathbf{m}}) - f(\mathbf{x})| < \varepsilon$ because the above implies $|\frac{\mathbf{k}}{\mathbf{m}} - \mathbf{x}| < \delta$. Therefore, the first sum on the right in 7.1.3 is no larger than

$$\sum_{\mathbf{k} \in G} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} \varepsilon \leq \sum_{\|\mathbf{k}\|_\infty \leq m} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m} - \mathbf{k}} \varepsilon = \varepsilon.$$

Letting $M \geq \max \{|f(\mathbf{x})| : \mathbf{x} \in [0, 1]^n\}$ it follows

$$\begin{aligned}
 & |p_m(\mathbf{x}) - f(\mathbf{x})| \\
 & \leq \varepsilon + 2M \sum_{\mathbf{k} \in G^C} \binom{\mathbf{m}}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m}-\mathbf{k}} \\
 & \leq \varepsilon + 2M \left(\frac{1}{\eta^2 m^2} \right)^n \sum_{\mathbf{k} \in G^C} \binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^n (k_j - mx_j)^2 \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m}-\mathbf{k}} \\
 & \leq \varepsilon + 2M \left(\frac{1}{\eta^2 m^2} \right)^n \sum_{\|\mathbf{k}\|_{\infty} \leq m} \binom{\mathbf{m}}{\mathbf{k}} \prod_{j=1}^n (k_j - mx_j)^2 \mathbf{x}^{\mathbf{k}} (1 - \mathbf{x})^{\mathbf{m}-\mathbf{k}}
 \end{aligned}$$

because on G^C ,

$$\frac{(k_j - mx_j)^2}{\eta^2 m^2} < 1, \quad j = 1, \dots, n.$$

Now by Lemma 7.1.2,

$$|p_m(\mathbf{x}) - f(\mathbf{x})| \leq \varepsilon + 2M \left(\frac{1}{\eta^2 m^2} \right)^n \left(\frac{m}{4} \right)^n.$$

Therefore, since the right side does not depend on $\mathbf{x}$, it follows

$$\limsup_{m \rightarrow \infty} \|p_m - f\|_{[0,1]^n} \leq \varepsilon$$

and since ε is arbitrary, this shows $\lim_{m \rightarrow \infty} \|p_m - f\|_{[0,1]^n} = 0$. This proves the lemma.

The following is not surprising.

Lemma 7.1.4 *Let f be a continuous function defined on $[-M, M]^n$. Then there exists a sequence of polynomials, $\{p_m\}$ converging uniformly to f on $[-M, M]^n$.*

Proof: Let $h(t) = -M + 2Mt$ so $h : [0, 1] \rightarrow [-M, M]$ and let $\mathbf{h}(\mathbf{t}) \equiv (h(t_1), \dots, h(t_n))$. Therefore, $f \circ \mathbf{h}$ is a continuous function defined on $[0, 1]^n$. From Lemma 7.1.3 there exists a polynomial, $p(\mathbf{t})$ such that $\|p_m - f \circ \mathbf{h}\|_{[0,1]^n} < \frac{1}{m}$. Now for $\mathbf{x} \in [-M, M]^n$, $\mathbf{h}^{-1}(\mathbf{x}) = (h^{-1}(x_1), \dots, h^{-1}(x_n))$ and so

$$\|p_m \circ \mathbf{h}^{-1} - f\|_{[-M,M]^n} = \|p_m - f \circ \mathbf{h}\|_{[0,1]^n} < \frac{1}{m}.$$

But $h^{-1}(x) = \frac{x}{2M} + \frac{1}{2}$ and so p_m is still a polynomial. This proves the lemma.

The classical version of the Weierstrass approximation theorem involved showing that a continuous function of one variable defined on a closed and bounded interval is the uniform limit of a sequence of polynomials. This is certainly included as a special case of the above. Now recall the Tietze extension theorem found on Page 153. In the general version about to be presented, the set on which f is defined is just a compact subset of $\mathbb{R}^n$, not the Cartesian product of intervals. For convenience here is the Tietze extension theorem.

Theorem 7.1.5 *Let M be a closed nonempty subset of a metric space (X, d) and let $f : M \rightarrow [a, b]$ be continuous at every point of M . Then there exists a function, g continuous on all of X which coincides with f on M such that $g(X) \subseteq [a, b]$.*

The Weierstrass approximation theorem follows.

Theorem 7.1.6 *Let K be a compact set in $\mathbb{R}^n$ and let f be a continuous function defined on K . Then there exists a sequence of polynomials $\{p_m\}$ converging uniformly to f on K .*

Proof: Choose M large enough that $K \subseteq [-M, M]^n$ and let $\tilde{f}$ denote a continuous function defined on all of $[-M, M]^n$ such that $\tilde{f} = f$ on K . Such an extension exists by the Tietze extension theorem, Theorem 7.1.5 applied to the real and imaginary parts of f . By Lemma 7.1.4 there exists a sequence of polynomials, $\{p_m\}$ defined on $[-M, M]^n$ such that $\|\tilde{f} - p_m\|_{[-M, M]^n} \rightarrow 0$. Therefore, $\|\tilde{f} - p_m\|_K \rightarrow 0$ also. This proves the theorem.

7.2 Stone Weierstrass Theorem

7.2.1 The Case Of Compact Sets

There is a profound generalization of the Weierstrass approximation theorem due to Stone.

Definition 7.2.1 *$\mathcal{A}$ is an algebra of functions if $\mathcal{A}$ is a vector space and if whenever $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$.*

To begin with assume that the field of scalars is $\mathbb{R}$. This will be generalized later. Theorem 7.1.6 implies the following very special case.

Corollary 7.2.2 *The polynomials are dense in $C([a, b])$.*

The next result is the key to the profound generalization of the Weierstrass theorem due to Stone in which an interval will be replaced by a compact or locally compact set and polynomials will be replaced with elements of an algebra satisfying certain axioms.

Corollary 7.2.3 *On the interval $[-M, M]$, there exist polynomials p_n such that*

$$p_n(0) = 0$$

and

$$\lim_{n \rightarrow \infty} \|p_n - |\cdot|\|_{\infty} = 0.$$

Proof: By Corollary 7.2.2 there exists a sequence of polynomials, $\{\tilde{p}_n\}$ such that $\tilde{p}_n \rightarrow |\cdot|$ uniformly. Then let $p_n(t) \equiv \tilde{p}_n(t) - \tilde{p}_n(0)$. This proves the corollary.

Definition 7.2.4 An algebra of functions, $\mathcal{A}$ defined on A , annihilates no point of A if for all $x \in A$, there exists $g \in \mathcal{A}$ such that $g(x) \neq 0$. The algebra separates points if whenever $x_1 \neq x_2$, then there exists $g \in \mathcal{A}$ such that $g(x_1) \neq g(x_2)$.

The following generalization is known as the Stone Weierstrass approximation theorem.

Theorem 7.2.5 Let A be a compact topological space and let $\mathcal{A} \subseteq C(A; \mathbb{R})$ be an algebra of functions which separates points and annihilates no point. Then $\mathcal{A}$ is dense in $C(A; \mathbb{R})$.

Proof: First here is a lemma.

Lemma 7.2.6 Let c_1 and c_2 be two real numbers and let $x_1 \neq x_2$ be two points of A . Then there exists a function $f_{x_1 x_2}$ such that

$$f_{x_1 x_2}(x_1) = c_1, f_{x_1 x_2}(x_2) = c_2.$$

Proof of the lemma: Let $g \in \mathcal{A}$ satisfy

$$g(x_1) \neq g(x_2).$$

Such a g exists because the algebra separates points. Since the algebra annihilates no point, there exist functions h and k such that

$$h(x_1) \neq 0, k(x_2) \neq 0.$$

Then let

$$u \equiv gh - g(x_2)h, v \equiv gk - g(x_1)k.$$

It follows that $u(x_1) \neq 0$ and $u(x_2) = 0$ while $v(x_2) \neq 0$ and $v(x_1) = 0$. Let

$$f_{x_1 x_2} \equiv \frac{c_1 u}{u(x_1)} + \frac{c_2 v}{v(x_2)}.$$

This proves the lemma. Now continue the proof of Theorem 7.2.5.

First note that $\overline{\mathcal{A}}$ satisfies the same axioms as $\mathcal{A}$ but in addition to these axioms, $\overline{\mathcal{A}}$ is closed. The closure of $\mathcal{A}$ is taken with respect to the usual norm on $C(A)$,

$$\|f\|_\infty \equiv \max \{|f(x)| : x \in A\}.$$

Suppose $f \in \overline{\mathcal{A}}$ and suppose M is large enough that

$$\|f\|_\infty < M.$$

Using Corollary 7.2.3, let p_n be a sequence of polynomials such that

$$\|p_n - \cdot\|_\infty \rightarrow 0, p_n(0) = 0.$$

It follows that $p_n \circ f \in \overline{\mathcal{A}}$ and so $|f| \in \overline{\mathcal{A}}$ whenever $f \in \overline{\mathcal{A}}$. Also note that

$$\begin{aligned}\max(f, g) &= \frac{|f - g| + (f + g)}{2} \\ \min(f, g) &= \frac{(f + g) - |f - g|}{2}.\end{aligned}$$

Therefore, this shows that if $f, g \in \overline{\mathcal{A}}$ then

$$\max(f, g), \min(f, g) \in \overline{\mathcal{A}}.$$

By induction, if $f_i, i = 1, 2, \dots, m$ are in $\overline{\mathcal{A}}$ then

$$\max(f_i, i = 1, 2, \dots, m), \min(f_i, i = 1, 2, \dots, m) \in \overline{\mathcal{A}}.$$

Now let $h \in C(A; \mathbb{R})$ and let $x \in A$. Use Lemma 7.2.6 to obtain f_{xy} , a function of $\overline{\mathcal{A}}$ which agrees with h at x and y . Letting $\varepsilon > 0$, there exists an open set $U(y)$ containing y such that

$$f_{xy}(z) > h(z) - \varepsilon \text{ if } z \in U(y).$$

Since A is compact, let $U(y_1), \dots, U(y_l)$ cover A . Let

$$f_x \equiv \max(f_{xy_1}, f_{xy_2}, \dots, f_{xy_l}).$$

Then $f_x \in \overline{\mathcal{A}}$ and

$$f_x(z) > h(z) - \varepsilon$$

for all $z \in A$ and $f_x(x) = h(x)$. This implies that for each $x \in A$ there exists an open set $V(x)$ containing x such that for $z \in V(x)$,

$$f_x(z) < h(z) + \varepsilon.$$

Let $V(x_1), \dots, V(x_m)$ cover A and let

$$f \equiv \min(f_{x_1}, \dots, f_{x_m}).$$

Therefore,

$$f(z) < h(z) + \varepsilon$$

for all $z \in A$ and since $f_x(z) > h(z) - \varepsilon$ for all $z \in A$, it follows

$$f(z) > h(z) - \varepsilon$$

also and so

$$|f(z) - h(z)| < \varepsilon$$

for all z . Since ε is arbitrary, this shows $h \in \overline{\mathcal{A}}$ and proves $\overline{\mathcal{A}} = C(A; \mathbb{R})$. This proves the theorem.

7.2.2 The Case Of Locally Compact Sets

Definition 7.2.7 Let (X, τ) be a locally compact Hausdorff space. $C_0(X)$ denotes the space of real or complex valued continuous functions defined on X with the property that if $f \in C_0(X)$, then for each $\varepsilon > 0$ there exists a compact set K such that $|f(x)| < \varepsilon$ for all $x \notin K$. Define

$$\|f\|_\infty = \sup \{|f(x)| : x \in X\}.$$

Lemma 7.2.8 For (X, τ) a locally compact Hausdorff space with the above norm, $C_0(X)$ is a complete space.

Proof: Let $(\tilde{X}, \tilde{\tau})$ be the one point compactification described in Lemma 6.6.20.

$$D \equiv \left\{ f \in C(\tilde{X}) : f(\infty) = 0 \right\}.$$

Then D is a closed subspace of $C(\tilde{X})$. For $f \in C_0(X)$,

$$\tilde{f}(x) \equiv \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}$$

and let $\theta : C_0(X) \rightarrow D$ be given by $\theta f = \tilde{f}$. Then θ is one to one and onto and also satisfies $\|f\|_\infty = \|\theta f\|_\infty$. Now D is complete because it is a closed subspace of a complete space and so $C_0(X)$ with $\|\cdot\|_\infty$ is also complete. This proves the lemma.

The above refers to functions which have values in $\mathbb{C}$ but the same proof works for functions which have values in any complete normed linear space.

In the case where the functions in $C_0(X)$ all have real values, I will denote the resulting space by $C_0(X; \mathbb{R})$ with similar meanings in other cases.

With this lemma, the generalization of the Stone Weierstrass theorem to locally compact sets is as follows.

Theorem 7.2.9 Let $\mathcal{A}$ be an algebra of functions in $C_0(X; \mathbb{R})$ where (X, τ) is a locally compact Hausdorff space which separates the points and annihilates no point. Then $\mathcal{A}$ is dense in $C_0(X; \mathbb{R})$.

Proof: Let $(\tilde{X}, \tilde{\tau})$ be the one point compactification as described in Lemma 6.6.20. Let $\tilde{\mathcal{A}}$ denote all finite linear combinations of the form

$$\left\{ \sum_{i=1}^n c_i \tilde{f}_i + c_0 : f \in \mathcal{A}, c_i \in \mathbb{R} \right\}$$

where for $f \in C_0(X; \mathbb{R})$,

$$\tilde{f}(x) \equiv \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}.$$

Then $\tilde{\mathcal{A}}$ is obviously an algebra of functions in $C(\tilde{X}; \mathbb{R})$. It separates points because this is true of $\mathcal{A}$. Similarly, it annihilates no point because of the inclusion of c_0 an arbitrary element of $\mathbb{R}$ in the definition above. Therefore from Theorem 7.2.5, $\tilde{\mathcal{A}}$ is dense in $C(\tilde{X}; \mathbb{R})$. Letting $f \in C_0(X; \mathbb{R})$, it follows $\tilde{f} \in C(\tilde{X}; \mathbb{R})$ and so there exists a sequence $\{h_n\} \subseteq \tilde{\mathcal{A}}$ such that h_n converges uniformly to $\tilde{f}$. Now h_n is of the form $\sum_{i=1}^n c_i^n \tilde{f}_i^n + c_0^n$ and since $\tilde{f}(\infty) = 0$, you can take each $c_0^n = 0$ and so this has shown the existence of a sequence of functions in $\mathcal{A}$ such that it converges uniformly to f . This proves the theorem.

7.2.3 The Case Of Complex Valued Functions

What about the general case where $C_0(X)$ consists of complex valued functions and the field of scalars is $\mathbb{C}$ rather than $\mathbb{R}$? The following is the version of the Stone Weierstrass theorem which applies to this case. You have to assume that for $f \in \mathcal{A}$ it follows $\bar{f} \in \mathcal{A}$. Such an algebra is called self adjoint.

Theorem 7.2.10 *Suppose $\mathcal{A}$ is an algebra of functions in $C_0(X)$, where X is a locally compact Hausdorff space, which separates the points, annihilates no point, and has the property that if $f \in \mathcal{A}$, then $\bar{f} \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C_0(X)$.*

Proof: Let $\text{Re } \mathcal{A} \equiv \{\text{Re } f : f \in \mathcal{A}\}$, $\text{Im } \mathcal{A} \equiv \{\text{Im } f : f \in \mathcal{A}\}$. First I will show that $\mathcal{A} = \text{Re } \mathcal{A} + i \text{Im } \mathcal{A} = \text{Im } \mathcal{A} + i \text{Re } \mathcal{A}$. Let $f \in \mathcal{A}$. Then

$$f = \frac{1}{2}(f + \bar{f}) + \frac{1}{2}(f - \bar{f}) = \text{Re } f + i \text{Im } f \in \text{Re } \mathcal{A} + i \text{Im } \mathcal{A}$$

and so $\mathcal{A} \subseteq \text{Re } \mathcal{A} + i \text{Im } \mathcal{A}$. Also

$$f = \frac{1}{2i}(if + i\bar{f}) - \frac{i}{2}(if + \overline{if}) = \text{Im}(if) + i \text{Re}(if) \in \text{Im } \mathcal{A} + i \text{Re } \mathcal{A}$$

This proves one half of the desired equality. Now suppose $h \in \text{Re } \mathcal{A} + i \text{Im } \mathcal{A}$. Then $h = \text{Re } g_1 + i \text{Im } g_2$ where $g_i \in \mathcal{A}$. Then since $\text{Re } g_1 = \frac{1}{2}(g_1 + \bar{g}_1)$, it follows $\text{Re } g_1 \in \mathcal{A}$. Similarly $\text{Im } g_2 \in \mathcal{A}$. Therefore, $h \in \mathcal{A}$. The case where $h \in \text{Im } \mathcal{A} + i \text{Re } \mathcal{A}$ is similar. This establishes the desired equality.

Now $\text{Re } \mathcal{A}$ and $\text{Im } \mathcal{A}$ are both real algebras. I will show this now. First consider $\text{Im } \mathcal{A}$. It is obvious this is a real vector space. It only remains to verify that the product of two functions in $\text{Im } \mathcal{A}$ is in $\text{Im } \mathcal{A}$. Note that from the first part, $\text{Re } \mathcal{A}, \text{Im } \mathcal{A}$ are both subsets of $\mathcal{A}$ because, for example, if $u \in \text{Im } \mathcal{A}$ then $u + 0 \in \text{Im } \mathcal{A} + i \text{Re } \mathcal{A} = \mathcal{A}$. Therefore, if $v, w \in \text{Im } \mathcal{A}$, both iv and w are in $\mathcal{A}$ and so $\text{Im}(ivw) = vw$ and $ivw \in \mathcal{A}$. Similarly, $\text{Re } \mathcal{A}$ is an algebra.

Both $\text{Re } \mathcal{A}$ and $\text{Im } \mathcal{A}$ must separate the points. Here is why: If $x_1 \neq x_2$, then there exists $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$. If $\text{Im } f(x_1) \neq \text{Im } f(x_2)$, this shows there is a function in $\text{Im } \mathcal{A}$, $\text{Im } f$ which separates these two points. If $\text{Im } f$ fails to separate the two points, then $\text{Re } f$ must separate the points and so you could consider $\text{Im}(if)$ to get a function in $\text{Im } \mathcal{A}$ which separates these points. This shows $\text{Im } \mathcal{A}$ separates the points. Similarly $\text{Re } \mathcal{A}$ separates the points.

Neither $\operatorname{Re} \mathcal{A}$ nor $\operatorname{Im} \mathcal{A}$ annihilate any point. This is easy to see because if x is a point there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$. Thus either $\operatorname{Re} f(x) \neq 0$ or $\operatorname{Im} f(x) \neq 0$. If $\operatorname{Im} f(x) \neq 0$, this shows this point is not annihilated by $\operatorname{Im} \mathcal{A}$. If $\operatorname{Im} f(x) = 0$, consider $\operatorname{Im}(if)(x) = \operatorname{Re} f(x) \neq 0$. Similarly, $\operatorname{Re} \mathcal{A}$ does not annihilate any point.

It follows from Theorem 7.2.9 that $\operatorname{Re} \mathcal{A}$ and $\operatorname{Im} \mathcal{A}$ are dense in the real valued functions of $C_0(X)$. Let $f \in C_0(X)$. Then there exists $\{h_n\} \subseteq \operatorname{Re} \mathcal{A}$ and $\{g_n\} \subseteq \operatorname{Im} \mathcal{A}$ such that $h_n \rightarrow \operatorname{Re} f$ uniformly and $g_n \rightarrow \operatorname{Im} f$ uniformly. Therefore, $h_n + ig_n \in \mathcal{A}$ and it converges to f uniformly. This proves the theorem.

7.3 Exercises

1. Let $(X, \tau), (Y, \eta)$ be topological spaces and let $A \subseteq X$ be compact. Then if $f : X \rightarrow Y$ is continuous, show that $f(A)$ is also compact.
2. $\uparrow$ In the context of Problem 1, suppose $\mathbb{R} = Y$ where the usual topology is placed on $\mathbb{R}$. Show f achieves its maximum and minimum on A .
3. Let V be an open set in $\mathbb{R}^n$. Show there is an increasing sequence of compact sets, K_m , such that $V = \cup_{m=1}^{\infty} K_m$. **Hint:** Let

$$C_m \equiv \left\{ \mathbf{x} \in \mathbb{R}^n : \operatorname{dist}(\mathbf{x}, V^C) \geq \frac{1}{m} \right\}$$

where

$$\operatorname{dist}(\mathbf{x}, S) \equiv \inf \{ |\mathbf{y} - \mathbf{x}| \text{ such that } \mathbf{y} \in S \}.$$

Consider $K_m \equiv C_m \cap \overline{B(\mathbf{0}, m)}$.

4. Let $B(X; \mathbb{R}^n)$ be the space of functions $\mathbf{f}$, mapping X to $\mathbb{R}^n$ such that

$$\sup \{ |\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X \} < \infty.$$

Show $B(X; \mathbb{R}^n)$ is a complete normed linear space if

$$\|\mathbf{f}\| \equiv \sup \{ |\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X \}.$$

5. Let $\alpha \in [0, 1]$. Define, for X a compact subset of $\mathbb{R}^p$,

$$C^\alpha(X; \mathbb{R}^n) \equiv \{ \mathbf{f} \in C(X; \mathbb{R}^n) : \rho_\alpha(\mathbf{f}) + \|\mathbf{f}\| \equiv \|\mathbf{f}\|_\alpha < \infty \}$$

where

$$\|\mathbf{f}\| \equiv \sup \{ |\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X \}$$

and

$$\rho_\alpha(\mathbf{f}) \equiv \sup \left\{ \frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha} : \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y} \right\}.$$

Show that $(C^\alpha(X; \mathbb{R}^n), \|\cdot\|_\alpha)$ is a complete normed linear space.

6. Let $\{\mathbf{f}_n\}_{n=1}^\infty \subseteq C^\alpha(X; \mathbb{R}^n)$ where X is a compact subset of $\mathbb{R}^p$ and suppose

$$\|\mathbf{f}_n\|_\alpha \leq M$$

for all n . Show there exists a subsequence, n_k , such that $\mathbf{f}_{n_k}$ converges in $C(X; \mathbb{R}^n)$. The given sequence is called precompact when this happens. (This also shows the embedding of $C^\alpha(X; \mathbb{R}^n)$ into $C(X; \mathbb{R}^n)$ is a compact embedding.)

7. Use the general Stone Weierstrass approximation theorem to prove Theorem 7.1.6.
8. Let (X, d) be a metric space where d is a bounded metric. Let $\mathcal{C}$ denote the collection of closed subsets of X . For $A, B \in \mathcal{C}$, define

$$\rho(A, B) \equiv \inf \{ \delta > 0 : A_\delta \supseteq B \text{ and } B_\delta \supseteq A \}$$

where for a set S ,

$$S_\delta \equiv \{x : \text{dist}(x, S) \equiv \inf \{d(x, s) : s \in S\} \leq \delta\}.$$

Show $x \rightarrow \text{dist}(x, S)$ is continuous and that therefore, S_δ is a closed set containing S . Also show that ρ is a metric on $\mathcal{C}$. This is called the Hausdorff metric.

9. $\uparrow$ Suppose (X, d) is a compact metric space. Show $(\mathcal{C}, \rho)$ is a complete metric space. **Hint:** Show first that if $W_n \downarrow W$ where W_n is closed, then $\rho(W_n, W) \rightarrow 0$. Now let $\{A_n\}$ be a Cauchy sequence in $\mathcal{C}$. Then if $\varepsilon > 0$ there exists N such that when $m, n \geq N$, then $\rho(A_n, A_m) < \varepsilon$. Therefore, for each $n \geq N$,

$$(A_n)_\varepsilon \supseteq \overline{\bigcup_{k=n}^\infty A_k}.$$

Let $A \equiv \bigcap_{n=1}^\infty \overline{\bigcup_{k=n}^\infty A_k}$. By the first part, there exists $N_1 > N$ such that for $n \geq N_1$,

$$\rho(\overline{\bigcup_{k=n}^\infty A_k}, A) < \varepsilon, \text{ and } (A_n)_\varepsilon \supseteq \overline{\bigcup_{k=n}^\infty A_k}.$$

Therefore, for such n , $A_\varepsilon \supseteq W_n \supseteq A_n$ and $(W_n)_\varepsilon \supseteq (A_n)_\varepsilon \supseteq A$ because

$$(A_n)_\varepsilon \supseteq \overline{\bigcup_{k=n}^\infty A_k} \supseteq A.$$

10. $\uparrow$ Let X be a compact metric space. Show $(\mathcal{C}, \rho)$ is compact. **Hint:** Let $\mathcal{D}_n$ be a 2^{-n} net for X . Let $\mathcal{K}_n$ denote finite unions of sets of the form $\overline{B(p, 2^{-n})}$ where $p \in \mathcal{D}_n$. Show $\mathcal{K}_n$ is a $2^{-(n-1)}$ net for $(\mathcal{C}, \rho)$.

Part II

Real And Abstract Analysis

Abstract Measure And Integration

8.1 σ Algebras

This chapter is on the basics of measure theory and integration. A measure is a real valued mapping from some subset of the power set of a given set which has values in $[0, \infty]$. Many apparently different things can be considered as measures and also there is an integral defined. By discussing this in terms of axioms and in a very abstract setting, many different topics can be considered in terms of one general theory. For example, it will turn out that sums are included as an integral of this sort. So is the usual integral as well as things which are often thought of as being in between sums and integrals.

Let Ω be a set and let $\mathcal{F}$ be a collection of subsets of Ω satisfying

$$\emptyset \in \mathcal{F}, \Omega \in \mathcal{F}, \quad (8.1.1)$$

$$E \in \mathcal{F} \text{ implies } E^C \equiv \Omega \setminus E \in \mathcal{F},$$

$$\text{If } \{E_n\}_{n=1}^{\infty} \subseteq \mathcal{F}, \text{ then } \cup_{n=1}^{\infty} E_n \in \mathcal{F}. \quad (8.1.2)$$

Definition 8.1.1 *A collection of subsets of a set, Ω , satisfying Formulas 8.1.1-8.1.2 is called a σ algebra.*

As an example, let Ω be any set and let $\mathcal{F} = \mathcal{P}(\Omega)$, the set of all subsets of Ω (power set). This obviously satisfies Formulas 8.1.1-8.1.2.

Lemma 8.1.2 *Let $\mathcal{C}$ be a set whose elements are σ algebras of subsets of Ω . Then $\cap \mathcal{C}$ is a σ algebra also.*

Be sure to verify this lemma. It follows immediately from the above definitions but it is important for you to check the details.

Example 8.1.3 *Let τ denote the collection of all open sets in $\mathbb{R}^n$ and let $\sigma(\tau) \equiv$ intersection of all σ algebras that contain τ . $\sigma(\tau)$ is called the σ algebra of Borel sets. In general, for a collection of sets, Σ , $\sigma(\Sigma)$ is the smallest σ algebra which contains Σ .*

This is a very important σ algebra and it will be referred to frequently as the Borel sets. Attempts to describe a typical Borel set are more trouble than they are worth and it is not easy to do so. Rather, one uses the definition just given in the example. Note, however, that all countable intersections of open sets and countable unions of closed sets are Borel sets. Such sets are called G_δ and F_σ respectively.

Definition 8.1.4 Let $\mathcal{F}$ be a σ algebra of sets of Ω and let $\mu : \mathcal{F} \rightarrow [0, \infty]$. μ is called a measure if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad (8.1.3)$$

whenever the E_i are disjoint sets of $\mathcal{F}$. The triple, $(\Omega, \mathcal{F}, \mu)$ is called a measure space and the elements of $\mathcal{F}$ are called the measurable sets. $(\Omega, \mathcal{F}, \mu)$ is a finite measure space when $\mu(\Omega) < \infty$.

Note that the above definition immediately implies that if $E_i \in \mathcal{F}$ and the sets E_i are not necessarily disjoint,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

To see this, let $F_1 \equiv E_1$, $F_2 \equiv E_2 \setminus E_1, \dots, F_n \equiv E_n \setminus \bigcup_{i=1}^{n-1} E_i$, then the sets F_i are disjoint sets in $\mathcal{F}$ and

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

because of the fact that each $E_i \supseteq F_i$ and so

$$\mu(E_i) = \mu(F_i) + \mu(E_i \setminus F_i)$$

which implies $\mu(E_i) \geq \mu(F_i)$.

The following theorem is the basis for most of what is done in the theory of measure and integration. It is a very simple result which follows directly from the above definition.

Theorem 8.1.5 Let $\{E_m\}_{m=1}^{\infty}$ be a sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$. Then if $\dots E_n \subseteq E_{n+1} \subseteq E_{n+2} \subseteq \dots$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} \mu(E_n) \quad (8.1.4)$$

and if $\dots E_n \supseteq E_{n+1} \supseteq E_{n+2} \supseteq \dots$ and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \rightarrow \infty} \mu(E_n). \quad (8.1.5)$$

Stated more succinctly, $E_k \uparrow E$ implies $\mu(E_k) \uparrow \mu(E)$ and $E_k \downarrow E$ with $\mu(E_1) < \infty$ implies $\mu(E_k) \downarrow \mu(E)$.

Proof: First note that $\cap_{i=1}^{\infty} E_i = (\cup_{i=1}^{\infty} E_i^C)^C \in \mathcal{F}$ so $\cap_{i=1}^{\infty} E_i$ is measurable. Also note that for A and B sets of $\mathcal{F}$, $A \setminus B \equiv (A^C \cup B)^C \in \mathcal{F}$. To show 8.1.4, note that 8.1.4 is obviously true if $\mu(E_k) = \infty$ for any k . Therefore, assume $\mu(E_k) < \infty$ for all k . Thus

$$\mu(E_{k+1} \setminus E_k) + \mu(E_k) = \mu(E_{k+1})$$

and so

$$\mu(E_{k+1} \setminus E_k) = \mu(E_{k+1}) - \mu(E_k).$$

Also,

$$\bigcup_{k=1}^{\infty} E_k = E_1 \cup \bigcup_{k=1}^{\infty} (E_{k+1} \setminus E_k)$$

and the sets in the above union are disjoint. Hence by 8.1.3,

$$\begin{aligned} \mu(\cup_{i=1}^{\infty} E_i) &= \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} \setminus E_k) = \mu(E_1) \\ &\quad + \sum_{k=1}^{\infty} (\mu(E_{k+1}) - \mu(E_k)) \\ &= \mu(E_1) + \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(E_{k+1}) - \mu(E_k) = \lim_{n \rightarrow \infty} \mu(E_{n+1}). \end{aligned}$$

This shows part 8.1.4.

To verify 8.1.5,

$$\mu(E_1) = \mu(\cap_{i=1}^{\infty} E_i) + \mu(E_1 \setminus \cap_{i=1}^{\infty} E_i)$$

since $\mu(E_1) < \infty$, it follows $\mu(\cap_{i=1}^{\infty} E_i) < \infty$. Also, $E_1 \setminus \cap_{i=1}^n E_i \uparrow E_1 \setminus \cap_{i=1}^{\infty} E_i$ and so by 8.1.4,

$$\begin{aligned} \mu(E_1) - \mu(\cap_{i=1}^{\infty} E_i) &= \mu(E_1 \setminus \cap_{i=1}^{\infty} E_i) = \lim_{n \rightarrow \infty} \mu(E_1 \setminus \cap_{i=1}^n E_i) \\ &= \mu(E_1) - \lim_{n \rightarrow \infty} \mu(\cap_{i=1}^n E_i) = \mu(E_1) - \lim_{n \rightarrow \infty} \mu(E_n), \end{aligned}$$

Hence, subtracting $\mu(E_1)$ from both sides,

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(\cap_{i=1}^{\infty} E_i).$$

This proves the theorem.

It is convenient to allow functions to take the value $+\infty$. You should think of $+\infty$, usually referred to as ∞ as something out at the right end of the real line and its only importance is the notion of sequences converging to it. $x_n \rightarrow \infty$ exactly when for all $l \in \mathbb{R}$, there exists N such that if $n \geq N$, then

$$x_n > l.$$

This is what it means for a sequence to converge to ∞ . Don't think of ∞ as a number. It is just a convenient symbol which allows the consideration of some limit operations more simply. Similar considerations apply to $-\infty$ but this value is not of very great interest. In fact the set of most interest is the complex numbers or some vector space. Therefore, this topic is not considered.

Lemma 8.1.6 *Let $f : \Omega \rightarrow (-\infty, \infty]$ where $\mathcal{F}$ is a σ algebra of subsets of Ω . Then the following are equivalent.*

$$\begin{aligned} f^{-1}((d, \infty]) &\in \mathcal{F} \text{ for all finite } d, \\ f^{-1}((-\infty, d)) &\in \mathcal{F} \text{ for all finite } d, \\ f^{-1}([d, \infty]) &\in \mathcal{F} \text{ for all finite } d, \\ f^{-1}((-\infty, d]) &\in \mathcal{F} \text{ for all finite } d, \\ f^{-1}((a, b)) &\in \mathcal{F} \text{ for all } a < b, -\infty < a < b < \infty. \end{aligned}$$

Proof: First note that the first and the third are equivalent. To see this, observe

$$f^{-1}([d, \infty]) = \bigcap_{n=1}^{\infty} f^{-1}((d - 1/n, \infty]),$$

and so if the first condition holds, then so does the third.

$$f^{-1}((d, \infty]) = \bigcup_{n=1}^{\infty} f^{-1}([d + 1/n, \infty]),$$

and so if the third condition holds, so does the first.

Similarly, the second and fourth conditions are equivalent. Now

$$f^{-1}((-\infty, d]) = (f^{-1}((d, \infty]))^C$$

so the first and fourth conditions are equivalent. Thus the first four conditions are equivalent and if any of them hold, then for $-\infty < a < b < \infty$,

$$f^{-1}((a, b)) = f^{-1}((-\infty, b)) \cap f^{-1}((a, \infty]) \in \mathcal{F}.$$

Finally, if the last condition holds,

$$f^{-1}([d, \infty]) = \left(\bigcup_{k=1}^{\infty} f^{-1}((-k + d, d)) \right)^C \in \mathcal{F}$$

and so the third condition holds. Therefore, all five conditions are equivalent. This proves the lemma.

This lemma allows for the following definition of a measurable function having values in $(-\infty, \infty]$.

Definition 8.1.7 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \rightarrow (-\infty, \infty]$. Then f is said to be measurable if any of the equivalent conditions of Lemma 8.1.6 hold. When the σ algebra, $\mathcal{F}$ equals the Borel σ algebra, $\mathcal{B}$, the function is called Borel measurable. More generally, if $f : \Omega \rightarrow X$ where X is a topological space, f is said to be measurable if $f^{-1}(U) \in \mathcal{F}$ whenever U is open.*

Theorem 8.1.8 *Let f_n and f be functions mapping Ω to $(-\infty, \infty]$ where $\mathcal{F}$ is a σ algebra of measurable sets of Ω . Then if f_n is measurable, and $f(\omega) = \lim_{n \rightarrow \infty} f_n(\omega)$, it follows that f is also measurable. (Pointwise limits of measurable functions are measurable.)*

Proof: First it is shown $f^{-1}((a, b)) \in \mathcal{F}$. Let $V_m \equiv (a + \frac{1}{m}, b - \frac{1}{m})$ and $\bar{V}_m = [a + \frac{1}{m}, b - \frac{1}{m}]$. Then for all m , $V_m \subseteq (a, b)$ and

$$(a, b) = \cup_{m=1}^{\infty} V_m = \cup_{m=1}^{\infty} \bar{V}_m.$$

Note that $V_m \neq \emptyset$ for all m large enough. Since f is the pointwise limit of f_n ,

$$f^{-1}(V_m) \subseteq \{\omega : f_k(\omega) \in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\bar{V}_m).$$

You should note that the expression in the middle is of the form

$$\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m).$$

Therefore,

$$\begin{aligned} f^{-1}((a, b)) &= \cup_{m=1}^{\infty} f^{-1}(V_m) \subseteq \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^{\infty} f^{-1}(\bar{V}_m) = f^{-1}((a, b)). \end{aligned}$$

It follows $f^{-1}((a, b)) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. This shows f is measurable.

The following theorem considers the case of functions which have values in a metric space. Its proof is similar to the proof of the above.

Theorem 8.1.9 *Let $\{f_n\}$ be a sequence of measurable functions mapping Ω to (X, d) where (X, d) is a metric space and $(\Omega, \mathcal{F})$ is a measure space. Suppose also that $f(\omega) = \lim_{n \rightarrow \infty} f_n(\omega)$ for all ω . Then f is also a measurable function.*

Proof: It is required to show $f^{-1}(U)$ is measurable for all U open. Let

$$V_m \equiv \left\{ x \in U : \text{dist}(x, U^C) > \frac{1}{m} \right\}.$$

Thus

$$V_m \subseteq \left\{ x \in U : \text{dist}(x, U^C) \geq \frac{1}{m} \right\}$$

and $V_m \subseteq \bar{V}_m \subseteq V_{m+1}$ and $\cup_m V_m = U$. Then since V_m is open,

$$f^{-1}(V_m) = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m)$$

and so

$$\begin{aligned} f^{-1}(U) &= \cup_{m=1}^{\infty} f^{-1}(V_m) \\ &= \cup_{m=1}^{\infty} \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} f_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^{\infty} f^{-1}(\bar{V}_m) = f^{-1}(U) \end{aligned}$$

which shows $f^{-1}(U)$ is measurable. This proves the theorem.

Now here is a simple observation.

Observation 8.1.10 Let $f : \Omega \rightarrow X$ where X is some topological space. Suppose

$$f(\omega) = \sum_{k=1}^m x_k \chi_{A_k}(\omega)$$

where each $x_k \in X$ and the A_k are disjoint measurable sets. (Such functions are often referred to as simple functions.) The sum means the function has value x_k on set A_k . Then f is measurable.

Proof: Letting U be open, $f^{-1}(U) = \cup \{A_k : x_k \in U\}$, a finite union of measurable sets.

There is also a very interesting theorem due to Kuratowski [44] which is presented next.

Theorem 8.1.11 Let E be a compact metric space and let $(\Omega, \mathcal{F})$ be a measure space. Suppose $\psi : E \times \Omega \rightarrow \mathbb{R}$ has the property that $x \rightarrow \psi(x, \omega)$ is continuous and $\omega \rightarrow \psi(x, \omega)$ is measurable. Then there exists a measurable function, f having values in E such that

$$\psi(f(\omega), \omega) = \sup_{x \in E} \psi(x, \omega).$$

Furthermore, $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable.

Proof: Let C_1 be a 2^{-1} net of E . Suppose $C_1, \dots, C_m$ have been chosen such that C_k is a 2^{-k} net and $C_{i+1} \supseteq C_i$ for all i . Then consider $E \setminus \bigcup \{B(x, 2^{-(m+1)}) : x \in C_m\}$. If this set is empty, let $C_{m+1} = C_m$. If it is nonempty, let $\{y_i\}_{i=1}^r$ be a $2^{-(m+1)}$ net for this compact set. Then let $C_{m+1} = C_m \cup \{y_i\}_{i=1}^r$. It follows $\{C_m\}_{m=1}^\infty$ satisfies C_m is a 2^{-m} net and $C_m \subseteq C_{m+1}$.

Let $\{x_k^1\}_{k=1}^{m(1)}$ equal C_1 . Let

$$A_1^1 \equiv \left\{ \omega : \psi(x_1^1, \omega) = \max_k \psi(x_k^1, \omega) \right\}$$

For $\omega \in A_1^1$, define $s_1(\omega) \equiv x_1^1$. Next let

$$A_2^1 \equiv \left\{ \omega \notin A_1^1 : \psi(x_2^1, \omega) = \max_k \psi(x_k^1, \omega) \right\}$$

and let $s_1(\omega) \equiv x_2^1$ on A_2^1 . Continue in this way to obtain a simple function, s_1 such that

$$\psi(s_1(\omega), \omega) = \max \{ \psi(x, \omega) : x \in C_1 \}$$

and s_1 has values in C_1 .

Suppose $s_1(\omega), s_2(\omega), \dots, s_m(\omega)$ are simple functions with the property that if $m > 1$,

$$\begin{aligned} |s_k(\omega) - s_{k+1}(\omega)| &< 2^{-k}, \\ \psi(s_k(\omega), \omega) &= \max \{ \psi(x, \omega) : x \in C_k \} \\ s_k &\text{ has values in } C_k \end{aligned}$$

for each $k+1 \leq m$, only the second and third assertions holding if $m=1$. Letting $C_m = \{x_k\}_{k=1}^N$, it follows $s_m(\omega)$ is of the form

$$s_m(\omega) = \sum_{k=1}^N x_k \chi_{A_k}(\omega), \quad A_i \cap A_j = \emptyset. \quad (8.1.6)$$

Denote by $\{y_{1i}\}_{i=1}^{n_1}$ those points of C_{m+1} which are contained in $B(x_1, 2^{-m})$. Letting A_k play the role of Ω in the first step in which s_1 was constructed, for each $\omega \in A_1$ let $s_{m+1}(\omega)$ be a simple function which has one of the values y_{1i} and satisfies

$$\psi(s_{m+1}(\omega), \omega) = \max_{i \leq n_1} \psi(y_{1i}, \omega)$$

for each $\omega \in A_1$. Next let $\{y_{2i}\}_{i=1}^{n_2}$ be those points of C_{m+1} different than $\{y_{1i}\}_{i=1}^{n_1}$ which are contained in $B(x_2, 2^{-m})$. Then define $s_{m+1}(\omega)$ on A_2 to have values taken from $\{y_{2i}\}_{i=1}^{n_2}$ and

$$\psi(s_{m+1}(\omega), \omega) = \max_{i \leq n_2} \psi(y_{2i}, \omega)$$

for each $\omega \in A_2$. Continuing this way defines s_{m+1} on all of Ω and it satisfies

$$|s_m(\omega) - s_{m+1}(\omega)| < 2^{-m} \text{ for all } \omega \in \Omega \quad (8.1.7)$$

It remains to verify

$$\psi(s_{m+1}(\omega), \omega) = \max \{ \psi(x, \omega) : x \in C_{m+1} \}. \quad (8.1.8)$$

To see this is so, pick $\omega \in \Omega$. Let

$$\max \{ \psi(x, \omega) : x \in C_{m+1} \} = \psi(y_j, \omega) \quad (8.1.9)$$

where $y_j \in C_{m+1}$ and out of all the balls $B(x_l, 2^{-m})$, the first one which contains y_j is $B(x_k, 2^{-m})$. Then by the construction, $s_{m+1}(\omega) = y_j$. This and 8.1.9 verifies 8.1.8.

From 8.1.7 it follows $s_m(\omega)$ converges uniformly on Ω to a measurable function, $f(\omega)$. Then from the construction, $\psi(f(\omega), \omega) \geq \psi(s_m(\omega), \omega)$ for all m and ω . Now pick $\omega \in \Omega$ and let z be such that $\psi(z, \omega) = \max_{x \in E} \psi(x, \omega)$. Letting $y_k \rightarrow z$ where $y_k \in C_k$, it follows from continuity of ψ in the first argument that

$$\begin{aligned} \max_{x \in E} \psi(x, \omega) &= \psi(z, \omega) = \lim_{k \rightarrow \infty} \psi(y_k, \omega) \\ &\leq \lim_{m \rightarrow \infty} \psi(s_m(\omega), \omega) = \psi(f(\omega), \omega) \leq \max_{x \in E} \psi(x, \omega). \end{aligned}$$

To show $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable, note that since E is compact, there exists a countable dense subset, D . Then using continuity of ψ in the first argument,

$$\begin{aligned} \psi(f(\omega), \omega) &= \sup_{x \in E} \psi(x, \omega) \\ &= \sup_{x \in D} \psi(x, \omega) \end{aligned}$$

which equals a measurable function of ω because D is countable. This proves the theorem.

Theorem 8.1.12 *Let $\mathcal{B}$ consist of open cubes of the form*

$$Q_{\mathbf{x}} \equiv \prod_{i=1}^n (x_i - \delta, x_i + \delta)$$

where δ is a positive rational number and $\mathbf{x} \in \mathbb{Q}^n$. Then every open set in $\mathbb{R}^n$ can be written as a countable union of open cubes from $\mathcal{B}$. Furthermore, $\mathcal{B}$ is a countable set.

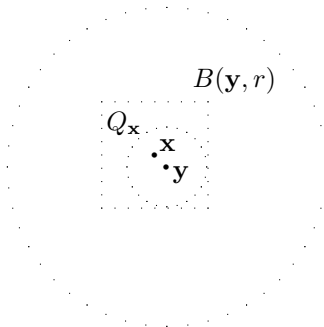
Proof: Let U be an open set and let $\mathbf{y} \in U$. Since U is open, $B(\mathbf{y}, r) \subseteq U$ for some $r > 0$ and it can be assumed $r/\sqrt{n} \in \mathbb{Q}$. Let

$$\mathbf{x} \in B\left(\mathbf{y}, \frac{r}{10\sqrt{n}}\right) \cap \mathbb{Q}^n$$

and consider the cube, $Q_{\mathbf{x}} \in \mathcal{B}$ defined by

$$Q_{\mathbf{x}} \equiv \prod_{i=1}^n (x_i - \delta, x_i + \delta)$$

where $\delta = r/4\sqrt{n}$. The following picture is roughly illustrative of what is taking place.



Then the diameter of $Q_{\mathbf{x}}$ equals

$$\left(n \left(\frac{r}{2\sqrt{n}} \right)^2 \right)^{1/2} = \frac{r}{2}$$

and so, if $\mathbf{z} \in Q_{\mathbf{x}}$, then

$$\begin{aligned} |\mathbf{z} - \mathbf{y}| &\leq |\mathbf{z} - \mathbf{x}| + |\mathbf{x} - \mathbf{y}| \\ &< \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

Consequently, $Q_{\mathbf{x}} \subseteq U$. Now also,

$$\left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} < \frac{r}{10\sqrt{n}}$$

and so it follows that for each i ,

$$|x_i - y_i| < \frac{r}{4\sqrt{n}}$$

since otherwise the above inequality would not hold. Therefore, $\mathbf{y} \in Q_{\mathbf{x}} \subseteq U$. Now let $\mathcal{B}_U$ denote those sets of $\mathcal{B}$ which are contained in U . Then $\cup \mathcal{B}_U = U$.

To see $\mathcal{B}$ is countable, note there are countably many choices for $\mathbf{x}$ and countably many choices for δ . This proves the theorem.

Recall that $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous means $g^{-1}(\text{open set}) = \text{an open set}$. In particular $g^{-1}((a, b))$ must be an open set.

Theorem 8.1.13 *Let $f_i : \Omega \rightarrow \mathbb{R}$ for $i = 1, \dots, n$ be measurable functions and let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous where $\mathbf{f} \equiv (f_1 \cdots f_n)^T$. Then $g \circ \mathbf{f}$ is a measurable function from Ω to $\mathbb{R}$.*

Proof: First it is shown

$$(g \circ \mathbf{f})^{-1}((a, b)) \in \mathcal{F}.$$

Now $(g \circ \mathbf{f})^{-1}((a, b)) = \mathbf{f}^{-1}(g^{-1}((a, b)))$ and since g is continuous, it follows that $g^{-1}((a, b))$ is an open set which is denoted as U for convenience. Now by Theorem 8.1.12 above, it follows there are countably many open cubes, $\{Q_k\}$ such that

$$U = \cup_{k=1}^{\infty} Q_k$$

where each Q_k is a cube of the form

$$Q_k = \prod_{i=1}^n (x_i - \delta, x_i + \delta).$$

Now

$$\mathbf{f}^{-1} \left(\prod_{i=1}^n (x_i - \delta, x_i + \delta) \right) = \cap_{i=1}^n f_i^{-1}((x_i - \delta, x_i + \delta)) \in \mathcal{F}$$

and so

$$\begin{aligned} (g \circ \mathbf{f})^{-1}((a, b)) &= \mathbf{f}^{-1}(g^{-1}((a, b))) = \mathbf{f}^{-1}(U) \\ &= \mathbf{f}^{-1}(\cup_{k=1}^{\infty} Q_k) = \cup_{k=1}^{\infty} \mathbf{f}^{-1}(Q_k) \in \mathcal{F}. \end{aligned}$$

This proves the theorem.

Corollary 8.1.14 *Sums, products, and linear combinations of measurable functions are measurable.*

Proof: To see the product of two measurable functions is measurable, let $g(x, y) = xy$, a continuous function defined on $\mathbb{R}^2$. Thus if you have two measurable functions, f_1 and f_2 defined on Ω ,

$$g \circ (f_1, f_2)(\omega) = f_1(\omega) f_2(\omega)$$

and so $\omega \rightarrow f_1(\omega) f_2(\omega)$ is measurable. Similarly you can show the sum of two measurable functions is measurable by considering $g(x, y) = x + y$ and you can show a linear combination of two measurable functions is measurable by considering $g(x, y) = ax + by$. More than two functions can also be considered as well.

The message of this corollary is that starting with measurable real valued functions you can combine them in pretty much any way you want and you end up with a measurable function.

Here is some notation which will be used whenever convenient.

Definition 8.1.15 *Let $f : \Omega \rightarrow [-\infty, \infty]$. Define*

$$[\alpha < f] \equiv \{\omega \in \Omega : f(\omega) > \alpha\} \equiv f^{-1}((\alpha, \infty])$$

with obvious modifications for the symbols $[\alpha \leq f]$, $[\alpha \geq f]$, $[\alpha \geq f \geq \beta]$, etc.

Definition 8.1.16 *For a set E ,*

$$\mathcal{X}_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{if } \omega \notin E. \end{cases}$$

This is called the characteristic function of E . Sometimes this is called the indicator function which I think is better terminology since the term characteristic function has another meaning. Note that this “indicates” whether a point, ω is contained in E . It is exactly when the function has the value 1.

Theorem 8.1.17 (Egoroff) *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space,*

$$(\mu(\Omega) < \infty)$$

and let f_n, f be complex valued functions such that $\operatorname{Re} f_n, \operatorname{Im} f_n$ are all measurable and

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$$

for all $\omega \notin E$ where $\mu(E) = 0$. Then for every $\varepsilon > 0$, there exists a set,

$$F \supseteq E, \mu(F) < \varepsilon,$$

such that f_n converges uniformly to f on F^C .

Proof: First suppose $E = \emptyset$ so that convergence is pointwise everywhere. It follows then that $\operatorname{Re} f$ and $\operatorname{Im} f$ are pointwise limits of measurable functions and are therefore measurable. Let $E_{km} = \{\omega \in \Omega : |f_n(\omega) - f(\omega)| \geq 1/m \text{ for some } n > k\}$. Note that

$$|f_n(\omega) - f(\omega)| = \sqrt{(\operatorname{Re} f_n(\omega) - \operatorname{Re} f(\omega))^2 + (\operatorname{Im} f_n(\omega) - \operatorname{Im} f(\omega))^2}$$

and so, By Theorem 8.1.13,

$$\left[|f_n - f| \geq \frac{1}{m} \right]$$

is measurable. Hence E_{km} is measurable because

$$E_{km} = \bigcup_{n=k+1}^{\infty} \left[|f_n - f| \geq \frac{1}{m} \right].$$

For fixed m , $\bigcap_{k=1}^{\infty} E_{km} = \emptyset$ because f_n converges to f . Therefore, if $\omega \in \Omega$ there exists k such that if $n > k$, $|f_n(\omega) - f(\omega)| < \frac{1}{m}$ which means $\omega \notin E_{km}$. Note also that

$$E_{km} \supseteq E_{(k+1)m}.$$

Since $\mu(E_{1m}) < \infty$, Theorem 8.1.5 on Page 178 implies

$$0 = \mu\left(\bigcap_{k=1}^{\infty} E_{km}\right) = \lim_{k \rightarrow \infty} \mu(E_{km}).$$

Let $k(m)$ be chosen such that $\mu(E_{k(m)m}) < \varepsilon 2^{-m}$ and let

$$F = \bigcup_{m=1}^{\infty} E_{k(m)m}.$$

Then $\mu(F) < \varepsilon$ because

$$\mu(F) \leq \sum_{m=1}^{\infty} \mu(E_{k(m)m}) < \sum_{m=1}^{\infty} \varepsilon 2^{-m} = \varepsilon$$

Now let $\eta > 0$ be given and pick m_0 such that $m_0^{-1} < \eta$. If $\omega \in F^C$, then

$$\omega \in \bigcap_{m=1}^{\infty} E_{k(m)m}^C.$$

Hence $\omega \in E_{k(m_0)m_0}^C$ so

$$|f_n(\omega) - f(\omega)| < 1/m_0 < \eta$$

for all $n > k(m_0)$. This holds for all $\omega \in F^C$ and so f_n converges uniformly to f on F^C .

Now if $E \neq \emptyset$, consider $\{\mathcal{X}_{E^C} f_n\}_{n=1}^{\infty}$. Each $\mathcal{X}_{E^C} f_n$ has real and imaginary parts measurable and the sequence converges pointwise to $\mathcal{X}_E f$ everywhere. Therefore, from the first part, there exists a set of measure less than ε , F such that on F^C , $\{\mathcal{X}_{E^C} f_n\}$ converges uniformly to $\mathcal{X}_{E^C} f$. Therefore, on $(E \cup F)^C$, $\{f_n\}$ converges uniformly to f . This proves the theorem.

Finally here is a comment about notation.

Definition 8.1.18 *Something happens for μ a.e. ω said as μ almost everywhere, if there exists a set E with $\mu(E) = 0$ and the thing takes place for all $\omega \notin E$. Thus $f(\omega) = g(\omega)$ a.e. if $f(\omega) = g(\omega)$ for all $\omega \notin E$ where $\mu(E) = 0$. A measure space, $(\Omega, \mathcal{F}, \mu)$ is σ finite if there exist measurable sets, Ω_n such that $\mu(\Omega_n) < \infty$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$.*

8.2 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \dots\}$. Let $\mathcal{F} = \mathcal{P}(\mathbb{N})$ and let $\mu(S)$ = number of elements in S . Thus $\mu(\{1\}) = 1 = \mu(\{2\})$, $\mu(\{1, 2\}) = 2$, etc. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. It is called counting measure. What functions are measurable in this case?
2. Let Ω be any uncountable set and let $\mathcal{F} = \{A \subseteq \Omega : \text{either } A \text{ or } A^C \text{ is countable}\}$. Let $\mu(A) = 1$ if A is uncountable and $\mu(A) = 0$ if A is countable. Show $(\Omega, \mathcal{F}, \mu)$ is a measure space. This is a well known bad example.
3. Let $\mathcal{F}$ be a σ algebra of subsets of Ω and suppose $\mathcal{F}$ has infinitely many elements. Show that $\mathcal{F}$ is uncountable. **Hint:** You might try to show there exists a countable sequence of disjoint sets of $\mathcal{F}$, $\{A_i\}$. It might be easiest to verify this by contradiction if it doesn't exist rather than a direct construction. Once this has been done, you can define a map, θ , from $\mathcal{P}(\mathbb{N})$ into $\mathcal{F}$ which is one to one by $\theta(S) = \bigcup_{i \in S} A_i$. Then argue $\mathcal{P}(\mathbb{N})$ is uncountable and so $\mathcal{F}$ is also uncountable.
4. Prove Lemma 8.1.2.
5. g is Borel measurable if whenever U is open, $g^{-1}(U)$ is Borel. Let $\mathbf{f} : \Omega \rightarrow \mathbb{R}^n$ and let $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathcal{F}$ is a σ algebra of sets of Ω . Suppose $\mathbf{f}$ is measurable and g is Borel measurable. Show $g \circ \mathbf{f}$ is measurable. To say g is Borel measurable means $g^{-1}(\text{open set}) = (\text{Borel set})$ where a Borel set is one of those sets in the smallest σ algebra containing the open sets of $\mathbb{R}^n$. See Lemma 8.1.2. **Hint:** You should show, using Theorem 8.1.12 that $\mathbf{f}^{-1}(\text{open set}) \in \mathcal{F}$. Now let

$$\mathcal{S} \equiv \{E \subseteq \mathbb{R}^n : \mathbf{f}^{-1}(E) \in \mathcal{F}\}$$

By what you just showed, $\mathcal{S}$ contains the open sets. Now verify $\mathcal{S}$ is a σ algebra. Argue that from the definition of the Borel sets, it follows $\mathcal{S}$ contains the Borel sets.

6. Let $(\Omega, \mathcal{F})$ be a measure space and suppose $f : \Omega \rightarrow \mathbb{C}$. Then f is said to be measurable if

$$f^{-1}(\text{open set}) \in \mathcal{F}.$$

Show f is measurable if and only if $\text{Re } f$ and $\text{Im } f$ are measurable real-valued functions. Thus it suffices to define a complex valued function to be measurable if the real and imaginary parts are measurable. **Hint:** Argue that

$f^{-1}((a, b) + i(c, d)) = (\operatorname{Re} f)^{-1}((a, b)) \cap (\operatorname{Im} f)^{-1}((c, d))$. Then use Theorem 8.1.12 to verify that if $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable, it follows f is. Conversely, argue that $(\operatorname{Re} f)^{-1}((a, b)) = f^{-1}((a, b) + i\mathbb{R})$ with a similar formula holding for $\operatorname{Im} f$.

7. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define $\bar{\mu} : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ by

$$\bar{\mu}(A) = \inf\{\mu(B) : B \supseteq A, B \in \mathcal{F}\}.$$

Show $\bar{\mu}$ satisfies

$$\begin{aligned} \bar{\mu}(\emptyset) &= 0, \text{ if } A \subseteq B, \bar{\mu}(A) \leq \bar{\mu}(B), \\ \bar{\mu}(\cup_{i=1}^{\infty} A_i) &\leq \sum_{i=1}^{\infty} \bar{\mu}(A_i), \mu(A) = \bar{\mu}(A) \text{ if } A \in \mathcal{F}. \end{aligned}$$

If $\bar{\mu}$ satisfies these conditions, it is called an outer measure. This shows every measure determines an outer measure on the power set.

8. Let $\{E_i\}$ be a sequence of measurable sets with the property that

$$\sum_{i=1}^{\infty} \mu(E_i) < \infty.$$

Let $S = \{\omega \in \Omega \text{ such that } \omega \in E_i \text{ for infinitely many values of } i\}$. Show $\mu(S) = 0$ and S is measurable. This is part of the Borel Cantelli lemma.

Hint: Write S in terms of intersections and unions. Something is in S means that for every n there exists $k > n$ such that it is in E_k . Remember the tail of a convergent series is small.

9. $\uparrow$ Let f_n, f be measurable functions. f_n converges in measure if

$$\lim_{n \rightarrow \infty} \mu(x \in \Omega : |f(x) - f_n(x)| \geq \varepsilon) = 0$$

for each fixed $\varepsilon > 0$. Prove the theorem of F. Riesz. If f_n converges to f in measure, then there exists a subsequence $\{f_{n_k}\}$ which converges to f a.e.

Hint: Choose n_1 such that

$$\mu(x : |f(x) - f_{n_1}(x)| \geq 1) < 1/2.$$

Choose $n_2 > n_1$ such that

$$\mu(x : |f(x) - f_{n_2}(x)| \geq 1/2) < 1/2^2,$$

$n_3 > n_2$ such that

$$\mu(x : |f(x) - f_{n_3}(x)| \geq 1/3) < 1/2^3,$$

etc. Now consider what it means for $f_{n_k}(x)$ to fail to converge to $f(x)$. Then use Problem 8.

8.3 The Abstract Lebesgue Integral

8.3.1 Preliminary Observations

This section is on the Lebesgue integral and the major convergence theorems which are the reason for studying it. In all that follows μ will be a measure defined on a σ algebra $\mathcal{F}$ of subsets of Ω . $0 \cdot \infty = 0$ is always defined to equal zero. This is a meaningless expression and so it can be defined arbitrarily but a little thought will soon demonstrate that this is the right definition in the context of measure theory. To see this, consider the zero function defined on $\mathbb{R}$. What should the integral of this function equal? Obviously, by an analogy with the Riemann integral, it should equal zero. Formally, it is zero times the length of the set or infinity. This is why this convention will be used.

Lemma 8.3.1 *Let $f(a, b) \in [-\infty, \infty]$ for $a \in A$ and $b \in B$ where A, B are sets. Then*

$$\sup_{a \in A} \sup_{b \in B} f(a, b) = \sup_{b \in B} \sup_{a \in A} f(a, b).$$

Proof: Note that for all a, b , $f(a, b) \leq \sup_{b \in B} \sup_{a \in A} f(a, b)$ and therefore, for all a ,

$$\sup_{b \in B} f(a, b) \leq \sup_{b \in B} \sup_{a \in A} f(a, b).$$

Therefore,

$$\sup_{a \in A} \sup_{b \in B} f(a, b) \leq \sup_{b \in B} \sup_{a \in A} f(a, b).$$

Repeating the same argument interchanging a and b , gives the conclusion of the lemma.

Lemma 8.3.2 *If $\{A_n\}$ is an increasing sequence in $[-\infty, \infty]$, then $\sup \{A_n\} = \lim_{n \rightarrow \infty} A_n$.*

The following lemma is useful also and this is a good place to put it. First $\{b_j\}_{j=1}^{\infty}$ is an enumeration of the a_{ij} if

$$\cup_{j=1}^{\infty} \{b_j\} = \cup_{i,j} \{a_{ij}\}.$$

In other words, the countable set, $\{a_{ij}\}_{i,j=1}^{\infty}$ is listed as $b_1, b_2, \dots$.

Lemma 8.3.3 *Let $a_{ij} \geq 0$. Then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$. Also if $\{b_j\}_{j=1}^{\infty}$ is any enumeration of the a_{ij} , then $\sum_{j=1}^{\infty} b_j = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$.*

Proof: First note there is no trouble in defining these sums because the a_{ij} are all nonnegative. If a sum diverges, it only diverges to ∞ and so ∞ is written as the answer.

$$\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \geq \sup_n \sum_{j=1}^{\infty} \sum_{i=1}^n a_{ij} = \sup_n \lim_{m \rightarrow \infty} \sum_{j=1}^m \sum_{i=1}^n a_{ij}$$

$$= \sup_n \lim_{m \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^m a_{ij} = \sup_n \sum_{i=1}^n \sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}. \quad (8.3.10)$$

Interchanging the i and j in the above argument the first part of the lemma is proved.

Finally, note that for all p ,

$$\sum_{j=1}^p b_j \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$$

and so $\sum_{j=1}^{\infty} b_j \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$. Now let $m, n > 1$ be given. Then

$$\sum_{i=1}^m \sum_{j=1}^n a_{ij} \leq \sum_{j=1}^p b_j$$

where p is chosen large enough that $\{b_1, \dots, b_p\} \supseteq \{a_{ij} : i \leq m \text{ and } j \leq n\}$. Therefore, since such a p exists for any choice of m, n , it follows that for any m, n ,

$$\sum_{i=1}^m \sum_{j=1}^n a_{ij} \leq \sum_{j=1}^{\infty} b_j.$$

Therefore, taking the limit as $n \rightarrow \infty$,

$$\sum_{i=1}^m \sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j$$

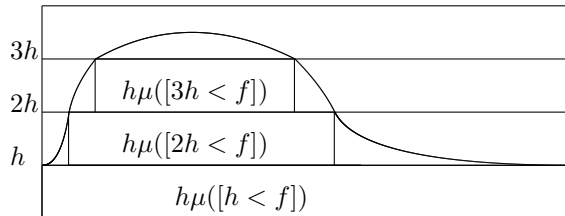
and finally, taking the limit as $m \rightarrow \infty$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \leq \sum_{j=1}^{\infty} b_j$$

proving the lemma.

8.3.2 Definition Of The Lebesgue Integral For Nonnegative Measurable Functions

The following picture illustrates the idea used to define the Lebesgue integral to be like the area under a curve.



You can see that by following the procedure illustrated in the picture and letting h get smaller, you would expect to obtain better approximations to the area under the curve¹ although all these approximations would likely be too small. Therefore, define

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < f])$$

Lemma 8.3.4 *The following inequality holds.*

$$\sum_{i=1}^{\infty} h\mu([ih < f]) \leq \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right).$$

Also, it suffices to consider only h smaller than a given positive number in the above definition of the integral.

Proof:

Let $N \in \mathbb{N}$.

$$\begin{aligned} \sum_{i=1}^{2N} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) &= \sum_{i=1}^{2N} \frac{h}{2}\mu([ih < 2f]) \\ &= \sum_{i=1}^N \frac{h}{2}\mu([(2i-1)h < 2f]) + \sum_{i=1}^N \frac{h}{2}\mu([2ih < 2f]) \\ &= \sum_{i=1}^N \frac{h}{2}\mu\left(\left[\frac{(2i-1)}{2}h < f\right]\right) + \sum_{i=1}^N \frac{h}{2}\mu([ih < f]) \\ &\geq \sum_{i=1}^N \frac{h}{2}\mu([ih < f]) + \sum_{i=1}^N \frac{h}{2}\mu([ih < f]) = \sum_{i=1}^N h\mu([ih < f]). \end{aligned}$$

Now letting $N \rightarrow \infty$ yields the claim of the lemma.

To verify the last claim, suppose $M < \int f d\mu$ and let $\delta > 0$ be given. Then there exists $h > 0$ such that

$$M < \sum_{i=1}^{\infty} h\mu([ih < f]) \leq \int f d\mu.$$

By the first part of this lemma,

$$M < \sum_{i=1}^{\infty} \frac{h}{2}\mu\left(\left[i\frac{h}{2} < f\right]\right) \leq \int f d\mu$$

¹Note the difference between this picture and the one usually drawn in calculus courses where the little rectangles are upright rather than on their sides. This illustrates a fundamental philosophical difference between the Riemann and the Lebesgue integrals. With the Riemann integral intervals are measured. With the Lebesgue integral, it is inverse images of intervals which are measured.

and continuing to apply the first part,

$$M < \sum_{i=1}^{\infty} \frac{h}{2^n} \mu \left(\left[i \frac{h}{2^n} < f \right] \right) \leq \int f d\mu.$$

Choose n large enough that $h/2^n < \delta$. It follows

$$M < \sup_{\delta > h > 0} \sum_{i=1}^{\infty} h \mu([ih < f]) \leq \int f d\mu.$$

Since M is arbitrary, this proves the last claim.

8.3.3 The Lebesgue Integral For Nonnegative Simple Functions

Definition 8.3.5 A function, s , is called simple if it is a measurable real valued function and has only finitely many values. These values will never be $\pm\infty$. Thus a simple function is one which may be written in the form

$$s(\omega) = \sum_{i=1}^n c_i \chi_{E_i}(\omega)$$

where the sets, E_i are disjoint and measurable. s takes the value c_i at E_i .

Note that by taking the union of some of the E_i in the above definition, you can assume that the numbers, c_i are the distinct values of s . Simple functions are important because it will turn out to be very easy to take their integrals as shown in the following lemma.

Lemma 8.3.6 Let $s(\omega) = \sum_{i=1}^p a_i \chi_{E_i}(\omega)$ be a nonnegative simple function with the a_i the distinct non zero values of s . Then

$$\int s d\mu = \sum_{i=1}^p a_i \mu(E_i). \quad (8.3.11)$$

Also, for any nonnegative measurable function, f , if $\lambda \geq 0$, then

$$\int \lambda f d\mu = \lambda \int f d\mu. \quad (8.3.12)$$

Proof: Consider 8.3.11 first. Without loss of generality, you can assume $0 < a_1 < a_2 < \dots < a_p$ and that $\mu(E_i) < \infty$. Let $\varepsilon > 0$ be given and let

$$\delta_1 \sum_{i=1}^p \mu(E_i) < \varepsilon.$$

Pick $\delta < \delta_1$ such that for $h < \delta$ it is also true that

$$h < \frac{1}{2} \min(a_1, a_2 - a_1, a_3 - a_2, \dots, a_n - a_{n-1}).$$

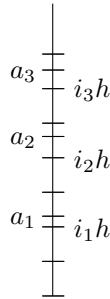
Then for $0 < h < \delta$

$$\begin{aligned} \sum_{k=1}^{\infty} h \mu([s > kh]) &= \sum_{k=1}^{\infty} h \sum_{i=k}^{\infty} \mu([ih < s \leq (i+1)h]) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^i h \mu([ih < s \leq (i+1)h]) \\ &= \sum_{i=1}^{\infty} i h \mu([ih < s \leq (i+1)h]). \end{aligned} \quad (8.3.13)$$

Because of the choice of h there exist positive integers, i_k such that $i_1 < i_2 < \dots, < i_p$ and

$$\begin{aligned} i_1 h &< a_1 \leq (i_1 + 1)h < \dots < i_2 h < a_2 < \\ &< (i_2 + 1)h < \dots < i_p h < a_p \leq (i_p + 1)h \end{aligned}$$

Then in the sum of 8.3.13 the only terms which are nonzero are those for which $i \in \{i_1, i_2, \dots, i_p\}$. To see this, you might consider the following picture.



When ih and $(i+1)h$ are both in between two of the a_i the set $[ih < s \leq (i+1)h]$ must be empty because the only values of the function are one of the a_i . At an i_k , $i_k h$ is smaller than a_k while $(i_k + 1)h$ is at least as large. Therefore, the set $[ih < s \leq (i+1)h]$ equals E_k and so

$$\mu([i_k h < s \leq (i_k + 1)h]) = \mu(E_k).$$

Therefore,

$$\sum_{k=1}^{\infty} h \mu([s > kh]) = \sum_{k=1}^p i_k h \mu(E_k).$$

It follows that for all h this small,

$$\begin{aligned} 0 &< \sum_{k=1}^p a_k \mu(E_k) - \sum_{k=1}^{\infty} h \mu([s > kh]) \\ &= \sum_{k=1}^p a_k \mu(E_k) - \sum_{k=1}^p i_k h \mu(E_k) \leq h \sum_{k=1}^p \mu(E_k) < \varepsilon. \end{aligned}$$

Taking the inf for h this small and using Lemma 8.3.4,

$$\begin{aligned} 0 &\leq \sum_{k=1}^p a_k \mu(E_k) - \sup_{\delta > h > 0} \sum_{k=1}^{\infty} h \mu([s > kh]) \\ &= \sum_{k=1}^p a_k \mu(E_k) - \int s d\mu \leq \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves the first part.

To verify 8.3.12 Note the formula is obvious if $\lambda = 0$ because then $[ih < \lambda f] = \emptyset$ for all $i > 0$. Assume $\lambda > 0$. Then

$$\begin{aligned} \int \lambda f d\mu &\equiv \sup_{h > 0} \sum_{i=1}^{\infty} h \mu([ih < \lambda f]) \\ &= \sup_{h > 0} \sum_{i=1}^{\infty} h \mu([ih/\lambda < f]) \\ &= \sup_{h > 0} \lambda \sum_{i=1}^{\infty} (h/\lambda) \mu([i(h/\lambda) < f]) \\ &= \lambda \int f d\mu. \end{aligned}$$

This proves the lemma.

Lemma 8.3.7 *Let the nonnegative simple function, s be defined as*

$$s(\omega) = \sum_{i=1}^n c_i \chi_{E_i}(\omega)$$

where the c_i are not necessarily distinct but the E_i are disjoint. It follows that

$$\int s = \sum_{i=1}^n c_i \mu(E_i).$$

Proof: Let the values of s be $\{a_1, \dots, a_m\}$. Therefore, since the E_i are disjoint, each a_i equal to one of the c_j . Let $A_i \equiv \cup \{E_j : c_j = a_i\}$. Then from Lemma 8.3.6

it follows that

$$\begin{aligned}\int s &= \sum_{i=1}^m a_i \mu(A_i) = \sum_{i=1}^m a_i \sum_{\{j: c_j = a_i\}} \mu(E_j) \\ &= \sum_{i=1}^m \sum_{\{j: c_j = a_i\}} c_j \mu(E_j) = \sum_{i=1}^n c_i \mu(E_i).\end{aligned}$$

This proves the lemma.

Note that $\int s$ could equal $+\infty$ if $\mu(A_k) = \infty$ and $a_k > 0$ for some k , but $\int s$ is well defined because $s \geq 0$. Recall that $0 \cdot \infty = 0$.

Lemma 8.3.8 *If $a, b \geq 0$ and if s and t are nonnegative simple functions, then*

$$\int as + bt = a \int s + b \int t.$$

Proof: Let

$$s(\omega) = \sum_{i=1}^n \alpha_i \chi_{A_i}(\omega), \quad t(\omega) = \sum_{j=1}^m \beta_j \chi_{B_j}(\omega)$$

where α_i are the distinct values of s and the β_j are the distinct values of t . Clearly $as + bt$ is a nonnegative simple function because it is measurable and has finitely many values. Also,

$$(as + bt)(\omega) = \sum_{j=1}^m \sum_{i=1}^n (a\alpha_i + b\beta_j) \chi_{A_i \cap B_j}(\omega)$$

where the sets $A_i \cap B_j$ are disjoint. By Lemma 8.3.7,

$$\begin{aligned}\int as + bt &= \sum_{j=1}^m \sum_{i=1}^n (a\alpha_i + b\beta_j) \mu(A_i \cap B_j) \\ &= a \sum_{i=1}^n \alpha_i \mu(A_i) + b \sum_{j=1}^m \beta_j \mu(B_j) \\ &= a \int s + b \int t.\end{aligned}$$

This proves the lemma.

8.3.4 Simple Functions And Measurable Functions

There is a fundamental theorem about the relationship of simple functions to measurable functions given in the next theorem.

Theorem 8.3.9 *Let $f \geq 0$ be measurable. Then there exists a sequence of nonnegative simple functions $\{s_n\}$ satisfying*

$$0 \leq s_n(\omega) \quad (8.3.14)$$

$$\cdots s_n(\omega) \leq s_{n+1}(\omega) \cdots$$

$$f(\omega) = \lim_{n \rightarrow \infty} s_n(\omega) \text{ for all } \omega \in \Omega. \quad (8.3.15)$$

If f is bounded the convergence is actually uniform.

Proof: Letting $I \equiv \{\omega : f(\omega) = \infty\}$, define

$$t_n(\omega) = \sum_{k=0}^{2^n} \frac{k}{n} \chi_{[k/n \leq f < (k+1)/n]}(\omega) + n \chi_I(\omega).$$

Then $t_n(\omega) \leq f(\omega)$ for all ω and $\lim_{n \rightarrow \infty} t_n(\omega) = f(\omega)$ for all ω . This is because $t_n(\omega) = n$ for $\omega \in I$ and if $f(\omega) \in [0, \frac{2^n+1}{n})$, then

$$0 \leq f(\omega) - t_n(\omega) \leq \frac{1}{n}. \quad (8.3.16)$$

Thus whenever $\omega \notin I$, the above inequality will hold for all n large enough. Let

$$s_1 = t_1, \quad s_2 = \max(t_1, t_2), \quad s_3 = \max(t_1, t_2, t_3), \cdots$$

Then the sequence $\{s_n\}$ satisfies 8.3.14-8.3.15.

To verify the last claim, note that in this case the term $n \chi_I(\omega)$ is not present. Therefore, for all n large enough, 8.3.16 holds for all ω . Thus the convergence is uniform. This proves the theorem.

Although it is not needed here, there is a similar theorem which applies to measurable functions which have values in a separable metric space. In this context, a simple function is one which is of the form

$$\sum_{k=1}^m x_k \chi_{E_k}(\omega)$$

where the E_k are disjoint measurable sets and the x_k are in X . I am abusing notation somewhat by using a sum. You can't add in a general metric space. The symbol means the function has value x_k on the set E_k .

Theorem 8.3.10 *Let $(\Omega, \mathcal{F})$ be a measure space and let $f : \Omega \rightarrow X$ where (X, d) is a separable metric space. Then f is a measurable function if and only if there exists a sequence of simple functions, $\{f_n\}$ such that for each $\omega \in \Omega$ and $n \in \mathbb{N}$,*

$$d(f_n(\omega), f(\omega)) \geq d(f_{n+1}(\omega), f(\omega)) \quad (8.3.17)$$

and

$$\lim_{n \rightarrow \infty} d(f_n(\omega), f(\omega)) = 0. \quad (8.3.18)$$

Proof: Let $D = \{x_k\}_{k=1}^\infty$ be a countable dense subset of X . First suppose f is measurable. Then since in a metric space every open set is the countable intersection of closed sets, it follows $f^{-1}(\text{closed set}) \in \mathcal{F}$. Now let $D_n = \{x_k\}_{k=1}^n$. Let

$$A_1 \equiv \left\{ \omega : d(x_1, f(\omega)) = \min_{k \leq n} d(x_k, f(\omega)) \right\}$$

That is, A_1 are those ω such that $f(\omega)$ is approximated best out of D_n by x_1 . Why is this a measurable set? It is because $\omega \rightarrow d(x, f(\omega))$ is a real valued measurable function, being the composition of a continuous function, $y \rightarrow d(x, y)$ and a measurable function, $\omega \rightarrow f(\omega)$. Next let

$$A_2 \equiv \left\{ \omega \notin A_1 : d(x_2, f(\omega)) = \min_{k \leq n} d(x_k, f(\omega)) \right\}$$

and continue in this manner obtaining disjoint measurable sets, $\{A_k\}_{k=1}^n$ such that for $\omega \in A_k$ the best approximation to $f(\omega)$ from D_n is x_k . Then

$$f_n(\omega) \equiv \sum_{k=1}^n x_k \chi_{A_k}(\omega).$$

Note

$$\min_{k \leq n+1} d(x_k, f(\omega)) \leq \min_{k \leq n} d(x_k, f(\omega))$$

and so this verifies 8.3.17. It remains to verify 8.3.18.

Let $\varepsilon > 0$ be given and pick $\omega \in \Omega$. Then there exists $x_n \in D$ such that $d(x_n, f(\omega)) < \varepsilon$. It follows from the construction that $d(f_n(\omega), f(\omega)) \leq d(x_n, f(\omega)) < \varepsilon$. This proves the first half.

Now suppose the existence of the sequence of simple functions as described above. Each f_n is a measurable function because $f_n^{-1}(U) = \cup \{A_k : x_k \in U\}$. Therefore, the conclusion that f is measurable follows from Theorem 8.1.9 on Page 181.

In the context of this more general notion of measurable function having values in a metric space, here is a version of Egoroff's theorem.

Theorem 8.3.11 (Egoroff) *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space,*

$$(\mu(\Omega) < \infty)$$

and let f_n, f be X valued measurable functions where X is a separable metric space and for all $\omega \notin E$ where $\mu(E) = 0$

$$f_n(\omega) \rightarrow f(\omega)$$

Then for every $\varepsilon > 0$, there exists a set,

$$F \supseteq E, \mu(F) < \varepsilon,$$

such that f_n converges uniformly to f on F^C .

Proof: First suppose $E = \emptyset$ so that convergence is pointwise everywhere. Let

$$E_{km} = \{\omega \in \Omega : d(f_n(\omega), f(\omega)) \geq 1/m \text{ for some } n > k\}.$$

Claim: $[\omega : d(f_n(\omega), f(\omega)) \geq \frac{1}{m}]$ is measurable.

Proof of claim: Let $\{x_k\}_{k=1}^{\infty}$ be a countable dense subset of X and let r denote a positive rational number, $\mathbb{Q}^+$. Then

$$\begin{aligned} & \cup_{k \in \mathbb{N}, r \in \mathbb{Q}^+} f_n^{-1}(B(x_k, r)) \cap f^{-1}\left(B\left(x_k, \frac{1}{m} - r\right)\right) \\ &= \left[d(f, f_n) < \frac{1}{m} \right] \end{aligned} \quad (8.3.19)$$

Here is why. If ω is in the set on the left, then $d(f_n(\omega), x_k) < r$ and

$$d(f(\omega), x_k) < \frac{1}{m} - r.$$

Therefore,

$$d(f(\omega), f_n(\omega)) < r + \frac{1}{m} - r = \frac{1}{m}.$$

Thus the left side is contained in the right. Now let ω be in the right side. That is $d(f_n(\omega), f(\omega)) < \frac{1}{m}$. Choose $2r < \frac{1}{m} - d(f_n(\omega), f(\omega))$ and pick $x_k \in B(f_n(\omega), r)$. Then

$$\begin{aligned} d(f(\omega), x_k) &\leq d(f(\omega), f_n(\omega)) + d(f_n(\omega), x_k) \\ &< \frac{1}{m} - 2r + r = \frac{1}{m} - r \end{aligned}$$

Thus $\omega \in f_n^{-1}(B(x_k, r)) \cap f^{-1}(B(x_k, \frac{1}{m} - r))$ and so ω is in the left side. Thus the two sets are equal. Now the set on the left in 8.3.19 is measurable because it is a countable union of measurable sets. This proves the claim since

$$\left[\omega : d(f_n(\omega), f(\omega)) \geq \frac{1}{m} \right]$$

is the complement of this measurable set.

Hence E_{km} is measurable because

$$E_{km} = \cup_{n=k+1}^{\infty} \left[\omega : d(f_n(\omega), f(\omega)) \geq \frac{1}{m} \right].$$

For fixed m , $\cap_{k=1}^{\infty} E_{km} = \emptyset$ because $f_n(\omega)$ converges to $f(\omega)$. Therefore, if $\omega \in \Omega$ there exists k such that if $n > k$, $|f_n(\omega) - f(\omega)| < \frac{1}{m}$ which means $\omega \notin E_{km}$. Note also that

$$E_{km} \supseteq E_{(k+1)m}.$$

Since $\mu(E_{1m}) < \infty$, Theorem 8.1.5 on Page 178 implies

$$0 = \mu(\cap_{k=1}^{\infty} E_{km}) = \lim_{k \rightarrow \infty} \mu(E_{km}).$$

Let $k(m)$ be chosen such that $\mu(E_{k(m)m}) < \varepsilon 2^{-m}$ and let

$$F = \bigcup_{m=1}^{\infty} E_{k(m)m}.$$

Then $\mu(F) < \varepsilon$ because

$$\mu(F) \leq \sum_{m=1}^{\infty} \mu(E_{k(m)m}) < \sum_{m=1}^{\infty} \varepsilon 2^{-m} = \varepsilon$$

Now let $\eta > 0$ be given and pick m_0 such that $m_0^{-1} < \eta$. If $\omega \in F^C$, then

$$\omega \in \bigcap_{m=1}^{\infty} E_{k(m)m}^C.$$

Hence $\omega \in E_{k(m_0)m_0}^C$ so

$$d(f(\omega), f_n(\omega)) < 1/m_0 < \eta$$

for all $n > k(m_0)$. This holds for all $\omega \in F^C$ and so f_n converges uniformly to f on F^C .

Now if $E \neq \emptyset$, consider $\{\mathcal{X}_{E^C} f_n\}_{n=1}^{\infty}$. Then $\mathcal{X}_{E^C} f_n$ is measurable and the sequence converges pointwise to $\mathcal{X}_E f$ everywhere. Therefore, from the first part, there exists a set of measure less than ε , F such that on F^C , $\{\mathcal{X}_{E^C} f_n\}$ converges uniformly to $\mathcal{X}_{E^C} f$. Therefore, on $(E \cup F)^C$, $\{f_n\}$ converges uniformly to f . This proves the theorem.

8.3.5 The Monotone Convergence Theorem

The following is called the monotone convergence theorem. This theorem and related convergence theorems are the reason for using the Lebesgue integral.

Theorem 8.3.12 (*Monotone Convergence theorem*) *Let f have values in $[0, \infty]$ and suppose $\{f_n\}$ is a sequence of nonnegative measurable functions having values in $[0, \infty]$ and satisfying*

$$\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega) \text{ for each } \omega.$$

$$\cdots f_n(\omega) \leq f_{n+1}(\omega) \cdots$$

Then f is measurable and

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Proof: From Lemmas 8.3.1 and 8.3.2,

$$\begin{aligned}
 \int f d\mu &\equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < f]) \\
 &= \sup_{h>0} \sup_k \sum_{i=1}^k h\mu([ih < f]) \\
 &= \sup_{h>0} \sup_k \sup_m \sum_{i=1}^k h\mu([ih < f_m]) \\
 &= \sup_m \sup_{h>0} \sum_{i=1}^{\infty} h\mu([ih < f_m]) \\
 &\equiv \sup_m \int f_m d\mu \\
 &= \lim_{m \rightarrow \infty} \int f_m d\mu.
 \end{aligned}$$

The third equality follows from the observation that

$$\lim_{m \rightarrow \infty} \mu([ih < f_m]) = \mu([ih < f])$$

which follows from Theorem 8.1.5 since the sets, $[ih < f_m]$ are increasing in m and their union equals $[ih < f]$. This proves the theorem.

To illustrate what goes wrong without the Lebesgue integral, consider the following example.

Example 8.3.13 Let $\{r_n\}$ denote the rational numbers in $[0, 1]$ and let

$$f_n(t) \equiv \begin{cases} 1 & \text{if } t \notin \{r_1, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n(t) \uparrow f(t)$ where f is the function which is one on the rationals and zero on the irrationals. Each f_n is Riemann integrable (why?) but f is not Riemann integrable. Therefore, you can't write $\int f dx = \lim_{n \rightarrow \infty} \int f_n dx$.

A meta-mathematical observation related to this type of example is this. If you can choose your functions, you don't need the Lebesgue integral. The Riemann integral is just fine. It is when you can't choose your functions and they come to you as pointwise limits that you really need the superior Lebesgue integral or at least something more general than the Riemann integral. The Riemann integral is entirely adequate for evaluating the seemingly endless lists of boring problems found in calculus books.

8.3.6 Other Definitions

To review and summarize the above, if $f \geq 0$ is measurable,

$$\int f d\mu \equiv \sup_{h>0} \sum_{i=1}^{\infty} h\mu([f > ih]) \quad (8.3.20)$$

another way to get the same thing for $\int f d\mu$ is to take an increasing sequence of nonnegative simple functions, $\{s_n\}$ with $s_n(\omega) \rightarrow f(\omega)$ and then by monotone convergence theorem,

$$\int f d\mu = \lim_{n \rightarrow \infty} \int s_n$$

where if $s_n(\omega) = \sum_{j=1}^m c_j \chi_{E_j}(\omega)$,

$$\int s_n d\mu = \sum_{j=1}^m c_j \mu(E_j).$$

Similarly this also shows that for such nonnegative measurable function,

$$\int f d\mu = \sup \left\{ \int s : 0 \leq s \leq f, s \text{ simple} \right\}$$

which is the usual way of defining the Lebesgue integral for nonnegative simple functions in most books. I have done it differently because this approach led to an easier proof of the Monotone convergence theorem. Here is an equivalent definition of the integral. The fact it is well defined has been discussed above.

Definition 8.3.14 For s a nonnegative simple function,

$$s(\omega) = \sum_{k=1}^n c_k \chi_{E_k}(\omega), \quad \int s = \sum_{k=1}^n c_k \mu(E_k).$$

For f a nonnegative measurable function,

$$\int f d\mu = \sup \left\{ \int s : 0 \leq s \leq f, s \text{ simple} \right\}.$$

8.3.7 Fatou's Lemma

Sometimes the limit of a sequence does not exist. There are two more general notions known as $\limsup$ and $\liminf$ which do always exist in some sense. These notions are dependent on the following lemma.

Lemma 8.3.15 Let $\{a_n\}$ be an increasing (decreasing) sequence in $[-\infty, \infty]$. Then $\lim_{n \rightarrow \infty} a_n$ exists.

Proof: Suppose first $\{a_n\}$ is increasing. Recall this means $a_n \leq a_{n+1}$ for all n . If the sequence is bounded above, then it has a least upper bound and so $a_n \rightarrow a$ where a is its least upper bound. If the sequence is not bounded above, then for every $l \in \mathbb{R}$, it follows l is not an upper bound and so eventually, $a_n > l$. But this is what is meant by $a_n \rightarrow \infty$. The situation for decreasing sequences is completely similar.

Now take any sequence, $\{a_n\} \subseteq [-\infty, \infty]$ and consider the sequence $\{A_n\}$ where

$$A_n \equiv \inf \{a_k : k \geq n\}.$$

Then as n increases, the set of numbers whose inf is being taken is getting smaller. Therefore, A_n is an increasing sequence and so it must converge. Similarly, if $B_n \equiv \sup \{a_k : k \geq n\}$, it follows B_n is decreasing and so $\{B_n\}$ also must converge. With this preparation, the following definition can be given.

Definition 8.3.16 Let $\{a_n\}$ be a sequence of points in $[-\infty, \infty]$. Then define

$$\liminf_{n \rightarrow \infty} a_n \equiv \lim_{n \rightarrow \infty} \inf \{a_k : k \geq n\}$$

and

$$\limsup_{n \rightarrow \infty} a_n \equiv \lim_{n \rightarrow \infty} \sup \{a_k : k \geq n\}$$

In the case of functions having values in $[-\infty, \infty]$,

$$\left(\liminf_{n \rightarrow \infty} f_n \right) (\omega) \equiv \lim_{n \rightarrow \infty} \inf (f_n(\omega)).$$

A similar definition applies to $\limsup_{n \rightarrow \infty} f_n$.

Lemma 8.3.17 Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$. Then $\lim_{n \rightarrow \infty} a_n$ exists if and only if

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$$

and in this case, the limit equals the common value of these two numbers.

Proof: Suppose first $\lim_{n \rightarrow \infty} a_n = a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, $a_n \in (a - \varepsilon, a + \varepsilon)$ for all n large enough, say $n \geq N$. Therefore, both $\inf \{a_k : k \geq n\}$ and $\sup \{a_k : k \geq n\}$ are contained in $[a - \varepsilon, a + \varepsilon]$ whenever $n \geq N$. It follows $\limsup_{n \rightarrow \infty} a_n$ and $\liminf_{n \rightarrow \infty} a_n$ are both in $[a - \varepsilon, a + \varepsilon]$, showing

$$\left| \liminf_{n \rightarrow \infty} a_n - \limsup_{n \rightarrow \infty} a_n \right| < 2\varepsilon.$$

Since ε is arbitrary, the two must be equal and they both must equal a . Next suppose $\lim_{n \rightarrow \infty} a_n = \infty$. Then if $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$l \leq a_n$$

and therefore, for such n ,

$$l \leq \inf \{a_k : k \geq n\} \leq \sup \{a_k : k \geq n\}$$

and this shows, since l is arbitrary that

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \infty.$$

The case for $-\infty$ is similar.

Conversely, suppose $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = a$. Suppose first that $a \in \mathbb{R}$. Then, letting $\varepsilon > 0$ be given, there exists N such that if $n \geq N$,

$$\sup \{a_k : k \geq n\} - \inf \{a_k : k \geq n\} < \varepsilon$$

therefore, if $k, m > N$, and $a_k > a_m$,

$$|a_k - a_m| = a_k - a_m \leq \sup \{a_k : k \geq n\} - \inf \{a_k : k \geq n\} < \varepsilon$$

showing that $\{a_n\}$ is a Cauchy sequence. Therefore, it converges to $a \in \mathbb{R}$, and as in the first part, the $\liminf$ and $\limsup$ both equal a . If $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \infty$, then given $l \in \mathbb{R}$, there exists N such that for $n \geq N$,

$$\inf_{n > N} a_n > l.$$

Therefore, $\lim_{n \rightarrow \infty} a_n = \infty$. The case for $-\infty$ is similar. This proves the lemma.

The next theorem, known as Fatou's lemma is another important theorem which justifies the use of the Lebesgue integral.

Theorem 8.3.18 (*Fatou's lemma*) *Let f_n be a nonnegative measurable function with values in $[0, \infty]$. Let $g(\omega) = \liminf_{n \rightarrow \infty} f_n(\omega)$. Then g is measurable and*

$$\int g d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

In other words,

$$\int \left(\liminf_{n \rightarrow \infty} f_n \right) d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu$$

Proof: Let $g_n(\omega) = \inf \{f_k(\omega) : k \geq n\}$. Then

$$g_n^{-1}([a, \infty]) = \cap_{k=n}^{\infty} f_k^{-1}([a, \infty]) \in \mathcal{F}.$$

Thus g_n is measurable by Lemma 8.1.6 on Page 180. Also $g(\omega) = \lim_{n \rightarrow \infty} g_n(\omega)$ so g is measurable because it is the pointwise limit of measurable functions. Now the functions g_n form an increasing sequence of nonnegative measurable functions so the monotone convergence theorem applies. This yields

$$\int g d\mu = \lim_{n \rightarrow \infty} \int g_n d\mu \leq \liminf_{n \rightarrow \infty} \int f_n d\mu.$$

The last inequality holding because

$$\int g_n d\mu \leq \int f_n d\mu.$$

(Note that it is not known whether $\lim_{n \rightarrow \infty} \int f_n d\mu$ exists.) This proves the Theorem.

8.3.8 The Righteous Algebraic Desires Of The Lebesgue Integral

The monotone convergence theorem shows the integral wants to be linear. This is the essential content of the next theorem.

Theorem 8.3.19 *Let f, g be nonnegative measurable functions and let a, b be non-negative numbers. Then*

$$\int (af + bg) d\mu = a \int f d\mu + b \int g d\mu. \quad (8.3.21)$$

Proof: By Theorem 8.3.9 on Page 197 there exist sequences of nonnegative simple functions, $s_n \rightarrow f$ and $t_n \rightarrow g$. Then by the monotone convergence theorem and Lemma 8.3.8,

$$\begin{aligned} \int (af + bg) d\mu &= \lim_{n \rightarrow \infty} \int as_n + bt_n d\mu \\ &= \lim_{n \rightarrow \infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\ &= a \int f d\mu + b \int g d\mu. \end{aligned}$$

As long as you are allowing functions to take the value $+\infty$, you cannot consider something like $f + (-g)$ and so you can't very well expect a satisfactory statement about the integral being linear until you restrict yourself to functions which have values in a vector space. This is discussed next.

8.4 The Space L^1

The functions considered here have values in $\mathbb{C}$, a vector space.

Definition 8.4.1 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose $f : \Omega \rightarrow \mathbb{C}$. Then f is said to be measurable if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are measurable real valued functions.*

Definition 8.4.2 *A complex simple function will be a function which is of the form*

$$s(\omega) = \sum_{k=1}^n c_k \chi_{E_k}(\omega)$$

where $c_k \in \mathbb{C}$ and $\mu(E_k) < \infty$. For s a complex simple function as above, define

$$I(s) \equiv \sum_{k=1}^n c_k \mu(E_k).$$

Lemma 8.4.3 *The definition, 8.4.2 is well defined. Furthermore, I is linear on the vector space of complex simple functions. Also the triangle inequality holds,*

$$|I(s)| \leq I(|s|).$$

Proof: Suppose $\sum_{k=1}^n c_k \mathcal{X}_{E_k}(\omega) = 0$. Does it follow that $\sum_k c_k \mu(E_k) = 0$? The supposition implies

$$\sum_{k=1}^n \operatorname{Re} c_k \mathcal{X}_{E_k}(\omega) = 0, \quad \sum_{k=1}^n \operatorname{Im} c_k \mathcal{X}_{E_k}(\omega) = 0. \quad (8.4.22)$$

Choose λ large and positive so that $\lambda + \operatorname{Re} c_k \geq 0$. Then adding $\sum_k \lambda \mathcal{X}_{E_k}$ to both sides of the first equation above,

$$\sum_{k=1}^n (\lambda + \operatorname{Re} c_k) \mathcal{X}_{E_k}(\omega) = \sum_{k=1}^n \lambda \mathcal{X}_{E_k}$$

and by Lemma 8.3.8 on Page 196, it follows upon taking $\int$ of both sides that

$$\sum_{k=1}^n (\lambda + \operatorname{Re} c_k) \mu(E_k) = \sum_{k=1}^n \lambda \mu(E_k)$$

which implies $\sum_{k=1}^n \operatorname{Re} c_k \mu(E_k) = 0$. Similarly, $\sum_{k=1}^n \operatorname{Im} c_k \mu(E_k) = 0$ and so $\sum_{k=1}^n c_k \mu(E_k) = 0$. Thus if

$$\sum_j c_j \mathcal{X}_{E_j} = \sum_k d_k \mathcal{X}_{F_k}$$

then $\sum_j c_j \mathcal{X}_{E_j} + \sum_k (-d_k) \mathcal{X}_{F_k} = 0$ and so the result just established verifies $\sum_j c_j \mu(E_j) - \sum_k d_k \mu(F_k) = 0$ which proves I is well defined.

That I is linear is now obvious. It only remains to verify the triangle inequality.

Let s be a simple function,

$$s = \sum_j c_j \mathcal{X}_{E_j}$$

Then pick $\theta \in \mathbb{C}$ such that $\theta I(s) = |I(s)|$ and $|\theta| = 1$. Then from the triangle inequality for sums of complex numbers,

$$\begin{aligned} |I(s)| &= \theta I(s) = I(\theta s) = \sum_j \theta c_j \mu(E_j) \\ &= \left| \sum_j \theta c_j \mu(E_j) \right| \leq \sum_j |\theta c_j| \mu(E_j) = I(|s|). \end{aligned}$$

This proves the lemma.

With this lemma, the following is the definition of $L^1(\Omega)$.

Definition 8.4.4 $f \in L^1(\Omega)$ means there exists a sequence of complex simple functions, $\{s_n\}$ such that

$$\lim_{m,n \rightarrow \infty} I(|s_n - s_m|) = \lim_{n,m \rightarrow \infty} \int |s_n - s_m| d\mu = 0 \quad (8.4.23)$$

Then

$$I(f) \equiv \lim_{n \rightarrow \infty} I(s_n). \quad (8.4.24)$$

Lemma 8.4.5 Definition 8.4.4 is well defined.

Proof: There are several things which need to be verified. First suppose 8.4.23. Then by Lemma 8.4.3

$$|I(s_n) - I(s_m)| = |I(s_n - s_m)| \leq I(|s_n - s_m|)$$

and for m, n large enough this last is given to be small so $\{I(s_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so it converges. This verifies the limit in 8.4.24 at least exists. It remains to consider another sequence $\{t_n\}$ having the same properties as $\{s_n\}$ and verifying $I(f)$ determined by this other sequence is the same. By Lemma 8.4.3 and Fatou's lemma, Theorem 8.3.18 on Page 204,

$$\begin{aligned} |I(s_n) - I(t_n)| &\leq I(|s_n - t_n|) = \int |s_n - t_n| d\mu \\ &\leq \int |s_n - f| + |f - t_n| d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int |s_n - s_k| d\mu + \liminf_{k \rightarrow \infty} \int |t_n - t_k| d\mu < \varepsilon \end{aligned}$$

whenever n is large enough. Since ε is arbitrary, this shows the limit from using the t_n is the same as the limit from using s_n . This proves the lemma.

What if f has values in $[0, \infty)$? Earlier $\int f d\mu$ was defined for such functions and now $I(f)$ has been defined. Are they the same? If so, I can be regarded as an extension of $\int f d\mu$ to a larger class of functions.

Lemma 8.4.6 Suppose f has values in $[0, \infty)$ and $f \in L^1(\Omega)$. Then f is measurable and

$$I(f) = \int f d\mu.$$

Proof: Since f is the pointwise limit of a sequence of complex simple functions, $\{s_n\}$ having the properties described in Definition 8.4.4, it follows $f(\omega) = \lim_{n \rightarrow \infty} \operatorname{Re} s_n(\omega)$ and so f is measurable. Also

$$\int \left| (\operatorname{Re} s_n)^+ - (\operatorname{Re} s_m)^+ \right| d\mu \leq \int |\operatorname{Re} s_n - \operatorname{Re} s_m| d\mu \leq \int |s_n - s_m| d\mu$$

where $x^+ \equiv \frac{1}{2}(|x| + x)$, the positive part of the real number, x .² Thus there is no loss of generality in assuming $\{s_n\}$ is a sequence of complex simple functions having values in $[0, \infty)$. Then since for such complex simple functions, $I(s) = \int s d\mu$,

$$\begin{aligned}
 \left| I(f) - \int f d\mu \right| &\leq |I(f) - I(s_n)| + \left| \int s_n d\mu - \int f d\mu \right| \\
 &< \varepsilon + \left| \int_{[s_n - f \geq 0]} s_n d\mu - \int_{[s_n - f \geq 0]} f d\mu \right. \\
 &\quad \left. + \int_{[s_n - f < 0]} s_n d\mu - \int_{[s_n - f < 0]} f d\mu \right| \\
 &\leq \varepsilon + \left| \int_{[s_n - f \geq 0]} (s_n - f) d\mu \right| + \left| \int_{[s_n - f < 0]} (s_n - f) d\mu \right| \\
 &\leq \varepsilon + \int_{[s_n - f \geq 0]} |s_n - f| d\mu + \int_{[s_n - f < 0]} |s_n - f| d\mu \\
 &= \varepsilon + \int |s_n - f| d\mu
 \end{aligned}$$

whenever n is large enough. But by Fatou's lemma, Theorem 8.3.18 on Page 204, the last term is no larger than

$$\liminf_{k \rightarrow \infty} \int |s_n - s_k| d\mu < \varepsilon$$

whenever n is large enough. Since ε is arbitrary, this shows $I(f) = \int f d\mu$ as claimed.

As explained above, I can be regarded as an extension of $\int d\mu$ so from now on, the usual symbol, $\int d\mu$ will be used. It is now easy to verify $\int d\mu$ is linear on $L^1(\Omega)$.

Theorem 8.4.7 $\int d\mu$ is linear on $L^1(\Omega)$ and $L^1(\Omega)$ is a complex vector space. If $f \in L^1(\Omega)$, then $\operatorname{Re} f$, $\operatorname{Im} f$, and $|f|$ are all in $L^1(\Omega)$. Furthermore, for $f \in L^1(\Omega)$,

$$\int f d\mu = \int (\operatorname{Re} f)^+ d\mu - \int (\operatorname{Re} f)^- d\mu + i \left(\int (\operatorname{Im} f)^+ d\mu - \int (\operatorname{Im} f)^- d\mu \right)$$

Also the triangle inequality holds,

$$\left| \int f d\mu \right| \leq \int |f| d\mu$$

²The negative part of the real number x is defined to be $x^- \equiv \frac{1}{2}(|x| - x)$. Thus $|x| = x^+ + x^-$ and $x = x^+ - x^-$.

Proof: First it is necessary to verify that $L^1(\Omega)$ is really a vector space because it makes no sense to speak of linear maps without having these maps defined on a vector space. Let f, g be in $L^1(\Omega)$ and let $a, b \in \mathbb{C}$. Then let $\{s_n\}$ and $\{t_n\}$ be sequences of complex simple functions associated with f and g respectively as described in Definition 8.4.4. Consider $\{as_n + bt_n\}$, another sequence of complex simple functions. Then $as_n(\omega) + bt_n(\omega) \rightarrow af(\omega) + bg(\omega)$ for each ω . Also, from Lemma 8.4.3

$$\int |as_n + bt_n - (as_m + bt_m)| d\mu \leq |a| \int |s_n - s_m| d\mu + |b| \int |t_n - t_m| d\mu$$

and the sum of the two terms on the right converge to zero as $m, n \rightarrow \infty$. Thus $af + bg \in L^1(\Omega)$. Also

$$\begin{aligned} \int (af + bg) d\mu &= \lim_{n \rightarrow \infty} \int (as_n + bt_n) d\mu \\ &= \lim_{n \rightarrow \infty} \left(a \int s_n d\mu + b \int t_n d\mu \right) \\ &= a \lim_{n \rightarrow \infty} \int s_n d\mu + b \lim_{n \rightarrow \infty} \int t_n d\mu \\ &= a \int f d\mu + b \int g d\mu. \end{aligned}$$

If $\{s_n\}$ is a sequence of complex simple functions described in Definition 8.4.4 corresponding to f , then $\{|s_n|\}$ is a sequence of complex simple functions satisfying the conditions of Definition 8.4.4 corresponding to $|f|$. This is because $|s_n(\omega)| \rightarrow |f(\omega)|$ and

$$\int ||s_n| - |s_m|| d\mu \leq \int |s_m - s_n| d\mu$$

with this last expression converging to 0 as $m, n \rightarrow \infty$. Thus $|f| \in L^1(\Omega)$. Also, by similar reasoning, $\{\operatorname{Re} s_n\}$ and $\{\operatorname{Im} s_n\}$ correspond to $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively in the manner described by Definition 8.4.4 showing that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $L^1(\Omega)$. Now $(\operatorname{Re} f)^+ = \frac{1}{2}(|\operatorname{Re} f| + \operatorname{Re} f)$ and $(\operatorname{Re} f)^- = \frac{1}{2}(|\operatorname{Re} f| - \operatorname{Re} f)$ so both of these functions are in $L^1(\Omega)$. Similar formulas establish that $(\operatorname{Im} f)^+$ and $(\operatorname{Im} f)^-$ are in $L^1(\Omega)$.

The formula follows from the observation that

$$f = (\operatorname{Re} f)^+ - (\operatorname{Re} f)^- + i \left((\operatorname{Im} f)^+ - (\operatorname{Im} f)^- \right)$$

and the fact shown first that $\int d\mu$ is linear.

To verify the triangle inequality, let $\{s_n\}$ be complex simple functions for f as in Definition 8.4.4. Then

$$\left| \int f d\mu \right| = \lim_{n \rightarrow \infty} \left| \int s_n d\mu \right| \leq \lim_{n \rightarrow \infty} \int |s_n| d\mu = \int |f| d\mu.$$

This proves the theorem.

The following description of $L^1(\Omega)$ is the version most often used because it is easy to verify the conditions for it.

Corollary 8.4.8 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let $f : \Omega \rightarrow \mathbb{C}$. Then $f \in L^1(\Omega)$ if and only if f is measurable and $\int |f| d\mu < \infty$.*

Proof: Suppose $f \in L^1(\Omega)$. Then from Definition 8.4.4, it follows both real and imaginary parts of f are measurable. Just take real and imaginary parts of s_n and observe the real and imaginary parts of f are limits of the real and imaginary parts of s_n respectively. Why is $\int |f| d\mu < \infty$? It follows from Theorem 8.4.7. Recall why this was so. Let $\{s_n\}$ be a sequence of simple functions attached to f as in the definition of what it means to be L^1 . Then from the definition of $I(s)$ for s simple,

$$|I(|s_n|) - I(|s_m|)| \leq I(|s_n - s_m|)$$

which converges to 0. Since $\{I(|s_n|)\}$ is a Cauchy sequence, it is bounded by a constant C and also $\{|s_n|\}$ is a sequence of simple functions of the right sort which converges pointwise to $|f|$ and so by definition,

$$\int |f| d\mu = I(f) = \lim_{n \rightarrow \infty} I(|s_n|) \leq C.$$

This shows the only if part.

The more interesting part is the if part. Suppose then that f is measurable and $\int |f| d\mu < \infty$. Suppose first that f has values in $[0, \infty)$. It is necessary to obtain the sequence of complex simple functions. By Theorem 8.3.9, there exists a sequence of nonnegative simple functions, $\{s_n\}$ such that $s_n(\omega) \uparrow f(\omega)$. Then by the monotone convergence theorem,

$$\lim_{n \rightarrow \infty} \int (2f - (f - s_n)) d\mu = \int 2f d\mu$$

and so

$$\lim_{n \rightarrow \infty} \int (f - s_n) d\mu = 0.$$

Letting m be large enough, it follows $\int (f - s_m) d\mu < \varepsilon$ and so if $n > m$

$$\int |s_m - s_n| d\mu \leq \int |f - s_m| d\mu < \varepsilon.$$

Therefore, $f \in L^1(\Omega)$ because $\{s_n\}$ is a suitable sequence.

The general case follows from considering positive and negative parts of real and imaginary parts of f . These are each measurable and nonnegative and their integral is finite so each is in $L^1(\Omega)$ by what was just shown. Thus

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i(\operatorname{Im} f^+ - \operatorname{Im} f^-)$$

and so $f \in L^1(\Omega)$. This proves the corollary.

Theorem 8.4.9 (*Dominated Convergence theorem*) Let $f_n \in L^1(\Omega)$ and suppose

$$f(\omega) = \lim_{n \rightarrow \infty} f_n(\omega),$$

and there exists a measurable function g , with values in $[0, \infty]$,³ such that

$$|f_n(\omega)| \leq g(\omega) \text{ and } \int g(\omega) d\mu < \infty.$$

Then $f \in L^1(\Omega)$ and

$$\int f d\mu = \lim_{n \rightarrow \infty} \int f_n d\mu.$$

Proof: f is measurable by Theorem 8.1.8. Since $|f| \leq g$, it follows that

$$f \in L^1(\Omega) \text{ and } |f - f_n| \leq 2g.$$

By Fatou's lemma (Theorem 8.3.18),

$$\begin{aligned} \int 2g d\mu &\leq \liminf_{n \rightarrow \infty} \int 2g - |f - f_n| d\mu \\ &= \int 2g d\mu - \limsup_{n \rightarrow \infty} \int |f - f_n| d\mu. \end{aligned}$$

Subtracting $\int 2g d\mu$,

$$0 \leq -\limsup_{n \rightarrow \infty} \int |f - f_n| d\mu.$$

Hence

$$\begin{aligned} 0 &\geq \limsup_{n \rightarrow \infty} \left(\int |f - f_n| d\mu \right) \geq \limsup_{n \rightarrow \infty} \left| \int f d\mu - \int f_n d\mu \right| \\ &\geq \liminf_{n \rightarrow \infty} \left| \int f d\mu - \int f_n d\mu \right| \geq 0. \end{aligned}$$

This proves the theorem by Lemma 8.3.17 on Page 203 because the $\limsup$ and $\liminf$ are equal.

Corollary 8.4.10 Suppose $f_n \in L^1(\Omega)$ and $f(\omega) = \lim_{n \rightarrow \infty} f_n(\omega)$. Suppose also there exist measurable functions, g_n, g with values in $[0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \int g_n d\mu = \int g d\mu,$$

$g_n(\omega) \rightarrow g(\omega)$ μ a.e. and both $\int g_n d\mu$ and $\int g d\mu$ are finite. Also suppose $|f_n(\omega)| \leq g_n(\omega)$. Then

$$\lim_{n \rightarrow \infty} \int |f - f_n| d\mu = 0.$$

³Note that, since g is allowed to have the value ∞ , it is not known that $g \in L^1(\Omega)$.

Proof: It is just like the above. This time $g + g_n - |f - f_n| \geq 0$ and so by Fatou's lemma,

$$\begin{aligned} & \int 2gd\mu - \limsup_{n \rightarrow \infty} \int |f - f_n| d\mu = \\ & \liminf_{n \rightarrow \infty} \int (g_n + g) - \limsup_{n \rightarrow \infty} \int |f - f_n| d\mu \\ & = \liminf_{n \rightarrow \infty} \int ((g_n + g) - |f - f_n|) d\mu \geq \int 2gd\mu \end{aligned}$$

and so $-\limsup_{n \rightarrow \infty} \int |f - f_n| d\mu \geq 0$.

Definition 8.4.11 Let E be a measurable subset of Ω .

$$\int_E f d\mu \equiv \int f \chi_E d\mu.$$

If $L^1(E)$ is written, the σ algebra is defined as

$$\{E \cap A : A \in \mathcal{F}\}$$

and the measure is μ restricted to this smaller σ algebra. Clearly, if $f \in L^1(\Omega)$, then

$$f \chi_E \in L^1(E)$$

and if $f \in L^1(E)$, then letting $\tilde{f}$ be the 0 extension of f off of E , it follows $\tilde{f} \in L^1(\Omega)$.

8.5 Vitali Convergence Theorem

The Vitali convergence theorem is a convergence theorem which in the case of a finite measure space is superior to the dominated convergence theorem.

Definition 8.5.1 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $\mathfrak{S} \subseteq L^1(\Omega)$. $\mathfrak{S}$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathfrak{S}$

$$|\int_E f d\mu| < \varepsilon \text{ whenever } \mu(E) < \delta.$$

Lemma 8.5.2 If $\mathfrak{S}$ is uniformly integrable, then $|\mathfrak{S}| \equiv \{|f| : f \in \mathfrak{S}\}$ is uniformly integrable. Also $\mathfrak{S}$ is uniformly integrable if $\mathfrak{S}$ is finite.

Proof: Let $\varepsilon > 0$ be given and suppose $\mathfrak{S}$ is uniformly integrable. First suppose the functions are real valued. Let δ be such that if $\mu(E) < \delta$, then

$$\left| \int_E f d\mu \right| < \frac{\varepsilon}{2}$$

for all $f \in \mathfrak{S}$. Let $\mu(E) < \delta$. Then if $f \in \mathfrak{S}$,

$$\begin{aligned} \int_E |f| d\mu &\leq \int_{E \cap [f \leq 0]} (-f) d\mu + \int_{E \cap [f > 0]} f d\mu \\ &= \left| \int_{E \cap [f \leq 0]} f d\mu \right| + \left| \int_{E \cap [f > 0]} f d\mu \right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

In general, if $\mathfrak{S}$ is a uniformly integrable set of complex valued functions, the inequalities,

$$\left| \int_E \operatorname{Re} f d\mu \right| \leq \left| \int_E f d\mu \right|, \quad \left| \int_E \operatorname{Im} f d\mu \right| \leq \left| \int_E f d\mu \right|,$$

imply $\operatorname{Re} \mathfrak{S} \equiv \{\operatorname{Re} f : f \in \mathfrak{S}\}$ and $\operatorname{Im} \mathfrak{S} \equiv \{\operatorname{Im} f : f \in \mathfrak{S}\}$ are also uniformly integrable. Therefore, applying the above result for real valued functions to these sets of functions, it follows $|\mathfrak{S}|$ is uniformly integrable also.

For the last part, it suffices to verify a single function in $L^1(\Omega)$ is uniformly integrable. To do so, note that from the dominated convergence theorem,

$$\lim_{R \rightarrow \infty} \int_{[|f| > R]} |f| d\mu = 0.$$

Let $\varepsilon > 0$ be given and choose R large enough that $\int_{[|f| > R]} |f| d\mu < \frac{\varepsilon}{2}$. Now let $\mu(E) < \frac{\varepsilon}{2R}$. Then

$$\begin{aligned} \int_E |f| d\mu &= \int_{E \cap [|f| \leq R]} |f| d\mu + \int_{E \cap [|f| > R]} |f| d\mu \\ &< R\mu(E) + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves the lemma.

The following theorem is Vitali's convergence theorem.

Theorem 8.5.3 *Let $\{f_n\}$ be a uniformly integrable set of complex valued functions, $\mu(\Omega) < \infty$, and $f_n(x) \rightarrow f(x)$ a.e. where f is a measurable complex valued function. Then $f \in L^1(\Omega)$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} |f_n - f| d\mu = 0. \quad (8.5.25)$$

Proof: First it will be shown that $f \in L^1(\Omega)$. By uniform integrability, there exists $\delta > 0$ such that if $\mu(E) < \delta$, then

$$\int_E |f_n| d\mu < 1$$

for all n . By Egoroff's theorem, there exists a set, E of measure less than δ such that on E^C , $\{f_n\}$ converges uniformly. Therefore, for p large enough, and $n > p$,

$$\int_{E^C} |f_p - f_n| d\mu < 1$$

which implies

$$\int_{E^C} |f_n| d\mu < 1 + \int_{\Omega} |f_p| d\mu.$$

Then since there are only finitely many functions, f_n with $n \leq p$, there exists a constant, M_1 such that for all n ,

$$\int_{E^C} |f_n| d\mu < M_1.$$

But also,

$$\begin{aligned} \int_{\Omega} |f_m| d\mu &= \int_{E^C} |f_m| d\mu + \int_E |f_m| \\ &\leq M_1 + 1 \equiv M. \end{aligned}$$

Therefore, by Fatou's lemma,

$$\int_{\Omega} |f| d\mu \leq \liminf_{n \rightarrow \infty} \int |f_n| d\mu \leq M,$$

showing that $f \in L^1$ as hoped.

Now $\mathfrak{S} \cup \{f\}$ is uniformly integrable so there exists $\delta_1 > 0$ such that if $\mu(E) < \delta_1$, then $\int_E |g| d\mu < \varepsilon/3$ for all $g \in \mathfrak{S} \cup \{f\}$. By Egoroff's theorem, there exists a set, F with $\mu(F) < \delta_1$ such that f_n converges uniformly to f on F^C . Therefore, there exists N such that if $n > N$, then

$$\int_{F^C} |f - f_n| d\mu < \frac{\varepsilon}{3}.$$

It follows that for $n > N$,

$$\begin{aligned} \int_{\Omega} |f - f_n| d\mu &\leq \int_{F^C} |f - f_n| d\mu + \int_F |f| d\mu + \int_F |f_n| d\mu \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which verifies 8.5.25.

8.6 Exercises

1. Let $\Omega = \mathbb{N} = \{1, 2, \dots\}$ and $\mu(S)$ = number of elements in S . If

$$f : \Omega \rightarrow \mathbb{C}$$

what is meant by $\int f d\mu$? Which functions are in $L^1(\Omega)$? Which functions are measurable?

2. Show that for $f \geq 0$ and measurable, $\int f d\mu \equiv \lim_{h \rightarrow 0+} \sum_{i=1}^{\infty} h\mu([ih < f])$.
3. For the measure space of Problem 1, give an example of a sequence of nonnegative measurable functions $\{f_n\}$ converging pointwise to a function f , such that inequality is obtained in Fatou's lemma.
4. Fill in all the details of the proof of Lemma 8.5.2.
5. Let $\sum_{i=1}^n c_i \chi_{E_i}(\omega) = s(\omega)$ be a nonnegative simple function for which the c_i are the distinct nonzero values. Show with the aid of the monotone convergence theorem that the two definitions of the Lebesgue integral given in the chapter are equivalent.
6. Suppose (Ω, μ) is a finite measure space and $\mathfrak{S} \subseteq L^1(\Omega)$. Show $\mathfrak{S}$ is uniformly integrable and bounded in $L^1(\Omega)$ if there exists an increasing function h which satisfies

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = 0, \sup \left\{ \int_{\Omega} h(|f|) d\mu : f \in \mathfrak{S} \right\} < \infty.$$

$\mathfrak{S}$ is bounded if there is some number, M such that

$$\int |f| d\mu \leq M$$

for all $f \in \mathfrak{S}$.

7. Let $\{a_n\}, \{b_n\}$ be sequences in $[-\infty, \infty]$ and $a \in \mathbb{R}$. Show

$$\liminf_{n \rightarrow \infty} (a - a_n) = a - \limsup_{n \rightarrow \infty} a_n.$$

This was used in the proof of the Dominated convergence theorem. Also show

$$\limsup_{n \rightarrow \infty} (-a_n) = -\liminf_{n \rightarrow \infty} (a_n)$$

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$$

provided no sum is of the form $\infty - \infty$. Also show strict inequality can hold in the inequality. State and prove corresponding statements for $\liminf$.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose $f, g : \Omega \rightarrow (-\infty, \infty]$ are measurable. Prove the sets

$$\{\omega : f(\omega) < g(\omega)\} \text{ and } \{\omega : f(\omega) = g(\omega)\}$$

are measurable. **Hint:** The easy way to do this is to write

$$\{\omega : f(\omega) < g(\omega)\} = \cup_{r \in \mathbb{Q}} [f < r] \cap [g > r].$$

Note that $l(x, y) = x - y$ is not continuous on $(-\infty, \infty]$ so the obvious idea doesn't work.

9. Let $\{f_n\}$ be a sequence of real or complex valued measurable functions. Let

$$S = \{\omega : \{f_n(\omega)\} \text{ converges}\}.$$

Show S is measurable. **Hint:** You might try to exhibit the set where f_n converges in terms of countable unions and intersections using the definition of a Cauchy sequence.

10. Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and let f be a nonnegative measurable function defined on Ω . Also let $\phi : [0, \infty) \rightarrow [0, \infty)$ be strictly increasing and have a continuous derivative and $\phi(0) = 0$. Suppose f is bounded and that $0 \leq \phi(f(\omega)) \leq M$ for some number, M . Show that

$$\int_{\Omega} \phi(f) d\mu = \int_0^{\infty} \phi'(s) \mu([s < f]) ds,$$

where the integral on the right is the ordinary improper Riemann integral. **Hint:** First note that $s \rightarrow \phi'(s) \mu([s < f])$ is Riemann integrable because ϕ' is continuous and $s \rightarrow \mu([s < f])$ is a nonincreasing function, hence Riemann integrable. From the second description of the Lebesgue integral and the assumption that $\phi(f(\omega)) \leq M$, argue that for $[M/h]$ the greatest integer less than M/h ,

$$\begin{aligned} \int_{\Omega} \phi(f) d\mu &= \sup_{h>0} \sum_{i=1}^{[M/h]} h\mu([ih < \phi(f)]) \\ &= \sup_{h>0} \sum_{i=1}^{[M/h]} h\mu([\phi^{-1}(ih) < f]) \\ &= \sup_{h>0} \sum_{i=1}^{[M/h]} \frac{h\Delta_i}{\Delta_i} \mu([\phi^{-1}(ih) < f]) \end{aligned}$$

where $\Delta_i = (\phi^{-1}(ih) - \phi^{-1}((i-1)h))$. Now use the mean value theorem to write

$$\begin{aligned} \Delta_i &= (\phi^{-1})'(t_i) h \\ &= \frac{1}{\phi'(\phi^{-1}(t_i))} h \end{aligned}$$

for some t_i between $(i-1)h$ and ih . Therefore, the right side is of the form

$$\sup_h \sum_{i=1}^{[M/h]} \phi'(\phi^{-1}(t_i)) \Delta_i \mu([\phi^{-1}(ih) < f])$$

where $\phi^{-1}(t_i) \in (\phi^{-1}((i-1)h), \phi^{-1}(ih))$. Argue that if t_i were replaced with ih , this would be a Riemann sum for the Riemann integral

$$\int_0^{\phi^{-1}(M)} \phi'(t) \mu([t < f]) dt = \int_0^{\infty} \phi'(t) \mu([t < f]) dt.$$

11. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and suppose f_n converges uniformly to f and that f_n is in $L^1(\Omega)$. When is

$$\lim_{n \rightarrow \infty} \int f_n d\mu = \int f d\mu?$$

12. Suppose $u_n(t)$ is a differentiable function for $t \in (a, b)$ and suppose that for $t \in (a, b)$,

$$|u_n(t)|, |u'_n(t)| < K_n$$

where $\sum_{n=1}^{\infty} K_n < \infty$. Show

$$\left(\sum_{n=1}^{\infty} u_n(t)\right)' = \sum_{n=1}^{\infty} u'_n(t).$$

Hint: This is an exercise in the use of the dominated convergence theorem and the mean value theorem.

13. Show that $\{\sum_{i=1}^{\infty} 2^{-n} \mu([i2^{-n} < f])\}$ for f a nonnegative measurable function is an increasing sequence. Could you define

$$\int f d\mu \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} 2^{-n} \mu([i2^{-n} < f])$$

and would it be equivalent to the above definitions of the Lebesgue integral?

14. Suppose $\{f_n\}$ is a sequence of nonnegative measurable functions defined on a measure space, $(\Omega, \mathcal{S}, \mu)$. Show that

$$\int \sum_{k=1}^{\infty} f_k d\mu = \sum_{k=1}^{\infty} \int f_k d\mu.$$

Hint: Use the monotone convergence theorem along with the fact the integral is linear.

The Construction Of Measures

9.1 Outer Measures

What are some examples of measure spaces? In this chapter, a general procedure is discussed called the method of outer measures. It is due to Caratheodory (1918). This approach shows how to obtain measure spaces starting with an outer measure. This will then be used to construct measures determined by positive linear functionals.

Definition 9.1.1 *Let Ω be a nonempty set and let $\mu : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ satisfy*

$$\mu(\emptyset) = 0,$$

$$\text{If } A \subseteq B, \text{ then } \mu(A) \leq \mu(B),$$

$$\mu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Such a function is called an outer measure. For $E \subseteq \Omega$, E is μ measurable if for all $S \subseteq \Omega$,

$$\mu(S) = \mu(S \setminus E) + \mu(S \cap E). \quad (9.1.1)$$

To help in remembering 9.1.1, think of a measurable set, E , as a process which divides a given set into two pieces, the part in E and the part not in E as in 9.1.1. In the Bible, there are four incidents recorded in which a process of division resulted in more stuff than was originally present.¹ Measurable sets are exactly

¹1 Kings 17, 2 Kings 4, Mathew 14, and Mathew 15 all contain such descriptions. The stuff involved was either oil, bread, flour or fish. In mathematics such things have also been done with sets. In the book by Bruckner Bruckner and Thompson there is an interesting discussion of the Banach Tarski paradox which says it is possible to divide a ball in $\mathbb{R}^3$ into five disjoint pieces and assemble the pieces to form two disjoint balls of the same size as the first. The details can be found in: The Banach Tarski Paradox by Wagon, Cambridge University press. 1985. It is known that all such examples must involve the axiom of choice.

those for which no such miracle occurs. You might think of the measurable sets as the nonmiraculous sets. The idea is to show that they form a σ algebra on which the outer measure, μ is a measure.

First here is a definition and a lemma.

Definition 9.1.2 $(\mu|S)(A) \equiv \mu(S \cap A)$ for all $A \subseteq \Omega$. Thus $\mu|S$ is the name of a new outer measure, called μ restricted to S .

The next lemma indicates that the property of measurability is not lost by considering this restricted measure.

Lemma 9.1.3 *If A is μ measurable, then A is $\mu|S$ measurable.*

Proof: Suppose A is μ measurable. It is desired to show that for all $T \subseteq \Omega$,

$$(\mu|S)(T) = (\mu|S)(T \cap A) + (\mu|S)(T \setminus A).$$

Thus it is desired to show

$$\mu(S \cap T) = \mu(T \cap A \cap S) + \mu(T \cap S \cap A^C). \quad (9.1.2)$$

But 9.1.2 holds because A is μ measurable. Apply Definition 9.1.1 to $S \cap T$ instead of S .

If A is $\mu|S$ measurable, it does not follow that A is μ measurable. Indeed, if you believe in the existence of non measurable sets, you could let $A = S$ for such a μ non measurable set and verify that S is $\mu|S$ measurable.

The next theorem is the main result on outer measures. It is a very general result which applies whenever one has an outer measure on the power set of any set. This theorem will be referred to as Caratheodory's procedure in the rest of the book.

Theorem 9.1.4 *The collection of μ measurable sets, $\mathcal{S}$, forms a σ algebra and*

$$\text{If } F_i \in \mathcal{S}, F_i \cap F_j = \emptyset, \text{ then } \mu(\cup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mu(F_i). \quad (9.1.3)$$

If $F_1 \cdots F_n \subseteq F_{n+1} \subseteq \cdots$, then if $F = \cup_{n=1}^{\infty} F_n$ and $F_n \in \mathcal{S}$, it follows that

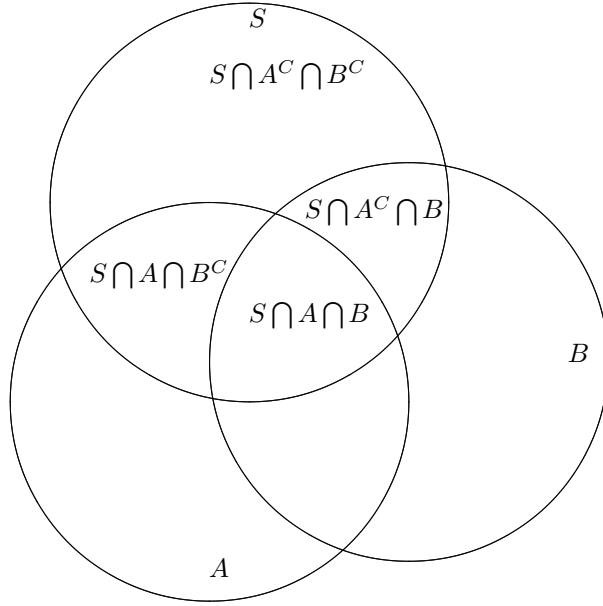
$$\mu(F) = \lim_{n \rightarrow \infty} \mu(F_n). \quad (9.1.4)$$

If $F_1 \cdots F_n \supseteq F_{n+1} \supseteq \cdots$, and if $F = \cap_{n=1}^{\infty} F_n$ for $F_n \in \mathcal{S}$ then if $\mu(F_1) < \infty$,

$$\mu(F) = \lim_{n \rightarrow \infty} \mu(F_n). \quad (9.1.5)$$

Also, $(\mathcal{S}, \mu)$ is complete. By this it is meant that if $F \in \mathcal{S}$ and if $E \subseteq \Omega$ with $\mu(E \setminus F) + \mu(F \setminus E) = 0$, then $E \in \mathcal{S}$.

Proof: First note that $\emptyset$ and Ω are obviously in $\mathcal{S}$. Now suppose $A, B \in \mathcal{S}$. I will show $A \setminus B \equiv A \cap B^C$ is in $\mathcal{S}$. To do so, consider the following picture.



Since μ is subadditive,

$$\mu(S) \leq \mu(S \cap A \cap B^C) + \mu(A \cap B \cap S) + \mu(S \cap B \cap A^C) + \mu(S \cap A^C \cap B^C).$$

Now using $A, B \in \mathcal{S}$,

$$\begin{aligned} \mu(S) &\leq \mu(S \cap A \cap B^C) + \mu(S \cap A \cap B) + \mu(S \cap B \cap A^C) + \mu(S \cap A^C \cap B^C) \\ &= \mu(S \cap A) + \mu(S \cap A^C) = \mu(S) \end{aligned}$$

It follows equality holds in the above. Now observe using the picture if you like that

$$(A \cap B \cap S) \cup (S \cap B \cap A^C) \cup (S \cap A^C \cap B^C) = S \setminus (A \setminus B)$$

and therefore,

$$\begin{aligned} \mu(S) &= \mu(S \cap A \cap B^C) + \mu(A \cap B \cap S) + \mu(S \cap B \cap A^C) + \mu(S \cap A^C \cap B^C) \\ &\geq \mu(S \cap (A \setminus B)) + \mu(S \setminus (A \setminus B)). \end{aligned}$$

Therefore, since S is arbitrary, this shows $A \setminus B \in \mathcal{S}$.

Since $\Omega \in \mathcal{S}$, this shows that $A \in \mathcal{S}$ if and only if $A^C \in \mathcal{S}$. Now if $A, B \in \mathcal{S}$, $A \cup B = (A^C \cap B^C)^C = (A^C \setminus B)^C \in \mathcal{S}$. By induction, if $A_1, \dots, A_n \in \mathcal{S}$, then so is $\cup_{i=1}^n A_i$. If $A, B \in \mathcal{S}$, with $A \cap B = \emptyset$,

$$\mu(A \cup B) = \mu((A \cup B) \cap A) + \mu((A \cup B) \setminus A) = \mu(A) + \mu(B).$$

By induction, if $A_i \cap A_j = \emptyset$ and $A_i \in \mathcal{S}$, $\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$.

Now let $A = \cup_{i=1}^\infty A_i$ where $A_i \cap A_j = \emptyset$ for $i \neq j$.

$$\sum_{i=1}^\infty \mu(A_i) \geq \mu(A) \geq \mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$

Since this holds for all n , you can take the limit as $n \rightarrow \infty$ and conclude,

$$\sum_{i=1}^\infty \mu(A_i) = \mu(A)$$

which establishes 9.1.3. Part 9.1.4 follows from part 9.1.3 just as in the proof of Theorem 8.1.5 on Page 178. That is, letting $F_0 \equiv \emptyset$, use part 9.1.3 to write

$$\begin{aligned} \mu(F) &= \mu(\cup_{k=1}^\infty (F_k \setminus F_{k-1})) = \sum_{k=1}^\infty \mu(F_k \setminus F_{k-1}) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n (\mu(F_k) - \mu(F_{k-1})) = \lim_{n \rightarrow \infty} \mu(F_n). \end{aligned}$$

In order to establish 9.1.5, let the F_n be as given there. Then, since $(F_1 \setminus F_n)$ increases to $(F_1 \setminus F)$, 9.1.4 implies

$$\lim_{n \rightarrow \infty} (\mu(F_1) - \mu(F_n)) = \mu(F_1 \setminus F).$$

Now $\mu(F_1 \setminus F) + \mu(F) \geq \mu(F_1)$ and so $\mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$. Hence

$$\lim_{n \rightarrow \infty} (\mu(F_1) - \mu(F_n)) = \mu(F_1 \setminus F) \geq \mu(F_1) - \mu(F)$$

which implies

$$\lim_{n \rightarrow \infty} \mu(F_n) \leq \mu(F).$$

But since $F \subseteq F_n$,

$$\mu(F) \leq \lim_{n \rightarrow \infty} \mu(F_n)$$

and this establishes 9.1.5. Note that it was assumed $\mu(F_1) < \infty$ because $\mu(F_1)$ was subtracted from both sides.

It remains to show $\mathcal{S}$ is closed under countable unions. Recall that if $A \in \mathcal{S}$, then $A^C \in \mathcal{S}$ and $\mathcal{S}$ is closed under finite unions. Let $A_i \in \mathcal{S}$, $A = \cup_{i=1}^\infty A_i$, $B_n = \cup_{i=1}^n A_i$. Then

$$\begin{aligned} \mu(S) &= \mu(S \cap B_n) + \mu(S \setminus B_n) \\ &= (\mu \lfloor S)(B_n) + (\mu \lfloor S)(B_n^C). \end{aligned} \tag{9.1.6}$$

By Lemma 9.1.3 B_n is $(\mu \lfloor S)$ measurable and so is B_n^C . I want to show $\mu(S) \geq \mu(S \setminus A) + \mu(S \cap A)$. If $\mu(S) = \infty$, there is nothing to prove. Assume $\mu(S) < \infty$.

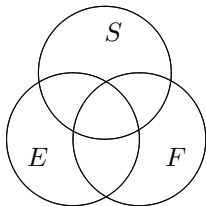
Then apply Parts 9.1.5 and 9.1.4 to the outer measure, $\mu|_S$ in 9.1.6 and let $n \rightarrow \infty$. Thus

$$B_n \uparrow A, \quad B_n^C \downarrow A^C$$

and this yields $\mu(S) = (\mu|_S)(A) + (\mu|_S)(A^C) = \mu(S \cap A) + \mu(S \setminus A)$.

Therefore $A \in \mathcal{S}$ and this proves Parts 9.1.3, 9.1.4, and 9.1.5. It remains to prove the last assertion about the measure being complete.

Let $F \in \mathcal{S}$ and let $\mu(E \setminus F) + \mu(F \setminus E) = 0$. Consider the following picture.



Then referring to this picture and using $F \in \mathcal{S}$,

$$\begin{aligned} \mu(S) &\leq \mu(S \cap E) + \mu(S \setminus E) \\ &\leq \mu(S \cap E \cap F) + \mu((S \cap E) \setminus F) + \mu(S \setminus F) + \mu(F \setminus E) \\ &\leq \mu(S \cap F) + \mu(E \setminus F) + \mu(S \setminus F) + \mu(F \setminus E) \\ &= \mu(S \cap F) + \mu(S \setminus F) = \mu(S) \end{aligned}$$

Hence $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$ and so $E \in \mathcal{S}$. This shows that $(\mathcal{S}, \mu)$ is complete and proves the theorem.

Completeness usually occurs in the following form. $E \subseteq F \in \mathcal{S}$ and $\mu(F) = 0$. Then $E \in \mathcal{S}$.

Where do outer measures come from? One way to obtain an outer measure is to start with a measure μ , defined on a σ algebra of sets, $\mathcal{S}$, and use the following definition of the outer measure induced by the measure.

Definition 9.1.5 Let μ be a measure defined on a σ algebra of sets, $\mathcal{S} \subseteq \mathcal{P}(\Omega)$. Then the outer measure induced by μ , denoted by $\bar{\mu}$ is defined on $\mathcal{P}(\Omega)$ as

$$\bar{\mu}(E) = \inf\{\mu(F) : F \in \mathcal{S} \text{ and } F \supseteq E\}.$$

A measure space, $(\mathcal{S}, \Omega, \mu)$ is σ finite if there exist measurable sets, Ω_i with $\mu(\Omega_i) < \infty$ and $\Omega = \cup_{i=1}^{\infty} \Omega_i$.

You should prove the following lemma.

Lemma 9.1.6 If $(\mathcal{S}, \Omega, \mu)$ is σ finite then there exist disjoint measurable sets, $\{B_n\}$ such that $\mu(B_n) < \infty$ and $\cup_{n=1}^{\infty} B_n = \Omega$.

The following lemma deals with the outer measure generated by a measure which is σ finite. It says that if the given measure is σ finite and complete then no new measurable sets are gained by going to the induced outer measure and then considering the measurable sets in the sense of Caratheodory.

Lemma 9.1.7 *Let $(\Omega, \mathcal{S}, \mu)$ be any measure space and let $\bar{\mu} : \mathcal{P}(\Omega) \rightarrow [0, \infty]$ be the outer measure induced by μ . Then $\bar{\mu}$ is an outer measure as claimed and if $\bar{\mathcal{S}}$ is the set of $\bar{\mu}$ measurable sets in the sense of Caratheodory, then $\bar{\mathcal{S}} \supseteq \mathcal{S}$ and $\bar{\mu} = \mu$ on $\mathcal{S}$. Furthermore, if μ is σ finite and $(\Omega, \mathcal{S}, \mu)$ is complete, then $\bar{\mathcal{S}} = \mathcal{S}$.*

Proof: It is easy to see that $\bar{\mu}$ is an outer measure. Let $E \in \mathcal{S}$. The plan is to show $E \in \bar{\mathcal{S}}$ and $\bar{\mu}(E) = \mu(E)$. To show this, let $S \subseteq \Omega$ and then show

$$\bar{\mu}(S) \geq \bar{\mu}(S \cap E) + \bar{\mu}(S \setminus E). \quad (9.1.7)$$

This will verify that $E \in \bar{\mathcal{S}}$. If $\bar{\mu}(S) = \infty$, there is nothing to prove, so assume $\bar{\mu}(S) < \infty$. Thus there exists $T \in \mathcal{S}$, $T \supseteq S$, and

$$\begin{aligned} \bar{\mu}(S) &> \mu(T) - \varepsilon = \mu(T \cap E) + \mu(T \setminus E) - \varepsilon \\ &\geq \bar{\mu}(T \cap E) + \bar{\mu}(T \setminus E) - \varepsilon \\ &\geq \bar{\mu}(S \cap E) + \bar{\mu}(S \setminus E) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves 9.1.7 and verifies $\mathcal{S} \subseteq \bar{\mathcal{S}}$. Now if $E \in \mathcal{S}$ and $V \supseteq E$ with $V \in \mathcal{S}$, $\mu(E) \leq \mu(V)$. Hence, taking inf, $\mu(E) \leq \bar{\mu}(E)$. But also $\mu(E) \geq \bar{\mu}(E)$ since $E \in \mathcal{S}$ and $E \supseteq E$. Hence

$$\bar{\mu}(E) \leq \mu(E) \leq \bar{\mu}(E).$$

Next consider the claim about not getting any new sets from the outer measure in the case the measure space is σ finite and complete.

Suppose first $F \in \bar{\mathcal{S}}$ and $\bar{\mu}(F) < \infty$. Then there exists $E \in \mathcal{S}$ such that $E \supseteq F$ and $\mu(E) = \bar{\mu}(F)$. Since $\bar{\mu}(F) < \infty$,

$$\bar{\mu}(E \setminus F) = \mu(E) - \bar{\mu}(F) = 0.$$

Then there exists $D \supseteq E \setminus F$ such that $D \in \mathcal{S}$ and $\mu(D) = \bar{\mu}(E \setminus F) = 0$. Then by completeness of $\mathcal{S}$, it follows $E \setminus F \in \mathcal{S}$ and so

$$E = (E \setminus F) \cup F$$

Hence $F = E \setminus (E \setminus F) \in \mathcal{S}$. In the general case where $\bar{\mu}(F)$ is not known to be finite, let $\mu(B_n) < \infty$, with $B_n \cap B_m = \emptyset$ for all $n \neq m$ and $\cup_n B_n = \Omega$. Apply what was just shown to $F \cap B_n$, obtaining each of these is in $\mathcal{S}$. Then $F = \cup_n F \cap B_n \in \mathcal{S}$. This proves the Lemma.

9.2 Regular Measures

Usually Ω is not just a set. It is also a topological space. It is very important to consider how the measure is related to this topology.

Definition 9.2.1 Let μ be a measure on a σ algebra $\mathcal{S}$, of subsets of Ω , where (Ω, τ) is a topological space. μ is a Borel measure if $\mathcal{S}$ contains all Borel sets. μ is called outer regular if μ is Borel and for all $E \in \mathcal{S}$,

$$\mu(E) = \inf\{\mu(V) : V \text{ is open and } V \supseteq E\}.$$

μ is called inner regular if μ is Borel and

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, \text{ and } K \text{ is compact}\}.$$

If the measure is both outer and inner regular, it is called regular.

There is an interesting situation in which regularity is obtained automatically. To save on words, let $\mathcal{B}(E)$ denote the σ algebra of Borel sets in E , a closed subset of $\mathbb{R}^n$. It is a very interesting fact that every finite measure on $\mathcal{B}(E)$ must be regular.

Lemma 9.2.2 Let μ be a finite measure defined on $\mathcal{B}(E)$ where E is a closed subset of $\mathbb{R}^n$. Then for every $F \in \mathcal{B}(E)$,

$$\mu(F) = \sup\{\mu(K) : K \subseteq F, K \text{ is closed}\}$$

$$\mu(F) = \inf\{\mu(V) : V \supseteq F, V \text{ is open}\}$$

Proof: For convenience, I will call a measure which satisfies the above two conditions “almost regular”. It would be regular if closed were replaced with compact. First note every open set is the countable union of compact sets and every closed set is the countable intersection of open sets. Here is why. Let V be an open set and let

$$K_k \equiv \{x \in V : \text{dist}(x, V^C) \geq 1/k\}.$$

Then clearly the union of the K_k equals V and each is closed because $x \rightarrow \text{dist}(x, S)$ is always a continuous function whenever S is any nonempty set. Next, for K closed let

$$V_k \equiv \{x \in E : \text{dist}(x, K) < 1/k\}.$$

Clearly the intersection of the V_k equals K . Therefore, letting V denote an open set and K a closed set,

$$\begin{aligned} \mu(V) &= \sup\{\mu(K) : K \subseteq V \text{ and } K \text{ is closed}\} \\ \mu(K) &= \inf\{\mu(V) : V \supseteq K \text{ and } V \text{ is open}\}. \end{aligned}$$

Also since V is open and K is closed,

$$\begin{aligned} \mu(V) &= \inf\{\mu(U) : U \supseteq V \text{ and } U \text{ is open}\} \\ \mu(K) &= \sup\{\mu(L) : L \subseteq K \text{ and } L \text{ is closed}\} \end{aligned}$$

In words, μ is almost regular on open and closed sets. Let

$$\mathcal{F} \equiv \{F \in \mathcal{B}(E) \text{ such that } \mu \text{ is almost regular on } F\}.$$

Then $\mathcal{F}$ contains the open sets. I want to show $\mathcal{F}$ is a σ algebra and then it will follow $\mathcal{F} = \mathcal{B}(E)$.

First I will show $\mathcal{F}$ is closed with respect to complements. Let $F \in \mathcal{F}$. Then since μ is finite and F is inner regular, there exists $K \subseteq F$ such that $\mu(F \setminus K) < \varepsilon$. But $K^C \setminus F^C = F \setminus K$ and so $\mu(K^C \setminus F^C) < \varepsilon$ showing that F^C is outer regular. I have just approximated the measure of F^C with the measure of K^C , an open set containing F^C . A similar argument works to show F^C is inner regular. You start with $V \supseteq F$ such that $\mu(V \setminus F) < \varepsilon$, note $F^C \setminus V^C = V \setminus F$, and then conclude $\mu(F^C \setminus V^C) < \varepsilon$, thus approximating F^C with the closed subset, V^C .

Next I will show $\mathcal{F}$ is closed with respect to taking countable unions. Let $\{F_k\}$ be a sequence of sets in $\mathcal{F}$. Then since $F_k \in \mathcal{F}$, there exist $\{K_k\}$ such that $K_k \subseteq F_k$ and $\mu(F_k \setminus K_k) < \varepsilon/2^{k+1}$. First choose m large enough that

$$\mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m F_k)) < \frac{\varepsilon}{2}.$$

Then

$$\mu((\cup_{k=1}^m F_k) \setminus (\cup_{k=1}^m K_k)) \leq \sum_{k=1}^m \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}$$

and so

$$\begin{aligned} \mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m K_k)) &\leq \mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m F_k)) \\ &\quad + \mu((\cup_{k=1}^m F_k) \setminus (\cup_{k=1}^m K_k)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Since μ is outer regular on F_k , there exists V_k such that $\mu(V_k \setminus F_k) < \varepsilon/2^k$. Then

$$\begin{aligned} \mu((\cup_{k=1}^{\infty} V_k) \setminus (\cup_{k=1}^{\infty} F_k)) &\leq \sum_{k=1}^{\infty} \mu(V_k \setminus F_k) \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \end{aligned}$$

and this completes the demonstration that $\mathcal{F}$ is a σ algebra. This proves the lemma.

Theorem 9.2.3 *Let μ be a finite measure defined on $\mathcal{B}(E)$ where E is a closed subset of $\mathbb{R}^n$. Then μ is regular.*

Proof: From Lemma 9.2.2 μ is outer regular. Now let $F \in \mathcal{B}(E)$. Then since μ is finite, there exists $K \subseteq F$ such that K is closed, $K \subseteq F$, and

$$\mu(F) < \mu(K) + \varepsilon.$$

Then let $K_k \equiv K \cap \overline{B(0, k)}$. Thus K_k is a closed and bounded, hence compact set and $\cup_{k=1}^{\infty} K_k = K$. Therefore, for all k large enough,

$$\begin{aligned} & \mu(F) \\ & < \mu(K_k) + \varepsilon \\ & < \sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ compact} \} + \varepsilon \\ & \leq \mu(F) + \varepsilon \end{aligned}$$

Since ε was arbitrary, it follows

$$\sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ compact} \} = \mu(F).$$

This proves the theorem.

It will be assumed in what follows that (Ω, τ) is a locally compact Hausdorff space. This means it is Hausdorff: If $p, q \in \Omega$ such that $p \neq q$, there exist open sets, U_p and U_q containing p and q respectively such that $U_p \cap U_q = \emptyset$ and Locally compact: There exists a basis of open sets for the topology, $\mathcal{B}$ such that for each $U \in \mathcal{B}$, $\overline{U}$ is compact. Recall $\mathcal{B}$ is a basis for the topology if $\cup \mathcal{B} = \Omega$ and if every open set in τ is the union of sets of $\mathcal{B}$. Also recall a Hausdorff space is normal if whenever H and C are two closed sets, there exist disjoint open sets, U_H and U_C containing H and C respectively. A regular space is one which has the property that if p is a point not in H , a closed set, then there exist disjoint open sets, U_p and U_H containing p and H respectively.

9.3 Urysohn's lemma

Urysohn's lemma which characterizes normal spaces is a very important result which is useful in general topology and in the construction of measures. Because it is somewhat technical a proof is given for the part which is needed.

Theorem 9.3.1 (*Urysohn*) *Let (X, τ) be normal and let $H \subseteq U$ where H is closed and U is open. Then there exists $g : X \rightarrow [0, 1]$ such that g is continuous, $g(x) = 1$ on H and $g(x) = 0$ if $x \notin U$.*

Proof: Let $D \equiv \{r_n\}_{n=1}^{\infty}$ be the rational numbers in $(0, 1)$. Choose V_{r_1} an open set such that

$$H \subseteq V_{r_1} \subseteq \overline{V_{r_1}} \subseteq U.$$

This can be done by applying the assumption that X is normal to the disjoint closed sets, H and U^C , to obtain open sets V and W with

$$H \subseteq V, U^C \subseteq W, \text{ and } V \cap W = \emptyset.$$

Then

$$H \subseteq V \subseteq \overline{V}, \overline{V} \cap U^C = \emptyset$$

and so let $V_{r_1} = V$.

Suppose $V_{r_1}, \dots, V_{r_k}$ have been chosen and list the rational numbers $r_1, \dots, r_k$ in order,

$$r_{l_1} < r_{l_2} < \dots < r_{l_k} \text{ for } \{l_1, \dots, l_k\} = \{1, \dots, k\}.$$

If $r_{k+1} > r_{l_k}$ then letting $p = r_{l_k}$, let $V_{r_{k+1}}$ satisfy

$$\overline{V}_p \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq U.$$

If $r_{k+1} \in (r_{l_i}, r_{l_{i+1}})$, let $p = r_{l_i}$ and let $q = r_{l_{i+1}}$. Then let $V_{r_{k+1}}$ satisfy

$$\overline{V}_p \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq V_q.$$

If $r_{k+1} < r_{l_1}$, let $p = r_{l_1}$ and let $V_{r_{k+1}}$ satisfy

$$H \subseteq V_{r_{k+1}} \subseteq \overline{V}_{r_{k+1}} \subseteq V_p.$$

Thus there exist open sets V_r for each $r \in \mathbb{Q} \cap (0, 1)$ with the property that if $r < s$,

$$H \subseteq V_r \subseteq \overline{V}_r \subseteq V_s \subseteq \overline{V}_s \subseteq U.$$

Now let

$$f(x) = \inf\{t \in D : x \in V_t\}, \quad f(x) \equiv 1 \text{ if } x \notin \bigcup_{t \in D} V_t.$$

(Recall $D = \mathbb{Q} \cap (0, 1)$.) I claim f is continuous.

$$f^{-1}([0, a)) = \cup\{V_t : t < a, t \in D\},$$

an open set.

Next consider $x \in f^{-1}([0, a])$ so $f(x) \leq a$. If $t > a$, then $x \notin V_t$ because if not, then

$$\inf\{t \in D : x \in V_t\} > a.$$

Thus

$$f^{-1}([0, a]) = \cap\{V_t : t > a\} = \cap\{\overline{V}_t : t > a\}$$

which is a closed set. If $a = 1$, $f^{-1}([0, 1]) = f^{-1}([0, a]) = X$. Therefore,

$$f^{-1}((a, 1]) = X \setminus f^{-1}([0, a]) = \text{open set}.$$

It follows f is continuous. Clearly $f(x) = 0$ on H . If $x \in U^C$, then $x \notin V_t$ for any $t \in D$ so $f(x) = 1$ on U^C . Let $g(x) = 1 - f(x)$. This proves the theorem.

In any metric space there is a much easier proof of the conclusion of Urysohn's lemma which applies.

Lemma 9.3.2 *Let S be a nonempty subset of a metric space, (X, d) . Define*

$$f(x) \equiv \text{dist}(x, S) \equiv \inf\{d(x, y) : y \in S\}.$$

Then f is continuous.

Proof: Consider $|f(x) - f(x_1)|$ and suppose without loss of generality that $f(x_1) \geq f(x)$. Then choose $y \in S$ such that $f(x) + \varepsilon > d(x, y)$. Then

$$\begin{aligned} |f(x_1) - f(x)| &= f(x_1) - f(x) \leq f(x_1) - d(x, y) + \varepsilon \\ &\leq d(x_1, y) - d(x, y) + \varepsilon \\ &\leq d(x, x_1) + d(x, y) - d(x, y) + \varepsilon \\ &= d(x_1, x) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $|f(x_1) - f(x)| \leq d(x_1, x)$ and this proves the lemma.

Theorem 9.3.3 (*Urysohn's lemma for metric space*) *Let H be a closed subset of an open set, U in a metric space, (X, d) . Then there exists a continuous function, $g : X \rightarrow [0, 1]$ such that $g(x) = 1$ for all $x \in H$ and $g(x) = 0$ for all $x \notin U$.*

Proof: If $x \notin C$, a closed set, then $\text{dist}(x, C) > 0$ because if not, there would exist a sequence of points of C converging to x and it would follow that $x \in C$. Therefore, $\text{dist}(x, H) + \text{dist}(x, U^C) > 0$ for all $x \in X$. Now define a continuous function, g as

$$g(x) \equiv \frac{\text{dist}(x, U^C)}{\text{dist}(x, H) + \text{dist}(x, U^C)}.$$

It is easy to see this verifies the conclusions of the theorem and this proves the theorem.

Theorem 9.3.4 *Every compact Hausdorff space is normal.*

Proof: First it is shown that X , is regular. Let H be a closed set and let $p \notin H$. Then for each $h \in H$, there exists an open set U_h containing p and an open set V_h containing h such that $U_h \cap V_h = \emptyset$. Since H must be compact, it follows there are finitely many of the sets $V_h, V_{h_1}, \dots, V_{h_n}$ such that $H \subseteq \cup_{i=1}^n V_{h_i}$. Then letting $U = \cap_{i=1}^n U_{h_i}$ and $V = \cup_{i=1}^n V_{h_i}$, it follows that $p \in U$, $H \subseteq V$ and $U \cap V = \emptyset$. Thus X is regular as claimed.

Next let K and H be disjoint nonempty closed sets. Using regularity of X , for every $k \in K$, there exists an open set U_k containing k and an open set V_k containing H such that these two open sets have empty intersection. Thus $H \cap \overline{U_k} = \emptyset$. Finitely many of the $U_k, U_{k_1}, \dots, U_{k_p}$ cover K and so $\cup_{i=1}^p \overline{U_{k_i}}$ is a closed set which has empty intersection with H . Therefore, $K \subseteq \cup_{i=1}^p U_{k_i}$ and $H \subseteq (\cup_{i=1}^p \overline{U_{k_i}})^C$. This proves the theorem.

A useful construction when dealing with locally compact Hausdorff spaces is the notion of the one point compactification of the space discussed earlier. However, it is reviewed here for the sake of convenience or in case you have not read the earlier treatment.

Definition 9.3.5 *Suppose (X, τ) is a locally compact Hausdorff space. Then let $\tilde{X} \equiv X \cup \{\infty\}$ where ∞ is just the name of some point which is not in X which is*

called the point at infinity. A basis for the topology $\tilde{\tau}$ for $\tilde{X}$ is

$$\tau \cup \{K^C \text{ where } K \text{ is a compact subset of } X\}.$$

The complement is taken with respect to $\tilde{X}$ and so the open sets, K^C are basic open sets which contain ∞ .

The reason this is called a compactification is contained in the next lemma.

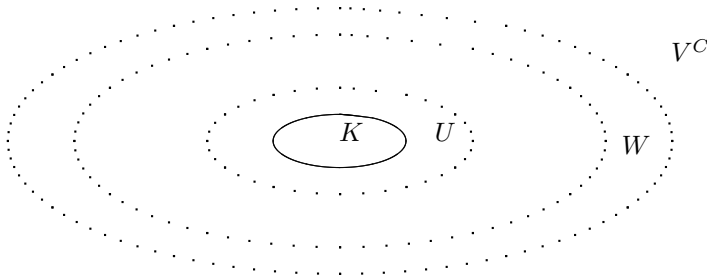
Lemma 9.3.6 *If (X, τ) is a locally compact Hausdorff space, then $(\tilde{X}, \tilde{\tau})$ is a compact Hausdorff space. Also if U is an open set of $\tilde{\tau}$, then $U \setminus \{\infty\}$ is an open set of τ .*

Proof: Since (X, τ) is a locally compact Hausdorff space, it follows $(\tilde{X}, \tilde{\tau})$ is a Hausdorff topological space. The only case which needs checking is the one of $p \in X$ and ∞ . Since (X, τ) is locally compact, there exists an open set of τ , U having compact closure which contains p . Then $p \in U$ and $\infty \in \overline{U}^C$ and these are disjoint open sets containing the points, p and ∞ respectively. Now let $\mathcal{C}$ be an open cover of $\tilde{X}$ with sets from $\tilde{\tau}$. Then ∞ must be in some set, U_∞ from $\mathcal{C}$, which must contain a set of the form K^C where K is a compact subset of X . Then there exist sets from $\mathcal{C}$, $U_1, \dots, U_r$ which cover K . Therefore, a finite subcover of $\tilde{X}$ is $U_1, \dots, U_r, U_\infty$.

To see the last claim, suppose U contains ∞ since otherwise there is nothing to show. Notice that if C is a compact set, then $X \setminus C$ is an open set. Therefore, if $x \in U \setminus \{\infty\}$, and if $\tilde{X} \setminus C$ is a basic open set contained in U containing ∞ , then if x is in this basic open set of $\tilde{X}$, it is also in the open set $X \setminus C \subseteq U \setminus \{\infty\}$. If x is not in any basic open set of the form $\tilde{X} \setminus C$ then x is contained in an open set of τ which is contained in $U \setminus \{\infty\}$. Thus $U \setminus \{\infty\}$ is indeed open in τ .

Theorem 9.3.7 *Let X be a locally compact Hausdorff space, and let K be a compact subset of the open set V . Then there exists a continuous function, $f : X \rightarrow [0, 1]$, such that f equals 1 on K and $\overline{\{x : f(x) \neq 0\}} \equiv \text{spt}(f)$ is a compact subset of V .*

Proof: Let $\tilde{X}$ be the space just described. Then K and V are respectively closed and open in $\tilde{\tau}$. By Theorem 9.3.4 there exist open sets in $\tilde{\tau}$, U , and W such that $K \subseteq U, \infty \in V^C \subseteq W$, and $U \cap W = U \cap (W \setminus \{\infty\}) = \emptyset$.



Thus $W \setminus \{\infty\}$ is an open set in the original topological space which contains V^C , U is an open set in the original topological space which contains K , and $W \setminus \{\infty\}$ and U are disjoint.

Now for each $x \in K$, let U_x be a basic open set whose closure is compact and such that

$$x \in U_x \subseteq U.$$

Thus $\overline{U_x}$ must have empty intersection with V^C because the open set, $W \setminus \{\infty\}$ contains no points of U_x . Since K is compact, there are finitely many of these sets, $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ which cover K . Now let $H \equiv \cup_{i=1}^n U_{x_i}$.

Claim: $\overline{H} = \cup_{i=1}^n \overline{U_{x_i}}$

Proof of claim: Suppose $p \in \overline{H}$. If $p \notin \cup_{i=1}^n \overline{U_{x_i}}$ then it follows $p \notin \overline{U_{x_i}}$ for each i . Therefore, there exists an open set, R_i containing p such that R_i contains no other points of U_{x_i} . Therefore, $R = \cap_{i=1}^n R_i$ is an open set containing p which contains no other points of $\cup_{i=1}^n U_{x_i} = H$, a contradiction. Therefore, $\overline{H} \subseteq \cup_{i=1}^n \overline{U_{x_i}}$. On the other hand, if $p \in \overline{U_{x_i}}$ then p is obviously in $\overline{H}$ so this proves the claim.

From the claim, $K \subseteq H \subseteq \overline{H} \subseteq V$ and $\overline{H}$ is compact because it is the finite union of compact sets. By Urysohn's lemma, there exists f_1 continuous on $\overline{H}$ which has values in $[0, 1]$ such that f_1 equals 1 on K and equals 0 off H . Let f denote the function which extends f_1 to be 0 off $\overline{H}$. Then for $\alpha > 0$, the continuity of f_1 implies there exists U open in the topological space such that

$$f^{-1}((-\infty, \alpha)) = f_1^{-1}((-\infty, \alpha)) \cup \overline{H}^C = (U \cap \overline{H}) \cup \overline{H}^C = U \cup \overline{H}^C$$

an open set. If $\alpha \leq 0$,

$$f^{-1}((-\infty, \alpha)) = \emptyset$$

an open set. If $\alpha > 0$, there exists an open set U such that

$$f^{-1}((\alpha, \infty)) = f_1^{-1}((\alpha, \infty)) = U \cap \overline{H} = U \cap H$$

because U must be a subset of H since by definition $f = 0$ off H . If $\alpha \leq 0$, then

$$f^{-1}((\alpha, \infty)) = X,$$

an open set. Thus f is continuous and $\text{spt}(f) \subseteq \overline{H}$, a compact subset of V . This proves the theorem.

In fact, the conclusion of the above theorem could be used to prove that the topological space is locally compact. However, this is not needed here.

In case you would like a more elementary proof which does not use the one point compactification idea, here is such a proof.

Theorem 9.3.8 *Let X be a locally compact Hausdorff space, and let K be a compact subset of the open set V . Then there exists a continuous function, $f : X \rightarrow [0, 1]$, such that f equals 1 on K and $\overline{\{x : f(x) \neq 0\}} \equiv \text{spt}(f)$ is a compact subset of V .*

Proof: To begin with, here is a claim. This claim is obvious in the case of a metric space but requires some proof in this more general case.

Claim: If $k \in K$ then there exists an open set U_k containing k such that $\overline{U_k}$ is contained in V .

Proof of claim: Since X is locally compact, there exists a basis of open sets whose closures are compact, $\mathcal{U}$. Denote by $\mathcal{C}$ the set of all $U \in \mathcal{U}$ which contain k and let $\mathcal{C}'$ denote the set of all closures of these sets of $\mathcal{C}$ intersected with the closed set V^C . Thus $\mathcal{C}'$ is a collection of compact sets. I will argue that there are finitely many of the sets of $\mathcal{C}'$ which have empty intersection. If not, then $\mathcal{C}'$ has the finite intersection property and so there exists a point p in all of them. Since X is a Hausdorff space, there exist disjoint basic open sets from $\mathcal{U}$, A, B such that $k \in A$ and $p \in B$. Therefore, $p \notin \overline{A}$ contrary to the above requirement that p be in all such sets. It follows there are sets $A_1, \dots, A_m$ in $\mathcal{C}$ such that

$$V^C \cap \overline{A_1} \cap \dots \cap \overline{A_m} = \emptyset$$

Let $U_k = A_1 \cap \dots \cap A_m$. Then $\overline{U_k} \subseteq \overline{A_1} \cap \dots \cap \overline{A_m}$ and so it has empty intersection with V . Thus it is contained in V . Also $\overline{U_k}$ is a closed subset of the compact set $\overline{A_1}$ so it is compact. This proves the claim.

Now to complete the proof of the theorem, since K is compact, there are finitely many U_k of the sort just described which cover $K, U_{k_1}, \dots, U_{k_r}$. Let

$$H = \cup_{i=1}^r U_{k_i}$$

so it follows

$$\overline{H} = \cup_{i=1}^r \overline{U_{k_i}}$$

and so $K \subseteq H \subseteq \overline{H} \subseteq V$ and $\overline{H}$ is a compact set. By Urysohn's lemma, there exists f_1 continuous on $\overline{H}$ which has values in $[0, 1]$ such that f_1 equals 1 on K and equals 0 off H . Let f denote the function which extends f_1 to be 0 off $\overline{H}$. Then for $\alpha > 0$, the continuity of f_1 implies there exists U open in the topological space such that

$$f^{-1}((-\infty, \alpha)) = f_1^{-1}((-\infty, \alpha)) \cup \overline{H}^C = (U \cap \overline{H}) \cup \overline{H}^C = U \cup \overline{H}^C$$

an open set. If $\alpha \leq 0$,

$$f^{-1}((-\infty, \alpha)) = \emptyset$$

an open set. If $\alpha > 0$, there exists an open set U such that

$$f^{-1}((\alpha, \infty)) = f_1^{-1}((\alpha, \infty)) = U \cap \overline{H} = U \cap H$$

because U must be a subset of H since by definition $f = 0$ off H . If $\alpha \leq 0$, then

$$f^{-1}((\alpha, \infty)) = X,$$

an open set. Thus f is continuous and $\text{spt}(f) \subseteq \overline{H}$, a compact subset of V . This proves the theorem.

Definition 9.3.9 Define $\text{spt}(f)$ (support of f) to be the closure of the set $\{x : f(x) \neq 0\}$. If V is an open set, $C_c(V)$ will be the set of continuous functions f , defined on Ω having $\text{spt}(f) \subseteq V$. Thus in Theorem 9.3.7 or 9.3.8, $f \in C_c(V)$.

Definition 9.3.10 If K is a compact subset of an open set, V , then $K \prec \phi \prec V$ if

$$\phi \in C_c(V), \phi(K) = \{1\}, \phi(\Omega) \subseteq [0, 1],$$

where Ω denotes the whole topological space considered. Also for $\phi \in C_c(\Omega)$, $K \prec \phi$ if

$$\phi(\Omega) \subseteq [0, 1] \text{ and } \phi(K) = 1.$$

and $\phi \prec V$ if

$$\phi(\Omega) \subseteq [0, 1] \text{ and } \text{spt}(\phi) \subseteq V.$$

Theorem 9.3.11 (Partition of unity) Let K be a compact subset of a locally compact Hausdorff topological space satisfying Theorem 9.3.7 or 9.3.8 and suppose

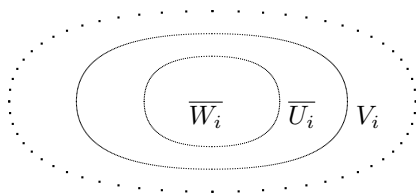
$$K \subseteq V = \cup_{i=1}^n V_i, V_i \text{ open}.$$

Then there exist $\psi_i \prec V_i$ with

$$\sum_{i=1}^n \psi_i(x) = 1$$

for all $x \in K$.

Proof: Let $K_1 = K \setminus \cup_{i=2}^n V_i$. Thus K_1 is compact and $K_1 \subseteq V_1$. Let $K_1 \subseteq W_1 \subseteq \overline{W}_1 \subseteq V_1$ with $\overline{W}_1$ compact. To obtain W_1 , use Theorem 9.3.7 or 9.3.8 to get f such that $K_1 \prec f \prec V_1$ and let $W_1 \equiv \{x : f(x) \neq 0\}$. Thus $W_1, V_2, \dots, V_n$ covers K and $\overline{W}_1 \subseteq V_1$. Let $K_2 = K \setminus (\cup_{i=3}^n V_i \cup W_1)$. Then K_2 is compact and $K_2 \subseteq V_2$. Let $K_2 \subseteq W_2 \subseteq \overline{W}_2 \subseteq V_2$, $\overline{W}_2$ compact. Continue this way finally obtaining $W_1, \dots, W_n$, $K \subseteq W_1 \cup \dots \cup W_n$, and $\overline{W}_i \subseteq V_i$, $\overline{W}_i$ compact. Now let $\overline{W}_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i$, $\overline{U}_i$ compact.



By Theorem 9.3.7 or 9.3.8, let $\overline{U}_i \prec \phi_i \prec V_i$, $\cup_{i=1}^n \overline{W}_i \prec \gamma \prec \cup_{i=1}^n U_i$. Define

$$\psi_i(x) = \begin{cases} \gamma(x)\phi_i(x)/\sum_{j=1}^n \phi_j(x) & \text{if } \sum_{j=1}^n \phi_j(x) \neq 0, \\ 0 & \text{if } \sum_{j=1}^n \phi_j(x) = 0. \end{cases}$$

If x is such that $\sum_{j=1}^n \phi_j(x) = 0$, then $x \notin \cup_{i=1}^n \overline{U}_i$. Consequently $\gamma(y) = 0$ for all y near x and so $\psi_i(y) = 0$ for all y near x . Hence ψ_i is continuous at such x .

If $\sum_{j=1}^n \phi_j(x) \neq 0$, this situation persists near x and so ψ_i is continuous at such points. Therefore ψ_i is continuous. If $x \in K$, then $\gamma(x) = 1$ and so $\sum_{j=1}^n \psi_j(x) = 1$. Clearly $0 \leq \psi_i(x) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The following corollary won't be needed immediately but is of considerable interest later.

Corollary 9.3.12 *If H is a compact subset of V_i , there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Theorem 9.3.11.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i$, $\phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H .

9.4 Positive Linear Functionals

Definition 9.4.1 *Let (Ω, τ) be a topological space. $L : C_c(\Omega) \rightarrow \mathbb{C}$ is called a positive linear functional if L is linear,*

$$L(af_1 + bf_2) = aLf_1 + bLf_2,$$

and if $Lf \geq 0$ whenever $f \geq 0$.

Theorem 9.4.2 *(Riesz representation theorem) Let (Ω, τ) be a locally compact Hausdorff space and let L be a positive linear functional on $C_c(\Omega)$. Then there exists a σ algebra $\mathcal{S}$ containing the Borel sets and a unique measure μ , defined on $\mathcal{S}$, such that*

$$\mu \text{ is complete,} \tag{9.4.8}$$

$$\mu(K) < \infty \text{ for all } K \text{ compact,} \tag{9.4.9}$$

$$\mu(F) = \sup\{\mu(K) : K \subseteq F, K \text{ compact}\},$$

for all F open and for all $F \in \mathcal{S}$ with $\mu(F) < \infty$,

$$\mu(F) = \inf\{\mu(V) : V \supseteq F, V \text{ open}\}$$

for all $F \in \mathcal{S}$, and

$$\int f d\mu = Lf \text{ for all } f \in C_c(\Omega). \tag{9.4.10}$$

The plan is to define an outer measure and then to show that it, together with the σ algebra of sets measurable in the sense of Caratheodory, satisfies the conclusions of the theorem. Always, K will be a compact set and V will be an open set.

Definition 9.4.3 $\mu(V) \equiv \sup\{Lf : f \prec V\}$ for V open, $\mu(\emptyset) = 0$. $\mu(E) \equiv \inf\{\mu(V) : V \supseteq E\}$ for arbitrary sets E .

Lemma 9.4.4 μ is a well-defined outer measure.

Proof: First it is necessary to verify μ is well defined because there are two descriptions of it on open sets. Suppose then that $\mu_1(V) \equiv \inf\{\mu(U) : U \supseteq V \text{ and } U \text{ is open}\}$. It is required to verify that $\mu_1(V) = \mu(V)$ where μ is given as $\sup\{Lf : f \prec V\}$. If $U \supseteq V$, then $\mu(U) \geq \mu(V)$ directly from the definition. Hence from the definition of μ_1 , it follows $\mu_1(V) \geq \mu(V)$. On the other hand, $V \supseteq V$ and so $\mu_1(V) \leq \mu(V)$. This verifies μ is well defined.

It remains to show that μ is an outer measure. Let $V = \cup_{i=1}^{\infty} V_i$ and let $f \prec V$. Then $\text{spt}(f) \subseteq \cup_{i=1}^n V_i$ for some n . Let $\psi_i \prec V_i$, $\sum_{i=1}^n \psi_i = 1$ on $\text{spt}(f)$.

$$Lf = \sum_{i=1}^n L(f\psi_i) \leq \sum_{i=1}^n \mu(V_i) \leq \sum_{i=1}^{\infty} \mu(V_i).$$

Hence

$$\mu(V) \leq \sum_{i=1}^{\infty} \mu(V_i)$$

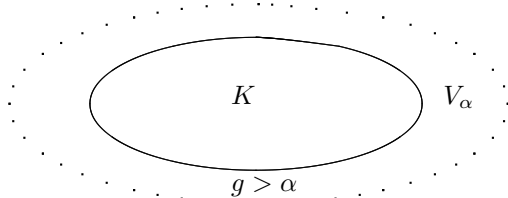
since $f \prec V$ is arbitrary. Now let $E = \cup_{i=1}^{\infty} E_i$. Is $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$? Without loss of generality, it can be assumed $\mu(E_i) < \infty$ for each i since if not so, there is nothing to prove. Let $V_i \supseteq E_i$ with $\mu(E_i) + \varepsilon 2^{-i} > \mu(V_i)$.

$$\mu(E) \leq \mu(\cup_{i=1}^{\infty} V_i) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq \varepsilon + \sum_{i=1}^{\infty} \mu(E_i).$$

Since ε was arbitrary, $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$ which proves the lemma.

Lemma 9.4.5 Let K be compact, $g \geq 0$, $g \in C_c(\Omega)$, and $g = 1$ on K . Then $\mu(K) \leq Lg$. Also $\mu(K) < \infty$ whenever K is compact.

Proof: Let $\alpha \in (0, 1)$ and $V_\alpha = \{x : g(x) > \alpha\}$ so $V_\alpha \supseteq K$ and let $h \prec V_\alpha$.



Then $h \leq 1$ on V_α while $g\alpha^{-1} \geq 1$ on V_α and so $g\alpha^{-1} \geq h$ which implies $L(g\alpha^{-1}) \geq Lh$ and that therefore, since L is linear,

$$Lg \geq \alpha Lh.$$

Since $h \prec V_\alpha$ is arbitrary, and $K \subseteq V_\alpha$,

$$Lg \geq \alpha \mu(V_\alpha) \geq \alpha \mu(K).$$

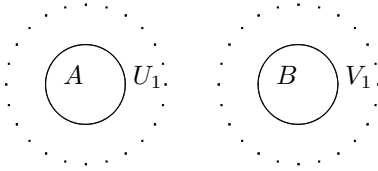
Letting $\alpha \uparrow 1$ yields $Lg \geq \mu(K)$. This proves the first part of the lemma. The second assertion follows from this and Theorem 9.3.7. If K is given, let

$$K \prec g \prec \Omega$$

and so from what was just shown, $\mu(K) \leq Lg < \infty$. This proves the lemma.

Lemma 9.4.6 *If A and B are disjoint compact subsets of Ω , then $\mu(A \cup B) = \mu(A) + \mu(B)$.*

Proof: By Theorem 9.3.7 or 9.3.8, there exists $h \in C_c(\Omega)$ such that $A \prec h \prec B^C$. Let $U_1 = h^{-1}((\frac{1}{2}, 1])$, $V_1 = h^{-1}([0, \frac{1}{2}))$. Then $A \subseteq U_1$, $B \subseteq V_1$ and $U_1 \cap V_1 = \emptyset$.



From Lemma 9.4.5 $\mu(A \cup B) < \infty$ and so there exists an open set, W such that

$$W \supseteq A \cup B, \mu(A \cup B) + \varepsilon > \mu(W).$$

Now let $U = U_1 \cap W$ and $V = V_1 \cap W$. Then

$$U \supseteq A, V \supseteq B, U \cap V = \emptyset, \text{ and } \mu(A \cup B) + \varepsilon \geq \mu(W) \geq \mu(U \cup V).$$

Let $A \prec f \prec U$, $B \prec g \prec V$. Then by Lemma 9.4.5,

$$\mu(A \cup B) + \varepsilon \geq \mu(U \cup V) \geq L(f + g) = Lf + Lg \geq \mu(A) + \mu(B).$$

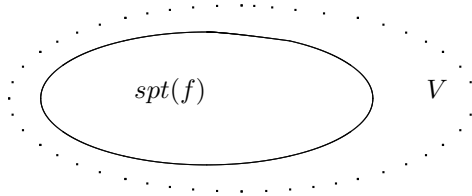
Since $\varepsilon > 0$ is arbitrary, this proves the lemma.

From Lemma 9.4.5 the following lemma is obtained.

Lemma 9.4.7 *Let $f \in C_c(\Omega)$, $f(\Omega) \subseteq [0, 1]$. Then $\mu(\text{spt}(f)) \geq Lf$. Also, every open set, V satisfies*

$$\mu(V) = \sup \{ \mu(K) : K \subseteq V \}.$$

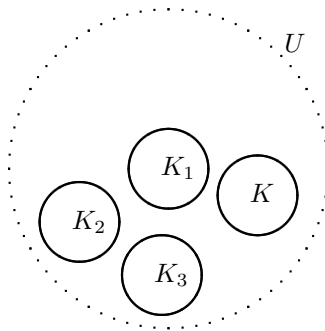
Proof: Let $V \supseteq \text{spt}(f)$ and let $\text{spt}(f) \prec g \prec V$. Then $Lf \leq Lg \leq \mu(V)$ because $f \leq g$. Since this holds for all $V \supseteq \text{spt}(f)$, $Lf \leq \mu(\text{spt}(f))$ by definition of μ .



Finally, let V be open and let $l < \mu(V)$. Then from the definition of μ , there exists $f \prec V$ such that $L(f) > l$. Therefore, $l < \mu(\text{spt}(f)) \leq \mu(V)$ and so this shows the claim about inner regularity of the measure on an open set.

Lemma 9.4.8 *If K is compact there exists V open, $V \supseteq K$, such that $\mu(V \setminus K) \leq \varepsilon$. If V is open with $\mu(V) < \infty$, then there exists a compact set, $K \subseteq V$ with $\mu(V \setminus K) \leq \varepsilon$.*

Proof: Let K be compact. Then from the definition of μ , there exists an open set U , with $\mu(U) < \infty$ and $U \supseteq K$. If $\mu(U \setminus K) \leq \varepsilon$, stop. The open set has been found. If $\mu(U \setminus K) > \varepsilon$, then from Lemma 9.4.7 there exists $K_1 \subseteq U \setminus K$ with $\mu(K_1) > \varepsilon$. Now let $U \setminus K_1$ play the role of U in the above argument. If $\mu((U \setminus K_1) \setminus K) \leq \varepsilon$, stop. Otherwise $\mu((U \setminus K_1) \setminus K) > \varepsilon$ and there exists $K_2 \subseteq (U \setminus K_1) \setminus K$ such that $\mu(K_2) > \varepsilon$. continue this way. You obtain a sequence of disjoint compact sets contained in U such that $\mu(K_j) > \varepsilon$.



Therefore, the process eventually stops and you obtain $\mu((U \cup \cup_{j=1}^m K_j) \setminus K) \leq \varepsilon$ since otherwise, by Lemma 9.4.6,

$$\mu(U) \geq \mu(K \cup \cup_{i=1}^r K_i) = \mu(K) + \sum_{i=1}^r \mu(K_i) \geq r\varepsilon$$

for all r , contradicting $\mu(U) < \infty$. This demonstrates the first part of the lemma.

To show the second part, employ a similar construction. Suppose $\mu(V \setminus K) > \varepsilon$ for all $K \subseteq V$. Then $\mu(V) > \varepsilon$ so there exists $K_1 \subseteq V$ with $\mu(K_1) > \varepsilon$. If $K_1 \cdots K_n$, disjoint, compact subsets of V have been chosen, $\mu(V \setminus \cup_{i=1}^n K_i) > \varepsilon$ and so there exists $K_{n+1} \subseteq V \setminus \cup_{i=1}^n K_i$ such that $\mu(K_{n+1}) > \varepsilon$. In this way there exists a sequence of disjoint compact subsets of V , $\{K_i\}$ with $\mu(K_i) > \varepsilon$. Thus for any m , $K_1 \cdots K_m$ are all contained in V and are disjoint and compact. By Lemma 9.4.6

$$\mu(V) \geq \mu(\cup_{i=1}^m K_i) = \sum_{i=1}^m \mu(K_i) > m\varepsilon$$

for all m , a contradiction to $\mu(V) < \infty$. This proves the second part.

Lemma 9.4.9 *Let $\mathcal{S}$ be the σ algebra of μ measurable sets in the sense of Caratheodory. Then $\mathcal{S} \supseteq$ Borel sets and μ is inner regular on every open set and for every $E \in \mathcal{S}$ with $\mu(E) < \infty$.*

Proof: Define

$$\mathcal{S}_1 = \{E \subseteq \Omega : E \cap K \in \mathcal{S}\}$$

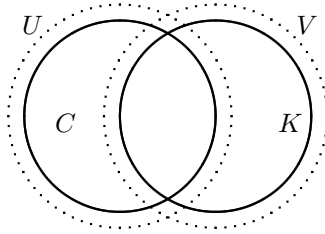
for all compact K .

First it will be shown the compact sets are in $\mathcal{S}$. From this it will follow the closed sets are in $\mathcal{S}_1$. Then you show $\mathcal{S}_1$ is a σ algebra and so it contains the Borel sets. Finally, it is shown that $\mathcal{S}_1 = \mathcal{S}$ and then the inner regularity assertion is established.

Let V be an open set with $\mu(V) < \infty$. I will show that

$$\mu(V) \geq \mu(V \setminus C) + \mu(V \cap C).$$

By Lemma 9.4.8, there exists an open set U containing C and a compact subset of V , K , such that $\mu(V \setminus K) < \varepsilon$ and $\mu(U \setminus C) < \varepsilon$.



Then by Lemma 9.4.6,

$$\begin{aligned} \mu(V) &\geq \mu(K) \geq \mu((K \setminus U) \cup (K \cap C)) \\ &= \mu(K \setminus U) + \mu(K \cap C) \\ &\geq \mu(V \setminus C) + \mu(V \cap C) - 3\varepsilon \end{aligned} \tag{9.4.11}$$

To see the last step,

$$\begin{aligned} \mu(V \setminus C) &\leq \mu(K \setminus U) + \mu(V \setminus K) + \mu(U \setminus C) \\ &\leq \mu(K \setminus U) + 2\varepsilon \end{aligned}$$

while

$$\begin{aligned} \mu(V \cap C) &\leq \mu(K \cap C) + \mu(V \setminus K) \\ &\leq \mu(K \cap C) + \varepsilon \end{aligned}$$

Since ε is arbitrary, it follows from 9.4.11

$$\mu(V) = \mu(V \setminus C) + \mu(V \cap C) \tag{9.4.12}$$

whenever C is compact and V is open. (If $\mu(V) = \infty$, it is obvious that $\mu(V) \geq \mu(V \setminus C) + \mu(V \cap C)$ and it is always the case that $\mu(V) \leq \mu(V \setminus C) + \mu(V \cap C)$.)

Of course 9.4.12 is exactly what needs to be shown for arbitrary S in place of V . It suffices to consider only S having $\mu(S) < \infty$. If $S \subseteq \Omega$, with $\mu(S) < \infty$, let $V \supseteq S$, $\mu(S) + \varepsilon > \mu(V)$. Then from what was just shown, if C is compact,

$$\begin{aligned} \varepsilon + \mu(S) &> \mu(V) = \mu(V \setminus C) + \mu(V \cap C) \\ &\geq \mu(S \setminus C) + \mu(S \cap C). \end{aligned}$$

Since ε is arbitrary, this shows the compact sets are in $\mathcal{S}$. As discussed above, this verifies the closed sets are in $\mathcal{S}_1$.

Therefore, if $E \in \mathcal{S}$ and K is a compact set, it follows $K \cap E \in \mathcal{S}$ and so $\mathcal{S}_1 \supseteq \mathcal{S}$.

To see that $\mathcal{S}_1$ is closed with respect to taking complements, let $E \in \mathcal{S}_1$ and K a compact set.

$$K = (E^C \cap K) \cup (E \cap K).$$

Then from the fact, just established, that the compact sets are in $\mathcal{S}$,

$$E^C \cap K = K \setminus (E \cap K) \in \mathcal{S}.$$

$\mathcal{S}_1$ is closed under countable unions because if K is a compact set and $E_n \in \mathcal{S}_1$,

$$K \cap \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} K \cap E_n \in \mathcal{S}$$

because it is a countable union of sets of $\mathcal{S}$. Thus $\mathcal{S}_1$ is a σ algebra which contains the Borel sets since it contains the closed sets.

The next task is to show $\mathcal{S}_1 = \mathcal{S}$. Let $E \in \mathcal{S}_1$ and let V be an open set with $\mu(V) < \infty$ and choose $K \subseteq V$ such that $\mu(V \setminus K) < \varepsilon$. Then since $E \in \mathcal{S}_1$, it follows $E \cap K \in \mathcal{S}$ and

$$\begin{aligned} \mu(V) &= \mu(V \setminus (K \cap E)) + \mu(V \cap (K \cap E)) \\ &\geq \mu(V \setminus E) + \mu(V \cap E) - \varepsilon \end{aligned}$$

because

$$\mu(V \cap (K \cap E)) + \overbrace{\mu(V \setminus K)}^{< \varepsilon} \geq \mu(V \cap E)$$

Since ε is arbitrary,

$$\mu(V) = \mu(V \setminus E) + \mu(V \cap E)$$

which would show $E \in \mathcal{S}$ if V were an arbitrary set. Now let $S \subseteq \Omega$ be such an arbitrary set. If $\mu(S) = \infty$, then $\mu(S) = \mu(S \cap E) + \mu(S \setminus E)$. If $\mu(S) < \infty$, let

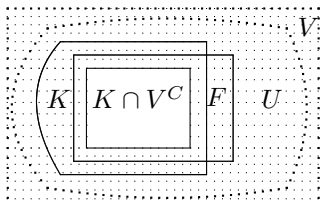
$$V \supseteq S, \mu(S) + \varepsilon \geq \mu(V).$$

Then

$$\mu(S) + \varepsilon \geq \mu(V) = \mu(V \setminus E) + \mu(V \cap E) \geq \mu(S \setminus E) + \mu(S \cap E).$$

Since ε is arbitrary, this shows that $E \in \mathcal{S}$ and so $\mathcal{S}_1 = \mathcal{S}$. Thus $\mathcal{S} \supseteq$ Borel sets as claimed.

From Lemma 9.4.7 and the definition of μ it follows μ is inner regular on all open sets. It remains to show that $\mu(F) = \sup\{\mu(K) : K \subseteq F\}$ for all $F \in \mathcal{S}$ with $\mu(F) < \infty$. It might help to refer to the following crude picture to keep things straight. It also might not help. I am not sure.



In this picture the shaded area is V .

Let $\mu(F) < \infty$ and let U be an open set, $U \supseteq F$, $\mu(U) < \infty$. Let V be open, $V \supseteq U \setminus F$, and $\mu(V \setminus (U \setminus F)) < \varepsilon$. This can be obtained because μ is a measure on $\mathcal{S}$.

$$\mu(V) = \mu(U \setminus F) + \mu(V \setminus (U \setminus F))$$

Thus from the definition of μ , there exists V such that it contains $U \setminus F$ and

$$\mu(U \setminus F) + \varepsilon > \mu(V).$$

and so

$$\mu(V \setminus (U \setminus F)) = \mu(V) - \mu(U \setminus F) < \varepsilon.$$

Also,

$$\begin{aligned} V \setminus (U \setminus F) &= V \cap (U \cap F^C)^C \\ &= V \cap [U^C \cup F] \\ &= (V \cap F) \cup (V \cap U^C) \\ &\supseteq V \cap F \end{aligned}$$

and so

$$\mu(V \cap F) \leq \mu(V \setminus (U \setminus F)) < \varepsilon.$$

Since $V \supseteq U \cap F^C$, $V^C \subseteq U^C \cup F$ so $U \cap V^C \subseteq U \cap F = F$. Hence $U \cap V^C$ is a subset of F . Now let $K \subseteq U$, $\mu(U \setminus K) < \varepsilon$. Thus $K \cap V^C$ is a compact subset of F and

$$\begin{aligned} \mu(F) &= \mu(V \cap F) + \mu(F \setminus V) \\ &< \varepsilon + \mu(F \setminus V) \leq \varepsilon + \mu(U \cap V^C) \leq 2\varepsilon + \mu(K \cap V^C). \end{aligned}$$

Since ε is arbitrary, this proves the second part of the lemma. Formula 9.4.9 of this theorem was established earlier.

It remains to show μ satisfies 9.4.10.

Lemma 9.4.10 $\int f d\mu = Lf$ for all $f \in C_c(\Omega)$.

Proof: Let $f \in C_c(\Omega)$, f real-valued, and suppose $f(\Omega) \subseteq [a, b]$. Choose $t_0 < a$ and let $t_0 < t_1 < \cdots < t_n = b$, $t_i - t_{i-1} < \varepsilon$. Let

$$E_i = f^{-1}((t_{i-1}, t_i]) \cap \text{spt}(f). \quad (9.4.13)$$

Note that $\cup_{i=1}^n E_i$ is a closed set and in fact

$$\cup_{i=1}^n E_i = \text{spt}(f) \quad (9.4.14)$$

since $\Omega = \cup_{i=1}^n f^{-1}((t_{i-1}, t_i])$. Let $V_i \supseteq E_i$, V_i is open and let V_i satisfy

$$f(x) < t_i + \varepsilon \text{ for all } x \in V_i, \quad (9.4.15)$$

$$\mu(V_i \setminus E_i) < \varepsilon/n.$$

By Theorem 9.3.11 there exists $h_i \in C_c(\Omega)$ such that

$$h_i \prec V_i, \quad \sum_{i=1}^n h_i(x) = 1 \text{ on } \text{spt}(f).$$

Now note that for each i ,

$$f(x)h_i(x) \leq h_i(x)(t_i + \varepsilon).$$

(If $x \in V_i$, this follows from 9.4.15. If $x \notin V_i$ both sides equal 0.) Therefore,

$$\begin{aligned} Lf &= L\left(\sum_{i=1}^n fh_i\right) \leq L\left(\sum_{i=1}^n h_i(t_i + \varepsilon)\right) \\ &= \sum_{i=1}^n (t_i + \varepsilon)L(h_i) \\ &= \sum_{i=1}^n (|t_0| + t_i + \varepsilon)L(h_i) - |t_0|L\left(\sum_{i=1}^n h_i\right). \end{aligned}$$

Now note that $|t_0| + t_i + \varepsilon \geq 0$ and so from the definition of μ and Lemma 9.4.5, this is no larger than

$$\begin{aligned} &\sum_{i=1}^n (|t_0| + t_i + \varepsilon)\mu(V_i) - |t_0|\mu(\text{spt}(f)) \\ &\leq \sum_{i=1}^n (|t_0| + t_i + \varepsilon)(\mu(E_i) + \varepsilon/n) - |t_0|\mu(\text{spt}(f)) \\ &\leq |t_0|\sum_{i=1}^n \mu(E_i) + |t_0|\varepsilon + \sum_{i=1}^n t_i\mu(E_i) + \varepsilon(|t_0| + |b|) \end{aligned}$$

$$+\varepsilon \sum_{i=1}^n \mu(E_i) + \varepsilon^2 - |t_0| \mu(\text{spt}(f)).$$

From 9.4.14 and 9.4.13, the first and last terms cancel. Therefore this is no larger than

$$\begin{aligned} (2|t_0| + |b| + \mu(\text{spt}(f)) + \varepsilon)\varepsilon + \sum_{i=1}^n t_{i-1} \mu(E_i) + \varepsilon \mu(\text{spt}(f)) \\ \leq \int f d\mu + (2|t_0| + |b| + 2\mu(\text{spt}(f)) + \varepsilon)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$Lf \leq \int f d\mu \tag{9.4.16}$$

for all $f \in C_c(\Omega)$, f real. Hence equality holds in 9.4.16 because $L(-f) \leq -\int f d\mu$ so $L(f) \geq \int f d\mu$. Thus $Lf = \int f d\mu$ for all $f \in C_c(\Omega)$. Just apply the result for real functions to the real and imaginary parts of f . This proves the Lemma.

This gives the existence part of the Riesz representation theorem.

It only remains to prove uniqueness. Suppose both μ_1 and μ_2 are measures on $\mathcal{S}$ satisfying the conclusions of the theorem. Then if K is compact and $V \supseteq K$, let $K \prec f \prec V$. Then

$$\mu_1(K) \leq \int f d\mu_1 = Lf = \int f d\mu_2 \leq \mu_2(V).$$

Thus $\mu_1(K) \leq \mu_2(K)$ for all K . Similarly, the inequality can be reversed and so it follows the two measures are equal on compact sets. By the assumption of inner regularity on open sets, the two measures are also equal on all open sets. By outer regularity, they are equal on all sets of $\mathcal{S}$. This proves the theorem.

An important example of a locally compact Hausdorff space is any metric space in which the closures of balls are compact. For example, $\mathbb{R}^n$ with the usual metric is an example of this. Not surprisingly, more can be said in this important special case.

Theorem 9.4.11 *Let (Ω, τ) be a metric space in which the closures of the balls are compact and let L be a positive linear functional defined on $C_c(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 9.3.7 or 9.3.8 and in addition the property that μ is regular. The same conclusion follows if (Ω, τ) is a compact Hausdorff space.*

Theorem 9.4.12 *Let (Ω, τ) be a metric space in which the closures of the balls are compact and let L be a positive linear functional defined on $C_c(\Omega)$. Then there exists a measure representing the positive linear functional which satisfies all the conclusions of Theorem 9.3.7 or 9.3.8 and in addition the property that μ is regular. The same conclusion follows if (Ω, τ) is a compact Hausdorff space.*

Proof: Let μ and $\mathcal{S}$ be as described in Theorem 9.4.2. The outer regularity comes automatically as a conclusion of Theorem 9.4.2. It remains to verify inner regularity. Let $F \in \mathcal{S}$ and let $l < k < \mu(F)$. Now let $z \in \Omega$ and $\Omega_n = \overline{B(z, n)}$ for $n \in \mathbb{N}$. Thus $F \cap \Omega_n \uparrow F$. It follows that for n large enough,

$$k < \mu(F \cap \Omega_n) \leq \mu(F).$$

Since $\mu(F \cap \Omega_n) < \infty$ it follows there exists a compact set, K such that $K \subseteq F \cap \Omega_n \subseteq F$ and

$$l < \mu(K) \leq \mu(F).$$

This proves inner regularity. In case (Ω, τ) is a compact Hausdorff space, the conclusion of inner regularity follows from Theorem 9.4.2. This proves the theorem.

The proof of the above yields the following corollary.

Corollary 9.4.13 *Let (Ω, τ) be a locally compact Hausdorff space and suppose μ defined on a σ algebra, $\mathcal{S}$ represents the positive linear functional L where L is defined on $C_c(\Omega)$ in the sense of Theorem 9.3.7 or 9.3.8. Suppose also that there exist $\Omega_n \in \mathcal{S}$ such that $\Omega = \cup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty$. Then μ is regular.*

The following is on the uniqueness of the σ algebra in some cases.

Definition 9.4.14 *Let (Ω, τ) be a locally compact Hausdorff space and let L be a positive linear functional defined on $C_c(\Omega)$ such that the complete measure defined by the Riesz representation theorem for positive linear functionals is inner regular. Then this is called a Radon measure. Thus a Radon measure is complete, and regular.*

Corollary 9.4.15 *Let (Ω, τ) be a locally compact Hausdorff space which is also σ compact meaning*

$$\Omega = \cup_{n=1}^{\infty} \Omega_n, \quad \Omega_n \text{ is compact,}$$

and let L be a positive linear functional defined on $C_c(\Omega)$. Then if $(\mu_1, \mathcal{S}_1)$, and $(\mu_2, \mathcal{S}_2)$ are two Radon measures, together with their σ algebras which represent L then the two σ algebras are equal and the two measures are equal.

Proof: Suppose $(\mu_1, \mathcal{S}_1)$ and $(\mu_2, \mathcal{S}_2)$ both work. It will be shown the two measures are equal on every compact set. Let K be compact and let V be an open set containing K . Then let $K \prec f \prec V$. Then

$$\mu_1(K) = \int_K d\mu_1 \leq \int f d\mu_1 = L(f) = \int f d\mu_2 \leq \mu_2(V).$$

Therefore, taking the infimum over all V containing K implies $\mu_1(K) \leq \mu_2(K)$. Reversing the argument shows $\mu_1(K) = \mu_2(K)$. This also implies the two measures are equal on all open sets because they are both inner regular on open sets. It is being assumed the two measures are regular. Now let $F \in \mathcal{S}_1$ with $\mu_1(F) < \infty$. Then there exist sets, H, G such that $H \subseteq F \subseteq G$ such that H is the countable

union of compact sets and G is a countable intersection of open sets such that $\mu_1(G) = \mu_1(H)$ which implies $\mu_1(G \setminus H) = 0$. Now $G \setminus H$ can be written as the countable intersection of sets of the form $V_k \setminus K_k$ where V_k is open, $\mu_1(V_k) < \infty$ and K_k is compact. From what was just shown, $\mu_2(V_k \setminus K_k) = \mu_1(V_k \setminus K_k)$ so it follows $\mu_2(G \setminus H) = 0$ also. Since μ_2 is complete, and G and H are in $\mathcal{S}_2$, it follows $F \in \mathcal{S}_2$ and $\mu_2(F) = \mu_1(F)$. Now for arbitrary F possibly having $\mu_1(F) = \infty$, consider $F \cap \Omega_n$. From what was just shown, this set is in $\mathcal{S}_2$ and $\mu_2(F \cap \Omega_n) = \mu_1(F \cap \Omega_n)$. Taking the union of these $F \cap \Omega_n$ gives $F \in \mathcal{S}_2$ and also $\mu_1(F) = \mu_2(F)$. This shows $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Similarly, $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

The following lemma is often useful.

Lemma 9.4.16 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space where Ω is a topological space. Suppose μ is a Radon measure and f is measurable with respect to $\mathcal{F}$. Then there exists a Borel measurable function, g , such that $g = f$ a.e.*

Proof: Assume without loss of generality that $f \geq 0$. Then let $s_n \uparrow f$ pointwise. Say

$$s_n(\omega) = \sum_{k=1}^{P_n} c_k^n \mathcal{X}_{E_k^n}(\omega)$$

where $E_k^n \in \mathcal{F}$. By the outer regularity of μ , there exists a Borel set, $F_k^n \supseteq E_k^n$ such that $\mu(F_k^n) = \mu(E_k^n)$. In fact F_k^n can be assumed to be a G_δ set. Let

$$t_n(\omega) \equiv \sum_{k=1}^{P_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Then t_n is Borel measurable and $t_n(\omega) = s_n(\omega)$ for all $\omega \notin N_n$ where $N_n \in \mathcal{F}$ is a set of measure zero. Now let $N \equiv \bigcup_{n=1}^{\infty} N_n$. Then N is a set of measure zero and if $\omega \notin N$, then $t_n(\omega) \rightarrow f(\omega)$. Let $N' \supseteq N$ where N' is a Borel set and $\mu(N') = 0$. Then $t_n \mathcal{X}_{(N')^c}$ converges pointwise to a Borel measurable function, g , and $g(\omega) = f(\omega)$ for all $\omega \notin N'$. Therefore, $g = f$ a.e. and this proves the lemma.

9.5 One Dimensional Lebesgue Measure

To obtain one dimensional Lebesgue measure, you use the positive linear functional L given by

$$Lf = \int f(x) dx$$

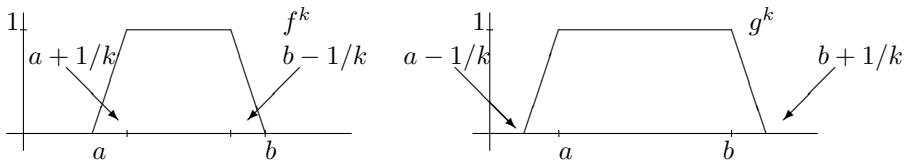
whenever $f \in C_c(\mathbb{R})$. Lebesgue measure, denoted by m is the measure obtained from the Riesz representation theorem such that

$$\int f dm = Lf = \int f(x) dx.$$

From this it is easy to verify that

$$m([a, b]) = m((a, b)) = b - a. \quad (9.5.17)$$

This will be done in general a little later but for now, consider the following picture of functions, f^k and g^k . Note that $f^k \leq \mathcal{X}_{(a,b)} \leq \mathcal{X}_{[a,b]} \leq g^k$.



Then considering lower sums and upper sums in the inequalities on the ends,

$$\begin{aligned} \left(b - a - \frac{2}{k}\right) &\leq \int f^k dx = \int f^k dm \leq m((a, b)) \leq m([a, b]) \\ &= \int \mathcal{X}_{[a,b]} dm \leq \int g^k dm = \int g^k dx \leq \left(b - a + \frac{2}{k}\right). \end{aligned}$$

From this the claim in 9.5.17 follows.

9.6 One Dimensional Lebesgue Stieltjes Measure

This is just a generalization of Lebesgue measure. Instead of the functional,

$$Lf \equiv \int f(x) dx, \quad f \in C_c(\mathbb{R}),$$

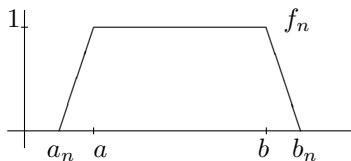
you use the functional

$$Lf \equiv \int f(x) dF(x) \quad f \in C_c(\mathbb{R}),$$

where F is an increasing function defined on $\mathbb{R}$. By Theorem 3.3.4 this functional is easily seen to be well defined. Therefore, by the Riesz representation theorem there exists a unique Radon measure μ representing the functional. Thus

$$\int_{\mathbb{R}} f d\mu = \int f dF$$

for all $f \in C_c(\mathbb{R})$. Now consider what this measure does to intervals. To begin with, consider what it does to the closed interval, $[a, b]$. The following picture may help.



In this picture $\{a_n\}$ increases to a and b_n decreases to b . Also suppose a, b are points of continuity of F . Therefore,

$$F(b) - F(a) \leq Lf_n = \int_{\mathbb{R}} f_n d\mu \leq F(b_n) - F(a_n)$$

Passing to the limit and using the dominated convergence theorem, this shows

$$\mu([a, b]) = F(b) - F(a) = F(b+) - F(a-).$$

Next suppose a, b are arbitrary, maybe not points of continuity of F . Then letting a_n and b_n be as in the above picture which are points of continuity of F ,

$$\begin{aligned} \mu([a, b]) &= \lim_{n \rightarrow \infty} \mu([a_n, b_n]) = \lim_{n \rightarrow \infty} F(b_n) - F(a_n) \\ &= F(b+) - F(a-). \end{aligned}$$

In particular $\mu(a) = F(a+) - F(a-)$ and so

$$\begin{aligned} \mu((a, b)) &= F(b+) - F(a-) - (F(a+) - F(a-)) \\ &\quad - (F(b+) - F(b-)) \\ &= F(b-) - F(a+). \end{aligned}$$

This shows what μ does to intervals. This is stated as the following proposition.

Proposition 9.6.1 *Let μ be the measure representing the functional*

$$Lf \equiv \int f dF, \quad f \in C_c(\mathbb{R})$$

for F an increasing function defined on $\mathbb{R}$. Then

$$\begin{aligned} \mu([a, b]) &= F(b+) - F(a-) \\ \mu((a, b)) &= F(b-) - F(a+) \\ \mu(a) &= F(a+) - F(a-). \end{aligned}$$

Observation 9.6.2 *Note that all the above would work as well if*

$$Lf \equiv \int f dF, \quad f \in C_c([0, \infty))$$

where F is continuous at 0 and ν is the measure representing this functional. This is because you could just extend $F(x)$ to equal $F(0)$ for $x \leq 0$ and apply the above to the extended F . In this case, $\nu([0, b]) = F(b+) - F(0)$.

9.7 The Distribution Function

There is an interesting connection between the Lebesgue integral of a nonnegative function with something called the distribution function.

Definition 9.7.1 Let $f \geq 0$ and suppose f is measurable. The distribution function is the function defined by

$$t \rightarrow \mu([t < f]).$$

Lemma 9.7.2 If $\{f_n\}$ is an increasing sequence of functions converging pointwise to f then

$$\mu([f > t]) = \lim_{n \rightarrow \infty} \mu([f_n > t])$$

Proof: The sets, $[f_n > t]$ are increasing and their union is $[f > t]$ because if $f(\omega) > t$, then for all n large enough, $f_n(\omega) > t$ also. Therefore, from Theorem 8.1.5 on Page 178 the desired conclusion follows.

Lemma 9.7.3 Suppose $s \geq 0$ is a measurable simple function,

$$s(\omega) \equiv \sum_{k=1}^n a_k \chi_{E_k}(\omega)$$

where the a_k are the distinct nonzero values of s , $a_1 < a_2 < \dots < a_n$. Suppose ϕ is a C^1 function defined on $[0, \infty)$ which has the property that $\phi(0) = 0$, $\phi'(t) > 0$ for all t . Then

$$\int_0^\infty \phi'(t) \mu([s > t]) dm = \int \phi(s) d\mu.$$

Proof: First note that if $\mu(E_k) = \infty$ for any k then both sides equal ∞ and so without loss of generality, assume $\mu(E_k) < \infty$ for all k . Letting $a_0 \equiv 0$, the left side equals

$$\begin{aligned} \sum_{k=1}^n \int_{a_{k-1}}^{a_k} \phi'(t) \mu([s > t]) dm(t) &= \sum_{k=1}^n \int_{a_{k-1}}^{a_k} \phi'(t) \sum_{i=k}^n \mu(E_i) dm \\ &= \sum_{k=1}^n \sum_{i=k}^n \mu(E_i) \int_{a_{k-1}}^{a_k} \phi'(t) dm \\ &= \sum_{k=1}^n \sum_{i=k}^n \mu(E_i) (\phi(a_k) - \phi(a_{k-1})) \\ &= \sum_{i=1}^n \mu(E_i) \sum_{k=1}^i (\phi(a_k) - \phi(a_{k-1})) \\ &= \sum_{i=1}^n \mu(E_i) \phi(a_i) = \int \phi(s) d\mu. \end{aligned}$$

This proves the lemma.

With this lemma the next theorem which is the main result follows easily.

Theorem 9.7.4 *Let $f \geq 0$ be measurable and let ϕ be a C^1 function defined on $[0, \infty)$ which satisfies $\phi'(t) > 0$ for all $t > 0$ and $\phi(0) = 0$. Then*

$$\int \phi(f) d\mu = \int_0^\infty \phi'(t) \mu([f > t]) dm.$$

Proof: By Theorem 8.3.9 on Page 197 there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ which converges pointwise to f . By the monotone convergence theorem and Lemma 9.7.2,

$$\begin{aligned} \int \phi(f) d\mu &= \lim_{n \rightarrow \infty} \int \phi(s_n) d\mu = \lim_{n \rightarrow \infty} \int_0^\infty \phi'(t) \mu([s_n > t]) dm \\ &= \int_0^\infty \phi'(t) \mu([f > t]) dm \end{aligned}$$

This proves the theorem.

This theorem can be generalized to a situation in which ϕ is only increasing and continuous. In the generalization I will replace the symbol ϕ with F to coincide with earlier notation.

Lemma 9.7.5 *Suppose $s \geq 0$ is a measurable simple function,*

$$s(\omega) \equiv \sum_{k=1}^n a_k \chi_{E_k}(\omega)$$

where the a_k are the distinct nonzero values of s , $a_1 < a_2 < \dots < a_n$. Suppose F is an increasing function defined on $[0, \infty)$, $F(0) = 0$, F being continuous at 0 from the right and continuous at every a_k . Then letting μ be a measure and $(\Omega, \mathcal{F}, \mu)$ a measure space,

$$\int_{(0, \infty]} \mu([s > t]) d\nu = \int_\Omega F(s) d\mu.$$

where the integral on the left is the Lebesgue integral for the measure ν given as the Radon measure representing the functional

$$\int_0^\infty g dF$$

for $g \in C_c([0, \infty))$.

Proof: This follows from the following computation and Proposition 9.6.1. Since F is continuous at 0 and the values a_k ,

$$\int_0^\infty \mu([s > t]) d\nu(t) = \sum_{k=1}^n \int_{(a_{k-1}, a_k]} \mu([s > t]) d\nu(t)$$

$$\begin{aligned}
&= \sum_{k=1}^n \int_{(a_{k-1}, a_k]} \sum_{j=k}^n \mu(E_j) dF(t) = \sum_{j=1}^n \mu(E_j) \sum_{k=1}^j \nu((a_{k-1}, a_k]) \\
&= \sum_{j=1}^n \mu(E_j) \sum_{k=1}^j (F(a_k) - F(a_{k-1})) = \sum_{j=1}^n \mu(E_j) F(a_j) \equiv \int_{\Omega} F(s) d\mu
\end{aligned}$$

This proves the lemma.

Now here is the generalization to nonnegative measurable f .

Theorem 9.7.6 *Let $f \geq 0$ be measurable with respect to $\mathcal{F}$ where $(\Omega, \mathcal{F}, \mu)$ a measure space, and let F be an increasing continuous function defined on $[0, \infty)$ and $F(0) = 0$. Then*

$$\int_{\Omega} F(f) d\mu = \int_{(0, \infty]} \mu([f > t]) d\nu(t)$$

where ν is the Radon measure representing

$$Lg = \int_0^{\infty} g dF$$

for $g \in C_c([0, \infty))$.

Proof: By Theorem 8.3.9 on Page 197 there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ which converges pointwise to f . By the monotone convergence theorem and Lemma 9.7.5,

$$\begin{aligned}
\int_{\Omega} F(f) d\mu &= \lim_{n \rightarrow \infty} \int_{\Omega} F(s_n) d\mu = \lim_{n \rightarrow \infty} \int_{(0, \infty]} \mu([s_n > t]) d\nu \\
&= \int_{(0, \infty]} \mu([f > t]) d\nu
\end{aligned}$$

This proves the theorem.

Note that the function $t \rightarrow \mu([f > t])$ is a decreasing function. Therefore, one can make sense of an improper Riemann Stieltjes integral $\int_0^{\infty} \mu([f > t]) dF(t)$. With more work, one can have this equal to the corresponding Lebesgue integral above.

There is a very interesting and important inequality called the good lambda inequality (I am not sure if there is a bad lambda inequality.) which follows from the above theory of distribution functions. It involves the inequality

$$\mu([f > \beta\lambda] \cap [g \leq \delta\lambda]) \leq \phi(\delta) \mu([f > \lambda])$$

for $\beta > 1$, nonnegative functions f, g and is supposed to hold for all small positive δ and $\phi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Note the left side is small when g is large and f is small. The inequality involves dominating an integral involving f with one involving g as described below.

Theorem 9.7.7 *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let F be a continuous increasing function defined on $[0, \infty)$ such that $F(0) = 0$. Suppose also that for all $\alpha > 1$, there exists a constant C_α such that for all $x \in [0, \infty)$,*

$$F(\alpha x) \leq C_\alpha F(x).$$

Also suppose f, g are nonnegative measurable functions and there exists $\beta > 1$ such that for all $\lambda > 0$ and $1 > \delta > 0$,

$$\mu([f > \beta\lambda] \cap [g \leq \delta\lambda]) \leq \phi(\delta) \mu([f > \lambda]) \quad (9.7.18)$$

where $\lim_{\delta \rightarrow 0+} \phi(\delta) = 0$ and ϕ is increasing. Under these conditions, there exists a constant C depending only on β, ϕ such that

$$\int_{\Omega} F(f(\omega)) d\mu(\omega) \leq C \int_{\Omega} F(g(\omega)) d\mu(\omega).$$

Proof: This follows from Theorem 9.7.6. By assumption there exists a constant C_β such that for all $y \geq 0$,

$$F(y) = F\left(\beta \frac{y}{\beta}\right) \leq C_\beta^{-1} F\left(\frac{y}{\beta}\right)$$

and so for all $y \geq 0$,

$$F\left(\frac{y}{\beta}\right) \geq C_\beta F(y). \quad (9.7.19)$$

Let ν be the measure representing the functional $Lg = \int_0^\infty g dF$ for $g \in C_c([0, \infty))$. Integrating both sides gives

$$\int_{[0, \infty)} \mu([f > \beta\lambda] \cap [g \leq \delta\lambda]) d\nu(\lambda) \leq \phi(\delta) \int_{[0, \infty)} \mu([f > \lambda]) d\nu(\lambda)$$

and so by Theorem 9.7.6 applied to the right side,

$$\int_{[0, \infty)} \mu([f/\beta > \lambda] \cap [g/\delta \leq \lambda]) d\nu(\lambda) \leq \phi(\delta) \int_{\Omega} F(f(\omega)) d\mu(\omega)$$

Consider the left side.

$$\begin{aligned} & \int_{[0, \infty)} \mu([f/\beta > \lambda] \cap [g/\delta \leq \lambda]) d\nu(\lambda) \\ &= \int_{[0, \infty)} \int_{\Omega} \mathcal{X}_{([f/\beta > \lambda] \cap [g/\delta \leq \lambda])}(\omega) d\mu d\nu \end{aligned} \quad (9.7.20)$$

Consider the function

$$k(\lambda, \omega) \equiv \mathcal{X}_{([f/\beta > \lambda] \cap [g/\delta \leq \lambda])}(\omega)$$

this is product measurable because you can approximate f, g with increasing sequences of simple functions and replacing f, g respectively with the corresponding function from the sequence, the resulting function is obviously product measurable. Therefore, this function, being the limit of those functions, must also be product measurable. Thus it makes sense to interchange the order of integration in 9.7.20 which implies 9.7.20 equals

$$\begin{aligned} & \int_{\Omega} \int_{[0, \infty)} \mathcal{X}_{([f/\beta > \lambda] \cap [g/\delta \leq \lambda])}(\omega) d\nu d\mu \\ &= \int_{\Omega} \left(\int_{g(\omega)/\delta}^{f(\omega)/\beta} d\nu \right)_+ d\mu. \end{aligned}$$

By Proposition 9.6.1 this reduces to

$$\int_{\Omega} \left(F\left(\frac{f(\omega)}{\beta}\right) - F\left(\frac{g(\omega)}{\delta}\right) \right)_+ d\mu$$

where the $+$ indicates the positive part of the expression. Then by 9.7.19

$$\begin{aligned} & \geq \int_{\Omega} F\left(\frac{f(\omega)}{\beta}\right) d\mu - \int_{\Omega} F\left(\frac{g(\omega)}{\delta}\right) d\mu \\ & \geq C_{\beta} \int_{\Omega} F(f(\omega)) d\mu - \int_{\Omega} F\left(\frac{g(\omega)}{\delta}\right) d\mu. \end{aligned}$$

It follows

$$\phi(\delta) \int_{\Omega} F(f(\omega)) d\mu(\omega) \geq C_{\beta} \int_{\Omega} F(f(\omega)) d\mu - \int_{\Omega} F\left(\frac{g(\omega)}{\delta}\right) d\mu$$

and so if δ is picked so small that $C_{\beta}/2 > \phi(\delta)$, it follows

$$\frac{C_{\beta}}{2} \int_{\Omega} F(f(\omega)) d\mu \leq \int_{\Omega} F\left(\frac{g(\omega)}{\delta}\right) d\mu \leq C_{1/\delta} \int_{\Omega} F(g(\omega)) d\mu$$

and this proves the theorem with $C = 2C_{1/\delta}/C_{\beta}$.

9.8 Product Measures

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be two complete measure spaces. In this section consider the problem of defining a product measure, $\bar{\mu} \times \bar{\nu}$ which is defined on a σ algebra of sets of $X \times Y$ such that $(\bar{\mu} \times \bar{\nu})(E \times F) = \mu(E)\nu(F)$ whenever $E \in \mathcal{S}$ and $F \in \mathcal{T}$. I found the following approach to product measures in [24] and they say they got it from [26].

Definition 9.8.1 Let $\mathcal{R}$ denote the set of countable unions of sets of the form $A \times B$, where $A \in \mathcal{S}$ and $B \in \mathcal{T}$ (Sets of the form $A \times B$ are referred to as measurable rectangles) and also let

$$\rho(A \times B) = \mu(A)\nu(B) \tag{9.8.21}$$

More generally, define

$$\rho(E) \equiv \int \int \mathcal{X}_E(x, y) d\mu d\nu \quad (9.8.22)$$

whenever E is such that

$$x \rightarrow \mathcal{X}_E(x, y) \text{ is } \mu \text{ measurable for all } y \quad (9.8.23)$$

and

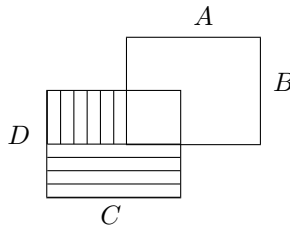
$$y \rightarrow \int \mathcal{X}_E(x, y) d\mu \text{ is } \nu \text{ measurable.} \quad (9.8.24)$$

Note that if $E = A \times B$ as above, then

$$\begin{aligned} \int \int \mathcal{X}_E(x, y) d\mu d\nu &= \int \int \mathcal{X}_{A \times B}(x, y) d\mu d\nu \\ &= \int \int \mathcal{X}_A(x) \mathcal{X}_B(y) d\mu d\nu = \mu(A) \nu(B) = \rho(E) \end{aligned}$$

and so there is no contradiction between 9.8.22 and 9.8.21.

The first goal is to show that for $Q \in \mathcal{R}$, 9.8.23 and 9.8.24 both hold. That is, $x \rightarrow \mathcal{X}_Q(x, y)$ is μ measurable for all y and $y \rightarrow \int \mathcal{X}_Q(x, y) d\mu$ is ν measurable. This is done so that it is possible to speak of $\rho(Q)$. The following lemma will be the fundamental result which will make this possible. First here is a picture.



Lemma 9.8.2 Given $C \times D$ and $\{A_i \times B_i\}_{i=1}^n$, there exist finitely many disjoint rectangles, $\{C'_i \times D'_i\}_{i=1}^p$ such that none of these sets intersect any of the $A_i \times B_i$, each set is contained in $C \times D$ and

$$(\cup_{i=1}^n A_i \times B_i) \cup (\cup_{k=1}^p C'_k \times D'_k) = (C \times D) \cup (\cup_{i=1}^n A_i \times B_i).$$

Proof: From the above picture, you see that

$$(C \times D) \setminus (A_1 \times B_1) = C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)$$

and these last two sets are disjoint, have empty intersection with $A_1 \times B_1$, and

$$(C \times (D \setminus B_1) \cup (C \setminus A_1) \times (D \cap B_1)) \cup (\cup_{i=1}^n A_i \times B_i) = (C \times D) \cup (\cup_{i=1}^n A_i \times B_i)$$

Now suppose disjoint sets, $\left\{\tilde{C}_i \times \tilde{D}_i\right\}_{i=1}^m$ have been obtained, each being a subset of $C \times D$ such that

$$\left(\cup_{i=1}^n A_i \times B_i\right) \cup \left(\cup_{k=1}^m \tilde{C}_k \times \tilde{D}_k\right) = \left(\cup_{i=1}^n A_i \times B_i\right) \cup (C \times D)$$

and for all k , $\tilde{C}_k \times \tilde{D}_k$ has empty intersection with each set of $\{A_i \times B_i\}_{i=1}^p$. Then using the same procedure, replace each of $\tilde{C}_k \times \tilde{D}_k$ with finitely many disjoint rectangles such that none of these intersect $A_{p+1} \times B_{p+1}$ while preserving the union of all the sets involved. The process stops when you have gotten to n . This proves the lemma.

Lemma 9.8.3 *If $Q = \cup_{i=1}^{\infty} A_i \times B_i \in \mathcal{R}$, then there exist disjoint sets, of the form $A'_i \times B'_i$ such that $Q = \cup_{i=1}^{\infty} A'_i \times B'_i$, each $A'_i \times B'_i$ is a subset of some $A_i \times B_i$, and $A'_i \in \mathcal{S}$ while $B'_i \in \mathcal{T}$. Also, the intersection of finitely many sets of $\mathcal{R}$ is a set of $\mathcal{R}$. For ρ defined in 9.8.22, it follows that 9.8.23 and 9.8.24 hold for any element of $\mathcal{R}$. Furthermore,*

$$\rho(Q) = \sum_i \mu(A'_i) \nu(B'_i) = \sum_i \rho(A'_i \times B'_i).$$

Proof: Let Q be given as above. Let $A'_1 \times B'_1 = A_1 \times B_1$. By Lemma 9.8.2, it is possible to replace $A_2 \times B_2$ with finitely many disjoint rectangles, $\{A'_i \times B'_i\}_{i=2}^{m_2}$ such that none of these rectangles intersect $A'_1 \times B'_1$, each is a subset of $A_2 \times B_2$, and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_2} A'_i \times B'_i\right) \cup \left(\cup_{k=3}^{\infty} A_k \times B_k\right)$$

Now suppose disjoint rectangles, $\{A'_i \times B'_i\}_{i=1}^{m_p}$ have been obtained such that each rectangle is a subset of $A_k \times B_k$ for some $k \leq p$ and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_p} A'_i \times B'_i\right) \cup \left(\cup_{k=p+1}^{\infty} A_k \times B_k\right).$$

By Lemma 9.8.2 again, there exist disjoint rectangles $\{A'_i \times B'_i\}_{i=m_p+1}^{m_{p+1}}$ such that each is contained in $A_{p+1} \times B_{p+1}$, none have intersection with any of $\{A'_i \times B'_i\}_{i=1}^{m_p}$ and

$$\cup_{i=1}^{\infty} A_i \times B_i = \left(\cup_{i=1}^{m_{p+1}} A'_i \times B'_i\right) \cup \left(\cup_{k=p+2}^{\infty} A_k \times B_k\right).$$

Note that no change is made in $\{A'_i \times B'_i\}_{i=1}^{m_p}$. Continuing this way proves the existence of the desired sequence of disjoint rectangles, each of which is a subset of at least one of the original rectangles and such that

$$Q = \cup_{i=1}^{\infty} A'_i \times B'_i.$$

It remains to verify $x \rightarrow \mathcal{X}_Q(x, y)$ is μ measurable for all y and

$$y \rightarrow \int \mathcal{X}_Q(x, y) d\mu$$

is ν measurable whenever $Q \in \mathcal{R}$. Let $Q \equiv \cup_{i=1}^{\infty} A_i \times B_i \in \mathcal{R}$. Then by the first part of this lemma, there exists $\{A'_i \times B'_i\}_{i=1}^{\infty}$ such that the sets are disjoint and $\cup_{i=1}^{\infty} A'_i \times B'_i = Q$. Therefore, since the sets are disjoint,

$$\mathcal{X}_Q(x, y) = \sum_{i=1}^{\infty} \mathcal{X}_{A'_i \times B'_i}(x, y) = \sum_{i=1}^{\infty} \mathcal{X}_{A'_i}(x) \mathcal{X}_{B'_i}(y).$$

It follows $x \rightarrow \mathcal{X}_Q(x, y)$ is measurable. Now by the monotone convergence theorem,

$$\begin{aligned} \int \mathcal{X}_Q(x, y) d\mu &= \int \sum_{i=1}^{\infty} \mathcal{X}_{A'_i}(x) \mathcal{X}_{B'_i}(y) d\mu \\ &= \sum_{i=1}^{\infty} \mathcal{X}_{B'_i}(y) \int \mathcal{X}_{A'_i}(x) d\mu \\ &= \sum_{i=1}^{\infty} \mathcal{X}_{B'_i}(y) \mu(A'_i). \end{aligned}$$

It follows $y \rightarrow \int \mathcal{X}_Q(x, y) d\mu$ is measurable and so by the monotone convergence theorem again,

$$\begin{aligned} \int \int \mathcal{X}_Q(x, y) d\mu d\nu &= \int \sum_{i=1}^{\infty} \mathcal{X}_{B'_i}(y) \mu(A'_i) d\nu \\ &= \sum_{i=1}^{\infty} \int \mathcal{X}_{B'_i}(y) \mu(A'_i) d\nu \\ &= \sum_{i=1}^{\infty} \nu(B'_i) \mu(A'_i). \end{aligned} \tag{9.8.25}$$

This shows the measurability conditions, 9.8.23 and 9.8.24 hold for $Q \in \mathcal{R}$ and also establishes the formula for $\rho(Q)$, 9.8.25.

If $\cup_i A_i \times B_i$ and $\cup_j C_j \times D_j$ are two sets of $\mathcal{R}$, then their intersection is

$$\cup_i \cup_j (A_i \cap C_j) \times (B_i \cap D_j)$$

a countable union of measurable rectangles. Thus finite intersections of sets of $\mathcal{R}$ are in $\mathcal{R}$. This proves the lemma.

Now note that from the definition of $\mathcal{R}$ if you have a sequence of elements of $\mathcal{R}$ then their union is also in $\mathcal{R}$. The next lemma will enable the definition of an outer measure.

Lemma 9.8.4 *Suppose $\{R_i\}_{i=1}^{\infty}$ is a sequence of sets of $\mathcal{R}$ then*

$$\rho(\cup_{i=1}^{\infty} R_i) \leq \sum_{i=1}^{\infty} \rho(R_i).$$

Proof: Let $R_i = \cup_{j=1}^{\infty} A_j^i \times B_j^i$. Using Lemma 9.8.3, let $\{A'_m \times B'_m\}_{m=1}^{\infty}$ be a sequence of disjoint rectangles each of which is contained in some $A_j^i \times B_j^i$ for some i, j such that

$$\cup_{i=1}^{\infty} R_i = \cup_{m=1}^{\infty} A'_m \times B'_m.$$

Now define

$$S_i \equiv \{m : A'_m \times B'_m \subseteq A_j^i \times B_j^i \text{ for some } j\}.$$

It is not important to consider whether some m might be in more than one S_i . The important thing to notice is that

$$\cup_{m \in S_i} A'_m \times B'_m \subseteq \cup_{j=1}^{\infty} A_j^i \times B_j^i = R_i.$$

Then by Lemma 9.8.3,

$$\begin{aligned} \rho(\cup_{i=1}^{\infty} R_i) &= \sum_m \rho(A'_m \times B'_m) \\ &\leq \sum_{i=1}^{\infty} \sum_{m \in S_i} \rho(A'_m \times B'_m) \\ &\leq \sum_{i=1}^{\infty} \rho(\cup_{m \in S_i} A'_m \times B'_m) \leq \sum_{i=1}^{\infty} \rho(R_i). \end{aligned}$$

This proves the lemma.

So far, there is no measure and no σ algebra. However, the next step is to define an outer measure which will lead to a measure on a σ algebra of measurable sets from the Caratheodory procedure. When this is done, it will be shown that this measure can be computed using ρ which implies the important Fubini theorem.

Now it is possible to define an outer measure.

Definition 9.8.5 For $S \subseteq X \times Y$, define

$$(\overline{\mu \times \nu})(S) \equiv \inf \{\rho(R) : S \subseteq R, R \in \mathcal{R}\}. \quad (9.8.26)$$

The following proposition is interesting but is not needed in the development which follows. It gives a different description of $(\overline{\mu \times \nu})$.

Proposition 9.8.6 $(\overline{\mu \times \nu})(S) = \inf \{\sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : S \subseteq \cup_{i=1}^{\infty} A_i \times B_i\}$

Proof: Let $\lambda(S) \equiv \inf \{\sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : S \subseteq \cup_{i=1}^{\infty} A_i \times B_i\}$. Suppose $S \subseteq \cup_{i=1}^{\infty} A_i \times B_i \equiv Q \in \mathcal{R}$. Then by Lemma 9.8.3, $Q = \cup_i A'_i \times B'_i$ where these rectangles are disjoint. Thus by this lemma, $\rho(Q) = \sum_{i=1}^{\infty} \mu(A'_i) \nu(B'_i) \geq \lambda(S)$ and so $\lambda(S) \leq (\overline{\mu \times \nu})(S)$. If $\lambda(S) = \infty$, this shows $\lambda(S) = (\overline{\mu \times \nu})(S)$. Suppose then that $\lambda(S) < \infty$ and $\lambda(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i)$ where $Q = \cup_{i=1}^{\infty} A_i \times B_i \supseteq S$. Then by Lemma 9.8.3 again, $\cup_{i=1}^{\infty} A_i \times B_i = \cup_{i=1}^{\infty} A'_i \times B'_i$ where the primed rectangles are disjoint, each is a subset of some $A_i \times B_i$ and so

$$\lambda(S) + \varepsilon \geq \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) \geq \sum_{i=1}^{\infty} \mu(A'_i) \nu(B'_i) = \rho(Q) \geq (\overline{\mu \times \nu})(S).$$

Since ε is arbitrary, this shows $\lambda(S) \geq (\overline{\mu \times \nu})(S)$ and this proves the proposition.

Lemma 9.8.7 $\overline{\mu \times \nu}$ is an outer measure on $X \times Y$ and for $R \in \mathcal{R}$

$$(\overline{\mu \times \nu})(R) = \rho(R). \quad (9.8.27)$$

Proof: First consider 9.8.27. Since $R \supseteq R$, it follows $\rho(R) \geq (\overline{\mu \times \nu})(R)$. On the other hand, if $Q \in \mathcal{R}$ and $Q \supseteq R$, then $\rho(Q) \geq \rho(R)$ and so, taking the infimum on the left yields $(\overline{\mu \times \nu})(R) \geq \rho(R)$. This shows 9.8.27.

It is necessary to show that if $S \subseteq T$, then

$$(\overline{\mu \times \nu})(S) \leq (\overline{\mu \times \nu})(T), \quad (9.8.28)$$

$$(\overline{\mu \times \nu})(\cup_{i=1}^{\infty} S_i) \leq \sum_{i=1}^{\infty} (\overline{\mu \times \nu})(S_i). \quad (9.8.29)$$

To do this, note that 9.8.28 is obvious. To verify 9.8.29, note that it is obvious if $(\overline{\mu \times \nu})(S_i) = \infty$ for any i . Therefore, assume $(\overline{\mu \times \nu})(S_i) < \infty$. Then letting $\varepsilon > 0$ be given, there exist $R_i \in \mathcal{R}$ such that

$$(\overline{\mu \times \nu})(S_i) + \frac{\varepsilon}{2^i} > \rho(R_i), \quad R_i \supseteq S_i.$$

Then by Lemma 9.8.4, 9.8.27, and the observation that $\cup_{i=1}^{\infty} R_i \in \mathcal{R}$,

$$\begin{aligned} (\overline{\mu \times \nu})(\cup_{i=1}^{\infty} S_i) &\leq (\overline{\mu \times \nu})(\cup_{i=1}^{\infty} R_i) \\ &= \rho(\cup_{i=1}^{\infty} R_i) \leq \sum_{i=1}^{\infty} \rho(R_i) \\ &\leq \sum_{i=1}^{\infty} \left((\overline{\mu \times \nu})(S_i) + \frac{\varepsilon}{2^i} \right) \\ &= \left(\sum_{i=1}^{\infty} (\overline{\mu \times \nu})(S_i) \right) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves the lemma.

By Caratheodory's procedure, it follows there is a σ algebra of subsets of $X \times Y$, denoted here by $\overline{\mathcal{S}} \times \overline{\mathcal{T}}$ such that $(\overline{\mu \times \nu})$ is a complete measure on this σ algebra. The first thing to note is that every rectangle is in this σ algebra.

Lemma 9.8.8 Every rectangle is $(\overline{\mu \times \nu})$ measurable.

Proof: Let $S \subseteq X \times Y$. The following inequality must be established.

$$(\overline{\mu \times \nu})(S) \geq (\overline{\mu \times \nu})(S \cap (A \times B)) + (\overline{\mu \times \nu})(S \setminus (A \times B)). \quad (9.8.30)$$

The following claim will be used to establish this inequality.

Claim: Let $P, A \times B \in \mathcal{R}$. Then

$$\rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) = \rho(P).$$

Proof of the claim: From Lemma 9.8.3, $P = \bigcup_{i=1}^{\infty} A'_i \times B'_i$ where the $A'_i \times B'_i$ are disjoint. Therefore,

$$P \cap (A \times B) = \bigcup_{i=1}^{\infty} (A \cap A'_i) \times (B \cap B'_i)$$

while

$$P \setminus (A \times B) = \bigcup_{i=1}^{\infty} (A'_i \setminus A) \times B'_i \cup \bigcup_{i=1}^{\infty} (A \cap A'_i) \times (B'_i \setminus B).$$

Since all of the sets in the above unions are disjoint,

$$\begin{aligned} & \rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) = \\ & \int \int \sum_{i=1}^{\infty} \mathcal{X}_{(A \cap A'_i)}(x) \mathcal{X}_{(B \cap B'_i)}(y) d\mu d\nu + \int \int \sum_{i=1}^{\infty} \mathcal{X}_{(A'_i \setminus A)}(x) \mathcal{X}_{B'_i}(y) d\mu d\nu \\ & \quad + \int \int \sum_{i=1}^{\infty} \mathcal{X}_{A \cap A'_i}(x) \mathcal{X}_{B'_i \setminus B}(y) d\mu d\nu \\ & = \sum_{i=1}^{\infty} \mu(A \cap A'_i) \nu(B \cap B'_i) + \mu(A'_i \setminus A) \nu(B'_i) + \mu(A \cap A'_i) \nu(B'_i \setminus B) \\ & = \sum_{i=1}^{\infty} \mu(A \cap A'_i) \nu(B'_i) + \mu(A'_i \setminus A) \nu(B'_i) = \sum_{i=1}^{\infty} \mu(A'_i) \nu(B'_i) = \rho(P). \end{aligned}$$

This proves the claim.

Now continuing to verify 9.8.30, without loss of generality, $(\overline{\mu \times \nu})(S)$ can be assumed finite. Let $P \supseteq S$ for $P \in \mathcal{R}$ and

$$(\overline{\mu \times \nu})(S) + \varepsilon > \rho(P).$$

Then from the claim,

$$\begin{aligned} (\overline{\mu \times \nu})(S) + \varepsilon & > \rho(P) = \rho(P \cap (A \times B)) + \rho(P \setminus (A \times B)) \\ & \geq (\overline{\mu \times \nu})(S \cap (A \times B)) + (\overline{\mu \times \nu})(S \setminus (A \times B)). \end{aligned}$$

Since $\varepsilon > 0$ this shows $A \times B$ is $\overline{\mu \times \nu}$ measurable as claimed.

Lemma 9.8.9 *Let $\mathcal{R}_1$ be defined as the set of all countable intersections of sets of $\mathcal{R}$. Then if $S \subseteq X \times Y$, there exists $R \in \mathcal{R}_1$ for which it makes sense to write $\rho(R)$ because 9.8.23 and 9.8.24 hold such that*

$$(\overline{\mu \times \nu})(S) = \rho(R). \quad (9.8.31)$$

Also, every element of $\mathcal{R}_1$ is $\overline{\mu \times \nu}$ measurable.

Proof: Consider 9.8.31. Let $S \subseteq X \times Y$. If $(\overline{\mu \times \nu})(S) = \infty$, let $R = X \times Y$ and it follows $\rho(X \times Y) = \infty = (\overline{\mu \times \nu})(S)$. Assume then that $(\overline{\mu \times \nu})(S) < \infty$.

Therefore, there exists $P_n \in \mathcal{R}$ such that $P_n \supseteq S$ and

$$(\overline{\mu \times \nu})(S) \leq \rho(P_n) < (\overline{\mu \times \nu})(S) + 1/n. \quad (9.8.32)$$

Let $Q_n = \cap_{i=1}^n P_i \in \mathcal{R}$. Define

$$P \equiv \cap_{i=1}^{\infty} Q_i \supseteq S.$$

Then 9.8.32 holds with Q_n in place of P_n . It is clear that

$$x \rightarrow \mathcal{X}_P(x, y) \text{ is } \mu \text{ measurable}$$

because this function is the pointwise limit of functions for which this is so. It remains to consider whether $y \rightarrow \int \mathcal{X}_P(x, y) d\mu$ is ν measurable. First observe $Q_n \supseteq Q_{n+1}$, $\mathcal{X}_{Q_i} \leq \mathcal{X}_{P_i}$, and

$$\rho(Q_1) = \rho(P_1) = \int \int \mathcal{X}_{P_1}(x, y) d\mu d\nu < \infty. \quad (9.8.33)$$

Therefore, there exists a set of ν measure 0, N , such that if $y \notin N$, then

$$\int \mathcal{X}_{P_1}(x, y) d\mu < \infty.$$

It follows from the dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \mathcal{X}_{N^c}(y) \int \mathcal{X}_{Q_n}(x, y) d\mu = \mathcal{X}_{N^c}(y) \int \mathcal{X}_P(x, y) d\mu$$

and so

$$y \rightarrow \mathcal{X}_{N^c}(y) \int \mathcal{X}_P(x, y) d\mu$$

is also measurable. By completeness of ν ,

$$y \rightarrow \int \mathcal{X}_P(x, y) d\mu$$

must also be ν measurable and so it makes sense to write

$$\int \int \mathcal{X}_P(x, y) d\mu d\nu$$

for every $P \in \mathcal{R}_1$. Also, by the dominated convergence theorem,

$$\begin{aligned} \int \int \mathcal{X}_P(x, y) d\mu d\nu &= \int \mathcal{X}_{N^c}(y) \int \mathcal{X}_P(x, y) d\mu d\nu \\ &= \lim_{n \rightarrow \infty} \int \mathcal{X}_{N^c}(y) \int \mathcal{X}_{Q_n}(x, y) d\mu d\nu \\ &= \lim_{n \rightarrow \infty} \int \int \mathcal{X}_{Q_n}(x, y) d\mu d\nu \\ &= \lim_{n \rightarrow \infty} \rho(Q_n) \in [(\overline{\mu \times \nu})(S), (\overline{\mu \times \nu})(S) + 1/n] \end{aligned}$$

for all n . Therefore,

$$\rho(P) \equiv \int \int \mathcal{X}_P(x, y) d\mu d\nu = (\overline{\mu \times \nu})(S).$$

The sets of $\mathcal{R}_1$ are $\overline{\mu \times \nu}$ measurable because these sets are countable intersections of countable unions of rectangles and Lemma 9.8.8 verifies the rectangles are $\overline{\mu \times \nu}$ measurable. This proves the Lemma.

The following theorem is the main result.

Theorem 9.8.10 *Let $E \subseteq X \times Y$ be $\overline{\mu \times \nu}$ measurable and suppose $(\overline{\mu \times \nu})(E) < \infty$. Then*

$$x \rightarrow \mathcal{X}_E(x, y) \text{ is } \mu \text{ measurable a.e. } y.$$

Modifying $\mathcal{X}_E$ on a set of measure zero, it is possible to write

$$\int \mathcal{X}_E(x, y) d\mu.$$

The function,

$$y \rightarrow \int \mathcal{X}_E(x, y) d\mu$$

is ν measurable and

$$(\overline{\mu \times \nu})(E) = \int \int \mathcal{X}_E(x, y) d\mu d\nu.$$

Similarly,

$$(\overline{\mu \times \nu})(E) = \int \int \mathcal{X}_E(x, y) d\nu d\mu.$$

Proof: By Lemma 9.8.9, there exists $R \in \mathcal{R}_1$ such that

$$\rho(R) = (\overline{\mu \times \nu})(E), \quad R \supseteq E.$$

Therefore, since R is $\overline{\mu \times \nu}$ measurable and $\rho(R) = (\overline{\mu \times \nu})(R)$, it follows

$$(\overline{\mu \times \nu})(R \setminus E) = 0.$$

By Lemma 9.8.9 again, there exists $P \supseteq R \setminus E$ with $P \in \mathcal{R}_1$ and

$$\rho(P) = (\overline{\mu \times \nu})(R \setminus E) = 0.$$

Thus

$$\int \int \mathcal{X}_P(x, y) d\mu d\nu = 0. \tag{9.8.34}$$

Since $P \in \mathcal{R}_1$ Lemma 9.8.9 implies $x \rightarrow \mathcal{X}_P(x, y)$ is μ measurable and it follows from the above there exists a set of ν measure zero, N such that if $y \notin N$, then $\int \mathcal{X}_P(x, y) d\mu = 0$. Therefore, by completeness of ν ,

$$x \rightarrow \mathcal{X}_{N^c}(y) \mathcal{X}_{R \setminus E}(x, y)$$

is μ measurable and

$$\int \mathcal{X}_{N^C}(y) \mathcal{X}_{R \setminus E}(x, y) d\mu = 0. \quad (9.8.35)$$

Now also

$$\mathcal{X}_{N^C}(y) \mathcal{X}_R(x, y) = \mathcal{X}_{N^C}(y) \mathcal{X}_{R \setminus E}(x, y) + \mathcal{X}_{N^C}(y) \mathcal{X}_E(x, y) \quad (9.8.36)$$

and this shows that

$$x \rightarrow \mathcal{X}_{N^C}(y) \mathcal{X}_E(x, y)$$

is μ measurable because it is the difference of two functions with this property. Then by 9.8.35 it follows

$$\int \mathcal{X}_{N^C}(y) \mathcal{X}_E(x, y) d\mu = \int \mathcal{X}_{N^C}(y) \mathcal{X}_R(x, y) d\mu.$$

The right side of this equation equals a ν measurable function and so the left side which equals it is also a ν measurable function. It follows from completeness of ν that $y \rightarrow \int \mathcal{X}_E(x, y) d\mu$ is ν measurable because for y outside of a set of ν measure zero, N it equals $\int \mathcal{X}_R(x, y) d\mu$. Therefore,

$$\begin{aligned} \int \int \mathcal{X}_E(x, y) d\mu d\nu &= \int \int \mathcal{X}_{N^C}(y) \mathcal{X}_E(x, y) d\mu d\nu \\ &= \int \int \mathcal{X}_{N^C}(y) \mathcal{X}_R(x, y) d\mu d\nu \\ &= \int \int \mathcal{X}_R(x, y) d\mu d\nu \\ &= \rho(R) = (\overline{\mu \times \nu})(E). \end{aligned}$$

In all the above there would be no change in writing $d\nu d\mu$ instead of $d\mu d\nu$. The same result would be obtained. This proves the theorem.

Now let $f : X \times Y \rightarrow [0, \infty]$ be $\overline{\mu \times \nu}$ measurable and

$$\int f d(\overline{\mu \times \nu}) < \infty. \quad (9.8.37)$$

Let $s(x, y) \equiv \sum_{i=1}^m c_i \mathcal{X}_{E_i}(x, y)$ be a nonnegative simple function with c_i being the nonzero values of s and suppose

$$0 \leq s \leq f.$$

Then from the above theorem,

$$\int s d(\overline{\mu \times \nu}) = \int \int s d\mu d\nu$$

In which

$$\int s d\mu = \int \mathcal{X}_{N^C}(y) s d\mu$$

for N a set of ν measure zero such that $y \rightarrow \int \mathcal{X}_{N^c}(y) s d\mu$ is ν measurable. This follows because 9.8.37 implies $(\overline{\mu \times \nu})(E_i) < \infty$. Now let $s_n \uparrow f$ where s_n is a nonnegative simple function and

$$\int s_n d(\overline{\mu \times \nu}) = \int \int \mathcal{X}_{N_n^c}(y) s_n(x, y) d\mu d\nu$$

where

$$y \rightarrow \int \mathcal{X}_{N_n^c}(y) s_n(x, y) d\mu$$

is ν measurable. Then let $N \equiv \cup_{n=1}^{\infty} N_n$. It follows N is a set of ν measure zero. Thus

$$\int s_n d(\overline{\mu \times \nu}) = \int \int \mathcal{X}_{N^c}(y) s_n(x, y) d\mu d\nu$$

and letting $n \rightarrow \infty$, the monotone convergence theorem implies

$$\begin{aligned} \int f d(\overline{\mu \times \nu}) &= \int \int \mathcal{X}_{N^c}(y) f(x, y) d\mu d\nu \\ &= \int \int f(x, y) d\mu d\nu \end{aligned}$$

because of completeness of the measures, μ and ν . This proves Fubini's theorem.

Theorem 9.8.11 (Fubini) *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete measure spaces and let*

$$(\overline{\mu \times \nu})(E) \equiv \inf \left\{ \int \int \mathcal{X}_R(x, y) d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}^2$$

where $A_i \in \mathcal{S}$ and $B_i \in \mathcal{T}$. Then $\overline{\mu \times \nu}$ is an outer measure on the subsets of $X \times Y$ and the σ algebra of $\overline{\mu \times \nu}$ measurable sets, $\overline{\mathcal{S} \times \mathcal{T}}$, contains all measurable rectangles. If $f \geq 0$ is a $\overline{\mu \times \nu}$ measurable function satisfying

$$\int_{X \times Y} f d(\overline{\mu \times \nu}) < \infty, \quad (9.8.38)$$

then

$$\int_{X \times Y} f d(\overline{\mu \times \nu}) = \int_Y \int_X f d\mu d\nu,$$

where the iterated integral on the right makes sense because for ν a.e. y , $x \rightarrow f(x, y)$ is μ measurable and $y \rightarrow \int f(x, y) d\mu$ is ν measurable. Similarly,

$$\int_{X \times Y} f d(\overline{\mu \times \nu}) = \int_X \int_Y f d\nu d\mu.$$

²Recall this is the same as

$$\inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \nu(B_i) : E \subseteq \cup_{i=1}^{\infty} A_i \times B_i \right\}$$

in which the A_i and B_i are measurable.

In the case where $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are both σ finite, it is not necessary to assume 9.8.38.

Corollary 9.8.12 (*Fubini*) *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be complete measure spaces such that $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ are both σ finite and let*

$$(\overline{\mu \times \nu})(E) \equiv \inf \left\{ \int \int \chi_R(x, y) d\mu d\nu : E \subseteq R \in \mathcal{R} \right\}$$

where $A_i \in \mathcal{S}$ and $B_i \in \mathcal{T}$. Then $\overline{\mu \times \nu}$ is an outer measure. If $f \geq 0$ is a $\overline{\mu \times \nu}$ measurable function then

$$\int_{X \times Y} f d(\overline{\mu \times \nu}) = \int_Y \int_X f d\mu d\nu,$$

where the iterated integral on the right makes sense because for ν a.e. y , $x \rightarrow f(x, y)$ is μ measurable and $y \rightarrow \int f(x, y) d\mu$ is ν measurable. Similarly,

$$\int_{X \times Y} f d(\overline{\mu \times \nu}) = \int_X \int_Y f d\nu d\mu.$$

Proof: Let $\cup_{n=1}^{\infty} X_n = X$ and $\cup_{n=1}^{\infty} Y_n = Y$ where $X_n \in \mathcal{S}$, $Y_n \in \mathcal{T}$, $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$ for all n and $\mu(X_n) < \infty$, $\nu(Y_n) < \infty$. From Theorem 9.8.11 applied to X_n, Y_n and $f_m \equiv \min(f, m)$,

$$\int_{X_n \times Y_n} f_m d(\overline{\mu \times \nu}) = \int_{Y_n} \int_{X_n} f_m d\mu d\nu$$

Now take $m \rightarrow \infty$ and use the monotone convergence theorem to obtain

$$\int_{X_n \times Y_n} f d(\overline{\mu \times \nu}) = \int_{Y_n} \int_{X_n} f d\mu d\nu.$$

Then use the monotone convergence theorem again letting $n \rightarrow \infty$ to obtain the desired conclusion. The argument for the other order of integration is similar.

Corollary 9.8.13 *If $f \in L^1(X \times Y)$, then*

$$\int f d(\overline{\mu \times \nu}) = \int \int f(x, y) d\mu d\nu = \int \int f(x, y) d\nu d\mu.$$

If μ and ν are σ finite, then if f is $\overline{\mu \times \nu}$ measurable having complex values and either $\int \int |f| d\mu d\nu < \infty$ or $\int \int |f| d\nu d\mu < \infty$, then $\int |f| d(\overline{\mu \times \nu}) < \infty$ so $f \in L^1(X \times Y)$.

Proof: Without loss of generality, it can be assumed that f has real values. Then

$$f = \frac{|f| + f - (|f| - f)}{2}$$

and both $f^+ \equiv \frac{|f|+f}{2}$ and $f^- \equiv \frac{|f|-f}{2}$ are nonnegative and are less than $|f|$. Therefore, $\int g d(\overline{\mu \times \nu}) < \infty$ for $g = f^+$ and $g = f^-$ so the above theorem applies and

$$\begin{aligned} \int f d(\overline{\mu \times \nu}) &\equiv \int f^+ d(\overline{\mu \times \nu}) - \int f^- d(\overline{\mu \times \nu}) \\ &= \int \int f^+ d\mu d\nu - \int \int f^- d\mu d\nu \\ &= \int \int f d\mu d\nu. \end{aligned}$$

It remains to verify the last claim. Suppose s is a simple function,

$$s(x, y) \equiv \sum_{i=1}^m c_i \chi_{E_i} \leq |f|(x, y)$$

where the c_i are the nonzero values of s . Then

$$s\chi_{R_n} \leq |f|\chi_{R_n}$$

where $R_n \equiv X_n \times Y_n$ where $X_n \uparrow X$ and $Y_n \uparrow Y$ with $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. It follows, since the nonzero values of $s\chi_{R_n}$ are achieved on sets of finite measure,

$$\int s\chi_{R_n} d(\overline{\mu \times \nu}) = \int \int s\chi_{R_n} d\mu d\nu.$$

Letting $n \rightarrow \infty$ and applying the monotone convergence theorem, this yields

$$\int s d(\overline{\mu \times \nu}) = \int \int s d\mu d\nu. \quad (9.8.39)$$

Now let $s_n \uparrow |f|$ where s_n is a nonnegative simple function. From 9.8.39,

$$\int s_n d(\overline{\mu \times \nu}) = \int \int s_n d\mu d\nu.$$

Letting $n \rightarrow \infty$ and using the monotone convergence theorem, yields

$$\int |f| d(\overline{\mu \times \nu}) = \int \int |f| d\mu d\nu < \infty$$

9.9 Alternative Treatment Of Product Measure

9.9.1 Monotone Classes And Algebras

Measures are defined on σ algebras which are closed under countable unions. It is for this reason that the theory of measure and integration is so useful in dealing with limits of sequences. However, there is a more basic notion which involves only finite unions and differences.

Definition 9.9.1 $\mathcal{A}$ is said to be an algebra of subsets of a set, Z if $Z \in \mathcal{A}$, $\emptyset \in \mathcal{A}$, and when $E, F \in \mathcal{A}$, $E \cup F$ and $E \setminus F$ are both in $\mathcal{A}$.

It is important to note that if $\mathcal{A}$ is an algebra, then it is also closed under finite intersections. This is because $E \cap F = (E^C \cup F^C)^C \in \mathcal{A}$ since $E^C = Z \setminus E \in \mathcal{A}$ and $F^C = Z \setminus F \in \mathcal{A}$. Note that every σ algebra is an algebra but not the other way around.

Something satisfying the above definition is called an algebra because union is like addition, the set difference is like subtraction and intersection is like multiplication. Furthermore, only finitely many operations are done at a time and so there is nothing like a limit involved.

How can you recognize an algebra when you see one? The answer to this question is the purpose of the following lemma.

Lemma 9.9.2 Suppose $\mathcal{R}$ and $\mathcal{E}$ are subsets of $\mathcal{P}(Z)$ ³ such that $\mathcal{E}$ is defined as the set of all finite disjoint unions of sets of $\mathcal{R}$. Suppose also

$$\emptyset, Z \in \mathcal{R}$$

$$A \cap B \in \mathcal{R} \text{ whenever } A, B \in \mathcal{R},$$

$$A \setminus B \in \mathcal{E} \text{ whenever } A, B \in \mathcal{R}.$$

Then $\mathcal{E}$ is an algebra of sets of Z .

Proof: Note first that if $A \in \mathcal{R}$, then $A^C \in \mathcal{E}$ because $A^C = Z \setminus A$.

Now suppose that E_1 and E_2 are in $\mathcal{E}$,

$$E_1 = \cup_{i=1}^m R_i, \quad E_2 = \cup_{j=1}^n R_j$$

where the R_i are disjoint sets in $\mathcal{R}$ and the R_j are disjoint sets in $\mathcal{R}$. Then

$$E_1 \cap E_2 = \cup_{i=1}^m \cup_{j=1}^n R_i \cap R_j$$

which is clearly an element of $\mathcal{E}$ because no two of the sets in the union can intersect and by assumption they are all in $\mathcal{R}$. Thus by induction, finite intersections of sets of $\mathcal{E}$ are in $\mathcal{E}$. Consider the difference of two elements of $\mathcal{E}$ next.

If $E = \cup_{i=1}^n R_i \in \mathcal{E}$,

$$E^C = \cap_{i=1}^n R_i^C = \text{finite intersection of sets of } \mathcal{E}$$

which was just shown to be in $\mathcal{E}$. Now, if $E_1, E_2 \in \mathcal{E}$,

$$E_1 \setminus E_2 = E_1 \cap E_2^C \in \mathcal{E}$$

from what was just shown about finite intersections.

³Set of all subsets of Z

Finally consider finite unions of sets of $\mathcal{E}$. Let E_1 and E_2 be sets of $\mathcal{E}$. Then

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup E_2 \in \mathcal{E}$$

because $E_1 \setminus E_2$ consists of a finite disjoint union of sets of $\mathcal{R}$ and these sets must be disjoint from the sets of $\mathcal{R}$ whose union yields E_2 because $(E_1 \setminus E_2) \cap E_2 = \emptyset$. This proves the lemma.

The following corollary is particularly helpful in verifying the conditions of the above lemma.

Corollary 9.9.3 *Let $(Z_1, \mathcal{R}_1, \mathcal{E}_1)$ and $(Z_2, \mathcal{R}_2, \mathcal{E}_2)$ be as described in Lemma 9.9.2. Then $(Z_1 \times Z_2, \mathcal{R}, \mathcal{E})$ also satisfies the conditions of Lemma 9.9.2 if $\mathcal{R}$ is defined as*

$$\mathcal{R} \equiv \{R_1 \times R_2 : R_i \in \mathcal{R}_i\}$$

and

$$\mathcal{E} \equiv \{ \text{finite disjoint unions of sets of } \mathcal{R} \}.$$

Consequently, $\mathcal{E}$ is an algebra of sets.

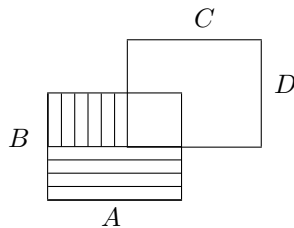
Proof: It is clear $\emptyset, Z_1 \times Z_2 \in \mathcal{R}$. Let $A \times B$ and $C \times D$ be two elements of $\mathcal{R}$.

$$A \times B \cap C \times D = A \cap C \times B \cap D \in \mathcal{R}$$

by assumption.

$$\begin{aligned} A \times B \setminus (C \times D) &= \\ A \times \overbrace{(B \setminus D)}^{\in \mathcal{E}_2} \cup \overbrace{(A \setminus C)}^{\in \mathcal{E}_1} \times \overbrace{(D \cap B)}^{\in \mathcal{R}_2} &= \\ = (A \times Q) \cup (P \times R) \end{aligned}$$

where $Q \in \mathcal{E}_2$, $P \in \mathcal{E}_1$, and $R \in \mathcal{R}_2$.



Since $A \times Q$ and $P \times R$ do not intersect, it follows the above expression is in $\mathcal{E}$ because each of these terms are. This proves the corollary.

Definition 9.9.4 $\mathcal{M} \subseteq \mathcal{P}(Z)$ is called a monotone class if

a.) $\cdots E_n \supseteq E_{n+1} \cdots$, $E = \bigcap_{n=1}^{\infty} E_n$, and $E_n \in \mathcal{M}$, then $E \in \mathcal{M}$.

b.) $\cdots E_n \subseteq E_{n+1} \cdots$, $E = \bigcup_{n=1}^{\infty} E_n$, and $E_n \in \mathcal{M}$, then $E \in \mathcal{M}$.

(In simpler notation, $E_n \downarrow E$ and $E_n \in \mathcal{M}$ implies $E \in \mathcal{M}$. $E_n \uparrow E$ and $E_n \in \mathcal{M}$ implies $E \in \mathcal{M}$.)

Theorem 9.9.5 (*Monotone Class theorem*) *Let $\mathcal{A}$ be an algebra of subsets of Z and let $\mathcal{M}$ be a monotone class containing $\mathcal{A}$. Then $\mathcal{M} \supseteq \sigma(\mathcal{A})$, the smallest σ -algebra containing $\mathcal{A}$.*

Proof: Consider all monotone classes which contain $\mathcal{A}$, and take their intersection. The result is still a monotone class which contains $\mathcal{A}$ and is therefore the smallest monotone class containing $\mathcal{A}$. Therefore, assume without loss of generality that $\mathcal{M}$ is the smallest monotone class containing $\mathcal{A}$ because if it is shown the smallest monotone class containing $\mathcal{A}$ contains $\sigma(\mathcal{A})$, then the given monotone class does also. To avoid more notation, let $\mathcal{M}$ denote this smallest monotone class.

The plan is to show $\mathcal{M}$ is a σ -algebra. It will then follow $\mathcal{M} \supseteq \sigma(\mathcal{A})$ because $\sigma(\mathcal{A})$ is defined as the intersection of all σ algebras which contain $\mathcal{A}$. For $A \in \mathcal{A}$, define

$$\mathcal{M}_A \equiv \{B \in \mathcal{M} \text{ such that } A \cup B \in \mathcal{M}\}.$$

Clearly $\mathcal{M}_A$ is a monotone class containing $\mathcal{A}$. Hence $\mathcal{M}_A \supseteq \mathcal{M}$ because $\mathcal{M}$ is the smallest such monotone class. But by construction, $\mathcal{M}_A \subseteq \mathcal{M}$. Therefore, $\mathcal{M} = \mathcal{M}_A$. This shows that $A \cup B \in \mathcal{M}$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{M}$. Now pick $B \in \mathcal{M}$ and define

$$\mathcal{M}_B \equiv \{D \in \mathcal{M} \text{ such that } D \cup B \in \mathcal{M}\}.$$

It was just shown that $\mathcal{A} \subseteq \mathcal{M}_B$. It is clear that $\mathcal{M}_B$ is a monotone class. Thus by a similar argument, $\mathcal{M}_B = \mathcal{M}$ and it follows that $D \cup B \in \mathcal{M}$ whenever $D \in \mathcal{M}$ and $B \in \mathcal{M}$. This shows $\mathcal{M}$ is closed under finite unions.

Next consider the difference of two sets. Let $A \in \mathcal{A}$

$$\mathcal{M}_A \equiv \{B \in \mathcal{M} \text{ such that } B \setminus A \text{ and } A \setminus B \in \mathcal{M}\}.$$

Then $\mathcal{M}_A$ is a monotone class containing $\mathcal{A}$. As before, $\mathcal{M} = \mathcal{M}_A$. Thus $B \setminus A$ and $A \setminus B$ are both in $\mathcal{M}$ whenever $A \in \mathcal{A}$ and $B \in \mathcal{M}$. Now pick $A \in \mathcal{M}$ and consider

$$\mathcal{M}_A \equiv \{B \in \mathcal{M} \text{ such that } B \setminus A \text{ and } A \setminus B \in \mathcal{M}\}.$$

It was just shown $\mathcal{M}_A$ contains $\mathcal{A}$. Now $\mathcal{M}_A$ is a monotone class and so $\mathcal{M}_A = \mathcal{M}$ as before.

Thus $\mathcal{M}$ is both a monotone class and an algebra. Hence, if $E \in \mathcal{M}$ then $Z \setminus E \in \mathcal{M}$. Next consider the question of whether $\mathcal{M}$ is a σ -algebra. If $E_i \in \mathcal{M}$ and $F_n = \cup_{i=1}^n E_i$, then $F_n \in \mathcal{M}$ and $F_n \uparrow \cup_{i=1}^\infty E_i$. Since $\mathcal{M}$ is a monotone class, $\cup_{i=1}^\infty E_i \in \mathcal{M}$ and so $\mathcal{M}$ is a σ -algebra. This proves the theorem.

9.9.2 Product Measure

Definition 9.9.6 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be two measure spaces. A measurable rectangle is a set $A \times B \subseteq X \times Y$ where $A \in \mathcal{S}$ and $B \in \mathcal{F}$. An elementary set will be any subset of $X \times Y$ which is a finite union of disjoint measurable rectangles. $\mathcal{S} \times \mathcal{F}$ will denote the smallest σ algebra of sets in $\mathcal{P}(X \times Y)$ containing all elementary sets.*

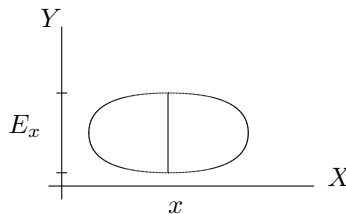
Example 9.9.7 It follows from Lemma 9.9.2 or more easily from Corollary 9.9.3 that the elementary sets form an algebra.

Definition 9.9.8 Let $E \subseteq X \times Y$,

$$E_x = \{y \in Y : (x, y) \in E\},$$

$$E^y = \{x \in X : (x, y) \in E\}.$$

These are called the x and y sections.



Theorem 9.9.9 If $E \in \mathcal{S} \times \mathcal{F}$, then $E_x \in \mathcal{F}$ and $E^y \in \mathcal{S}$ for all $x \in X$ and $y \in Y$.

Proof: Let

$$\mathcal{M} = \{E \subseteq \mathcal{S} \times \mathcal{F} \text{ such that for all } x \in X, E_x \in \mathcal{F}, \\ \text{and for all } y \in Y, E^y \in \mathcal{S}\}$$

Then $\mathcal{M}$ contains all measurable rectangles. If $E_i \in \mathcal{M}$,

$$(\cup_{i=1}^{\infty} E_i)_x = \cup_{i=1}^{\infty} (E_i)_x \in \mathcal{F}.$$

Similarly,

$$(\cup_{i=1}^{\infty} E_i)^y = \cup_{i=1}^{\infty} E_i^y \in \mathcal{S}.$$

It follows $\mathcal{M}$ is closed under countable unions.

If $E \in \mathcal{M}$,

$$(E^C)_x = (E_x)^C \in \mathcal{F}.$$

Similarly, $(E^C)^y \in \mathcal{S}$. Thus $\mathcal{M}$ is closed under complementation. Therefore $\mathcal{M}$ is a σ -algebra containing the elementary sets. Hence, $\mathcal{M} \supseteq \mathcal{S} \times \mathcal{F}$ because $\mathcal{S} \times \mathcal{F}$ is the smallest σ algebra containing these elementary sets. But $\mathcal{M} \subseteq \mathcal{S} \times \mathcal{F}$ by definition and so $\mathcal{M} = \mathcal{S} \times \mathcal{F}$. This proves the theorem.

It follows from Lemma 9.9.2 that the elementary sets form an algebra because clearly the intersection of two measurable rectangles is a measurable rectangle and

$$(A \times B) \setminus (A_0 \times B_0) = (A \setminus A_0) \times B \cup (A \cap A_0) \times (B \setminus B_0),$$

an elementary set.

Theorem 9.9.10 *If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ are both finite measure spaces ($\mu(X), \lambda(Y) < \infty$), then for every $E \in \mathcal{S} \times \mathcal{F}$,*

- a.) $x \rightarrow \lambda(E_x)$ is μ measurable, $y \rightarrow \mu(E^y)$ is λ measurable*
- b.) $\int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.*

Proof: Let

$$\mathcal{M} = \{E \in \mathcal{S} \times \mathcal{F} \text{ such that both a.) and b.) hold}\}.$$

Since μ and λ are both finite, the monotone convergence and dominated convergence theorems imply $\mathcal{M}$ is a monotone class.

Next I will argue $\mathcal{M}$ contains the elementary sets. Let

$$E = \cup_{i=1}^n A_i \times B_i$$

where the measurable rectangles, $A_i \times B_i$ are disjoint. Then

$$\begin{aligned} \lambda(E_x) &= \int_Y \mathcal{X}_E(x, y) d\lambda = \int_Y \sum_{i=1}^n \mathcal{X}_{A_i \times B_i}(x, y) d\lambda \\ &= \sum_{i=1}^n \int_Y \mathcal{X}_{A_i \times B_i}(x, y) d\lambda = \sum_{i=1}^n \mathcal{X}_{A_i}(x) \lambda(B_i) \end{aligned}$$

which is clearly μ measurable. Furthermore,

$$\int_X \lambda(E_x) d\mu = \int_X \sum_{i=1}^n \mathcal{X}_{A_i}(x) \lambda(B_i) d\mu = \sum_{i=1}^n \mu(A_i) \lambda(B_i).$$

Similarly,

$$\int_Y \mu(E^y) d\lambda = \sum_{i=1}^n \mu(A_i) \lambda(B_i)$$

and $y \rightarrow \mu(E^y)$ is λ measurable and this shows $\mathcal{M}$ contains the algebra of elementary sets. By the monotone class theorem, $\mathcal{M} = \mathcal{S} \times \mathcal{F}$. This proves the theorem.

One can easily extend this theorem to the case where the measure spaces are σ finite.

Theorem 9.9.11 *If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ are both σ finite measure spaces, then for every $E \in \mathcal{S} \times \mathcal{F}$,*

- a.) $x \rightarrow \lambda(E_x)$ is μ measurable, $y \rightarrow \mu(E^y)$ is λ measurable.*
- b.) $\int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.*

Proof: Let $X = \cup_{n=1}^{\infty} X_n, Y = \cup_{n=1}^{\infty} Y_n$ where,

$$X_n \subseteq X_{n+1}, Y_n \subseteq Y_{n+1}, \mu(X_n) < \infty, \lambda(Y_n) < \infty.$$

Let

$$\mathcal{S}_n = \{A \cap X_n : A \in \mathcal{S}\}, \mathcal{F}_n = \{B \cap Y_n : B \in \mathcal{F}\}.$$

Thus $(X_n, \mathcal{S}_n, \mu)$ and $(Y_n, \mathcal{F}_n, \lambda)$ are both finite measure spaces.

Claim: If $E \in \mathcal{S} \times \mathcal{F}$, then $E \cap (X_n \times Y_n) \in \mathcal{S}_n \times \mathcal{F}_n$.

Proof: Let

$$\mathcal{M}_n = \{E \in \mathcal{S} \times \mathcal{F} : E \cap (X_n \times Y_n) \in \mathcal{S}_n \times \mathcal{F}_n\}.$$

Clearly $\mathcal{M}_n$ contains the algebra of elementary sets. It is also clear that $\mathcal{M}_n$ is a monotone class. Thus $\mathcal{M}_n = \mathcal{S} \times \mathcal{F}$.

Now let $E \in \mathcal{S} \times \mathcal{F}$. By Theorem 9.9.10,

$$\int_{X_n} \lambda((E \cap (X_n \times Y_n))_x) d\mu = \int_{Y_n} \mu((E \cap (X_n \times Y_n))^y) d\lambda \quad (9.9.40)$$

where the integrands are measurable. Also

$$(E \cap (X_n \times Y_n))_x = \emptyset$$

if $x \notin X_n$ and a similar observation holds for the second integrand in 9.9.40 if $y \notin Y_n$. Therefore,

$$\begin{aligned} \int_X \lambda((E \cap (X_n \times Y_n))_x) d\mu &= \int_{X_n} \lambda((E \cap (X_n \times Y_n))_x) d\mu \\ &= \int_{Y_n} \mu((E \cap (X_n \times Y_n))^y) d\lambda \\ &= \int_Y \mu((E \cap (X_n \times Y_n))^y) d\lambda. \end{aligned}$$

Then letting $n \rightarrow \infty$, the monotone convergence theorem implies b.) and the measurability assertions of a.) are valid because

$$\begin{aligned} \lambda(E_x) &= \lim_{n \rightarrow \infty} \lambda((E \cap (X_n \times Y_n))_x) \\ \mu(E^y) &= \lim_{n \rightarrow \infty} \mu((E \cap (X_n \times Y_n))^y). \end{aligned}$$

This proves the theorem.

This theorem makes it possible to define product measure.

Definition 9.9.12 For $E \in \mathcal{S} \times \mathcal{F}$ and $(X, \mathcal{S}, \mu), (Y, \mathcal{F}, \lambda)$ σ finite, $(\mu \times \lambda)(E) \equiv \int_X \lambda(E_x) d\mu = \int_Y \mu(E^y) d\lambda$.

This definition is well defined because of Theorem 9.9.11.

Theorem 9.9.13 If $A \in \mathcal{S}$, $B \in \mathcal{F}$, then $(\mu \times \lambda)(A \times B) = \mu(A)\lambda(B)$, and $\mu \times \lambda$ is a measure on $\mathcal{S} \times \mathcal{F}$ called product measure.

Proof: The first assertion about the measure of a measurable rectangle was established above. Now suppose $\{E_i\}_{i=1}^\infty$ is a disjoint collection of sets of $\mathcal{S} \times \mathcal{F}$.

Then using the monotone convergence theorem along with the observation that $(E_i)_x \cap (E_j)_x = \emptyset$,

$$\begin{aligned}
 (\mu \times \lambda)(\cup_{i=1}^{\infty} E_i) &= \int_X \lambda((\cup_{i=1}^{\infty} E_i)_x) d\mu \\
 &= \int_X \lambda(\cup_{i=1}^{\infty} (E_i)_x) d\mu = \int_X \sum_{i=1}^{\infty} \lambda((E_i)_x) d\mu \\
 &= \sum_{i=1}^{\infty} \int_X \lambda((E_i)_x) d\mu \\
 &= \sum_{i=1}^{\infty} (\mu \times \lambda)(E_i)
 \end{aligned}$$

This proves the theorem.

The next theorem is one of several theorems due to Fubini and Tonelli. These theorems all have to do with interchanging the order of integration in a multiple integral.

Theorem 9.9.14 *Let $f : X \times Y \rightarrow [0, \infty]$ be measurable with respect to $\mathcal{S} \times \mathcal{F}$ and suppose μ and λ are σ finite. Then*

$$\int_{X \times Y} f d(\mu \times \lambda) = \int_X \int_Y f(x, y) d\lambda d\mu = \int_Y \int_X f(x, y) d\mu d\lambda \quad (9.9.41)$$

and all integrals make sense.

Proof: For $E \in \mathcal{S} \times \mathcal{F}$,

$$\int_Y \mathcal{X}_E(x, y) d\lambda = \lambda(E_x), \quad \int_X \mathcal{X}_E(x, y) d\mu = \mu(E^y).$$

Thus from Definition 9.9.12, 9.9.41 holds if $f = \mathcal{X}_E$. It follows that 9.9.41 holds for every nonnegative simple function. By Theorem 8.3.9 on Page 197, there exists an increasing sequence, $\{f_n\}$, of simple functions converging pointwise to f . Then

$$\begin{aligned}
 \int_Y f(x, y) d\lambda &= \lim_{n \rightarrow \infty} \int_Y f_n(x, y) d\lambda, \\
 \int_X f(x, y) d\mu &= \lim_{n \rightarrow \infty} \int_X f_n(x, y) d\mu.
 \end{aligned}$$

This follows from the monotone convergence theorem. Since

$$x \rightarrow \int_Y f_n(x, y) d\lambda$$

is measurable with respect to $\mathcal{S}$, it follows that $x \rightarrow \int_Y f(x, y) d\lambda$ is also measurable with respect to $\mathcal{S}$. A similar conclusion can be drawn about $y \rightarrow \int_X f(x, y) d\mu$. Thus the two iterated integrals make sense. Since 9.9.41 holds for f_n , another application of the Monotone Convergence theorem shows 9.9.41 holds for f . This proves the theorem.

Corollary 9.9.15 *Let $f : X \times Y \rightarrow \mathbb{C}$ be $\mathcal{S} \times \mathcal{F}$ measurable. Suppose either $\int_X \int_Y |f| d\lambda d\mu$ or $\int_Y \int_X |f| d\mu d\lambda < \infty$. Then $f \in L^1(X \times Y, \mu \times \lambda)$ and*

$$\int_{X \times Y} f d(\mu \times \lambda) = \int_X \int_Y f d\lambda d\mu = \int_Y \int_X f d\mu d\lambda \quad (9.9.42)$$

with all integrals making sense.

Proof: Suppose first that f is real valued. Apply Theorem 9.9.14 to f^+ and f^- . 9.9.42 follows from observing that $f = f^+ - f^-$; and that all integrals are finite. If f is complex valued, consider real and imaginary parts. This proves the corollary.

Suppose f is product measurable. From the above discussion, and breaking f down into a sum of positive and negative parts of real and imaginary parts and then using Theorem 8.3.9 on Page 197 on approximation by simple functions, it follows that whenever f is $\mathcal{S} \times \mathcal{F}$ measurable, $x \rightarrow f(x, y)$ is μ measurable, $y \rightarrow f(x, y)$ is λ measurable.

9.10 Completion Of Measures

Suppose $(\Omega, \mathcal{F}, \mu)$ is a measure space. Then it is always possible to enlarge the σ algebra and define a new measure $\bar{\mu}$ on this larger σ algebra such that $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ is a complete measure space. Recall this means that if $N \subseteq N' \in \bar{\mathcal{F}}$ and $\bar{\mu}(N') = 0$, then $N \in \bar{\mathcal{F}}$. The following theorem is the main result. The new measure space is called the completion of the measure space.

Theorem 9.10.1 *Let $(\Omega, \mathcal{F}, \mu)$ be a σ finite measure space. Then there exists a unique measure space, $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ satisfying*

1. $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ is a complete measure space.
2. $\bar{\mu} = \mu$ on $\mathcal{F}$
3. $\bar{\mathcal{F}} \supseteq \mathcal{F}$
4. For every $E \in \bar{\mathcal{F}}$ there exists $G \in \mathcal{F}$ such that $G \supseteq E$ and $\mu(G) = \bar{\mu}(E)$.
5. For every $E \in \bar{\mathcal{F}}$ there exists $F \in \mathcal{F}$ such that $F \subseteq E$ and $\mu(F) = \bar{\mu}(E)$.

Also for every $E \in \bar{\mathcal{F}}$ there exist sets $G, F \in \mathcal{F}$ such that $G \supseteq E \supseteq F$ and

$$\mu(G \setminus F) = \bar{\mu}(G \setminus F) = 0 \quad (9.10.43)$$

Proof: First consider the claim about uniqueness. Suppose $(\Omega, \mathcal{F}_1, \nu_1)$ and $(\Omega, \mathcal{F}_2, \nu_2)$ both work and let $E \in \mathcal{F}_1$. Also let $\mu(\Omega_n) < \infty, \dots \Omega_n \subseteq \Omega_{n+1} \dots$, and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. Define $E_n \equiv E \cap \Omega_n$. Then pick $G_n \supseteq E_n \supseteq F_n$ such that $\mu(G_n) =$

$\mu(F_n) = \nu_1(E_n)$. It follows $\mu(G_n \setminus F_n) = 0$. Then letting $G = \cup_n G_n$, $F \equiv \cup_n F_n$, it follows $G \supseteq E \supseteq F$ and

$$\begin{aligned} \mu(G \setminus F) &\leq \mu(\cup_n (G_n \setminus F_n)) \\ &\leq \sum_n \mu(G_n \setminus F_n) = 0. \end{aligned}$$

It follows that $\nu_2(G \setminus F) = 0$ also. Now $E \setminus F \subseteq G \setminus F$ and since $(\Omega, \mathcal{F}_2, \nu_2)$ is complete, it follows $E \setminus F \in \mathcal{F}_2$. Since $F \in \mathcal{F}_2$, it follows $E = (E \setminus F) \cup F \in \mathcal{F}_2$. Thus $\mathcal{F}_1 \subseteq \mathcal{F}_2$. Similarly $\mathcal{F}_2 \subseteq \mathcal{F}_1$. Now it only remains to verify $\nu_1 = \nu_2$. Thus let $E \in \mathcal{F}_1 = \mathcal{F}_2$ and let G and F be as just described. Since $\nu_i = \mu$ on $\mathcal{F}$,

$$\begin{aligned} \mu(F) &\leq \nu_1(E) \\ &= \nu_1(E \setminus F) + \nu_1(F) \\ &\leq \nu_1(G \setminus F) + \nu_1(F) \\ &= \nu_1(F) = \mu(F) \end{aligned}$$

Similarly $\nu_2(E) = \mu(F)$. This proves uniqueness. The construction has also verified 9.10.43.

Next define an outer measure, $\bar{\mu}$ on $\mathcal{P}(\Omega)$ as follows. For $S \subseteq \Omega$,

$$\bar{\mu}(S) \equiv \inf \{ \mu(E) : E \in \mathcal{F} \}.$$

Then it is clear $\bar{\mu}$ is increasing. It only remains to verify $\bar{\mu}$ is subadditive. Then let $S = \cup_{i=1}^{\infty} S_i$. If any $\bar{\mu}(S_i) = \infty$, there is nothing to prove so suppose $\bar{\mu}(S_i) < \infty$ for each i . Then there exist $E_i \in \mathcal{F}$ such that $E_i \supseteq S_i$ and

$$\bar{\mu}(S_i) + \varepsilon/2^i > \mu(E_i).$$

Then

$$\begin{aligned} \bar{\mu}(S) &= \bar{\mu}(\cup_i S_i) \\ &\leq \mu(\cup_i E_i) \leq \sum_i \mu(E_i) \\ &\leq \sum_i (\bar{\mu}(S_i) + \varepsilon/2^i) = \sum_i \bar{\mu}(S_i) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this verifies $\bar{\mu}$ is subadditive and is an outer measure as claimed.

Denote by $\bar{\mathcal{F}}$ the σ algebra of measurable sets in the sense of Caratheodory. Then it follows from the Caratheodory procedure, Theorem 9.1.4, on Page 220 that $(\Omega, \bar{\mathcal{F}}, \bar{\mu})$ is a complete measure space. This verifies 1.

Now let $E \in \mathcal{F}$. Then from the definition of $\bar{\mu}$, it follows

$$\bar{\mu}(E) \equiv \inf \{ \mu(F) : F \in \mathcal{F} \text{ and } F \supseteq E \} \leq \mu(E).$$

If $F \supseteq E$ and $F \in \mathcal{F}$, then $\mu(F) \geq \mu(E)$ and so $\mu(E)$ is a lower bound for all such $\mu(F)$ which shows that

$$\bar{\mu}(E) \equiv \inf \{ \mu(F) : F \in \mathcal{F} \text{ and } F \supseteq E \} \geq \mu(E).$$

This verifies 2.

Next consider 3. Let $E \in \mathcal{F}$ and let S be a set. I must show

$$\bar{\mu}(S) \geq \bar{\mu}(S \setminus E) + \bar{\mu}(S \cap E).$$

If $\bar{\mu}(S) = \infty$ there is nothing to show. Therefore, suppose $\bar{\mu}(S) < \infty$. Then from the definition of $\bar{\mu}$ there exists $G \supseteq S$ such that $G \in \mathcal{F}$ and $\mu(G) = \bar{\mu}(S)$. Then from the definition of $\bar{\mu}$,

$$\begin{aligned} \bar{\mu}(S) &\leq \bar{\mu}(S \setminus E) + \bar{\mu}(S \cap E) \\ &\leq \mu(G \setminus E) + \mu(G \cap E) \\ &= \mu(G) = \bar{\mu}(S) \end{aligned}$$

This verifies 3.

Claim 4 comes by the definition of $\bar{\mu}$ as used above. The only other case is when $\bar{\mu}(S) = \infty$. However, in this case, you can let $G = \Omega$.

It only remains to verify 5. Let the Ω_n be as described above and let $E \in \overline{\mathcal{F}}$ such that $E \subseteq \Omega_n$. By 4 there exists $H \in \mathcal{F}$ such that $H \subseteq \Omega_n$, $H \supseteq \Omega_n \setminus E$, and

$$\mu(H) = \bar{\mu}(\Omega_n \setminus E). \quad (9.10.44)$$

Then let $F \equiv \Omega_n \cap H^C$. It follows $F \subseteq E$ and

$$\begin{aligned} E \setminus F &= E \cap F^C = E \cap (H \cup \Omega_n^C) \\ &= E \cap H = H \setminus (\Omega_n \setminus E) \end{aligned}$$

Hence from 9.10.44

$$\bar{\mu}(E \setminus F) = \bar{\mu}(H \setminus (\Omega_n \setminus E)) = 0.$$

It follows

$$\bar{\mu}(E) = \bar{\mu}(F) = \mu(F).$$

In the case where $E \in \overline{\mathcal{F}}$ is arbitrary, not necessarily contained in some Ω_n , it follows from what was just shown that there exists $F_n \in \mathcal{F}$ such that $F_n \subseteq E \cap \Omega_n$ and

$$\mu(F_n) = \bar{\mu}(E \cap \Omega_n).$$

Letting $F \equiv \cup_n F_n$

$$\bar{\mu}(E \setminus F) \leq \bar{\mu}(\cup_n (E \cap \Omega_n \setminus F_n)) \leq \sum_n \bar{\mu}(E \cap \Omega_n \setminus F_n) = 0.$$

Therefore, $\bar{\mu}(E) = \mu(F)$ and this proves 5. This proves the theorem.

Now here is an interesting theorem about complete measure spaces.

Theorem 9.10.2 *Let $(\Omega, \mathcal{F}, \mu)$ be a complete measure space and let $f \leq g \leq h$ be functions having values in $[0, \infty]$. Suppose also that $f(\omega) = h(\omega)$ a.e. ω and that f and h are measurable. Then g is also measurable. If $(\Omega, \overline{\mathcal{F}}, \bar{\mu})$ is the completion*

of a σ finite measure space $(\Omega, \mathcal{F}, \mu)$ as described above in Theorem 9.10.1 then if f is measurable with respect to $\overline{\mathcal{F}}$ having values in $[0, \infty]$, it follows there exists g measurable with respect to $\mathcal{F}$, $g \leq f$, and a set $N \in \mathcal{F}$ with $\mu(N) = 0$ and $g = f$ on N^C . There also exists h measurable with respect to $\mathcal{F}$ such that $h \geq f$, and a set of measure zero, $M \in \mathcal{F}$ such that $f = h$ on M^C .

Proof: Let $\alpha \in \mathbb{R}$.

$$[f > \alpha] \subseteq [g > \alpha] \subseteq [h > \alpha]$$

Thus

$$[g > \alpha] = [f > \alpha] \cup ([g > \alpha] \setminus [f > \alpha])$$

and $[g > \alpha] \setminus [f > \alpha]$ is a measurable set because it is a subset of the set of measure zero,

$$[h > \alpha] \setminus [f > \alpha].$$

Now consider the last assertion. By Theorem 8.3.9 on Page 197 there exists an increasing sequence of nonnegative simple functions, $\{s_n\}$ measurable with respect to $\overline{\mathcal{F}}$ which converges pointwise to f . Letting

$$s_n(\omega) = \sum_{k=1}^{m_n} c_k^n \mathcal{X}_{E_k^n}(\omega) \quad (9.10.45)$$

be one of these simple functions, it follows from Theorem 9.10.1 there exist sets, $F_k^n \in \mathcal{F}$ such that $F_k^n \subseteq E_k^n$ and $\mu(F_k^n) = \overline{\mu}(E_k^n)$. Then let

$$t_n(\omega) \equiv \sum_{k=1}^{m_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Thus $t_n = s_n$ off a set of measure zero, $N_n \in \overline{\mathcal{F}}$, $t_n \leq s_n$. Let $N' \equiv \cup_n N_n$. Then by Theorem 9.10.1 again, there exists $N \in \mathcal{F}$ such that $N \supseteq N'$ and $\mu(N) = 0$. Consider the simple functions,

$$s'_n(\omega) \equiv t_n(\omega) \mathcal{X}_{N^C}(\omega).$$

It is an increasing sequence so let $g(\omega) = \lim_{n \rightarrow \infty} s'_n(\omega)$. It follows g is measurable with respect to $\mathcal{F}$ and equals f off N .

Finally, to obtain the function, $h \geq f$, in 9.10.45 use Theorem 9.10.1 to obtain the existence of $F_k^n \in \mathcal{F}$ such that $F_k^n \supseteq E_k^n$ and $\mu(F_k^n) = \overline{\mu}(E_k^n)$. Then let

$$t_n(\omega) \equiv \sum_{k=1}^{m_n} c_k^n \mathcal{X}_{F_k^n}(\omega).$$

Thus $t_n = s_n$ off a set of measure zero, $M_n \in \overline{\mathcal{F}}$, $t_n \geq s_n$, and t_n is measurable with respect to $\mathcal{F}$. Then define

$$s'_n = \max_{k \leq n} t_k.$$

It follows s'_n is an increasing sequence of $\mathcal{F}$ measurable nonnegative simple functions. Since each $s'_n \geq s_n$, it follows that if $h(\omega) = \lim_{n \rightarrow \infty} s'_n(\omega)$, then $h(\omega) \geq f(\omega)$. Also if $h(\omega) > f(\omega)$, then $\omega \in \cup_n M_n \equiv M'$, a set of $\overline{\mathcal{F}}$ having measure zero. By Theorem 9.10.1, there exists $M \supseteq M'$ such that $M \in \mathcal{F}$ and $\mu(M) = 0$. It follows $h = f$ off M . This proves the theorem.

9.11 Another Version Of Product Measures

9.11.1 General Theory

Given two finite measure spaces, $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$, there is a way to define a σ algebra of subsets of $X \times Y$, denoted by $\mathcal{F} \times \mathcal{S}$ and a measure, denoted by $\mu \times \nu$ defined on this σ algebra such that

$$\mu \times \nu(A \times B) = \mu(A) \nu(B)$$

whenever $A \in \mathcal{F}$ and $B \in \mathcal{S}$. This is naturally related to the concept of iterated integrals similar to what is used in calculus to evaluate a multiple integral. The approach is based on something called a π system, [18].

Definition 9.11.1 Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two measure spaces. A measurable rectangle is a set of the form $A \times B$ where $A \in \mathcal{F}$ and $B \in \mathcal{S}$.

Definition 9.11.2 Let Ω be a set and let $\mathcal{K}$ be a collection of subsets of Ω . Then $\mathcal{K}$ is called a π system if $\emptyset, \Omega \in \mathcal{K}$ and whenever $A, B \in \mathcal{K}$, it follows $A \cap B \in \mathcal{K}$.

Obviously an example of a π system is the set of measurable rectangles because

$$A \times B \cap A' \times B' = (A \cap A') \times (B \cap B').$$

The following is the fundamental lemma which shows these π systems are useful. This lemma is due to Dynkin.

Lemma 9.11.3 Let $\mathcal{K}$ be a π system of subsets of Ω , a set. Also let $\mathcal{G}$ be a collection of subsets of Ω which satisfies the following three properties.

1. $\mathcal{K} \subseteq \mathcal{G}$
2. If $A \in \mathcal{G}$, then $A^C \in \mathcal{G}$
3. If $\{A_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets from $\mathcal{G}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{G}$.

Then $\mathcal{G} \supseteq \sigma(\mathcal{K})$, where $\sigma(\mathcal{K})$ is the smallest σ algebra which contains $\mathcal{K}$.

Proof: First note that if

$$\mathcal{H} \equiv \{\mathcal{G} : 1 - 3 \text{ all hold}\}$$

then $\cap \mathcal{H}$ yields a collection of sets which also satisfies 1 - 3. Therefore, I will assume in the argument that $\mathcal{G}$ is the smallest collection satisfying 1 - 3. Let $A \in \mathcal{K}$ and define

$$\mathcal{G}_A \equiv \{B \in \mathcal{G} : A \cap B \in \mathcal{G}\}.$$

I want to show $\mathcal{G}_A$ satisfies 1 - 3 because then it must equal $\mathcal{G}$ since $\mathcal{G}$ is the smallest collection of subsets of Ω which satisfies 1 - 3. This will give the conclusion that for $A \in \mathcal{K}$ and $B \in \mathcal{G}$, $A \cap B \in \mathcal{G}$. This information will then be used to show that if $A, B \in \mathcal{G}$ then $A \cap B \in \mathcal{G}$. From this it will follow very easily that $\mathcal{G}$ is a σ algebra which will imply it contains $\sigma(\mathcal{K})$. Now here are the details of the argument.

Since $\mathcal{K}$ is given to be a π system, $\mathcal{K} \subseteq \mathcal{G}_A$. Property 3 is obvious because if $\{B_i\}$ is a sequence of disjoint sets in $\mathcal{G}_A$, then

$$A \cap \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A \cap B_i \in \mathcal{G}$$

because $A \cap B_i \in \mathcal{G}$ and the property 3 of $\mathcal{G}$.

It remains to verify Property 2 so let $B \in \mathcal{G}_A$. I need to verify that $B^C \in \mathcal{G}_A$. In other words, I need to show that $A \cap B^C \in \mathcal{G}$. However,

$$A \cap B^C = (A^C \cup (A \cap B))^C \in \mathcal{G}$$

Here is why. Since $B \in \mathcal{G}_A$, $A \cap B \in \mathcal{G}$ and since $A \in \mathcal{K} \subseteq \mathcal{G}$ it follows $A^C \in \mathcal{G}$ by assumption 2. It follows from assumption 3 the union of the disjoint sets, A^C and $(A \cap B)$ is in $\mathcal{G}$ and then from 2 the complement of their union is in $\mathcal{G}$. Thus $\mathcal{G}_A$ satisfies 1 - 3 and this implies since $\mathcal{G}$ is the smallest such, that $\mathcal{G}_A \supseteq \mathcal{G}$. However, $\mathcal{G}_A$ is constructed as a subset of $\mathcal{G}$. This proves that for every $B \in \mathcal{G}$ and $A \in \mathcal{K}$, $A \cap B \in \mathcal{G}$. Now pick $B \in \mathcal{G}$ and consider

$$\mathcal{G}_B \equiv \{A \in \mathcal{G} : A \cap B \in \mathcal{G}\}.$$

I just proved $\mathcal{K} \subseteq \mathcal{G}_B$. The other arguments are identical to show $\mathcal{G}_B$ satisfies 1 - 3 and is therefore equal to $\mathcal{G}$. This shows that whenever $A, B \in \mathcal{G}$ it follows $A \cap B \in \mathcal{G}$.

This implies $\mathcal{G}$ is a σ algebra. To show this, all that is left is to verify $\mathcal{G}$ is closed under countable unions because then it follows $\mathcal{G}$ is a σ algebra. Let $\{A_i\} \subseteq \mathcal{G}$. Then let $A'_1 = A_1$ and

$$\begin{aligned} A'_{n+1} &\equiv A_{n+1} \setminus (\bigcup_{i=1}^n A_i) \\ &= A_{n+1} \cap (\bigcap_{i=1}^n A_i^C) \\ &= \bigcap_{i=1}^n (A_{n+1} \cap A_i^C) \in \mathcal{G} \end{aligned}$$

because finite intersections of sets of $\mathcal{G}$ are in $\mathcal{G}$. Since the A'_i are disjoint, it follows

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i \in \mathcal{G}$$

Therefore, $\mathcal{G} \supseteq \sigma(\mathcal{K})$ and this proves the Lemma.

With this lemma, it is easy to define product measure.

Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be two finite measure spaces. Define $\mathcal{K}$ to be the set of measurable rectangles, $A \times B$, $A \in \mathcal{F}$ and $B \in \mathcal{S}$. Let

$$\mathcal{G} \equiv \left\{ E \subseteq X \times Y : \int_Y \int_X \chi_E d\mu d\nu = \int_X \int_Y \chi_E d\nu d\mu \right\} \quad (9.11.46)$$

where in the above, part of the requirement is for all integrals to make sense.

Then $\mathcal{K} \subseteq \mathcal{G}$. This is obvious.

Next I want to show that if $E \in \mathcal{G}$ then $E^C \in \mathcal{G}$. Observe $\chi_{E^C} = 1 - \chi_E$ and so

$$\begin{aligned} \int_Y \int_X \chi_{E^C} d\mu d\nu &= \int_Y \int_X (1 - \chi_E) d\mu d\nu \\ &= \int_X \int_Y (1 - \chi_E) d\nu d\mu \\ &= \int_X \int_Y \chi_{E^C} d\nu d\mu \end{aligned}$$

which shows that if $E \in \mathcal{G}$, then $E^C \in \mathcal{G}$.

Next I want to show $\mathcal{G}$ is closed under countable unions of disjoint sets of $\mathcal{G}$. Let $\{A_i\}$ be a sequence of disjoint sets from $\mathcal{G}$. Then

$$\begin{aligned} \int_Y \int_X \chi_{\bigcup_{i=1}^{\infty} A_i} d\mu d\nu &= \int_Y \int_X \sum_{i=1}^{\infty} \chi_{A_i} d\mu d\nu \\ &= \int_Y \sum_{i=1}^{\infty} \int_X \chi_{A_i} d\mu d\nu \\ &= \sum_{i=1}^{\infty} \int_Y \int_X \chi_{A_i} d\mu d\nu \\ &= \sum_{i=1}^{\infty} \int_X \int_Y \chi_{A_i} d\nu d\mu \\ &= \int_X \sum_{i=1}^{\infty} \int_Y \chi_{A_i} d\nu d\mu \\ &= \int_X \int_Y \sum_{i=1}^{\infty} \chi_{A_i} d\nu d\mu \\ &= \int_X \int_Y \chi_{\bigcup_{i=1}^{\infty} A_i} d\nu d\mu, \end{aligned} \quad (9.11.47)$$

the interchanges between the summation and the integral depending on the monotone convergence theorem. Thus $\mathcal{G}$ is closed with respect to countable disjoint unions.

From Lemma 9.11.3, $\mathcal{G} \supseteq \sigma(\mathcal{K})$. Also the computation in 9.11.47 implies that on $\sigma(\mathcal{K})$ one can define a measure, denoted by $\mu \times \nu$ and that for every $E \in \sigma(\mathcal{K})$,

$$(\mu \times \nu)(E) = \int_Y \int_X \chi_E d\mu d\nu = \int_X \int_Y \chi_E d\nu d\mu. \quad (9.11.48)$$

Now here is Fubini's theorem.

Theorem 9.11.4 *Let $f : X \times Y \rightarrow [0, \infty]$ be measurable with respect to the σ algebra, $\sigma(\mathcal{K})$ just defined and let $\mu \times \nu$ be the product measure of 9.11.48 where μ and ν are finite measures on $(X, \mathcal{F})$ and $(Y, \mathcal{S})$ respectively. Then*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu.$$

Proof: Let $\{s_n\}$ be an increasing sequence of $\sigma(\mathcal{K})$ measurable simple functions which converges pointwise to f . The above equation holds for s_n in place of f from what was shown above. The final result follows from passing to the limit and using the monotone convergence theorem. This proves the theorem.

The symbol, $\mathcal{F} \times \mathcal{S}$ denotes $\sigma(\mathcal{K})$.

Of course one can generalize right away to measures which are only σ finite.

Theorem 9.11.5 *Let $f : X \times Y \rightarrow [0, \infty]$ be measurable with respect to the σ algebra, $\sigma(\mathcal{K})$ just defined and let $\mu \times \nu$ be the product measure of 9.11.48 where μ and ν are σ finite measures on $(X, \mathcal{F})$ and $(Y, \mathcal{S})$ respectively. Then*

$$\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu.$$

Proof: Since the measures are σ finite, there exist increasing sequences of sets, $\{X_n\}$ and $\{Y_n\}$ such that $\mu(X_n) < \infty$ and $\nu(Y_n) < \infty$. Then μ and ν restricted to X_n and Y_n respectively are finite. Then from Theorem 9.11.4,

$$\int_{Y_n} \int_{X_n} f d\mu d\nu = \int_{X_n} \int_{Y_n} f d\nu d\mu$$

Passing to the limit yields

$$\int_Y \int_X f d\mu d\nu = \int_X \int_Y f d\nu d\mu$$

whenever f is as above. In particular, you could take $f = \chi_E$ where $E \in \mathcal{F} \times \mathcal{S}$ and define

$$(\mu \times \nu)(E) \equiv \int_Y \int_X \chi_E d\mu d\nu = \int_X \int_Y \chi_E d\nu d\mu.$$

Then just as in the proof of Theorem 9.11.4, the conclusion of this theorem is obtained. This proves the theorem.

It is also useful to note that all the above holds for $\prod_{i=1}^n X_i$ in place of $X \times Y$. You would simply modify the definition of $\mathcal{G}$ in 9.11.46 including all permutations for the iterated integrals and for $\mathcal{K}$ you would use sets of the form $\prod_{i=1}^n A_i$ where A_i is measurable. Everything goes through exactly as above. Thus the following is obtained.

Theorem 9.11.6 Let $\{(X_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n$ be σ finite measure spaces and let $\prod_{i=1}^n \mathcal{F}_i$ denote the smallest σ algebra which contains the measurable boxes of the form $\prod_{i=1}^n A_i$ where $A_i \in \mathcal{F}_i$. Then there exists a measure, λ defined on $\prod_{i=1}^n \mathcal{F}_i$ such that if $f : \prod_{i=1}^n X_i \rightarrow [0, \infty]$ is $\prod_{i=1}^n \mathcal{F}_i$ measurable, and $(i_1, \dots, i_n)$ is any permutation of $(1, \dots, n)$, then

$$\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}$$

9.11.2 Completion Of Product Measure Spaces

Using Theorem 9.10.2 it is easy to give a generalization to yield a theorem for the completion of product spaces.

Theorem 9.11.7 Let $\{(X_i, \mathcal{F}_i, \mu_i)\}_{i=1}^n$ be σ finite measure spaces and let $\prod_{i=1}^n \mathcal{F}_i$ denote the smallest σ algebra which contains the measurable boxes of the form $\prod_{i=1}^n A_i$ where $A_i \in \mathcal{F}_i$. Then there exists a measure, λ defined on $\prod_{i=1}^n \mathcal{F}_i$ such that if $f : \prod_{i=1}^n X_i \rightarrow [0, \infty]$ is $\prod_{i=1}^n \mathcal{F}_i$ measurable, and $(i_1, \dots, i_n)$ is any permutation of $(1, \dots, n)$, then

$$\int f d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}$$

Let $(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \bar{\lambda})$ denote the completion of this product measure space and let

$$f : \prod_{i=1}^n X_i \rightarrow [0, \infty]$$

be $\overline{\prod_{i=1}^n \mathcal{F}_i}$ measurable. Then there exists $N \in \prod_{i=1}^n \mathcal{F}_i$ such that $\lambda(N) = 0$ and a nonnegative function, f_1 measurable with respect to $\prod_{i=1}^n \mathcal{F}_i$ such that $f_1 = f$ off N and if $(i_1, \dots, i_n)$ is any permutation of $(1, \dots, n)$, then

$$\int f d\bar{\lambda} = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f_1 d\mu_{i_1} \cdots d\mu_{i_n}.$$

Furthermore, f_1 may be chosen to satisfy either $f_1 \leq f$ or $f_1 \geq f$.

Proof: This follows immediately from Theorem 9.11.6 and Theorem 9.10.2. By the second theorem, there exists a function $f_1 \geq f$ such that $f_1 = f$ for all $(x_1, \dots, x_n) \notin N$, a set of $\prod_{i=1}^n \mathcal{F}_i$ having measure zero. Then by Theorem 9.10.1 and Theorem 9.11.6

$$\int f d\bar{\lambda} = \int f_1 d\lambda = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f_1 d\mu_{i_1} \cdots d\mu_{i_n}.$$

To get $f_1 \leq f$, just use that part of Theorem 9.10.2.

Since $f_1 = f$ off a set of measure zero, I will dispense with the subscript. Also it is customary to write

$$\lambda = \mu_1 \times \cdots \times \mu_n$$

and

$$\bar{\lambda} = \overline{\mu_1 \times \cdots \times \mu_n}.$$

Thus in more standard notation, one writes

$$\int f d(\mu_1 \times \cdots \times \mu_n) = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}$$

This theorem is often referred to as Fubini's theorem. The next theorem is also called this.

Corollary 9.11.8 *Suppose $f \in L^1 \left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\mu_1 \times \cdots \times \mu_n} \right)$ where each X_i is a σ finite measure space. Then if $(i_1, \cdots, i_n)$ is any permutation of $(1, \cdots, n)$, it follows*

$$\int f d(\overline{\mu_1 \times \cdots \times \mu_n}) = \int_{X_{i_n}} \cdots \int_{X_{i_1}} f d\mu_{i_1} \cdots d\mu_{i_n}.$$

Proof: Just apply Theorem 9.11.7 to the positive and negative parts of the real and imaginary parts of f . This proves the theorem.

Here is another easy corollary.

Corollary 9.11.9 *Suppose in the situation of Corollary 9.11.8, $f = f_1$ off N , a set of $\prod_{i=1}^n \mathcal{F}_i$ having $\mu_1 \times \cdots \times \mu_n$ measure zero and that f_1 is a complex valued function measurable with respect to $\prod_{i=1}^n \mathcal{F}_i$. Suppose also that for some permutation of $(1, 2, \cdots, n), (j_1, \cdots, j_n)$*

$$\int_{X_{j_n}} \cdots \int_{X_{j_1}} |f_1| d\mu_{j_1} \cdots d\mu_{j_n} < \infty.$$

Then

$$f \in L^1 \left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\mu_1 \times \cdots \times \mu_n} \right)$$

and the conclusion of Corollary 9.11.8 holds.

Proof: Since $|f_1|$ is $\prod_{i=1}^n \mathcal{F}_i$ measurable, it follows from Theorem 9.11.6 that

$$\begin{aligned} \infty &> \int_{X_{j_n}} \cdots \int_{X_{j_1}} |f_1| d\mu_{j_1} \cdots d\mu_{j_n} \\ &= \int |f_1| d(\mu_1 \times \cdots \times \mu_n) \\ &= \int |f_1| d(\overline{\mu_1 \times \cdots \times \mu_n}) \\ &= \int |f| d(\overline{\mu_1 \times \cdots \times \mu_n}). \end{aligned}$$

Thus $f \in L^1 \left(\prod_{i=1}^n X_i, \overline{\prod_{i=1}^n \mathcal{F}_i}, \overline{\mu_1 \times \cdots \times \mu_n} \right)$ as claimed and the rest follows from Corollary 9.11.8. This proves the corollary.

The following lemma is also useful.

Lemma 9.11.10 *Let $(X, \mathcal{F}, \mu)$ and $(Y, \mathcal{S}, \nu)$ be σ finite complete measure spaces and suppose $f \geq 0$ is $\mathcal{F} \times \mathcal{S}$ measurable. Then for a.e. x ,*

$$y \rightarrow f(x, y)$$

is $\mathcal{S}$ measurable. Similarly for a.e. y ,

$$x \rightarrow f(x, y)$$

is $\mathcal{F}$ measurable.

Proof: By Theorem 9.10.2, there exist $\mathcal{F} \times \mathcal{S}$ measurable functions, g and h and a set, $N \in \mathcal{F} \times \mathcal{S}$ of $\mu \times \lambda$ measure zero such that $g \leq f \leq h$ and for $(x, y) \notin N$, it follows that $g(x, y) = h(x, y)$. Then

$$\int_X \int_Y g d\nu d\mu = \int_X \int_Y h d\nu d\mu$$

and so for a.e. x ,

$$\int_Y g d\nu = \int_Y h d\nu.$$

Then it follows that for these values of x , $g(x, y) = h(x, y)$ and so by Theorem 9.10.2 again and the assumption that $(Y, \mathcal{S}, \nu)$ is complete, $y \rightarrow f(x, y)$ is $\mathcal{S}$ measurable. The other claim is similar. This proves the lemma.

9.12 Disturbing Examples

There are examples which help to define what can be expected of product measures and Fubini type theorems. Three such examples are given in Rudin [58]. Some of the theorems given above are more general than those in this reference but the same examples are still useful for showing that the hypotheses of the above theorems are all necessary.

Example 9.12.1 *Let $\{a_n\}$ be an increasing sequence of numbers in $(0, 1)$ which converges to 1. Let $g_n \in C_c(a_n, a_{n+1})$ such that $\int g_n dx = 1$. Now for $(x, y) \in [0, 1) \times [0, 1)$ define*

$$f(x, y) \equiv \sum_{k=1}^{\infty} g_n(y) (g_n(x) - g_{n+1}(x)).$$

Note this is actually a finite sum for each such (x, y) . Therefore, this is a continuous function on $[0, 1) \times [0, 1)$. Now for a fixed y ,

$$\int_0^1 f(x, y) dx = \sum_{k=1}^{\infty} g_n(y) \int_0^1 (g_n(x) - g_{n+1}(x)) dx = 0$$

showing that $\int_0^1 \int_0^1 f(x, y) dx dy = \int_0^1 0 dy = 0$. Next fix x .

$$\int_0^1 f(x, y) dy = \sum_{k=1}^{\infty} (g_n(x) - g_{n+1}(x)) \int_0^1 g_n(y) dy = g_1(x).$$

Hence $\int_0^1 \int_0^1 f(x, y) dy dx = \int_0^1 g_1(x) dx = 1$. The iterated integrals are not equal. Note the function, g is not nonnegative even though it is measurable. In addition, neither $\int_0^1 \int_0^1 |f(x, y)| dx dy$ nor $\int_0^1 \int_0^1 |f(x, y)| dy dx$ is finite and so you can't apply Corollary 9.8.13. The problem here is the function is not nonnegative and is not absolutely integrable.

Example 9.12.2 This time let $\mu = m$, Lebesgue measure on $[0, 1]$ and let ν be counting measure on $[0, 1]$, in this case, the σ algebra is $\mathcal{P}([0, 1])$. Let l denote the line segment in $[0, 1] \times [0, 1]$ which goes from $(0, 0)$ to $(1, 1)$. Thus $l = (x, x)$ where $x \in [0, 1]$. Consider the outer measure of l in $\overline{m \times \nu}$. Let $l \subseteq \cup_k A_k \times B_k$ where A_k is Lebesgue measurable and B_k is a subset of $[0, 1]$. Let $\mathcal{B} \equiv \{k \in \mathbb{N} : \nu(B_k) = \infty\}$. If $m(\cup_{k \in \mathcal{B}} A_k)$ has measure zero, then there are uncountably many points of $[0, 1]$ outside of $\cup_{k \in \mathcal{B}} A_k$. For p one of these points, $(p, p) \in A_i \times B_i$ and $i \notin \mathcal{B}$. Thus each of these points is in $\cup_{i \notin \mathcal{B}} B_i$, a countable set because these B_i are each finite. But this is a contradiction because there need to be uncountably many of these points as just indicated. Thus $m(A_k) > 0$ for some $k \in \mathcal{B}$ and so $\overline{m \times \nu}(A_k \times B_k) = \infty$. It follows $\overline{m \times \nu}(l) = \infty$ and so l is $\overline{m \times \nu}$ measurable. Thus $\int \mathcal{X}_l(x, y) d\overline{m \times \nu} = \infty$ and so you cannot apply Fubini's theorem, Theorem 9.8.11. Since ν is not σ finite, you cannot apply the corollary to this theorem either. Thus there is no contradiction to the above theorems in the following observation.

$$\iint \mathcal{X}_l(x, y) d\nu dm = \int 1 dm = 1, \quad \int \int \mathcal{X}_l(x, y) dm d\nu = \int 0 d\nu = 0.$$

The problem here is that you have neither $\int f d\overline{m \times \nu} < \infty$ nor σ finite measure spaces.

The next example is far more exotic. It concerns the case where both iterated integrals make perfect sense but are unequal. In 1877 Cantor conjectured that the cardinality of the real numbers is the next size of infinity after countable infinity. This hypothesis is called the continuum hypothesis and it has never been proved or disproved⁴. Assuming this continuum hypothesis will provide the basis for the following example. It is due to Sierpinski.

Example 9.12.3 Let X be an uncountable set. It follows from the well ordering theorem which says every set can be well ordered which is presented in the appendix that X can be well ordered. Let $\omega \in X$ be the first element of X which is

⁴In 1940 it was shown by Godel that the continuum hypothesis cannot be disproved. In 1963 it was shown by Cohen that the continuum hypothesis cannot be proved. These assertions are based on the axiom of choice and the Zermelo Frankel axioms of set theory. This topic is far outside the scope of this book and this is only a hopefully interesting historical observation.

preceded by uncountably many points of X . Let Ω denote $\{x \in X : x < \omega\}$. Then Ω is uncountable but there is no smaller uncountable set. Thus by the continuum hypothesis, there exists a one to one and onto mapping, j which maps $[0, 1]$ onto Ω . Thus, for $x \in [0, 1]$, $j(x)$ is preceded by countably many points. Let $Q \equiv \{(x, y) \in [0, 1]^2 : j(x) < j(y)\}$ and let $f(x, y) = \chi_Q(x, y)$. Then

$$\int_0^1 f(x, y) dy = 1, \quad \int_0^1 f(x, y) dx = 0$$

In each case, the integrals make sense. In the first, for fixed x , $f(x, y) = 1$ for all but countably many y so the function of y is Borel measurable. In the second where y is fixed, $f(x, y) = 0$ for all but countably many x . Thus

$$\int_0^1 \int_0^1 f(x, y) dy dx = 1, \quad \int_0^1 \int_0^1 f(x, y) dx dy = 0.$$

The problem here must be that f is not $\overline{m \times m}$ measurable.

9.13 Exercises

1. Let $\Omega = \mathbb{N}$, the natural numbers and let $d(p, q) = |p - q|$, the usual distance in $\mathbb{R}$. Show that (Ω, d) the closures of the balls are compact. Now let $\Lambda f \equiv \sum_{k=1}^{\infty} f(k)$ whenever $f \in C_c(\Omega)$. Show this is a well defined positive linear functional on the space $C_c(\Omega)$. Describe the measure of the Riesz representation theorem which results from this positive linear functional. What if $\Lambda(f) = f(1)$? What measure would result from this functional? Which functions are measurable?
2. Verify that $\bar{\mu}$ defined in Lemma 9.1.7 is an outer measure.
3. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be increasing and right continuous. Let $\Lambda f \equiv \int f dF$ where the integral is the Riemann Stieltjes integral of f . Show the measure μ from the Riesz representation theorem satisfies

$$\begin{aligned} \mu([a, b]) &= F(b) - F(a-), \quad \mu((a, b]) = F(b) - F(a), \\ \mu([a, a]) &= F(a) - F(a-). \end{aligned}$$

4. Let Ω be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of Ω which is finite on compact sets. Show there exists a unique Radon measure, $\bar{\mu}$ which equals μ on the Borel sets.
5. $\uparrow$ Random vectors are measurable functions, $\mathbf{X}$, mapping a probability space, $(\Omega, P, \mathcal{F})$ to $\mathbb{R}^n$. Thus $\mathbf{X}(\omega) \in \mathbb{R}^n$ for each $\omega \in \Omega$ and P is a probability measure defined on the sets of $\mathcal{F}$, a σ algebra of subsets of Ω . For E a Borel set in $\mathbb{R}^n$, define

$$\mu(E) \equiv P(\mathbf{X}^{-1}(E)) \equiv \text{probability that } \mathbf{X} \in E.$$

Show this is a well defined measure on the Borel sets of $\mathbb{R}^n$ and use Problem 4 to obtain a Radon measure, $\lambda_{\mathbf{X}}$ defined on a σ algebra of sets of $\mathbb{R}^n$ including the Borel sets such that for E a Borel set, $\lambda_{\mathbf{X}}(E) = \text{Probability that } (\mathbf{X} \in E)$.

6. Suppose X and Y are metric spaces having compact closed balls. Show

$$(X \times Y, d_{X \times Y})$$

is also a metric space which has the closures of balls compact. Here

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) \equiv \max(d(x_1, x_2), d(y_1, y_2)).$$

Let

$$\mathcal{A} \equiv \{E \times F : E \text{ is a Borel set in } X, F \text{ is a Borel set in } Y\}.$$

Show $\sigma(\mathcal{A})$, the smallest σ algebra containing $\mathcal{A}$ contains the Borel sets. **Hint:** Show every open set in a metric space which has closed balls compact can be obtained as a countable union of compact sets. Next show this implies every open set can be obtained as a countable union of open sets of the form $U \times V$ where U is open in X and V is open in Y .

7. Suppose $(\Omega, \mathcal{S}, \mu)$ is a measure space which may not be complete. Could you obtain a complete measure space, $(\Omega, \bar{\mathcal{S}}, \mu_1)$ by simply letting $\bar{\mathcal{S}}$ consist of all sets of the form E where there exists $F \in \mathcal{S}$ such that $(F \setminus E) \cup (E \setminus F) \subseteq N$ for some $N \in \mathcal{S}$ which has measure zero and then let $\mu(E) = \mu_1(F)$?
8. If μ and ν are Radon measures defined on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, show $\overline{\mu \times \nu}$ is also a radon measure on $\mathbb{R}^{n+m}$. **Hint:** Show the $\overline{\mu \times \nu}$ measurable sets include the open sets using the observation that every open set in $\mathbb{R}^{n+m}$ is the countable union of sets of the form $U \times V$ where U and V are open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Next verify outer regularity by considering $A \times B$ for A, B measurable. Argue sets of $\mathcal{R}$ defined above have the property that they can be approximated in measure from above by open sets. Then verify the same is true of sets of $\mathcal{R}_1$. Finally conclude using an appropriate lemma that $\overline{\mu \times \nu}$ is inner regular as well.
9. Let $(\Omega, \mathcal{S}, \mu)$ be a σ finite measure space and let $f : \Omega \rightarrow [0, \infty)$ be measurable. Define

$$A \equiv \{(x, y) : y < f(x)\}$$

Verify that A is $\overline{\mu \times m}$ measurable. Show that

$$\int f d\mu = \int \int \mathcal{X}_A(x, y) d\mu dm = \int \mathcal{X}_A d\overline{\mu \times m}.$$

Lebesgue Measure

10.1 Basic Properties

Definition 10.1.1 Define the following positive linear functional for $f \in C_c(\mathbb{R}^n)$.

$$\Lambda f \equiv \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x}) dx_1 \cdots dx_n.$$

Then the measure representing this functional is Lebesgue measure.

The following lemma will help in understanding Lebesgue measure.

Lemma 10.1.2 Every open set in $\mathbb{R}^n$ is the countable disjoint union of half open boxes of the form

$$\prod_{i=1}^n (a_i, a_i + 2^{-k}]$$

where $a_i = l2^{-k}$ for some integers, l, k . The sides of these boxes are of equal length. One could also have half open boxes of the form

$$\prod_{i=1}^n [a_i, a_i + 2^{-k})$$

and the conclusion would be unchanged.

Proof: Let

$$\mathcal{C}_k = \left\{ \text{All half open boxes } \prod_{i=1}^n (a_i, a_i + 2^{-k}] \text{ where} \right.$$

$$\left. a_i = l2^{-k} \text{ for some integer } l. \right\}$$

Thus $\mathcal{C}_k$ consists of a countable disjoint collection of boxes whose union is $\mathbb{R}^n$. This is sometimes called a tiling of $\mathbb{R}^n$. Think of tiles on the floor of a bathroom and

you will get the idea. Note that each box has diameter no larger than $2^{-k}\sqrt{n}$. This is because if

$$\mathbf{x}, \mathbf{y} \in \prod_{i=1}^n (a_i, a_i + 2^{-k}],$$

then $|x_i - y_i| \leq 2^{-k}$. Therefore,

$$|\mathbf{x} - \mathbf{y}| \leq \left(\sum_{i=1}^n (2^{-k})^2 \right)^{1/2} = 2^{-k} \sqrt{n}.$$

Let U be open and let $\mathcal{B}_1 \equiv$ all sets of $\mathcal{C}_1$ which are contained in U . If $\mathcal{B}_1, \dots, \mathcal{B}_k$ have been chosen, $\mathcal{B}_{k+1} \equiv$ all sets of $\mathcal{C}_{k+1}$ contained in

$$U \setminus \bigcup (\cup_{i=1}^k \mathcal{B}_i).$$

Let $\mathcal{B}_\infty = \cup_{i=1}^\infty \mathcal{B}_i$. In fact $\cup \mathcal{B}_\infty = U$. Clearly $\cup \mathcal{B}_\infty \subseteq U$ because every box of every $\mathcal{B}_i$ is contained in U . If $p \in U$, let k be the smallest integer such that p is contained in a box from $\mathcal{C}_k$ which is also a subset of U . Thus

$$p \in \cup \mathcal{B}_k \subseteq \cup \mathcal{B}_\infty.$$

Hence $\mathcal{B}_\infty$ is the desired countable disjoint collection of half open boxes whose union is U . The last assertion about the other type of half open rectangle is obvious. This proves the lemma.

Now what does Lebesgue measure do to a rectangle, $\prod_{i=1}^n (a_i, b_i]$?

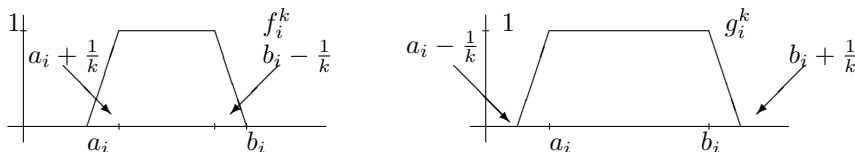
Lemma 10.1.3 *Let $R = \prod_{i=1}^n [a_i, b_i]$, $R_0 = \prod_{i=1}^n (a_i, b_i)$. Then*

$$m_n(R_0) = m_n(R) = \prod_{i=1}^n (b_i - a_i).$$

Proof: Let k be large enough that

$$a_i + 1/k < b_i - 1/k$$

for $i = 1, \dots, n$ and consider functions g_i^k and f_i^k having the following graphs.



Let

$$g^k(\mathbf{x}) = \prod_{i=1}^n g_i^k(x_i), \quad f^k(\mathbf{x}) = \prod_{i=1}^n f_i^k(x_i).$$

Then by elementary calculus along with the definition of Λ ,

$$\begin{aligned} \prod_{i=1}^n (b_i - a_i + 2/k) &\geq \Lambda g^k = \int g^k dm_n \geq m_n(R) \geq m_n(R_0) \\ &\geq \int f^k dm_n = \Lambda f^k \geq \prod_{i=1}^n (b_i - a_i - 2/k). \end{aligned}$$

Letting $k \rightarrow \infty$, it follows that

$$m_n(R) = m_n(R_0) = \prod_{i=1}^n (b_i - a_i).$$

This proves the lemma.

Lemma 10.1.4 *Let U be an open or closed set. Then $m_n(U) = m_n(\mathbf{x} + U)$.*

Proof: By Lemma 10.1.2 there is a sequence of disjoint half open rectangles, $\{R_i\}$ such that $\cup_i R_i = U$. Therefore, $\mathbf{x} + U = \cup_i (\mathbf{x} + R_i)$ and the $\mathbf{x} + R_i$ are also disjoint rectangles which are identical to the R_i but translated. From Lemma 10.1.3, $m_n(U) = \sum_i m_n(R_i) = \sum_i m_n(\mathbf{x} + R_i) = m_n(\mathbf{x} + U)$.

It remains to verify the lemma for a closed set. Let H be a closed bounded set first. Then $H \subseteq B(\mathbf{0}, R)$ for some R large enough. First note that $\mathbf{x} + H$ is a closed set. Thus

$$\begin{aligned} m_n(B(\mathbf{x}, R)) &= m_n(\mathbf{x} + H) + m_n((B(\mathbf{0}, R) + \mathbf{x}) \setminus (\mathbf{x} + H)) \\ &= m_n(\mathbf{x} + H) + m_n((B(\mathbf{0}, R) \setminus H) + \mathbf{x}) \\ &= m_n(\mathbf{x} + H) + m_n((B(\mathbf{0}, R) \setminus H)) \\ &= m_n(B(\mathbf{0}, R)) - m_n(H) + m_n(\mathbf{x} + H) \\ &= m_n(B(\mathbf{x}, R)) - m_n(H) + m_n(\mathbf{x} + H) \end{aligned}$$

the last equality because of the first part of the lemma which implies $m_n(B(\mathbf{x}, R)) = m_n(B(\mathbf{0}, R))$. Therefore, $m_n(\mathbf{x} + H) = m_n(H)$ as claimed. If H is not bounded, consider $H_m \equiv \overline{B(\mathbf{0}, m)} \cap H$. Then $m_n(\mathbf{x} + H_m) = m_n(H_m)$. Passing to the limit as $m \rightarrow \infty$ yields the result in general.

Theorem 10.1.5 *Lebesgue measure is translation invariant. That is*

$$m_n(E) = m_n(\mathbf{x} + E)$$

for all E Lebesgue measurable.

Proof: Suppose $m_n(E) < \infty$. By regularity of the measure, there exist sets G, H such that G is a countable intersection of open sets, H is a countable union of compact sets, $m_n(G \setminus H) = 0$, and $G \supseteq E \supseteq H$. Now $m_n(G) = m_n(G + \mathbf{x})$ and

$m_n(H) = m_n(H + \mathbf{x})$ which follows from Lemma 10.1.4 applied to the sets which are either intersected to form G or unioned to form H . Now

$$\mathbf{x} + H \subseteq \mathbf{x} + E \subseteq \mathbf{x} + G$$

and both $\mathbf{x} + H$ and $\mathbf{x} + G$ are measurable because they are either countable unions or countable intersections of measurable sets. Furthermore,

$$m_n(\mathbf{x} + G \setminus \mathbf{x} + H) = m_n(\mathbf{x} + G) - m_n(\mathbf{x} + H) = m_n(G) - m_n(H) = 0$$

and so by completeness of the measure, $\mathbf{x} + E$ is measurable. It follows

$$\begin{aligned} m_n(E) &= m_n(H) = m_n(\mathbf{x} + H) \leq m_n(\mathbf{x} + E) \\ &\leq m_n(\mathbf{x} + G) = m_n(G) = m_n(E). \end{aligned}$$

If $m_n(E)$ is not necessarily less than ∞ , consider $E_m \equiv B(\mathbf{0}, m) \cap E$. Then $m_n(E_m) = m_n(E_m + \mathbf{x})$ by the above. Letting $m \rightarrow \infty$ it follows $m_n(E) = m_n(E + \mathbf{x})$. This proves the theorem.

Corollary 10.1.6 *Let D be an $n \times n$ diagonal matrix and let U be an open set. Then*

$$m_n(DU) = |\det(D)| m_n(U).$$

Proof: If any of the diagonal entries of D equals 0 there is nothing to prove because then both sides equal zero. Therefore, it can be assumed none are equal to zero. Suppose these diagonal entries are $k_1, \dots, k_n$. From Lemma 10.1.2 there exist half open boxes, $\{R_i\}$ having all sides equal such that $U = \cup_i R_i$. Suppose one of these is $R_i = \prod_{j=1}^n (a_j, b_j]$, where $b_j - a_j = l_i$. Then $DR_i = \prod_{j=1}^n I_j$ where $I_j = (k_j a_j, k_j b_j]$ if $k_j > 0$ and $I_j = [k_j b_j, k_j a_j)$ if $k_j < 0$. Then the rectangles, DR_i are disjoint because D is one to one and their union is DU . Also,

$$m_n(DR_i) = \prod_{j=1}^n |k_j| l_i = |\det D| m_n(R_i).$$

Therefore,

$$m_n(DU) = \sum_{i=1}^{\infty} m_n(DR_i) = |\det(D)| \sum_{i=1}^{\infty} m_n(R_i) = |\det(D)| m_n(U).$$

and this proves the corollary.

From this the following corollary is obtained.

Corollary 10.1.7 *Let $M > 0$. Then $m_n(B(\mathbf{a}, Mr)) = M^n m_n(B(\mathbf{0}, r))$.*

Proof: By Lemma 10.1.4 there is no loss of generality in taking $\mathbf{a} = \mathbf{0}$. Let D be the diagonal matrix which has M in every entry of the main diagonal so $|\det(D)| = M^n$. Note that $DB(\mathbf{0}, r) = B(\mathbf{0}, Mr)$. By Corollary 10.1.6 $m_n(B(\mathbf{0}, Mr)) = m_n(DB(\mathbf{0}, r)) = M^n m_n(B(\mathbf{0}, r))$.

There are many norms on $\mathbb{R}^n$. Other common examples are

$$\|\mathbf{x}\|_\infty \equiv \max \{|x_k| : \mathbf{x} = (x_1, \dots, x_n)\}$$

or

$$\|\mathbf{x}\|_p \equiv \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}.$$

With $\|\cdot\|$ any norm for $\mathbb{R}^n$ you can define a corresponding ball in terms of this norm.

$$B(\mathbf{a}, r) \equiv \{\mathbf{x} \in \mathbb{R}^n \text{ such that } \|\mathbf{x} - \mathbf{a}\| < r\}$$

It follows from general considerations involving metric spaces presented earlier that these balls are open sets. Therefore, Corollary 10.1.7 has an obvious generalization.

Corollary 10.1.8 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Then for $M > 0$, $m_n(B(\mathbf{a}, Mr)) = M^n m_n(B(\mathbf{0}, r))$ where these balls are defined in terms of the norm $\|\cdot\|$.*

10.2 The Vitali Covering Theorem

The Vitali covering theorem is concerned with the situation in which a set is contained in the union of balls. You can imagine that it might be very hard to get disjoint balls from this collection of balls which would cover the given set. However, it is possible to get disjoint balls from this collection of balls which have the property that if each ball is enlarged appropriately, the resulting enlarged balls do cover the set. When this result is established, it is used to prove another form of this theorem in which the disjoint balls do not cover the set but they only miss a set of measure zero.

Recall the Hausdorff maximal principle, Theorem 1.4.2 on Page 26 which is proved to be equivalent to the axiom of choice in the appendix. For convenience, here it is:

Theorem 10.2.1 (Hausdorff Maximal Principle) *Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

I will use this Hausdorff maximal principle to give a very short and elegant proof of the Vitali covering theorem. This follows the treatment in Evans and Gariepy [24] which they got from another book. I am not sure who first did it this way but it is very nice because it is so short. In the following lemma and theorem, the balls will be either open or closed and determined by some norm on $\mathbb{R}^n$. When pictures are drawn, I shall draw them as though the norm is the usual norm but the results are unchanged for any norm. Also, I will write (in this section only) $B(\mathbf{a}, r)$ to indicate a set which satisfies

$$\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < r\} \subseteq B(\mathbf{a}, r) \subseteq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| \leq r\}$$

and $\widehat{B}(\mathbf{a}, r)$ to indicate the usual ball but with radius 5 times as large,

$$\{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < 5r\}.$$

Lemma 10.2.2 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and let $\mathcal{F}$ be a collection of balls determined by this norm. Suppose*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\text{if } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \quad (10.2.1)$$

$$\text{if } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \quad (10.2.2)$$

$\mathcal{G}$ is maximal with respect to 10.2.1 and 10.2.2.

Note that if there is no ball of $\mathcal{F}$ which has radius larger than k then $\mathcal{G} = \emptyset$.

Proof: Let $\mathcal{H} = \{\mathcal{B} \subseteq \mathcal{F} \text{ such that 10.2.1 and 10.2.2 hold}\}$. If there are no balls with radius larger than k then $\mathcal{H} = \emptyset$ and you let $\mathcal{G} = \emptyset$. In the other case, $\mathcal{H} \neq \emptyset$ because there exists $B(\mathbf{p}, r) \in \mathcal{F}$ with $r > k$. In this case, partially order $\mathcal{H}$ by set inclusion and use the Hausdorff maximal principle (see the appendix on set theory) to let $\mathcal{C}$ be a maximal chain in $\mathcal{H}$. Clearly $\cup \mathcal{C}$ satisfies 10.2.1 and 10.2.2 because if B_1 and B_2 are two balls from $\cup \mathcal{C}$ then since $\mathcal{C}$ is a chain, it follows there is some element of $\mathcal{C}$, $\mathcal{B}$ such that both B_1 and B_2 are elements of $\mathcal{B}$ and $\mathcal{B}$ satisfies 10.2.1 and 10.2.2. If $\cup \mathcal{C}$ is not maximal with respect to these two properties, then $\mathcal{C}$ was not a maximal chain because then there would exist $\mathcal{B} \supsetneq \cup \mathcal{C}$, that is, $\mathcal{B}$ contains $\mathcal{C}$ as a proper subset and $\{\mathcal{C}, \mathcal{B}\}$ would be a strictly larger chain in $\mathcal{H}$. Let $\mathcal{G} = \cup \mathcal{C}$.

Theorem 10.2.3 (Vitali) *Let $\mathcal{F}$ be a collection of balls and let*

$$A \equiv \cup\{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\widehat{B} : B \in \mathcal{G}\}.$$

Proof: Using Lemma 10.2.2, there exists $\mathcal{G}_1 \subseteq \mathcal{F} \equiv \mathcal{F}_0$ which satisfies

$$B(\mathbf{p}, r) \in \mathcal{G}_1 \text{ implies } r > \frac{M}{2}, \quad (10.2.3)$$

$$B_1, B_2 \in \mathcal{G}_1 \text{ implies } B_1 \cap B_2 = \emptyset, \quad (10.2.4)$$

$\mathcal{G}_1$ is maximal with respect to 10.2.3, and 10.2.4.

Suppose $\mathcal{G}_1, \dots, \mathcal{G}_m$ have been chosen, $m \geq 1$. Let

$$\mathcal{F}_m \equiv \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \cup\{\mathcal{G}_1 \cup \dots \cup \mathcal{G}_m\}\}.$$

Using Lemma 10.2.2, there exists $\mathcal{G}_{m+1} \subseteq \mathcal{F}_m$ such that

$$B(\mathbf{p}, r) \in \mathcal{G}_{m+1} \text{ implies } r > \frac{M}{2^{m+1}}, \quad (10.2.5)$$

$$B_1, B_2 \in \mathcal{G}_{m+1} \text{ implies } B_1 \cap B_2 = \emptyset, \quad (10.2.6)$$

$\mathcal{G}_{m+1}$ is a maximal subset of $\mathcal{F}_m$ with respect to 10.2.5 and 10.2.6.

Note it might be the case that $\mathcal{G}_{m+1} = \emptyset$ which happens if $\mathcal{F}_m = \emptyset$. Define

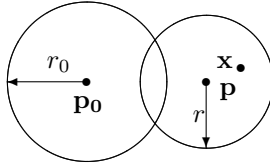
$$\mathcal{G} \equiv \bigcup_{k=1}^{\infty} \mathcal{G}_k.$$

Thus $\mathcal{G}$ is a collection of disjoint balls in $\mathcal{F}$. I must show $\{\widehat{B} : B \in \mathcal{G}\}$ covers A .

Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$ and let

$$\frac{M}{2^m} < r \leq \frac{M}{2^{m-1}}.$$

Then $B(\mathbf{p}, r)$ must intersect some set, $B(\mathbf{p}_0, r_0) \in \mathcal{G}_1 \cup \dots \cup \mathcal{G}_m$ since otherwise, $\mathcal{G}_m$ would fail to be maximal. Then $r_0 > \frac{M}{2^m}$ because all balls in $\mathcal{G}_1 \cup \dots \cup \mathcal{G}_m$ satisfy this inequality.



Then for $\mathbf{x} \in B(\mathbf{p}, r)$, the following chain of inequalities holds because $r \leq \frac{M}{2^{m-1}}$ and $r_0 > \frac{M}{2^m}$

$$\begin{aligned} |\mathbf{x} - \mathbf{p}_0| &\leq |\mathbf{x} - \mathbf{p}| + |\mathbf{p} - \mathbf{p}_0| \leq r + r_0 + r \\ &\leq \frac{2M}{2^{m-1}} + r_0 = \frac{4M}{2^m} + r_0 < 5r_0. \end{aligned}$$

Thus $B(\mathbf{p}, r) \subseteq \widehat{B}(\mathbf{p}_0, r_0)$ and this proves the theorem.

10.3 The Vitali Covering Theorem (Elementary Version)

The proof given here is from Basic Analysis [45]. It first considers the case of open balls and then generalizes to balls which may be neither open nor closed or closed.

Lemma 10.3.1 *Let $\mathcal{F}$ be a countable collection of balls satisfying*

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0$$

and let $k \in (0, \infty)$. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that

$$\text{If } B(\mathbf{p}, r) \in \mathcal{G} \text{ then } r > k, \quad (10.3.7)$$

$$\text{If } B_1, B_2 \in \mathcal{G} \text{ then } B_1 \cap B_2 = \emptyset, \quad (10.3.8)$$

$$\mathcal{G} \text{ is maximal with respect to 10.3.7 and 10.3.8.} \quad (10.3.9)$$

Proof: If no ball of $\mathcal{F}$ has radius larger than k , let $\mathcal{G} = \emptyset$. Assume therefore, that some balls have radius larger than k . Let $\mathcal{F} \equiv \{B_i\}_{i=1}^\infty$. Now let B_{n_1} be the first ball in the list which has radius greater than k . If every ball having radius larger than k intersects this one, then stop. The maximal set is just B_{n_1} . Otherwise, let B_{n_2} be the next ball having radius larger than k which is disjoint from B_{n_1} . Continue this way obtaining $\{B_{n_i}\}_{i=1}^\infty$, a finite or infinite sequence of disjoint balls having radius larger than k . Then let $\mathcal{G} \equiv \{B_{n_i}\}$. To see that $\mathcal{G}$ is maximal with respect to 10.3.7 and 10.3.8, suppose $B \in \mathcal{F}$, B has radius larger than k , and $\mathcal{G} \cup \{B\}$ satisfies 10.3.7 and 10.3.8. Then at some point in the process, B would have been chosen because it would be the ball of radius larger than k which has the smallest index. Therefore, $B \in \mathcal{G}$ and this shows $\mathcal{G}$ is maximal with respect to 10.3.7 and 10.3.8.

For the next lemma, for an open ball, $B = B(\mathbf{x}, r)$, denote by $\tilde{B}$ the open ball, $B(\mathbf{x}, 4r)$.

Lemma 10.3.2 *Let $\mathcal{F}$ be a collection of open balls, and let*

$$A \equiv \cup\{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup\{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup\{\tilde{B} : B \in \mathcal{G}\}.$$

Proof: Without loss of generality assume $\mathcal{F}$ is countable. This is because there is a countable subset of $\mathcal{F}$, $\mathcal{F}'$ such that $\cup\mathcal{F}' = A$. To see this, consider the set of balls having rational radii and centers having all components rational. This is a countable set of balls and you should verify that every open set is the union of balls of this form. Therefore, you can consider the subset of this set of balls consisting of those which are contained in some open set of $\mathcal{F}$, G so $\cup G = A$ and use the axiom of choice to define a subset of $\mathcal{F}$ consisting of a single set from $\mathcal{F}$ containing each set of G . Then this is $\mathcal{F}'$. The union of these sets equals A . Then consider $\mathcal{F}'$ instead of $\mathcal{F}$. Therefore, assume at the outset $\mathcal{F}$ is countable. By Lemma 10.3.1, there exists $\mathcal{G}_1 \subseteq \mathcal{F}$ which satisfies 10.3.7, 10.3.8, and 10.3.9 with $k = \frac{2M}{3}$.

Suppose $\mathcal{G}_1, \dots, \mathcal{G}_{m-1}$ have been chosen for $m \geq 2$. Let

$$\mathcal{F}_m = \{B \in \mathcal{F} : B \subseteq \mathbb{R}^n \setminus \overbrace{\cup \{\mathcal{G}_1 \cup \dots \cup \mathcal{G}_{m-1}\}}^{\text{union of the balls in these } \mathcal{G}_j} \}$$

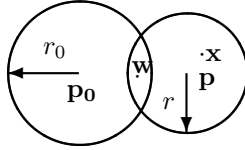
and using Lemma 10.3.1, let $\mathcal{G}_m$ be a maximal collection of disjoint balls from $\mathcal{F}_m$ with the property that each ball has radius larger than $(\frac{2}{3})^m M$. Let $\mathcal{G} \equiv \cup_{k=1}^{\infty} \mathcal{G}_k$. Let $\mathbf{x} \in B(\mathbf{p}, r) \in \mathcal{F}$. Choose m such that

$$\left(\frac{2}{3}\right)^m M < r \leq \left(\frac{2}{3}\right)^{m-1} M$$

Then $B(\mathbf{p}, r)$ must have nonempty intersection with some ball from $\mathcal{G}_1 \cup \dots \cup \mathcal{G}_m$ because if it didn't, then $\mathcal{G}_m$ would fail to be maximal. Denote by $B(\mathbf{p}_0, r_0)$ a ball in $\mathcal{G}_1 \cup \dots \cup \mathcal{G}_m$ which has nonempty intersection with $B(\mathbf{p}, r)$. Thus

$$r_0 > \left(\frac{2}{3}\right)^m M.$$

Consider the picture, in which $\mathbf{w} \in B(\mathbf{p}_0, r_0) \cap B(\mathbf{p}, r)$.



Then

$$\begin{aligned} |\mathbf{x} - \mathbf{p}_0| &\leq |\mathbf{x} - \mathbf{p}| + |\mathbf{p} - \mathbf{w}| + \overbrace{|\mathbf{w} - \mathbf{p}_0|}^{< r_0} \\ &< r + r + r_0 \leq 2 \overbrace{\left(\frac{2}{3}\right)^{m-1} M}^{< \frac{3}{2} r_0} + r_0 \\ &< 2 \left(\frac{3}{2}\right) r_0 + r_0 = 4r_0. \end{aligned}$$

This proves the lemma since it shows $B(\mathbf{p}, r) \subseteq B(\mathbf{p}_0, 4r_0)$.

With this Lemma consider a version of the Vitali covering theorem in which the balls do not have to be open. A ball centered at $\mathbf{x}$ of radius r will denote something which contains the open ball, $B(\mathbf{x}, r)$ and is contained in the closed ball, $\overline{B}(\mathbf{x}, r)$. Thus the balls could be open or they could contain some but not all of their boundary points.

Definition 10.3.3 Let B be a ball centered at $\mathbf{x}$ having radius r . Denote by $\widehat{B}$ the open ball, $B(\mathbf{x}, 5r)$.

Theorem 10.3.4 (Vitali) *Let $\mathcal{F}$ be a collection of balls, and let*

$$A \equiv \cup \{B : B \in \mathcal{F}\}.$$

Suppose

$$\infty > M \equiv \sup \{r : B(\mathbf{p}, r) \in \mathcal{F}\} > 0.$$

Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ consists of disjoint balls and

$$A \subseteq \cup \{\widehat{B} : B \in \mathcal{G}\}.$$

Proof: For B one of these balls, say $\overline{B(\mathbf{x}, r)} \supseteq B \supseteq B(\mathbf{x}, r)$, denote by B_1 , the ball $B(\mathbf{x}, \frac{5r}{4})$. Let $\mathcal{F}_1 \equiv \{B_1 : B \in \mathcal{F}\}$ and let A_1 denote the union of the balls in $\mathcal{F}_1$. Apply Lemma 10.3.2 to $\mathcal{F}_1$ to obtain

$$A_1 \subseteq \cup \{\widetilde{B}_1 : B_1 \in \mathcal{G}_1\}$$

where $\mathcal{G}_1$ consists of disjoint balls from $\mathcal{F}_1$. Now let $\mathcal{G} \equiv \{B \in \mathcal{F} : B_1 \in \mathcal{G}_1\}$. Thus $\mathcal{G}$ consists of disjoint balls from $\mathcal{F}$ because they are contained in the disjoint open balls, $\mathcal{G}_1$. Then

$$A \subseteq A_1 \subseteq \cup \{\widetilde{B}_1 : B_1 \in \mathcal{G}_1\} = \cup \{\widehat{B} : B \in \mathcal{G}\}$$

because for $B_1 = B(\mathbf{x}, \frac{5r}{4})$, it follows $\widetilde{B}_1 = B(\mathbf{x}, 5r) = \widehat{B}$. This proves the theorem.

10.4 Vitali Coverings

There is another version of the Vitali covering theorem which is also of great importance. In this one, balls from the original set of balls almost cover the set, leaving out only a set of measure zero. It is like packing a truck with stuff. You keep trying to fill in the holes with smaller and smaller things so as to not waste space. It is remarkable that you can avoid wasting any space at all when you are dealing with balls of any sort provided you can use arbitrarily small balls.

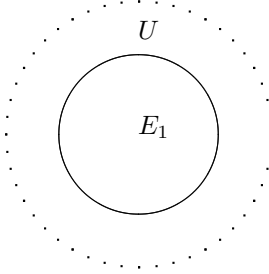
Definition 10.4.1 *Let $\mathcal{F}$ be a collection of balls that cover a set, E , which have the property that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Such a collection covers E in the sense of Vitali.*

In the following covering theorem, $\overline{m}_n$ denotes the outer measure determined by n dimensional Lebesgue measure.

Theorem 10.4.2 *Let $E \subseteq \mathbb{R}^n$ and suppose $0 < \overline{m}_n(E) < \infty$ where $\overline{m}_n$ is the outer measure determined by m_n , n dimensional Lebesgue measure, and let $\mathcal{F}$ be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}_n(E \setminus \cup_{j=1}^\infty B_j) = 0$.*

Proof: From the definition of outer measure there exists a Lebesgue measurable set, $E_1 \supseteq E$ such that $m_n(E_1) = \overline{m}_n(E)$. Now by outer regularity of Lebesgue measure, there exists U , an open set which satisfies

$$m_n(E_1) > (1 - 10^{-n})m_n(U), \quad U \supseteq E_1.$$



Each point of E is contained in balls of $\mathcal{F}$ of arbitrarily small radii and so there exists a covering of E with balls of $\mathcal{F}$ which are themselves contained in U . Therefore, by the Vitali covering theorem, there exist disjoint balls, $\{B_i\}_{i=1}^\infty \subseteq \mathcal{F}$ such that

$$E \subseteq \cup_{j=1}^\infty \widehat{B}_j, \quad B_j \subseteq U.$$

Therefore,

$$\begin{aligned} m_n(E_1) &= \overline{m}_n(E) \leq m_n\left(\cup_{j=1}^\infty \widehat{B}_j\right) \leq \sum_j m_n\left(\widehat{B}_j\right) \\ &= 5^n \sum_j m_n(B_j) = 5^n m_n\left(\cup_{j=1}^\infty B_j\right) \end{aligned}$$

Then E_1 and $\cup_{j=1}^\infty B_j$ are contained in U and so

$$\begin{aligned} m_n(E_1) &> (1 - 10^{-n})m_n(U) \\ &\geq (1 - 10^{-n})[m_n(E_1 \setminus \cup_{j=1}^\infty B_j) + m_n(\cup_{j=1}^\infty B_j)] \\ &\geq (1 - 10^{-n})[m_n(E_1 \setminus \cup_{j=1}^\infty B_j) + 5^{-n} \overbrace{m_n(E)}^{=m_n(E_1)}]. \end{aligned}$$

and so

$$(1 - (1 - 10^{-n})5^{-n})m_n(E_1) \geq (1 - 10^{-n})m_n(E_1 \setminus \cup_{j=1}^\infty B_j)$$

which implies

$$m_n(E_1 \setminus \cup_{j=1}^\infty B_j) \leq \frac{(1 - (1 - 10^{-n})5^{-n})}{(1 - 10^{-n})}m_n(E_1)$$

Now a short computation shows

$$0 < \frac{(1 - (1 - 10^{-n})5^{-n})}{(1 - 10^{-n})} < 1$$

Hence, denoting by θ_n a number such that

$$\frac{(1 - (1 - 10^{-n})5^{-n})}{(1 - 10^{-n})} < \theta_n < 1,$$

$$\overline{m}_n(E \setminus \cup_{j=1}^{\infty} B_j) \leq m_n(E_1 \setminus \cup_{j=1}^{\infty} B_j) < \theta_n m_n(E_1) = \theta_n \overline{m}_n(E)$$

Now using Theorem 8.1.5 on Page 178 there exists N_1 large enough that

$$\theta_n \overline{m}_n(E) \geq m_n(E_1 \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m}_n(E \setminus \cup_{j=1}^{N_1} B_j) \quad (10.4.10)$$

Let $\mathcal{F}_1 = \{B \in \mathcal{F} : B_j \cap B = \emptyset, j = 1, \dots, N_1\}$. If $E \setminus \cup_{j=1}^{N_1} B_j = \emptyset$, then $\mathcal{F}_1 = \emptyset$ and

$$\overline{m}_n(E \setminus \cup_{j=1}^{N_1} B_j) = 0$$

Therefore, in this case let $B_k = \emptyset$ for all $k > N_1$. Consider the case where

$$E \setminus \cup_{j=1}^{N_1} B_j \neq \emptyset.$$

In this case, since the balls are closed and $\mathcal{F}$ is a Vitali cover, $\mathcal{F}_1 \neq \emptyset$ and covers $E \setminus \cup_{j=1}^{N_1} B_j$ in the sense of Vitali. Repeat the same argument, letting $E \setminus \cup_{j=1}^{N_1} B_j$ play the role of E . (You pick a different E_1 whose measure equals the outer measure of $E \setminus \cup_{j=1}^{N_1} B_j$ and proceed as before.) Then choosing B_j for $j = N_1 + 1, \dots, N_2$ as in the above argument,

$$\theta_n \overline{m}_n(E \setminus \cup_{j=1}^{N_1} B_j) \geq \overline{m}_n(E \setminus \cup_{j=1}^{N_2} B_j)$$

and so from 10.4.10,

$$\theta_n^2 \overline{m}_n(E) \geq \overline{m}_n(E \setminus \cup_{j=1}^{N_2} B_j).$$

Continuing this way

$$\theta_n^k \overline{m}_n(E) \geq \overline{m}_n(E \setminus \cup_{j=1}^{N_k} B_j).$$

If it is ever the case that $E \setminus \cup_{j=1}^{N_k} B_j = \emptyset$, then as in the above argument,

$$\overline{m}_n(E \setminus \cup_{j=1}^{N_k} B_j) = 0.$$

Otherwise, the process continues and

$$\overline{m}_n(E \setminus \cup_{j=1}^{\infty} B_j) \leq \overline{m}_n(E \setminus \cup_{j=1}^{N_k} B_j) \leq \theta_n^k \overline{m}_n(E)$$

for every $k \in \mathbb{N}$. Therefore, the conclusion holds in this case also. This proves the Theorem.

There is an obvious corollary which removes the assumption that $0 < \overline{m}_n(E)$.

Corollary 10.4.3 *Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m}_n(E) < \infty$ where $\overline{m}_n$ is the outer measure determined by m_n , n dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}_n(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.*

Proof: If $0 = \overline{m}_n(E)$ you simply pick any ball from $\mathcal{F}$ for your collection of disjoint balls.

It is also not hard to remove the assumption that $\overline{m}_n(E) < \infty$.

Corollary 10.4.4 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}_n(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.*

Proof: Let $R_m \equiv (-m, m)^n$ be the open rectangle having sides of length $2m$ which is centered at $\mathbf{0}$ and let $R_0 = \emptyset$. Let $H_m \equiv \overline{R_m} \setminus R_m$. Since both $\overline{R_m}$ and R_m have the same measure, $(2m)^n$, it follows $m_n(H_m) = 0$. Now for all $k \in \mathbb{N}$, $R_k \subseteq \overline{R_k} \subseteq R_{k+1}$. Consider the disjoint open sets, $U_k \equiv R_{k+1} \setminus \overline{R_k}$. Thus $\mathbb{R}^n = \bigcup_{k=0}^\infty U_k \cup N$ where N is a set of measure zero equal to the union of the H_k . Let $\mathcal{F}_k$ denote those balls of $\mathcal{F}$ which are contained in U_k and let $E_k \equiv U_k \cap E$. Then from Theorem 10.4.2, there exists a sequence of disjoint balls, $D_k \equiv \{B_i^k\}_{i=1}^\infty$ of $\mathcal{F}_k$ such that $\overline{m}_n(E_k \setminus \bigcup_{j=1}^\infty B_j^k) = 0$. Letting $\{B_i\}_{i=1}^\infty$ be an enumeration of all the balls of $\bigcup_k D_k$, it follows that

$$\overline{m}_n(E \setminus \bigcup_{j=1}^\infty B_j) \leq m_n(N) + \sum_{k=1}^\infty \overline{m}_n(E_k \setminus \bigcup_{j=1}^\infty B_j^k) = 0.$$

Also, you don't have to assume the balls are closed.

Corollary 10.4.5 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}_n(E \setminus \bigcup_{j=1}^\infty B_j) = 0$.*

Proof: Let $\overline{\mathcal{F}}$ be the collection of closures of balls in $\mathcal{F}$. Then $\overline{\mathcal{F}}$ covers E in the sense of Vitali and so from Corollary 10.4.4 there exists a sequence of disjoint closed balls from $\overline{\mathcal{F}}$ satisfying $\overline{m}_n(E \setminus \bigcup_{i=1}^\infty \overline{B_i}) = 0$. Now boundaries of the balls, B_i have measure zero and so $\{B_i\}$ is a sequence of disjoint open balls satisfying $\overline{m}_n(E \setminus \bigcup_{i=1}^\infty B_i) = 0$. The reason for this is that

$$(E \setminus \bigcup_{i=1}^\infty B_i) \setminus (E \setminus \bigcup_{i=1}^\infty \overline{B_i}) \subseteq \bigcup_{i=1}^\infty \overline{B_i} \setminus \bigcup_{i=1}^\infty B_i \subseteq \bigcup_{i=1}^\infty \overline{B_i} \setminus B_i,$$

a set of measure zero. Therefore,

$$E \setminus \bigcup_{i=1}^\infty B_i \subseteq (E \setminus \bigcup_{i=1}^\infty \overline{B_i}) \cup (\bigcup_{i=1}^\infty \overline{B_i} \setminus B_i)$$

and so

$$\begin{aligned} \overline{m}_n(E \setminus \bigcup_{i=1}^\infty B_i) &\leq \overline{m}_n(E \setminus \bigcup_{i=1}^\infty \overline{B_i}) + m_n(\bigcup_{i=1}^\infty \overline{B_i} \setminus B_i) \\ &= \overline{m}_n(E \setminus \bigcup_{i=1}^\infty \overline{B_i}) = 0. \end{aligned}$$

This implies you can fill up an open set with balls which cover the open set in the sense of Vitali.

Corollary 10.4.6 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathcal{F}$ be a collection of closed or even open balls of bounded radii contained in U such that $\mathcal{F}$ covers U in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}_n(U \setminus \cup_{j=1}^\infty B_j) = 0$.*

10.5 Change Of Variables For Linear Maps

To begin with certain kinds of functions map measurable sets to measurable sets. It will be assumed that U is an open set in $\mathbb{R}^n$ and that $\mathbf{h} : U \rightarrow \mathbb{R}^n$ satisfies

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in U, \quad (10.5.11)$$

Lemma 10.5.1 *Let $\mathbf{h}$ satisfy 10.5.11. If $T \subseteq U$ and $m_n(T) = 0$, then $m_n(\mathbf{h}(T)) = 0$.*

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : \|D\mathbf{h}(\mathbf{x})\| < k\}$$

and let $\varepsilon > 0$ be given. Now by outer regularity, there exists an open set, V , containing T_k which is contained in U such that $m_n(V) < \varepsilon$. Let $\mathbf{x} \in T_k$. Then by differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so there exist arbitrarily small $r_{\mathbf{x}} < 1$ such that $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| \leq r_{\mathbf{x}}$, $|o(\mathbf{v})| < k|\mathbf{v}|$. Thus

$$\mathbf{h}(B(\mathbf{x}, r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 2kr_{\mathbf{x}}).$$

From the Vitali covering theorem there exists a countable disjoint sequence of these sets, $\{B(\mathbf{x}_i, r_i)\}_{i=1}^\infty$ such that $\{B(\mathbf{x}_i, 5r_i)\}_{i=1}^\infty = \{\widehat{B}_i\}_{i=1}^\infty$ covers T_k . Then letting $\overline{m}_n$ denote the outer measure determined by m_n ,

$$\begin{aligned} \overline{m}_n(\mathbf{h}(T_k)) &\leq \overline{m}_n\left(\mathbf{h}\left(\cup_{i=1}^\infty \widehat{B}_i\right)\right) \\ &\leq \sum_{i=1}^\infty \overline{m}_n\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^\infty m_n(B(\mathbf{h}(\mathbf{x}_i), 2kr_{\mathbf{x}_i})) \\ &= \sum_{i=1}^\infty m_n(B(\mathbf{x}_i, 2kr_{\mathbf{x}_i})) = (2k)^n \sum_{i=1}^\infty m_n(B(\mathbf{x}_i, r_{\mathbf{x}_i})) \\ &\leq (2k)^n m_n(V) \leq (2k)^n \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $m_n(\mathbf{h}(T_k)) = 0$. Now

$$m_n(\mathbf{h}(T)) = \lim_{k \rightarrow \infty} m_n(\mathbf{h}(T_k)) = 0.$$

This proves the lemma.

Lemma 10.5.2 *Let $\mathbf{h}$ satisfy 10.5.11. If S is a Lebesgue measurable subset of U , then $\mathbf{h}(S)$ is Lebesgue measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k)$, $k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F , which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = S_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 10.5.1, $m_n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is Lebesgue measurable because of completeness of Lebesgue measure. Now $\mathbf{h}(S) = \bigcup_{k=1}^{\infty} \mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is Lebesgue measurable. This proves the lemma.

In particular, this proves the following corollary.

Corollary 10.5.3 *Suppose A is an $n \times n$ matrix. Then if S is a Lebesgue measurable set, it follows AS is also a Lebesgue measurable set.*

Lemma 10.5.4 *Let R be unitary ($R^*R = RR^* = I$) and let V be a an open or closed set. Then $m_n(RV) = m_n(V)$.*

Proof: First assume V is a bounded open set. By Corollary 10.4.6 there is a disjoint sequence of closed balls, $\{B_i\}$ such that $V = \bigcup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and let r_i be the radius of B_i . Then by Lemma 10.5.1 $m_n(RV) = \sum_{i=1}^{\infty} m_n(RB_i)$. Now by invariance of translation of Lebesgue measure, this equals $\sum_{i=1}^{\infty} m_n(RB_i - R\mathbf{x}_i) = \sum_{i=1}^{\infty} m_n(RB(\mathbf{0}, r_i))$. Since R is unitary, it preserves all distances and so $RB(\mathbf{0}, r_i) = B(\mathbf{0}, r_i)$ and therefore,

$$m_n(RV) = \sum_{i=1}^{\infty} m_n(B(\mathbf{0}, r_i)) = \sum_{i=1}^{\infty} m_n(B_i) = m_n(V).$$

This proves the lemma in the case that V is bounded. Suppose now that V is just an open set. Let $V_k = V \cap B(\mathbf{0}, k)$. Then $m_n(RV_k) = m_n(V_k)$. Letting $k \rightarrow \infty$, this yields the desired conclusion. This proves the lemma in the case that V is open.

Suppose now that H is a closed and bounded set. Let $B(\mathbf{0}, R) \supseteq H$. Then letting $B = B(\mathbf{0}, R)$ for short,

$$\begin{aligned} m_n(RH) &= m_n(RB) - m_n(R(B \setminus H)) \\ &= m_n(B) - m_n(B \setminus H) = m_n(H). \end{aligned}$$

In general, let $H_m = H \cap \overline{B(\mathbf{0}, m)}$. Then from what was just shown, $m_n(RH_m) = m_n(H_m)$. Now let $m \rightarrow \infty$ to get the conclusion of the lemma in general. This proves the lemma.

Lemma 10.5.5 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let R be unitary. Then $m_n(RE) = m_n(E)$.*

Proof: First suppose E is bounded. Then there exist sets, G and H such that $H \subseteq E \subseteq G$ and H is the countable union of closed sets while G is the countable intersection of open sets such that $m_n(G \setminus H) = 0$. By Lemma 10.5.4 applied to these sets whose union or intersection equals H or G respectively, it follows

$$m_n(RG) = m_n(G) = m_n(H) = m_n(RH).$$

Therefore,

$$m_n(H) = m_n(RH) \leq m_n(RE) \leq m_n(RG) = m_n(G) = m_n(E) = m_n(H).$$

In the general case, let $E_m = E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \rightarrow \infty$.

Lemma 10.5.6 *Let V be an open or closed set in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Then $m_n(AV) = |\det(A)| m_n(V)$.*

Proof: Let RU be the right polar decomposition (Theorem 4.12.6 on Page 93) of A and let V be an open set. Then from Lemma 10.5.5,

$$m_n(AV) = m_n(RUV) = m_n(UV).$$

Now $U = Q^*DQ$ where D is a diagonal matrix such that $|\det(D)| = |\det(A)|$ and Q is unitary. Therefore,

$$m_n(AV) = m_n(Q^*DQV) = m_n(DQV).$$

Now QV is an open set and so by Corollary 10.1.6 on Page 288 and Lemma 10.5.4,

$$m_n(AV) = |\det(D)| m_n(QV) = |\det(D)| m_n(V) = |\det(A)| m_n(V).$$

This proves the lemma in case V is open.

Now let H be a closed set which is also bounded. First suppose $\det(A) = 0$. Then letting V be an open set containing H ,

$$m_n(AH) \leq m_n(AV) = |\det(A)| m_n(V) = 0$$

which shows the desired equation is obvious in the case where $\det(A) = 0$. Therefore, assume A is one to one. Since H is bounded, $H \subseteq B(\mathbf{0}, R)$ for some $R > 0$. Then letting $B = B(\mathbf{0}, R)$ for short,

$$\begin{aligned} m_n(AH) &= m_n(AB) - m_n(A(B \setminus H)) \\ &= |\det(A)| m_n(B) - |\det(A)| m_n(B \setminus H) = |\det(A)| m_n(H). \end{aligned}$$

If H is not bounded, apply the result just obtained to $H_m \equiv H \cap \overline{B(\mathbf{0}, m)}$ and then let $m \rightarrow \infty$.

With this preparation, the main result is the following theorem.

Theorem 10.5.7 *Let E be Lebesgue measurable set in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. Then $m_n(AE) = |\det(A)| m_n(E)$.*

Proof: First suppose E is bounded. Then there exist sets, G and H such that $H \subseteq E \subseteq G$ and H is the countable union of closed sets while G is the countable intersection of open sets such that $m_n(G \setminus H) = 0$. By Lemma 10.5.6 applied to these sets whose union or intersection equals H or G respectively, it follows

$$m_n(AG) = |\det(A)| m_n(G) = |\det(A)| m_n(H) = m_n(AH).$$

Therefore,

$$\begin{aligned} |\det(A)| m_n(E) &= |\det(A)| m_n(H) = m_n(AH) \leq m_n(AE) \\ &\leq m_n(AG) = |\det(A)| m_n(G) = |\det(A)| m_n(E). \end{aligned}$$

In the general case, let $E_m = E \cap B(\mathbf{0}, m)$ and apply what was just shown and let $m \rightarrow \infty$.

10.6 Change Of Variables For C^1 Functions

In this section theorems are proved which generalize the above to C^1 functions. More general versions can be seen in Kuttler [45], Kuttler [46], and Rudin [58]. There is also a very different approach to this theorem given in [45]. The more general version in [45] follows [58] and both are based on the Brouwer fixed point theorem and a very clever lemma presented in Rudin [58]. This same approach will be used later in this book to prove a different sort of change of variables theorem in which the functions are only Lipschitz. The proof will be based on a sequence of easy lemmas.

Lemma 10.6.1 *Let U and V be bounded open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $f \in C_c(V)$. Then*

$$\int_V f(\mathbf{y}) dm_n = \int_U f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n$$

Proof: First note $\mathbf{h}^{-1}(\text{spt}(f))$ is a closed subset of the bounded set, U and so it is compact. Thus $\mathbf{x} \rightarrow f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|$ is bounded and continuous.

Let $\mathbf{x} \in U$. By the assumption that $\mathbf{h}$ and $\mathbf{h}^{-1}$ are C^1 ,

$$\begin{aligned} \mathbf{h}(\mathbf{x} + \mathbf{v}) - \mathbf{h}(\mathbf{x}) &= D\mathbf{h}(\mathbf{x})\mathbf{v} + \mathbf{o}(\mathbf{v}) \\ &= D\mathbf{h}(\mathbf{x})(\mathbf{v} + D\mathbf{h}^{-1}(\mathbf{h}(\mathbf{x}))\mathbf{o}(\mathbf{v})) \\ &= D\mathbf{h}(\mathbf{x})(\mathbf{v} + \mathbf{o}(\mathbf{v})) \end{aligned}$$

and so if $r > 0$ is small enough then $B(\mathbf{x}, r)$ is contained in U and

$$\mathbf{h}(B(\mathbf{x}, r)) - \mathbf{h}(\mathbf{x}) =$$

$$\mathbf{h}(\mathbf{x} + B(\mathbf{0}, r)) - \mathbf{h}(\mathbf{x}) \subseteq D\mathbf{h}(\mathbf{x})(B(\mathbf{0}, (1 + \varepsilon)r)). \quad (10.6.12)$$

Making r still smaller if necessary, one can also obtain

$$|f(\mathbf{y}) - f(\mathbf{h}(\mathbf{x}))| < \varepsilon \quad (10.6.13)$$

for any $\mathbf{y} \in \mathbf{h}(B(\mathbf{x}, r))$ and also

$$|f(\mathbf{h}(\mathbf{x}_1))| |\det(D\mathbf{h}(\mathbf{x}_1))| - f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| < \varepsilon \quad (10.6.14)$$

whenever $\mathbf{x}_1 \in B(\mathbf{x}, r)$. The collection of such balls is a Vitali cover of U . By Corollary 10.4.6 there is a sequence of disjoint closed balls $\{B_i\}$ such that $U = \bigcup_{i=1}^{\infty} B_i \cup N$ where $m_n(N) = 0$. Denote by $\mathbf{x}_i$ the center of B_i and r_i the radius. Then by Lemma 10.5.1, the monotone convergence theorem, and 10.6.12 - 10.6.14,

$$\begin{aligned} \int_V f(\mathbf{y}) dm_n &= \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} f(\mathbf{y}) dm_n \\ &\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} f(\mathbf{h}(\mathbf{x}_i)) dm_n \\ &\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(\mathbf{h}(\mathbf{x}_i)) m_n(\mathbf{h}(B_i)) \\ &\leq \varepsilon m_n(V) + \sum_{i=1}^{\infty} f(\mathbf{h}(\mathbf{x}_i)) m_n(D\mathbf{h}(\mathbf{x}_i)(B(\mathbf{0}, (1 + \varepsilon)r_i))) \\ &= \varepsilon m_n(V) + (1 + \varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(\mathbf{h}(\mathbf{x}_i)) |\det(D\mathbf{h}(\mathbf{x}_i))| dm_n \\ &\leq \varepsilon m_n(V) + (1 + \varepsilon)^n \sum_{i=1}^{\infty} \left(\int_{B_i} f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n + \varepsilon m_n(B_i) \right) \\ &\leq \varepsilon m_n(V) + (1 + \varepsilon)^n \sum_{i=1}^{\infty} \int_{B_i} f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n + (1 + \varepsilon)^n \varepsilon m_n(U) \\ &= \varepsilon m_n(V) + (1 + \varepsilon)^n \int_U f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n + (1 + \varepsilon)^n \varepsilon m_n(U) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows

$$\int_V f(\mathbf{y}) dm_n \leq \int_U f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n \quad (10.6.15)$$

whenever $f \in C_c(V)$. Now $\mathbf{x} \mapsto f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))|$ is in $C_c(U)$ and so using the same argument with U and V switching roles and replacing $\mathbf{h}$ with $\mathbf{h}^{-1}$,

$$\begin{aligned} &\int_U f(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n \\ &\leq \int_V f(\mathbf{h}(\mathbf{h}^{-1}(\mathbf{y}))) |\det(D\mathbf{h}(\mathbf{h}^{-1}(\mathbf{y})))| |\det(D\mathbf{h}^{-1}(\mathbf{y}))| dm_n \\ &= \int_V f(\mathbf{y}) dm_n \end{aligned}$$

by the chain rule. This with 10.6.15 proves the lemma.

The next task is to relax the assumption that f is continuous.

Corollary 10.6.2 *Let U and V be bounded open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$ and $|\det(D\mathbf{h}(\mathbf{x}))|$ is bounded. Also let $E \subseteq V$ be measurable. Then*

$$\int_V \mathcal{X}_E(\mathbf{y}) dm_n = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n.$$

Proof: By regularity, there exist compact sets, K_k and open sets G_k such that

$$K_k \subseteq E \subseteq G_k$$

and $m_n(G_k \setminus K_k) < 2^{-k}$. By Theorem 9.3.7, there exist f_k such that $K_k \prec f_k \prec G_k$. Then $f_k(\mathbf{y}) \rightarrow \mathcal{X}_E(\mathbf{y})$ a.e. because if $\mathbf{y}$ is such that convergence fails, it must be the case that $\mathbf{y}$ is in $G_k \setminus K_k$ for infinitely many k and $\sum_k m_n(G_k \setminus K_k) < \infty$. This set equals

$$N = \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} G_k \setminus K_k$$

and so for each $m \in \mathbb{N}$

$$\begin{aligned} m_n(N) &\leq m_n(\cup_{k=m}^{\infty} G_k \setminus K_k) \\ &\leq \sum_{k=m}^{\infty} m_n(G_k \setminus K_k) < \sum_{k=m}^{\infty} 2^{-k} = 2^{-(m-1)} \end{aligned}$$

showing $m_n(N) = 0$.

Then $f_k(\mathbf{h}(\mathbf{x}))$ must converge to $\mathcal{X}_E(\mathbf{h}(\mathbf{x}))$ for all $\mathbf{x} \notin \mathbf{h}^{-1}(N)$, a set of measure zero by Lemma 10.5.1. Thus

$$\int_V f_k(\mathbf{y}) dm_n = \int_U f_k(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n.$$

By the dominated convergence theorem using a dominating function, $\mathcal{X}_V$ in the integral on the left and $\mathcal{X}_U |\det(D\mathbf{h})|$ on the right, it follows

$$\int_V \mathcal{X}_E(\mathbf{y}) dm_n = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n.$$

This proves the corollary.

You don't need to assume the open sets are bounded.

Corollary 10.6.3 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Also let $E \subseteq V$ be measurable. Then*

$$\int_V \mathcal{X}_E(\mathbf{y}) dm_n = \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n.$$

Proof: For each $\mathbf{x} \in U$, there exists $r_{\mathbf{x}}$ such that $\overline{B(\mathbf{x}, r_{\mathbf{x}})} \subseteq U$ and $r_{\mathbf{x}} < 1$. Then by the mean value inequality Theorem 5.13.4, it follows $\mathbf{h}(B(\mathbf{x}, r_{\mathbf{x}}))$ is also

bounded. This is a Vitali cover of U and so by Corollary 10.4.6 there is a sequence of these balls, $\{B_i\}$ such that they are disjoint, $\mathbf{h}(B_i)$ is also bounded and

$$m_n(U \setminus \cup_i B_i) = 0.$$

It follows from Lemma 10.5.1 that $\mathbf{h}(U \setminus \cup_i B_i)$ also has measure zero. Then from Corollary 10.6.2

$$\begin{aligned} \int_V \mathcal{X}_E(\mathbf{y}) dm_n &= \sum_i \int_{\mathbf{h}(B_i)} \mathcal{X}_{E \cap \mathbf{h}(B_i)}(\mathbf{y}) dm_n \\ &= \sum_i \int_{B_i} \mathcal{X}_E(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n \\ &= \int_U \mathcal{X}_E(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dm_n. \end{aligned}$$

This proves the corollary.

With this corollary, the main theorem follows.

Theorem 10.6.4 *Let U and V be open sets in $\mathbb{R}^n$ and let $\mathbf{h}, \mathbf{h}^{-1}$ be C^1 functions such that $\mathbf{h}(U) = V$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_V g(\mathbf{y}) dy = \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dx. \quad (10.6.16)$$

Proof: From Corollary 10.6.3, 10.6.16 holds for any nonnegative simple function in place of g . In general, let $\{s_k\}$ be an increasing sequence of simple functions which converges to g pointwise. Then from the monotone convergence theorem

$$\begin{aligned} \int_V g(\mathbf{y}) dy &= \lim_{k \rightarrow \infty} \int_V s_k dy = \lim_{k \rightarrow \infty} \int_U s_k(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dx \\ &= \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| dx. \end{aligned}$$

This proves the theorem.

This is a pretty good theorem but it isn't too hard to generalize it. In particular, it is not necessary to assume $\mathbf{h}^{-1}$ is C^1 .

Lemma 10.6.5 *Suppose V is an $n-1$ dimensional subspace of $\mathbb{R}^n$ and K is a compact subset of V . Then letting*

$$K_\varepsilon \equiv \cup_{\mathbf{x} \in K} B(\mathbf{x}, \varepsilon) = K + B(\mathbf{0}, \varepsilon),$$

it follows that

$$m_n(K_\varepsilon) \leq 2^n \varepsilon (\text{diam}(K) + \varepsilon)^{n-1}.$$

Proof: Let an orthonormal basis for V be $\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$ and let

$$\{\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{v}_n\}$$

be an orthonormal basis for $\mathbb{R}^n$. Now define a linear transformation, Q by $Q\mathbf{v}_i = \mathbf{e}_i$. Thus $QQ^* = Q^*Q = I$ and Q preserves all distances because

$$\left| Q \sum_i a_i \mathbf{e}_i \right|^2 = \left| \sum_i a_i \mathbf{v}_i \right|^2 = \sum_i |a_i|^2 = \left| \sum_i a_i \mathbf{e}_i \right|^2.$$

Letting $\mathbf{k}_0 \in K$, it follows $K \subseteq B(\mathbf{k}_0, \text{diam}(K))$ and so,

$$QK \subseteq B^{n-1}(Q\mathbf{k}_0, \text{diam}(QK)) = B^{n-1}(Q\mathbf{k}_0, \text{diam}(K))$$

where B^{n-1} refers to the ball taken with respect to the usual norm in $\mathbb{R}^{n-1}$. Every point of K_ε is within ε of some point of K and so it follows that every point of QK_ε is within ε of some point of QK . Therefore,

$$QK_\varepsilon \subseteq B^{n-1}(Q\mathbf{k}_0, \text{diam}(QK) + \varepsilon) \times (-\varepsilon, \varepsilon),$$

To see this, let $\mathbf{x} \in QK_\varepsilon$. Then there exists $\mathbf{k} \in QK$ such that $|\mathbf{k} - \mathbf{x}| < \varepsilon$. Therefore, $|(x_1, \dots, x_{n-1}) - (k_1, \dots, k_{n-1})| < \varepsilon$ and $|x_n - k_n| < \varepsilon$ and so $\mathbf{x}$ is contained in the set on the right in the above inclusion because $k_n = 0$. However, the measure of the set on the right is smaller than

$$[2(\text{diam}(QK) + \varepsilon)]^{n-1} (2\varepsilon) = 2^n [(\text{diam}(K) + \varepsilon)]^{n-1} \varepsilon.$$

This proves the lemma.

Note this is a very sloppy estimate. You can certainly do much better but this estimate is sufficient to prove Sard's lemma which follows.

Definition 10.6.6 *In any metric space, if $\mathbf{x}$ is a point of the metric space and S is a nonempty subset,*

$$\text{dist}(\mathbf{x}, S) \equiv \inf \{d(\mathbf{x}, s) : s \in S\}.$$

More generally, if T, S are two nonempty sets,

$$\text{dist}(S, T) \equiv \inf \{d(t, s) : s \in S, t \in T\}.$$

Lemma 10.6.7 *The function $\mathbf{x} \mapsto \text{dist}(\mathbf{x}, S)$ is continuous.*

Proof: Let $\mathbf{x}, \mathbf{y}$ be given. Suppose $\text{dist}(\mathbf{x}, S) \geq \text{dist}(\mathbf{y}, S)$ and pick $\mathbf{s} \in S$ such that $\text{dist}(\mathbf{y}, S) + \varepsilon \geq d(\mathbf{y}, \mathbf{s})$. Then

$$\begin{aligned} 0 &\leq \text{dist}(\mathbf{x}, S) - \text{dist}(\mathbf{y}, S) \leq \text{dist}(\mathbf{x}, S) - (d(\mathbf{y}, \mathbf{s}) - \varepsilon) \\ &\leq d(\mathbf{x}, \mathbf{s}) - d(\mathbf{y}, \mathbf{s}) + \varepsilon \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{s}) - d(\mathbf{y}, \mathbf{s}) + \varepsilon = d(\mathbf{x}, \mathbf{y}) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $|\text{dist}(\mathbf{x}, S) - \text{dist}(\mathbf{y}, S)| \leq d(\mathbf{x}, \mathbf{y})$. This proves the lemma.

Lemma 10.6.8 *Let $\mathbf{h}$ be a C^1 function defined on an open set, U and let K be a compact subset of U . Then if $\varepsilon > 0$ is given, there exists $r_1 > 0$ such that if $|\mathbf{v}| \leq r_1$, then for all $\mathbf{x} \in K$,*

$$|\mathbf{h}(\mathbf{x} + \mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon |\mathbf{v}|.$$

Proof: Let $0 < \delta < \text{dist}(K, U^C)$. Such a positive number exists because if there exists a sequence of points in K , $\{\mathbf{k}_k\}$ and points in U^C , $\{\mathbf{s}_k\}$ such that $|\mathbf{k}_k - \mathbf{s}_k| \rightarrow 0$, then you could take a subsequence, still denoted by k such that $\mathbf{k}_k \rightarrow \mathbf{k} \in K$ and then $\mathbf{s}_k \rightarrow \mathbf{k}$ also. But U^C is closed so $\mathbf{k} \in K \cap U^C$, a contradiction. Then

$$\begin{aligned} \frac{|\mathbf{h}(\mathbf{x} + \mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\leq \frac{\left| \int_0^1 D\mathbf{h}(\mathbf{x} + t\mathbf{v})\mathbf{v} dt - D\mathbf{h}(\mathbf{x})\mathbf{v} \right|}{|\mathbf{v}|} \\ &\leq \frac{\int_0^1 |D\mathbf{h}(\mathbf{x} + t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}| dt}{|\mathbf{v}|}. \end{aligned}$$

Now from uniform continuity of $D\mathbf{h}$ on the compact set, $\{\mathbf{x} : \text{dist}(\mathbf{x}, K) \leq \delta\}$ it follows there exists $r_1 < \delta$ such that if $|\mathbf{v}| \leq r_1$, then $\|D\mathbf{h}(\mathbf{x} + t\mathbf{v}) - D\mathbf{h}(\mathbf{x})\| < \varepsilon$ for every $\mathbf{x} \in K$. From the above formula, it follows that if $|\mathbf{v}| \leq r_1$,

$$\begin{aligned} \frac{|\mathbf{h}(\mathbf{x} + \mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}|}{|\mathbf{v}|} &\leq \frac{\int_0^1 |D\mathbf{h}(\mathbf{x} + t\mathbf{v})\mathbf{v} - D\mathbf{h}(\mathbf{x})\mathbf{v}| dt}{|\mathbf{v}|} \\ &< \frac{\int_0^1 \varepsilon |\mathbf{v}| dt}{|\mathbf{v}|} = \varepsilon. \end{aligned}$$

This proves the lemma.

A different proof of the following is in [45]. See also [46].

Lemma 10.6.9 (*Sard*) *Let U be an open set in $\mathbb{R}^n$ and let $\mathbf{h} : U \rightarrow \mathbb{R}^n$ be C^1 . Let*

$$Z \equiv \{\mathbf{x} \in U : \det D\mathbf{h}(\mathbf{x}) = 0\}.$$

Then $m_n(\mathbf{h}(Z)) = 0$.

Proof: Let $\{U_k\}_{k=1}^\infty$ be an increasing sequence of open sets whose closures are compact and whose union equals U and let $Z_k \equiv Z \cap \overline{U_k}$. To obtain such a sequence, let $U_k = \{\mathbf{x} \in U : \text{dist}(\mathbf{x}, U^C) < \frac{1}{k}\} \cap B(\mathbf{0}, k)$. First it is shown that $\mathbf{h}(Z_k)$ has measure zero. Let W be an open set contained in U_{k+1} which contains Z_k and satisfies

$$m_n(Z_k) + \varepsilon > m_n(W)$$

where here and elsewhere, $\varepsilon < 1$. Let

$$r = \text{dist}(\overline{U_k}, U_{k+1}^C)$$

and let $r_1 > 0$ be a constant as in Lemma 10.6.8 such that whenever $\mathbf{x} \in \overline{U_k}$ and $0 < |\mathbf{v}| \leq r_1$,

$$|\mathbf{h}(\mathbf{x} + \mathbf{v}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})\mathbf{v}| < \varepsilon |\mathbf{v}|. \quad (10.6.17)$$

Now the closures of balls which are contained in W and which have the property that their diameters are less than r_1 yield a Vitali covering of W . Therefore, by Corollary 10.4.6 there is a disjoint sequence of these closed balls, $\{\tilde{B}_i\}$ such that

$$W = \bigcup_{i=1}^{\infty} \tilde{B}_i \cup N$$

where N is a set of measure zero. Denote by $\{B_i\}$ those closed balls in this sequence which have nonempty intersection with Z_k , let d_i be the diameter of B_i , and let $\mathbf{z}_i$ be a point in $B_i \cap Z_k$. Since $\mathbf{z}_i \in Z_k$, it follows $D\mathbf{h}(\mathbf{z}_i) B(\mathbf{0}, d_i) = D_i$ where D_i is contained in a subspace, V which has dimension $n-1$ and the diameter of D_i is no larger than $2C_k d_i$ where

$$C_k \geq \max \{ \|D\mathbf{h}(\mathbf{x})\| : \mathbf{x} \in Z_k \}$$

Then by 10.6.17, if $\mathbf{z} \in B_i$,

$$\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{z}_i) \in D_i + B(\mathbf{0}, \varepsilon d_i) \subseteq \overline{D_i} + B(\mathbf{0}, \varepsilon d_i).$$

Thus

$$\mathbf{h}(B_i) \subseteq \mathbf{h}(\mathbf{z}_i) + \overline{D_i} + B(\mathbf{0}, \varepsilon d_i)$$

By Lemma 10.6.5

$$\begin{aligned} m_n(\mathbf{h}(B_i)) &\leq 2^n (2C_k d_i + \varepsilon d_i)^{n-1} \varepsilon d_i \\ &\leq d_i^n \left(2^n [2C_k + \varepsilon]^{n-1} \right) \varepsilon \\ &\leq C_{n,k} m_n(B_i) \varepsilon. \end{aligned}$$

Therefore, by Lemma 10.5.1

$$\begin{aligned} m_n(\mathbf{h}(Z_k)) &\leq \sum_i m_n(\mathbf{h}(B_i)) \leq C_{n,k} \varepsilon \sum_i m_n(B_i) \\ &\leq \varepsilon C_{n,k} m_n(W) \leq \varepsilon C_{n,k} (m_n(Z_k) + \varepsilon) \end{aligned}$$

Since ε is arbitrary, this shows $m_n(\mathbf{h}(Z_k)) = 0$ and so $0 = \lim_{k \rightarrow \infty} m_n(\mathbf{h}(Z_k)) = m_n(\mathbf{h}(Z))$.

With this important lemma, here is a generalization of Theorem 10.6.4.

Theorem 10.6.10 *Let U be an open set and let $\mathbf{h}$ be a $1-1$, C^1 function with values in $\mathbb{R}^n$. Then if g is a nonnegative Lebesgue measurable function,*

$$\int_{\mathbf{h}(U)} g(\mathbf{y}) d\mathbf{y} = \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| d\mathbf{x}. \quad (10.6.18)$$

Proof: Let $Z = \{\mathbf{x} : \det(D\mathbf{h}(\mathbf{x})) = 0\}$. Then by the inverse function theorem, $\mathbf{h}^{-1}$ is C^1 on $\mathbf{h}(U \setminus Z)$ and $\mathbf{h}(U \setminus Z)$ is an open set. Therefore, from Lemma 10.6.9 and Theorem 10.6.4,

$$\begin{aligned} \int_{\mathbf{h}(U)} g(\mathbf{y}) d\mathbf{y} &= \int_{\mathbf{h}(U \setminus Z)} g(\mathbf{y}) d\mathbf{y} = \int_{U \setminus Z} g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| d\mathbf{x} \\ &= \int_U g(\mathbf{h}(\mathbf{x})) |\det(D\mathbf{h}(\mathbf{x}))| d\mathbf{x}. \end{aligned}$$

This proves the theorem.

Of course the next generalization considers the case when $\mathbf{h}$ is not even one to one.

10.7 Mappings Which Are Not One To One

Now suppose $\mathbf{h}$ is only C^1 , not necessarily one to one. For

$$U_+ \equiv \{\mathbf{x} \in U : |\det D\mathbf{h}(\mathbf{x})| > 0\}$$

and Z the set where $|\det D\mathbf{h}(\mathbf{x})| = 0$, Lemma 10.6.9 implies $m_n(\mathbf{h}(Z)) = 0$. For $\mathbf{x} \in U_+$, the inverse function theorem implies there exists an open set $B_{\mathbf{x}}$ such that $\mathbf{x} \in B_{\mathbf{x}} \subseteq U_+$, $\mathbf{h}$ is one to one on $B_{\mathbf{x}}$.

Let $\{B_i\}$ be a countable subset of $\{B_{\mathbf{x}}\}_{\mathbf{x} \in U_+}$ such that $U_+ = \cup_{i=1}^{\infty} B_i$. Let $E_1 = B_1$. If $E_1, \dots, E_k$ have been chosen, $E_{k+1} = B_{k+1} \setminus \cup_{i=1}^k E_i$. Thus

$$\cup_{i=1}^{\infty} E_i = U_+, \quad \mathbf{h} \text{ is one to one on } E_i, \quad E_i \cap E_j = \emptyset,$$

and each E_i is a Borel set contained in the open set B_i . Now define

$$n(\mathbf{y}) \equiv \sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y}).$$

The set, $\mathbf{h}(E_i)$, $\mathbf{h}(Z)$ are measurable by Lemma 10.5.2. Thus $n(\cdot)$ is measurable.

Lemma 10.7.1 *Let $F \subseteq \mathbf{h}(U)$ be measurable. Then*

$$\int_{\mathbf{h}(U)} n(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathbf{y} = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| d\mathbf{x}.$$

Proof: Using Lemma 10.6.9 and the Monotone Convergence Theorem or Fubini's Theorem,

$$\begin{aligned} \int_{\mathbf{h}(U)} n(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathbf{y} &= \int_{\mathbf{h}(U)} \left(\sum_{i=1}^{\infty} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) + \overbrace{\mathcal{X}_{\mathbf{h}(Z)}(\mathbf{y})}^{m_n(\mathbf{h}(Z))=0} \right) \mathcal{X}_F(\mathbf{y}) d\mathbf{y} \\ &= \sum_{i=1}^{\infty} \int_{\mathbf{h}(U)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathbf{y} \\ &= \sum_{i=1}^{\infty} \int_{\mathbf{h}(B_i)} \mathcal{X}_{\mathbf{h}(E_i)}(\mathbf{y}) \mathcal{X}_F(\mathbf{y}) d\mathbf{y} \\ &= \sum_{i=1}^{\infty} \int_{B_i} \mathcal{X}_{E_i}(\mathbf{x}) \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| d\mathbf{x} \\ &= \sum_{i=1}^{\infty} \int_U \mathcal{X}_{E_i}(\mathbf{x}) \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| d\mathbf{x} \\ &= \int_U \sum_{i=1}^{\infty} \mathcal{X}_{E_i}(\mathbf{x}) \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| d\mathbf{x} \end{aligned}$$

$$= \int_{U_+} \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| dx = \int_U \mathcal{X}_F(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| dx.$$

This proves the lemma.

Definition 10.7.2 For $\mathbf{y} \in \mathbf{h}(U)$, define a function, $\#$, according to the formula
 $\#(\mathbf{y}) \equiv \text{number of elements in } \mathbf{h}^{-1}(\mathbf{y}).$

Observe that

$$\#(\mathbf{y}) = n(\mathbf{y}) \quad \text{a.e.} \quad (10.7.19)$$

because $n(\mathbf{y}) = \#(\mathbf{y})$ if $\mathbf{y} \notin \mathbf{h}(Z)$, a set of measure 0. Therefore, $\#$ is a measurable function.

Theorem 10.7.3 Let $g \geq 0$, g measurable, and let $\mathbf{h}$ be $C^1(U)$. Then

$$\int_{\mathbf{h}(U)} \#(\mathbf{y}) g(\mathbf{y}) d\mathbf{y} = \int_U g(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| dx. \quad (10.7.20)$$

Proof: From 10.7.19 and Lemma 10.7.1, 10.7.20 holds for all g , a nonnegative simple function. Approximating an arbitrary measurable nonnegative function, g , with an increasing pointwise convergent sequence of simple functions and using the monotone convergence theorem, yields 10.7.20 for an arbitrary nonnegative measurable function, g . This proves the theorem.

10.8 Lebesgue Measure And Iterated Integrals

The following is the main result.

Theorem 10.8.1 Let $f \geq 0$ and suppose f is a Lebesgue measurable function defined on $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} f dm_n < \infty$. Then

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

This will be accomplished by Fubini's theorem, Theorem 9.8.11 and the following lemma.

Lemma 10.8.2 $\overline{m_k \times m_{n-k}} = m_n$ on the m_n measurable sets.

Proof: First of all, let $R = \prod_{i=1}^n (a_i, b_i]$ be a measurable rectangle and let $R_k = \prod_{i=1}^k (a_i, b_i]$, $R_{n-k} = \prod_{i=k+1}^n (a_i, b_i]$. Then by Fubini's theorem,

$$\begin{aligned} \int \mathcal{X}_R d(\overline{m_k \times m_{n-k}}) &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \mathcal{X}_{R_k} \mathcal{X}_{R_{n-k}} dm_k dm_{n-k} \\ &= \int_{\mathbb{R}^k} \mathcal{X}_{R_k} dm_k \int_{\mathbb{R}^{n-k}} \mathcal{X}_{R_{n-k}} dm_{n-k} \\ &= \int \mathcal{X}_R dm_n \end{aligned}$$

and so $\overline{m_k \times m_{n-k}}$ and m_n agree on every half open rectangle. By Lemma 10.1.2 these two measures agree on every open set. Now if K is a compact set, then $K = \bigcap_{k=1}^{\infty} U_k$ where U_k is the open set, $K + B(\mathbf{0}, \frac{1}{k})$. Another way of saying this is $U_k \equiv \{\mathbf{x} : \text{dist}(\mathbf{x}, K) < \frac{1}{k}\}$ which is obviously open because $\mathbf{x} \rightarrow \text{dist}(\mathbf{x}, K)$ is a continuous function. Since K is the countable intersection of these decreasing open sets, each of which has finite measure with respect to either of the two measures, it follows that $\overline{m_k \times m_{n-k}}$ and m_n agree on all the compact sets. Now let E be a bounded Lebesgue measurable set. Then there are sets, H and G such that H is a countable union of compact sets, G a countable intersection of open sets, $H \subseteq E \subseteq G$, and $m_n(G \setminus H) = 0$. Then from what was just shown about compact and open sets, the two measures agree on G and on H . Therefore,

$$\begin{aligned} m_n(H) &= \overline{m_k \times m_{n-k}}(H) \leq \overline{m_k \times m_{n-k}}(E) \\ &\leq \overline{m_k \times m_{n-k}}(G) = m_n(E) = m_n(H) \end{aligned}$$

By completeness of the measure space for $\overline{m_k \times m_{n-k}}$, it follows E is $\overline{m_k \times m_{n-k}}$ measurable and

$$\overline{m_k \times m_{n-k}}(E) = m_n(E).$$

This proves the lemma.

You could also show that the two σ algebras are the same. However, this is not needed for the lemma or the theorem.

Proof of Theorem 10.8.1: By the lemma and Fubini's theorem, Theorem 9.8.11,

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^n} f d(\overline{m_k \times m_{n-k}}) = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

Corollary 10.8.3 *Let f be a nonnegative real valued measurable function. Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

Proof: Let $S_p \equiv \{\mathbf{x} \in \mathbb{R}^n : 0 \leq f(\mathbf{x}) \leq p\} \cap B(\mathbf{0}, p)$. Then $\int_{\mathbb{R}^n} f \chi_{S_p} dm_n < \infty$. Therefore, from Theorem 10.8.1,

$$\int_{\mathbb{R}^n} f \chi_{S_p} dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \chi_{S_p} f dm_{n-k} dm_k.$$

Now let $p \rightarrow \infty$ and use the Monotone convergence theorem and the Fubini Theorem 9.8.11 on Page 261.

Not surprisingly, the following corollary follows from this.

Corollary 10.8.4 *Let $f \in L^1(\mathbb{R}^n)$ where the measure is m_n . Then*

$$\int_{\mathbb{R}^n} f dm_n = \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} f dm_{n-k} dm_k.$$

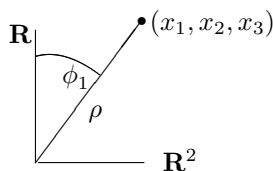
Proof: Apply Corollary 10.8.3 to the positive and negative parts of the real and imaginary parts of f .

10.9 Spherical Coordinates In Many Dimensions

Sometimes there is a need to deal with spherical coordinates in more than three dimensions. In this section, this concept is defined and formulas are derived for these coordinate systems. Recall polar coordinates are of the form

$$\begin{aligned}y_1 &= \rho \cos \theta \\y_2 &= \rho \sin \theta\end{aligned}$$

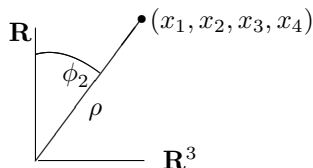
where $\rho > 0$ and $\theta \in [0, 2\pi)$. Here I am writing ρ in place of r to emphasize a pattern which is about to emerge. I will consider polar coordinates as spherical coordinates in two dimensions. I will also simply refer to such coordinate systems as polar coordinates regardless of the dimension. This is also the reason I am writing y_1 and y_2 instead of the more usual x and y . Now consider what happens when you go to three dimensions. The situation is depicted in the following picture.



From this picture, you see that $y_3 = \rho \cos \phi_1$. Also the distance between (y_1, y_2) and $(0, 0)$ is $\rho \sin(\phi_1)$. Therefore, using polar coordinates to write (y_1, y_2) in terms of θ and this distance,

$$\begin{aligned}y_1 &= \rho \sin \phi_1 \cos \theta, \\y_2 &= \rho \sin \phi_1 \sin \theta, \\y_3 &= \rho \cos \phi_1.\end{aligned}$$

where $\phi_1 \in [0, \pi]$. What was done is to replace ρ with $\rho \sin \phi_1$ and then to add in $y_3 = \rho \cos \phi_1$. Having done this, there is no reason to stop with three dimensions. Consider the following picture:



From this picture, you see that $y_4 = \rho \cos \phi_2$. Also the distance between (y_1, y_2, y_3) and $(0, 0, 0)$ is $\rho \sin(\phi_2)$. Therefore, using polar coordinates to write (y_1, y_2, y_3) in

terms of θ, ϕ_1 , and this distance,

$$\begin{aligned}y_1 &= \rho \sin \phi_2 \sin \phi_1 \cos \theta, \\y_2 &= \rho \sin \phi_2 \sin \phi_1 \sin \theta, \\y_3 &= \rho \sin \phi_2 \cos \phi_1, \\y_4 &= \rho \cos \phi_2\end{aligned}$$

where $\phi_2 \in [0, \pi]$.

Continuing this way, given spherical coordinates in $\mathbb{R}^n$, to get the spherical coordinates in $\mathbb{R}^{n+1}$, you let $y_{n+1} = \rho \cos \phi_{n-1}$ and then replace every occurrence of ρ with $\rho \sin \phi_{n-1}$ to obtain $y_1 \cdots y_n$ in terms of $\phi_1, \phi_2, \dots, \phi_{n-1}, \theta$, and ρ .

It is always the case that ρ measures the distance from the point in $\mathbb{R}^n$ to the origin in $\mathbb{R}^n$, $\mathbf{0}$. Each $\phi_i \in [0, \pi]$, and $\theta \in [0, 2\pi)$. It can be shown using math induction that these coordinates map $\prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi) \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$.

Theorem 10.9.1 *Let $\mathbf{y} = \mathbf{h}(\phi, \theta, \rho)$ be the spherical coordinate transformations in $\mathbb{R}^n$. Then letting $A = \prod_{i=1}^{n-2} [0, \pi] \times [0, 2\pi)$, it follows $\mathbf{h}$ maps $A \times (0, \infty)$ one to one onto $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Also $|\det D\mathbf{h}(\phi, \theta, \rho)|$ will always be of the form*

$$|\det D\mathbf{h}(\phi, \theta, \rho)| = \rho^{n-1} \Phi(\phi, \theta). \quad (10.9.21)$$

where Φ is a continuous function of ϕ and θ .¹ Furthermore whenever f is Lebesgue measurable and nonnegative,

$$\int_{\mathbb{R}^n} f(\mathbf{y}) d\mathbf{y} = \int_0^\infty \rho^{n-1} \int_A f(\mathbf{h}(\phi, \theta, \rho)) \Phi(\phi, \theta) d\phi d\theta d\rho \quad (10.9.22)$$

where here $d\phi d\theta$ denotes dm_{n-1} on A . The same formula holds if $f \in L^1(\mathbb{R}^n)$.

Proof: Formula 10.9.21 is obvious from the definition of the spherical coordinates. The first claim is also clear from the definition and math induction. It remains to verify 10.9.22. Let $A_0 \equiv \prod_{i=1}^{n-2} (0, \pi) \times (0, 2\pi)$. Then it is clear that $(A \setminus A_0) \times (0, \infty) \equiv N$ is a set of measure zero in $\mathbb{R}^n$. Therefore, from Lemma 10.5.1 it follows $\mathbf{h}(N)$ is also a set of measure zero. Therefore, using the change of variables theorem, Corollary 10.8.3, and Sard's lemma,

$$\begin{aligned}\int_{\mathbb{R}^n} f(\mathbf{y}) d\mathbf{y} &= \int_{\mathbb{R}^n \setminus \{\mathbf{0}\}} f(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^n \setminus (\{\mathbf{0}\} \cup \mathbf{h}(N))} f(\mathbf{y}) d\mathbf{y} \\&= \int_{A_0 \times (0, \infty)} f(\mathbf{h}(\phi, \theta, \rho)) \rho^{n-1} \Phi(\phi, \theta) dm_n \\&= \int_{\mathcal{X}_{A \times (0, \infty)}} (\phi, \theta, \rho) f(\mathbf{h}(\phi, \theta, \rho)) \rho^{n-1} \Phi(\phi, \theta) dm_n \\&= \int_0^\infty \rho^{n-1} \left(\int_A f(\mathbf{h}(\phi, \theta, \rho)) \Phi(\phi, \theta) d\phi d\theta \right) d\rho.\end{aligned}$$

¹Actually it is only a function of the first but this is not important in what follows.

Now the claim about $f \in L^1$ follows routinely from considering the positive and negative parts of the real and imaginary parts of f in the usual way. This proves the theorem.

Notation 10.9.2 *Often this is written differently. Note that from the spherical coordinate formulas, $f(\mathbf{h}(\phi, \theta, \rho)) = f(\rho\omega)$ where $|\omega| = 1$. Letting S^{n-1} denote the unit sphere, $\{\omega \in \mathbb{R}^n : |\omega| = 1\}$, the inside integral in the above formula is sometimes written as*

$$\int_{S^{n-1}} f(\rho\omega) d\sigma$$

where σ is a measure on S^{n-1} . See [45] for another description of this measure. It isn't an important issue here. Later in the book when integration on manifolds is discussed, more general considerations will be dealt with. Either 10.9.22 or the formula

$$\int_0^\infty \rho^{n-1} \left(\int_{S^{n-1}} f(\rho\omega) d\sigma \right) d\rho$$

will be referred to as polar coordinates and is very useful in establishing estimates. Here $\sigma(S^{n-1}) \equiv \int_A \Phi(\phi, \theta) d\phi d\theta$.

Example 10.9.3 *For what values of s is the integral $\int_{B(\mathbf{0}, R)} (1 + |\mathbf{x}|^2)^s dy$ bounded independent of R ? Here $B(\mathbf{0}, R)$ is the ball, $\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \leq R\}$.*

I think you can see immediately that s must be negative but exactly how negative? It turns out it depends on n and using polar coordinates, you can find just exactly what is needed. From the polar coordinates formula above,

$$\begin{aligned} \int_{B(\mathbf{0}, R)} (1 + |\mathbf{x}|^2)^s dy &= \int_0^R \int_{S^{n-1}} (1 + \rho^2)^s \rho^{n-1} d\sigma d\rho \\ &= C_n \int_0^R (1 + \rho^2)^s \rho^{n-1} d\rho \end{aligned}$$

Now the very hard problem has been reduced to considering an easy one variable problem of finding when

$$\int_0^R \rho^{n-1} (1 + \rho^2)^s d\rho$$

is bounded independent of R . You need $2s + (n - 1) < -1$ so you need $s < -n/2$.

10.10 The Brouwer Fixed Point Theorem

This seems to be a good place to present a short proof of one of the most important of all fixed point theorems. There are many approaches to this but one of the easiest and shortest I have ever seen is the one in Dunford and Schwartz [22]. This is what is presented here. In Evans [25] there is a different proof which depends on

integration theory. A good reference for an introduction to various kinds of fixed point theorems is the book by Smart [61]. This book also gives an entirely different approach to the Brouwer fixed point theorem.

The proof given here is based on the following lemma. Recall that the mixed partial derivatives of a C^2 function are equal. In the following lemma, and elsewhere, a comma followed by an index indicates the partial derivative with respect to the indicated variable. Thus, $f_{,j}$ will mean $\frac{\partial f}{\partial x_j}$. Also, write $D\mathbf{g}$ for the Jacobian matrix which is the matrix of $D\mathbf{g}$ taken with respect to the usual basis vectors in $\mathbb{R}^n$. Recall that for A an $n \times n$ matrix, $\text{cof}(A)_{ij}$ is the determinant of the matrix which results from deleting the i^{th} row and the j^{th} column and multiplying by $(-1)^{i+j}$.

Lemma 10.10.1 *Let $\mathbf{g} : U \rightarrow \mathbb{R}^n$ be C^2 where U is an open subset of $\mathbb{R}^n$. Then*

$$\sum_{j=1}^n \text{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\text{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Proof: From the cofactor expansion theorem,

$$\det(D\mathbf{g}) = \sum_{i=1}^n g_{i,j} \text{cof}(D\mathbf{g})_{ij}$$

and so

$$\frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}} = \text{cof}(D\mathbf{g})_{ij} \quad (10.10.23)$$

which shows the last claim of the lemma. Also

$$\delta_{kj} \det(D\mathbf{g}) = \sum_i g_{i,k} (\text{cof}(D\mathbf{g}))_{ij} \quad (10.10.24)$$

because if $k \neq j$ this is just the cofactor expansion of the determinant of a matrix in which the k^{th} and j^{th} columns are equal. Differentiate 10.10.24 with respect to x_j and sum on j . This yields

$$\sum_{r,s,j} \delta_{kj} \frac{\partial (\det D\mathbf{g})}{\partial g_{r,s}} g_{r,s,j} = \sum_{ij} g_{i,kj} (\text{cof}(D\mathbf{g}))_{ij} + \sum_{ij} g_{i,k} \text{cof}(D\mathbf{g})_{ij,j}.$$

Hence, using $\delta_{kj} = 0$ if $j \neq k$ and 10.10.23,

$$\sum_{rs} (\text{cof}(D\mathbf{g}))_{rs} g_{r,sk} = \sum_{rs} g_{r,ks} (\text{cof}(D\mathbf{g}))_{rs} + \sum_{ij} g_{i,k} \text{cof}(D\mathbf{g})_{ij,j}.$$

Subtracting the first sum on the right from both sides and using the equality of mixed partials,

$$\sum_i g_{i,k} \left(\sum_j (\text{cof}(D\mathbf{g}))_{ij,j} \right) = 0.$$

If $\det(g_{i,k}) \neq 0$ so that $(g_{i,k})$ is invertible, this shows $\sum_j (\text{cof}(D\mathbf{g}))_{ij,j} = 0$. If $\det(D\mathbf{g}) = 0$, let

$$g_k = g + \varepsilon_k I$$

where $\varepsilon_k \rightarrow 0$ and $\det(D\mathbf{g} + \varepsilon_k I) \equiv \det(D\mathbf{g}_k) \neq 0$. Then

$$\sum_j (\text{cof}(D\mathbf{g}))_{ij,j} = \lim_{k \rightarrow \infty} \sum_j (\text{cof}(D\mathbf{g}_k))_{ij,j} = 0$$

and this proves the lemma.

To prove the Brouwer fixed point theorem, first consider a version of it valid for C^2 mappings. This is the following lemma.

Lemma 10.10.2 *Let $B_r = \overline{B(\mathbf{0}, r)}$ and suppose $\mathbf{g}$ is a C^2 function defined on $\mathbb{R}^n$ which maps B_r to B_r . Then $\mathbf{g}(\mathbf{x}) = \mathbf{x}$ for some $\mathbf{x} \in B_r$.*

Proof: Suppose not. Then $|\mathbf{g}(\mathbf{x}) - \mathbf{x}|$ must be bounded away from zero on B_r . Let $a(\mathbf{x})$ be the larger of the two roots of the equation,

$$|\mathbf{x} + a(\mathbf{x})(\mathbf{x} - \mathbf{g}(\mathbf{x}))|^2 = r^2. \quad (10.10.25)$$

Thus

$$a(\mathbf{x}) = \frac{-\langle \mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x})) \rangle + \sqrt{\langle \mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x})) \rangle^2 + (r^2 - |\mathbf{x}|^2) |\mathbf{x} - \mathbf{g}(\mathbf{x})|^2}}{|\mathbf{x} - \mathbf{g}(\mathbf{x})|^2} \quad (10.10.26)$$

The expression under the square root sign is always nonnegative and it follows from the formula that $a(\mathbf{x}) \geq 0$. Therefore, $\langle \mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x})) \rangle \geq 0$ for all $\mathbf{x} \in B_r$. The reason for this is that $a(\mathbf{x})$ is the larger zero of a polynomial of the form $p(z) = |\mathbf{x}|^2 + z^2 |\mathbf{x} - \mathbf{g}(\mathbf{x})|^2 - 2z \langle \mathbf{x}, \mathbf{x} - \mathbf{g}(\mathbf{x}) \rangle$ and from the formula above, it is nonnegative. $-2\langle \mathbf{x}, \mathbf{x} - \mathbf{g}(\mathbf{x}) \rangle$ is the slope of the tangent line to $p(z)$ at $z = 0$. If $\mathbf{x} \neq \mathbf{0}$, then $|\mathbf{x}|^2 > 0$ and so this slope needs to be negative for the larger of the two zeros to be positive. If $\mathbf{x} = \mathbf{0}$, then $\langle \mathbf{x}, \mathbf{x} - \mathbf{g}(\mathbf{x}) \rangle = 0$.

Now define for $t \in [0, 1]$,

$$\mathbf{f}(t, \mathbf{x}) \equiv \mathbf{x} + ta(\mathbf{x})(\mathbf{x} - \mathbf{g}(\mathbf{x})).$$

The important properties of $\mathbf{f}(t, \mathbf{x})$ and $a(\mathbf{x})$ are that

$$a(\mathbf{x}) = 0 \text{ if } |\mathbf{x}| = r. \quad (10.10.27)$$

and

$$|\mathbf{f}(t, \mathbf{x})| = r \text{ for all } |\mathbf{x}| = r \quad (10.10.28)$$

These properties follow immediately from 10.10.26 and the above observation that for $\mathbf{x} \in B_r$, it follows $\langle \mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x})) \rangle \geq 0$.

Also from 10.10.26, a is a C^2 function near B_r . This is obvious from 10.10.26 as long as $|\mathbf{x}| < r$. However, even if $|\mathbf{x}| = r$ it is still true. To show this, it suffices to

verify the expression under the square root sign is positive. If this expression were not positive for some $|\mathbf{x}| = r$, then $(\mathbf{x}, (\mathbf{x} - \mathbf{g}(\mathbf{x}))) = 0$. Then also, since $\mathbf{g}(\mathbf{x}) \neq \mathbf{x}$,

$$\left| \frac{\mathbf{g}(\mathbf{x}) + \mathbf{x}}{2} \right| < r$$

and so

$$r^2 > \left(\mathbf{x}, \frac{\mathbf{g}(\mathbf{x}) + \mathbf{x}}{2} \right) = \frac{1}{2} (\mathbf{x}, \mathbf{g}(\mathbf{x})) + \frac{r^2}{2} = \frac{|\mathbf{x}|^2}{2} + \frac{r^2}{2} = r^2,$$

a contradiction. Therefore, the expression under the square root in 10.10.26 is always positive near B_r and so a is a C^2 function near B_r as claimed because the square root function is C^2 away from zero.

Now define

$$I(t) \equiv \int_{B_r} \det(D_2 \mathbf{f}(t, \mathbf{x})) dx.$$

Then

$$I(0) = \int_{B_r} dx = m_n(B_r) > 0. \quad (10.10.29)$$

Using the dominated convergence theorem one can differentiate $I(t)$ as follows.

$$\begin{aligned} I'(t) &= \int_{B_r} \sum_{ij} \frac{\partial \det(D_2 \mathbf{f}(t, \mathbf{x}))}{\partial f_{i,j}} \frac{\partial f_{i,j}}{\partial t} dx \\ &= \int_{B_r} \sum_{ij} \text{cof}(D_2 \mathbf{f})_{ij} \frac{\partial (a(\mathbf{x})(x_i - g_i(\mathbf{x})))}{\partial x_j} dx. \end{aligned}$$

Now from 10.10.27 $a(\mathbf{x}) = 0$ when $|\mathbf{x}| = r$ and so integration by parts and Lemma 10.10.1 yields

$$\begin{aligned} I'(t) &= \int_{B_r} \sum_{ij} \text{cof}(D_2 \mathbf{f})_{ij} \frac{\partial (a(\mathbf{x})(x_i - g_i(\mathbf{x})))}{\partial x_j} dx \\ &= - \int_{B_r} \sum_{ij} \text{cof}(D_2 \mathbf{f})_{ij,j} a(\mathbf{x})(x_i - g_i(\mathbf{x})) dx = 0. \end{aligned}$$

Therefore, $I(1) = I(0)$. However, from 10.10.25 it follows that for $t = 1$,

$$\sum_i f_i f_i = r^2$$

and so, $\sum_i f_{i,j} f_i = 0$ which implies since $|\mathbf{f}(1, \mathbf{x})| = r$ by 10.10.25, that $\det(f_{i,j}) = \det(D_2 \mathbf{f}(1, \mathbf{x})) = 0$ and so $I(1) = 0$, a contradiction to 10.10.29 since $I(1) = I(0)$. This proves the lemma.

The following theorem is the Brouwer fixed point theorem for a ball.

Theorem 10.10.3 *Let B_r be the above closed ball and let $\mathbf{f} : B_r \rightarrow B_r$ be continuous. Then there exists $\mathbf{x} \in B_r$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof: Let $\mathbf{f}_k(\mathbf{x}) \equiv \frac{\mathbf{f}(\mathbf{x})}{1+k^{-1}}$. Thus $\|\mathbf{f}_k - \mathbf{f}\| < \frac{r}{1+k}$ where

$$\|\mathbf{h}\| \equiv \max \{|\mathbf{h}(\mathbf{x})| : \mathbf{x} \in B_r\}.$$

Using the Weierstrass approximation theorem, there exists a polynomial $\mathbf{g}_k$ such that $\|\mathbf{g}_k - \mathbf{f}_k\| < \frac{r}{k+1}$. Then if $\mathbf{x} \in B_r$, it follows

$$\begin{aligned} |\mathbf{g}_k(\mathbf{x})| &\leq |\mathbf{g}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{x})| + |\mathbf{f}_k(\mathbf{x})| \\ &< \frac{r}{1+k} + \frac{kr}{1+k} = r \end{aligned}$$

and so $\mathbf{g}_k$ maps B_r to B_r . By Lemma 10.10.2 each of these $\mathbf{g}_k$ has a fixed point, $\mathbf{x}_k$ such that $\mathbf{g}_k(\mathbf{x}_k) = \mathbf{x}_k$. The sequence of points, $\{\mathbf{x}_k\}$ is contained in the compact set, B_r and so there exists a convergent subsequence still denoted by $\{\mathbf{x}_k\}$ which converges to a point, $\mathbf{x} \in B_r$. Then

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{x}| &\leq |\mathbf{f}(\mathbf{x}) - \mathbf{f}_k(\mathbf{x})| + |\mathbf{f}_k(\mathbf{x}) - \mathbf{f}_k(\mathbf{x}_k)| + \left| \mathbf{f}_k(\mathbf{x}_k) - \overbrace{\mathbf{g}_k(\mathbf{x}_k)}^{=\mathbf{x}_k} \right| + |\mathbf{x}_k - \mathbf{x}| \\ &\leq \frac{r}{1+k} + |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_k)| + \frac{r}{1+k} + |\mathbf{x}_k - \mathbf{x}|. \end{aligned}$$

Now let $k \rightarrow \infty$ in the right side to conclude $\mathbf{f}(\mathbf{x}) = \mathbf{x}$. This proves the theorem.

It is not surprising that the ball does not need to be centered at $\mathbf{0}$.

Corollary 10.10.4 *Let $\mathbf{f} : \overline{B}(\mathbf{a}, r) \rightarrow \overline{B}(\mathbf{a}, r)$ be continuous. Then there exists $\mathbf{x} \in \overline{B}(\mathbf{a}, r)$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof: Let $\mathbf{g} : B_r \rightarrow B_r$ be defined by $\mathbf{g}(\mathbf{y}) \equiv \mathbf{f}(\mathbf{y} + \mathbf{a}) - \mathbf{a}$. Then $\mathbf{g}$ is a continuous map from B_r to B_r . Therefore, there exists $\mathbf{y} \in B_r$ such that $\mathbf{g}(\mathbf{y}) = \mathbf{y}$. Therefore, $\mathbf{f}(\mathbf{y} + \mathbf{a}) - \mathbf{a} = \mathbf{y}$ and so letting $\mathbf{x} = \mathbf{y} + \mathbf{a}$, $\mathbf{f}$ also has a fixed point as claimed.

10.11 The Brouwer Fixed Point Theorem Another Proof

This proof is also based on Lemma 10.10.1. I found this proof of the Brouwer fixed point theorem or one close to it in Evans [25]. It is even shorter than the proof just presented. I think it might be easier to remember also. It is also based on Lemma 10.10.1 which is stated next for convenience.

Lemma 10.11.1 *Let $\mathbf{g} : U \rightarrow \mathbb{R}^n$ be C^2 where U is an open subset of $\mathbb{R}^n$. Then*

$$\sum_{j=1}^n \text{cof}(D\mathbf{g})_{ij,j} = 0,$$

where here $(D\mathbf{g})_{ij} \equiv g_{i,j} \equiv \frac{\partial g_i}{\partial x_j}$. Also, $\text{cof}(D\mathbf{g})_{ij} = \frac{\partial \det(D\mathbf{g})}{\partial g_{i,j}}$.

Definition 10.11.2 Let $\mathbf{h}$ be a function defined on an open set, $U \subseteq \mathbb{R}^n$. Then $\mathbf{h} \in C^k(\bar{U})$ if there exists a function $\mathbf{g}$ defined on an open set, W containing $\bar{U}$ such that $\mathbf{g} = \mathbf{h}$ on U and $\mathbf{g}$ is $C^k(W)$.

Lemma 10.11.3 There does not exist $\mathbf{h} \in C^2(\overline{B(\mathbf{0}, R)})$ such that $\mathbf{h} : \overline{B(\mathbf{0}, R)} \rightarrow \partial B(\mathbf{0}, R)$ which also has the property that $\mathbf{h}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial B(\mathbf{0}, R)$. Such a function is called a retraction.

Proof: Suppose such an $\mathbf{h}$ exists. Let $\lambda \in [0, 1]$ and let $\mathbf{p}_\lambda(\mathbf{x}) \equiv \mathbf{x} + \lambda(\mathbf{h}(\mathbf{x}) - \mathbf{x})$. This function, $\mathbf{p}_\lambda$ is a homotopy of the identity map and the retraction, $\mathbf{h}$. Let

$$I(\lambda) \equiv \int_{B(\mathbf{0}, R)} \det(D\mathbf{p}_\lambda(\mathbf{x})) dx.$$

Then using the dominated convergence theorem,

$$\begin{aligned} I'(\lambda) &= \int_{B(\mathbf{0}, R)} \sum_{i,j} \frac{\partial \det(D\mathbf{p}_\lambda(\mathbf{x}))}{\partial p_{\lambda i,j}} \frac{\partial p_{\lambda i,j}(\mathbf{x})}{\partial \lambda} \\ &= \int_{B(\mathbf{0}, R)} \sum_i \sum_j \frac{\partial \det(D\mathbf{p}_\lambda(\mathbf{x}))}{\partial p_{\lambda i,j}} (h_i(\mathbf{x}) - x_i)_{,j} dx \\ &= \int_{B(\mathbf{0}, R)} \sum_i \sum_j \text{cof}(D\mathbf{p}_\lambda(\mathbf{x}))_{ij} (h_i(\mathbf{x}) - x_i)_{,j} dx \end{aligned}$$

Now by assumption, $h_i(\mathbf{x}) = x_i$ on $\partial B(\mathbf{0}, R)$ and so one can integrate by parts and write

$$I'(\lambda) = - \sum_i \int_{B(\mathbf{0}, R)} \sum_j \text{cof}(D\mathbf{p}_\lambda(\mathbf{x}))_{ij,j} (h_i(\mathbf{x}) - x_i) dx = 0.$$

Therefore, $I(\lambda)$ equals a constant. However,

$$I(0) = m_n(B(\mathbf{0}, R)) > 0$$

but

$$I(1) = \int_{B(\mathbf{0}, 1)} \det(D\mathbf{h}(\mathbf{x})) dm_n = \int_{\partial B(\mathbf{0}, 1)} \#(\mathbf{y}) dm_n = 0$$

because from polar coordinates or other elementary reasoning, $m_n(\partial B(\mathbf{0}, 1)) = 0$. This proves the lemma.

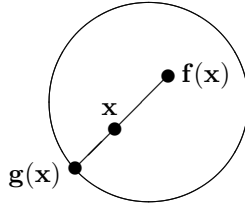
The following is the Brouwer fixed point theorem for C^2 maps.

Lemma 10.11.4 If $\mathbf{h} \in C^2(\overline{B(\mathbf{0}, R)})$ and $\mathbf{h} : \overline{B(\mathbf{0}, R)} \rightarrow \overline{B(\mathbf{0}, R)}$, then $\mathbf{h}$ has a fixed point, $\mathbf{x}$ such that $\mathbf{h}(\mathbf{x}) = \mathbf{x}$.

Proof: Suppose the lemma is not true. Then for all $\mathbf{x}$, $|\mathbf{x} - \mathbf{h}(\mathbf{x})| \neq 0$. Then define

$$\mathbf{g}(\mathbf{x}) = \mathbf{h}(\mathbf{x}) + \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} t(\mathbf{x})$$

where $t(\mathbf{x})$ is nonnegative and is chosen such that $\mathbf{g}(\mathbf{x}) \in \partial B(\mathbf{0}, R)$. This mapping is illustrated in the following picture.



If $\mathbf{x} \rightarrow t(\mathbf{x})$ is C^2 near $\overline{B(\mathbf{0}, R)}$, it will follow $\mathbf{g}$ is a C^2 retraction onto $\partial B(\mathbf{0}, R)$ contrary to Lemma 10.11.3. Thus $t(\mathbf{x})$ is the nonnegative solution to

$$H(\mathbf{x}, t) = |\mathbf{h}(\mathbf{x})|^2 + 2 \left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} \right) t + t^2 = R^2 \quad (10.11.30)$$

Then

$$H_t(\mathbf{x}, t) = 2 \left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} \right) + 2t.$$

If this is nonzero for all $\mathbf{x}$ near $\overline{B(\mathbf{0}, R)}$, it follows from the implicit function theorem that t is a C^2 function of $\mathbf{x}$. Then from 10.11.30

$$\begin{aligned} 2t &= -2 \left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} \right) \\ &\quad \pm \sqrt{4 \left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} \right)^2 - 4 \left(|\mathbf{h}(\mathbf{x})|^2 - R^2 \right)} \end{aligned}$$

and so

$$\begin{aligned} H_t(\mathbf{x}, t) &= 2t + 2 \left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} \right) \\ &= \pm \sqrt{4 \left(R^2 - |\mathbf{h}(\mathbf{x})|^2 \right) + 4 \left(\mathbf{h}(\mathbf{x}), \frac{\mathbf{x} - \mathbf{h}(\mathbf{x})}{|\mathbf{x} - \mathbf{h}(\mathbf{x})|} \right)^2} \end{aligned}$$

If $|\mathbf{h}(\mathbf{x})| < R$, this is nonzero. If $|\mathbf{h}(\mathbf{x})| = R$, then it is still nonzero unless

$$(\mathbf{h}(\mathbf{x}), \mathbf{x} - \mathbf{h}(\mathbf{x})) = 0.$$

But this cannot happen because the angle between $\mathbf{h}(\mathbf{x})$ and $\mathbf{x} - \mathbf{h}(\mathbf{x})$ cannot be $\pi/2$. Alternatively, if the above equals zero, you would need

$$(\mathbf{h}(\mathbf{x}), \mathbf{x}) = |\mathbf{h}(\mathbf{x})|^2 = R^2$$

which cannot happen unless $\mathbf{x} = \mathbf{h}(\mathbf{x})$ which is assumed not to happen. Therefore, $\mathbf{x} \rightarrow t(\mathbf{x})$ is C^2 near $\overline{B(\mathbf{0}, R)}$ and so $\mathbf{g}(\mathbf{x})$ given above contradicts Lemma 10.11.3. This proves the lemma.

Now it is easy to prove the Brouwer fixed point theorem.

Theorem 10.11.5 *Let $\mathbf{f} : \overline{B(\mathbf{0}, R)} \rightarrow \overline{B(\mathbf{0}, R)}$ be continuous. Then $\mathbf{f}$ has a fixed point.*

Proof: If this is not so, there exists $\varepsilon > 0$ such that for all $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$,

$$|\mathbf{x} - \mathbf{f}(\mathbf{x})| > \varepsilon.$$

By the Weierstrass approximation theorem, there exists $\mathbf{h}$, a polynomial such that

$$\max \left\{ |\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| : \mathbf{x} \in \overline{B(\mathbf{0}, R)} \right\} < \frac{\varepsilon}{2}.$$

Then for all $\mathbf{x} \in \overline{B(\mathbf{0}, R)}$,

$$|\mathbf{x} - \mathbf{h}(\mathbf{x})| \geq |\mathbf{x} - \mathbf{f}(\mathbf{x})| - |\mathbf{h}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}$$

contradicting Lemma 10.11.4. This proves the theorem.

Some Extension Theorems

11.1 Caratheodory Extension Theorem

The Caratheodory extension theorem is a fundamental result which makes possible the consideration of measures on infinite products among other things. The idea is that if a finite measure defined only on an algebra is trying to be a measure, then in fact it can be extended to a measure.

Definition 11.1.1 *Let $\mathcal{E}$ be an algebra of sets of Ω and let μ_0 be a finite measure on $\mathcal{E}$. This means μ_0 is finitely additive and if E_i, E are sets of $\mathcal{E}$ with the E_i disjoint and*

$$E = \cup_{i=1}^{\infty} E_i,$$

then

$$\mu_0(E) = \sum_{i=1}^{\infty} \mu_0(E_i)$$

while $\mu_0(\Omega) < \infty$.

In this definition, μ_0 is trying to be a measure and acts like one whenever possible. Under these conditions, μ_0 can be extended uniquely to a complete measure, μ , defined on a σ algebra of sets containing $\mathcal{E}$ such that μ agrees with μ_0 on $\mathcal{E}$. The following is the main result.

Theorem 11.1.2 *Let μ_0 be a measure on an algebra of sets, $\mathcal{E}$, which satisfies $\mu_0(\Omega) < \infty$. Then there exists a complete measure space $(\Omega, \mathcal{S}, \mu)$ such that*

$$\mu(E) = \mu_0(E)$$

for all $E \in \mathcal{E}$. Also if ν is any such measure which agrees with μ_0 on $\mathcal{E}$, then $\nu = \mu$ on $\sigma(\mathcal{E})$, the σ algebra generated by $\mathcal{E}$.

Proof: Define an outer measure as follows.

$$\mu(S) \equiv \inf \left\{ \sum_{i=1}^{\infty} \mu_0(E_i) : S \subseteq \cup_{i=1}^{\infty} E_i, E_i \in \mathcal{E} \right\}$$

Claim 1: μ is an outer measure.

Proof of Claim 1: Let $S \subseteq \bigcup_{i=1}^{\infty} S_i$ and let $S_i \subseteq \bigcup_{j=1}^{\infty} E_{ij}$, where

$$\mu(S_i) + \frac{\varepsilon}{2^i} \geq \sum_{j=1}^{\infty} \mu(E_{ij}).$$

Then

$$\mu(S) \leq \sum_i \sum_j \mu(E_{ij}) = \sum_i \left(\mu(S_i) + \frac{\varepsilon}{2^i} \right) = \sum_i \mu(S_i) + \varepsilon.$$

Since ε is arbitrary, this shows μ is an outer measure as claimed.

By the Caratheodory procedure, there exists a unique σ algebra, $\mathcal{S}$, consisting of the μ measurable sets such that

$$(\Omega, \mathcal{S}, \mu)$$

is a complete measure space. It remains to show μ extends μ_0 .

Claim 2: If $\mathcal{S}$ is the σ algebra of μ measurable sets, $\mathcal{S} \supseteq \mathcal{E}$ and $\mu = \mu_0$ on $\mathcal{E}$.

Proof of Claim 2: First observe that if $A \in \mathcal{E}$, then $\mu(A) \leq \mu_0(A)$ by definition. Letting

$$\mu(A) + \varepsilon > \sum_{i=1}^{\infty} \mu_0(E_i), \quad \bigcup_{i=1}^{\infty} E_i \supseteq A, \quad E_i \in \mathcal{E},$$

it follows

$$\mu(A) + \varepsilon > \sum_{i=1}^{\infty} \mu_0(E_i \cap A) \geq \mu_0(A)$$

since $A = \bigcup_{i=1}^{\infty} E_i \cap A$. Therefore, $\mu = \mu_0$ on $\mathcal{E}$.

Consider the assertion that $\mathcal{E} \subseteq \mathcal{S}$. Let $A \in \mathcal{E}$ and let $S \subseteq \Omega$ be any set. There exist sets $\{E_i\} \subseteq \mathcal{E}$ such that $\bigcup_{i=1}^{\infty} E_i \supseteq S$ but

$$\mu(S) + \varepsilon > \sum_{i=1}^{\infty} \mu(E_i).$$

Then

$$\begin{aligned} \mu(S) &\leq \mu(S \cap A) + \mu(S \setminus A) \\ &\leq \mu(\bigcup_{i=1}^{\infty} E_i \setminus A) + \mu(\bigcup_{i=1}^{\infty} (E_i \cap A)) \\ &\leq \sum_{i=1}^{\infty} \mu(E_i \setminus A) + \sum_{i=1}^{\infty} \mu(E_i \cap A) = \sum_{i=1}^{\infty} \mu(E_i) < \mu(S) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows $A \in \mathcal{S}$.

This has proved the existence part of the theorem. To verify uniqueness, Let

$$\mathcal{M} \equiv \{E \in \sigma(\mathcal{E}) : \mu(E) = \nu(E)\}.$$

Then $\mathcal{M}$ is given to contain $\mathcal{E}$ and is obviously a monotone class. Therefore by Theorem 9.9.5 on monotone classes, $\mathcal{M} = \sigma(\mathcal{E})$ and this proves the lemma.

The following lemma is also very significant.

Lemma 11.1.3 *Let M be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of M which is finite on compact sets. Then there exists a unique Radon measure, $\bar{\mu}$ which equals μ on the Borel sets. In particular μ must be both inner and outer regular on all Borel sets.*

Proof: Define a positive linear functional, $\Lambda(f) = \int f d\mu$. Let $\bar{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all f continuous,

$$\int f d\mu = \int f d\bar{\mu}.$$

If V is an open set, let $\{f_n\}$ be a sequence of continuous functions which is increasing and converges to χ_V pointwise. Then applying the monotone convergence theorem,

$$\int \chi_V d\mu = \mu(V) = \int \chi_V d\bar{\mu} = \bar{\mu}(V)$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a, n)$ be a ball of radius n and let E be a Borel set contained in this ball. Then by regularity of $\bar{\mu}$ there exist sets F, G such that G is a countable intersection of open sets and F is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\bar{\mu}(G \setminus F) = 0$. Now $\mu(G) = \bar{\mu}(G)$ and $\mu(F) = \bar{\mu}(F)$. Thus

$$\begin{aligned} \bar{\mu}(G \setminus F) + \bar{\mu}(F) &= \bar{\mu}(G) \\ &= \mu(G) = \mu(G \setminus F) + \mu(F) \end{aligned}$$

and so $\mu(G \setminus F) = \bar{\mu}(G \setminus F)$. It follows

$$\mu(E) = \mu(F) = \bar{\mu}(F) = \bar{\mu}(G) = \bar{\mu}(E).$$

If E is an arbitrary Borel set, then

$$\mu(E \cap B(a, n)) = \bar{\mu}(E \cap B(a, n))$$

and letting $n \rightarrow \infty$, this yields $\mu(E) = \bar{\mu}(E)$.

11.2 The Tychonoff Theorem

Sometimes it is necessary to consider infinite Cartesian products of topological spaces. When you have finitely many topological spaces in the product and each is compact, it can be shown that the Cartesian product is compact with the product topology. It turns out that the same thing holds for infinite products but you have to be careful how you define the topology. The first thing likely to come to mind by analogy with finite products is not the right way to do it.

First recall the Hausdorff maximal principle.

Theorem 11.2.1 (*Hausdorff maximal principle*) *Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

The main tool in the study of products of compact topological spaces is the Alexander subbasis theorem which is presented next. Recall a set is compact if every basic open cover admits a finite subcover. This was pretty easy to prove. However, there is a much smaller set of open sets called a subbasis which has this property. The proof of this result is much harder.

Definition 11.2.2 $\mathcal{S} \subseteq \tau$ is called a subbasis for the topology τ if the set $\mathcal{B}$ of finite intersections of sets of $\mathcal{S}$ is a basis for the topology, τ .

Theorem 11.2.3 *Let (X, τ) be a topological space and let $\mathcal{S} \subseteq \tau$ be a subbasis for τ . Then if $H \subseteq X$, H is compact if and only if every open cover of H consisting entirely of sets of $\mathcal{S}$ admits a finite subcover.*

Proof: The only if part is obvious because the subbasic sets are themselves open.

By Lemma 6.6.23 on Page 6.6.23, if every basic open cover admits a finite subcover then the set in question is compact. Suppose then that H is a subset of X having the property that subbasic open covers admit finite subcovers. Is H compact? Assume this is not so. Then what was just observed about basic covers implies there exists a basic open cover of H , $\mathcal{O}$, which admits no finite subcover. Let $\mathcal{F}$ be defined as

$$\{\mathcal{O} : \mathcal{O} \text{ is a basic open cover of } H \text{ which admits no finite subcover}\}.$$

The assumption is that $\mathcal{F}$ is nonempty. Partially order $\mathcal{F}$ by set inclusion and use the Hausdorff maximal principle to obtain a maximal chain, $\mathcal{C}$, of such open covers and let

$$\mathcal{D} = \cup \mathcal{C}.$$

If $\mathcal{D}$ admits a finite subcover, then since $\mathcal{C}$ is a chain and the finite subcover has only finitely many sets, some element of $\mathcal{C}$ would also admit a finite subcover, contrary to the definition of $\mathcal{F}$. Therefore, $\mathcal{D}$ admits no finite subcover. If $\mathcal{D}' \supsetneq \mathcal{D}$ and $\mathcal{D}'$ is a basic open cover of H , then $\mathcal{D}'$ has a finite subcover of H since otherwise, $\mathcal{C}$ would fail to be a maximal chain, being properly contained in $\mathcal{C} \cup \{\mathcal{D}'\}$. Every set of $\mathcal{D}$ is of the form

$$U = \cap_{i=1}^m B_i, \quad B_i \in \mathcal{S}$$

because they are all basic open sets. If it is the case that for all $U \in \mathcal{D}$ one of the B_i is found in $\mathcal{D}$, then replace each such U with the subbasic set from $\mathcal{D}$ containing it. But then this would be a subbasic open cover of H which by assumption would admit a finite subcover contrary to the properties of $\mathcal{D}$. Therefore, one of the sets of $\mathcal{D}$, denoted by U , has the property that

$$U = \cap_{i=1}^m B_i, \quad B_i \in \mathcal{S}$$

and no B_i is in $\mathcal{D}$. Thus $\mathcal{D} \cup \{B_i\}$ admits a finite subcover, for each of the above B_i because it is strictly larger than $\mathcal{D}$. Let this finite subcover corresponding to B_i be denoted by

$$V_1^i, \dots, V_{m_i}^i, B_i$$

Consider

$$\{U, V_j^i, j = 1, \dots, m_i, i = 1, \dots, m\}.$$

If $p \in H \setminus \cup\{V_j^i\}$, then $p \in B_i$ for each i and so $p \in U$. This is therefore a finite subcover of $\mathcal{D}$ contradicting the properties of $\mathcal{D}$. Therefore, $\mathcal{F}$ must be empty and by Lemma 6.6.23, this proves the theorem.

Let I be a set and suppose for each $i \in I$, (X_i, τ_i) is a nonempty topological space. The Cartesian product of the X_i , denoted by $\prod_{i \in I} X_i$, consists of the set of all choice functions defined on I which select a single element of each X_i . Thus $f \in \prod_{i \in I} X_i$ means for every $i \in I$, $f(i) \in X_i$. The axiom of choice says $\prod_{i \in I} X_i$ is nonempty. Let

$$P_j(A) = \prod_{i \in I} B_i$$

where $B_i = X_i$ if $i \neq j$ and $B_j = A$. A subbasis for a topology on the product space consists of all sets $P_j(A)$ where $A \in \tau_j$. (These sets have an open set from the topology of X_j in the j^{th} slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_i$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\prod \tau_i$.

It is tempting to define a basis for a topology to be sets of the form $\prod_{i \in I} A_i$ where A_i is open in X_i . This is not the same thing at all. Note that the basis just described has at most finitely many slots filled with an open set which is not the whole space. The thing just mentioned in which every slot may be filled by a proper open set is called the box topology and there exist people who are interested in it.

The Alexander subbasis theorem is used to prove the Tychonoff theorem which says that if each X_i is a compact topological space, then in the product topology, $\prod_{i \in I} X_i$ is also compact.

Theorem 11.2.4 *If (X_i, τ_i) is compact, then so is $(\prod_{i \in I} X_i, \prod \tau_i)$.*

Proof: By the Alexander subbasis theorem, the theorem will be proved if every subbasic open cover admits a finite subcover. Therefore, let $\mathcal{O}$ be a subbasic open cover of $\prod_{i \in I} X_i$. Let

$$\mathcal{O}_j = \{Q \in \mathcal{O} : Q = P_j(A) \text{ for some } A \in \tau_j\}.$$

Thus $\mathcal{O}_j$ consists of those sets of $\mathcal{O}$ which have a possibly proper subset of X_i only in the slot $i = j$. Let

$$\pi_j \mathcal{O}_j = \{A : P_j(A) \in \mathcal{O}_j\}.$$

Thus $\pi_j \mathcal{O}_j$ picks out those proper open subsets of X_j which occur in $\mathcal{O}_j$.

If no $\pi_j \mathcal{O}_j$ covers X_j , then by the axiom of choice, there exists

$$f \in \prod_{i \in I} X_i \setminus \cup \pi_i \mathcal{O}_i$$

Therefore, $f(j) \notin \cup \pi_j \mathcal{O}_j$ for each $j \in I$. Now f is a point of $\prod_{i \in I} X_i$ and so $f \in P_k(A) \in \mathcal{O}$ for some k . However, this is a contradiction it was shown that $f(k)$ is not an element of A . (A is one of the sets whose union makes up $\cup \pi_k \mathcal{O}_k$.) This contradiction shows that for some j , $\pi_j \mathcal{O}_j$ covers X_j . Thus

$$X_j = \cup \pi_j \mathcal{O}_j$$

and so by compactness of X_j , there exist $A_1, \dots, A_m$, sets in τ_j such that $X_j \subseteq \cup_{i=1}^m A_i$ and $P_j(A_i) \in \mathcal{O}$. Therefore, $\{P_j(A_i)\}_{i=1}^m$ covers $\prod_{i \in I} X_i$. By the Alexander subbasis theorem this proves $\prod_{i \in I} X_i$ is compact.

11.3 Kolmogorov Extension Theorem

Here $M_t \equiv [-\infty, \infty]^{n_t}$ where n_t is a positive integer. Let $\mathcal{A}_t$ denote finite disjoint unions the n_t dimensional boxes. Thus $\mathcal{A}_t$ denotes finite disjoint unions of sets of the form $\prod_{k=1}^{n_t} E_k$ where E_k is an interval. By Corollary 9.9.3 on Page 265 this is an algebra of sets. This algebra has the property that every set in $\mathcal{A}_t$ is the countable union of an increasing sequence of compact sets each of which is in $\mathcal{A}_t$.

I will denote a totally ordered index set, and the interest will be in building a measure on the product space, $\prod_{t \in I} M_t$. By the well ordering principle, you can always put an order on the index set so this order is no restriction. Also for X a topological space, $\mathcal{B}(X)$ will denote the Borel sets.

Let $J \subseteq I$. Then if $\mathbf{E} \equiv \prod_{t \in I} E_t$, define

$$\gamma_J \mathbf{E} \equiv \prod_{t \in I} F_t$$

where

$$F_t = \begin{cases} E_t & \text{if } t \in J \\ M_t & \text{if } t \notin J \end{cases}$$

Thus $\gamma_J \mathbf{E}$ leaves alone E_t for $t \in J$ and changes the other E_t into M_t . If $\gamma_J \mathbf{E} = \mathbf{E}$, then this means $E_t = M_t$ for all $t \notin J$. Also define for J a subset of I and $\mathbf{x} \in \prod_{t \in I} M_t$,

$$\pi_J \mathbf{x} \equiv \prod_{t \in J} x_t$$

so π_J is a continuous mapping from $\prod_{t \in I} M_t$ to $\prod_{t \in J} M_t$.

$$\pi_J \mathbf{E} \equiv \prod_{t \in J} E_t.$$

Definition 11.3.1 Now define for J a finite subset of I ,

$$\begin{aligned}\mathcal{R}_J &\equiv \left\{ \mathbf{E} = \prod_{t \in I} E_t : \gamma_J \mathbf{E} = \mathbf{E}, E_t \in \mathcal{A}_t \right\} \\ \mathcal{R} &\equiv \cup \{ \mathcal{R}_J : J \subseteq I, \text{ and } J \text{ finite} \}\end{aligned}$$

Thus $\mathcal{R}$ consists of those sets of $\prod_{t \in I} M_t$ for which every slot is filled with M_t except for a finite set, $J \subseteq I$ where the slots are filled with E_t , a set of $\mathcal{A}_t$. Define $\mathcal{E}$ as finite disjoint unions of sets of $\mathcal{R}$ and $\mathcal{E}_J$ is defined as finite disjoint unions of sets of $\mathcal{R}_J$.

First I show $\mathcal{E}$ is an algebra of sets.

Lemma 11.3.2 The sets, $\mathcal{E}, \mathcal{E}_J$ defined above form an algebra of sets of $\prod_{t \in I} M_t$.

Proof: First consider $\mathcal{R}_J$. If $\mathbf{A}, \mathbf{B} \in \mathcal{R}_J$, then $\mathbf{A} \cap \mathbf{B} \in \mathcal{R}_J$ also. Is $\mathbf{A} \setminus \mathbf{B}$ a finite disjoint union of sets of $\mathcal{R}_J$? Using induction on the size of J and Corollary 9.9.3 on Page 265 this follows. Thus finite disjoint unions of sets of $\mathcal{R}_J$ form an algebra since it is clear that $\emptyset$ and $\prod_{t \in I} M_t$ are both in $\mathcal{R}_J$. This shows $\mathcal{E}_J$ is an algebra.

Now suppose $\mathbf{A}, \mathbf{B} \in \mathcal{R}$. Then for some finite set, J ,

$$\gamma_J \mathbf{A} = \mathbf{A}, \gamma_J \mathbf{B} = \mathbf{B}.$$

Then from what was just shown,

$$\mathbf{A} \setminus \mathbf{B} \in \mathcal{E}, \mathbf{A} \cap \mathbf{B} \in \mathcal{R}.$$

By Lemma 9.9.2 on Page 264 this shows $\mathcal{E}$ is an algebra.

With this preparation here is the Kolmogorov extension theorem. In the statement and proof of the theorem, F_i, G_i , and E_i will denote Borel sets. Any list of indices from I will always be assumed to be taken in order. Thus, if $J \subseteq I$ and $J = (t_1, \dots, t_n)$, it will always be assumed $t_1 < t_2 < \dots < t_n$.

Theorem 11.3.3 For each finite set

$$J = (t_1, \dots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \dots t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ such that the following consistency condition holds. If

$$(t_1, \dots, t_n) \subseteq (s_1, \dots, s_p),$$

then

$$\nu_{t_1 \dots t_n} (F_{t_1} \times \dots \times F_{t_n}) = \nu_{s_1 \dots s_p} (G_{s_1} \times \dots \times G_{s_p}) \quad (11.3.1)$$

where if $s_i = t_j$, then $G_{s_i} = F_{t_j}$ and if s_i is not equal to any of the indices, t_k , then $G_{s_i} = M_{s_i}$. Then for $\mathcal{E}$ defined in Definition 11.3.1, there exists a probability measure, P and a σ algebra $\mathcal{F} = \sigma(\mathcal{E})$ such that

$$\left(\prod_{t \in I} M_t, P, \mathcal{F} \right)$$

is a probability space. Also there exist measurable functions, $X_s : \prod_{t \in I} M_t \rightarrow M_s$ defined as

$$X_s \mathbf{x} \equiv x_s$$

for each $s \in I$ such that for each $(t_1 \cdots t_n) \subseteq I$,

$$\begin{aligned} \nu_{t_1 \cdots t_n} (F_{t_1} \times \cdots \times F_{t_n}) &= P([X_{t_1} \in F_{t_1}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]) \\ &= P \left((X_{t_1}, \cdots, X_{t_n}) \in \prod_{j=1}^n F_{t_j} \right) = P \left(\prod_{t \in I} F_t \right) \end{aligned} \quad (11.3.2)$$

where $F_t = M_t$ for every $t \notin \{t_1 \cdots t_n\}$ and F_{t_i} is a Borel set. Also if f is a non-negative function of finitely many variables, $x_{t_1}, \cdots, x_{t_n}$, measurable with respect to $\mathcal{B} \left(\prod_{j=1}^n M_{t_j} \right)$, then f is also measurable with respect to $\mathcal{F}$ and

$$\begin{aligned} &\int_{M_{t_1} \times \cdots \times M_{t_n}} f(x_{t_1}, \cdots, x_{t_n}) d\nu_{t_1 \cdots t_n} \\ &= \int_{\prod_{t \in I} M_t} f(x_{t_1}, \cdots, x_{t_n}) dP \end{aligned} \quad (11.3.3)$$

Proof: Let $\mathcal{E}$ be the algebra of sets defined in Definition 11.3.1. I want to define a measure on $\mathcal{E}$. Suppose $\mathbf{E} = \mathbf{E}^1 \cup \mathbf{E}^2 \cup \cdots \cup \mathbf{E}^m$ where $\gamma_{J_k} \mathbf{E}^k = \mathbf{E}^k$ and the sets, $\mathbf{E}^k$ are disjoint. Thus $\mathbf{E} \in \mathcal{E}$. Let $(t_1^k \cdots t_{m_k}^k) = J_k$. Then letting

$$J = (s_1, \cdots, s_p) \supseteq \bigcup_{k=1}^m J_k$$

define

$$P_0(\mathbf{E}) \equiv \sum_{k=1}^m \nu_{s_1 \cdots s_p} \left(G_{s_1}^k \times \cdots \times G_{s_p}^k \right)$$

where $G_{s_i}^k = E_{t_j^k}^k$ in case $s_i = t_j^k$ and M_{s_i} otherwise. By the consistency condition, 11.3.1 this is well defined and equals

$$\sum_{k=1}^m \nu_{t_1^k \cdots t_{m_k}^k} \left(E_{t_1^k}^k \times \cdots \times E_{t_{m_k}^k}^k \right).$$

P_0 is clearly finitely additive because the ν_J are measures and one can pick J as large as desired. Also, from the definition,

$$P_0 \left(\prod_{t \in I} M_t \right) = \nu_{t_1} (M_{t_1}) = 1.$$

Next I will show P_0 is a finite measure on $\mathcal{E}$. After this it is only a matter of using the Caratheodory extension theorem to get the existence of the desired probability measure, P .

Claim: If $\mathbf{E}^n \downarrow \emptyset$, then $P_0(\mathbf{E}^n) \downarrow 0$.

Proof of the claim: If not, there exists a sequence such that although $\mathbf{E}^n \downarrow \emptyset$, $P_0(\mathbf{E}^n) \downarrow \varepsilon > 0$. Suppose first

$$\mathbf{E}^n = \prod_{t \in I} E_t^n$$

where $E_t^n = M_t$ except for $t \in J_n$, a finite subset of I . By the assumption that every set of $\mathcal{A}_t$ is the countable union of compact sets, there exists a compact set, $\prod_{t \in J_n} K_t^n$ such that

$$\prod_{t \in J_n} K_t^n \subseteq \prod_{t \in J_n} E_t^n$$

and

$$\nu_{J_n} \left(\prod_{t \in J_n} K_t^n \right) + \eta > \nu_{J_n} \left(\prod_{t \in J_n} E_t^n \right)$$

where η is a small positive number. Now let

$$\mathbf{K}^n = \prod_{t \in I} K_t \subseteq \prod_{t \in I} M_t$$

be such that for $t \notin J_n$, $K_t = M_t$. Doing this for each of the finitely many sets of the above form whose union is $\mathbf{E}^n$ and choosing η small enough, it follows there exists $\mathbf{K}^n \in \mathcal{E}$ such that

$$\mathbf{K}^n \subseteq \mathbf{E}^n$$

and

$$\begin{aligned} P_0(\mathbf{K}^n) + \frac{\varepsilon}{2^{n+2}} &= \nu_{J_n} \left(\prod_{t \in J_n} K_t^n \right) + \frac{\varepsilon}{2^{n+2}} \\ &> \nu_{J_n} \left(\prod_{t \in J_n} E_t^n \right) \equiv P_0(\mathbf{E}^n) \end{aligned}$$

By Tychonoff, theorem, $\mathbf{K}^n$ is compact. The interesting thing about these $\mathbf{K}^n$ is they have the finite intersection property. Here is why.

$$\begin{aligned} \varepsilon &\leq P_0(\cap_{k=1}^m \mathbf{K}^k) + P_0(\mathbf{E}^m \setminus \cap_{k=1}^m \mathbf{K}^k) \\ &\leq P_0(\cap_{k=1}^m \mathbf{K}^k) + P_0(\cup_{k=1}^m \mathbf{E}^k \setminus \mathbf{K}^k) \\ &< P_0(\cap_{k=1}^m \mathbf{K}^k) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+2}} < P_0(\cap_{k=1}^m \mathbf{K}^k) + \varepsilon/2 \end{aligned}$$

and so $P_0(\cap_{k=1}^n \mathbf{K}^k) > \varepsilon/2$. Now this yields a contradiction because this finite intersection property implies the intersection of all the $\mathbf{K}^n$ is nonempty, contradicting $\mathbf{E}^n \downarrow \emptyset$ since each $\mathbf{K}^n$ is contained in $\mathbf{E}^n$.

With the claim, it follows P_0 is a measure on $\mathcal{E}$. Here is why: If $\mathbf{E} = \cup_{k=1}^{\infty} \mathbf{E}^k$ where $\mathbf{E}, \mathbf{E}^k \in \mathcal{E}$, then $(\mathbf{E} \setminus \cup_{k=1}^n \mathbf{E}^k) \downarrow \emptyset$ and so

$$P_0(\cup_{k=1}^n \mathbf{E}^k) \rightarrow P_0(\mathbf{E}).$$

Hence if the $\mathbf{E}_k$ are disjoint, $P_0(\cup_{k=1}^n \mathbf{E}_k) = \sum_{k=1}^n P_0(\mathbf{E}_k) \rightarrow P_0(\mathbf{E})$.

Now to conclude the proof, apply the Caratheodory extension theorem to obtain P a probability measure which extends P_0 to $\sigma(\mathcal{E})$ the sigma algebra generated by $\mathcal{E}$.

Let $\mathcal{F} \equiv \sigma(\mathcal{E})$. It follows $(\prod_{t \in I} M_t, \mathcal{F}, P)$ is a probability measure space with the property that when $\gamma_J(\mathbf{E}) = \mathbf{E}$ for $J = (t_1 \cdots t_n)$ a finite subset of I ,

$$P(\mathbf{E}) = P_0(\mathbf{E}) = \nu_{t_1 \cdots t_n}(E_{t_1} \times \cdots \times E_{t_n}).$$

Next, let $(\prod_{t \in I} M_t, \mathcal{F}, P)$ be the probability space and for $\mathbf{x} \in \prod_{t \in I} M_t$ let $X_t(\mathbf{x}) = x_t$, the t^{th} entry of $\mathbf{x}$. ($x_t = \pi_t \mathbf{x}$). It follows X_t is measurable because if U is open in M_t , then $X_t^{-1}(U)$ has a U in the t^{th} slot and M_s everywhere else for $s \neq t$. Thus inverse images of open sets are measurable. Also, letting $(t_1 \cdots t_n)$ be a finite subset of I and $F_{t_1}, \cdots, F_{t_n}$ be Borel sets in $M_{t_1} \cdots M_{t_n}$ respectively,

$$\begin{aligned} P([X_{t_1} \in F_{t_1}] \cap [X_{t_2} \in F_{t_2}] \cap \cdots \cap [X_{t_n} \in F_{t_n}]) &= \\ P((X_{t_1}, X_{t_2}, \cdots, X_{t_n}) \in F_{t_1} \times \cdots \times F_{t_n}) &= P\left(\prod_{t \in I} F_t\right) \\ &= \nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) \end{aligned}$$

where $F_t \equiv M_t$ if $t \notin \{t_1 \cdots t_n\}$.

Finally consider the claim about the integrals. Suppose $f(x_{t_1}, \cdots, x_{t_n}) = \mathcal{X}_F$ where F is a Borel set of $\prod_{t \in J} M_t$ where $J = \{t_1, \cdots, t_n\}$. To begin with suppose

$$F = F_{t_1} \times \cdots \times F_{t_n} \tag{11.3.4}$$

where each F_{t_j} is in $\mathcal{A}_{t_j}$. Then

$$\begin{aligned} \int_{M_{t_1} \times \cdots \times M_{t_n}} \mathcal{X}_F(x_{t_1}, \cdots, x_{t_n}) d\nu_{t_1 \cdots t_n} &= \nu_{t_1 \cdots t_n}(F_{t_1} \times \cdots \times F_{t_n}) \\ &= P\left(\prod_{t \in I} F_t\right) = \int_{\prod_{t \in I} M_t} \mathcal{X}_{\prod_{t \in I} F_t}(\mathbf{x}) dP \\ &= \int_{\prod_{t \in I} M_t} \mathcal{X}_F(x_{t_1}, \cdots, x_{t_n}) dP \end{aligned} \tag{11.3.5}$$

where $F_t = M_t$ if $t \notin J$. Let $\mathcal{K}$ denote sets, F of the sort in 11.3.4. It is clearly a π system. Now let $\mathcal{G}$ denote those sets, F in $\mathcal{B}(\prod_{t \in J} M_t)$ such that 11.3.5 holds. Thus $\mathcal{G} \supseteq \mathcal{K}$. It is clear that $\mathcal{G}$ is closed with respect to countable disjoint unions and complements. Hence $\mathcal{G} \supseteq \sigma(\mathcal{K})$ but $\sigma(\mathcal{K}) = \mathcal{B}(\prod_{t \in J} M_t)$ because every open set in $\prod_{t \in J} M_t$ is the countable union of rectangles like 11.3.4. Therefore, 11.3.5 holds for every $F \in \mathcal{B}(\prod_{t \in J} M_t)$.

Passing to simple functions and then using the monotone convergence theorem yields the final claim of the theorem. This proves the theorem.

The next task is to consider the case where $M_t = (-\infty, \infty)^{n_t}$. To consider this case, here is a lemma which will allow this case to be deduced from the above theorem. In this lemma,

$$M'_t \equiv [-\infty, \infty]^{n_t}.$$

Lemma 11.3.4 *Let J be a finite subset of I . Then $\mathbf{U}$ is a Borel set in $\prod_{t \in J} M_t$ if and only if there exists a Borel set, $\mathbf{U}'$ in $\prod_{t \in J} M'_t$ such that $\mathbf{U} = \mathbf{U}' \cap \prod_{t \in J} M_t$.*

Proof: First suppose Borel is replaced with open. If $\mathbf{U}$ is an open set in $\prod_{t \in J} M_t$, then it is open in $\prod_{t \in J} M'_t$ from the definition of the open sets of $\prod_{t \in J} M'_t$ which consist of open sets of $\prod_{t \in J} M_t$ along with complements of compact sets which have the points $\pm\infty$ added in. Thus you can let $\mathbf{U}' = \mathbf{U}$ in this case. Next suppose $\mathbf{U} = \mathbf{U}' \cap \prod_{t \in J} M_t$ where $\mathbf{U}'$ is open in $\prod_{t \in J} M'_t$. I need show $\mathbf{U}$ is open in $\prod_{t \in J} M_t$. Letting $\mathbf{x} \in \mathbf{U}$, it follows $x_t \neq \pm\infty$ for each t . It follows from $\mathbf{U}'$ open in $\prod_{t \in J} M'_t$ that $x_t \in V'_t$ where V'_t is open in M'_t and

$$\mathbf{x} \in \prod_{t \in J} V'_t \subseteq \mathbf{U}'$$

Letting $V_t = V'_t \setminus \{\infty, -\infty\}$, it follows V_t is open in M_t and

$$\mathbf{x} \in \prod_{t \in J} V_t \subseteq \mathbf{U}' \cap \prod_{t \in J} M_t.$$

This shows that $\mathbf{U}$ is an open set in $\prod_{t \in J} M_t$.

Now let

$$\mathcal{G} \equiv \left\{ \mathbf{F} \text{ Borel in } \prod_{t \in J} M'_t \text{ such that } \mathbf{F} \cap \prod_{t \in J} M_t \text{ is Borel in } \prod_{t \in J} M_t \right\}$$

then from what was just shown $\mathcal{G}$ contains the open sets. It is also clearly a σ algebra. Hence $\mathcal{G}$ equals the Borel sets of $\prod_{t \in J} M'_t$.

It only remains to verify that any Borel set in $\prod_{t \in J} M_t$ is the intersection of a Borel set of $\prod_{t \in J} M'_t$ with $\prod_{t \in J} M_t$. Let

$$\mathcal{H} \equiv \left\{ \mathbf{F} \text{ Borel in } \prod_{t \in J} M_t \text{ such that } \mathbf{F} = \mathbf{F}' \cap \prod_{t \in J} M_t, \mathbf{F}' \text{ Borel in } \prod_{t \in J} M'_t \right\}$$

From the first part of the argument, $\mathcal{H}$ contains the open sets. Now let $\{\mathbf{F}_n\}$ be a sequence in $\mathcal{H}$. Thus $\mathbf{F}_n = \mathbf{F}'_n \cap \prod_{t \in J} M_t$ where $\mathbf{F}'_n$ is Borel in $\prod_{t \in J} M'_t$. Then $\cup_n \mathbf{F}_n = \cup_n \mathbf{F}'_n \cap \prod_{t \in J} M_t$ and $\cup_n \mathbf{F}'_n$ is Borel in $\prod_{t \in J} M'_t$. Thus $\mathcal{H}$ is closed under countable unions. Next let $\mathbf{F} \in \mathcal{H}$ and $\mathbf{F} = \mathbf{F}' \cap \prod_{t \in J} M_t$. Then $\mathbf{F}'^C \equiv \prod_{t \in J} M'_t \setminus \mathbf{F}'$ is Borel in $\prod_{t \in J} M'_t$ and $(\prod_{t \in J} M_t \setminus \mathbf{F}) = \mathbf{F}'^C \cap \prod_{t \in J} M_t$. Thus $\mathcal{H}$ is a σ algebra containing the open sets and so $\mathcal{H}$ equals the Borel sets in $\prod_{t \in J} M_t$. This proves the lemma.

Now here is the Kolmogorov extension theorem.

Theorem 11.3.5 (*Kolmogorov extension theorem*) *For each finite set*

$$J = (t_1, \dots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \dots t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ for $M_t = \mathbb{R}^{n_t}$ for n_t an integer, such that the following consistency condition holds. If

$$(t_1, \dots, t_n) \subseteq (s_1, \dots, s_p),$$

then

$$\nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) = \nu_{s_1 \dots s_p}(G_{s_1} \times \dots \times G_{s_p}) \quad (11.3.6)$$

where if $s_i = t_j$, then $G_{s_i} = F_{t_j}$ and if s_i is not equal to any of the indices, t_k , then $G_{s_i} = M_{s_i}$. Then for $\mathcal{E}$ defined as in Definition 11.3.1, adjusted so that $\pm\infty$ never appears as any endpoint of any interval, there exists a probability measure, P and a σ algebra $\mathcal{F} = \sigma(\mathcal{E})$ such that

$$\left(\prod_{t \in I} M_t, P, \mathcal{F} \right)$$

is a probability space. Also there exist measurable functions, $X_s : \prod_{t \in I} M_t \rightarrow M_s$ defined as

$$X_s \mathbf{x} \equiv x_s$$

for each $s \in I$ such that for each $(t_1 \dots t_n) \subseteq I$,

$$\begin{aligned} \nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) &= P([X_{t_1} \in F_{t_1}] \cap \dots \cap [X_{t_n} \in F_{t_n}]) \\ &= P\left((X_{t_1}, \dots, X_{t_n}) \in \prod_{j=1}^n F_{t_j}\right) = P\left(\prod_{t \in I} F_t\right) \end{aligned} \quad (11.3.7)$$

where $F_t = M_t$ for every $t \notin \{t_1 \dots t_n\}$ and F_{t_i} is a Borel set. Also if f is a non-negative function of finitely many variables, $x_{t_1}, \dots, x_{t_n}$, measurable with respect to $\mathcal{B}\left(\prod_{j=1}^n M_{t_j}\right)$, then f is also measurable with respect to $\mathcal{F}$ and

$$\begin{aligned} &\int_{M_{t_1} \times \dots \times M_{t_n}} f(x_{t_1}, \dots, x_{t_n}) d\nu_{t_1 \dots t_n} \\ &= \int_{\prod_{t \in I} M_t} f(x_{t_1}, \dots, x_{t_n}) dP \end{aligned} \quad (11.3.8)$$

Proof: Using Lemma 11.3.4, extend each measure, ν_J to M'_t , defined by adding in the points $\pm\infty$ at the ends, by letting $\nu_J(\mathbf{E}) \equiv \nu_J(\mathbf{E} \cap \prod_{t \in I} M_t)$ for all $\mathbf{E} \in \mathcal{B}(\prod_{t \in I} M'_t)$. Then apply Theorem 11.3.3 to these extended measures and use the definition of the extensions of each ν_J to replace each M'_t with M_t everywhere it occurs. This proves the theorem.

As a special case, you can obtain a version of product measure for possibly infinitely many factors. Suppose in the context of the above theorem that ν_t is a probability measure defined on the Borel sets of $M_t \equiv \mathbb{R}^{n_t}$ for n_t a positive integer, and let the measures, $\nu_{t_1 \dots t_n}$ be defined on the Borel sets of $\prod_{i=1}^n M_{t_i}$ by

$$\nu_{t_1 \dots t_n}(\mathbf{E}) \equiv (\nu_{t_1} \times \dots \times \nu_{t_n})(\mathbf{E}).$$

Then these measures satisfy the necessary consistency condition and so the Kolmogorov extension theorem given above can be applied to obtain a measure, P defined on a $(\prod_{t \in I} M_t, \mathcal{F})$ and measurable functions $X_s : \prod_{t \in I} M_t \rightarrow M_s$ such that for F_{t_i} a Borel set in M_{t_i} ,

$$\begin{aligned} P\left((X_{t_1}, \dots, X_{t_n}) \in \prod_{i=1}^n F_{t_i}\right) &= \nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) \\ &= \nu_{t_1}(F_{t_1}) \dots \nu_{t_n}(F_{t_n}). \end{aligned} \quad (11.3.9)$$

In particular, $P(X_t \in F_t) = \nu_t(F_t)$. Then P in the resulting probability space,

$$\left(\prod_{t \in I} M_t, \mathcal{F}, P\right)$$

will be denoted as $\prod_{t \in I} \nu_t$. This proves the following theorem which describes an infinite product measure.

Theorem 11.3.6 *Let M_t for $t \in I$ be given as in Theorem 11.3.5 and let ν_t be a Borel probability measure defined on the Borel sets of M_t . Then there exists a measure, P and a σ algebra $\mathcal{F} = \sigma(\mathcal{E})$ where $\mathcal{E}$ is given in Definition 11.3.1 such that $(\prod_t M_t, \mathcal{F}, P)$ is a probability space satisfying 11.3.9 whenever each F_{t_i} is a Borel set of M_{t_i} . This probability measure is sometimes denoted as $\prod_t \nu_t$.*

11.4 Exercises

1. Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be two finite measure spaces. A subset of $X \times Y$ is called a measurable rectangle if it is of the form $A \times B$ where $A \in \mathcal{S}$ and $B \in \mathcal{F}$. A subset of $X \times Y$ is called an elementary set if it is a finite disjoint union of measurable rectangles. Denote this set of functions by $\mathcal{E}$. Show that $\mathcal{E}$ is an algebra of sets.

2. †For $A \in \sigma(\mathcal{E})$, the smallest σ algebra containing $\mathcal{E}$, show that $x \rightarrow \mathcal{X}_A(x, y)$ is μ measurable and that

$$y \rightarrow \int \mathcal{X}_A(x, y) d\mu$$

is λ measurable. Show similar assertions hold for $y \rightarrow \mathcal{X}_A(x, y)$ and

$$x \rightarrow \int \mathcal{X}_A(x, y) d\lambda$$

and that

$$\int \int \mathcal{X}_A(x, y) d\mu d\lambda = \int \int \mathcal{X}_A(x, y) d\lambda d\mu. \quad (11.4.10)$$

Hint: Let $\mathcal{M} \equiv \{A \in \sigma(\mathcal{E}) : 11.4.10 \text{ holds}\}$ along with all relevant measurability assertions. Show $\mathcal{M}$ contains $\mathcal{E}$ and is a monotone class. Then apply the Theorem 9.9.5.

3. †For $A \in \sigma(\mathcal{E})$ define $(\mu \times \lambda)(A) \equiv \int \int \mathcal{X}_A(x, y) d\mu d\lambda$. Show that $(\mu \times \lambda)$ is a measure on $\sigma(\mathcal{E})$ and that whenever $f \geq 0$ is measurable with respect to $\sigma(\mathcal{E})$,

$$\int_{X \times Y} f d(\mu \times \lambda) = \int \int f(x, y) d\mu d\lambda = \int \int f(x, y) d\lambda d\mu.$$

This is a common approach to Fubini's theorem.

4. †Generalize the above version of Fubini's theorem to the case where the measure spaces are only σ finite.
5. †Suppose now that μ and λ are both complete σ finite measures. Let $\overline{(\mu \times \lambda)}$ denote the completion of this measure. Let the larger measure space be $(X \times Y, \overline{\sigma(\mathcal{E})}, \overline{(\mu \times \lambda)})$. Thus if $E \in \overline{\sigma(\mathcal{E})}$, it follows there exists a set $A \in \sigma(\mathcal{E})$ such that $E \cup N = A$ where $\overline{(\mu \times \lambda)}(N) = 0$. Now argue that for λ a.e. y , $x \rightarrow \mathcal{X}_N(x, y)$ is measurable because it is equal to zero μ a.e. and μ is complete. Therefore,

$$\int \int \mathcal{X}_N(x, y) d\mu d\lambda$$

makes sense and equals zero. Use to argue that for λ a.e. y , $x \rightarrow \mathcal{X}_E(x, y)$ is μ measurable and equals $\int \mathcal{X}_A(x, y) d\mu$. Then by completeness of λ , $y \rightarrow \int \mathcal{X}_E(x, y) d\mu$ is λ measurable and

$$\int \int \mathcal{X}_A(x, y) d\mu d\lambda = \int \int \mathcal{X}_E(x, y) d\mu d\lambda = \overline{(\mu \times \lambda)}(E).$$

Similarly

$$\int \int \mathcal{X}_E(x, y) d\lambda d\mu = \overline{(\mu \times \lambda)}(E).$$

Use this to give a generalization of the above Fubini theorem. Prove that if f is measurable with respect to the σ algebra, $\overline{\sigma(\mathcal{E})}$ and nonnegative, then

$$\int_{X \times Y} f d(\overline{\mu \times \lambda}) = \int \int f(x, y) d\mu d\lambda = \int \int f(x, y) d\lambda d\mu$$

where the iterated integrals make sense.

The L^p Spaces

12.1 Basic Inequalities And Properties

One of the main applications of the Lebesgue integral is to the study of various sorts of functions space. These are vector spaces whose elements are functions of various types. One of the most important examples of a function space is the space of measurable functions whose absolute values are p^{th} power integrable where $p \geq 1$. These spaces, referred to as L^p spaces, are very useful in applications. In the chapter $(\Omega, \mathcal{S}, \mu)$ will be a measure space.

Definition 12.1.1 Let $1 \leq p < \infty$. Define

$$L^p(\Omega) \equiv \{f : f \text{ is measurable and } \int_{\Omega} |f(\omega)|^p d\mu < \infty\}$$

In terms of the distribution function,

$$L^p(\Omega) = \{f : f \text{ is measurable and } \int_0^{\infty} pt^{p-1} \mu(|f| > t) dt < \infty\}$$

For each $p > 1$ define q by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Often one uses p' instead of q in this context.

$L^p(\Omega)$ is a vector space and has a norm. This is similar to the situation for $\mathbb{R}^n$ but the proof requires the following fundamental inequality.

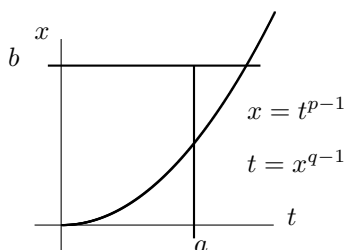
Theorem 12.1.2 (Holder's inequality) If f and g are measurable functions, then if $p > 1$,

$$\int |f| |g| d\mu \leq \left(\int |f|^p d\mu \right)^{\frac{1}{p}} \left(\int |g|^q d\mu \right)^{\frac{1}{q}}. \quad (12.1.1)$$

Proof: First here is a proof of Young's inequality .

Lemma 12.1.3 If $p > 1$, and $0 \leq a, b$ then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Proof: Consider the following picture:



From this picture, the sum of the area between the x axis and the curve added to the area between the t axis and the curve is at least as large as ab . Using beginning calculus, this is equivalent to the following inequality.

$$ab \leq \int_0^a t^{p-1} dt + \int_0^b x^{q-1} dx = \frac{a^p}{p} + \frac{b^q}{q}.$$

The above picture represents the situation which occurs when $p > 2$ because the graph of the function is concave up. If $2 \geq p > 1$ the graph would be concave down or a straight line. You should verify that the same argument holds in these cases just as well. In fact, the only thing which matters in the above inequality is that the function $x = t^{p-1}$ be strictly increasing.

Note equality occurs when $a^p = b^q$.

Here is an alternate proof.

Lemma 12.1.4 For $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

and equality occurs when if and only if $a^p = b^q$.

Proof: If $b = 0$, the inequality is obvious. Fix $b > 0$ and consider

$$f(a) \equiv \frac{a^p}{p} + \frac{b^q}{q} - ab.$$

Then $f'(a) = a^{p-1} - b$. This is negative when $a < b^{1/(p-1)}$ and is positive when $a > b^{1/(p-1)}$. Therefore, f has a minimum when $a = b^{1/(p-1)}$. In other words, when $a^p = b^{p/(p-1)} = b^q$ since $1/p + 1/q = 1$. Thus the minimum value of f is

$$\frac{b^q}{p} + \frac{b^q}{q} - b^{1/(p-1)}b = b^q - b^q = 0.$$

It follows $f \geq 0$ and this yields the desired inequality.

Proof of Holder's inequality: If either $\int |f|^p d\mu$ or $\int |g|^q d\mu$ equals ∞ , the inequality 12.1.1 is obviously valid because $\infty \geq$ anything. If either $\int |f|^p d\mu$ or

$\int |g|^p d\mu$ equals 0, then $f = 0$ a.e. or that $g = 0$ a.e. and so in this case the left side of the inequality equals 0 and so the inequality is therefore true. Therefore assume both $\int |f|^p d\mu$ and $\int |g|^p d\mu$ are less than ∞ and not equal to 0. Let

$$\left(\int |f|^p d\mu \right)^{1/p} = I(f)$$

and let $(\int |g|^p d\mu)^{1/q} = I(g)$. Then using the lemma,

$$\int \frac{|f|}{I(f)} \frac{|g|}{I(g)} d\mu \leq \frac{1}{p} \int \frac{|f|^p}{I(f)^p} d\mu + \frac{1}{q} \int \frac{|g|^q}{I(g)^q} d\mu = 1.$$

Hence,

$$\int |f| |g| d\mu \leq I(f) I(g) = \left(\int |f|^p d\mu \right)^{1/p} \left(\int |g|^q d\mu \right)^{1/q}.$$

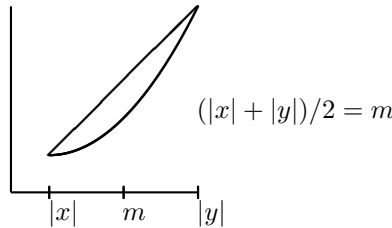
This proves Holder's inequality.

The following lemma will be needed.

Lemma 12.1.5 Suppose $x, y \in \mathbb{C}$. Then

$$|x + y|^p \leq 2^{p-1} (|x|^p + |y|^p).$$

Proof: The function $f(t) = t^p$ is concave up for $t \geq 0$ because $p > 1$. Therefore, the secant line joining two points on the graph of this function must lie above the graph of the function. This is illustrated in the following picture.



Now as shown above,

$$\left(\frac{|x| + |y|}{2} \right)^p \leq \frac{|x|^p + |y|^p}{2}$$

which implies

$$|x + y|^p \leq (|x| + |y|)^p \leq 2^{p-1} (|x|^p + |y|^p)$$

and this proves the lemma.

Note that if $y = \phi(x)$ is any function for which the graph of ϕ is concave up, you could get a similar inequality by the same argument.

Corollary 12.1.6 (*Minkowski inequality*) Let $1 \leq p < \infty$. Then

$$\left(\int |f + g|^p d\mu \right)^{1/p} \leq \left(\int |f|^p d\mu \right)^{1/p} + \left(\int |g|^p d\mu \right)^{1/p}. \quad (12.1.2)$$

Proof: If $p = 1$, this is obvious because it is just the triangle inequality. Let $p > 1$. Without loss of generality, assume

$$\left(\int |f|^p d\mu \right)^{1/p} + \left(\int |g|^p d\mu \right)^{1/p} < \infty$$

and $(\int |f + g|^p d\mu)^{1/p} \neq 0$ or there is nothing to prove. Therefore, using the above lemma,

$$\int |f + g|^p d\mu \leq 2^{p-1} \left(\int |f|^p + |g|^p d\mu \right) < \infty.$$

Now $|f(\omega) + g(\omega)|^p \leq |f(\omega) + g(\omega)|^{p-1} (|f(\omega)| + |g(\omega)|)$. Also, it follows from the definition of p and q that $p - 1 = \frac{p}{q}$. Therefore, using this and Holder's inequality,

$$\begin{aligned} & \int |f + g|^p d\mu \leq \\ & \int |f + g|^{p-1} |f| d\mu + \int |f + g|^{p-1} |g| d\mu \\ & = \int |f + g|^{\frac{p}{q}} |f| d\mu + \int |f + g|^{\frac{p}{q}} |g| d\mu \\ & \leq \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}} \left(\int |f|^p d\mu \right)^{\frac{1}{p}} + \left(\int |f + g|^p d\mu \right)^{\frac{1}{q}} \left(\int |g|^p d\mu \right)^{\frac{1}{p}}. \end{aligned}$$

Dividing both sides by $(\int |f + g|^p d\mu)^{\frac{1}{q}}$ yields 12.1.2. This proves the corollary.

The following follows immediately from the above.

Corollary 12.1.7 Let $f_i \in L^p(\Omega)$ for $i = 1, 2, \dots, n$. Then

$$\left(\int \left| \sum_{i=1}^n f_i \right|^p d\mu \right)^{1/p} \leq \sum_{i=1}^n \left(\int |f_i|^p d\mu \right)^{1/p}.$$

This shows that if $f, g \in L^p$, then $f + g \in L^p$. Also, it is clear that if a is a constant and $f \in L^p$, then $af \in L^p$ because

$$\int |af|^p d\mu = |a|^p \int |f|^p d\mu < \infty.$$

Thus L^p is a vector space and

a.) $(\int |f|^p d\mu)^{1/p} \geq 0$, $(\int |f|^p d\mu)^{1/p} = 0$ if and only if $f = 0$ a.e.

b.) $(\int |af|^p d\mu)^{1/p} = |a| (\int |f|^p d\mu)^{1/p}$ if a is a scalar.

c.) $(\int |f+g|^p d\mu)^{1/p} \leq (\int |f|^p d\mu)^{1/p} + (\int |g|^p d\mu)^{1/p}$.

$f \rightarrow (\int |f|^p d\mu)^{1/p}$ would define a norm if $(\int |f|^p d\mu)^{1/p} = 0$ implied $f = 0$. Unfortunately, this is not so because if $f = 0$ a.e. but is nonzero on a set of measure zero, $(\int |f|^p d\mu)^{1/p} = 0$ and this is not allowed. However, all the other properties of a norm are available and so a little thing like a set of measure zero will not prevent the consideration of L^p as a normed vector space if two functions in L^p which differ only on a set of measure zero are considered the same. That is, an element of L^p is really an equivalence class of functions where two functions are equivalent if they are equal a.e. With this convention, here is a definition.

Definition 12.1.8 Let $f \in L^p(\Omega)$. Define

$$\|f\|_p \equiv \|f\|_{L^p} \equiv \left(\int |f|^p d\mu \right)^{1/p}.$$

Then with this definition and using the convention that elements in L^p are considered to be the same if they differ only on a set of measure zero, $\|\cdot\|_p$ is a norm on $L^p(\Omega)$ because if $\|f\|_p = 0$ then $f = 0$ a.e. and so f is considered to be the zero function because it differs from 0 only on a set of measure zero.

The following is an important definition.

Definition 12.1.9 A complete normed linear space is called a Banach¹ space.

L^p is a Banach space. This is the next big theorem.

Theorem 12.1.10 The following hold for $L^p(\Omega)$

a.) $L^p(\Omega)$ is complete.

b.) If $\{f_n\}$ is a Cauchy sequence in $L^p(\Omega)$, then there exists $f \in L^p(\Omega)$ and a subsequence which converges a.e. to $f \in L^p(\Omega)$, and $\|f_n - f\|_p \rightarrow 0$.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $L^p(\Omega)$. This means that for every $\varepsilon > 0$ there exists N such that if $n, m \geq N$, then $\|f_n - f_m\|_p < \varepsilon$. Now select a subsequence as follows. Let n_1 be such that $\|f_n - f_m\|_p < 2^{-1}$ whenever $n, m \geq n_1$.

¹These spaces are named after Stefan Banach, 1892-1945. Banach spaces are the basic item of study in the subject of functional analysis and will be considered later in this book.

There is a recent biography of Banach, R. Kato, *The Life of Stefan Banach*, (A. Kostant and W. Wojcieszki, translators and editors) Birkhauser, Boston (1996). More information on Banach can also be found in a recent short article written by Douglas Henderson who is in the department of chemistry and biochemistry at BYU.

Banach was born in Austria, worked in Poland and died in the Ukraine but never moved. This is because borders kept changing. There is a rumor that he died in a German concentration camp which is apparently not true. It seems he died after the war of lung cancer.

He was an interesting character. He hated taking examinations so much that he did not receive his undergraduate university degree. Nevertheless, he did become a professor of mathematics due to his important research. He and some friends would meet in a cafe called the Scottish cafe where they wrote on the marble table tops until Banach's wife supplied them with a notebook which became the "Scottish notebook" and was eventually published.

Let n_2 be such that $n_2 > n_1$ and $\|f_n - f_m\|_p < 2^{-2}$ whenever $n, m \geq n_2$. If $n_1, \dots, n_k$ have been chosen, let $n_{k+1} > n_k$ and whenever $n, m \geq n_{k+1}$, $\|f_n - f_m\|_p < 2^{-(k+1)}$. The subsequence just mentioned is $\{f_{n_k}\}$. Thus, $\|f_{n_k} - f_{n_{k+1}}\|_p < 2^{-k}$. Let

$$g_{k+1} = f_{n_{k+1}} - f_{n_k}.$$

Then by the corollary to Minkowski's inequality,

$$\infty > \sum_{k=1}^{\infty} \|g_{k+1}\|_p \geq \sum_{k=1}^m \|g_{k+1}\|_p \geq \left\| \sum_{k=1}^m |g_{k+1}| \right\|_p$$

for all m . It follows that

$$\int \left(\sum_{k=1}^m |g_{k+1}| \right)^p d\mu \leq \left(\sum_{k=1}^{\infty} \|g_{k+1}\|_p \right)^p < \infty \quad (12.1.3)$$

for all m and so the monotone convergence theorem implies that the sum up to m in 12.1.3 can be replaced by a sum up to ∞ . Thus,

$$\int \left(\sum_{k=1}^{\infty} |g_{k+1}| \right)^p d\mu < \infty$$

which requires

$$\sum_{k=1}^{\infty} |g_{k+1}(x)| < \infty \text{ a.e. } x.$$

Therefore, $\sum_{k=1}^{\infty} g_{k+1}(x)$ converges for a.e. x because the functions have values in a complete space, $\mathbb{C}$, and this shows the partial sums form a Cauchy sequence. Now let x be such that this sum is finite. Then define

$$f(x) \equiv f_{n_1}(x) + \sum_{k=1}^{\infty} g_{k+1}(x) = \lim_{m \rightarrow \infty} f_{n_m}(x)$$

since $\sum_{k=1}^m g_{k+1}(x) = f_{n_{m+1}}(x) - f_{n_1}(x)$. Therefore there exists a set, E having measure zero such that

$$\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x)$$

for all $x \notin E$. Redefine f_{n_k} to equal 0 on E and let $f(x) = 0$ for $x \in E$. It then follows that $\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x)$ for all x . By Fatou's lemma, and the Minkowski inequality,

$$\begin{aligned} \|f - f_{n_k}\|_p &= \left(\int |f - f_{n_k}|^p d\mu \right)^{1/p} \leq \\ \liminf_{m \rightarrow \infty} \left(\int |f_{n_m} - f_{n_k}|^p d\mu \right)^{1/p} &= \liminf_{m \rightarrow \infty} \|f_{n_m} - f_{n_k}\|_p \leq \end{aligned}$$

$$\lim_{m \rightarrow \infty} \inf \sum_{j=k}^{m-1} \|f_{n_{j+1}} - f_{n_j}\|_p \leq \sum_{i=k}^{\infty} \|f_{n_{i+1}} - f_{n_i}\|_p \leq 2^{-(k-1)}. \quad (12.1.4)$$

Therefore, $f \in L^p(\Omega)$ because

$$\|f\|_p \leq \|f - f_{n_k}\|_p + \|f_{n_k}\|_p < \infty,$$

and $\lim_{k \rightarrow \infty} \|f_{n_k} - f\|_p = 0$. This proves b.).

This has shown f_{n_k} converges to f in $L^p(\Omega)$. It follows the original Cauchy sequence also converges to f in $L^p(\Omega)$. This is a general fact that if a subsequence of a Cauchy sequence converges, then so does the original Cauchy sequence. You should give a proof of this. This proves the theorem.

In working with the L^p spaces, the following inequality also known as Minkowski's inequality is very useful. It is similar to the Minkowski inequality for sums. To see this, replace the integral, $\int_X$ with a finite summation sign and you will see the usual Minkowski inequality or rather the version of it given in Corollary 12.1.7.

To prove this theorem first consider a special case of it in which technical considerations which shed no light on the proof are excluded.

Lemma 12.1.11 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be finite complete measure spaces and let f be $\overline{\mu \times \lambda}$ measurable and uniformly bounded. Then the following inequality is valid for $p \geq 1$.*

$$\int_X \left(\int_Y |f(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \geq \left(\int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}}. \quad (12.1.5)$$

Proof: Since f is bounded and $\mu(X), \lambda(Y) < \infty$,

$$\left(\int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}} < \infty.$$

Let

$$J(y) = \int_X |f(x, y)| d\mu.$$

Note there is no problem in writing this for a.e. y because f is product measurable. Then by Fubini's theorem,

$$\begin{aligned} \int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda &= \int_Y J(y)^{p-1} \int_X |f(x, y)| d\mu d\lambda \\ &= \int_X \int_Y J(y)^{p-1} |f(x, y)| d\lambda d\mu \end{aligned}$$

Now apply Holder's inequality in the last integral above and recall $p - 1 = \frac{p}{q}$. This yields

$$\begin{aligned}
 & \int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda \\
 & \leq \int_X \left(\int_Y J(y)^p d\lambda \right)^{\frac{1}{q}} \left(\int_Y |f(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \\
 & = \left(\int_Y J(y)^p d\lambda \right)^{\frac{1}{q}} \int_X \left(\int_Y |f(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \\
 & = \left(\int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda \right)^{\frac{1}{q}} \int_X \left(\int_Y |f(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu. \tag{12.1.6}
 \end{aligned}$$

Therefore, dividing both sides by the first factor in the above expression,

$$\left(\int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |f(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu. \tag{12.1.7}$$

Note that 12.1.7 holds even if the first factor of 12.1.6 equals zero. This proves the lemma.

Now consider the case where f is not assumed to be bounded and where the measure spaces are σ finite.

Theorem 12.1.12 *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{F}, \lambda)$ be σ -finite measure spaces and let f be product measurable. Then the following inequality is valid for $p \geq 1$.*

$$\int_X \left(\int_Y |f(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \geq \left(\int_Y \left(\int_X |f(x, y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}}. \tag{12.1.8}$$

Proof: Since the two measure spaces are σ finite, there exist measurable sets, X_m and Y_k such that $X_m \subseteq X_{m+1}$ for all m , $Y_k \subseteq Y_{k+1}$ for all k , and $\mu(X_m), \lambda(Y_k) < \infty$. Now define

$$f_n(x, y) \equiv \begin{cases} f(x, y) & \text{if } |f(x, y)| \leq n \\ n & \text{if } |f(x, y)| > n. \end{cases}$$

Thus f_n is uniformly bounded and product measurable. By the above lemma,

$$\int_{X_m} \left(\int_{Y_k} |f_n(x, y)|^p d\lambda \right)^{\frac{1}{p}} d\mu \geq \left(\int_{Y_k} \left(\int_{X_m} |f_n(x, y)| d\mu \right)^p d\lambda \right)^{\frac{1}{p}}. \tag{12.1.9}$$

Now observe that $|f_n(x, y)|$ increases in n and the pointwise limit is $|f(x, y)|$. Therefore, using the monotone convergence theorem in 12.1.9 yields the same inequality with f replacing f_n . Next let $k \rightarrow \infty$ and use the monotone convergence theorem again to replace Y_k with Y . Finally let $m \rightarrow \infty$ in what is left to obtain 12.1.8. This proves the theorem.

Note that the proof of this theorem depends on two manipulations, the interchange of the order of integration and Holder's inequality. Note that there is nothing to check in the case of double sums. Thus if $a_{ij} \geq 0$, it is always the case that

$$\left(\sum_j \left(\sum_i a_{ij} \right)^p \right)^{1/p} \leq \sum_i \left(\sum_j a_{ij}^p \right)^{1/p}$$

because the integrals in this case are just sums and $(i, j) \rightarrow a_{ij}$ is measurable.

The L^p spaces have many important properties.

12.2 Density Considerations

Theorem 12.2.1 *Let $p \geq 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a measure space. Then the simple functions are dense in $L^p(\Omega)$.*

Proof: Recall that a function, f , having values in $\mathbb{R}$ can be written in the form $f = f^+ - f^-$ where

$$f^+ = \max(0, f), \quad f^- = \max(0, -f).$$

Therefore, an arbitrary complex valued function, f is of the form

$$f = \operatorname{Re} f^+ - \operatorname{Re} f^- + i(\operatorname{Im} f^+ - \operatorname{Im} f^-).$$

If each of these nonnegative functions is approximated by a simple function, it follows f is also approximated by a simple function. Therefore, there is no loss of generality in assuming at the outset that $f \geq 0$.

Since f is measurable, Theorem 8.3.9 implies there is an increasing sequence of simple functions, $\{s_n\}$, converging pointwise to $f(x)$. Now

$$|f(x) - s_n(x)| \leq |f(x)|.$$

By the Dominated Convergence theorem,

$$0 = \lim_{n \rightarrow \infty} \int |f(x) - s_n(x)|^p d\mu.$$

Thus simple functions are dense in L^p .

Recall that for Ω a topological space, $C_c(\Omega)$ is the space of continuous functions with compact support in Ω . Also recall the following definition.

Definition 12.2.2 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose (Ω, τ) is also a topological space. Then $(\Omega, \mathcal{S}, \mu)$ is called a regular measure space if the σ algebra of Borel sets is contained in $\mathcal{S}$ and for all $E \in \mathcal{S}$,*

$$\mu(E) = \inf\{\mu(V) : V \supseteq E \text{ and } V \text{ open}\}$$

and if $\mu(E) < \infty$,

$$\mu(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}$$

and $\mu(K) < \infty$ for any compact set, K .

For example Lebesgue measure is an example of such a measure.

Lemma 12.2.3 *Let Ω be a metric space in which the closed balls are compact and let K be a compact subset of V , an open set. Then there exists a continuous function $f : \Omega \rightarrow [0, 1]$ such that $f(x) = 1$ for all $x \in K$ and $\text{spt}(f)$ is a compact subset of V . That is, $K \prec f \prec V$.*

Proof: Let $K \subseteq W \subseteq \overline{W} \subseteq V$ and $\overline{W}$ is compact. To obtain this list of inclusions consider a point in K , x , and take $B(x, r_x)$ a ball containing x such that $\overline{B(x, r_x)}$ is a compact subset of V . Next use the fact that K is compact to obtain the existence of a list, $\{B(x_i, r_{x_i}/2)\}_{i=1}^m$ which covers K . Then let

$$W \equiv \cup_{i=1}^m B\left(x_i, \frac{r_{x_i}}{2}\right).$$

It follows since this is a finite union that

$$\overline{W} = \overline{\cup_{i=1}^m B\left(x_i, \frac{r_{x_i}}{2}\right)}$$

and so $\overline{W}$, being a finite union of compact sets is itself a compact set. Also, from the construction

$$\overline{W} \subseteq \cup_{i=1}^m B(x_i, r_{x_i}).$$

Define f by

$$f(x) = \frac{\text{dist}(x, W^C)}{\text{dist}(x, K) + \text{dist}(x, W^C)}.$$

It is clear that f is continuous if the denominator is always nonzero. But this is clear because if $x \in W^C$ there must be a ball $B(x, r)$ such that this ball does not intersect K . Otherwise, x would be a limit point of K and since K is closed, $x \in K$. However, $x \notin K$ because $K \subseteq W$.

It is not necessary to be in a metric space to do this. You can accomplish the same thing using Urysohn's lemma.

Theorem 12.2.4 *Let $(\Omega, \mathcal{S}, \mu)$ be a regular measure space as in Definition 12.2.2 where the conclusion of Lemma 12.2.3 holds. Then $C_c(\Omega)$ is dense in $L^p(\Omega)$.*

Proof: First consider a measurable set, E where $\mu(E) < \infty$. Let $K \subseteq E \subseteq V$ where $\mu(V \setminus K) < \varepsilon$. Now let $K \prec h \prec V$. Then

$$\int |h - \chi_E|^p d\mu \leq \int \chi_{V \setminus K}^p d\mu = \mu(V \setminus K) < \varepsilon.$$

It follows that for each s a simple function in $L^p(\Omega)$, there exists $h \in C_c(\Omega)$ such that $\|s - h\|_p < \varepsilon$. This is because if

$$s(x) = \sum_{i=1}^m c_i \chi_{E_i}(x)$$

is a simple function in L^p where the c_i are the distinct nonzero values of s each $\mu(E_i) < \infty$ since otherwise $s \notin L^p$ due to the inequality

$$\int |s|^p d\mu \geq |c_i|^p \mu(E_i).$$

By Theorem 12.2.1, simple functions are dense in $L^p(\Omega)$, and so this proves the Theorem.

12.3 Separability

Theorem 12.3.1 *For $p \geq 1$ and μ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $\|f - g\|_p < \varepsilon$.*

Proof: Let Q be all functions of the form $c\chi_{[\mathbf{a}, \mathbf{b}]}$ where

$$[\mathbf{a}, \mathbf{b}] \equiv [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

and both a_i, b_i are rational, while c has rational real and imaginary parts. Let $\mathcal{D}$ be the set of all finite sums of functions in Q . Thus, $\mathcal{D}$ is countable. In fact $\mathcal{D}$ is dense in $L^p(\mathbb{R}^n, \mu)$. To prove this it is necessary to show that for every $f \in L^p(\mathbb{R}^n, \mu)$, there exists an element of $\mathcal{D}$, s such that $\|s - f\|_p < \varepsilon$. If it can be shown that for every $g \in C_c(\mathbb{R}^n)$ there exists $h \in \mathcal{D}$ such that $\|g - h\|_p < \varepsilon$, then this will suffice because if $f \in L^p(\mathbb{R}^n)$ is arbitrary, Theorem 12.2.4 implies there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p \leq \frac{\varepsilon}{2}$ and then there would exist $h \in C_c(\mathbb{R}^n)$ such that $\|h - g\|_p < \frac{\varepsilon}{2}$. By the triangle inequality,

$$\|f - h\|_p \leq \|h - g\|_p + \|g - f\|_p < \varepsilon.$$

Therefore, assume at the outset that $f \in C_c(\mathbb{R}^n)$.

Let $\mathcal{P}_m$ consist of all sets of the form $[\mathbf{a}, \mathbf{b}] \equiv \prod_{i=1}^n [a_i, b_i]$ where $a_i = j2^{-m}$ and $b_i = (j+1)2^{-m}$ for j an integer. Thus $\mathcal{P}_m$ consists of a tiling of $\mathbb{R}^n$ into half open rectangles having diameters $2^{-m}n^{\frac{1}{2}}$. There are countably many of these rectangles; so, let $\mathcal{P}_m = \{[\mathbf{a}_i, \mathbf{b}_i]\}_{i=1}^\infty$ and $\mathbb{R}^n = \cup_{i=1}^\infty [\mathbf{a}_i, \mathbf{b}_i]$. Let c_i^m be complex numbers with rational real and imaginary parts satisfying

$$\begin{aligned} |f(\mathbf{a}_i) - c_i^m| &< 2^{-m}, \\ |c_i^m| &\leq |f(\mathbf{a}_i)|. \end{aligned} \tag{12.3.10}$$

Let

$$s_m(\mathbf{x}) = \sum_{i=1}^{\infty} c_i^m \mathcal{X}_{[\mathbf{a}_i, \mathbf{b}_i]}(\mathbf{x}).$$

Since $f(\mathbf{a}_i) = 0$ except for finitely many values of i , the above is a finite sum. Then 12.3.10 implies $s_m \in \mathcal{D}$. If s_m converges uniformly to f then it follows $\|s_m - f\|_p \rightarrow 0$ because $|s_m| \leq |f|$ and so

$$\begin{aligned} \|s_m - f\|_p &= \left(\int |s_m - f|^p d\mu \right)^{1/p} \\ &= \left(\int_{\text{spt}(f)} |s_m - f|^p d\mu \right)^{1/p} \\ &\leq [\varepsilon m_n(\text{spt}(f))]^{1/p} \end{aligned}$$

whenever m is large enough.

Since $f \in C_c(\mathbb{R}^n)$ it follows that f is uniformly continuous and so given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{x} - \mathbf{y}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon/2$. Now let m be large enough that every box in $\mathcal{P}_m$ has diameter less than δ and also that $2^{-m} < \varepsilon/2$. Then if $[\mathbf{a}_i, \mathbf{b}_i]$ is one of these boxes of $\mathcal{P}_m$, and $\mathbf{x} \in [\mathbf{a}_i, \mathbf{b}_i]$,

$$|f(\mathbf{x}) - f(\mathbf{a}_i)| < \varepsilon/2$$

and

$$|f(\mathbf{a}_i) - c_i^m| < 2^{-m} < \varepsilon/2.$$

Therefore, using the triangle inequality, it follows that

$$|f(\mathbf{x}) - c_i^m| = |s_m(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$$

and since $\mathbf{x}$ is arbitrary, this establishes uniform convergence. This proves the theorem.

Here is an easier proof if you know the Weierstrass approximation theorem.

Theorem 12.3.2 For $p \geq 1$ and μ a Radon measure, $L^p(\mathbb{R}^n, \mu)$ is separable. Recall this means there exists a countable set, $\mathcal{D}$, such that if $f \in L^p(\mathbb{R}^n, \mu)$ and $\varepsilon > 0$, there exists $g \in \mathcal{D}$ such that $\|f - g\|_p < \varepsilon$.

Proof: Let $\mathcal{P}$ denote the set of all polynomials which have rational coefficients. Then $\mathcal{P}$ is countable. Let $\tau_k \in C_c((- (k+1), (k+1))^n)$ such that $[-k, k]^n \prec \tau_k \prec (- (k+1), (k+1))^n$. Let $\mathcal{D}_k$ denote the functions which are of the form, $p\tau_k$ where $p \in \mathcal{P}$. Thus $\mathcal{D}_k$ is also countable. Let $\mathcal{D} \equiv \cup_{k=1}^{\infty} \mathcal{D}_k$. It follows each function in $\mathcal{D}$ is in $C_c(\mathbb{R}^n)$ and so it is in $L^p(\mathbb{R}^n, \mu)$. Let $f \in L^p(\mathbb{R}^n, \mu)$. By regularity of μ there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_{L^p(\mathbb{R}^n, \mu)} < \frac{\varepsilon}{3}$. Let k be such that $\text{spt}(g) \subseteq (-k, k)^n$. Now by the Weierstrass approximation theorem there exists a polynomial q such that

$$\begin{aligned} \|g - q\|_{[-(k+1), k+1]^n} &\equiv \sup \{|g(\mathbf{x}) - q(\mathbf{x})| : \mathbf{x} \in [-(k+1), (k+1)]^n\} \\ &< \frac{\varepsilon}{3\mu((- (k+1), k+1)^n)}. \end{aligned}$$

It follows

$$\begin{aligned} \|g - \tau_k q\|_{[-(k+1), k+1]^n} &= \|\tau_k g - \tau_k q\|_{[-(k+1), k+1]^n} \\ &< \frac{\varepsilon}{3\mu(-(k+1), k+1)^n}. \end{aligned}$$

Without loss of generality, it can be assumed this polynomial has all rational coefficients. Therefore, $\tau_k q \in \mathcal{D}$.

$$\begin{aligned} \|g - \tau_k q\|_{L^p(\mathbb{R}^n)}^p &= \int_{(-(k+1), k+1)^n} |g(\mathbf{x}) - \tau_k(\mathbf{x}) q(\mathbf{x})|^p d\mu \\ &\leq \left(\frac{\varepsilon}{3\mu(-(k+1), k+1)^n} \right)^p \mu(-(k+1), k+1)^n \\ &< \left(\frac{\varepsilon}{3} \right)^p. \end{aligned}$$

It follows

$$\|f - \tau_k q\|_{L^p(\mathbb{R}^n, \mu)} \leq \|f - g\|_{L^p(\mathbb{R}^n, \mu)} + \|g - \tau_k q\|_{L^p(\mathbb{R}^n, \mu)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon.$$

This proves the theorem.

Corollary 12.3.3 *Let Ω be any μ measurable subset of $\mathbb{R}^n$ and let μ be a Radon measure. Then $L^p(\Omega, \mu)$ is separable. Here the σ algebra of measurable sets will consist of all intersections of measurable sets with Ω and the measure will be μ restricted to these sets.*

Proof: Let $\tilde{\mathcal{D}}$ be the restrictions of $\mathcal{D}$ to Ω . If $f \in L^p(\Omega)$, let F be the zero extension of f to all of $\mathbb{R}^n$. Let $\varepsilon > 0$ be given. By Theorem 12.3.1 or 12.3.2 there exists $s \in \mathcal{D}$ such that $\|F - s\|_p < \varepsilon$. Thus

$$\|s - f\|_{L^p(\Omega, \mu)} \leq \|s - F\|_{L^p(\mathbb{R}^n, \mu)} < \varepsilon$$

and so the countable set $\tilde{\mathcal{D}}$ is dense in $L^p(\Omega)$.

12.4 Continuity Of Translation

Definition 12.4.1 *Let f be a function defined on $U \subseteq \mathbb{R}^n$ and let $\mathbf{w} \in \mathbb{R}^n$. Then $f_{\mathbf{w}}$ will be the function defined on $\mathbf{w} + U$ by*

$$f_{\mathbf{w}}(\mathbf{x}) = f(\mathbf{x} - \mathbf{w}).$$

Theorem 12.4.2 *(Continuity of translation in L^p) Let $f \in L^p(\mathbb{R}^n)$ with the measure being Lebesgue measure. Then*

$$\lim_{\|\mathbf{w}\| \rightarrow 0} \|f_{\mathbf{w}} - f\|_p = 0.$$

Proof: Let $\varepsilon > 0$ be given and let $g \in C_c(\mathbb{R}^n)$ with $\|g - f\|_p < \frac{\varepsilon}{3}$. Since Lebesgue measure is translation invariant ($m_n(\mathbf{w} + E) = m_n(E)$),

$$\|g_{\mathbf{w}} - f_{\mathbf{w}}\|_p = \|g - f\|_p < \frac{\varepsilon}{3}.$$

You can see this from looking at simple functions and passing to the limit or you could use the change of variables formula to verify it.

Therefore

$$\begin{aligned} \|f - f_{\mathbf{w}}\|_p &\leq \|f - g\|_p + \|g - g_{\mathbf{w}}\|_p + \|g_{\mathbf{w}} - f_{\mathbf{w}}\|_p \\ &< \frac{2\varepsilon}{3} + \|g - g_{\mathbf{w}}\|_p. \end{aligned} \quad (12.4.11)$$

But $\lim_{|\mathbf{w}| \rightarrow 0} g_{\mathbf{w}}(\mathbf{x}) = g(\mathbf{x})$ uniformly in $\mathbf{x}$ because g is uniformly continuous. Now let B be a large ball containing $\text{spt}(g)$ and let δ_1 be small enough that $B(\mathbf{x}, \delta) \subseteq B$ whenever $\mathbf{x} \in \text{spt}(g)$. If $\varepsilon > 0$ is given there exists $\delta < \delta_1$ such that if $|\mathbf{w}| < \delta$, it follows that $|g(\mathbf{x} - \mathbf{w}) - g(\mathbf{x})| < \varepsilon/3 \left(1 + m_n(B)^{1/p}\right)$. Therefore,

$$\begin{aligned} \|g - g_{\mathbf{w}}\|_p &= \left(\int_B |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{w})|^p dm_n \right)^{1/p} \\ &\leq \varepsilon \frac{m_n(B)^{1/p}}{3 \left(1 + m_n(B)^{1/p}\right)} < \frac{\varepsilon}{3}. \end{aligned}$$

Therefore, whenever $|\mathbf{w}| < \delta$, it follows $\|g - g_{\mathbf{w}}\|_p < \frac{\varepsilon}{3}$ and so from 12.4.11 $\|f - f_{\mathbf{w}}\|_p < \varepsilon$. This proves the theorem.

Part of the argument of this theorem is significant enough to be stated as a corollary.

Corollary 12.4.3 *Suppose $g \in C_c(\mathbb{R}^n)$ and μ is a Radon measure on $\mathbb{R}^n$. Then*

$$\lim_{\mathbf{w} \rightarrow \mathbf{0}} \|g - g_{\mathbf{w}}\|_p = 0.$$

Proof: The proof of this follows from the last part of the above argument simply replacing m_n with μ . Translation invariance of the measure is not needed to draw this conclusion because of uniform continuity of g .

12.5 Mollifiers And Density Of Smooth Functions

Definition 12.5.1 *Let U be an open subset of $\mathbb{R}^n$. $C_c^\infty(U)$ is the vector space of all infinitely differentiable functions which equal zero for all $\mathbf{x}$ outside of some compact set contained in U . Similarly, $C_c^m(U)$ is the vector space of all functions which are m times continuously differentiable and whose support is a compact subset of U .*

Example 12.5.2 Let $U = B(\mathbf{z}, 2r)$

$$\psi(\mathbf{x}) = \begin{cases} \exp \left[\left(|\mathbf{x} - \mathbf{z}|^2 - r^2 \right)^{-1} \right] & \text{if } |\mathbf{x} - \mathbf{z}| < r, \\ 0 & \text{if } |\mathbf{x} - \mathbf{z}| \geq r. \end{cases}$$

Then a little work shows $\psi \in C_c^\infty(U)$. The following also is easily obtained.

Lemma 12.5.3 Let U be any open set. Then $C_c^\infty(U) \neq \emptyset$.

Proof: Pick $\mathbf{z} \in U$ and let r be small enough that $B(\mathbf{z}, 2r) \subseteq U$. Then let $\psi \in C_c^\infty(B(\mathbf{z}, 2r)) \subseteq C_c^\infty(U)$ be the function of the above example.

Definition 12.5.4 Let $U = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < 1\}$. A sequence $\{\psi_m\} \subseteq C_c^\infty(U)$ is called a mollifier (This is sometimes called an approximate identity if the differentiability is not included.) if

$$\psi_m(\mathbf{x}) \geq 0, \psi_m(\mathbf{x}) = 0, \text{ if } |\mathbf{x}| \geq \frac{1}{m},$$

and $\int \psi_m(\mathbf{x}) = 1$. Sometimes it may be written as $\{\psi_\varepsilon\}$ where ψ_ε satisfies the above conditions except $\psi_\varepsilon(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq \varepsilon$. In other words, ε takes the place of $1/m$ and in everything that follows $\varepsilon \rightarrow 0$ instead of $m \rightarrow \infty$.

As before, $\int f(\mathbf{x}, \mathbf{y}) d\mu(\mathbf{y})$ will mean $\mathbf{x}$ is fixed and the function $\mathbf{y} \rightarrow f(\mathbf{x}, \mathbf{y})$ is being integrated. To make the notation more familiar, dx is written instead of $dm_n(x)$.

Example 12.5.5 Let

$$\psi \in C_c^\infty(B(0, 1)) \quad (B(0, 1) = \{\mathbf{x} : |\mathbf{x}| < 1\})$$

with $\psi(\mathbf{x}) \geq 0$ and $\int \psi dm = 1$. Let $\psi_m(\mathbf{x}) = c_m \psi(m\mathbf{x})$ where c_m is chosen in such a way that $\int \psi_m dm = 1$. By the change of variables theorem $c_m = m^n$.

Definition 12.5.6 A function, f , is said to be in $L_{loc}^1(\mathbb{R}^n, \mu)$ if f is μ measurable and if $|f|\chi_K \in L^1(\mathbb{R}^n, \mu)$ for every compact set, K . Here μ is a Radon measure on $\mathbb{R}^n$. Usually $\mu = m_n$, Lebesgue measure. When this is so, write $L_{loc}^1(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$, etc. If $f \in L_{loc}^1(\mathbb{R}^n, \mu)$, and $g \in C_c(\mathbb{R}^n)$,

$$f * g(\mathbf{x}) \equiv \int f(\mathbf{y})g(\mathbf{x} - \mathbf{y})d\mu.$$

The following lemma will be useful in what follows. It says that one of these very irregular functions in $L_{loc}^1(\mathbb{R}^n, \mu)$ is smoothed out by convolving with a mollifier.

Lemma 12.5.7 Let $f \in L_{loc}^1(\mathbb{R}^n, \mu)$, and $g \in C_c^\infty(\mathbb{R}^n)$. Then $f * g$ is an infinitely differentiable function. Here μ is a Radon measure on $\mathbb{R}^n$.

Proof: Consider the difference quotient for calculating a partial derivative of $f * g$.

$$\frac{f * g(\mathbf{x} + t\mathbf{e}_j) - f * g(\mathbf{x})}{t} = \int f(\mathbf{y}) \frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x} - \mathbf{y})}{t} d\mu(y).$$

Using the fact that $g \in C_c^\infty(\mathbb{R}^n)$, the quotient,

$$\frac{g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x} - \mathbf{y})}{t},$$

is uniformly bounded. To see this easily, use Theorem 5.13.4 on Page 125 to get the existence of a constant, M depending on

$$\max \{ \|Dg(\mathbf{x})\| : \mathbf{x} \in \mathbb{R}^n \}$$

such that

$$|g(\mathbf{x} + t\mathbf{e}_j - \mathbf{y}) - g(\mathbf{x} - \mathbf{y})| \leq M|t|$$

for any choice of $\mathbf{x}$ and $\mathbf{y}$. Therefore, there exists a dominating function for the integrand of the above integral which is of the form $C|f(\mathbf{y})|\mathcal{X}_K$ where K is a compact set depending on the support of g . It follows the limit of the difference quotient above passes inside the integral as $t \rightarrow 0$ and

$$\frac{\partial}{\partial x_j} (f * g)(\mathbf{x}) = \int f(\mathbf{y}) \frac{\partial}{\partial x_j} g(\mathbf{x} - \mathbf{y}) d\mu(y).$$

Now letting $\frac{\partial}{\partial x_j} g$ play the role of g in the above argument, partial derivatives of all orders exist. A similar use of the dominated convergence theorem shows all these partial derivatives are also continuous. This proves the lemma.

Theorem 12.5.8 *Let K be a compact subset of an open set, U . Then there exists a function, $h \in C_c^\infty(U)$, such that $h(\mathbf{x}) = 1$ for all $\mathbf{x} \in K$ and $h(\mathbf{x}) \in [0, 1]$ for all $\mathbf{x}$.*

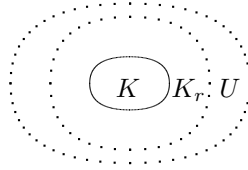
Proof: Let $r > 0$ be small enough that $K + B(\mathbf{0}, 3r) \subseteq U$. The symbol, $K + B(\mathbf{0}, 3r)$ means

$$\{\mathbf{k} + \mathbf{x} : \mathbf{k} \in K \text{ and } \mathbf{x} \in B(\mathbf{0}, 3r)\}.$$

Thus this is simply a way to write

$$\cup \{B(\mathbf{k}, 3r) : \mathbf{k} \in K\}.$$

Think of it as fattening up the set, K . Let $K_r = K + B(0, r)$. A picture of what is happening follows.



Consider $\mathcal{X}_{K_r} * \psi_m$ where ψ_m is a mollifier. Let m be so large that $\frac{1}{m} < r$. Then from the definition of what is meant by a convolution, and using that ψ_m has support in $B(0, \frac{1}{m})$, $\mathcal{X}_{K_r} * \psi_m = 1$ on K and that its support is in $K + B(0, 3r)$. Now using Lemma 12.5.7, $\mathcal{X}_{K_r} * \psi_m$ is also infinitely differentiable. Therefore, let $h = \mathcal{X}_{K_r} * \psi_m$.

The following corollary will be used later.

Corollary 12.5.9 *Let K be a compact set in $\mathbb{R}^n$ and let $\{U_i\}_{i=1}^\infty$ be an open cover of K . Then there exist functions, $\psi_k \in C_c^\infty(U_i)$ such that $\psi_i \prec U_i$ and*

$$\sum_{i=1}^{\infty} \psi_i(\mathbf{x}) = 1.$$

If K_1 is a compact subset of U_1 there exist such functions such that also $\psi_1(\mathbf{x}) = 1$ for all $\mathbf{x} \in K_1$.

Proof: This follows from a repeat of the proof of Theorem 9.3.11 on Page 233, replacing the lemma used in that proof with Theorem 12.5.8.

Theorem 12.5.10 *For each $p \geq 1$, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$. Here the measure is Lebesgue measure.*

Proof: Let $f \in L^p(\mathbb{R}^n)$ and let $\varepsilon > 0$ be given. Choose $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \frac{\varepsilon}{2}$. This can be done by using Theorem 12.2.4. Now let

$$g_m(\mathbf{x}) = g * \psi_m(\mathbf{x}) \equiv \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y}) = \int g(\mathbf{y}) \psi_m(\mathbf{x} - \mathbf{y}) dm_n(\mathbf{y})$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 12.5.7 $g_m \in C_c^\infty(\mathbb{R}^n)$. It vanishes if $\mathbf{x} \notin \text{spt}(g) + B(0, \frac{1}{m})$.

$$\begin{aligned} \|g - g_m\|_p &= \left(\int |g(\mathbf{x}) - \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\ &\leq \left(\int \left(\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \right)^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\ &\leq \int \left(\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \\ &= \int_{B(0, \frac{1}{m})} \|g - g_{\mathbf{y}}\|_p \psi_m(\mathbf{y}) dm_n(\mathbf{y}) < \frac{\varepsilon}{2} \end{aligned}$$

whenever m is large enough. This follows from Corollary 12.4.3. Theorem 12.1.12 was used to obtain the third inequality. There is no measurability problem because the function

$$(\mathbf{x}, \mathbf{y}) \rightarrow |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y})$$

is continuous. Thus when m is large enough,

$$\|f - g_m\|_p \leq \|f - g\|_p + \|g - g_m\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the theorem.

This is a very remarkable result. Functions in $L^p(\mathbb{R}^n)$ don't need to be continuous anywhere and yet every such function is very close in the L^p norm to one which is infinitely differentiable having compact support. The same result holds for $L^p(U)$ for U an open set. This is the next corollary.

Corollary 12.5.11 *Let U be an open set. For each $p \geq 1$, $C_c^\infty(U)$ is dense in $L^p(U)$. Here the measure is Lebesgue measure.*

Proof: Let $f \in L^p(U)$ and let $\varepsilon > 0$ be given. Choose $g \in C_c(U)$ such that $\|f - g\|_p < \frac{\varepsilon}{2}$. This is possible because Lebesgue measure restricted to the open set, U is regular. Thus the existence of such a g follows from Theorem 12.2.4. Now let

$$g_m(\mathbf{x}) = g * \psi_m(\mathbf{x}) \equiv \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y}) = \int g(\mathbf{y}) \psi_m(\mathbf{x} - \mathbf{y}) dm_n(\mathbf{y})$$

where $\{\psi_m\}$ is a mollifier. It follows from Lemma 12.5.7 $g_m \in C_c^\infty(U)$ for all m sufficiently large. It vanishes if $\mathbf{x} \notin \text{spt}(g) + B(0, \frac{1}{m})$. Then

$$\begin{aligned} \|g - g_m\|_p &= \left(\int |g(\mathbf{x}) - \int g(\mathbf{x} - \mathbf{y}) \psi_m(\mathbf{y}) dm_n(\mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\ &\leq \left(\int \left(\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \right)^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \\ &\leq \int \left(\int |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})|^p dm_n(\mathbf{x}) \right)^{\frac{1}{p}} \psi_m(\mathbf{y}) dm_n(\mathbf{y}) \\ &= \int_{B(0, \frac{1}{m})} \|g - g_{\mathbf{y}}\|_p \psi_m(\mathbf{y}) dm_n(\mathbf{y}) < \frac{\varepsilon}{2} \end{aligned}$$

whenever m is large enough. This follows from Corollary 12.4.3. Theorem 12.1.12 was used to obtain the third inequality. There is no measurability problem because the function

$$(\mathbf{x}, \mathbf{y}) \rightarrow |g(\mathbf{x}) - g(\mathbf{x} - \mathbf{y})| \psi_m(\mathbf{y})$$

is continuous. Thus when m is large enough,

$$\|f - g_m\|_p \leq \|f - g\|_p + \|g - g_m\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the corollary.

Another thing should probably be mentioned. If you have had a course in complex analysis, you may be wondering whether these infinitely differentiable functions having compact support have anything to do with analytic functions which also have infinitely many derivatives. The answer is no! Recall that if an analytic function has a limit point in the set of zeros then it is identically equal to zero. Thus these functions in $C_c^\infty(\mathbb{R}^n)$ are not analytic. This is a strictly real analysis phenomenon and has absolutely nothing to do with the theory of functions of a complex variable.

12.6 Exercises

1. Let E be a Lebesgue measurable set in $\mathbb{R}$. Suppose $m(E) > 0$. Consider the set

$$E - E = \{x - y : x \in E, y \in E\}.$$

Show that $E - E$ contains an interval. **Hint:** Let

$$f(x) = \int \chi_E(t) \chi_E(x+t) dt.$$

Note f is continuous at 0 and $f(0) > 0$ and use continuity of translation in L^p .

2. Establish the inequality $\|fg\|_r \leq \|f\|_p \|g\|_q$ whenever $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.
3. Let $(\Omega, \mathcal{S}, \mu)$ be counting measure on $\mathbb{N}$. Thus $\Omega = \mathbb{N}$ and $\mathcal{S} = \mathcal{P}(\mathbb{N})$ with $\mu(S) = \text{number of things in } S$. Let $1 \leq p \leq q$. Show that in this case,

$$L^1(\mathbb{N}) \subseteq L^p(\mathbb{N}) \subseteq L^q(\mathbb{N}).$$

Hint: This is real easy if you consider what $\int_\Omega f d\mu$ equals. How are the norms related?

4. Consider the function, $f(x, y) = \frac{x^{p-1}}{py} + \frac{y^{q-1}}{qx}$ for $x, y > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show directly that $f(x, y) \geq 1$ for all such x, y and show this implies $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.
5. Give an example of a sequence of functions in $L^p(\mathbb{R})$ which converges to zero in L^p but does not converge pointwise to 0. Does this contradict the proof of the theorem that L^p is complete?
6. Let K be a bounded subset of $L^p(\mathbb{R}^n)$ and suppose that there exists G such that $\bar{G}$ is compact with

$$\int_{\mathbb{R}^n \setminus \bar{G}} |u(\mathbf{x})|^p dx < \varepsilon^p$$

and for all $\varepsilon > 0$, there exist a $\delta > 0$ and such that if $|\mathbf{h}| < \delta$, then

$$\int |u(\mathbf{x} + \mathbf{h}) - u(\mathbf{x})|^p dx < \varepsilon^p$$

for all $u \in K$. Show that K is precompact in $L^p(\mathbb{R}^n)$. **Hint:** Let ϕ_k be a mollifier and consider

$$K_k \equiv \{u * \phi_k : u \in K\}.$$

Verify the conditions of the Ascoli Arzela theorem for these functions defined on $\overline{G}$ and show there is an ε net for each $\varepsilon > 0$. Can you modify this to let an arbitrary open set take the place of $\mathbb{R}^n$?

7. Let (Ω, d) be a metric space and suppose also that $(\Omega, \mathcal{S}, \mu)$ is a regular measure space such that $\mu(\Omega) < \infty$ and let $f \in L^1(\Omega)$ where f has complex values. Show that for every $\varepsilon > 0$, there exists an open set of measure less than ε , denoted here by V and a continuous function, g defined on Ω such that $f = g$ on V^C . Thus, aside from a set of small measure, f is continuous. If $|f(\omega)| \leq M$, show that it can be assumed that $|g(\omega)| \leq M$. This is called Lusin's theorem. **Hint:** Use Theorems 12.2.4 and 12.1.10 to obtain a sequence of functions in $C_c(\Omega)$, $\{g_n\}$ which converges pointwise a.e. to f and then use Egoroff's theorem to obtain a small set, W of measure less than $\varepsilon/2$ such that convergence is uniform on W^C . Now let F be a closed subset of W^C such that $\mu(W^C \setminus F) < \varepsilon/2$. Let $V = F^C$. Thus $\mu(V) < \varepsilon$ and on $F = V^C$, the convergence of $\{g_n\}$ is uniform showing that the restriction of f to V^C is continuous. Now use the Tietze extension theorem.

8. Let $\phi_m \in C_c^\infty(\mathbb{R}^n)$, $\phi_m(\mathbf{x}) \geq 0$, and $\int_{\mathbb{R}^n} \phi_m(\mathbf{y}) d\mathbf{y} = 1$ with

$$\lim_{m \rightarrow \infty} \sup \{|\mathbf{x}| : \mathbf{x} \in \text{spt}(\phi_m)\} = 0.$$

Show if $f \in L^p(\mathbb{R}^n)$, $\lim_{m \rightarrow \infty} f * \phi_m = f$ in $L^p(\mathbb{R}^n)$.

9. Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be convex. This means

$$\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$$

whenever $\lambda \in [0, 1]$. Verify that if $x < y < z$, then $\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(z) - \phi(y)}{z - y}$ and that $\frac{\phi(z) - \phi(x)}{z - x} \leq \frac{\phi(z) - \phi(y)}{z - y}$. Show if $s \in \mathbb{R}$ there exists λ such that $\phi(s) \leq \phi(t) + \lambda(s - t)$ for all t . Show that if ϕ is convex, then ϕ is continuous.

10. $\uparrow$ Prove Jensen's inequality. If $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is convex, $\mu(\Omega) = 1$, and $f : \Omega \rightarrow \mathbb{R}$ is in $L^1(\Omega)$, then $\phi(\int_{\Omega} f d\mu) \leq \int_{\Omega} \phi(f) d\mu$. **Hint:** Let $s = \int_{\Omega} f d\mu$ and use Problem 9.
11. Let $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$, let $f \in L^p(\mathbb{R})$, $g \in L^{p'}(\mathbb{R})$. Show $f * g$ is uniformly continuous on $\mathbb{R}$ and $|(f * g)(x)| \leq \|f\|_{L^p} \|g\|_{L^{p'}}$. **Hint:** You need to consider why $f * g$ exists and then this follows from the definition of convolution and continuity of translation in L^p .
12. $B(p, q) = \int_0^1 x^{p-1} (1 - x)^{q-1} dx$, $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$ for $p, q > 0$. The first of these is called the beta function, while the second is the gamma function. Show a.) $\Gamma(p + 1) = p\Gamma(p)$; b.) $\Gamma(p)\Gamma(q) = B(p, q)\Gamma(p + q)$.

13. Let $f \in C_c(0, \infty)$ and define $F(x) = \frac{1}{x} \int_0^x f(t) dt$. Show

$$\|F\|_{L^p(0, \infty)} \leq \frac{p}{p-1} \|f\|_{L^p(0, \infty)} \quad \text{whenever } p > 1.$$

Hint: Argue there is no loss of generality in assuming $f \geq 0$ and then assume this is so. Integrate $\int_0^\infty |F(x)|^p dx$ by parts as follows:

$$\int_0^\infty F^p dx = \overbrace{x F^p|_0^\infty}^{\text{show} = 0} - p \int_0^\infty x F^{p-1} F' dx.$$

Now show $x F' = f - F$ and use this in the last integral. Complete the argument by using Holder's inequality and $p - 1 = p/q$.

14. $\uparrow$ Now suppose $f \in L^p(0, \infty)$, $p > 1$, and f not necessarily in $C_c(0, \infty)$. Show that $F(x) = \frac{1}{x} \int_0^x f(t) dt$ still makes sense for each $x > 0$. Show the inequality of Problem 13 is still valid. This inequality is called Hardy's inequality. **Hint:** To show this, use the above inequality along with the density of $C_c(0, \infty)$ in $L^p(0, \infty)$.
15. Suppose $f, g \geq 0$. When does equality hold in Holder's inequality?
16. Prove Vitali's Convergence theorem: Let $\{f_n\}$ be uniformly integrable and complex valued, $\mu(\Omega) < \infty$, $f_n(x) \rightarrow f(x)$ a.e. where f is measurable. Then $f \in L^1$ and $\lim_{n \rightarrow \infty} \int_\Omega |f_n - f| d\mu = 0$. **Hint:** Use Egoroff's theorem to show $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$. This yields a different and easier proof than what was done earlier. See Theorem 8.5.3 on Page 213.
17. $\uparrow$ Show the Vitali Convergence theorem implies the Dominated Convergence theorem for finite measure spaces but there exist examples where the Vitali convergence theorem works and the dominated convergence theorem does not.
18. $\uparrow$ Suppose $\mu(\Omega) < \infty$, $\{f_n\} \subseteq L^1(\Omega)$, and

$$\int_\Omega h(|f_n|) d\mu < C$$

for all n where h is a continuous, nonnegative function satisfying

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = \infty.$$

Show $\{f_n\}$ is uniformly integrable. In applications, this often occurs in the form of a bound on $\|f_n\|_p$.

19. $\uparrow$ Sometimes, especially in books on probability, a different definition of uniform integrability is used than that presented here. A set of functions, $\mathfrak{S}$,

defined on a finite measure space, $(\Omega, \mathcal{S}, \mu)$ is said to be uniformly integrable if for all $\varepsilon > 0$ there exists $\alpha > 0$ such that for all $f \in \mathfrak{S}$,

$$\int_{\{|f| \geq \alpha\}} |f| d\mu \leq \varepsilon.$$

Show that this definition is equivalent to the definition of uniform integrability given earlier in Definition 8.5.1 on Page 212 with the addition of the condition that there is a constant, $C < \infty$ such that

$$\int |f| d\mu \leq C$$

for all $f \in \mathfrak{S}$.

20. $f \in L^\infty(\Omega, \mu)$ if there exists a set of measure zero, E , and a constant $C < \infty$ such that $|f(x)| \leq C$ for all $x \notin E$.

$$\|f\|_\infty \equiv \inf\{C : |f(x)| \leq C \text{ a.e.}\}.$$

Show $\|\cdot\|_\infty$ is a norm on $L^\infty(\Omega, \mu)$ provided f and g are identified if $f(x) = g(x)$ a.e. Show $L^\infty(\Omega, \mu)$ is complete. **Hint:** You might want to show that $\{|f| > \|f\|_\infty\}$ has measure zero so $\|f\|_\infty$ is the smallest number at least as large as $|f(x)|$ for a.e. x . Thus $\|f\|_\infty$ is one of the constants, C in the above.

21. Suppose $f \in L^\infty \cap L^1$. Show $\lim_{p \rightarrow \infty} \|f\|_{L^p} = \|f\|_\infty$. **Hint:**

$$(\|f\|_\infty - \varepsilon)^p \mu(\{|f| > \|f\|_\infty - \varepsilon\}) \leq \int_{\{|f| > \|f\|_\infty - \varepsilon\}} |f|^p d\mu \leq$$

$$\int |f|^p d\mu = \int |f|^{p-1} |f| d\mu \leq \|f\|_\infty^{p-1} \int |f| d\mu.$$

Now raise both ends to the $1/p$ power and take $\liminf$ and $\limsup$ as $p \rightarrow \infty$. You should get $\|f\|_\infty - \varepsilon \leq \liminf \|f\|_p \leq \limsup \|f\|_p \leq \|f\|_\infty$

22. Suppose $\mu(\Omega) < \infty$. Show that if $1 \leq p < q$, then $L^q(\Omega) \subseteq L^p(\Omega)$. **Hint** Use Holder's inequality.
23. Show $L^1(\mathbb{R}) \not\subseteq L^2(\mathbb{R})$ and $L^2(\mathbb{R}) \not\subseteq L^1(\mathbb{R})$ if Lebesgue measure is used. **Hint:** Consider $1/\sqrt{x}$ and $1/x$.
24. Suppose that $\theta \in [0, 1]$ and $r, s, q > 0$ with

$$\frac{1}{q} = \frac{\theta}{r} + \frac{1-\theta}{s}.$$

show that

$$\left(\int |f|^q d\mu\right)^{1/q} \leq \left(\left(\int |f|^r d\mu\right)^{1/r}\right)^\theta \left(\left(\int |f|^s d\mu\right)^{1/s}\right)^{1-\theta}.$$

If $q, r, s \geq 1$ this says that

$$\|f\|_q \leq \|f\|_r^\theta \|f\|_s^{1-\theta}.$$

Using this, show that

$$\ln(\|f\|_q) \leq \theta \ln(\|f\|_r) + (1 - \theta) \ln(\|f\|_s).$$

Hint:

$$\int |f|^q d\mu = \int |f|^{q\theta} |f|^{q(1-\theta)} d\mu.$$

Now note that $1 = \frac{\theta q}{r} + \frac{q(1-\theta)}{s}$ and use Holder's inequality.

25. Suppose f is a function in $L^1(\mathbb{R})$ and f is infinitely differentiable. Is $f' \in L^1(\mathbb{R})$? **Hint:** What if $\phi \in C_c^\infty(0, 1)$ and $f(x) = \phi(2^n(x - n))$ for $x \in (n, n + 1)$, $f(x) = 0$ if $x < 0$?

Banach Spaces

13.1 Theorems Based On Baire Category

13.1.1 Baire Category Theorem

Some examples of Banach spaces that have been discussed up to now are $\mathbb{R}^n$, $\mathbb{C}^n$, and $L^p(\Omega)$. Theorems about general Banach spaces are proved in this chapter. The main theorems to be presented here are the uniform boundedness theorem, the open mapping theorem, the closed graph theorem, and the Hahn Banach Theorem. The first three of these theorems come from the Baire category theorem which is about to be presented. They are topological in nature. The Hahn Banach theorem has nothing to do with topology. Banach spaces are all normed linear spaces and as such, they are all metric spaces because a normed linear space may be considered as a metric space with $d(x, y) \equiv \|x - y\|$. You can check that this satisfies all the axioms of a metric. As usual, if every Cauchy sequence converges, the metric space is called complete.

Definition 13.1.1 *A complete normed linear space is called a Banach space.*

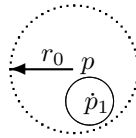
The following remarkable result is called the Baire category theorem. To get an idea of its meaning, imagine you draw a line in the plane. The complement of this line is an open set and is dense because every point, even those on the line, are limit points of this open set. Now draw another line. The complement of the two lines is still open and dense. Keep drawing lines and looking at the complements of the union of these lines. You always have an open set which is dense. Now what if there were countably many lines? The Baire category theorem implies the complement of the union of these lines is dense. In particular it is nonempty. Thus you cannot write the plane as a countable union of lines. This is a rather rough description of this very important theorem. The precise statement and proof follow.

Theorem 13.1.2 *Let (X, d) be a complete metric space and let $\{U_n\}_{n=1}^\infty$ be a sequence of open subsets of X satisfying $\overline{U_n} = X$ (U_n is dense). Then $D \equiv \bigcap_{n=1}^\infty U_n$ is a dense subset of X .*

Proof: Let $p \in X$ and let $r_0 > 0$. I need to show $D \cap B(p, r_0) \neq \emptyset$. Since U_1 is dense, there exists $p_1 \in U_1 \cap B(p, r_0)$, an open set. Let $p_1 \in B(p_1, r_1) \subseteq \overline{B(p_1, r_1)} \subseteq U_1 \cap B(p, r_0)$ and $r_1 < 2^{-1}$. This is possible because $U_1 \cap B(p, r_0)$ is an open set and so there exists r_1 such that $B(p_1, 2r_1) \subseteq U_1 \cap B(p, r_0)$. But

$$B(p_1, r_1) \subseteq \overline{B(p_1, r_1)} \subseteq B(p_1, 2r_1)$$

because $\overline{B(p_1, r_1)} = \{x \in X : d(x, p_1) \leq r_1\}$. (Why?)



There exists $p_2 \in U_2 \cap B(p_1, r_1)$ because U_2 is dense. Let

$$p_2 \in B(p_2, r_2) \subseteq \overline{B(p_2, r_2)} \subseteq U_2 \cap B(p_1, r_1) \subseteq U_1 \cap U_2 \cap B(p, r_0).$$

and let $r_2 < 2^{-2}$. Continue in this way. Thus

$$r_n < 2^{-n},$$

$$\overline{B(p_n, r_n)} \subseteq U_1 \cap U_2 \cap \dots \cap U_n \cap B(p, r_0),$$

$$\overline{B(p_n, r_n)} \subseteq B(p_{n-1}, r_{n-1}).$$

The sequence, $\{p_n\}$ is a Cauchy sequence because all terms of $\{p_k\}$ for $k \geq n$ are contained in $B(p_n, r_n)$, a set whose diameter is no larger than 2^{-n} . Since X is complete, there exists p_∞ such that

$$\lim_{n \rightarrow \infty} p_n = p_\infty.$$

Since all but finitely many terms of $\{p_n\}$ are in $\overline{B(p_m, r_m)}$, it follows that $p_\infty \in \overline{B(p_m, r_m)}$ for each m . Therefore,

$$p_\infty \in \bigcap_{m=1}^{\infty} \overline{B(p_m, r_m)} \subseteq \bigcap_{i=1}^{\infty} U_i \cap B(p, r_0).$$

This proves the theorem.

The following corollary is also called the Baire category theorem.

Corollary 13.1.3 *Let X be a complete metric space and suppose $X = \bigcup_{i=1}^{\infty} F_i$ where each F_i is a closed set. Then for some i , interior $F_i \neq \emptyset$.*

Proof: If all F_i has empty interior, then F_i^C would be a dense open set. Therefore, from Theorem 13.1.2, it would follow that

$$\emptyset = (\bigcup_{i=1}^{\infty} F_i)^C = \bigcap_{i=1}^{\infty} F_i^C \neq \emptyset.$$

The set D of Theorem 13.1.2 is called a G_δ set because it is the countable intersection of open sets. Thus D is a dense G_δ set.

Recall that a norm satisfies:

- a.) $\|x\| \geq 0$, $\|x\| = 0$ if and only if $x = 0$.
- b.) $\|x + y\| \leq \|x\| + \|y\|$.
- c.) $\|cx\| = |c| \|x\|$ if c is a scalar and $x \in X$.

From the definition of continuity, it follows easily that a function is continuous if

$$\lim_{n \rightarrow \infty} x_n = x$$

implies

$$\lim_{n \rightarrow \infty} f(x_n) = f(x).$$

Theorem 13.1.4 *Let X and Y be two normed linear spaces and let $L : X \rightarrow Y$ be linear ($L(ax + by) = aL(x) + bL(y)$ for a, b scalars and $x, y \in X$). The following are equivalent*

- a.) L is continuous at 0
- b.) L is continuous
- c.) There exists $K > 0$ such that $\|Lx\|_Y \leq K \|x\|_X$ for all $x \in X$ (L is bounded).

Proof: a.) $\Rightarrow$ b.) Let $x_n \rightarrow x$. It is necessary to show that $Lx_n \rightarrow Lx$. But $(x_n - x) \rightarrow 0$ and so from continuity at 0, it follows

$$L(x_n - x) = Lx_n - Lx \rightarrow 0$$

so $Lx_n \rightarrow Lx$. This shows a.) implies b.).

b.) $\Rightarrow$ c.) Since L is continuous, L is continuous at 0. Hence $\|Lx\|_Y < 1$ whenever $\|x\|_X \leq \delta$ for some δ . Therefore, suppressing the subscript on the $\|$,

$$\left\| L \left(\frac{\delta x}{\|x\|} \right) \right\| \leq 1.$$

Hence

$$\|Lx\| \leq \frac{1}{\delta} \|x\|.$$

c.) $\Rightarrow$ a.) follows from the inequality given in c.).

Definition 13.1.5 *Let $L : X \rightarrow Y$ be linear and continuous where X and Y are normed linear spaces. Denote the set of all such continuous linear maps by $\mathcal{L}(X, Y)$ and define*

$$\|L\| = \sup\{\|Lx\| : \|x\| \leq 1\}. \quad (13.1.1)$$

This is called the operator norm.

Note that from Theorem 13.1.4 $\|L\|$ is well defined because of part c.) of that Theorem.

The next lemma follows immediately from the definition of the norm and the assumption that L is linear.

Lemma 13.1.6 *With $\|L\|$ defined in 13.1.1, $\mathcal{L}(X, Y)$ is a normed linear space. Also $\|Lx\| \leq \|L\| \|x\|$.*

Proof: Let $x \neq 0$ then $x/\|x\|$ has norm equal to 1 and so

$$\left\| L \left(\frac{x}{\|x\|} \right) \right\| \leq \|L\|.$$

Therefore, multiplying both sides by $\|x\|$, $\|Lx\| \leq \|L\| \|x\|$. This is obviously a linear space. It remains to verify the operator norm really is a norm. First of all, if $\|L\| = 0$, then $Lx = 0$ for all $\|x\| \leq 1$. It follows that for any $x \neq 0$, $0 = L \left(\frac{x}{\|x\|} \right)$ and so $Lx = 0$. Therefore, $L = 0$. Also, if c is a scalar,

$$\|cL\| = \sup_{\|x\| \leq 1} \|cL(x)\| = |c| \sup_{\|x\| \leq 1} \|Lx\| = |c| \|L\|.$$

It remains to verify the triangle inequality. Let $L, M \in \mathcal{L}(X, Y)$.

$$\begin{aligned} \|L + M\| &\equiv \sup_{\|x\| \leq 1} \|(L + M)(x)\| \leq \sup_{\|x\| \leq 1} (\|Lx\| + \|Mx\|) \\ &\leq \sup_{\|x\| \leq 1} \|Lx\| + \sup_{\|x\| \leq 1} \|Mx\| = \|L\| + \|M\|. \end{aligned}$$

This shows the operator norm is really a norm as hoped. This proves the lemma.

For example, consider the space of linear transformations defined on $\mathbb{R}^n$ having values in $\mathbb{R}^m$. The fact the transformation is linear automatically imparts continuity to it. You should give a proof of this fact. Recall that every such linear transformation can be realized in terms of matrix multiplication.

Thus, in finite dimensions the algebraic condition that an operator is linear is sufficient to imply the topological condition that the operator is continuous. The situation is not so simple in infinite dimensional spaces such as $C(X; \mathbb{R}^n)$. This explains the imposition of the topological condition of continuity as a criterion for membership in $\mathcal{L}(X, Y)$ in addition to the algebraic condition of linearity.

Theorem 13.1.7 *If Y is a Banach space, then $\mathcal{L}(X, Y)$ is also a Banach space.*

Proof: Let $\{L_n\}$ be a Cauchy sequence in $\mathcal{L}(X, Y)$ and let $x \in X$.

$$\|L_n x - L_m x\| \leq \|x\| \|L_n - L_m\|.$$

Thus $\{L_n x\}$ is a Cauchy sequence. Let

$$Lx = \lim_{n \rightarrow \infty} L_n x.$$

Then, clearly, L is linear because if x_1, x_2 are in X , and a, b are scalars, then

$$\begin{aligned} L(ax_1 + bx_2) &= \lim_{n \rightarrow \infty} L_n(ax_1 + bx_2) \\ &= \lim_{n \rightarrow \infty} (aL_nx_1 + bL_nx_2) \\ &= aLx_1 + bLx_2. \end{aligned}$$

Also L is continuous. To see this, note that $\{\|L_n\|\}$ is a Cauchy sequence of real numbers because $|\|L_n\| - \|L_m\|| \leq \|L_n - L_m\|$. Hence there exists $K > \sup\{\|L_n\| : n \in \mathbb{N}\}$. Thus, if $x \in X$,

$$\|Lx\| = \lim_{n \rightarrow \infty} \|L_nx\| \leq K\|x\|.$$

This proves the theorem.

13.1.2 Uniform Boundedness Theorem

The next big result is sometimes called the Uniform Boundedness theorem, or the Banach-Steinhaus theorem. This is a very surprising theorem which implies that for a collection of bounded linear operators, if they are bounded pointwise, then they are also bounded uniformly. As an example of a situation in which pointwise bounded does not imply uniformly bounded, consider the functions $f_\alpha(x) \equiv \mathcal{X}_{(\alpha,1)}(x)x^{-1}$ for $\alpha \in (0,1)$. Clearly each function is bounded and the collection of functions is bounded at each point of $(0,1)$, but there is no bound for all these functions taken together. One problem is that $(0,1)$ is not a Banach space. Therefore, the functions cannot be linear.

Theorem 13.1.8 *Let X be a Banach space and let Y be a normed linear space. Let $\{L_\alpha\}_{\alpha \in \Lambda}$ be a collection of elements of $\mathcal{L}(X, Y)$. Then one of the following happens.*

- a.) $\sup\{\|L_\alpha\| : \alpha \in \Lambda\} < \infty$
- b.) *There exists a dense G_δ set, D , such that for all $x \in D$,*

$$\sup\{\|L_\alpha x\| : \alpha \in \Lambda\} = \infty.$$

Proof: For each $n \in \mathbb{N}$, define

$$U_n = \{x \in X : \sup\{\|L_\alpha x\| : \alpha \in \Lambda\} > n\}.$$

Then U_n is an open set because if $x \in U_n$, then there exists $\alpha \in \Lambda$ such that

$$\|L_\alpha x\| > n$$

But then, since L_α is continuous, this situation persists for all y sufficiently close to x , say for all $y \in B(x, \delta)$. Then $B(x, \delta) \subseteq U_n$ which shows U_n is open.

Case b.) is obtained from Theorem 13.1.2 if each U_n is dense.

The other case is that for some n , U_n is not dense. If this occurs, there exists x_0 and $r > 0$ such that for all $x \in B(x_0, r)$, $\|L_\alpha x\| \leq n$ for all α . Now if $y \in$

$B(0, r)$, $x_0 + y \in B(x_0, r)$. Consequently, for all such y , $\|L_\alpha(x_0 + y)\| \leq n$. This implies that for all $\alpha \in \Lambda$ and $\|y\| < r$,

$$\|L_\alpha y\| \leq n + \|L_\alpha(x_0)\| \leq 2n.$$

Therefore, if $\|y\| \leq 1$, $\|\frac{r}{2}y\| < r$ and so for all α ,

$$\|L_\alpha\left(\frac{r}{2}y\right)\| \leq 2n.$$

Now multiplying by $r/2$ it follows that whenever $\|y\| \leq 1$, $\|L_\alpha(y)\| \leq 4n/r$. Hence case a.) holds.

13.1.3 Open Mapping Theorem

Another remarkable theorem which depends on the Baire category theorem is the open mapping theorem. Unlike Theorem 13.1.8 it requires both X and Y to be Banach spaces.

Theorem 13.1.9 *Let X and Y be Banach spaces, let $L \in \mathcal{L}(X, Y)$, and suppose L is onto. Then L maps open sets onto open sets.*

To aid in the proof, here is a lemma.

Lemma 13.1.10 *Let a and b be positive constants and suppose*

$$B(0, a) \subseteq \overline{L(B(0, b))}.$$

Then

$$\overline{L(B(0, b))} \subseteq L(B(0, 2b)).$$

Proof of Lemma 13.1.10: Let $y \in \overline{L(B(0, b))}$. There exists $x_1 \in B(0, b)$ such that $\|y - Lx_1\| < \frac{a}{2}$. Now this implies

$$2y - 2Lx_1 \in B(0, a) \subseteq \overline{L(B(0, b))}.$$

Thus $2y - 2Lx_1 \in \overline{L(B(0, b))}$ just like y was. Therefore, there exists $x_2 \in B(0, b)$ such that $\|2y - 2Lx_1 - Lx_2\| < a/2$. Hence $\|4y - 4Lx_1 - 2Lx_2\| < a$, and there exists $x_3 \in B(0, b)$ such that $\|4y - 4Lx_1 - 2Lx_2 - Lx_3\| < a/2$. Continuing in this way, there exist $x_1, x_2, x_3, x_4, \dots$ in $B(0, b)$ such that

$$\|2^n y - \sum_{i=1}^n 2^{n-(i-1)} L(x_i)\| < a$$

which implies

$$\|y - \sum_{i=1}^n 2^{-(i-1)} L(x_i)\| = \|y - L\left(\sum_{i=1}^n 2^{-(i-1)}(x_i)\right)\| < 2^{-n}a \quad (13.1.2)$$

Now consider the partial sums of the series, $\sum_{i=1}^{\infty} 2^{-(i-1)} x_i$.

$$\left\| \sum_{i=m}^n 2^{-(i-1)} x_i \right\| \leq b \sum_{i=m}^{\infty} 2^{-(i-1)} = b 2^{-m+2}.$$

Therefore, these partial sums form a Cauchy sequence and so since X is complete, there exists $x = \sum_{i=1}^{\infty} 2^{-(i-1)} x_i$. Letting $n \rightarrow \infty$ in 13.1.2 yields $\|y - Lx\| = 0$. Now

$$\begin{aligned} \|x\| &= \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n 2^{-(i-1)} x_i \right\| \\ &\leq \lim_{n \rightarrow \infty} \sum_{i=1}^n 2^{-(i-1)} \|x_i\| < \lim_{n \rightarrow \infty} \sum_{i=1}^n 2^{-(i-1)} b = 2b. \end{aligned}$$

This proves the lemma.

Proof of Theorem 13.1.9: $Y = \cup_{n=1}^{\infty} \overline{L(B(0, n))}$. By Corollary 13.1.3, the set, $\overline{L(B(0, n_0))}$ has nonempty interior for some n_0 . Thus $B(y, r) \subseteq \overline{L(B(0, n_0))}$ for some y and some $r > 0$. Since L is linear $B(-y, r) \subseteq \overline{L(B(0, n_0))}$ also. Here is why. If $z \in B(-y, r)$, then $-z \in B(y, r)$ and so there exists $x_n \in B(0, n_0)$ such that $Lx_n \rightarrow -z$. Therefore, $L(-x_n) \rightarrow z$ and $-x_n \in B(0, n_0)$ also. Therefore $z \in \overline{L(B(0, n_0))}$. Then it follows that

$$\begin{aligned} B(0, r) &\subseteq B(y, r) + B(-y, r) \\ &\equiv \{y_1 + y_2 : y_1 \in B(y, r) \text{ and } y_2 \in B(-y, r)\} \\ &\subseteq \overline{L(B(0, 2n_0))} \end{aligned}$$

The reason for the last inclusion is that from the above, if $y_1 \in B(y, r)$ and $y_2 \in B(-y, r)$, there exists $x_n, z_n \in B(0, n_0)$ such that

$$Lx_n \rightarrow y_1, \quad Lz_n \rightarrow y_2.$$

Therefore,

$$\|x_n + z_n\| \leq 2n_0$$

and so $(y_1 + y_2) \in \overline{L(B(0, 2n_0))}$.

By Lemma 13.1.10, $\overline{L(B(0, 2n_0))} \subseteq L(B(0, 4n_0))$ which shows

$$B(0, r) \subseteq L(B(0, 4n_0)).$$

Letting $a = r(4n_0)^{-1}$, it follows, since L is linear, that $B(0, a) \subseteq L(B(0, 1))$. It follows since L is linear,

$$L(B(0, r)) \supseteq B(0, ar). \quad (13.1.3)$$

Now let U be open in X and let $x + B(0, r) = B(x, r) \subseteq U$. Using 13.1.3,

$$\begin{aligned} L(U) &\supseteq L(x + B(0, r)) \\ &= Lx + L(B(0, r)) \supseteq Lx + B(0, ar) = B(Lx, ar). \end{aligned}$$

Hence

$$Lx \in B(Lx, ar) \subseteq L(U).$$

which shows that every point, $Lx \in LU$, is an interior point of LU and so LU is open. This proves the theorem.

This theorem is surprising because it implies that if $|\cdot|$ and $\|\cdot\|$ are two norms with respect to which a vector space X is a Banach space such that $|\cdot| \leq K\|\cdot\|$, then there exists a constant k , such that $\|\cdot\| \leq k|\cdot|$. This can be useful because sometimes it is not clear how to compute k when all that is needed is its existence. To see the open mapping theorem implies this, consider the identity map $\text{id } x = x$. Then $\text{id} : (X, \|\cdot\|) \rightarrow (X, |\cdot|)$ is continuous and onto. Hence id is an open map which implies id^{-1} is continuous. Theorem 13.1.4 gives the existence of the constant k .

13.1.4 Closed Graph Theorem

Definition 13.1.11 Let $f : D \rightarrow E$. The set of all ordered pairs of the form $\{(x, f(x)) : x \in D\}$ is called the graph of f .

Definition 13.1.12 If X and Y are normed linear spaces, make $X \times Y$ into a normed linear space by using the norm $\|(x, y)\| = \max(\|x\|, \|y\|)$ along with component-wise addition and scalar multiplication. Thus $a(x, y) + b(z, w) \equiv (ax + bz, ay + bw)$.

There are other ways to give a norm for $X \times Y$. For example, you could define $\|(x, y)\| = \|x\| + \|y\|$

Lemma 13.1.13 The norm defined in Definition 13.1.12 on $X \times Y$ along with the definition of addition and scalar multiplication given there make $X \times Y$ into a normed linear space.

Proof: The only axiom for a norm which is not obvious is the triangle inequality. Therefore, consider

$$\begin{aligned} \|(x_1, y_1) + (x_2, y_2)\| &= \|(x_1 + x_2, y_1 + y_2)\| \\ &= \max(\|x_1 + x_2\|, \|y_1 + y_2\|) \\ &\leq \max(\|x_1\| + \|x_2\|, \|y_1\| + \|y_2\|) \\ &\leq \max(\|x_1\|, \|y_1\|) + \max(\|x_2\|, \|y_2\|) \\ &= \|(x_1, y_1)\| + \|(x_2, y_2)\|. \end{aligned}$$

It is obvious $X \times Y$ is a vector space from the above definition. This proves the lemma.

Lemma 13.1.14 If X and Y are Banach spaces, then $X \times Y$ with the norm and vector space operations defined in Definition 13.1.12 is also a Banach space.

Proof: The only thing left to check is that the space is complete. But this follows from the simple observation that $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$ if and only if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y respectively. Thus if $\{(x_n, y_n)\}$ is a Cauchy sequence in $X \times Y$, it follows there exist x and y such that $x_n \rightarrow x$ and $y_n \rightarrow y$. But then from the definition of the norm, $(x_n, y_n) \rightarrow (x, y)$.

Lemma 13.1.15 *Every closed subspace of a Banach space is a Banach space.*

Proof: If $F \subseteq X$ where X is a Banach space and $\{x_n\}$ is a Cauchy sequence in F , then since X is complete, there exists a unique $x \in X$ such that $x_n \rightarrow x$. However this means $x \in \overline{F} = F$ since F is closed.

Definition 13.1.16 *Let X and Y be Banach spaces and let $D \subseteq X$ be a subspace. A linear map $L : D \rightarrow Y$ is said to be closed if its graph is a closed subspace of $X \times Y$. Equivalently, L is closed if $x_n \rightarrow x$ and $Lx_n \rightarrow y$ implies $x \in D$ and $y = Lx$.*

Note the distinction between closed and continuous. If the operator is closed the assertion that $y = Lx$ only follows if it is known that the sequence $\{Lx_n\}$ converges. In the case of a continuous operator, the convergence of $\{Lx_n\}$ follows from the assumption that $x_n \rightarrow x$. It is not always the case that a mapping which is closed is necessarily continuous. Consider the function $f(x) = \tan(x)$ if x is not an odd multiple of $\frac{\pi}{2}$ and $f(x) \equiv 0$ at every odd multiple of $\frac{\pi}{2}$. Then the graph is closed and the function is defined on $\mathbb{R}$ but it clearly fails to be continuous. Of course this function is not linear. You could also consider the map,

$$\frac{d}{dx} : \{y \in C^1([0, 1]) : y(0) = 0\} \equiv D \rightarrow C([0, 1]).$$

where the norm is the uniform norm on $C([0, 1])$, $\|y\|_\infty$. If $y \in D$, then

$$y(x) = \int_0^x y'(t) dt.$$

Therefore, if $\frac{dy_n}{dx} \rightarrow f \in C([0, 1])$ and if $y_n \rightarrow y$ in $C([0, 1])$ it follows that

$$\begin{array}{ccc} y_n(x) & = & \int_0^x \frac{dy_n(t)}{dx} dt \\ \downarrow & & \downarrow \\ y(x) & = & \int_0^x f(t) dt \end{array}$$

and so by the fundamental theorem of calculus $f(x) = y'(x)$ and so the mapping is closed. It is obviously not continuous because it takes $y(x)$ and $y(x) + \frac{1}{n} \sin(nx)$ to two functions which are far from each other even though these two functions are very close in $C([0, 1])$. Furthermore, it is not defined on the whole space, $C([0, 1])$.

The next theorem, the closed graph theorem, gives conditions under which closed implies continuous.

Theorem 13.1.17 *Let X and Y be Banach spaces and suppose $L : X \rightarrow Y$ is closed and linear. Then L is continuous.*

Proof: Let G be the graph of L . $G = \{(x, Lx) : x \in X\}$. By Lemma 13.1.15 it follows that G is a Banach space. Define $P : G \rightarrow X$ by $P(x, Lx) = x$. P maps the Banach space G onto the Banach space X and is continuous and linear. By the open mapping theorem, P maps open sets onto open sets. Since P is also one to one, this says that P^{-1} is continuous. Thus $\|P^{-1}x\| \leq K\|x\|$. Hence

$$\|Lx\| \leq \max(\|x\|, \|Lx\|) \leq K\|x\|$$

By Theorem 13.1.4 on Page 363, this shows L is continuous and proves the theorem.

The following corollary is quite useful. It shows how to obtain a new norm on the domain of a closed operator such that the domain with this new norm becomes a Banach space.

Corollary 13.1.18 *Let $L : D \subseteq X \rightarrow Y$ where X, Y are a Banach spaces, and L is a closed operator. Then define a new norm on D by*

$$\|x\|_D \equiv \|x\|_X + \|Lx\|_Y.$$

Then D with this new norm is a Banach space.

Proof: If $\{x_n\}$ is a Cauchy sequence in D with this new norm, it follows both $\{x_n\}$ and $\{Lx_n\}$ are Cauchy sequences and therefore, they converge. Since L is closed, $x_n \rightarrow x$ and $Lx_n \rightarrow Lx$ for some $x \in D$. Thus $\|x_n - x\|_D \rightarrow 0$.

13.2 Hahn Banach Theorem

The closed graph, open mapping, and uniform boundedness theorems are the three major topological theorems in functional analysis. The other major theorem is the Hahn-Banach theorem which has nothing to do with topology. Before presenting this theorem, here are some preliminaries about partially ordered sets.

13.2.1 Partially Ordered Sets

Definition 13.2.1 *Let $\mathcal{F}$ be a nonempty set. $\mathcal{F}$ is called a partially ordered set if there is a relation, denoted here by $\leq$, such that*

$$x \leq x \text{ for all } x \in \mathcal{F}.$$

$$\text{If } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\mathcal{C} \subseteq \mathcal{F}$ is said to be a chain if every two elements of $\mathcal{C}$ are related. This means that if $x, y \in \mathcal{C}$, then either $x \leq y$ or $y \leq x$. Sometimes a chain is called a totally ordered set. $\mathcal{C}$ is said to be a maximal chain if whenever $\mathcal{D}$ is a chain containing $\mathcal{C}$, $\mathcal{D} = \mathcal{C}$.

The most common example of a partially ordered set is the power set of a given set with $\subseteq$ being the relation. It is also helpful to visualize partially ordered sets as trees. Two points on the tree are related if they are on the same branch of the tree and one is higher than the other. Thus two points on different branches would not be related although they might both be larger than some point on the trunk. You might think of many other things which are best considered as partially ordered sets. Think of food for example. You might find it difficult to determine which of two favorite pies you like better although you may be able to say very easily that you would prefer either pie to a dish of lard topped with whipped cream and mustard. The following theorem is equivalent to the axiom of choice. For a discussion of this, see the appendix on the subject.

Theorem 13.2.2 (*Hausdorff Maximal Principle*) *Let $\mathcal{F}$ be a nonempty partially ordered set. Then there exists a maximal chain.*

13.2.2 Gauge Functions And Hahn Banach Theorem

Definition 13.2.3 *Let X be a real vector space $\rho : X \rightarrow \mathbb{R}$ is called a gauge function if*

$$\begin{aligned}\rho(x + y) &\leq \rho(x) + \rho(y), \\ \rho(ax) &= a\rho(x) \text{ if } a \geq 0.\end{aligned}\tag{13.2.4}$$

Suppose M is a subspace of X and $z \notin M$. Suppose also that f is a linear real-valued function having the property that $f(x) \leq \rho(x)$ for all $x \in M$. Consider the problem of extending f to $M \oplus \mathbb{R}z$ such that if F is the extended function, $F(y) \leq \rho(y)$ for all $y \in M \oplus \mathbb{R}z$ and F is linear. Since F is to be linear, it suffices to determine how to define $F(z)$. Letting $a > 0$, it is required to define $F(z)$ such that the following hold for all $x, y \in M$.

$$\begin{aligned}\overbrace{F(x)}^{f(x)} + aF(z) &= F(x + az) \leq \rho(x + az), \\ \overbrace{F(y)}^{f(y)} - aF(z) &= F(y - az) \leq \rho(y - az).\end{aligned}\tag{13.2.5}$$

Now if these inequalities hold for all y/a , they hold for all y because M is given to be a subspace. Therefore, multiplying by a^{-1} 13.2.4 implies that what is needed is to choose $F(z)$ such that for all $x, y \in M$,

$$f(x) + F(z) \leq \rho(x + z), \quad f(y) - \rho(y - z) \leq F(z)$$

and that if $F(z)$ can be chosen in this way, this will satisfy 13.2.5 for all x, y and the problem of extending f will be solved. Hence it is necessary to choose $F(z)$ such that for all $x, y \in M$

$$f(y) - \rho(y - z) \leq F(z) \leq \rho(x + z) - f(x).\tag{13.2.6}$$

Is there any such number between $f(y) - \rho(y - z)$ and $\rho(x + z) - f(x)$ for every pair $x, y \in M$? This is where $f(x) \leq \rho(x)$ on M and that f is linear is used. For $x, y \in M$,

$$\begin{aligned} & \rho(x + z) - f(x) - [f(y) - \rho(y - z)] \\ &= \rho(x + z) + \rho(y - z) - (f(x) + f(y)) \\ &\geq \rho(x + y) - f(x + y) \geq 0. \end{aligned}$$

Therefore there exists a number between

$$\sup \{f(y) - \rho(y - z) : y \in M\}$$

and

$$\inf \{\rho(x + z) - f(x) : x \in M\}$$

Choose $F(z)$ to satisfy 13.2.6. This has proved the following lemma.

Lemma 13.2.4 *Let M be a subspace of X , a real linear space, and let ρ be a gauge function on X . Suppose $f : M \rightarrow \mathbb{R}$ is linear, $z \notin M$, and $f(x) \leq \rho(x)$ for all $x \in M$. Then f can be extended to $M \oplus \mathbb{R}z$ such that, if F is the extended function, F is linear and $F(x) \leq \rho(x)$ for all $x \in M \oplus \mathbb{R}z$.*

With this lemma, the Hahn Banach theorem can be proved.

Theorem 13.2.5 (*Hahn Banach theorem*) *Let X be a real vector space, let M be a subspace of X , let $f : M \rightarrow \mathbb{R}$ be linear, let ρ be a gauge function on X , and suppose $f(x) \leq \rho(x)$ for all $x \in M$. Then there exists a linear function, $F : X \rightarrow \mathbb{R}$, such that*

- a.) $F(x) = f(x)$ for all $x \in M$
- b.) $F(x) \leq \rho(x)$ for all $x \in X$.

Proof: Let $\mathcal{F} = \{(V, g) : V \supseteq M, V \text{ is a subspace of } X, g : V \rightarrow \mathbb{R} \text{ is linear, } g(x) = f(x) \text{ for all } x \in M, \text{ and } g(x) \leq \rho(x) \text{ for } x \in V\}$. Then $(M, f) \in \mathcal{F}$ so $\mathcal{F} \neq \emptyset$. Define a partial order by the following rule.

$$(V, g) \leq (W, h)$$

means

$$V \subseteq W \text{ and } h(x) = g(x) \text{ if } x \in V.$$

By Theorem 13.2.2, there exists a maximal chain, $\mathcal{C} \subseteq \mathcal{F}$. Let $Y = \cup\{V : (V, g) \in \mathcal{C}\}$ and let $h : Y \rightarrow \mathbb{R}$ be defined by $h(x) = g(x)$ where $x \in V$ and $(V, g) \in \mathcal{C}$. This is well defined because if $x \in V_1$ and V_2 where (V_1, g_1) and (V_2, g_2) are both in the chain, then since $\mathcal{C}$ is a chain, the two elements are related. Therefore, $g_1(x) = g_2(x)$. Also h is linear because if $ax + by \in Y$, then $x \in V_1$ and $y \in V_2$ where (V_1, g_1) and (V_2, g_2) are elements of $\mathcal{C}$. Therefore, letting V denote the larger of the two V_i ,

and g be the function that goes with V , it follows $ax + by \in V$ where $(V, g) \in \mathcal{C}$. Therefore,

$$\begin{aligned} h(ax + by) &= g(ax + by) \\ &= ag(x) + bg(y) \\ &= ah(x) + bh(y). \end{aligned}$$

Also, $h(x) = g(x) \leq \rho(x)$ for any $x \in Y$ because for such x , $x \in V$ where $(V, g) \in \mathcal{C}$.

Is $Y = X$? If not, there exists $z \in X \setminus Y$ and there exists an extension of h to $Y \oplus \mathbb{R}z$ using Lemma 13.2.4. Letting $\bar{h}$ denote this extended function, contradicts the maximality of $\mathcal{C}$. Indeed, $\mathcal{C} \cup \{(Y \oplus \mathbb{R}z, \bar{h})\}$ would be a longer chain. This proves the Hahn Banach theorem.

This is the original version of the theorem. There is also a version of this theorem for complex vector spaces which is based on a trick.

13.2.3 The Complex Version Of The Hahn Banach Theorem

Corollary 13.2.6 (*Hahn Banach*) *Let M be a subspace of a complex normed linear space, X , and suppose $f : M \rightarrow \mathbb{C}$ is linear and satisfies $|f(x)| \leq K\|x\|$ for all $x \in M$. Then there exists a linear function, F , defined on all of X such that $F(x) = f(x)$ for all $x \in M$ and $|F(x)| \leq K\|x\|$ for all x .*

Proof: First note $f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x)$ and so

$$\operatorname{Re} f(ix) + i \operatorname{Im} f(ix) = f(ix) = if(x) = i \operatorname{Re} f(x) - \operatorname{Im} f(x).$$

Therefore, $\operatorname{Im} f(x) = -\operatorname{Re} f(ix)$, and

$$f(x) = \operatorname{Re} f(x) - i \operatorname{Re} f(ix).$$

This is important because it shows it is only necessary to consider $\operatorname{Re} f$ in understanding f . Now it happens that $\operatorname{Re} f$ is linear with respect to real scalars so the above version of the Hahn Banach theorem applies. This is shown next.

If c is a real scalar

$$\operatorname{Re} f(cx) - i \operatorname{Re} f(icx) = cf(x) = c \operatorname{Re} f(x) - ic \operatorname{Re} f(ix).$$

Thus $\operatorname{Re} f(cx) = c \operatorname{Re} f(x)$. Also,

$$\begin{aligned} \operatorname{Re} f(x + y) - i \operatorname{Re} f(i(x + y)) &= f(x + y) \\ &= f(x) + f(y) \\ &= \operatorname{Re} f(x) - i \operatorname{Re} f(ix) + \operatorname{Re} f(y) - i \operatorname{Re} f(iy). \end{aligned}$$

Equating real parts, $\operatorname{Re} f(x + y) = \operatorname{Re} f(x) + \operatorname{Re} f(y)$. Thus $\operatorname{Re} f$ is linear with respect to real scalars as hoped.

Consider X as a real vector space and let $\rho(x) \equiv K\|x\|$. Then for all $x \in M$,

$$|\operatorname{Re} f(x)| \leq |f(x)| \leq K\|x\| = \rho(x).$$

From Theorem 13.2.5, $\operatorname{Re} f$ may be extended to a function, h which satisfies

$$\begin{aligned} h(ax + by) &= ah(x) + bh(y) \text{ if } a, b \in \mathbb{R} \\ h(x) &\leq K\|x\| \text{ for all } x \in X. \end{aligned}$$

Actually, $|h(x)| \leq K\|x\|$. The reason for this is that $h(-x) = -h(x) \leq K\|-x\| = K\|x\|$ and therefore, $h(x) \geq -K\|x\|$. Let

$$F(x) \equiv h(x) - ih(ix).$$

By arguments similar to the above, F is linear.

$$\begin{aligned} F(ix) &= h(ix) - ih(-x) \\ &= ih(x) + h(x) \\ &= i(h(x) - ih(ix)) = iF(x). \end{aligned}$$

If c is a real scalar,

$$\begin{aligned} F(cx) &= h(cx) - ih(icx) \\ &= ch(x) - cih(ix) = cF(x) \end{aligned}$$

Now

$$\begin{aligned} F(x + y) &= h(x + y) - ih(i(x + y)) \\ &= h(x) + h(y) - ih(ix) - ih(iy) \\ &= F(x) + F(y). \end{aligned}$$

Thus

$$\begin{aligned} F((a + ib)x) &= F(ax) + F(ibx) \\ &= aF(x) + ibF(x) \\ &= (a + ib)F(x). \end{aligned}$$

This shows F is linear as claimed.

Now $wF(x) = |F(x)|$ for some $|w| = 1$. Therefore

$$\begin{aligned} |F(x)| &= wF(x) = h(wx) - \overbrace{ih(iwx)}^{\text{must equal zero}} = h(wx) \\ &= |h(wx)| \leq K\|wx\| = K\|x\|. \end{aligned}$$

This proves the corollary.

13.2.4 The Dual Space And Adjoint Operators

Definition 13.2.7 Let X be a Banach space. Denote by X' the space of continuous linear functions which map X to the field of scalars. Thus $X' = \mathcal{L}(X, \mathbb{F})$. By Theorem 13.1.7 on Page 364, X' is a Banach space. Remember with the norm defined on $\mathcal{L}(X, \mathbb{F})$,

$$\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$$

X' is called the dual space.

Definition 13.2.8 Let X and Y be Banach spaces and suppose $L \in \mathcal{L}(X, Y)$. Then define the adjoint map in $\mathcal{L}(Y', X')$, denoted by L^* , by

$$L^*y^*(x) \equiv y^*(Lx)$$

for all $y^* \in Y'$.

The following diagram is a good one to help remember this definition.

$$\begin{array}{ccc} & L^* & \\ X' & \leftarrow & Y' \\ X & \xrightarrow{L} & Y \end{array}$$

This is a generalization of the adjoint of a linear transformation on an inner product space. Recall

$$(Ax, y) = (x, A^*y)$$

What is being done here is to generalize this algebraic concept to arbitrary Banach spaces. There are some issues which need to be discussed relative to the above definition. First of all, it must be shown that $L^*y^* \in X'$. Also, it will be useful to have the following lemma which is a useful application of the Hahn Banach theorem.

Lemma 13.2.9 Let X be a normed linear space and let $x \in X \setminus V$ where V is a closed subspace of X . Then there exists $x^* \in X'$ such that $x^*(x) = \|x\|$, $x^*(V) = \{0\}$, and

$$\|x^*\| \leq \frac{1}{\text{dist}(x, V)}$$

In the case that $V = \{0\}$, $\|x^*\| = 1$.

Proof: Let $f : \mathbb{F}x + V \rightarrow \mathbb{F}$ be defined by $f(\alpha x + v) = \alpha\|x\|$. First it is necessary to show f is well defined and continuous. If $\alpha_1 x + v_1 = \alpha_2 x + v_2$ then if $\alpha_1 \neq \alpha_2$, then $x \in V$ which is assumed not to happen so f is well defined. It remains to show f is continuous. Suppose then that $\alpha_n x + v_n \rightarrow 0$. It is necessary to show $\alpha_n \rightarrow 0$. If this does not happen, then there exists a subsequence, still denoted by α_n such that $|\alpha_n| \geq \delta > 0$. Then $x + (1/\alpha_n)v_n \rightarrow 0$ contradicting the assumption

that $x \notin V$ and V is a closed subspace. Hence f is continuous on $\mathbb{F}x + V$. Being a little more careful,

$$\|f\| = \sup_{\|\alpha x + v\| \leq 1} |f(\alpha x + v)| = \sup_{|\alpha| \|x + (v/\alpha)\| \leq 1} |\alpha| \|x\| = \frac{1}{\text{dist}(x, V)} \|x\|$$

By the Hahn Banach theorem, there exists $x^* \in X'$ such that $x^* = f$ on $\mathbb{F}x + V$. Thus $x^*(x) = \|x\|$ and also

$$\|x^*\| \leq \|f\| = \frac{1}{\text{dist}(x, V)}$$

In case $V = \{0\}$, the result follows from the above or alternatively,

$$\|f\| \equiv \sup_{\|\alpha x\| \leq 1} |f(\alpha x)| = \sup_{|\alpha| \leq 1/\|x\|} |\alpha| \|x\| = 1$$

and so, in this case, $\|x^*\| \leq \|f\| = 1$. Since $x^*(x) = \|x\|$ it follows

$$\|x^*\| \geq \left| x^* \left(\frac{x}{\|x\|} \right) \right| = \frac{\|x\|}{\|x\|} = 1.$$

Thus $\|x^*\| = 1$ and this proves the lemma.

Theorem 13.2.10 *Let $L \in \mathcal{L}(X, Y)$ where X and Y are Banach spaces. Then*

- a.) $L^* \in \mathcal{L}(Y', X')$ as claimed and $\|L^*\| = \|L\|$.
- b.) If L maps one to one onto a closed subspace of Y , then L^* is onto.
- c.) If L maps onto a dense subset of Y , then L^* is one to one.

Proof: It is routine to verify L^*y^* and L^* are both linear. This follows immediately from the definition. As usual, the interesting thing concerns continuity.

$$\|L^*y^*\| = \sup_{\|x\| \leq 1} |L^*y^*(x)| = \sup_{\|x\| \leq 1} |y^*(Lx)| \leq \|y^*\| \|L\|.$$

Thus L^* is continuous as claimed and $\|L^*\| \leq \|L\|$.

By Lemma 13.2.9, there exists $y_x^* \in Y'$ such that $\|y_x^*\| = 1$ and $y_x^*(Lx) = \|Lx\|$. Therefore,

$$\begin{aligned} \|L^*\| &= \sup_{\|y^*\| \leq 1} \|L^*y^*\| = \sup_{\|y^*\| \leq 1} \sup_{\|x\| \leq 1} |L^*y^*(x)| \\ &= \sup_{\|y^*\| \leq 1} \sup_{\|x\| \leq 1} |y^*(Lx)| = \sup_{\|x\| \leq 1} \sup_{\|y^*\| \leq 1} |y^*(Lx)| \\ &\geq \sup_{\|x\| \leq 1} |y_x^*(Lx)| = \sup_{\|x\| \leq 1} \|Lx\| = \|L\| \end{aligned}$$

showing that $\|L^*\| \geq \|L\|$ and this shows part a.).

If L is one to one and onto a closed subset of Y , then $L(X)$ being a closed subspace of a Banach space, is itself a Banach space and so the open mapping theorem implies $L^{-1} : L(X) \rightarrow X$ is continuous. Hence

$$\|x\| = \|L^{-1}Lx\| \leq \|L^{-1}\| \|Lx\|$$

Now let $x^* \in X'$ be given. Define $f \in \mathcal{L}(L(X), \mathbb{C})$ by $f(Lx) = x^*(x)$. The function, f is well defined because if $Lx_1 = Lx_2$, then since L is one to one, it follows $x_1 = x_2$ and so $f(L(x_1)) = x^*(x_1) = x^*(x_2) = f(L(x_2))$. Also, f is linear because

$$\begin{aligned} f(aL(x_1) + bL(x_2)) &= f(L(ax_1 + bx_2)) \\ &\equiv x^*(ax_1 + bx_2) \\ &= ax^*(x_1) + bx^*(x_2) \\ &= af(L(x_1)) + bf(L(x_2)). \end{aligned}$$

In addition to this,

$$|f(Lx)| = |x^*(x)| \leq \|x^*\| \|x\| \leq \|x^*\| \|L^{-1}\| \|Lx\|$$

and so the norm of f on $L(X)$ is no larger than $\|x^*\| \|L^{-1}\|$. By the Hahn Banach theorem, there exists an extension of f to an element $y^* \in Y'$ such that $\|y^*\| \leq \|x^*\| \|L^{-1}\|$. Then

$$L^*y^*(x) = y^*(Lx) = f(Lx) = x^*(x)$$

so $L^*y^* = x^*$ because this holds for all x . Since x^* was arbitrary, this shows L^* is onto and proves b.).

Consider the last assertion. Suppose $L^*y^* = 0$. Is $y^* = 0$? In other words is $y^*(y) = 0$ for all $y \in Y$? Pick $y \in Y$. Since $L(X)$ is dense in Y , there exists a sequence, $\{Lx_n\}$ such that $Lx_n \rightarrow y$. But then by continuity of y^* , $y^*(y) = \lim_{n \rightarrow \infty} y^*(Lx_n) = \lim_{n \rightarrow \infty} L^*y^*(x_n) = 0$. Since $y^*(y) = 0$ for all y , this implies $y^* = 0$ and so L^* is one to one.

Corollary 13.2.11 *Suppose X and Y are Banach spaces, $L \in \mathcal{L}(X, Y)$, and L is one to one and onto. Then L^* is also one to one and onto.*

There exists a natural mapping, called the James map from a normed linear space, X , to the dual of the dual space which is described in the following definition.

Definition 13.2.12 *Define $J : X \rightarrow X''$ by $J(x)(x^*) = x^*(x)$.*

Theorem 13.2.13 *The map, J , has the following properties.*

- a.) J is one to one and linear.
 - b.) $\|Jx\| = \|x\|$ and $\|J\| = 1$.
 - c.) $J(X)$ is a closed subspace of X'' if X is complete.
- Also if $x^* \in X'$,

$$\|x^*\| = \sup \{|x^{**}(x^*)| : \|x^{**}\| \leq 1, x^{**} \in X''\}.$$

Proof:

$$\begin{aligned} J(ax + by)(x^*) &\equiv x^*(ax + by) \\ &= ax^*(x) + bx^*(y) \\ &= (aJ(x) + bJ(y))(x^*). \end{aligned}$$

Since this holds for all $x^* \in X'$, it follows that

$$J(ax + by) = aJ(x) + bJ(y)$$

and so J is linear. If $Jx = 0$, then by Lemma 13.2.9 there exists x^* such that $x^*(x) = \|x\|$ and $\|x^*\| = 1$. Then

$$0 = J(x)(x^*) = x^*(x) = \|x\|.$$

This shows a.).

To show b.), let $x \in X$ and use Lemma 13.2.9 to obtain $x^* \in X'$ such that $x^*(x) = \|x\|$ with $\|x^*\| = 1$. Then

$$\begin{aligned} \|x\| &\geq \sup\{|y^*(x)| : \|y^*\| \leq 1\} \\ &= \sup\{|J(x)(y^*)| : \|y^*\| \leq 1\} = \|Jx\| \\ &\geq |J(x)(x^*)| = |x^*(x)| = \|x\| \end{aligned}$$

Therefore, $\|Jx\| = \|x\|$ as claimed. Therefore,

$$\|J\| = \sup\{\|Jx\| : \|x\| \leq 1\} = \sup\{\|x\| : \|x\| \leq 1\} = 1.$$

This shows b.).

To verify c.), use b.). If $Jx_n \rightarrow y^{**} \in X''$ then by b.), x_n is a Cauchy sequence converging to some $x \in X$ because

$$\|x_n - x_m\| = \|Jx_n - Jx_m\|$$

and $\{Jx_n\}$ is a Cauchy sequence. Then $Jx = \lim_{n \rightarrow \infty} Jx_n = y^{**}$.

Finally, to show the assertion about the norm of x^* , use what was just shown applied to the James map from X' to X''' still referred to as J .

$$\begin{aligned} \|x^*\| &= \sup\{|x^*(x)| : \|x\| \leq 1\} = \sup\{|J(x)(x^*)| : \|Jx\| \leq 1\} \\ &\leq \sup\{|x^{**}(x^*)| : \|x^{**}\| \leq 1\} = \sup\{|J(x^*)(x^{**})| : \|x^{**}\| \leq 1\} \\ &\equiv \|Jx^*\| = \|x^*\|. \end{aligned}$$

This proves the theorem.

Definition 13.2.14 When J maps X onto X'' , X is called *reflexive*.

It happens the L^p spaces are reflexive whenever $p > 1$. This is shown later.

13.3 Approximation With Sums

Theorem 13.3.1 Let X be a Banach space and let $V = \text{span}(x_1, \dots, x_n)$. Then V is a closed subspace of X .

Proof: Without loss of generality, it can be assumed $\{x_1, \dots, x_n\}$ is linearly independent. Otherwise, delete those vectors which are in the span of the others till a linearly independent set is obtained. Let

$$x = \lim_{p \rightarrow \infty} \sum_{k=1}^n c_k^p x_k \in \bar{V}. \quad (13.3.7)$$

First suppose $\mathbf{c}^p \equiv (c_1^p, \dots, c_n^p)$ is not bounded in $\mathbb{F}^n$. Then $\mathbf{d}^p \equiv \mathbf{c}^p / \|\mathbf{c}^p\|_{\mathbb{F}^n}$ is a unit vector in $\mathbb{F}^n$ and so there exists a subsequence, still denoted by $\mathbf{d}^p$ which converges to $\mathbf{d}$ where $|\mathbf{d}| = 1$. Then

$$\mathbf{0} = \lim_{p \rightarrow \infty} \frac{x}{\|\mathbf{c}^p\|} = \lim_{p \rightarrow \infty} \sum_{k=1}^n d_k^p x_k = \sum_{k=1}^n d_k x_k$$

where $\sum_k |d_k|^2 = 1$ in contradiction to the linear independence of the $\{x_1, \dots, x_n\}$. Hence it must be the case that $\mathbf{c}^p$ is bounded in $\mathbb{F}^n$. Then taking a subsequence, still denoted as p , it can be assumed $\mathbf{c}^p \rightarrow \mathbf{c}$ and then in 13.3.7 it follows

$$x = \sum_{k=1}^n c_k x_k \in \text{span}(x_1, \dots, x_n).$$

This proves the Theorem.

Lemma 13.3.2 *Let E be a separable Banach space. Then there exists an increasing sequence of subspaces, $\{F_n\}$ such that $\dim(F_{n+1}) - \dim(F_n) \leq 1$ and equals 1 for all n if the dimension of E is infinite. Also $\cup_{n=1}^{\infty} F_n$ is dense in E . In the case where E is infinite dimensional, $F_n = \text{span}(e_1, \dots, e_n)$ where for each n*

$$\text{dist}(e_{n+1}, F_n) \geq \frac{1}{2} \quad (13.3.8)$$

and defining,

$$G_k \equiv \text{span}(\{e_j : j \neq k\})$$

$$\text{dist}(e_k, G_k) \geq \frac{1}{4}. \quad (13.3.9)$$

Proof: Since E is separable, so is $\partial B(0, 1)$, the boundary of the unit ball. Let $\{w_k\}_{k=1}^{\infty}$ be a countable dense subset of $\partial B(0, 1)$.

Let $e_1 = w_1$. Let $F_1 = \mathbb{F}e_1$. Suppose F_n has been obtained and equals $\text{span}(e_1, \dots, e_n)$ where $\{e_1, \dots, e_n\}$ is independent, $\|e_k\| = 1$, and

$$\text{dist}(e_n, \text{span}(e_1, \dots, e_{n-1})) \geq \frac{1}{2}.$$

For each n , F_n is closed by Theorem 13.3.1.

If F_n contains $\{w_k\}_{k=1}^{\infty}$, let $F_m = F_n$ for all $m > n$. Otherwise, pick $w \in \{w_k\}$ to be the point of $\{w_k\}_{k=1}^{\infty}$ having the smallest subscript which is not contained in

F_n . Then w is at a positive distance, λ from F_n because F_n is closed. Therefore, there exists $y \in F_n$ such that $\lambda \leq \|y - w\| \leq 2\lambda$. Let $e_{n+1} = \frac{w-y}{\|w-y\|}$. It follows

$$w = \|w - y\| e_{n+1} + y \in \text{span}(e_1, \dots, e_{n+1}) \equiv F_{n+1}$$

Then if $x \in \text{span}(e_1, \dots, e_n)$,

$$\begin{aligned} \|e_{n+1} - x\| &= \left\| \frac{w-y}{\|w-y\|} - x \right\| \\ &= \left\| \frac{w-y}{\|w-y\|} - \frac{\|w-y\| x}{\|w-y\|} \right\| \\ &\geq \frac{1}{2\lambda} \|w-y - \|w-y\| x\| \\ &\geq \frac{\lambda}{2\lambda} = \frac{1}{2}. \end{aligned}$$

This has shown the existence of an increasing sequence of subspaces, $\{F_n\}$ as described above. It remains to show the union of these subspaces is dense. First note that the union of these subspaces must contain the $\{w_k\}_{k=1}^\infty$ because if w_m is missing, then it would contradict the construction at the m^{th} step. That one should have been chosen. However, $\{w_k\}_{k=1}^\infty$ is dense in $\partial B(0, 1)$. If $x \in E$ and $x \neq 0$, then $\frac{x}{\|x\|} \in \partial B(0, 1)$ then there exists

$$w_m \in \{w_k\}_{k=1}^\infty \subseteq \cup_{n=1}^\infty F_n$$

such that $\left\| w_m - \frac{x}{\|x\|} \right\| < \frac{\varepsilon}{\|x\|}$. But then

$$\| \|x\| w_m - x \| < \varepsilon$$

and so $\|x\| w_m$ is a point of $\cup_{n=1}^\infty F_n$ which is within ε of x . This proves $\cup_{n=1}^\infty F_n$ is dense as desired. 13.3.8 follows from the construction. It remains to verify 13.3.9.

Let $y \in G_k$. Thus for some n ,

$$y = \sum_{j=1}^{k-1} c_j e_j + \sum_{j=k+1}^n c_j e_j$$

and I need to show $\|y - e_k\| \geq 1/4$. Without loss of generality, $c_n \neq 0$ and $n > k$. Suppose 13.3.9 does not hold for some such y so that

$$\left\| e_k - \left(\sum_{j=1}^{k-1} c_j e_j + \sum_{j=k+1}^n c_j e_j \right) \right\| < \frac{1}{4}. \quad (13.3.10)$$

Then from the construction,

$$\begin{aligned} \frac{1}{4} &> |c_n| \left\| e_k - \left(\sum_{j=1}^{k-1} (c_j/c_n) e_j + \sum_{j=k+1}^{n-1} (c_j/c_n) e_j + e_n \right) \right\| \\ &\geq |c_n| \frac{1}{2} \end{aligned}$$

and so $|c_n| < 1/2$. Consider the left side of 13.3.10. By the construction

$$\begin{aligned} & \left\| c_n (e_k - e_n) + (1 - c_n) e_k - \left(\sum_{j=1}^{k-1} c_j e_j + \sum_{j=k+1}^{n-1} c_j e_j \right) \right\| \\ & \geq |1 - c_n| - |c_n| \left\| (e_k - e_n) - \left(\sum_{j=1}^{k-1} (c_j/c_n) e_j + \sum_{j=k+1}^{n-1} (c_j/c_n) e_j \right) \right\| \\ & \geq |1 - c_n| - |c_n| \frac{1}{2} \geq 1 - \frac{3}{2} |c_n| > 1 - \frac{3}{2} \frac{1}{2} = \frac{1}{4}, \end{aligned}$$

a contradiction. This proves the desired estimate. This proves the lemma.

The following is the main result.

Theorem 13.3.3 *Let X be a separable Banach space. Then there exists a sequence, $\{e_k^*\}_{k=1}^\infty \subseteq X'$ and $\{e_k\}$ as in Lemma 13.3.2 such that for every $x \in X$,*

$$x = \sum_{k=1}^{\infty} e_k^*(x) e_k.$$

Proof: The case where X is finite dimensional is left to the reader. Assume then that $\dim(X) = \infty$. Then letting $\{F_n\}$ be the sequence of closed finite dimensional subspaces described in Lemma 13.3.2, $F_n = \text{span}(e_1, \dots, e_n)$. Also let G_k be as defined there,

$$G_k \equiv \text{span}(\{e_j : j \neq k\}).$$

Then, as shown in Lemma 13.3.2,

$$\text{dist}(e_k, \overline{G_k}) \geq 1/4.$$

By Lemma 13.2.9 there exists $e_k^* \in X'$ such that

$$e_k^*(e_1) = \|e_k\|_X, \quad e_k^*(G_k) = \{0\}.$$

Now let $x \in X$ be given. By Lemma 13.3.2 there exist scalars $\{c_k\}$ such that

$$x = \sum_{k=1}^{\infty} c_k e_k.$$

Then doing e_m^* to both sides, yields

$$e_m^*(x) = \sum_{k=1}^{\infty} c_k e_m^*(e_k) = c_m$$

This proves the theorem.

13.4 Weak And Weak * Topologies

13.4.1 Basic Definitions

Let X be a Banach space and let X' be its dual space.¹ For A' a finite subset of X' , denote by $\rho_{A'}$ the function defined on X

$$\rho_{A'}(x) \equiv \max_{x^* \in A'} |x^*(x)| \quad (13.4.11)$$

and also let $B_{A'}(x, r)$ be defined by

$$B_{A'}(x, r) \equiv \{y \in X : \rho_{A'}(y - x) < r\} \quad (13.4.12)$$

Then certain things are obvious. First of all, if $a \in \mathbb{F}$ and $x, y \in X$,

$$\begin{aligned} \rho_{A'}(x + y) &\leq \rho_{A'}(x) + \rho_{A'}(y), \\ \rho_{A'}(ax) &= |a| \rho_{A'}(x). \end{aligned}$$

Similarly, letting A be a finite subset of X , denote by ρ_A the function defined on X'

$$\rho_A(x^*) \equiv \max_{x \in A} |x^*(x)| \quad (13.4.13)$$

and let $B_A(x^*, r)$ be defined by

$$B_A(x^*, r) \equiv \{y^* \in X' : \rho_A(y^* - x^*) < r\}. \quad (13.4.14)$$

It is also clear that

$$\begin{aligned} \rho_A(x^* + y^*) &\leq \rho_A(x^*) + \rho_A(y^*), \\ \rho_A(ax^*) &= |a| \rho_A(x^*). \end{aligned}$$

Lemma 13.4.1 *The sets, $B_{A'}(x, r)$ where A' is a finite subset of X' and $x \in X$ form a basis for a topology on X known as the weak topology. The sets $B_A(x^*, r)$ where A is a finite subset of X and $x^* \in X'$ form a basis for a topology on X' known as the weak * topology.*

Proof: The two assertions are very similar. I will verify the one for the weak topology. The union of these sets, $B_{A'}(x, r)$ for $x \in X$ and $r > 0$ is all of X . Now suppose z is contained in the intersection of two of these sets. Say

$$z \in B_{A'}(x, r) \cap B_{A'_1}(x_1, r_1)$$

Then let $C' = A' \cup A'_1$ and let

$$0 < \delta \leq \min \left(r - \rho_{A'}(z - x), r_1 - \rho_{A'_1}(z - x_1) \right).$$

¹Actually, all this works in much more general settings than this.

Consider $y \in B_{C'}(z, \delta)$. Then

$$r - \rho_{A'}(z - x) \geq \delta > \rho_{C'}(y - z) \geq \rho_{A'}(y - z)$$

and so

$$r > \rho_{A'}(y - z) + \rho_{A'}(z - x) \geq \rho_{A'}(y - x)$$

which shows $y \in B_{A'}(x, r)$. Similar reasoning shows $y \in B_{A'_1}(x_1, r_1)$ and so

$$B_{C'}(z, \delta) \subseteq B_{A'}(x, r) \cap B_{A'_1}(x_1, r_1).$$

Therefore, the weak topology consists of the union of all sets of the form $B_A(x, r)$.

13.4.2 Banach Alaoglu Theorem

Why does anyone care about these topologies? The short answer is that in the weak * topology, closed unit ball in X' is compact. This is not true in the normal topology. This wonderful result is the Banach Alaoglu theorem. First recall the notion of the product topology, and the Tychonoff theorem, Theorem 11.2.4 on Page 325 which are stated here for convenience.

Definition 13.4.2 Let I be a set and suppose for each $i \in I$, (X_i, τ_i) is a nonempty topological space. The Cartesian product of the X_i , denoted by $\prod_{i \in I} X_i$, consists of the set of all choice functions defined on I which select a single element of each X_i . Thus $f \in \prod_{i \in I} X_i$ means for every $i \in I$, $f(i) \in X_i$. The axiom of choice says $\prod_{i \in I} X_i$ is nonempty. Let

$$P_j(A) = \prod_{i \in I} B_i$$

where $B_i = X_i$ if $i \neq j$ and $B_j = A$. A subbasis for a topology on the product space consists of all sets $P_j(A)$ where $A \in \tau_j$. (These sets have an open set from the topology of X_j in the j^{th} slot and the whole space in the other slots.) Thus a basis consists of finite intersections of these sets. Note that the intersection of two of these basic sets is another basic set and their union yields $\prod_{i \in I} X_i$. Therefore, they satisfy the condition needed for a collection of sets to serve as a basis for a topology. This topology is called the product topology and is denoted by $\prod \tau_i$.

Theorem 13.4.3 If (X_i, τ_i) is compact, then so is $(\prod_{i \in I} X_i, \prod \tau_i)$.

The Banach Alaoglu theorem is as follows.

Theorem 13.4.4 Let B' be the closed unit ball in X' . Then B' is compact in the weak * topology.

Proof: By the Tychonoff theorem, Theorem 13.4.3

$$P \equiv \prod_{x \in X} \overline{B(0, \|x\|)}$$

is compact in the product topology where the topology on $\overline{B(0, \|x\|)}$ is the usual topology of $\mathbb{F}$. Recall P is the set of functions which map a point, $x \in X$ to a point in $\overline{B(0, \|x\|)}$. Therefore, $B' \subseteq P$. Also the basic open sets in the weak $*$ topology on B' are obtained as the intersection of basic open sets in the product topology of P to B' and so it suffices to show B' is a closed subset of P . Suppose then that $f \in P \setminus B'$. It follows f cannot be linear. There are two ways this can happen. One way is that for some x, y

$$f(x+y) \neq f(x) + f(y)$$

for some $x, y \in X$. However, if g is close enough to f at the three points, $x+y, x$, and y , the above inequality will hold for g in place of f . In other words there is a basic open set containing f such that for all g in this basic open set, $g \notin B'$. A similar consideration applies in case $f(\lambda x) \neq \lambda f(x)$ for some scalar, λ and x . Since $P \setminus B'$ is open, it follows B' is a closed subset of P and is therefore, compact. This proves the theorem.

Sometimes one can consider the weak $*$ topology in terms of a metric space.

Theorem 13.4.5 *If $K \subseteq X'$ is compact in the weak $*$ topology and X is separable then there exists a metric, d , on K such that if τ_d is the topology on K induced by d and if τ is the topology on K induced by the weak $*$ topology of X' , then $\tau = \tau_d$. Thus one can consider K with the weak $*$ topology as a metric space.*

Proof: Let $D = \{x_n\}$ be the dense countable subset. The metric is

$$d(f, g) \equiv \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_{x_n}(f-g)}{1 + \rho_{x_n}(f-g)}$$

where $\rho_{x_n}(f) = |f(x_n)|$. Clearly $d(f, g) = d(g, f) \geq 0$. If $d(f, g) = 0$, then this requires $f(x_n) = g(x_n)$ for all $x_n \in D$. Since f and g are continuous and D is dense, this requires that $f(x) = g(x)$ for all x . It is routine to verify the triangle inequality from the easy to establish inequality,

$$\frac{x}{1+x} + \frac{y}{1+y} \geq \frac{x+y}{1+x+y},$$

valid whenever $x, y \geq 0$. Therefore this is a metric. Now for each n

$$g \rightarrow \frac{\rho_{x_n}(f-g)}{1 + \rho_{x_n}(f-g)}$$

is a continuous function from (K, τ) to $[0, \infty)$ and also the above sum defining d converges uniformly. It follows

$$g \rightarrow d(f, g)$$

is continuous. Therefore, the ball with respect to d ,

$$B_d(f, r) \equiv \{g \in K : d(g, f) < r\}$$

is open in τ which implies $\tau_d \subseteq \tau$.

Now suppose $U \in \tau$. Then $K \setminus U$ is closed in K . Hence, $K \setminus U$ is compact in τ because it is a closed subset of the compact set K . It follows that $K \setminus U$ is compact with respect to τ_d because $\tau_d \subseteq \tau$. But (K, τ_d) is a Hausdorff space and so $K \setminus U$ must be closed with respect to τ_d . This implies $U \in \tau_d$. Thus $\tau \subseteq \tau_d$ and this proves $\tau = \tau_d$. This proves the theorem.

The fact that this set with the weak * topology can be considered a metric space is very significant because if a point is a limit point in a metric space, one can extract a convergent sequence.

Corollary 13.4.6 *If X is separable and $K \subseteq X'$ is compact in the weak * topology, then K is sequentially compact. That is, if $\{f_n\}_{n=1}^\infty \subseteq K$, then there exists a subsequence f_{n_k} and $f \in K$ such that for all $x \in X$,*

$$\lim_{k \rightarrow \infty} f_{n_k}(x) = f(x).$$

Proof: By Theorem 13.4.5, K is a metric space for the metric described there and it is compact. Therefore by the characterization of compact metric spaces, Proposition 6.2.5 on Page 142, K is sequentially compact. This proves the corollary.

13.4.3 Eberlein Smulian Theorem

Next consider the weak topology. The most interesting results have to do with a reflexive Banach space. The following lemma ties together the weak and weak * topologies in the case of a reflexive Banach space.

Lemma 13.4.7 *Let $J : X \rightarrow X''$ be the James map*

$$Jx(f) \equiv f(x)$$

and let X be reflexive so that J is onto. Then J is a homeomorphism of $(X, \text{weak topology})$ and $(X'', \text{weak * topology})$. This means J is one to one, onto, and both J and J^{-1} are continuous.

Proof: Let $f \in X'$ and let

$$B_f(x, r) \equiv \{y : |f(x) - f(y)| < r\}.$$

Thus $B_f(x, r)$ is a subbasic set for the weak topology on X . Now by the definition of J ,

$$y \in B_f(x, r) \text{ if and only if } |Jy(f) - Jx(f)| < r$$

$$\text{if and only if } Jy \in B_f(Jx, r) \equiv$$

$$\{y^{**} \in X'' : |y^{**}(f) - J(x)(f)| < r\},$$

a subbasic set for the weak * topology on X'' . Since J^{-1} and J are one to one and onto and map subbasic sets to subbasic sets, it follows that J is a homeomorphism. This proves the Lemma.

The following is an easy corollary.

Corollary 13.4.8 *If X is a reflexive Banach space, then the closed unit ball is weakly compact.*

Proof: Let B be the closed unit ball. Then $B = J^{-1}(B^{**})$ where B^{**} is the unit ball in X'' which is compact in the weak $*$ topology. Therefore B is weakly compact because J^{-1} is continuous.

Corollary 13.4.9 *Let X be a reflexive Banach space. If $K \subseteq X$ is compact in the weak topology and X' is separable then there exists a metric d , on K such that if τ_d is the topology on K induced by d and if τ is the topology on K induced by the weak topology of X , then $\tau = \tau_d$. Thus one can consider K with the weak topology as a metric space.*

Proof: This follows from Theorem 13.4.5 and Lemma 13.4.7. Lemma 13.4.7 implies $J(K)$ is compact in X'' . Then since X' is separable, there is a metric, d'' on $J(K)$ which delivers the weak $*$ topology. Let $d(x, y) \equiv d''(Jx, Jy)$. Then

$$(K, \tau_d) \xrightarrow{J} (J(K), \tau_{d''}) \xrightarrow{id} (J(K), \tau_{\text{weak } *}) \xrightarrow{J^{-1}} (K, \tau_{\text{weak}})$$

and all the maps are homeomorphisms.

Next is the Eberlein Smulian theorem which states that a Banach space is reflexive if and only if the closed unit ball is weakly sequentially compact. Actually, only half the theorem is proved here, the more useful only if part. The book by Yoshida [67] has the complete theorem discussed. First here is an interesting lemma for its own sake.

Lemma 13.4.10 *A closed subspace of a reflexive Banach space is reflexive.*

Proof: Let Y be the closed subspace of the reflexive space, X . Consider the following diagram

$$\begin{array}{ccc} Y'' & \xrightarrow{i^{**} 1-1} & X'' \\ Y' & \xleftarrow{i^* \text{ onto}} & X' \\ Y & \xrightarrow{i} & X \end{array}$$

This diagram follows from Theorem 13.2.10 on Page 376, the theorem on adjoints. Now let $y^{**} \in Y''$. Then $i^{**}y^{**} = J_X(y)$ because X is reflexive. I want to show that $y \in Y$. If it is not in Y then since Y is closed, there exists $x^* \in X'$ such that $x^*(y) \neq 0$ but $x^*(Y) = 0$. Then $i^*x^* = 0$. Hence

$$0 = y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = J(y)(x^*) = x^*(y) \neq 0,$$

a contradiction. Hence $y \in Y$. Letting J_Y denote the James map from Y to Y'' and $x^* \in X'$,

$$\begin{aligned} y^{**}(i^*x^*) &= i^{**}y^{**}(x^*) = J_X(y)(x^*) \\ &= x^*(y) = x^*(iy) = i^*x^*(y) = J_Y(y)(i^*x^*) \end{aligned}$$

Since i^* is onto, this shows $y^{**} = J_Y(y)$ and this proves the lemma.

Theorem 13.4.11 (Eberlein Smulian) *The closed unit ball in a reflexive Banach space X , is weakly sequentially compact. By this is meant that if $\{x_n\}$ is contained in the closed unit ball, there exists a subsequence, $\{x_{n_k}\}$ and $x \in X$ such that for all $x^* \in X'$,*

$$x^*(x_{n_k}) \rightarrow x^*(x).$$

Proof: Let $\{x_n\} \subseteq B \equiv \overline{B(0,1)}$. Let Y be the closure of the linear span of $\{x_n\}$. Thus Y is a separable. It is reflexive because it is a closed subspace of a reflexive space so the above lemma applies. By the Banach Alaoglu theorem, the closed unit ball B^* in Y' is weak * compact. Also by Theorem 13.4.5, B^* is a metric space with a suitable metric. Thus B^* is complete and totally bounded with respect to this metric and it follows that B^* with the weak * topology is separable. This implies Y' is also separable in the weak * topology. To see this, let $\{y_n^*\} \equiv D$ be a weak * dense set in B^* and let $y^* \in Y'$. Let p be a large enough positive rational number that $y^*/p \in B^*$. Then if A is any finite set from Y , there exists $y_n^* \in D$ such that $\rho_A(y^*/p - y_n^*) < \frac{\varepsilon}{p}$. It follows $py_n^* \in B_A(y^*, \varepsilon)$ showing that rational multiples of D are weak * dense in Y' . Since Y is reflexive, the weak and weak * topologies on Y' coincide and so Y' is weakly separable. Since Y' is separable, Corollary 13.4.6 implies B^{**} , the closed unit ball in Y'' is weak * sequentially compact. Then by Lemma 13.4.7 B , the unit ball in Y , is weakly sequentially compact. It follows there exists a subsequence x_{n_k} , of the sequence $\{x_n\}$ and a point $x \in Y$, such that for all $f \in Y'$,

$$f(x_{n_k}) \rightarrow f(x).$$

Now if $x^* \in X'$, and i is the inclusion map of Y into X ,

$$x^*(x_{n_k}) = i^*x^*(x_{n_k}) \rightarrow i^*x^*(x) = x^*(x).$$

which shows x_{n_k} converges weakly and this shows the unit ball in X is weakly sequentially compact.

Corollary 13.4.12 *Let $\{x_n\}$ be any bounded sequence in a reflexive Banach space, X . Then there exists $x \in X$ and a subsequence, $\{x_{n_k}\}$ such that for all $x^* \in X'$,*

$$\lim_{k \rightarrow \infty} x^*(x_{n_k}) = x^*(x)$$

Proof: If a subsequence, x_{n_k} has $\|x_{n_k}\| \rightarrow 0$, then the conclusion follows. Simply let $x = 0$. Suppose then that $\|x_n\|$ is bounded away from 0. That is, $\|x_n\| \in [\delta, C]$. Take a subsequence such that $\|x_{n_k}\| \rightarrow a$. Then consider $x_{n_k}/\|x_{n_k}\|$. By the Eberlein Smulian theorem, this subsequence has a further subsequence, $x_{n_{k_j}}/\|x_{n_{k_j}}\|$ which converges weakly to $x \in B$ where B is the closed unit ball. It follows from routine considerations that $x_{n_{k_j}} \rightarrow ax$ weakly. This proves the corollary.

13.5 Exercises

1. Is $\mathbb{N}$ a G_δ set? What about $\mathbb{Q}$? What about a countable dense subset of a complete metric space?
2. $\uparrow$ Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a function. Define the oscillation of a function in $B(x, r)$ by $\omega_r f(x) = \sup\{|f(z) - f(y)| : y, z \in B(x, r)\}$. Define the oscillation of the function at the point, x by $\omega f(x) = \lim_{r \rightarrow 0} \omega_r f(x)$. Show f is continuous at x if and only if $\omega f(x) = 0$. Then show the set of points where f is continuous is a G_δ set (try $U_n = \{x : \omega f(x) < \frac{1}{n}\}$). Does there exist a function continuous at only the rational numbers? Does there exist a function continuous at every irrational and discontinuous elsewhere? **Hint:** Suppose D is any countable set, $D = \{d_i\}_{i=1}^\infty$, and define the function, $f_n(x)$ to equal zero for every $x \notin \{d_1, \dots, d_n\}$ and 2^{-n} for x in this finite set. Then consider $g(x) \equiv \sum_{n=1}^\infty f_n(x)$. Show that this series converges uniformly.
3. Let $f \in C([0, 1])$ and suppose $f'(x)$ exists. Show there exists a constant, K , such that $|f(x) - f(y)| \leq K|x - y|$ for all $y \in [0, 1]$. Let $U_n = \{f \in C([0, 1]) \text{ such that for each } x \in [0, 1] \text{ there exists } y \in [0, 1] \text{ such that } |f(x) - f(y)| > n|x - y|\}$. Show that U_n is open and dense in $C([0, 1])$ where for $f \in C([0, 1])$,

$$\|f\| \equiv \sup\{|f(x)| : x \in [0, 1]\}.$$

Show that $\cap_n U_n$ is a dense G_δ set of nowhere differentiable continuous functions. Thus every continuous function is uniformly close to one which is nowhere differentiable.

4. $\uparrow$ Suppose $f(x) = \sum_{k=1}^\infty u_k(x)$ where the convergence is uniform and each u_k is a polynomial. Is it reasonable to conclude that $f'(x) = \sum_{k=1}^\infty u'_k(x)$? The answer is no. Use Problem 3 and the Weierstrass approximation theorem do show this.
5. Let X be a normed linear space. We say $A \subseteq X$ is “weakly bounded” if for each $x^* \in X'$, $\sup\{|x^*(x)| : x \in A\} < \infty$, while A is bounded if $\sup\{\|x\| : x \in A\} < \infty$. Show A is weakly bounded if and only if it is bounded.
6. Let X and Y be two Banach spaces. Define the norm

$$\|(x, y)\| \equiv \|x\|_X + \|y\|_Y.$$

Show this is a norm on $X \times Y$ which is equivalent to the norm given in the chapter for $X \times Y$. Can you do the same for the norm defined for $p > 1$ by

$$\|(x, y)\| \equiv (\|x\|_X^p + \|y\|_Y^p)^{1/p}?$$

7. Let f be a 2π periodic locally integrable function on $\mathbb{R}$. The Fourier series for f is given by

$$\sum_{k=-\infty}^{\infty} a_k e^{ikx} \equiv \lim_{n \rightarrow \infty} \sum_{k=-n}^n a_k e^{ikx} \equiv \lim_{n \rightarrow \infty} S_n f(x)$$

where

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x) dx.$$

Show

$$S_n f(x) = \int_{-\pi}^{\pi} D_n(x-y) f(y) dy$$

where

$$D_n(t) = \frac{\sin((n + \frac{1}{2})t)}{2\pi \sin(\frac{t}{2})}.$$

Verify that $\int_{-\pi}^{\pi} D_n(t) dt = 1$. Also show that if $g \in L^1(\mathbb{R})$, then

$$\lim_{a \rightarrow \infty} \int_{\mathbb{R}} g(x) \sin(ax) dx = 0.$$

This last is called the Riemann Lebesgue lemma. **Hint:** For the last part, assume first that $g \in C_c^\infty(\mathbb{R})$ and integrate by parts. Then exploit density of the set of functions in $L^1(\mathbb{R})$.

8. ↑ It turns out that the Fourier series sometimes converges to the function pointwise. Suppose f is 2π periodic and Hölder continuous. That is $|f(x) - f(y)| \leq K|x - y|^\theta$ where $\theta \in (0, 1]$. Show that if f is like this, then the Fourier series converges to f at every point. Next modify your argument to show that if at every point, x , $|f(x+) - f(y)| \leq K|x - y|^\theta$ for y close enough to x and larger than x and $|f(x-) - f(y)| \leq K|x - y|^\theta$ for every y close enough to x and smaller than x , then $S_n f(x) \rightarrow \frac{f(x+) + f(x-)}{2}$, the midpoint of the jump of the function. **Hint:** Use Problem 7.
9. ↑ Let $Y = \{f \text{ such that } f \text{ is continuous, defined on } \mathbb{R}, \text{ and } 2\pi \text{ periodic}\}$. Define $\|f\|_Y = \sup\{|f(x)| : x \in [-\pi, \pi]\}$. Show that $(Y, \|\cdot\|_Y)$ is a Banach space. Let $x \in \mathbb{R}$ and define $L_n(f) = S_n f(x)$. Show $L_n \in Y'$ but $\lim_{n \rightarrow \infty} \|L_n\| = \infty$. Show that for each $x \in \mathbb{R}$, there exists a dense G_δ subset of Y such that for f in this set, $|S_n f(x)|$ is unbounded. Finally, show there is a dense G_δ subset of Y having the property that $|S_n f(x)|$ is unbounded on the rational numbers. **Hint:** To do the first part, let $f(y)$ approximate $\text{sgn}(D_n(x-y))$. Here $\text{sgn } r = 1$ if $r > 0$, -1 if $r < 0$ and 0 if $r = 0$. This rules out one possibility of the uniform boundedness principle. After this, show the countable intersection of dense G_δ sets must also be a dense G_δ set.
10. Let $\alpha \in (0, 1]$. Define, for X a compact subset of $\mathbb{R}^p$,

$$C^\alpha(X; \mathbb{R}^n) \equiv \{\mathbf{f} \in C(X; \mathbb{R}^n) : \rho_\alpha(\mathbf{f}) + \|\mathbf{f}\| \equiv \|\mathbf{f}\|_\alpha < \infty\}$$

where

$$\|\mathbf{f}\| \equiv \sup\{|\mathbf{f}(\mathbf{x})| : \mathbf{x} \in X\}$$

and

$$\rho_\alpha(\mathbf{f}) \equiv \sup\left\{\frac{|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha} : \mathbf{x}, \mathbf{y} \in X, \mathbf{x} \neq \mathbf{y}\right\}.$$

Show that $(C^\alpha(X; \mathbb{R}^n), \|\cdot\|_\alpha)$ is a complete normed linear space. This is called a Holder space. What would this space consist of if $\alpha > 1$?

11. ↑ Now recall Problem 10 about the Holder spaces. Let X be the Holder functions which are periodic of period 2π . Define $L_n f(x) = S_n f(x)$ where $L_n : X \rightarrow Y$ for Y given in Problem 9. Show $\|L_n\|$ is bounded independent of n . Conclude that $L_n f \rightarrow f$ in Y for all $f \in X$. In other words, for the Holder continuous and 2π periodic functions, the Fourier series converges to the function uniformly. **Hint:** $L_n f(x)$ is given by

$$L_n f(x) = \int_{-\pi}^{\pi} D_n(y) f(x-y) dy$$

where $f(x-y) = f(x) + g(x, y)$ where $|g(x, y)| \leq C|y|^\alpha$. Use the fact the Dirichlet kernel integrates to one to write

$$\begin{aligned} \left| \int_{-\pi}^{\pi} D_n(y) f(x-y) dy \right| &\leq \overbrace{\left| \int_{-\pi}^{\pi} D_n(y) f(x) dy \right|}^{=|f(x)|} \\ &+ C \left| \int_{-\pi}^{\pi} \sin\left(\left(n + \frac{1}{2}\right)y\right) (g(x, y) / \sin(y/2)) dy \right| \end{aligned}$$

Show the functions, $y \rightarrow g(x, y) / \sin(y/2)$ are bounded in L^1 independent of x and get a uniform bound on $\|L_n\|$. Now use a similar argument to show $\{L_n f\}$ is equicontinuous in addition to being uniformly bounded. If $L_n f$ fails to converge to f uniformly, then there exists $\varepsilon > 0$ and a subsequence, n_k such that $\|L_{n_k} f - f\|_\infty \geq \varepsilon$ where this is the norm in Y or equivalently the sup norm on $[-\pi, \pi]$. By the Arzela Ascoli theorem, there is a further subsequence, $L_{n_{k_l}} f$ which converges uniformly on $[-\pi, \pi]$. But by Problem 8 $L_n f(x) \rightarrow f(x)$.

12. Let X be a normed linear space and let M be a convex open set containing 0. Define

$$\rho(x) = \inf\{t > 0 : \frac{x}{t} \in M\}.$$

Show ρ is a gauge function defined on X . This particular example is called a Minkowski functional. It is of fundamental importance in the study of locally convex topological vector spaces. A set, M , is convex if $\lambda x + (1 - \lambda)y \in M$ whenever $\lambda \in [0, 1]$ and $x, y \in M$.

13. ↑ The Hahn Banach theorem can be used to establish separation theorems. Let M be an open convex set containing 0. Let $x \notin M$. Show there exists $x^* \in X'$ such that $\operatorname{Re} x^*(x) \geq 1 > \operatorname{Re} x^*(y)$ for all $y \in M$. **Hint:** If $y \in M$, $\rho(y) < 1$.

Show this. If $x \notin M$, $\rho(x) \geq 1$. Try $f(\alpha x) = \alpha \rho(x)$ for $\alpha \in \mathbb{R}$. Then extend f to the whole space using the Hahn Banach theorem and call the result F , show F is continuous, then fix it so F is the real part of $x^* \in X'$.

14. A Banach space is said to be strictly convex if whenever $\|x\| = \|y\|$ and $x \neq y$, then

$$\left\| \frac{x+y}{2} \right\| < \|x\|.$$

$F : X \rightarrow X'$ is said to be a duality map if it satisfies the following: a.) $\|F(x)\| = \|x\|$. b.) $F(x)(x) = \|x\|^2$. Show that if X' is strictly convex, then such a duality map exists. The duality map is an attempt to duplicate some of the features of the Riesz map in Hilbert space which is discussed in the chapter on Hilbert space. **Hint:** For an arbitrary Banach space, let

$$F(x) \equiv \left\{ x^* : \|x^*\| \leq \|x\| \text{ and } x^*(x) = \|x\|^2 \right\}$$

Show $F(x) \neq \emptyset$ by using the Hahn Banach theorem on $f(\alpha x) = \alpha \|x\|^2$. Next show $F(x)$ is closed and convex. Finally show that you can replace the inequality in the definition of $F(x)$ with an equal sign. Now use strict convexity to show there is only one element in $F(x)$.

15. Prove the following theorem which is an improved version of the open mapping theorem, [20]. Let X and Y be Banach spaces and let $A \in \mathcal{L}(X, Y)$. Then the following are equivalent.

$$AX = Y,$$

A is an open map.

There exists a constant M such that for every $y \in Y$, there exists $x \in X$ with $y = Ax$ and

$$\|x\| \leq M \|y\|.$$

Note this gives the equivalence between A being onto and A being an open map. The open mapping theorem says that if A is onto then it is open.

16. Suppose $D \subseteq X$ and D is dense in X . Suppose $L : D \rightarrow Y$ is linear and $\|Lx\| \leq K\|x\|$ for all $x \in D$. Show there is a unique extension of L , $\tilde{L}$, defined on all of X with $\|\tilde{L}x\| \leq K\|x\|$ and $\tilde{L}$ is linear. You do not get uniqueness when you use the Hahn Banach theorem. Therefore, in the situation of this problem, it is better to use this result.
17. $\uparrow$ A Banach space is uniformly convex if whenever $\|x_n\|, \|y_n\| \leq 1$ and $\|x_n + y_n\| \rightarrow 2$, it follows that $\|x_n - y_n\| \rightarrow 0$. Show uniform convexity implies strict convexity (See Problem 14). **Hint:** Suppose it is not strictly convex. Then there exist $\|x\|$ and $\|y\|$ both equal to 1 and $\left\| \frac{x+y}{2} \right\| = 1$ consider $x_n \equiv x$ and $y_n \equiv y$, and use the conditions for uniform convexity to get a contradiction. It can be shown that L^p is uniformly convex whenever $\infty > p > 1$. See Hewitt and Stromberg [31] or Ray [56].

18. Show that a closed subspace of a reflexive Banach space is reflexive. **Hint:** The proof of this is an exercise in the use of the Hahn Banach theorem. Let Y be the closed subspace of the reflexive space X and let $y^{**} \in Y''$. Then $i^{**}y^{**} \in X''$ and so $i^{**}y^{**} = Jx$ for some $x \in X$ because X is reflexive. Now argue that $x \in Y$ as follows. If $x \notin Y$, then there exists x^* such that $x^*(Y) = 0$ but $x^*(x) \neq 0$. Thus, $i^*x^* = 0$. Use this to get a contradiction. When you know that $x = y \in Y$, the Hahn Banach theorem implies i^* is onto Y' and for all $x^* \in X'$,

$$y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = Jx(x^*) = x^*(x) = x^*(iy) = i^*x^*(y).$$

19. We say that x_n converges weakly to x if for every $x^* \in X'$, $x^*(x_n) \rightarrow x^*(x)$. $x_n \rightharpoonup x$ denotes weak convergence. Show that if $\|x_n - x\| \rightarrow 0$, then $x_n \rightharpoonup x$.
20. $\uparrow$ Show that if X is uniformly convex, then if $x_n \rightharpoonup x$ and $\|x_n\| \rightarrow \|x\|$, it follows $\|x_n - x\| \rightarrow 0$. **Hint:** Use Lemma 13.2.9 to obtain $f \in X'$ with $\|f\| = 1$ and $f(x) = \|x\|$. See Problem 17 for the definition of uniform convexity. Now by the weak convergence, you can argue that if $x \neq 0$, $f(x_n/\|x_n\|) \rightarrow f(x/\|x\|)$. You also might try to show this in the special case where $\|x_n\| = \|x\| = 1$.
21. Suppose $L \in \mathcal{L}(X, Y)$ and $M \in \mathcal{L}(Y, Z)$. Show $ML \in \mathcal{L}(X, Z)$ and that $(ML)^* = L^*M^*$.
22. Let X and Y be Banach spaces and suppose $f \in \mathcal{L}(X, Y)$ is compact. Recall this means that if B is a bounded set in X , then $f(B)$ has compact closure in Y . Show that f^* is also a compact map. **Hint:** Take a bounded subset of Y' , S . You need to show $f^*(S)$ is totally bounded. You might consider using the Ascoli Arzela theorem on the functions of S applied to $f(B)$ where B is the closed unit ball in X .

Locally Convex Topological Vector Spaces

14.1 Fundamental Considerations

The right context to consider certain topics like separation theorems is in locally convex topological vector spaces, a generalization of normed linear spaces. Let X be a vector space and let Ψ be a collection of functions defined on X such that if $\rho \in \Psi$,

$$\begin{aligned}\rho(x + y) &\leq \rho(x) + \rho(y), \\ \rho(ax) &= |a| \rho(x) \text{ if } a \in \mathbb{F}, \\ \rho(x) &\geq 0,\end{aligned}$$

where $\mathbb{F}$ denotes the field of scalars, either $\mathbb{R}$ or $\mathbb{C}$, assumed to be $\mathbb{C}$ unless otherwise specified. These functions are called seminorms because it is not necessarily true that $x = 0$ when $\rho(x) = 0$. A basis for a topology, $\mathcal{B}$, is defined as follows.

Definition 14.1.1 *For A a finite subset of Ψ and $r > 0$,*

$$B_A(x, r) \equiv \{y \in X : \rho(x - y) < r \text{ for all } \rho \in A\}.$$

Then

$$\mathcal{B} \equiv \{B_A(x, r) : x \in X, r > 0, \text{ and } A \subseteq \Psi, A \text{ finite}\}.$$

That this really is a basis is the content of the next theorem.

Theorem 14.1.2 *$\mathcal{B}$ is the basis for a topology.*

Proof: I need to show that if $B_A(x, r_1)$ and $B_B(y, r_2)$ are two elements of $\mathcal{B}$ and if $z \in B_A(x, r_1) \cap B_B(y, r_2)$, then there exists $U \in \mathcal{B}$ such that

$$z \in U \subseteq B_A(x, r_1) \cap B_B(y, r_2).$$

Let

$$r = \min(\min\{(r_1 - \rho(z - x)) : \rho \in A\},$$

$$\min\{(r_2 - \rho(z - y)) : \rho \in B\}$$

and consider $B_{A \cup B}(z, r)$. If w belongs to this set, then for $\rho \in A$,

$$\rho(w - z) < r_1 - \rho(z - x).$$

Hence

$$\rho(w - x) \leq \rho(w - z) + \rho(z - x) < r_1$$

for each $\rho \in A$ and so $B_{A \cup B}(z, r) \subseteq B_A(x, r_1)$. Similarly, $B_{A \cup B}(z, r) \subseteq B_B(y, r_2)$. This proves the theorem.

Let τ be the topology consisting of unions of all subsets of $\mathcal{B}$. Then (X, τ) is a locally convex topological vector space.

Theorem 14.1.3 *The vector space operations of addition and scalar multiplication are continuous. More precisely,*

$$+ : X \times X \rightarrow X, \quad \cdot : \mathbb{F} \times X \rightarrow X$$

are continuous.

Proof: It suffices to show $+^{-1}(B)$ is open in $X \times X$ and $\cdot^{-1}(B)$ is open in $\mathbb{F} \times X$ if B is of the form

$$B = \{y \in X : \rho(y - x) < r\}$$

because finite intersections of such sets form the basis $\mathcal{B}$. (This collection of sets is a subbasis.) Suppose $u + v \in B$ where B is described above. Then

$$\rho(u + v - x) < \lambda r$$

for some $\lambda < 1$. Consider

$$B_\rho(u, \delta) \times B_\rho(v, \delta).$$

If (u_1, v_1) is in this set, then

$$\begin{aligned} \rho(u_1 + v_1 - x) &\leq \rho(u + v - x) + \rho(u_1 - u) + \rho(v_1 - v) \\ &< \lambda r + 2\delta. \end{aligned}$$

Let δ be positive but small enough that

$$2\delta + \lambda r < r.$$

Thus this choice of δ shows that $+^{-1}(B)$ is open and this shows $+$ is continuous.

Now suppose $\alpha z \in B$. Then

$$\rho(\alpha z - x) < \lambda r < r$$

for some $\lambda \in (0, 1)$. Let $\delta > 0$ be small enough that $\delta < 1$ and also

$$\lambda r + \delta(\rho(z) + 1) + \delta|\alpha| < r.$$

Then consider $(\beta, w) \in B(\alpha, \delta) \times B_\rho(z, \delta)$.

$$\begin{aligned} \rho(\beta w - x) - \rho(\alpha z - x) &\leq \rho(\beta w - \alpha z) \\ &\leq |\beta - \alpha| \rho(w) + \rho(w - z) |\alpha| \\ &\leq |\beta - \alpha| (\rho(z) + 1) + \rho(w - z) |\alpha| \\ &< \delta (\rho(z) + 1) + \delta |\alpha|. \end{aligned}$$

Hence

$$\rho(\beta w - x) < \lambda r + \delta (\rho(z) + 1) + \delta |\alpha| < r$$

and so

$$B(\alpha, \delta) \times B_\rho(z, \delta) \subseteq \cdot^{-1}(B).$$

This proves the theorem.

Theorem 14.1.4 *Let x be given and let $f_x(y) = x + y$. Then f_x is 1-1, onto, and continuous. If $\alpha \neq 0$ and $g_\alpha(x) = \alpha x$, then g_α is also 1-1 onto and continuous.*

Proof: The assertions about 1-1 and onto are obvious. It remains to show f_x and g_α are continuous. Let $B = B_\rho(z, r)$ and consider $f_x^{-1}(B)$. Then it is easy to see that

$$f_x^{-1}(B) = B_\rho(z - x, r)$$

and so f_x is continuous. To see that g_α is continuous, note that

$$g_\alpha^{-1}(B) = B_\rho\left(\frac{z}{\alpha}, \frac{r}{|\alpha|}\right).$$

This proves the theorem.

As in the case of a normed linear space, the vector space of continuous linear functionals, is denoted by X' .

Definition 14.1.5 *Define, for A a finite subset of Ψ ,*

$$\rho_A(x) = \max\{\rho(x) : \rho \in A\}.$$

The following theorem is the equivalent to the earlier theorems concerning continuous linear functionals on normed linear spaces.

Theorem 14.1.6 *The following are equivalent for f , a linear function mapping X to $\mathbb{F}$.*

$$f \text{ is continuous at } 0. \quad (14.1.1)$$

For some $A \subseteq \Psi$, A finite,

$$|f(x)| \leq C \rho_A(x) \quad (14.1.2)$$

for all $x \in X$ where the constant may depend on A but is independent of x .

$$f \text{ is continuous at } x \quad (14.1.3)$$

for all x .

Proof: Clearly 14.1.3 implies 14.1.1. Suppose 14.1.1. Then

$$0 = f(0) \in B(0, 1) \subseteq \mathbb{F}.$$

Since f is continuous at 0, $0 \in f^{-1}(B(0, 1))$ and there exists an open set $V \in \tau$ such that

$$0 \in V \subseteq f^{-1}(B(0, 1)).$$

Then $0 \in B_A(0, r) \subseteq V$ for some r and some $A \subseteq \Psi$, A finite. Hence

$$|f(y)| < 1 \text{ if } \rho_A(y) < r.$$

Since f is linear

$$|f(x)| \leq \frac{2}{r} \rho_A(x).$$

To see this, note that if $x \neq 0$, then

$$\frac{rx}{2\rho_A(x)} \in B_A(0, r)$$

and so

$$\frac{|f(rx)|}{2\rho_A(x)} \leq 1$$

which shows that 14.1.1 implies 14.1.2.

Now suppose 14.1.2 and suppose $f(x) \in V$, an open set in $\mathbb{F}$. Then

$$f(x) \in B(f(x), r) \subseteq V$$

for some $r > 0$. Suppose $\rho_A(x - y) < r(C_A + 1)^{-1}$. Then

$$|f(x) - f(y)| = |f(x - y)| \leq C_A \rho_A(y - x) < r.$$

Hence

$$f\left(B_A\left(x, r(C_A + 1)^{-1}\right)\right) \subseteq B(f(x), r) \subseteq V.$$

Thus f is continuous at x . This proves the theorem.

What are some examples of locally convex topological vector spaces? It is obvious that any normed linear space is such an example. More generally, here is a theorem which shows how to make any vector space into a locally convex topological vector space.

Theorem 14.1.7 *Let X be a vector space and let Y be a vector space of linear functionals defined on X . For each $y \in Y$, define*

$$\rho_y(x) \equiv |y(x)|.$$

Then the collection of seminorms $\{\rho_y\}_{y \in Y}$ defined on X makes X into a locally convex topological vector space and $Y = X'$.

Proof: Clearly $\{\rho_y\}_{y \in Y}$ is a collection of seminorms defined on X ; so, X supplied with the topology induced by this collection of seminorms is a locally convex topological vector space. Is $Y = X'$?

Let $y \in Y$, let $U \subseteq \mathbb{F}$ be open and let $x \in y^{-1}(U)$. Then $B(y(x), r) \subseteq U$ for some $r > 0$. Letting $A = \{y\}$, it is easy to see from the definition that $B_A(x, r) \subseteq y^{-1}(U)$ and so $y^{-1}(U)$ is an open set as desired. Thus, $Y \subseteq X'$.

Now suppose $z \in X'$. Then by 14.1.2, there exists a finite subset of Y , $A \equiv \{y_1, \dots, y_n\}$, such that

$$|z(x)| \leq C\rho_A(x).$$

Let

$$\pi(x) \equiv (y_1(x), \dots, y_n(x))$$

and let f be a linear map from $\pi(X)$ to $\mathbb{F}$ defined by

$$f(\pi x) \equiv z(x).$$

(This is well defined because if $\pi(x) = \pi(x_1)$, then $y_i(x) = y_i(x_1)$ for $i = 1, \dots, n$ and so

$$\rho_A(x - x_1) = 0.$$

Thus,

$$|z(x_1) - z(x)| = |z(x_1 - x)| \leq C\rho_A(x - x_1) = 0.)$$

Extend f to all of $\mathbb{F}^n$ and denote the resulting linear map by F . Then there exists a vector

$$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$$

with $\alpha_i = F(\mathbf{e}_i)$ such that

$$F(\beta) = \alpha \cdot \beta.$$

Hence for each $x \in X$,

$$z(x) = f(\pi x) = F(\pi x) = \sum_{i=1}^n \alpha_i y_i(x)$$

and so

$$z = \sum_{i=1}^n \alpha_i y_i \in Y.$$

This proves the theorem.

14.2 Separation Theorems

It will always be assumed that X is a locally convex topological vector space. A set, K , is said to be convex if whenever $x, y \in K$,

$$\lambda x + (1 - \lambda)y \in K$$

for all $\lambda \in [0, 1]$.

Definition 14.2.1 Let U be an open convex set containing 0 and define

$$m(x) \equiv \inf\{t > 0 : x/t \in U\}.$$

This is called a Minkowski functional.

Proposition 14.2.2 Let X be a locally convex topological vector space. Then m is defined on X and satisfies

$$m(x+y) \leq m(x) + m(y) \quad (14.2.4)$$

$$m(\lambda x) = \lambda m(x) \quad \text{if } \lambda > 0. \quad (14.2.5)$$

Thus, m is a gauge function on X .

Proof: Let $x \in X$ be arbitrary. There exists $A \subseteq \Psi$ such that

$$0 \in B_A(0, r) \subseteq U.$$

Then

$$\frac{rx}{2\rho_A(x)} \in B_A(0, r) \subseteq U$$

which implies

$$\frac{2\rho_A(x)}{r} \geq m(x). \quad (14.2.6)$$

Thus $m(x)$ is defined on X .

Let $x/t \in U$, $y/s \in U$. Then since U is convex,

$$\frac{x+y}{t+s} = \left(\frac{t}{t+s}\right)\left(\frac{x}{t}\right) + \left(\frac{s}{t+s}\right)\left(\frac{y}{s}\right) \in U.$$

It follows that

$$m(x+y) \leq t+s.$$

Choosing s, t such that $t - \varepsilon < m(x)$ and $s - \varepsilon < m(y)$,

$$m(x+y) \leq m(x) + m(y) + 2\varepsilon.$$

Since ε is arbitrary, this shows 14.2.4. It remains to show 14.2.5. Let $x/t \in U$. Then if $\lambda > 0$,

$$\frac{\lambda x}{\lambda t} \in U$$

and so $m(\lambda x) \leq \lambda t$. Thus $m(\lambda x) \leq \lambda m(x)$ for all $\lambda > 0$. Hence

$$m(x) = m(\lambda^{-1}\lambda x) \leq \lambda^{-1}m(\lambda x) \leq \lambda^{-1}\lambda m(x) = m(x)$$

and so

$$\lambda m(x) = m(\lambda x).$$

This proves the proposition.

Lemma 14.2.3 *Let U be an open convex set containing 0 and let $q \notin U$. Then there exists $f \in X'$ such that*

$$\operatorname{Re} f(q) > \operatorname{Re} f(x)$$

for all $x \in U$.

Proof: Let m be the Minkowski functional just defined and let

$$F(cq) = cm(q)$$

for $c \in \mathbb{R}$. If $c > 0$ then

$$F(cq) = m(cq)$$

while if $c \leq 0$,

$$F(cq) = cm(q) \leq 0 \leq m(cq).$$

By the Hahn Banach theorem, F has an extension, g , defined on all of X satisfying

$$g(x+y) = g(x) + g(y), \quad g(cx) = cg(x)$$

for all $c \in \mathbb{R}$, and

$$g(x) \leq m(x).$$

Thus, $g(-x) \leq m(-x)$ and so

$$-m(-x) \leq g(x) \leq m(x).$$

It follows as in 14.2.6 that for some $A \subseteq \Psi$, A finite, and $r > 0$,

$$\begin{aligned} |g(x)| &\leq m(x) + m(-x) \\ &\leq \frac{2}{r}\rho_A(x) + \frac{2}{r}\rho_A(-x) = \frac{4}{r}\rho_A(x) \end{aligned}$$

because

$$\rho_A(-x) = |-1|\rho_A(x) = \rho_A(x).$$

Hence g is continuous by Theorem 14.1.6. Now define

$$f(x) \equiv g(x) - ig(ix).$$

Thus f is linear and continuous so $f \in X'$ and $\operatorname{Re} f(x) = g(x)$. But for $x \in U$, Theorem 14.1.3 implies that $x/t \in U$ for some $t < 1$ and so $m(x) < 1$. Since U is convex and $0 \in U$, it follows $q/t \notin U$ if $t < 1$ because if it were,

$$q = t\left(\frac{q}{t}\right) + (1-t)0 \in U.$$

Therefore, $m(q) \geq 1$ and for $x \in U$,

$$\operatorname{Re} f(x) = g(x) \leq m(x) < 1 \leq m(q) = g(q) = \operatorname{Re} f(q)$$

and this proves the lemma.

Theorem 14.2.4 *Let K be closed and convex in a locally convex topological vector space and let $p \notin K$. Then there exists a real number, c , and $f \in X'$ such that*

$$\operatorname{Re} f(p) > c > \operatorname{Re} f(k)$$

for all $k \in K$.

Proof: Since K is closed, and $p \notin K$, there exists a finite subset of Ψ, A , and a positive $r > 0$ such that

$$K \cap B_A(p, 2r) = \emptyset.$$

Pick $k_0 \in K$ and let

$$U = K + B_A(0, r) - k_0, \quad q = p - k_0.$$

It follows that U is an open convex set containing 0 and $q \notin U$. Therefore, by Lemma 14.2.3, there exists $f \in X'$ such that

$$\operatorname{Re} f(p - k_0) = \operatorname{Re} f(q) > \operatorname{Re} f(k + e - k_0) \quad (14.2.7)$$

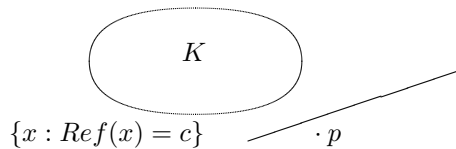
for all $k \in K$ and $e \in B_A(0, r)$. If $\operatorname{Re} f(e) = 0$ for all $e \in B_A(0, r)$, then $\operatorname{Re} f = 0$ and 14.2.7 could not hold. Therefore, $\operatorname{Re} f(e) > 0$ for some $e \in B_A(0, r)$ and so,

$$\operatorname{Re} f(p) > \operatorname{Re} f(k) + \operatorname{Re} f(e)$$

for all $k \in K$. Let $c_1 \equiv \sup \{\operatorname{Re} f(k) : k \in K\}$. Then for all $k \in K$,

$$\operatorname{Re} f(p) \geq c_1 + \operatorname{Re} f(e) > c_1 + \frac{\operatorname{Re} f(e)}{2} > \operatorname{Re} f(k).$$

Let $c = c_1 + \frac{\operatorname{Re} f(e)}{2}$. This proves the theorem.



Note that if the field of scalars comes from $\mathbb{R}$ rather than $\mathbb{C}$ there is no essential change to the above conclusions. Just eliminate all references to the real part.

14.2.1 Convex Functionals

As an important application, this theorem gives the basis for proving something about lower semicontinuity of functionals.

Definition 14.2.5 *Let X be a Banach space and let $\phi : X \rightarrow (0, \infty]$ be convex and lower semicontinuous. This means whenever $x \in X$ and $\lim_{n \rightarrow \infty} x_n = x$,*

$$\phi(x) \leq \liminf_{n \rightarrow \infty} \phi(x_n).$$

Also assume ϕ is not identically equal to ∞ .

Lemma 14.2.6 *Let X, Y be two Banach spaces. Then letting*

$$\|(x, y)\| \equiv \max(\|x\|_X, \|y\|_Y),$$

it follows $X \times Y$ is a Banach space and $\phi \in (X \times Y)'$ if and only if there exist $x^* \in X'$ and $y^* \in Y'$ such that

$$\phi((x, y)) = x^*(x) + y^*(y).$$

The topology coming from this norm is called the strong topology.

Proof: Most of these conclusions are obvious. In particular it is clear $X \times Y$ is a Banach space with the given norm. Let $\phi \in (X \times Y)'$. Also let $\pi_X(x, y) \equiv (x, 0)$ and $\pi_Y(x, y) \equiv (0, y)$. Then each of π_X and π_Y is continuous and

$$\begin{aligned} \phi((x, y)) &= \phi(\pi_X + \pi_Y)((x, y)) \\ &= \phi((x, 0)) + \phi((0, y)). \end{aligned}$$

Thus $\phi \circ \pi_X$ and $\phi \circ \pi_Y$ are both continuous and their sum equals ϕ . Let $x^*(x) \equiv \phi \circ \pi_X(x, 0)$ and let $y^* \equiv \phi \circ \pi_Y(x, 0)$. Then it is clear both x^* and y^* are continuous and linear defined on X and Y respectively. Also, if $(x^*, y^*) \in X' \times Y'$, then if $\phi((x, y)) \equiv x^*(x) + y^*(y)$, it follows $\phi \in (X \times Y)'$. This proves the lemma.

Lemma 14.2.7 *Let ϕ be a functional as described in Definition 14.2.5. Then ϕ is lower semicontinuous if and only if the epigraph of ϕ is closed in $X \times \mathbb{R}$ with the strong topology. Here the epigraph is defined as*

$$\text{epi}(\phi) \equiv \{(x, y) : y \geq \phi(x)\}.$$

In this case the functional is called strongly lower semicontinuous.

Proof: First suppose $\text{epi}(\phi)$ is closed and suppose $x_n \rightarrow x$. Let $l < \phi(x)$. Then $(x, l) \notin \text{epi}(\phi)$ and so there exists $\delta > 0$ such that if $|x - y| < \delta$ and $|\alpha - l| < \delta$, then $\alpha < \phi(y)$. This implies that if $|x - y| < \delta$ and $\alpha < l + \delta$, then the above holds. Therefore, $(x_n, \phi(x_n))$, being in $\text{epi}(\phi)$ cannot satisfy both conditions,

$$|x_n - x| < \delta, \phi(x_n) < l + \delta.$$

However, for all n large enough, the first condition is satisfied. Consequently, for all n large enough, $\phi(x_n) \geq l + \delta \geq l$. Thus

$$\liminf_{n \rightarrow \infty} \phi(x_n) \geq l$$

and since $l < \phi(x)$ is arbitrary, it follows

$$\liminf_{n \rightarrow \infty} \phi(x_n) \geq \phi(x).$$

Next suppose the condition about the $\liminf$. If $\text{epi}(\phi)$ is not closed, then there exists $(x, l) \notin \text{epi}(\phi)$ which is a limit point of points of $\text{epi}(\phi)$. Thus there exists $(x_n, l_n) \in \text{epi}(\phi)$ such that $(x_n, l_n) \rightarrow (x, l)$ and so

$$l = \lim_{n \rightarrow \infty} \inf l_n \geq \lim_{n \rightarrow \infty} \inf \phi(x_n) \geq \phi(x),$$

contradicting $(x, l) \notin \text{epi}(\phi)$. This proves the lemma.

Definition 14.2.8 Let ϕ be convex and defined on X , a Banach space. Then ϕ is said to be weakly lower semicontinuous if $\text{epi}(\phi)$ is closed in $X \times \mathbb{R}$ where a basis for the topology of $X \times \mathbb{R}$ consists of sets of the form $U \times (a, b)$ for U a weakly open set in X .

Theorem 14.2.9 Let ϕ be a lower semicontinuous convex functional as described in Definition 14.2.5 and let X be a real Banach space. Then ϕ is also weakly lower semicontinuous.

Proof: By Lemma 14.2.7 $\text{epi}(\phi)$ is closed in $X \times \mathbb{R}$ with the strong topology as well as being convex. Letting $(z, l) \notin \text{epi}(\phi)$, it follows from Theorem 14.2.4 and Lemma 14.2.6 there exists $(x^*, \alpha) \in X' \times \mathbb{R}$ such that for some c

$$x^*(z) + \alpha l > c > x^*(x) + \alpha \beta$$

whenever $\beta \geq \phi(x)$. Consider $B_{\{(x^*, \alpha)\}}((z, l), r)$ where r is chosen so small that if $(y, \gamma) \in B_{\{(x^*, \alpha)\}}((z, l), r)$, then

$$x^*(y) + \alpha \gamma > c.$$

This shows that the complement of $\text{epi}(\phi)$ is weakly open and this proves the theorem.

Corollary 14.2.10 Let ϕ be a lower semicontinuous convex functional as described in Definition 14.2.5 and let X be a real Banach space. Then if x_n converges weakly to x , it follows that

$$\phi(x) \leq \lim_{n \rightarrow \infty} \inf \phi(x_n).$$

Proof: Let $l < \phi(x)$ so that $(x, l) \notin \text{epi}(\phi)$. Then by Theorem 14.2.9 there exists $B \times (-\infty, l + \delta)$ such that B is a weakly open set in X containing x and

$$B \times (-\infty, l + \delta) \subseteq \text{epi}(\phi)^C.$$

Thus $(x_n, \phi(x_n)) \notin B \times (-\infty, l + \delta)$ for all n . However, $x_n \in B$ for all n large enough. Therefore, for those values of n , it must be the case that $\phi(x_n) \notin (-\infty, l + \delta)$ and so

$$\lim_{n \rightarrow \infty} \inf \phi(x_n) \geq l + \delta \geq l$$

which shows, since $l < \phi(x)$ is arbitrary that

$$\lim_{n \rightarrow \infty} \inf \phi(x_n) \geq \phi(x).$$

This proves the corollary.

14.2.2 More Separation Theorems

There are other separation theorems which can be proved in a similar way. The next theorem considers the separation of an open convex set from a convex set.

Theorem 14.2.11 *Let A and B be disjoint, convex and nonempty sets with B open. Then there exists $f \in X'$ such that*

$$\operatorname{Re} f(a) < \operatorname{Re} f(b)$$

for all $a \in A$ and $b \in B$.

Proof: Let $b_0 \in B, a_0 \in A$. Then the set

$$B - A + a_0 - b_0$$

is open, convex, contains 0, and does not contain $a_0 - b_0$. By Lemma 14.2.3 there exists $f \in X'$ such that

$$\operatorname{Re} f(a_0 - b_0) > \operatorname{Re} f(b - a + a_0 - b_0)$$

for all $a \in A$ and $b \in B$. Therefore, for all $a \in A, b \in B$,

$$\operatorname{Re} f(b) > \operatorname{Re} f(a).$$

Before giving another separation theorem, here is a lemma.

Lemma 14.2.12 *If B is convex, then $\operatorname{int}(B) \equiv \text{union of all open sets contained in } B$ is convex. Also, if $\operatorname{int}(B) \neq \emptyset$, then $B \subseteq \overline{\operatorname{int}(B)}$.*

Proof: Suppose $x, y \in \operatorname{int}(B)$. Then there exists $r > 0$ and a finite set $A \subseteq \Psi$ such that

$$B_A(x, r), B_A(y, r) \subseteq B.$$

Let

$$V \equiv \cup_{\lambda \in [0,1]} \lambda B_A(x, r) + (1 - \lambda) B_A(y, r).$$

Then V is open, $V \subseteq B$, and if $\lambda \in [0, 1]$, then

$$\lambda x + (1 - \lambda)y \in V \subseteq B.$$

Therefore, $\operatorname{int}(B)$ is convex as claimed.

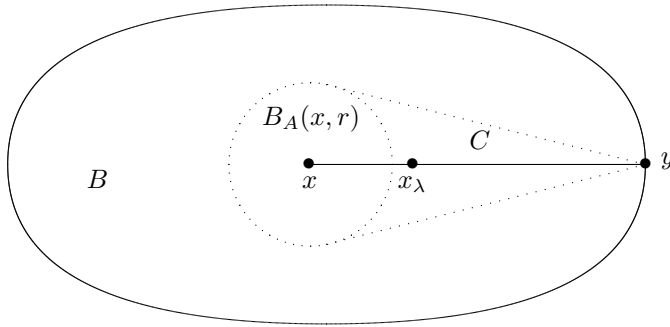
Now let $y \in B$ and $x \in \operatorname{int}(B)$. Let

$$x \in B_A(x, r) \subseteq \operatorname{int}(B)$$

and let $x_\lambda \equiv (1 - \lambda)x + \lambda y$. Define the open cone,

$$C \equiv \cup_{\lambda \in [0,1]} B_A(x_\lambda, (1 - \lambda)r).$$

Thus C is represented in the following picture.



I claim $C \subseteq B$ as suggested in the picture. To see this, let

$$z \in B_A(x_\lambda, (1 - \lambda)r), \lambda \in (0, 1).$$

Then

$$\rho_A(z - x_\lambda) < (1 - \lambda)r$$

and so

$$\rho_A\left(\frac{z}{1 - \lambda} - x - \frac{\lambda y}{1 - \lambda}\right) < r.$$

Therefore,

$$\frac{z}{1 - \lambda} - \frac{\lambda y}{1 - \lambda} \in B_A(x, r) \subseteq B.$$

It follows

$$(1 - \lambda)\left(\frac{z}{1 - \lambda} - \frac{\lambda y}{1 - \lambda}\right) + \lambda y = z \in B$$

and so $C \subseteq B$ as claimed. Now this shows $x_\lambda \in \text{int}(B)$ and $\lim_{\lambda \rightarrow 1} x_\lambda = y$. Thus, $y \in \overline{\text{int}(B)}$ and this proves the lemma.

Corollary 14.2.13 *Let A, B be convex, nonempty sets. Suppose $\text{int}(B) \neq \emptyset$ and $A \cap \text{int}(B) = \emptyset$. Then there exists $f \in X'$, $f \neq 0$, such that for all $a \in A$ and $b \in B$,*

$$\text{Re } f(b) \geq \text{Re } f(a).$$

Proof: By Theorem 14.2.11, there exists $f \in X'$ such that for all $b \in \text{int}(B)$, and $a \in A$,

$$\text{Re } f(b) > \text{Re } f(a).$$

Thus, in particular, $f \neq 0$. By Lemma 14.2.12, if $b \in B$ and $a \in A$,

$$\text{Re } f(b) \geq \text{Re } f(a).$$

This proves the theorem.

Lemma 14.2.14 *If X is a topological Hausdorff space then compact implies closed.*

Proof: Let K be compact and suppose K^C is not open. Then there exists $p \in K^C$ such that

$$V_p \cap K \neq \emptyset$$

for all open sets V_p containing p . Let

$$\mathcal{C} = \{(\overline{V_p})^C : V_p \text{ is an open set containing } p\}.$$

Then $\mathcal{C}$ is an open cover of K because if $q \in K$, there exist disjoint open sets V_p and V_q containing p and q respectively. Thus $q \in (\overline{V_p})^C$. This is an example of an open cover of K which has no finite subcover, contradicting the assumption that K is compact. This proves the lemma.

Lemma 14.2.15 *If X is a locally convex topological vector space, and if every point is a closed set, then the seminorms and X' separate the points. This means if $x \neq y$, then for some $\rho \in \Psi$,*

$$\rho(x - y) \neq 0$$

and for some $f \in X'$,

$$f(x) \neq f(y).$$

In this case, X is a Hausdorff space.

Proof: Let $x \neq y$. Then by Theorem 14.2.4, there exists $f \in X'$ such that $f(x) \neq f(y)$. Thus X' separates the points. Since $f \in X'$, Theorem 14.1.6 implies

$$|f(z)| \leq C\rho_A(z)$$

for some A a finite subset of Ψ . Thus

$$0 < |f(x - y)| \leq C\rho_A(x - y)$$

and so $\rho(x - y) \neq 0$ for some $\rho \in A \subseteq \Psi$. Now to show X is Hausdorff, let

$$0 < r < \rho(x - y) 2^{-1}.$$

Then the two disjoint open sets containing x and y respectively are

$$B_\rho(x, r) \text{ and } B_\rho(y, r).$$

This proves the lemma.

14.3 The Weak And Weak* Topologies

The weak and weak * topologies are examples which make the underlying vector space into a topological vector space. This section gives a description of these topologies. Unless otherwise specified, X is a locally convex topological vector space. For G a finite subset of X' define $\delta_G : X \rightarrow [0, \infty)$ by

$$\delta_G(x) = \max\{|f(x)| : f \in G\}.$$

Lemma 14.3.1 *The functions δ_G for G a finite subset of X' are seminorms and the sets*

$$B_G(x, r) \equiv \{y \in X : \delta_G(x - y) < r\}$$

form a basis for a topology on X . Furthermore, X with this topology is a locally convex topological vector space. If each point in X is a closed set, then the same is true of X with respect to this new topology.

Proof: It is obvious that the functions δ_G are seminorms and therefore the proof that the sets $B_G(x, r)$ form a basis for a topology is the same as in Theorem 14.1.2. To see every point is a closed set in this new topology, assuming this is true for X with the original topology, use Lemma 14.2.15 to assert X' separates the points. Let $x \in X$ and let $y \neq x$. There exists $f \in X'$ such that $f(x) \neq f(y)$. Let $G = \{f\}$ and consider

$$B_G(y, |f(x - y)|/2).$$

Then this open set does not contain x . Thus $\{x\}^C$ is open and so $\{x\}$ is closed. This proves the Lemma.

This topology for X is called the weak topology for X . For F a finite subset of X , define $\gamma_F : X' \rightarrow [0, \infty)$ by

$$\gamma_F(f) = \max\{|f(x)| : x \in F\}.$$

Lemma 14.3.2 *The functions γ_F for F a finite subset of X are seminorms and the sets*

$$B_F(f, r) \equiv \{g \in X' : \gamma_F(f - g) < r\}$$

form a basis for a topology on X' . Furthermore, X' with this topology is a locally convex topological vector space having the property that every point is a closed set.

Proof: The proof is similar to that of Lemma 14.3.1 but there is a difference in the part where every point is shown to be a closed set. Let $f \in X'$ and let $g \neq f$. Thus there exists $x \in X$ such that $f(x) \neq g(x)$. Let $F = \{x\}$. Then

$$B_F(g, |(f - g)(x)|/2)$$

contains g but not f . Thus $\{f\}^C$ is open and so $\{f\}$ is closed. This proves the lemma. Note that it was not necessary to assume points in X are closed sets to get this.

The topology for X' just described is called the weak $*$ topology. In terms of Theorem 14.1.7 the weak topology is obtained by letting $Y = X'$ in that theorem while the weak $*$ topology is obtained by letting $Y = X$ with the understanding that X is a vector space of linear functionals on X' defined by

$$x(x^*) \equiv x^*(x).$$

By Theorem 14.2.4, there is a useful result which follows immediately.

Theorem 14.3.3 *Let K be closed and convex in a Banach space X . Then it is also weakly closed. Furthermore, if $p \notin K$, there exists $f \in X'$ such that*

$$\operatorname{Re} f(p) > c > \operatorname{Re} f(k) \quad (14.3.8)$$

for all $k \in K$. If K^* is closed and convex in the dual of a Banach space, X' , then it is also weak $*$ closed.

Proof: By Theorem 14.2.4 there exists $f \in X'$ such that 14.3.8 holds. Therefore, letting $A = \{f\}$, it follows that for r small enough, $B_A(p, r) \cap K = \emptyset$. Thus K is weakly closed. This establishes the first part.

For the second part, the seminorms for the weak $*$ topology are determined from X and the continuous linear functionals are of the form $x^* \rightarrow x^*(x)$ where $x \in X$. Thus if $p^* \notin K^*$, it follows from Theorem 14.2.4 there exists $x \in X$ such that

$$\operatorname{Re} p^*(x) > c > \operatorname{Re} k^*(x)$$

for all $k^* \in K^*$. Therefore, letting $A = \{x\}$, $B_A(p^*, r) \cap K^* = \emptyset$ whenever r is small enough and this shows K^* is weak $*$ closed. This proves the theorem.

14.4 The Tychonoff Fixed Point Theorem

The objective in this section is to present a proof of the Tychonoff fixed point theorem, one of the most useful of the generalizations of the Brouwer fixed point theorem. In this section, X will be a locally convex topological vector space in which every point is a closed set. Let $\mathcal{B}$ be the basis described earlier and let $\mathcal{B}_0$ consist of all sets of $\mathcal{B}$ which are of the form $B_A(0, r)$ where A is a finite subset of Ψ as described earlier. Note that for $U \in \mathcal{B}_0$, $U = -U$ and U is convex. Also, if $U \in \mathcal{B}_0$, there exists $V \in \mathcal{B}_0$ such that

$$V + V \subseteq U$$

where

$$V + V \equiv \{v_1 + v_2 : v_i \in V\}.$$

To see this, note

$$B_A(0, r/2) + B_A(0, r/2) \subseteq B(0, r).$$

We let K be a closed convex subset of X and let

f be continuous, $f : K \rightarrow K$, and $\overline{f(K)}$ is compact.

Lemma 14.4.1 *For each $U \in \mathcal{B}_0$, there exists a finite set of points*

$$\{y_1 \cdots y_n\} \subseteq \overline{f(K)}$$

and continuous functions ψ_i defined on $\overline{f(K)}$ such that for $x \in \overline{f(K)}$,

$$\sum_{i=1}^n \psi_i(x) = 1, \quad (14.4.9)$$

$$\psi_i(x) = 0 \text{ if } x \notin y_i + U, \psi_i(x) > 0 \text{ if } x \in y_i + U.$$

If

$$f_U(x) \equiv \sum_{i=1}^n y_i \psi_i(f(x)), \quad (14.4.10)$$

then whenever $x \in K$,

$$f(x) - f_U(x) \in U.$$

Proof: Let $U = B_A(0, r)$. Using the compactness of $\overline{f(K)}$, there exists

$$\{y_1 \cdots y_n\} \subseteq \overline{f(K)}$$

such that

$$\{y_i + U\}_{i=1}^n$$

covers $\overline{f(K)}$. Let

$$\phi_i(y) \equiv (r - \rho_A(y - y_i))^+.$$

Thus $\phi_i(y) > 0$ if $y \in y_i + U$ and $\phi_i(y) = 0$ if $y \notin y_i + U$. For $x \in \overline{f(K)}$, let

$$\psi_i(x) \equiv \phi_i(x) \left(\sum_{j=1}^n \phi_j(x) \right)^{-1}.$$

Then 14.4.9 is satisfied. Now let f_U be given by 14.4.10 for $x \in K$. For such x ,

$$\begin{aligned} f(x) - f_U(x) &= \sum_{\{i: f(x) - y_i \in U\}} (f(x) - y_i) \psi_i(f(x)) \\ &\quad + \sum_{\{i: f(x) - y_i \notin U\}} (f(x) - y_i) \psi_i(f(x)) \\ &= \sum_{\{i: f(x) - y_i \in U\}} (f(x) - y_i) \psi_i(f(x)) = \\ &\quad \sum_{\{i: f(x) - y_i \in U\}} (f(x) - y_i) \psi_i(f(x)) + \sum_{\{i: f(x) - y_i \notin U\}} 0 \psi_i(f(x)) \in U \end{aligned}$$

because $0 \in U$, U is convex, and 14.4.9. This proves the lemma.

We think of f_U as an approximation to f .

Lemma 14.4.2 *For each $U \in \mathcal{B}_0$, there exists $x_U \in \text{convex hull of } \overline{f(K)}$ such that*

$$f_U(x_U) = x_U.$$

Proof: If $f_U(x_U) = x_U$ and

$$x_U = \sum_{i=1}^n a_i y_i$$

for $\sum_{i=1}^n a_i = 1$, we need

$$\sum_{j=1}^n y_j \psi_j \left(f \left(\sum_{i=1}^n a_i y_i \right) \right) = \sum_{j=1}^n a_j y_j.$$

This will be satisfied if for each $j = 1, \dots, n$,

$$a_j = \psi_j \left(f \left(\sum_{i=1}^n a_i y_i \right) \right); \quad (14.4.11)$$

so, let

$$\Sigma_{n-1} \equiv \left\{ \mathbf{a} \in \mathbb{R}^n : \sum_{i=1}^n a_i = 1, a_i \geq 0 \right\}$$

and let $h : \Sigma_{n-1} \rightarrow \Sigma_{n-1}$ be given by

$$h(\mathbf{a})_j \equiv \psi_j \left(f \left(\sum_{i=1}^n a_i y_i \right) \right).$$

Since h is continuous, the Brouwer fixed point theorem applies and we see there exists a fixed point for h which is a solution to 14.4.11. This proves the lemma.

Theorem 14.4.3 *Let K be a closed and convex subset of X , a locally convex topological vector space in which every point is closed. Let $f : K \rightarrow K$ be continuous and suppose $\overline{f(K)}$ is compact. Then f has a fixed point.*

Proof: First consider the following claim which will yield a candidate for the fixed point.

Claim: There exists $x \in \overline{f(K)}$ with the property that if $V \in \mathcal{B}_0$, there exists $U \subseteq V$, $U \in \mathcal{B}_0$, such that

$$f(x_U) \in x + V.$$

Proof of the claim: If no such x exists, then for each $x \in \overline{f(K)}$, there exists $V_x \in \mathcal{B}_0$ such that whenever $U \subseteq V_x$, with $U \in \mathcal{B}_0$,

$$f(x_U) \notin x + V_x.$$

Since $\overline{f(K)}$ is compact, there exist $x_1, \dots, x_n \in \overline{f(K)}$ such that

$$\{x_i + V_{x_i}\}_{i=1}^n$$

cover $\overline{f(K)}$. Let

$$U \in \mathcal{B}_0, U \subseteq \cap_{i=1}^n V_{x_i}$$

and consider x_U .

$$f(x_U) \in x_i + V_{x_i}$$

for some i but $U \subseteq V_{x_i}$, a contradiction. This shows the claim.

Now I show x is the desired fixed point. Let $W \in \mathcal{B}_0$ and let $V \in \mathcal{B}_0$ with

$$V + V + V \subseteq W.$$

Since f is continuous at x , there exists $V_0 \in \mathcal{B}_0$ such that

$$V_0 + V_0 \subseteq V$$

and if

$$y - x \in V_0 + V_0,$$

then

$$f(x) - f(y) \in V.$$

Using the claim, let $U \in \mathcal{B}_0$, $U \subseteq V_0$, such that

$$f(x_U) \in x + V_0.$$

Then

$$\begin{aligned} x - x_U &= x - f(x_U) + f(x_U) - f_U(x_U) \in V_0 + U \\ &\subseteq V_0 + V_0 \subseteq V \end{aligned}$$

and so

$$\begin{aligned} f(x) - x &= f(x) - f(x_U) + f(x_U) - f_U(x_U) + f_U(x_U) - x \\ &= f(x) - f(x_U) + f(x_U) - f_U(x_U) + x_U - x \\ &\subseteq V + U + V \subseteq W. \end{aligned}$$

Since $W \in \mathcal{B}_0$ is arbitrary, it follows from Lemma 14.2.15 that $f(x) - x = 0$. This proves the theorem.

In the case where X is normed linear space, this fixed point theorem is called the Schauder fixed point theorem. As an example of the usefulness of this fixed point theorem, consider the following application to the theory of ordinary differential equations. In the context of this theorem, $X = C([0, T]; \mathbb{R}^n)$, a Banach space with norm given by

$$\|\mathbf{x}\| \equiv \max \{|\mathbf{x}(t)| : t \in [0, T]\}.$$

Theorem 14.4.4 *Let $\mathbf{f} : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous and suppose there exists $L > 0$ such that for all $\lambda \in (0, 1)$, if*

$$\mathbf{x}' = \lambda \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (14.4.12)$$

for all $t \in [0, T]$, then $\|\mathbf{x}\| < L$. Then there exists a solution to

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \quad (14.4.13)$$

for $t \in [0, T]$.

Proof: Let

$$N\mathbf{x}(t) \equiv \int_0^t \mathbf{f}(s, \mathbf{x}(s)) ds.$$

Thus a solution to the initial value problem exists if there exists a solution to

$$\mathbf{x}_0 + N(\mathbf{x}) = \mathbf{x}.$$

Let

$$m \equiv \max \left\{ \|\mathbf{f}(t, \mathbf{x})\| : (t, \mathbf{x}) \in [0, T] \times \overline{B(0, L)} \right\}, \quad M \equiv |\mathbf{x}_0| + mT$$

and let

$$K \equiv \{\mathbf{x} \in C(0, T; \mathbb{R}^n) \text{ such that } \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \|\mathbf{x}\| \leq M\}.$$

Now define

$$A\mathbf{x} \equiv \begin{cases} \mathbf{x}_0 + N\mathbf{x} & \text{if } \|N\mathbf{x}\| \leq M - |\mathbf{x}_0|, \\ \mathbf{x}_0 + \frac{(M - |\mathbf{x}_0|)N\mathbf{x}}{\|N\mathbf{x}\|} & \text{if } \|N\mathbf{x}\| > M - |\mathbf{x}_0|. \end{cases}$$

Then A is continuous and maps X to K . Also $A(K)$ is equicontinuous because

$$\mathbf{x}_0 + N\mathbf{x}(t) - (\mathbf{x}_0 + N\mathbf{x}(t_1)) = \int_{t_1}^t \mathbf{f}(s, \mathbf{x}(s)) ds$$

and the integrand is bounded. Thus $\overline{A(K)}$ is a compact set in X by the Ascoli Arzela theorem. By the Schauder fixed point theorem, A has a fixed point, $\mathbf{x} \in K$.

If $\|N(\mathbf{x})\| > M - |\mathbf{x}_0|$, then

$$\mathbf{x}_0 + \lambda N(\mathbf{x}) = \mathbf{x}$$

where

$$\lambda = \frac{(M - |\mathbf{x}_0|)}{\|N\mathbf{x}\|} < 1$$

and so 14.4.12 holds. Therefore, by the assumed estimate on the solutions to 14.4.12, it follows that

$$\|\mathbf{x}\| < L$$

and so $\|N\mathbf{x}\| \leq mT = M - |\mathbf{x}_0|$, a contradiction. Therefore, it must be the case that

$$\|N(\mathbf{x})\| \leq M - |\mathbf{x}_0|$$

which implies that

$$\mathbf{x}_0 + N(\mathbf{x}) = \mathbf{x}.$$

Since this is equivalent to 14.4.13, this proves the theorem.

Note that existence for solutions to 14.4.12 is not assumed, only estimates of possible solutions. These estimates are called *a-priori* estimates. Also note this is a global existence theorem, not a local one for a solution defined on only a small interval.

14.5 Set-Valued Maps

In the abstract theory of partial differential equations and variational inequalities, it is important to consider set-valued maps from a Banach space to the power set of its dual. In this section we give an introduction to this theory by proving a general result on surjectivity for a class of such operators.

To begin with, if $A : X \rightarrow \mathcal{P}(Y)$ is a set-valued map, define the graph of A by

$$G(A) \equiv \{(x, y) : y \in Ax\}.$$

First consider a map A which maps $\mathbb{C}^n$ to $\mathcal{P}(\mathbb{C}^n)$ which satisfies

$$A\mathbf{x} \text{ is compact and convex.} \quad (14.5.14)$$

and also the condition that if O is open and $O \supseteq A\mathbf{x}$, then there exists $\delta > 0$ such that if

$$\mathbf{y} \in B(\mathbf{x}, \delta), \text{ then } A\mathbf{y} \subseteq O. \quad (14.5.15)$$

This last condition is sometimes referred to as upper semicontinuity. In words, A is upper semicontinuous and has values which are compact and convex.

Lemma 14.5.1 *Let A satisfy 14.5.14 and 14.5.15. Then AK is a subset of a compact set whenever K is compact. Also the graph of A is closed.*

Proof: Let $\mathbf{x} \in K$. Then $A\mathbf{x}$ is compact and contained in some open set whose closure is compact, $U_{\mathbf{x}}$. By assumption 14.5.15 there exists an open set $V_{\mathbf{x}}$ containing $\mathbf{x}$ such that if $\mathbf{y} \in V_{\mathbf{x}}$, then $A\mathbf{y} \subseteq U_{\mathbf{x}}$. Let $V_{\mathbf{x}_1}, \dots, V_{\mathbf{x}_m}$ cover K . Then $AK \subseteq \bigcup_{k=1}^m \overline{U_{\mathbf{x}_k}}$, a compact set. To see the graph of A is closed, let $\mathbf{x}_k \rightarrow \mathbf{x}$, $\mathbf{y}_k \rightarrow \mathbf{y}$ where $\mathbf{y}_k \in A\mathbf{x}_k$. Then letting $O = A\mathbf{x} + B(\mathbf{0}, r)$ it follows from 14.5.15 that $\mathbf{y}_k \in A\mathbf{x}_k \subseteq O$ for all k large enough. Therefore, $\mathbf{y} \in A\mathbf{x} + B(\mathbf{0}, 2r)$ and since $r > 0$ is arbitrary and $A\mathbf{x}$ is closed it follows $\mathbf{y} \in A\mathbf{x}$.

The next lemma is an application of the Brouwer fixed point theorem. First define an n simplex, denoted by $[\mathbf{x}_0, \dots, \mathbf{x}_n]$, to be the convex hull of the $n + 1$ points, $\{\mathbf{x}_0, \dots, \mathbf{x}_n\}$ where $\{\mathbf{x}_i - \mathbf{x}_0\}_{i=1}^n$ are independent. Thus

$$[\mathbf{x}_0, \dots, \mathbf{x}_n] \equiv \left\{ \sum_{i=1}^n t_i \mathbf{x}_i : \sum_{i=1}^n t_i = 1, t_i \geq 0 \right\}.$$

If $n \leq 2$, the simplex is a triangle, line segment, or point. If $n \leq 3$, it is a tetrahedron, triangle, line segment or point. A collection of simplices is a tiling of $\mathbb{R}^n$ if $\mathbb{R}^n$ is contained in their union and if S_1, S_2 are two simplices in the tiling, with

$$S_j = [\mathbf{x}_0^j, \dots, \mathbf{x}_n^j],$$

then

$$S_1 \cap S_2 = [\mathbf{x}_{k_0}, \dots, \mathbf{x}_{k_r}]$$

where

$$\{\mathbf{x}_{k_0}, \dots, \mathbf{x}_{k_r}\} \subseteq \{\mathbf{x}_0^1, \dots, \mathbf{x}_n^1\} \cap \{\mathbf{x}_0^2, \dots, \mathbf{x}_n^2\}$$

or else the two simplices do not intersect. The collection of simplices is said to be locally finite if, for every point, there exists a ball containing that point which also intersects only finitely many of the simplices in the collection. It is left to the reader to verify that for each $\varepsilon > 0$, there exists a locally finite tiling of $\mathbb{R}^n$ which is composed of simplices which have diameters less than ε .

Lemma 14.5.2 *Suppose $A : \mathbb{C}^n \rightarrow \mathcal{P}(\mathbb{C}^n)$ satisfies 14.5.14 and 14.5.15 and K is a nonempty compact convex set in $\mathbb{C}^n$. Then if $\mathbf{y} \in \mathbb{C}^n$ there exists $(\mathbf{x}, \mathbf{w}) \in G(A)$ such that $\mathbf{x} \in K$ and*

$$\operatorname{Re}(\mathbf{y} - \mathbf{w}, \mathbf{z} - \mathbf{x}) \leq 0$$

for all $\mathbf{z} \in K$.

Proof: Tile $\mathbb{C}^n$ with $2n$ simplices such that the collection is locally finite and each simplex has diameter less than $\varepsilon < 1$. This collection of simplices is determined by a countable collection of vertices. For each vertex, $\mathbf{x}$, pick $A_\varepsilon \mathbf{x} \in A\mathbf{x}$ and define A_ε on all of $\mathbb{C}^n$ by the following rule. If

$$\mathbf{x} \in [\mathbf{x}_0, \dots, \mathbf{x}_{2n}],$$

so $\mathbf{x} = \sum_{i=0}^{2n} t_i \mathbf{x}_i$, then

$$A_\varepsilon \mathbf{x} \equiv \sum_{k=0}^{2n} t_k A_\varepsilon \mathbf{x}_k.$$

Thus A_ε is a continuous map defined on $\mathbb{C}^n$ thanks to the local finiteness of the collection of simplices. Let P_K denote the projection on the convex set K . By the Brouwer fixed point theorem, there exists a fixed point, $\mathbf{x}_\varepsilon \in K$ such that

$$P_K(\mathbf{y} - A_\varepsilon \mathbf{x}_\varepsilon + \mathbf{x}_\varepsilon) = \mathbf{x}_\varepsilon.$$

By Corollary 15.1.9 this requires

$$\operatorname{Re}(\mathbf{y} - A_\varepsilon \mathbf{x}_\varepsilon, \mathbf{z} - \mathbf{x}_\varepsilon) \leq 0$$

for all $\mathbf{z} \in K$.

Suppose $\mathbf{x}_\varepsilon \in [\mathbf{x}_0^\varepsilon, \dots, \mathbf{x}_{2n}^\varepsilon]$ so $\mathbf{x}_\varepsilon = \sum_{k=0}^{2n} t_k^\varepsilon \mathbf{x}_k^\varepsilon$. Then since $\mathbf{x}_\varepsilon$ is contained in K , a compact set, and the diameter of each simplex is less than 1, it follows that

$A_\varepsilon \mathbf{x}_k^\varepsilon$ is contained in $\overline{A(K + B(\mathbf{0}, 1))}$, which is contained in a compact set thanks to Lemma 14.5.1. From the Heine Borel theorem, there exists a sequence $\varepsilon \rightarrow 0$ such that

$$t_k^\varepsilon \rightarrow t_k, \mathbf{x}_\varepsilon \rightarrow \mathbf{x}, A_\varepsilon \mathbf{x}_k^\varepsilon \rightarrow \mathbf{y}_k$$

for $k = 0, \dots, 2n$. Since the diameter of the simplex containing $\mathbf{x}_\varepsilon$ converges to 0, it follows

$$\mathbf{x}_k^\varepsilon \rightarrow \mathbf{x}, A_\varepsilon \mathbf{x}_k^\varepsilon \rightarrow \mathbf{y}_k.$$

Since the graph of A is closed and $A_\varepsilon \mathbf{x}_k^\varepsilon \in A\mathbf{x}_k^\varepsilon$, this implies $\mathbf{y}_k \in A\mathbf{x}$. Since $A\mathbf{x}$ is convex,

$$\sum_{k=1}^{2n} t_k \mathbf{y}_k \in A\mathbf{x}.$$

Hence for all $\mathbf{z} \in K$,

$$\begin{aligned} \operatorname{Re} \left(\mathbf{y} - \sum_{k=1}^{2n} t_k \mathbf{y}_k, \mathbf{z} - \mathbf{x} \right) &= \lim_{\varepsilon \rightarrow 0} \operatorname{Re} \left(\mathbf{y} - \sum_{k=1}^{2n} t_k^\varepsilon A_\varepsilon \mathbf{x}_k^\varepsilon, \mathbf{z} - \mathbf{x}_\varepsilon \right) \\ &= \lim_{\varepsilon \rightarrow 0} \operatorname{Re} (\mathbf{y} - A_\varepsilon \mathbf{x}_\varepsilon, \mathbf{z} - \mathbf{x}_\varepsilon) \leq 0. \end{aligned}$$

Let $\mathbf{w} = \sum_{k=1}^{2n} t_k \mathbf{y}_k$. This proves the lemma.

Lemma 14.5.3 *Suppose in addition to 14.5.14 and 14.5.15, A is coercive,*

$$\lim_{|\mathbf{x}| \rightarrow \infty} \inf \left\{ \frac{\operatorname{Re} (\mathbf{y}, \mathbf{x})}{|\mathbf{x}|} : \mathbf{y} \in A\mathbf{x} \right\} = \infty.$$

Then A is onto.

Proof: Let $\mathbf{y} \in \mathbb{C}^n$ and let $K_r \equiv \overline{B(\mathbf{0}, r)}$. By Lemma 14.5.2 there exists $\mathbf{x}_r \in K_r$ and $\mathbf{w}_r \in A\mathbf{x}_r$ such that

$$\operatorname{Re} (\mathbf{y} - \mathbf{w}_r, \mathbf{z} - \mathbf{x}_r) \leq 0 \quad (14.5.16)$$

for all $\mathbf{z} \in K_r$. Letting $\mathbf{z} = \mathbf{0}$,

$$\operatorname{Re} (\mathbf{w}_r, \mathbf{x}_r) \leq \operatorname{Re} (\mathbf{y}, \mathbf{x}_r).$$

Therefore,

$$\inf \left\{ \frac{\operatorname{Re} (\mathbf{w}, \mathbf{x}_r)}{|\mathbf{x}_r|} : \mathbf{w} \in A\mathbf{x}_r \right\} \leq |\mathbf{y}|.$$

It follows from the assumption of coercivity that $|\mathbf{x}_r|$ is bounded independent of r . Therefore, picking r strictly larger than this bound, 14.5.16 implies

$$\operatorname{Re} (\mathbf{y} - \mathbf{w}_r, \mathbf{v}) \leq 0$$

for all $\mathbf{v}$ in some open ball containing $\mathbf{0}$. Therefore, for all $\mathbf{v}$ in this ball

$$\operatorname{Re} (\mathbf{y} - \mathbf{w}_r, \mathbf{v}) = 0$$

and hence this holds for all $\mathbf{v} \in \mathbb{C}^n$ and so $\mathbf{y} = \mathbf{w}_r \in A\mathbf{x}_r$. This proves the lemma.

Lemma 14.5.4 *Let F be a finite dimensional Banach space of dimension n , and let T be a mapping from F to $\mathcal{P}(F')$ such that 14.5.14 and 14.5.15 both hold for F' in place of $\mathbb{C}^n$. Then if T is also coercive,*

$$\lim_{\|\mathbf{u}\| \rightarrow \infty} \inf \left\{ \frac{\operatorname{Re} \mathbf{y}^*(\mathbf{u})}{\|\mathbf{u}\|} : \mathbf{y}^* \in T\mathbf{u} \right\} = \infty, \quad (14.5.17)$$

it follows T is onto.

Proof: Let $|\cdot|$ be an equivalent norm for F such that there is an isometry of $\mathbb{C}^n$ and F, θ . Now define $A : \mathbb{C}^n \rightarrow \mathcal{P}(\mathbb{C}^n)$ by $A\mathbf{x} \equiv \theta^* T \theta \mathbf{x}$.

$$\begin{array}{ccc} \mathcal{P}(F') & \xrightarrow{\theta^*} & \mathbb{C}^n \\ T \uparrow & \circ & \uparrow A \\ F & \xleftarrow{\theta} & \mathbb{C}^n \end{array}$$

Thus $\mathbf{y} \in A\mathbf{x}$ means that there exists $\mathbf{z}^* \in T\theta\mathbf{x}$ such that

$$(\mathbf{w}, \mathbf{y})_{\mathbb{C}^n} = \mathbf{z}^*(\theta\mathbf{w})$$

for all $\mathbf{w} \in \mathbb{C}^n$. Then A satisfies the conditions of Lemma 14.5.3 and so A is onto. Consequently T is also onto.

With these lemmas, it is possible to prove a very useful result about a class of mappings which map a reflexive Banach space to the power set of its dual space. For more theorems about these mappings and their applications, see [52]. In the discussion below, we will use the symbol, $\rightharpoonup$, to denote weak convergence.

Definition 14.5.5 *Let V be a Reflexive Banach space. We say $T : V \rightarrow \mathcal{P}(V')$ is pseudomonotone if the following conditions hold.*

$$Tu \text{ is closed, nonempty, convex, and bounded.} \quad (14.5.18)$$

If F is a finite dimensional subspace of V , then if $u \in F$ and $W \supseteq Tu$ for W a weakly open set in V' , then there exists $\delta > 0$ such that

$$v \in B(u, \delta) \cap F \text{ implies } Tv \subseteq W. \quad (14.5.19)$$

If $u_k \rightharpoonup u$ and if $u_k^* \in Tu_k$ is such that

$$\limsup_{k \rightarrow \infty} \operatorname{Re} u_k^*(u_k - u) \leq 0,$$

then for all $v \in V$, there exists $u^*(v) \in Tu$ such that

$$\liminf_{k \rightarrow \infty} \operatorname{Re} u_k^*(u_k - v) \geq \operatorname{Re} u^*(v)(u - v). \quad (14.5.20)$$

We say T is coercive if

$$\lim_{\|v\| \rightarrow \infty} \inf \left\{ \frac{\operatorname{Re} z^*(v)}{\|v\|} : z^* \in Tv \right\} = \infty. \quad (14.5.21)$$

Theorem 14.5.6 *Let V be a reflexive Banach space and let $T : V \rightarrow \mathcal{P}(V')$ be pseudomonotone and coercive. Then T is onto.*

Proof: Let $\mathcal{F}$ be the set of finite subspaces of V and let $F \in \mathcal{F}$. Then define T_F as

$$T_F \equiv i_F^* T i_F$$

where here i_F is the identity map from F to V . Then T_F satisfies the conditions of Lemma 14.5.4 and so T_F is onto $\mathcal{P}(F')$. Let $w^* \in V'$. Then since T_F is onto, there exists $u_F \in F$ such that

$$i_F^* w^* \in i_F^* T i_F u_F.$$

Thus for each finite dimensional subspace, F , there exists $u_F \in F$ such that for all $v \in F$,

$$w^*(v) = u_F^*(v), \quad u_F^* \in T u_F. \quad (14.5.22)$$

Replacing v with u_F , in 14.5.22,

$$\frac{u_F^*(u_F)}{\|u_F\|} = \frac{w^*(u_F)}{\|u_F\|} \leq \|w^*\|.$$

Therefore, the assumption that T is coercive implies $\{u_F : F \in \mathcal{F}\}$ is bounded in V . Now define

$$W_F \equiv \cup \{u_{F'} : F' \supseteq F\}.$$

Then W_F is bounded and if $\overline{W_F} \equiv$ weak closure of W_F , then

$$\{\overline{W_F} : F \in \mathcal{F}\}$$

is a collection of nonempty weakly compact (since V is reflexive) sets having the finite intersection property because $W_F \neq \emptyset$ for each F . Thus there exists

$$u \in \cap \{\overline{W_F} : F \in \mathcal{F}\}.$$

I will show $w^* \in Tu$. If $w^* \notin Tu$, a closed convex set, there exists $v \in V$ such that

$$\operatorname{Re} w^*(u - v) < \operatorname{Re} u^*(u - v) \quad (14.5.23)$$

for all $u^* \in Tu$. This follows from the separation theorems. (These theorems imply there exists $z \in V$ such that

$$\operatorname{Re} w^*(z) < \operatorname{Re} u^*(z)$$

for all $u^* \in Tu$. Define $u - v \equiv z$.) Now let $F \supseteq \{u, v\}$. Since $u \in \overline{W_F}$, a weakly sequentially compact set, there exists a sequence, $\{u_k\}$, such that

$$u_k \rightharpoonup u, \quad u_k \in W_F.$$

Then since $F \supseteq \{u, v\}$, there exists $u_k^* \in Tu_k$ such that

$$u_k^*(u_k - u) = w^*(u_k - u).$$

Therefore,

$$\limsup_{k \rightarrow \infty} \operatorname{Re} u_k^* (u_k - u) = \limsup_{k \rightarrow \infty} \operatorname{Re} w^* (u_k - u) = 0.$$

It follows by the assumption T is pseudomonotone; the following holds for the v defined above in 14.5.23.

$$\liminf_{k \rightarrow \infty} \operatorname{Re} u_k^* (u_k - v) \geq \operatorname{Re} u^* (v) (u - v), \quad u^* (v) \in Tu.$$

But since $v \in F$, $\operatorname{Re} u_k^* (u_k - v) = \operatorname{Re} w^* (u_k - v)$ and so

$$\liminf_{k \rightarrow \infty} \operatorname{Re} u_k^* (u_k - v) = \liminf_{k \rightarrow \infty} \operatorname{Re} w^* (u_k - v) = \operatorname{Re} w^* (u - v),$$

so from 14.5.23,

$$\begin{aligned} \operatorname{Re} w^* (u - v) &= \liminf_{k \rightarrow \infty} \operatorname{Re} u_k^* (u_k - v) \\ &\geq \operatorname{Re} u^* (v) (u - v) > \operatorname{Re} w^* (u - v), \end{aligned}$$

a contradiction. Thus, $w^* \in Tu$ and this proves the theorem.

This is an interesting theorem, but one might wonder if there are easy to verify examples of such possibly set valued mappings. In what follows consider only real spaces because the essential ideas are included in this case which is also the case of most use in applications.

Definition 14.5.7 Let V be a real reflexive Banach space and let $f : V \rightarrow \mathbb{R}$ be a locally Lipschitz function, meaning that f is Lipschitz near every point of V although f need not be Lipschitz on all of V . Under these conditions,

$$f^0(x, y) \equiv \lim_{\mu \rightarrow 0+} \sup_{h \rightarrow 0} \frac{f(x + h + \mu y) - f(x + h)}{\mu} \quad (14.5.24)$$

and $\partial f(x) \subseteq X'$ is defined by

$$\partial f(x) \equiv \{x^* \in X' : x^*(y) \leq f^0(x, y) \text{ for all } y \in X\}. \quad (14.5.25)$$

The set just described is called the generalized gradient. In 14.5.24 we mean the following by the right hand side.

$$\lim_{(r, \delta) \rightarrow (0, 0)} \sup \left\{ \frac{f(x + h + \mu y) - f(x + h)}{\mu} : \mu \in (0, r), h \in B(0, \delta) \right\}$$

I will show, following [52], that these generalized gradients of locally Lipschitz functions are sometimes pseudomonotone. First here is a lemma.

Lemma 14.5.8 Let f be as described in the above definition. Then $\partial f(x)$ is a closed, bounded, convex, and non empty subset of V' . Furthermore, for $x^* \in \partial f(x)$,

$$\|x^*\| \leq \operatorname{Lip}_x(f). \quad (14.5.26)$$

Proof: It is left as an exercise to verify the assertions that $\partial f(x)$ is closed, and convex. It follows directly from the definition. To verify this set is bounded, let $Lip_x(f)$ denote a Lipschitz constant valid near $x \in V$ and let $x^* \in \partial f(x)$. Then choosing y with $\|y\| = 1$ and $x^*(y) \geq \frac{1}{2}\|x^*\|$,

$$\frac{1}{2}\|x^*\| = x^*(y) \leq f^0(x, y). \quad (14.5.27)$$

Also, for small μ and h ,

$$\left| \frac{f(x+h+\mu y) - f(x+h)}{\mu} \right| \leq Lip_x(f)\|y\| = Lip_x(f).$$

Therefore, $f^0(x, y) \leq Lip_x(f)$ and so 14.5.27 shows $\|x^*\| \leq 2Lip_x(f)$.

The interesting part of this Lemma is that $\partial f(x) \neq \emptyset$. To verify this first note that the definition of f^0 implies that $y \rightarrow f^0(x, y)$ is a gauge function. Now fix $y \in V$ and define on $\mathbb{R}y$ a linear map x_0^* by

$$x_0^*(\alpha y) \equiv \alpha f^0(x, y).$$

Then if $\alpha \geq 0$,

$$x_0^*(\alpha y) = \alpha f^0(x, y) = f^0(x, \alpha y).$$

If $\alpha < 0$,

$$\begin{aligned} x_0^*(\alpha y) &\equiv \alpha f^0(x, y) = \\ \lim_{\mu \rightarrow 0+} \inf_{h \rightarrow 0} \frac{(-\alpha)f(x+h) - (-\alpha)f(x+h+\mu y)}{\mu} &= \\ (-\alpha) \lim_{\mu \rightarrow 0+} \inf_{h \rightarrow 0} \frac{f(x+h-\mu y) - f(x+h)}{\mu} &\leq \\ (-\alpha)f^0(x, -y) &= f^0(x, \alpha y). \end{aligned}$$

Therefore, $x_0^*(\alpha y) \leq f^0(x, \alpha y)$ for all α . By the Hahn Banach theorem there is an extension of x_0^* to all of V , x^* which satisfies,

$$x^*(y) \leq f^0(x, y)$$

for all y . It remains to verify x^* is continuous. This follows easily from

$$|x^*(y)| = \max(x^*(-y), x^*(y)) \leq$$

$$\max(f^0(x, y), f^0(x, -y)) \leq Lip_x(f)\|y\|,$$

which verifies 14.5.26 and proves the lemma.

This lemma has verified the first condition needed in the definition of pseudomonotone. The next lemma verifies that these generalized subgradients satisfy the second of the conditions needed in the definition. In fact somewhat more than is needed in the definition is shown.

Lemma 14.5.9 *Let U be weakly open in V' and suppose $\partial f(x) \subseteq U$. Then $\partial f(z) \subseteq U$ whenever z is close enough to x .*

Proof: Suppose to the contrary there exists $z_n \rightarrow x$ but $z_n^* \in \partial f(z_n) \setminus U$. From the first lemma, we may assert that $\|z_n^*\| \leq 2\text{Lip}(f)$ for all n large enough. Therefore, there is a subsequence, still denoted by n such that z_n^* converges weakly to $z^* \notin U$.

Claim: $f^0(x, y) \geq \limsup_{n \rightarrow \infty} f^0(x_n, y)$.

Proof of the claim: There exists $\delta > 0$ such that if $\mu, \|h\| < \delta$, then

$$\varepsilon + f^0(x, y) \geq \frac{f(x + h + \mu y) - f(x + h)}{\mu}.$$

Thus, for $\|h\| < \delta$,

$$\varepsilon + f^0(x, y) \geq \frac{f(x_n + (x - x_n) + h + \mu y) - f(x_n + (x - x_n) + h)}{\mu}.$$

Now let $\|h'\| < \frac{\delta}{2}$ and let n be so large that $\|x - x_n\| < \frac{\delta}{2}$. Suppose $\|h'\| < \frac{\delta}{2}$. Then choosing $h \equiv h' - (x - x_n)$, it follows the above inequality holds because $\|h\| < \delta$. Therefore, if $\|h'\| < \frac{\delta}{2}$, and n is sufficiently large,

$$\varepsilon + f^0(x, y) \geq \frac{f(x_n + h' + \mu y) - f(x_n + h')}{\mu}.$$

Consequently, for all n large enough,

$$\varepsilon + f^0(x, y) \geq f^0(x_n, y)$$

which proves the claim.

Now with the claim,

$$z^*(y) = \limsup_{n \rightarrow \infty} z_n^*(y) \leq \limsup_{n \rightarrow \infty} f^0(x_n, y) \leq f^0(x, y)$$

so $z^* \in \partial f(x)$ contradicting the assumption that $z^* \notin U$. This proves the lemma.

It is necessary to assume more on f^0 in order to obtain the third axiom defining pseudomonotone. The following theorem describes the situation.

Theorem 14.5.10 *Let $f : V \rightarrow V'$ be locally Lipschitz and suppose it satisfies the condition that whenever*

$$x_n \text{ converges weakly to } x$$

and

$$\limsup_{n \rightarrow \infty} f^0(x_n, x - x_n) \geq 0$$

it follows that

$$\limsup_{n \rightarrow \infty} f^0(x_n, z - x_n) \leq f^0(x, z - x)$$

for all $z \in V$. Then ∂f is pseudomonotone.

Proof: 14.5.18 and 14.5.19 both are satisfied thanks to Lemmas 14.5.7 and 14.5.8. It remains to verify 14.5.20. To do so, I will adopt the convention that $x^* \in \partial f(x)$. Suppose

$$\limsup_{n \rightarrow \infty} x_n^*(x_n - x) \leq 0. \quad (14.5.28)$$

This implies $\liminf_{n \rightarrow \infty} x_n^*(x - x_n) \geq 0$. Thus,

$$\begin{aligned} 0 &\leq \liminf_{n \rightarrow \infty} x_n^*(x - x_n) \leq \liminf f^0(x_n, x - x_n) \\ &\leq \limsup_{n \rightarrow \infty} f^0(x_n, x - x_n), \end{aligned}$$

which implies, by the above assumption that for all z ,

$$\limsup x_n^*(z - x_n) \leq \limsup f^0(x_n, z - x_n) \leq f^0(x, z - x). \quad (14.5.29)$$

In particular, this holds for $z = x$ and this implies $\limsup x_n^*(x - x_n) \leq 0$ which along with 14.5.28 yields

$$\lim_{n \rightarrow \infty} x_n^*(x_n - x) = 0 \quad (14.5.30)$$

Now let z be arbitrary. There exists a subsequence, n_k , depending on z such that

$$\lim_{k \rightarrow \infty} x_{n_k}^*(x_{n_k} - z) = \liminf x_{n_k}^*(x_{n_k} - z).$$

Now from Lemma 14.5.8 and its proof, the $\|x_n^*\|$ are all bounded by $Lip_x(f)$ whenever n is large enough. Therefore, there is a further subsequence, still denoted by n_k such that

$$x_{n_k}^* \text{ converges weakly to } x^*(z).$$

We need to verify that $x^*(z) \in \partial f(x)$. To do so, let y be arbitrary. Then from the definition,

$$x_n^*(y - x_n) \leq f^0(x_n, y - x_n). \quad (14.5.31)$$

From 14.5.30, we can take the lim sup of both sides and obtain, using 14.5.29

$$x^*(z)(y - x) \leq \limsup f^0(x_n, y - x_n) \leq f^0(x, y - x).$$

Since y is arbitrary, this shows $x^*(z) \in \partial f(x)$ and proves the theorem.

Hilbert Spaces

15.1 Basic Theory

Definition 15.1.1 *Let X be a vector space. An inner product is a mapping from $X \times X$ to $\mathbb{C}$ if X is complex and from $X \times X$ to $\mathbb{R}$ if X is real, denoted by (x, y) which satisfies the following.*

$$(x, x) \geq 0, (x, x) = 0 \text{ if and only if } x = 0, \quad (15.1.1)$$

$$(x, y) = \overline{(y, x)}. \quad (15.1.2)$$

For $a, b \in \mathbb{C}$ and $x, y, z \in X$,

$$(ax + by, z) = a(x, z) + b(y, z). \quad (15.1.3)$$

Note that 15.1.2 and 15.1.3 imply $(x, ay + bz) = \bar{a}(x, y) + \bar{b}(x, z)$. Such a vector space is called an inner product space.

The Cauchy Schwarz inequality is fundamental for the study of inner product spaces.

Theorem 15.1.2 (*Cauchy Schwarz*) *In any inner product space*

$$|(x, y)| \leq \|x\| \|y\|.$$

Proof: Let $\omega \in \mathbb{C}$, $|\omega| = 1$, and $\bar{\omega}(x, y) = |(x, y)| = \operatorname{Re}(x, y\omega)$. Let

$$F(t) = (x + ty\omega, x + ty\omega).$$

If $y = 0$ there is nothing to prove because

$$(x, 0) = (x, 0 + 0) = (x, 0) + (x, 0)$$

and so $(x, 0) = 0$. Thus, it can be assumed $y \neq 0$. Then from the axioms of the inner product,

$$F(t) = \|x\|^2 + 2t \operatorname{Re}(x, \omega y) + t^2 \|y\|^2 \geq 0.$$

This yields

$$||x||^2 + 2t|(x, y)| + t^2||y||^2 \geq 0.$$

Since this inequality holds for all $t \in \mathbb{R}$, it follows from the quadratic formula that

$$4|(x, y)|^2 - 4||x||^2||y||^2 \leq 0.$$

This yields the conclusion and proves the theorem.

Proposition 15.1.3 *For an inner product space, $||x|| \equiv (x, x)^{1/2}$ does specify a norm.*

Proof: All the axioms are obvious except the triangle inequality. To verify this,

$$\begin{aligned} ||x + y||^2 &\equiv (x + y, x + y) \equiv ||x||^2 + ||y||^2 + 2 \operatorname{Re}(x, y) \\ &\leq ||x||^2 + ||y||^2 + 2|(x, y)| \\ &\leq ||x||^2 + ||y||^2 + 2||x||||y|| = (||x|| + ||y||)^2. \end{aligned}$$

The following lemma is called the parallelogram identity.

Lemma 15.1.4 *In an inner product space,*

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2.$$

The proof, a straightforward application of the inner product axioms, is left to the reader.

Lemma 15.1.5 *For $x \in H$, an inner product space,*

$$||x|| = \sup_{||y|| \leq 1} |(x, y)| \quad (15.1.4)$$

Proof: By the Cauchy Schwarz inequality, if $x \neq 0$,

$$||x|| \geq \sup_{||y|| \leq 1} |(x, y)| \geq \left(x, \frac{x}{||x||} \right) = ||x||.$$

It is obvious that 15.1.4 holds in the case that $x = 0$.

Definition 15.1.6 *A Hilbert space is an inner product space which is complete. Thus a Hilbert space is a Banach space in which the norm comes from an inner product as described above.*

In Hilbert space, one can define a projection map onto closed convex nonempty sets.

Definition 15.1.7 *A set, K , is convex if whenever $\lambda \in [0, 1]$ and $x, y \in K$, $\lambda x + (1 - \lambda)y \in K$.*

Theorem 15.1.8 *Let K be a closed convex nonempty subset of a Hilbert space, H , and let $x \in H$. Then there exists a unique point $Px \in K$ such that $\|Px - x\| \leq \|y - x\|$ for all $y \in K$.*

Proof: Consider uniqueness. Suppose that z_1 and z_2 are two elements of K such that for $i = 1, 2$,

$$\|z_i - x\| \leq \|y - x\| \quad (15.1.5)$$

for all $y \in K$. Also, note that since K is convex,

$$\frac{z_1 + z_2}{2} \in K.$$

Therefore, by the parallelogram identity,

$$\begin{aligned} \|z_1 - x\|^2 &\leq \left\| \frac{z_1 + z_2}{2} - x \right\|^2 = \left\| \frac{z_1 - x}{2} + \frac{z_2 - x}{2} \right\|^2 \\ &= 2\left(\left\| \frac{z_1 - x}{2} \right\|^2 + \left\| \frac{z_2 - x}{2} \right\|^2\right) - \left\| \frac{z_1 - z_2}{2} \right\|^2 \\ &= \frac{1}{2} \|z_1 - x\|^2 + \frac{1}{2} \|z_2 - x\|^2 - \left\| \frac{z_1 - z_2}{2} \right\|^2 \\ &\leq \|z_1 - x\|^2 - \left\| \frac{z_1 - z_2}{2} \right\|^2, \end{aligned}$$

where the last inequality holds because of 15.1.5 letting $z_i = z_2$ and $y = z_1$. Hence $z_1 = z_2$ and this shows uniqueness.

Now let $\lambda = \inf\{\|x - y\| : y \in K\}$ and let y_n be a minimizing sequence. This means $\{y_n\} \subseteq K$ satisfies $\lim_{n \rightarrow \infty} \|x - y_n\| = \lambda$. Now the following follows from properties of the norm.

$$\|y_n - x + y_m - x\|^2 = 4\left(\left\| \frac{y_n + y_m}{2} - x \right\|^2\right)$$

Then by the parallelogram identity, and convexity of K , $\frac{y_n + y_m}{2} \in K$, and so

$$\begin{aligned} \|(y_n - x) - (y_m - x)\|^2 &= 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4\overbrace{\left(\left\| \frac{y_n + y_m}{2} - x \right\|^2\right)}^{=\|y_n - x + y_m - x\|^2} \\ &\leq 2(\|y_n - x\|^2 + \|y_m - x\|^2) - 4\lambda^2. \end{aligned}$$

Since $\|x - y_n\| \rightarrow \lambda$, this shows $\{y_n - x\}$ is a Cauchy sequence. Thus also $\{y_n\}$ is a Cauchy sequence. Since H is complete, $y_n \rightarrow y$ for some $y \in H$ which must be in K because K is closed. Therefore

$$\|x - y\| = \lim_{n \rightarrow \infty} \|x - y_n\| = \lambda.$$

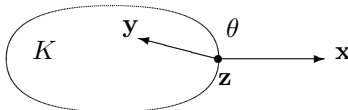
Let $Px = y$.

Corollary 15.1.9 *Let K be a closed, convex, nonempty subset of a Hilbert space, H , and let $x \in H$. Then for $z \in K$, $z = Px$ if and only if*

$$\operatorname{Re}(x - z, y - z) \leq 0 \quad (15.1.6)$$

for all $y \in K$.

Before proving this, consider what it says in the case where the Hilbert space is $\mathbb{R}^n$.



Condition 15.1.6 says the angle, θ , shown in the diagram is always obtuse. Remember from calculus, the sign of $\mathbf{x} \cdot \mathbf{y}$ is the same as the sign of the cosine of the included angle between $\mathbf{x}$ and $\mathbf{y}$. Thus, in finite dimensions, the conclusion of this corollary says that $z = Px$ exactly when the angle of the indicated angle is obtuse. Surely the picture suggests this is reasonable.

The inequality 15.1.6 is an example of a variational inequality and this corollary characterizes the projection of x onto K as the solution of this variational inequality.

Proof of Corollary: Let $z \in K$ and let $y \in K$ also. Since K is convex, it follows that if $t \in [0, 1]$,

$$z + t(y - z) = (1 - t)z + ty \in K.$$

Furthermore, every point of K can be written in this way. (Let $t = 1$ and $y \in K$.) Therefore, $z = Px$ if and only if for all $y \in K$ and $t \in [0, 1]$,

$$\|x - (z + t(y - z))\|^2 = \|(x - z) - t(y - z)\|^2 \geq \|x - z\|^2$$

for all $t \in [0, 1]$ and $y \in K$ if and only if for all $t \in [0, 1]$ and $y \in K$

$$\|x - z\|^2 + t^2 \|y - z\|^2 - 2t \operatorname{Re}(x - z, y - z) \geq \|x - z\|^2$$

If and only if for all $t \in [0, 1]$,

$$t^2 \|y - z\|^2 - 2t \operatorname{Re}(x - z, y - z) \geq 0. \quad (15.1.7)$$

Now this is equivalent to 15.1.7 holding for all $t \in (0, 1)$. Therefore, dividing by $t \in (0, 1)$, 15.1.7 is equivalent to

$$t \|y - z\|^2 - 2 \operatorname{Re}(x - z, y - z) \geq 0$$

for all $t \in (0, 1)$ which is equivalent to 15.1.6. This proves the corollary.

Corollary 15.1.10 *Let K be a nonempty convex closed subset of a Hilbert space, H . Then the projection map, P is continuous. In fact,*

$$|Px - Py| \leq |x - y|.$$

Proof: Let $x, x' \in H$. Then by Corollary 15.1.9,

$$\operatorname{Re}(x' - Px', Px - Px') \leq 0, \operatorname{Re}(x - Px, Px' - Px) \leq 0$$

Hence

$$\begin{aligned} 0 &\leq \operatorname{Re}(x - Px, Px - Px') - \operatorname{Re}(x' - Px', Px - Px') \\ &= \operatorname{Re}(x - x', Px - Px') - |Px - Px'|^2 \end{aligned}$$

and so

$$|Px - Px'|^2 \leq |x - x'| |Px - Px'|.$$

This proves the corollary.

The next corollary is a more general form for the Brouwer fixed point theorem.

Corollary 15.1.11 *Let $\mathbf{f} : K \rightarrow K$ where K is a convex compact subset of $\mathbb{R}^n$. Then $\mathbf{f}$ has a fixed point.*

Proof: Let $K \subseteq \overline{B(\mathbf{0}, R)}$ and let P be the projection map onto K . Then consider the map $\mathbf{f} \circ P$ which maps $\overline{B(\mathbf{0}, R)}$ to $\overline{B(\mathbf{0}, R)}$ and is continuous. By the Brouwer fixed point theorem for balls, this map has a fixed point. Thus there exists $\mathbf{x}$ such that

$$\mathbf{f} \circ P(\mathbf{x}) = \mathbf{x}$$

Now the equation also requires $\mathbf{x} \in K$ and so $P(\mathbf{x}) = \mathbf{x}$. Hence $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.

Definition 15.1.12 *Let H be a vector space and let U and V be subspaces. $U \oplus V = H$ if every element of H can be written as a sum of an element of U and an element of V in a unique way.*

The case where the closed convex set is a closed subspace is of special importance and in this case the above corollary implies the following.

Corollary 15.1.13 *Let K be a closed subspace of a Hilbert space, H , and let $x \in H$. Then for $z \in K$, $z = Px$ if and only if*

$$(x - z, y) = 0 \tag{15.1.8}$$

for all $y \in K$. Furthermore, $H = K \oplus K^\perp$ where

$$K^\perp \equiv \{x \in H : (x, k) = 0 \text{ for all } k \in K\}$$

and

$$\|x\|^2 = \|x - Px\|^2 + \|Px\|^2. \tag{15.1.9}$$

Proof: Since K is a subspace, the condition 15.1.6 implies $\operatorname{Re}(x - z, y) \leq 0$ for all $y \in K$. Replacing y with $-y$, it follows $\operatorname{Re}(x - z, -y) \leq 0$ which implies $\operatorname{Re}(x - z, y) \geq 0$ for all $y \in K$. Therefore, $\operatorname{Re}(x - z, y) = 0$ for all $y \in K$. Now let

$|\alpha| = 1$ and $\alpha(x - z, y) = |(x - z, y)|$. Since K is a subspace, it follows $\bar{\alpha}y \in K$ for all $y \in K$. Therefore,

$$0 = \operatorname{Re}(x - z, \bar{\alpha}y) = (x - z, \bar{\alpha}y) = \alpha(x - z, y) = |(x - z, y)|.$$

This shows that $z = Px$, if and only if 15.1.8.

For $x \in H$, $x = x - Px + Px$ and from what was just shown, $x - Px \in K^\perp$ and $Px \in K$. This shows that $K^\perp + K = H$. Is there only one way to write a given element of H as a sum of a vector in K with a vector in $K^\perp$? Suppose $y + z = y_1 + z_1$ where $z, z_1 \in K^\perp$ and $y, y_1 \in K$. Then $(y - y_1) = (z_1 - z)$ and so from what was just shown, $(y - y_1, y - y_1) = (y - y_1, z_1 - z) = 0$ which shows $y_1 = y$ and consequently $z_1 = z$. Finally, letting $z = Px$,

$$\begin{aligned} \|x\|^2 &= (x - z + z, x - z + z) = \|x - z\|^2 + (x - z, z) + (z, x - z) + \|z\|^2 \\ &= \|x - z\|^2 + \|z\|^2 \end{aligned}$$

This proves the corollary.

The following theorem is called the Riesz representation theorem for the dual of a Hilbert space. If $z \in H$ then define an element $f \in H'$ by the rule $(x, z) \equiv f(x)$. It follows from the Cauchy Schwarz inequality and the properties of the inner product that $f \in H'$. The Riesz representation theorem says that all elements of H' are of this form.

Theorem 15.1.14 *Let H be a Hilbert space and let $f \in H'$. Then there exists a unique $z \in H$ such that*

$$f(x) = (x, z) \tag{15.1.10}$$

for all $x \in H$.

Proof: Letting $y, w \in H$ the assumption that f is linear implies

$$f(yf(w) - f(y)w) = f(w)f(y) - f(y)f(w) = 0$$

which shows that $yf(w) - f(y)w \in f^{-1}(0)$, which is a closed subspace of H since f is continuous. If $f^{-1}(0) = H$, then f is the zero map and $z = 0$ is the unique element of H which satisfies 15.1.10. If $f^{-1}(0) \neq H$, pick $u \notin f^{-1}(0)$ and let $w \equiv u - Pu \neq 0$. Thus Corollary 15.1.13 implies $(y, w) = 0$ for all $y \in f^{-1}(0)$. In particular, let $y = xf(w) - f(x)w$ where $x \in H$ is arbitrary. Therefore,

$$0 = (f(w)x - f(x)w, w) = f(w)(x, w) - f(x)\|w\|^2.$$

Thus, solving for $f(x)$ and using the properties of the inner product,

$$f(x) = (x, \frac{\overline{f(w)}w}{\|w\|^2})$$

Let $z = \overline{f(w)}w/\|w\|^2$. This proves the existence of z . If $f(x) = (x, z_i)$ $i = 1, 2$, for all $x \in H$, then for all $x \in H$, then $(x, z_1 - z_2) = 0$ which implies, upon taking $x = z_1 - z_2$ that $z_1 = z_2$. This proves the theorem.

If $R : H \rightarrow H'$ is defined by $Rx(y) \equiv (y, x)$, the Riesz representation theorem above states this map is onto. This map is called the Riesz map. It is routine to show R is linear and $|Rx| = |x|$.

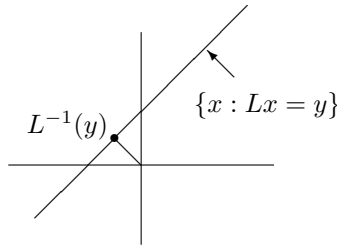
15.2 The Hilbert Space $L(U)$

Let $L \in \mathcal{L}(U, H)$. Then one can consider the image of $L, L(U)$ as a Hilbert space. This is another interesting application of Theorem 15.1.8. First here is a definition which involves abominable and atrociously misleading notation which nevertheless seems to be well accepted.

Definition 15.2.1 Let $L \in \mathcal{L}(U, H)$, the bounded linear maps from U to H for U, H Hilbert spaces. For $y \in L(U)$, let $L^{-1}y$ denote the unique vector in

$$\{x : Lx = y\} \equiv M_y$$

which is closest in U to 0.



Note this is a good definition because $\{x : Lx = y\}$ is closed thanks to the continuity of L and it is obviously convex. Thus Theorem 15.1.8 applies. With this definition define an inner product on $L(U)$ as follows. For $y, z \in L(U)$,

$$(y, z)_{L(U)} \equiv (L^{-1}y, L^{-1}z)_U$$

The notation is abominable because $L^{-1}(y)$ is the normal notation for M_y .

With the above definition, here is the main result.

Theorem 15.2.2 Let U, H be Hilbert spaces and let $L \in \mathcal{L}(U, H)$. Then Definition 15.2.1 makes $L(U)$ into a Hilbert space. Furthermore there is a constant C independent of $x \in U$ such that

$$C \|Lx\|_{L(U)} \geq \|Lx\|_H$$

If U is separable, so is $L(U)$. Also $(L^{-1}(y), x) = 0$ for all $x \in \ker(L)$.

Proof: First it is necessary to verify this is an inner product. It is clear $(y, z)_{L(U)} = (z, y)_{L(U)}$. What about the property of being linear in the first argument? Is

$$(ay_1 + by_2, z)_{L(U)} = a(y_1, z)_{L(U)} + b(y_2, z)_{L(U)}? \quad (15.2.11)$$

By Corollary 15.1.9

$$\operatorname{Re} (0 - L^{-1}(y), z - L^{-1}(y))_U \leq 0 \quad (15.2.12)$$

for all $z \in L^{-1}(y)$. Also it is clear that

$$M_y = L^{-1}(y) + \ker(L)$$

(This is because if $z \in M_y$ so that $Lz = y$, then

$$L(L^{-1}(y) - z) = 0$$

and so $L^{-1}(y) - z \in \ker(L)$.) Thus from 15.2.12,

$$\operatorname{Re}(0 - L^{-1}(y), x)_U \leq 0$$

for all $x \in \ker(L)$ and so in fact

$$(L^{-1}(y), x)_U = 0 \quad (15.2.13)$$

for all $x \in \ker(L)$. Therefore, returning to 15.2.11

$$(ay_1 + by_2, z)_{L(U)} \equiv (L^{-1}(ay_1 + by_2), L^{-1}z)_U \quad (15.2.14)$$

Also

$$L^{-1}(ay_1 + by_2) - (aL^{-1}(y_1) + bL^{-1}(y_2)) \in \ker(L) \quad (15.2.15)$$

and so the right side of 15.2.14 equals

$$\begin{aligned} & \overbrace{(L^{-1}(ay_1 + by_2) - (aL^{-1}(y_1) + bL^{-1}(y_2)), L^{-1}z)_U}^{=0} \\ & + ((aL^{-1}(y_1) + bL^{-1}(y_2)), L^{-1}z)_U \end{aligned}$$

which by 15.2.15 and 15.2.13 equals

$$\begin{aligned} & ((aL^{-1}(y_1) + bL^{-1}(y_2)), L^{-1}z)_U \\ & \equiv a(L^{-1}(y_1), L^{-1}(z)) + b(L^{-1}(y_2), L^{-1}(z)) \end{aligned}$$

which shows the proposed inner product is linear in the first argument.

If $(y, y)_{L(U)} = 0$, does it follow $y = 0$? This is easy because the inner product is defined as

$$(y, y)_{L(U)} = (L^{-1}y, L^{-1}y)_U = 0$$

and so $L^{-1}y = 0$ which requires $y = 0$.

Why is $L(U)$ a Hilbert space? Let $\{y_n\}$ be a Cauchy sequence in $L(U)$. Why does it converge? I claim $\{L^{-1}(y_n)\}$ is a Cauchy sequence in U .

$$\begin{aligned} & (L^{-1}(y_n) - L^{-1}(y_m), L^{-1}(y_n) - L^{-1}(y_m))_U \\ & = (L^{-1}(y_n) - L^{-1}(y_m), (L^{-1}(y_n) - L^{-1}(y_m)) - L^{-1}(y_n - y_m))_U \quad (15.2.16) \\ & + (L^{-1}(y_n) - L^{-1}(y_m), L^{-1}(y_n - y_m))_U \end{aligned}$$

Since $(L^{-1}(y_n) - L^{-1}(y_m)) - L^{-1}(y_n - y_m) \in \ker(L)$, it follows from 15.2.13 this term in 15.2.16 equals 0. Then also

$$\begin{aligned} (L^{-1}(y_n) - L^{-1}(y_m), L^{-1}(y_n - y_m))_U &= (L^{-1}(y_n - y_m), L^{-1}(y_n - y_m))_U \\ &+ (L^{-1}(y_n) - L^{-1}(y_m) - L^{-1}(y_n - y_m), L^{-1}(y_n - y_m))_U \end{aligned} \quad (15.2.17)$$

and by 15.2.13 again this last term equals 0. Therefore,

$$\begin{aligned} &(L^{-1}(y_n) - L^{-1}(y_m), L^{-1}(y_n) - L^{-1}(y_m))_U \\ &= (L^{-1}(y_n - y_m), L^{-1}(y_n - y_m))_U \\ &\equiv \|y_n - y_m\|_{L(U)}^2 \end{aligned} \quad (15.2.18)$$

and it follows $\{L^{-1}(y_n)\}$ is a Cauchy sequence as claimed. Therefore, there exists $z \in U$ such that

$$L^{-1}(y_n) \rightarrow z, \quad (15.2.19)$$

the convergence in U and therefore, by continuity of L ,

$$y_n \rightarrow Lz \in L(U),$$

the convergence taking place in H . Does y_n also converge to Lz in the inner product of $L(U)$? By the same reasoning which led to 15.2.18

$$\begin{aligned} \|y_n - Lz\|_{L(U)}^2 &\equiv (L^{-1}(y_n - L(z)), L^{-1}(y_n - L(z)))_U \\ &= (L^{-1}(y_n) - L^{-1}(L(z)), L^{-1}(y_n) - L^{-1}(L(z)))_U \\ &= (L^{-1}(y_n) - z, L^{-1}(y_n) - z)_U \end{aligned}$$

and this last expression converges to 0 by 15.2.19.

Also assuming $L \neq 0$,

$$\|Lx\|_{L(U)}^2 \equiv (L^{-1}(Lx), L^{-1}(Lx))_U = \|x\|_U^2 \geq \frac{1}{\|L\|^2} \|Lx\|_H^2$$

and this proves the claim about the norm. If $L = 0$ there is nothing to prove in this case.

Next suppose U is separable. Why is $L(U)$ separable? Let $\{x_n\}$ be a countable dense subset of U . I claim $\{Lx_n\}$ is a countable dense subset in $L(U)$. This is because

$$\|Lx - Lx_n\|_{L(U)}^2 \equiv (L^{-1}(Lx - Lx_n), L^{-1}(Lx - Lx_n))_U$$

and by 15.2.13, this equals

$$\begin{aligned} &(L^{-1}(Lx - Lx_n) - (x - x_n), L^{-1}(Lx - Lx_n))_U \\ &+ (x - x_n, L^{-1}(Lx - Lx_n)) \end{aligned}$$

$$\begin{aligned}
&= (x - x_n, L^{-1}(Lx - Lx_n)) \\
&= (x - x_n, L^{-1}(Lx - Lx_n) - (x - x_n)) \\
&\quad + (x - x_n, x - x_n)_U
\end{aligned}$$

Thus the density of $\{x_n\}$ implies the density of $\{Lx_n\}$ in $L(U)$. This proves the theorem.

Note what it really says is that the norm in $L(U)$ corresponds to the norm in U .

Now here are some other very interesting results. I am following [55].

Lemma 15.2.3 *Let $L \in \mathcal{L}(U, H)$. Then $L\left(\overline{B(0, r)}\right)$ is closed and convex.*

Proof: It is clear this is convex since L is linear. Why is it closed? $\overline{B(0, r)}$ is compact in the weak topology by the Banach Alaoglu theorem, Theorem 13.4.4 on Page 383. Furthermore, L is continuous with respect to the weak topologies on U and H so $L\left(\overline{B(0, r)}\right)$ is weakly compact because it is the continuous image of a compact set. Therefore, it must also be weakly closed because the weak topology Hausdorff because in this topology which comes from seminorms each point is a closed set thanks to Lemma 14.3.2 on Page 406 and so you can apply the separation theorem, Theorem 14.2.4 on Page 400 to obtain a separating functional. Thus if $x \neq y$, there exists $f \in H'$ such that $\operatorname{Re} f(y) > c > \operatorname{Re} f(x)$ and so taking

$$2r < \min(c - \operatorname{Re} f(x), \operatorname{Re} f(y) - c),$$

$$B_f(x, r) \cap B_f(y, r) = \emptyset$$

where

$$B_f(x, r) \equiv \{y \in H : |f(x - y)| < r\}$$

is an example of a basic open set in the weak topology.

Now suppose $p \notin L\left(\overline{B(0, r)}\right)$. Since the set is weakly closed and convex, it follows by Theorem 14.2.4 and the Riesz representation theorem for Hilbert space there exists $z \in H$ such that

$$\operatorname{Re}(p, z) > c > \operatorname{Re}(Lx, z)$$

for all $x \in \overline{B(0, r)}$. Therefore, p cannot be a strong limit point because if it were, there would exist $x_n \in \overline{B(0, r)}$ such that $Lx_n \rightarrow p$ which would require $\operatorname{Re}(Lx_n, z) \rightarrow \operatorname{Re}(p, z)$ which is prevented by the above inequality. This proves the lemma.

Now here is a very interesting result about showing that $T_1(U_1) = T_2(U_2)$ where U_i is a Hilbert space and $T_i \in \mathcal{L}(U_i, H)$.

Theorem 15.2.4 *Let $U_i, i = 1, 2$ and H be Hilbert spaces and let $T_i \in \mathcal{L}(U_i, H)$. If there exists $c \geq 0$ such that for all $x \in H$*

$$\|T_1^*x\|_1 \leq c \|T_2^*x\|_2 \quad (15.2.20)$$

then

$$T_1 \left(\overline{B(0, 1)} \right) \subseteq T_2 \left(\overline{B(0, c)} \right) \quad (15.2.21)$$

and so $T_1(U_1) \subseteq T_2(U_2)$. If $\|T_1^*x\|_1 = \|T_2^*x\|_2$ for all $x \in H$, then $T_1(U_1) = T_2(U_2)$ and in addition to this,

$$\|T_1^{-1}x\|_1 = \|T_2^{-1}x\|_2 \quad (15.2.22)$$

for all $x \in T_1(U_1) = T_2(U_2)$. In this theorem, T_i^{-1} refers to Definition 15.2.1.

Proof: Consider the first claim. If it is not so, then there exists $u_0, \|u_0\|_1 \leq 1$ but

$$T_1(u_0) \notin T_2 \left(\overline{B(0, c)} \right)$$

the latter set being a closed convex nonempty set thanks to Lemma 15.2.3. Then by the separation theorem, Theorem 14.2.4 there exists $z \in H$ such that

$$\operatorname{Re}(T_1(u_0), z)_H > 1 > \operatorname{Re}(T_2(v), z)_H$$

for all $\|v\|_2 \leq c$. Therefore, replacing v with $v\theta$ where θ is a suitable complex number having modulus 1, it follows

$$\|T_1^*z\| > 1 > |(v, T_2^*z)_{U_2}| \quad (15.2.23)$$

for all $\|v\|_2 \leq c$. If $c = 0$ this gives a contradiction immediately because of 15.2.20. Assume then that $c > 0$. Then from 15.2.23,

$$\|T_2^*z\|_{U_2} = \sup_{\|v\| \leq 1} |(v, T_2^*z)_{U_2}| \leq \frac{1}{c} < \frac{1}{c} \|T_1^*z\|$$

which contradicts 15.2.20. Therefore, it is clear $T_1(U_1) \subseteq T_2(U_2)$.

Now consider the second claim. The first part shows $T_1(U_1) = T_2(U_2)$. Denote by $u_i \in U_i$, the point $T_i^{-1}x$. Without loss of generality, it can be assumed $x \neq 0$ because if $x = 0$, then the definition of T_i^{-1} gives $T_i^{-1}(x) = 0$. Thus for $x \neq 0$ neither u_i can equal 0. I need to verify that $\|u_1\|_1 = \|u_2\|_2$. Suppose then that this is not so. Say $\|u_1\|_1 > \|u_2\|_2 > 0$.

$$\frac{x}{\|u_2\|_2} = T_2 \left(\frac{u_2}{\|u_2\|_2} \right) \in T_2 \left(\overline{B(0, 1)} \right)$$

But from the first part of the theorem this equals $T_1 \left(\overline{B(0, 1)} \right)$ and so there exists $u'_1 \in \overline{B(0, 1)}$ such that

$$\frac{x}{\|u_2\|_2} = T_1 u'_1$$

Hence

$$T_1 \left(u'_1 - \frac{u_1}{\|u_2\|_2} \right) = \frac{x}{\|u_2\|_2} - \frac{x}{\|u_2\|_2} = 0.$$

From Theorem 15.2.2 this implies

$$\begin{aligned} 0 &= \left(u_1, u'_1 - \frac{u_1}{\|u_2\|_2} \right) \leq \|u_1\|_1 \|u'_1\|_1 - \|u_1\|_1 \frac{\|u_1\|_1}{\|u_2\|_2} \\ &= \|u_1\|_1 \left(\|u'_1\|_1 - \frac{\|u_1\|_1}{\|u_2\|_2} \right) \leq \|u_1\|_1 \left(1 - \frac{\|u_1\|_1}{\|u_2\|_2} \right) \end{aligned}$$

which is a contradiction because it was assumed $\frac{\|u_1\|_1}{\|u_2\|_2} > 1$. This proves the theorem.

15.3 Approximations In Hilbert Space

The Gram Schmidt process applies in any Hilbert space.

Theorem 15.3.1 *Let $\{x_1, \dots, x_n\}$ be a basis for M a subspace of H a Hilbert space. Then there exists an orthonormal basis for M , $\{u_1, \dots, u_n\}$ which has the property that for each $k \leq n$, $\text{span}(x_1, \dots, x_k) = \text{span}(u_1, \dots, u_k)$. Also if $\{x_1, \dots, x_n\} \subseteq H$, then*

$$\text{span}(x_1, \dots, x_n)$$

is a closed subspace.

Proof: Let $\{x_1, \dots, x_n\}$ be a basis for M . Let $u_1 \equiv x_1 / |x_1|$. Thus for $k = 1$, $\text{span}(u_1) = \text{span}(x_1)$ and $\{u_1\}$ is an orthonormal set. Now suppose for some $k < n$, $u_1, \dots, u_k$ have been chosen such that $(u_j \cdot u_l) = \delta_{jl}$ and $\text{span}(x_1, \dots, x_k) = \text{span}(u_1, \dots, u_k)$. Then define

$$u_{k+1} \equiv \frac{x_{k+1} - \sum_{j=1}^k (x_{k+1} \cdot u_j) u_j}{\left| x_{k+1} - \sum_{j=1}^k (x_{k+1} \cdot u_j) u_j \right|}, \quad (15.3.24)$$

where the denominator is not equal to zero because the x_j form a basis and so

$$x_{k+1} \notin \text{span}(x_1, \dots, x_k) = \text{span}(u_1, \dots, u_k)$$

Thus by induction,

$$u_{k+1} \in \text{span}(u_1, \dots, u_k, x_{k+1}) = \text{span}(x_1, \dots, x_k, x_{k+1}).$$

Also, $x_{k+1} \in \text{span}(u_1, \dots, u_k, u_{k+1})$ which is seen easily by solving 15.3.24 for x_{k+1} and it follows

$$\text{span}(x_1, \dots, x_k, x_{k+1}) = \text{span}(u_1, \dots, u_k, u_{k+1}).$$

If $l \leq k$,

$$\begin{aligned}
 (u_{k+1} \cdot u_l) &= C \left((x_{k+1} \cdot u_l) - \sum_{j=1}^k (x_{k+1} \cdot u_j) (u_j \cdot u_l) \right) \\
 &= C \left((x_{k+1} \cdot u_l) - \sum_{j=1}^k (x_{k+1} \cdot u_j) \delta_{lj} \right) \\
 &= C ((x_{k+1} \cdot u_l) - (x_{k+1} \cdot u_l)) = 0.
 \end{aligned}$$

The vectors, $\{u_j\}_{j=1}^n$, generated in this way are therefore an orthonormal basis because each vector has unit length.

Consider the second claim about finite dimensional subspaces. Without loss of generality, assume $\{x_1, \dots, x_n\}$ is linearly independent. If it is not, delete vectors until a linearly independent set is obtained. Then by the first part, $\text{span}(x_1, \dots, x_n) = \text{span}(u_1, \dots, u_n) \equiv M$ where the u_i are an orthonormal set of vectors. Suppose $\{y_k\} \subseteq M$ and $y_k \rightarrow y \in H$. Is $y \in M$? Let

$$y_k \equiv \sum_{j=1}^n c_j^k u_j$$

Then let $\mathbf{c}^k \equiv (c_1^k, \dots, c_n^k)^T$. Then

$$\begin{aligned}
 |\mathbf{c}^k - \mathbf{c}^l|^2 &\equiv \sum_{j=1}^n |c_j^k - c_j^l|^2 = \left(\sum_{j=1}^n (c_j^k - c_j^l) u_j, \sum_{j=1}^n (c_j^k - c_j^l) u_j \right) \\
 &= \|y_k - y_l\|^2
 \end{aligned}$$

which shows $\{\mathbf{c}^k\}$ is a Cauchy sequence in $\mathbb{F}^n$ and so it converges to $\mathbf{c} \in \mathbb{F}^n$. Thus

$$y = \lim_{k \rightarrow \infty} y_k = \lim_{k \rightarrow \infty} \sum_{j=1}^n c_j^k u_j = \sum_{j=1}^n c_j u_j \in M.$$

This completes the proof.

Theorem 15.3.2 *Let M be the span of $\{u_1, \dots, u_n\}$ in a Hilbert space, H and let $y \in H$. Then Py is given by*

$$Py = \sum_{k=1}^n (y, u_k) u_k \quad (15.3.25)$$

and the distance is given by

$$\sqrt{|y|^2 - \sum_{k=1}^n |(y, u_k)|^2}. \quad (15.3.26)$$

Proof:

$$\begin{aligned} \left(y - \sum_{k=1}^n (y, u_k) u_k, u_p \right) &= (y, u_p) - \sum_{k=1}^n (y, u_k) (u_k, u_p) \\ &= (y, u_p) - (y, u_p) = 0 \end{aligned}$$

It follows that

$$\left(y - \sum_{k=1}^n (y, u_k) u_k, u \right) = 0$$

for all $u \in M$ and so by Corollary 15.1.13 this verifies 15.3.25.

The square of the distance, d is given by

$$\begin{aligned} d^2 &= \left(y - \sum_{k=1}^n (y, u_k) u_k, y - \sum_{k=1}^n (y, u_k) u_k \right) \\ &= |y|^2 - 2 \sum_{k=1}^n |(y, u_k)|^2 + \sum_{k=1}^n |(y, u_k)|^2 \end{aligned}$$

and this shows 15.3.26.

What if the subspace is the span of vectors which are not orthonormal? There is a very interesting formula for the distance between a point of a Hilbert space and a finite dimensional subspace spanned by an arbitrary basis.

Definition 15.3.3 Let $\{x_1, \dots, x_n\} \subseteq H$, a Hilbert space. Define

$$\mathcal{G}(x_1, \dots, x_n) \equiv \begin{pmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{pmatrix} \quad (15.3.27)$$

Thus the ij^{th} entry of this matrix is (x_i, x_j) . This is sometimes called the Gram matrix. Also define $G(x_1, \dots, x_n)$ as the determinant of this matrix, also called the Gram determinant.

$$G(x_1, \dots, x_n) \equiv \begin{vmatrix} (x_1, x_1) & \cdots & (x_1, x_n) \\ \vdots & & \vdots \\ (x_n, x_1) & \cdots & (x_n, x_n) \end{vmatrix} \quad (15.3.28)$$

The theorem is the following.

Theorem 15.3.4 Let $M = \text{span}(x_1, \dots, x_n) \subseteq H$, a Real Hilbert space where $\{x_1, \dots, x_n\}$ is a basis and let $y \in H$. Then letting d be the distance from y to M ,

$$d^2 = \frac{G(x_1, \dots, x_n, y)}{G(x_1, \dots, x_n)}. \quad (15.3.29)$$

Proof: By Theorem 15.3.1 M is a closed subspace of H . Let $\sum_{k=1}^n \alpha_k x_k$ be the element of M which is closest to y . Then by Corollary 15.1.13,

$$\left(y - \sum_{k=1}^n \alpha_k x_k, x_p \right) = 0$$

for each $p = 1, 2, \dots, n$. This yields the system of equations,

$$(y, x_p) = \sum_{k=1}^n (x_p, x_k) \alpha_k, p = 1, 2, \dots, n \quad (15.3.30)$$

Also by Corollary 15.1.13,

$$\|y\|^2 = \overbrace{\left\| y - \sum_{k=1}^n \alpha_k x_k \right\|^2}^{d^2} + \left\| \sum_{k=1}^n \alpha_k x_k \right\|^2$$

and so, using 15.3.30,

$$\begin{aligned} \|y\|^2 &= d^2 + \sum_j \left(\sum_k \alpha_k (x_k, x_j) \right) \alpha_j \\ &= d^2 + \sum_j (y, x_j) \alpha_j \end{aligned} \quad (15.3.31)$$

$$\equiv d^2 + \mathbf{y}_x^T \boldsymbol{\alpha} \quad (15.3.32)$$

in which

$$\mathbf{y}_x^T \equiv ((y, x_1), \dots, (y, x_n)), \quad \boldsymbol{\alpha}^T \equiv (\alpha_1, \dots, \alpha_n).$$

Then 15.3.30 and 15.3.31 imply the following system

$$\begin{pmatrix} \mathcal{G}(x_1, \dots, x_n) & \mathbf{0} \\ \mathbf{y}_x^T & 1 \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ d^2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}_x \\ \|y\|^2 \end{pmatrix}$$

By Cramer's rule,

$$\begin{aligned} d^2 &= \frac{\det \begin{pmatrix} \mathcal{G}(x_1, \dots, x_n) & \mathbf{y}_x \\ \mathbf{y}_x^T & \|y\|^2 \end{pmatrix}}{\det \begin{pmatrix} \mathcal{G}(x_1, \dots, x_n) & \mathbf{0} \\ \mathbf{y}_x^T & 1 \end{pmatrix}} \\ &= \frac{\det \begin{pmatrix} \mathcal{G}(x_1, \dots, x_n) & \mathbf{y}_x \\ \mathbf{y}_x^T & \|y\|^2 \end{pmatrix}}{\det(\mathcal{G}(x_1, \dots, x_n))} \\ &= \frac{\det(\mathcal{G}(x_1, \dots, x_n, y))}{\det(\mathcal{G}(x_1, \dots, x_n))} = \frac{G(x_1, \dots, x_n, y)}{G(x_1, \dots, x_n)} \end{aligned}$$

and this proves the theorem.

15.4 The Müntz Theorem

Recall the polynomials are dense in $C([0, 1])$. This is a consequence of the Weierstrass approximation theorem. Now consider finite linear combinations of the functions, t^{p_k} where $\{p_0, p_1, p_2, \dots\}$ is a sequence of nonnegative real numbers, $p_0 \equiv 0$. The Müntz theorem says this set, S of finite linear combinations is dense in $C([0, 1])$ exactly when $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$. There are two versions of this theorem, one for density of S in $L^2(0, 1)$ and one for $C([0, 1])$. The presentation follows Cheney [17].

Recall the Cauchy identity presented earlier, Theorem 4.8.1 on Page 78 which is stated here for convenience.

Theorem 15.4.1 *The following identity holds.*

$$\prod_{i,j} (a_i + b_j) \begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & \cdots & \frac{1}{a_n+b_n} \end{vmatrix} = \prod_{j < i} (a_i - a_j)(b_i - b_j). \quad (15.4.33)$$

Lemma 15.4.2 *Let $m, p_1, \dots, p_n$ be distinct real numbers larger than $-1/2$. Thus the functions, $f_m(x) \equiv x^m, f_{p_j}(x) \equiv x^{p_j}$ are all in $L^2(0, 1)$. Let $M = \text{span}(f_{p_1}, \dots, f_{p_n})$. Then the L^2 distance, d between f_m and M is*

$$d = \frac{1}{\sqrt{2m+1}} \prod_{j=1}^n \frac{|m - p_j|}{m + p_j + 1}$$

Proof: By Theorem 15.3.4

$$d^2 = \frac{G(f_{p_1}, \dots, f_{p_n}, f_m)}{G(f_{p_1}, \dots, f_{p_n})}.$$

$$(f_{p_i}, f_{p_j}) = \int_0^1 x^{p_i} x^{p_j} dx = \frac{1}{1 + p_i + p_j}$$

Therefore,

$$d^2 = \frac{\begin{vmatrix} \frac{1}{1+p_1+p_1} & \frac{1}{1+p_1+p_2} & \cdots & \frac{1}{1+p_1+p_n} & \frac{1}{1+m_1+p_1} \\ \frac{1}{1+p_2+p_1} & \frac{1}{1+p_2+p_2} & \cdots & \frac{1}{1+p_2+p_n} & \frac{1}{1+m+p_2} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{1+p_n+p_1} & \frac{1}{1+p_n+p_2} & \cdots & \frac{1}{1+p_n+p_n} & \frac{1}{1+p_n+m} \\ \frac{1}{1+m+p_1} & \frac{1}{1+m+p_2} & \cdots & \frac{1}{1+m+p_n} & \frac{1}{1+m+m} \end{vmatrix}}{\begin{vmatrix} \frac{1}{1+p_1+p_1} & \frac{1}{1+p_1+p_2} & \cdots & \frac{1}{1+p_1+p_n} \\ \frac{1}{1+p_2+p_1} & \frac{1}{1+p_2+p_2} & \cdots & \frac{1}{1+p_2+p_n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{1+p_n+p_1} & \frac{1}{1+p_n+p_2} & \cdots & \frac{1}{1+p_n+p_n} \end{vmatrix}}$$

Now from the Cauchy identity, letting $a_i = p_i + \frac{1}{2}$ and $b_j = \frac{1}{2} + p_j$ with $p_{n+1} = m$, the numerator of the above equals

$$\begin{aligned} & \frac{\prod_{j < i \leq n+1} (p_i - p_j) (p_i - p_j)}{\prod_{i,j \leq n+1} (p_i + p_j + 1)} \\ &= \frac{\prod_{k=1}^n (m - p_k)^2 \prod_{j < i \leq n} (p_i - p_j)^2}{\prod_{i=1}^n (m + p_i + 1) \prod_{j=1}^n (m + p_j + 1) \prod_{i,j \leq n} (p_i + p_j + 1) (2m + 1)} \\ &= \frac{\prod_{k=1}^n (m - p_k)^2 \prod_{j < i \leq n} (p_i - p_j)^2}{\prod_{i=1}^n (m + p_i + 1)^2 \prod_{i,j \leq n} (p_i + p_j + 1) (2m + 1)} \end{aligned}$$

while the denominator equals

$$\frac{\prod_{j < i \leq n} (p_i - p_j)^2}{\prod_{i,j \leq n} (p_i + p_j + 1)}$$

Therefore,

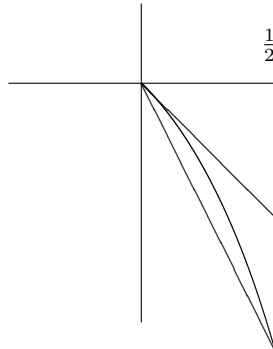
$$\begin{aligned} d^2 &= \frac{\left(\frac{\prod_{k=1}^n (m - p_k)^2 \prod_{j < i \leq n} (p_i - p_j)^2}{\prod_{i=1}^n (m + p_i + 1)^2 \prod_{i,j \leq n} (p_i + p_j + 1) (2m + 1)} \right)}{\left(\frac{\prod_{j < i \leq n} (p_i - p_j)^2}{\prod_{i,j \leq n} (p_i + p_j + 1)} \right)} \\ &= \frac{\prod_{k=1}^n (m - p_k)^2}{\prod_{i=1}^n (m + p_i + 1)^2 (2m + 1)} \end{aligned}$$

which shows

$$d = \frac{1}{\sqrt{2m+1}} \prod_{k=1}^n \frac{|m - p_k|}{m + p_k + 1}.$$

and this proves the lemma.

The following lemma relates an infinite sum to a product. First consider the graph of $\ln(1-x)$ for $x \in [0, \frac{1}{2}]$. Here is a rough sketch with two lines, $y = -x$ which lies above the graph of $\ln(1-x)$ and $y = -2x$ which lies below.



Lemma 15.4.3 *Let $a_n \neq 1, a_n > 0$, and $\lim_{n \rightarrow \infty} a_n = 0$. Then*

$$\prod_{k=1}^{\infty} (1 - a_n) \equiv \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 - a_n) = 0$$

if and only if

$$\sum_{n=1}^{\infty} a_n = +\infty.$$

Proof: Without loss of generality, you can assume $a_n < 1/2$ because the two conditions are determined by the values of a_n for n large. By the above sketch the following is obtained.

$$\ln \prod_{k=1}^n (1 - a_k) = \sum_{k=1}^n \ln (1 - a_k) \in \left[-2 \sum_{k=1}^n a_k, -\sum_{k=1}^n a_k \right].$$

Therefore,

$$e^{-2 \sum_{k=1}^n a_k} \leq \prod_{k=1}^n (1 - a_k) \leq e^{-\sum_{k=1}^n a_k}$$

The conclusion follows.

The following is Müntz's first theorem.

Theorem 15.4.4 *Let $\{p_n\}$ be a sequence of real numbers larger than $-1/2$ such that $\lim_{n \rightarrow \infty} p_n = \infty$. Let S denote the set of finite linear combinations of the functions, $\{x^{p_1}, x^{p_2}, \dots\}$. Then S is dense in $L^2(0, 1)$ if and only if*

$$\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty.$$

Proof: The polynomials are dense in $L^2(0, 1)$ and so S is dense in $L^2(0, 1)$ if and only if for every $\varepsilon > 0$ there exists a function f from S such that for each integer $m \geq 0$, $\left(\int_0^1 |f(x) - x^m|^2 dx \right)^{1/2} < \varepsilon$. This happens if and only if for all n large enough, the distance in $L^2(0, 1)$ between the function, $x \rightarrow x^m$ and $\text{span}(x^{p_1}, x^{p_2}, \dots, x^{p_n})$ is less than ε . However, from Lemma 15.4.2 this distance equals

$$\begin{aligned} & \frac{1}{\sqrt{2m+1}} \prod_{k=1}^n \frac{|m - p_k|}{m + p_k + 1} \\ &= \frac{1}{\sqrt{2m+1}} \prod_{k=1}^n \left(1 - \frac{|m - p_k|}{m + p_k + 1} \right) \end{aligned}$$

Thus S is dense if and only if

$$\prod_{k=1}^{\infty} \left(1 - \left(1 - \frac{|m - p_k|}{m + p_k + 1} \right) \right) = 0$$

which, by Lemma 15.4.3, happens if and only if

$$\sum_{k=1}^{\infty} \left(1 - \frac{|m - p_k|}{m + p_k + 1} \right) = +\infty$$

But this sum equals

$$\sum_{k=1}^{\infty} \left(\frac{m + p_k + 1 - |m - p_k|}{m + p_k + 1} \right)$$

which has the same convergence properties as $\sum \frac{1}{p_k}$ by the limit comparison test. This proves the theorem.

The following is Müntz's second theorem.

Theorem 15.4.5 *Let S be finite linear combinations of $\{1, x^{p_1}, x^{p_2}, \dots\}$ where $p_j \geq 1$ and $\lim_{n \rightarrow \infty} p_n = \infty$. Then S is dense in $C([0, 1])$ if and only if $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$.*

Proof: If S is dense in $C([0, 1])$ then S must also be dense in $L^2(0, 1)$ and so by Theorem 15.4.4 $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$.

Suppose then that $\sum_{k=1}^{\infty} \frac{1}{p_k} = \infty$ so that by Theorem 15.4.4, S is dense in $L^2(0, 1)$. The theorem will be proved if it is shown that for all m a nonnegative integer,

$$\max \{|x^m - f(x)| : x \in [0, 1]\} < \varepsilon$$

for some $f \in S$. This is true if $m = 0$ because $1 \in S$. Suppose then that $m > 0$. Let S' denote finite linear combinations of the functions

$$\{x^{p_1-1}, x^{p_2-1}, \dots\}.$$

These functions are also dense in $L^2(0, 1)$ because $\sum \frac{1}{p_k-1} = \infty$ by the limit comparison test. Then by Theorem 15.4.4 there exists $f \in S'$ such that

$$\left(\int_0^1 |f(x) - mx^{m-1}|^2 dx \right)^{1/2} < \varepsilon.$$

Thus $F(x) \equiv \int_0^x f(t) dt \in S$ and

$$\begin{aligned} |F(x) - x^m| &= \left| \int_0^x (f(t) - mt^{m-1}) dt \right| \\ &\leq \int_0^x |f(t) - mt^{m-1}| dt \\ &\leq \left(\int_0^1 |f(t) - mt^{m-1}|^2 dt \right)^{1/2} \left(\int_0^1 dx \right)^{1/2} \\ &< \varepsilon \end{aligned}$$

and this proves the theorem.

15.5 Orthonormal Sets

The concept of an orthonormal set of vectors is a generalization of the notion of the standard basis vectors of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition 15.5.1 Let H be a Hilbert space. $S \subseteq H$ is called an orthonormal set if $\|x\| = 1$ for all $x \in S$ and $(x, y) = 0$ if $x, y \in S$ and $x \neq y$. For any set, D ,

$$D^\perp \equiv \{x \in H : (x, d) = 0 \text{ for all } d \in D\}.$$

If S is a set, $\text{span}(S)$ is the set of all finite linear combinations of vectors from S .

You should verify that $D^\perp$ is always a closed subspace of H .

Theorem 15.5.2 In any separable Hilbert space, H , there exists a countable orthonormal set, $S = \{x_i\}$ such that the span of these vectors is dense in H . Furthermore, if $\text{span}(S)$ is dense, then for $x \in H$,

$$x = \sum_{i=1}^{\infty} (x, x_i) x_i \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^n (x, x_i) x_i. \quad (15.5.34)$$

Proof: Let $\mathcal{F}$ denote the collection of all orthonormal subsets of H . $\mathcal{F}$ is nonempty because $\{x\} \in \mathcal{F}$ where $\|x\| = 1$. The set, $\mathcal{F}$ is a partially ordered set with the order given by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathfrak{C}$ in $\mathcal{F}$. Then let $S \equiv \cup \mathfrak{C}$. It follows S must be a maximal orthonormal set of vectors. Why? It remains to verify that S is countable $\text{span}(S)$ is dense, and the condition, 15.5.34 holds. To see S is countable note that if $x, y \in S$, then

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2 \operatorname{Re}(x, y) = \|x\|^2 + \|y\|^2 = 2.$$

Therefore, the open sets, $B(x, \frac{1}{2})$ for $x \in S$ are disjoint and cover S . Since H is assumed to be separable, there exists a point from a countable dense set in each of these disjoint balls showing there can only be countably many of the balls and that consequently, S is countable as claimed.

It remains to verify 15.5.34 and that $\text{span}(S)$ is dense. If $\overline{\text{span}(S)}$ is not dense, then $\text{span}(S)$ is a closed proper subspace of H and letting $y \notin \overline{\text{span}(S)}$,

$$z \equiv \frac{y - Py}{\|y - Py\|} \in \text{span}(S)^\perp.$$

But then $S \cup \{z\}$ would be a larger orthonormal set of vectors contradicting the maximality of S .

It remains to verify 15.5.34. Let $S = \{x_i\}_{i=1}^{\infty}$ and consider the problem of choosing the constants, c_k in such a way as to minimize the expression

$$\left\| x - \sum_{k=1}^n c_k x_k \right\|^2 =$$

$$\|x\|^2 + \sum_{k=1}^n |c_k|^2 - \sum_{k=1}^n \overline{c_k} (x, x_k) - \sum_{k=1}^n c_k \overline{(x, x_k)}.$$

This equals

$$\|x\|^2 + \sum_{k=1}^n |c_k - (x, x_k)|^2 - \sum_{k=1}^n |(x, x_k)|^2$$

and therefore, this minimum is achieved when $c_k = (x, x_k)$ and equals

$$\|x\|^2 - \sum_{k=1}^n |(x, x_k)|^2$$

Now since $\text{span}(S)$ is dense, there exists n large enough that for some choice of constants, c_k ,

$$\left\| x - \sum_{k=1}^n c_k x_k \right\|^2 < \varepsilon.$$

However, from what was just shown,

$$\left\| x - \sum_{i=1}^n (x, x_i) x_i \right\|^2 \leq \left\| x - \sum_{k=1}^n c_k x_k \right\|^2 < \varepsilon$$

showing that $\lim_{n \rightarrow \infty} \sum_{i=1}^n (x, x_i) x_i = x$ as claimed. This proves the theorem.

The proof of this theorem contains the following corollary.

Corollary 15.5.3 *Let S be any orthonormal set of vectors and let*

$$\{x_1, \dots, x_n\} \subseteq S.$$

Then if $x \in H$

$$\left\| x - \sum_{k=1}^n c_k x_k \right\|^2 \geq \left\| x - \sum_{i=1}^n (x, x_i) x_i \right\|^2$$

for all choices of constants, c_k . In addition to this, Bessel's inequality

$$\|x\|^2 \geq \sum_{k=1}^n |(x, x_k)|^2.$$

If S is countable and $\text{span}(S)$ is dense, then letting $\{x_i\}_{i=1}^\infty = S$, 15.5.34 follows.

15.6 Fourier Series, An Example

In this section consider the Hilbert space, $L^2(0, 2\pi)$ with the inner product,

$$(f, g) \equiv \int_0^{2\pi} f \overline{g} dm.$$

This is a Hilbert space because of the theorem which states the L^p spaces are complete, Theorem 12.1.10 on Page 341. An example of an orthonormal set of functions in $L^2(0, 2\pi)$ is

$$\phi_n(x) \equiv \frac{1}{\sqrt{2\pi}} e^{inx}$$

for n an integer. Is it true that the span of these functions is dense in $L^2(0, 2\pi)$?

Theorem 15.6.1 *Let $S = \{\phi_n\}_{n \in \mathbb{Z}}$. Then $\text{span}(S)$ is dense in $L^2(0, 2\pi)$.*

Proof: By regularity of Lebesgue measure, it follows from Theorem 12.2.4 that $C_c(0, 2\pi)$ is dense in $L^2(0, 2\pi)$. Therefore, it suffices to show that for $g \in C_c(0, 2\pi)$, then for every $\varepsilon > 0$ there exists $h \in \text{span}(S)$ such that $\|g - h\|_{L^2(0, 2\pi)} < \varepsilon$.

Let T denote the points of $\mathbb{C}$ which are of the form e^{it} for $t \in \mathbb{R}$. Let $\mathcal{A}$ denote the algebra of functions consisting of polynomials in z and $1/z$ for $z \in T$. Thus a typical such function would be one of the form

$$\sum_{k=-m}^m c_k z^k$$

for m chosen large enough. This algebra separates the points of T because it contains the function, $p(z) = z$. It annihilates no point of t because it contains the constant function 1. Furthermore, it has the property that for $f \in \mathcal{A}$, $\bar{f} \in \mathcal{A}$. By the Stone Weierstrass approximation theorem, Theorem 7.2.10 on Page 171, $\mathcal{A}$ is dense in $C(T)$. Now for $g \in C_c(0, 2\pi)$, extend g to all of $\mathbb{R}$ to be 2π periodic. Then letting $G(e^{it}) \equiv g(t)$, it follows G is well defined and continuous on T . Therefore, there exists $H \in \mathcal{A}$ such that for all $t \in \mathbb{R}$,

$$|H(e^{it}) - G(e^{it})| < \varepsilon^2/2\pi.$$

Thus $H(e^{it})$ is of the form

$$H(e^{it}) = \sum_{k=-m}^m c_k (e^{it})^k = \sum_{k=-m}^m c_k e^{ikt} \in \text{span}(S).$$

Let $h(t) = \sum_{k=-m}^m c_k e^{ikt}$. Then

$$\begin{aligned} \left(\int_0^{2\pi} |g - h|^2 dx \right)^{1/2} &\leq \left(\int_0^{2\pi} \max\{|g(t) - h(t)| : t \in [0, 2\pi]\} dx \right)^{1/2} \\ &= \left(\int_0^{2\pi} \max\{|G(e^{it}) - H(e^{it})| : t \in [0, 2\pi]\} dx \right)^{1/2} \\ &< \left(\int_0^{2\pi} \frac{\varepsilon^2}{2\pi} \right)^{1/2} = \varepsilon. \end{aligned}$$

This proves the theorem.

Corollary 15.6.2 For $f \in L^2(0, 2\pi)$,

$$\lim_{m \rightarrow \infty} \left\| f - \sum_{k=-m}^m (f, \phi_k) \phi_k \right\|_{L^2(0, 2\pi)}$$

Proof: This follows from Theorem 15.5.2 on Page 440.

15.7 Compact Operators

15.7.1 Compact Operators In Hilbert Space

Definition 15.7.1 Let $A \in \mathcal{L}(H, H)$ where H is a Hilbert space. Then $|(Ax, y)| \leq \|A\| \|x\| \|y\|$ and so the map, $x \rightarrow (Ax, y)$ is continuous and linear. By the Riesz representation theorem, there exists a unique element of H , denoted by A^*y such that

$$(Ax, y) = (x, A^*y).$$

It is clear $y \rightarrow A^*y$ is linear and continuous. A^* is called the adjoint of A . A is a self adjoint operator if $A = A^*$. Thus for a self adjoint operator, $(Ax, y) = (x, Ay)$ for all $x, y \in H$. A is a compact operator if whenever $\{x_k\}$ is a bounded sequence, there exists a convergent subsequence of $\{Ax_k\}$. Equivalently, A maps bounded sets to sets whose closures are compact.

The big result is called the Hilbert Schmidt theorem. It is a generalization to arbitrary Hilbert spaces of standard finite dimensional results having to do with diagonalizing a symmetric matrix. There is another statement and proof of this theorem around Page 719.

Theorem 15.7.2 Let A be a compact self adjoint operator defined on a Hilbert space, H . Then there exists a countable set of eigenvalues, $\{\lambda_i\}$ and an orthonormal set of eigenvectors, u_i satisfying

$$\lambda_i \text{ is real, } |\lambda_n| \geq |\lambda_{n+1}|, \quad Au_i = \lambda_i u_i, \quad (15.7.35)$$

and either

$$\lim_{n \rightarrow \infty} \lambda_n = 0, \quad (15.7.36)$$

or for some n ,

$$\text{span}(u_1, \dots, u_n) = H. \quad (15.7.37)$$

In any case,

$$\text{span}(\{u_i\}_{i=1}^{\infty}) \text{ is dense in } A(H). \quad (15.7.38)$$

and for all $x \in H$,

$$Ax = \sum_{k=1}^{\infty} \lambda_k (x, u_k) u_k. \quad (15.7.39)$$

This sequence of eigenvectors and eigenvalues also satisfies

$$|\lambda_n| = \|A_n\|, \quad (15.7.40)$$

and

$$A_n : H_n \rightarrow H_n. \quad (15.7.41)$$

where $H \equiv H_1$ and $H_n \equiv \{u_1, \dots, u_{n-1}\}^\perp$ and A_n is the restriction of A to H_n .

Proof: If $A = 0$ then pick $u \in H$ with $\|u\| = 1$ and let $\lambda_1 = 0$. Since $A(H) = 0$ it follows the span of u is dense in $A(H)$ and this proves the theorem in this uninteresting case.

Assume from now on $A \neq 0$. Let λ_1 be real and $\lambda_1^2 \equiv \|A\|^2$. From the definition of $\|A\|$ there exists $x_n, \|x_n\| = 1$, and $\|Ax_n\| \rightarrow \|A\| = |\lambda_1|$. Now it is clear that A^2 is also a compact self adjoint operator. Consider

$$((\lambda_1^2 - A^2)x_n, x_n) = \lambda_1^2 - \|Ax_n\|^2 \rightarrow 0.$$

Since A is compact, there exists a subsequence of $\{x_n\}$ still denoted by $\{x_n\}$ such that Ax_n converges to some element of H . Thus since $\lambda_1^2 - A^2$ satisfies

$$((\lambda_1^2 - A^2)y, y) \geq 0$$

in addition to being self adjoint, it follows $x, y \rightarrow ((\lambda_1^2 - A^2)x, y)$ satisfies all the axioms for an inner product except for the one which says that $(z, z) = 0$ only if $z = 0$. Therefore, the Cauchy Schwarz inequality may be used to write

$$\begin{aligned} |((\lambda_1^2 - A^2)x_n, y)| &\leq ((\lambda_1^2 - A^2)y, y)^{1/2} ((\lambda_1^2 - A^2)x_n, x_n)^{1/2} \\ &\leq e_n \|y\|. \end{aligned}$$

where $e_n \rightarrow 0$ as $n \rightarrow \infty$. Therefore, taking the sup over all $\|y\| \leq 1$,

$$\lim_{n \rightarrow \infty} \|(\lambda_1^2 - A^2)x_n\| = 0.$$

Since A^2x_n converges, it follows since $\lambda_1 \neq 0$ that $\{x_n\}$ is a Cauchy sequence converging to x with $\|x\| = 1$. Therefore, $A^2x_n \rightarrow A^2x$ and so

$$\|(\lambda_1^2 - A^2)x\| = 0.$$

Now

$$(\lambda_1 I - A)(\lambda_1 I + A)x = (\lambda_1 I + A)(\lambda_1 I - A)x = 0.$$

If $(\lambda_1 I - A)x = 0$, let $u_1 \equiv x$. If $(\lambda_1 I - A)x = y \neq 0$, let $u_1 \equiv \frac{y}{\|y\|}$.

Suppose $\{u_1, \dots, u_n\}$ is such that $Au_k = \lambda_k u_k$ and $|\lambda_k| \geq |\lambda_{k+1}|$, $|\lambda_k| = \|A_k\|$ and $A_k : H_k \rightarrow H_k$ for $k \leq n$. If

$$\text{span}(u_1, \dots, u_n) = H$$

this yields the conclusion of the theorem in the situation of 15.7.37. Therefore, assume the span of these vectors is always a proper subspace of H . It is shown next that $A_{n+1} : H_{n+1} \rightarrow H_{n+1}$. Let

$$y \in H_{n+1} \equiv \{u_1, \dots, u_n\}^\perp$$

Then for $k \leq n$

$$(Ay, u_k) = (y, Au_k) = \lambda_k (y, u_k) = 0,$$

showing $A_{n+1} : H_{n+1} \rightarrow H_{n+1}$ as claimed. There are two cases. Either $\lambda_n = 0$ or it is not. In the case where $\lambda_n = 0$ it follows $A_n = 0$. Every element of H is the sum of one in $\text{span}(u_1, \dots, u_n)$ and one in $\text{span}(u_1, \dots, u_n)^\perp$. (note $\text{span}(u_1, \dots, u_n)$ is a closed subspace.) Thus, if $x \in H$, $x = y + z$ where $y \in \text{span}(u_1, \dots, u_n)$ and $z \in \text{span}(u_1, \dots, u_n)^\perp$ and $Az = 0$. Say $y = \sum_{j=1}^n c_j u_j$. Then

$$\begin{aligned} Ax &= Ay = \sum_{j=1}^n c_j Au_j \\ &= \sum_{j=1}^n c_j \lambda_j u_j \in \text{span}(u_1, \dots, u_n). \end{aligned}$$

The conclusion of the theorem holds in this case because the above equation holds if with $c_i = (x, u_i)$.

Now consider the case where $\lambda_n \neq 0$. In this case repeat the above argument used to find u_{n+1} and λ_{n+1} for the operator, A_{n+1} . This yields $u_{n+1} \in H_{n+1} \equiv \{u_1, \dots, u_n\}^\perp$ such that

$$\|u_{n+1}\| = 1, \|Au_{n+1}\| = |\lambda_{n+1}| = \|A_{n+1}\| \leq \|A_n\| = |\lambda_n|$$

and if it is ever the case that $\lambda_n = 0$, it follows from the above argument that the conclusion of the theorem is obtained.

I claim $\lim_{n \rightarrow \infty} \lambda_n = 0$. If this were not so, then for some $\varepsilon > 0$, $0 < \varepsilon = \lim_{n \rightarrow \infty} |\lambda_n|$ but then

$$\begin{aligned} \|Au_n - Au_m\|^2 &= \|\lambda_n u_n - \lambda_m u_m\|^2 \\ &= |\lambda_n|^2 + |\lambda_m|^2 \geq 2\varepsilon^2 \end{aligned}$$

and so there would not exist a convergent subsequence of $\{Au_k\}_{k=1}^\infty$ contrary to the assumption that A is compact. This verifies the claim that $\lim_{n \rightarrow \infty} \lambda_n = 0$.

It remains to verify that $\text{span}(\{u_i\})$ is dense in $A(H)$. If $w \in \text{span}(\{u_i\})^\perp$ then $w \in H_n$ for all n and so for all n ,

$$\|Aw\| \leq \|A_n\| \|w\| \leq |\lambda_n| \|w\|.$$

Therefore, $Aw = 0$. Now every vector from H can be written as a sum of one from

$$\text{span}(\{u_i\})^\perp = \overline{\text{span}(\{u_i\})}^\perp$$

and one from $\overline{\text{span}(\{u_i\})}$. Therefore, if $x \in H$, $x = y + w$ where $y \in \overline{\text{span}(\{u_i\})}$ and $w \in \text{span}(\{u_i\})^\perp$. It follows $Aw = 0$. Also, since $y \in \overline{\text{span}(\{u_i\})}$, there exist constants, c_k and n such that

$$\left\| y - \sum_{k=1}^n c_k u_k \right\| < \varepsilon.$$

Therefore, from Corollary 15.5.3,

$$\left\| y - \sum_{k=1}^n (y, u_k) u_k \right\| = \left\| y - \sum_{k=1}^n (x, u_k) u_k \right\| < \varepsilon.$$

Therefore,

$$\begin{aligned} \|A\| \varepsilon &> \left\| A \left(y - \sum_{k=1}^n (x, u_k) u_k \right) \right\| \\ &= \left\| Ax - \sum_{k=1}^n (x, u_k) \lambda_k u_k \right\|. \end{aligned}$$

Since ε is arbitrary, this shows $\text{span}(\{u_i\})$ is dense in $A(H)$ and also implies 15.7.39. This proves the theorem.

Define $v \otimes u \in \mathcal{L}(H, H)$ by

$$v \otimes u(x) = (x, u) v,$$

then 15.7.39 is of the form

$$A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k$$

This is the content of the following corollary.

Corollary 15.7.3 *The main conclusion of the above theorem can be written as*

$$A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k$$

where the convergence of the partial sums takes place in the operator norm.

Proof: Using 15.7.39

$$\begin{aligned}
& \left| \left(\left(A - \sum_{k=1}^n \lambda_k u_k \otimes u_k \right) x, y \right) \right| \\
&= \left| \left(Ax - \sum_{k=1}^n \lambda_k (x, u_k) u_k, y \right) \right| \\
&= \left| \left(\sum_{k=n}^{\infty} \lambda_k (x, u_k) u_k, y \right) \right| \\
&= \left| \sum_{k=n}^{\infty} \lambda_k (x, u_k) (u_k, y) \right| \\
&\leq |\lambda_n| \left(\sum_{k=n}^{\infty} |(x, u_k)|^2 \right)^{1/2} \left(\sum_{k=n}^{\infty} |(y, u_k)|^2 \right)^{1/2} \\
&\leq |\lambda_n| \|x\| \|y\|
\end{aligned}$$

It follows

$$\left\| \left(A - \sum_{k=1}^n \lambda_k u_k \otimes u_k \right) (x) \right\| \leq |\lambda_n| \|x\|$$

and this proves the corollary.

Corollary 15.7.4 *Let A be a compact self adjoint operator defined on a separable Hilbert space, H . Then there exists a countable set of eigenvalues, $\{\lambda_i\}$ and an orthonormal set of eigenvectors, v_i satisfying*

$$Av_i = \lambda_i v_i, \|v_i\| = 1, \quad (15.7.42)$$

$$\text{span}(\{v_i\}_{i=1}^{\infty}) \text{ is dense in } H. \quad (15.7.43)$$

Furthermore, if $\lambda_i \neq 0$, the space, $V_{\lambda_i} \equiv \{x \in H : Ax = \lambda_i x\}$ is finite dimensional.

Proof: In the proof of the above theorem, let $W \equiv \overline{\text{span}(\{u_i\})}^{\perp}$. By Theorem 15.5.2, there is an orthonormal set of vectors, $\{w_i\}_{i=1}^{\infty}$ whose span is dense in W . As shown in the proof of the above theorem, $Aw = 0$ for all $w \in W$. Let $\{v_i\}_{i=1}^{\infty} = \{u_i\}_{i=1}^{\infty} \cup \{w_i\}_{i=1}^{\infty}$.

It remains to verify the space, V_{λ_i} , is finite dimensional. First observe that $A : V_{\lambda_i} \rightarrow V_{\lambda_i}$. Since A is continuous, it follows that $A : \overline{V_{\lambda_i}} \rightarrow \overline{V_{\lambda_i}}$. Thus A is a compact self adjoint operator on $\overline{V_{\lambda_i}}$ and by Theorem 15.7.2, 15.7.37 holds because the only eigenvalue is λ_i . This proves the corollary.

Note the last claim of this corollary holds independent of the separability of H . This proves the corollary.

Suppose $\lambda \notin \{\lambda_n\}$ and $\lambda \neq 0$. Then the above formula for A , 15.7.39, yields an interesting formula for $(A - \lambda I)^{-1}$. Note first that since $\lim_{n \rightarrow \infty} \lambda_n = 0$, it follows that $\lambda_n^2 / (\lambda_n - \lambda)^2$ must be bounded, say by a positive constant, M .

Corollary 15.7.5 *Let A be a compact self adjoint operator and let $\lambda \notin \{\lambda_n\}_{n=1}^{\infty}$ and $\lambda \neq 0$ where the λ_n are the eigenvalues of A . Then*

$$(A - \lambda I)^{-1} x = -\frac{1}{\lambda} x + \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_k - \lambda} (x, u_k) u_k. \quad (15.7.44)$$

Proof: Let $m < n$. Then since the $\{u_k\}$ form an orthonormal set,

$$\begin{aligned} \left| \sum_{k=m}^n \frac{\lambda_k}{\lambda_k - \lambda} (x, u_k) u_k \right| &= \left(\sum_{k=m}^n \left(\frac{\lambda_k}{\lambda_k - \lambda} \right)^2 |(x, u_k)|^2 \right)^{1/2} \\ &\leq M \left(\sum_{k=m}^n |(x, u_k)|^2 \right)^{1/2}. \end{aligned} \quad (15.7.45)$$

But from Bessel's inequality,

$$\sum_{k=1}^{\infty} |(x, u_k)|^2 \leq \|x\|^2$$

and so for m large enough, the first term in 15.7.45 is smaller than ε . This shows the infinite series in 15.7.44 converges. It is now routine to verify that the formula in 15.7.44 is the inverse.

15.7.2 Nuclear Operators

Definition 15.7.6 *A self adjoint operator $A \in \mathcal{L}(H, H)$ for H a separable Hilbert space is called a nuclear operator if for some complete orthonormal set, $\{e_k\}$,*

$$\sum_{k=1}^{\infty} |(Ae_k, e_k)| < \infty$$

To begin with here is an interesting lemma.

Lemma 15.7.7 *Suppose $\{A_n\}$ is a sequence of compact operators in $\mathcal{L}(X, Y)$ for two Banach spaces, X and Y and suppose $A \in \mathcal{L}(X, Y)$ and*

$$\lim_{n \rightarrow \infty} \|A - A_n\| = 0.$$

Then A is also compact.

Proof: Let B be a bounded set in X such that $\|b\| \leq C$ for all $b \in B$. I need to verify AB is totally bounded. Suppose then it is not. Then there exists $\varepsilon > 0$ and a sequence, $\{Ab_i\}$ where $b_i \in B$ and

$$\|Ab_i - Ab_j\| \geq \varepsilon$$

whenever $i \neq j$. Then let n be large enough that

$$\|A - A_n\| \leq \frac{\varepsilon}{4C}.$$

Then

$$\begin{aligned} \|A_n b_i - A_n b_j\| &= \|A b_i - A b_j + (A_n - A) b_i - (A_n - A) b_j\| \\ &\geq \|A b_i - A b_j\| - \|(A_n - A) b_i\| - \|(A_n - A) b_j\| \\ &\geq \|A b_i - A b_j\| - \frac{\varepsilon}{4C} C - \frac{\varepsilon}{4C} C \geq \frac{\varepsilon}{2}, \end{aligned}$$

a contradiction to A_n being compact. This proves the lemma.

Then one can prove the following lemma. In this lemma, $A \geq 0$ will mean $(Ax, x) \geq 0$.

Lemma 15.7.8 *Let $A \geq 0$ be a nuclear operator defined on a separable Hilbert space, H . Then A is compact and also, whenever $\{e_k\}$ is a complete orthonormal set,*

$$A = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (A e_i, e_j) e_i \otimes e_j.$$

Proof: First consider the formula. Since A is given to be continuous,

$$Ax = A \left(\sum_{j=1}^{\infty} (x, e_j) e_j \right) = \sum_{j=1}^{\infty} (x, e_j) A e_j,$$

the series converging because

$$x = \sum_{j=1}^{\infty} (x, e_j) e_j$$

Then also since A is self adjoint,

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (A e_i, e_j) e_i \otimes e_j (x) &\equiv \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} (A e_i, e_j) (x, e_j) e_i \\ &= \sum_{j=1}^{\infty} (x, e_j) \sum_{i=1}^{\infty} (A e_i, e_j) e_i \\ &= \sum_{j=1}^{\infty} (x, e_j) \sum_{i=1}^{\infty} (A e_j, e_i) e_i \\ &= \sum_{j=1}^{\infty} (x, e_j) A e_j \end{aligned}$$

Next consider the claim that A is compact. Let $C_A \equiv \left(\sum_{j=1}^{\infty} |(Ae_j, e_j)| \right)^{1/2}$. Let A_n be defined by

$$A_n \equiv \sum_{j=1}^{\infty} \sum_{i=1}^n (Ae_i, e_j) (e_i \otimes e_j).$$

Then A_n has values in $\text{span}(e_1, \dots, e_n)$ and so it must be a compact operator because bounded sets in a finite dimensional space must be precompact. Then

$$\begin{aligned} |(Ax - A_n x, y)| &= \left| \sum_{j=1}^{\infty} \sum_{i=n+1}^{\infty} (Ae_i e_j)(y, e_j)(e_i, x) \right| \\ &= \left| \sum_{j=1}^{\infty} (y, e_j) \sum_{i=n+1}^{\infty} (Ae_i e_j)(e_i, x) \right| \\ &\leq \left| \sum_{j=1}^{\infty} |(y, e_j)| (Ae_j, e_j)^{1/2} \sum_{i=n+1}^{\infty} (Ae_i e_i)^{1/2} |(e_i, x)| \right| \\ &\leq \left(\sum_{j=1}^{\infty} |(y, e_j)|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |(Ae_j, e_j)| \right)^{1/2} \\ &\quad \cdot \left(\sum_{i=n+1}^{\infty} |(x, e_i)|^2 \right)^{1/2} \left(\sum_{i=n+1}^{\infty} |(Ae_i e_i)| \right)^{1/2} \\ &\leq |y| |x| C_A \left(\sum_{i=n+1}^{\infty} |(Ae_i, e_i)| \right)^{1/2} \end{aligned}$$

and this shows that if n is sufficiently large,

$$|((A - A_n)x, y)| \leq \varepsilon |x| |y|.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|A - A_n\| = 0$$

and so A is the limit in operator norm of finite rank bounded linear operators, each of which is compact. Therefore, A is also compact.

Definition 15.7.9 *The trace of a nuclear operator $A \in \mathcal{L}(H, H)$ such that $A \geq 0$ is defined to equal*

$$\sum_{k=1}^{\infty} (Ae_k, e_k)$$

where $\{e_k\}$ is an orthonormal basis for the Hilbert space, H .

Theorem 15.7.10 *Definition 15.7.9 is well defined and equals $\sum_{j=1}^{\infty} \lambda_j$ where the λ_j are the eigenvalues of A .*

Proof: Suppose $\{u_k\}$ is some other orthonormal basis. Then

$$e_k = \sum_{j=1}^{\infty} u_j (e_k, u_j)$$

By Lemma 15.7.8 A is compact and so

$$A = \sum_{k=1}^{\infty} \lambda_k u_k \otimes u_k$$

where the u_k are the orthonormal eigenvectors of A which form a complete orthonormal set. Then

$$\begin{aligned} \sum_{k=1}^{\infty} (Ae_k, e_k) &= \sum_{k=1}^{\infty} \left(A \left(\sum_{j=1}^{\infty} u_j (e_k, u_j) \right), \sum_{j=1}^{\infty} u_j (e_k, u_j) \right) \\ &= \sum_{k=1}^{\infty} \sum_{ij} (Au_j, u_i) (e_k, u_j) (u_i, e_k) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (Au_j, u_j) |(e_k, u_j)|^2 \\ &= \sum_{j=1}^{\infty} (Au_j, u_j) \sum_{k=1}^{\infty} |(e_k, u_j)|^2 = \sum_{j=1}^{\infty} (Au_j, u_j) |u_j|^2 \\ &= \sum_{j=1}^{\infty} (Au_j, u_j) = \sum_{j=1}^{\infty} \lambda_j \end{aligned}$$

and this proves the theorem.

This is just like it is for a matrix. Recall the trace of a matrix is the sum of the eigenvalues.

It is also easy to see that in any separable Hilbert space, there exist nuclear operators. Let $\sum_{k=1}^{\infty} |\lambda_k| < \infty$. Then let $\{e_k\}$ be a complete orthonormal set of vectors. Let

$$A \equiv \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k.$$

It is not too hard to verify this works.

Much more can be said about nuclear operators.

15.7.3 Hilbert Schmidt Operators

Definition 15.7.11 *Let H and G be two separable Hilbert spaces and let T map H to G be linear. Then T is called a Hilbert Schmidt operator if there exists some*

orthonormal basis for H , $\{e_j\}$ such that

$$\sum_j \|Te_j\|^2 < \infty.$$

The collection of all such linear maps will be denoted by $\mathcal{L}_2(H, G)$.

Theorem 15.7.12 $\mathcal{L}_2(H, G) \subseteq \mathcal{L}(H, G)$ and $\mathcal{L}_2(H, G)$ is a separable Hilbert space with norm given by

$$\|T\|_{\mathcal{L}_2} \equiv \left(\sum_k \|Te_k\|^2 \right)^{1/2}$$

where $\{e_k\}$ is some orthonormal basis for H . Also $\mathcal{L}_2(H, G) \subseteq \mathcal{L}(H, G)$ and

$$\|T\| \leq \|T\|_{\mathcal{L}_2}. \quad (15.7.46)$$

All Hilbert Schmidt operators are compact. Also for $X \in H$ and $Y \in G$, $X \otimes Y \in \mathcal{L}_2(H, G)$ and

$$\|X \otimes Y\|_{\mathcal{L}_2} = \|X\|_H \|Y\|_G \quad (15.7.47)$$

Proof: First I want to show $\mathcal{L}_2(H, G) \subseteq \mathcal{L}(H, G)$ and $\|T\| \leq \|T\|_{\mathcal{L}_2}$. Pick an orthonormal basis for H , $\{e_k\}$ and an orthonormal basis for G , $\{f_k\}$. Then letting

$$x = \sum_{k=1}^n x_k e_k,$$

$$Tx = T \left(\sum_{k=1}^n x_k e_k \right) = \sum_{k=1}^n x_k T(e_k)$$

where $x_k \equiv (x, e_k)$. Therefore using Minkowski's inequality,

$$\begin{aligned}
 \|Tx\| &= \left(\sum_{k=1}^{\infty} |(Tx, f_k)|^2 \right)^{1/2} \\
 &= \left(\sum_{k=1}^{\infty} \left| \left(\sum_{j=1}^n x_j T e_j, f_k \right) \right|^2 \right)^{1/2} \\
 &= \left(\sum_{k=1}^{\infty} \left| \sum_{j=1}^n (x_j T e_j, e_k) \right|^2 \right)^{1/2} \\
 &\leq \sum_{j=1}^n \left(\sum_{k=1}^{\infty} |(x_j T e_j, e_k)|^2 \right)^{1/2} \\
 &\leq \sum_{j=1}^n |x_j| \left(\sum_{k=1}^{\infty} |(T e_j, e_k)|^2 \right)^{1/2} \\
 &= \sum_{j=1}^n |x_j| \|T e_j\| \leq \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \|T\|_{\mathcal{L}_2} \\
 &= \|x\| \|T\|_{\mathcal{L}_2}
 \end{aligned}$$

Therefore, since finite sums of the form $\sum_{k=1}^n x_k e_k$ are dense in H , it follows $T \in \mathcal{L}(H, G)$ and $\|T\| \leq \|T\|_{\mathcal{L}_2}$.

Next consider the norm. I need to verify the norm does not depend on the choice of orthonormal basis. Let $\{f_k\}$ be an orthonormal basis for G . Then for $\{e_k\}$ an orthonormal basis for H ,

$$\begin{aligned}
 \sum_k \|T e_k\|^2 &= \sum_k \sum_j |(T e_k, f_j)|^2 = \sum_k \sum_j |(e_k, T^* f_j)|^2 \\
 &= \sum_j \sum_k |(e_k, T^* f_j)|^2 = \sum_j \|T^* f_j\|^2.
 \end{aligned}$$

The above computation makes sense because it was just shown that T is continuous. The same result would be obtained for any other orthonormal basis $\{e'_j\}$ and this shows the norm is at least well defined. It is clear this does indeed satisfy the axioms of a norm and this proves the above claims.

It only remains to verify $\mathcal{L}_2(H, G)$ is a separable Hilbert space. It is clear it is an inner product space because you only have to pick an orthonormal basis, $\{e_k\}$ and define the inner product as

$$(S, T) \equiv \sum_k (S e_k, T e_k).$$

The only remaining issue is the completeness. Suppose then that $\{T_n\}$ is a Cauchy sequence in $\mathcal{L}_2(H, G)$. Then from 15.7.46 $\{T_n\}$ is a Cauchy sequence in $\mathcal{L}(H, G)$ and so there exists a unique T such that $\lim_{n \rightarrow \infty} \|T_n - T\| = 0$. Then it only remains to verify $T \in \mathcal{L}_2(H, G)$. But by Fatou's lemma,

$$\begin{aligned} \sum_k \|Te_k\|^2 &\leq \liminf_{n \rightarrow \infty} \sum_k \|T_n e_k\|^2 \\ &= \liminf_{n \rightarrow \infty} \|T_n\|_{\mathcal{L}_2}^2 < \infty. \end{aligned}$$

All that remains is to verify $\mathcal{L}_2(H, G)$ is separable and these Hilbert Schmidt operators are compact. I will show an orthonormal basis for $\mathcal{L}_2(H, G)$ is $\{f_j \otimes e_k\}$ where $\{f_k\}$ is an orthonormal basis for G and $\{e_k\}$ is an orthonormal basis for H . Here, for $f \in G$ and $e \in H$,

$$f \otimes e(x) \equiv (x, e)f.$$

I need to show $f_j \otimes e_k \in \mathcal{L}_2(H, G)$ and that it is an orthonormal basis for $\mathcal{L}_2(H, G)$ as claimed.

$$\sum_k \|f_j \otimes e_i(e_k)\|^2 = \sum_k \|f_j \delta_{ik}\|^2 = \|f_j\|^2 = 1 < \infty$$

so each of these operators is in $\mathcal{L}_2(H, G)$. Next I show they are orthonormal.

$$\begin{aligned} (f_j \otimes e_k, f_s \otimes e_r) &= \sum_p (f_j \otimes e_k(e_p), f_s \otimes e_r(e_p)) \\ &= \sum_p \delta_{rp} \delta_{kp} (f_j, f_s) = \sum_p \delta_{rp} \delta_{kp} \delta_{js} \end{aligned}$$

If $j = s$ and $k = r$ this reduces to 1. Otherwise, this gives 0. Thus these operators are orthonormal. Now let $T \in \mathcal{L}_2(H, G)$. Consider

$$T_n \equiv \sum_{i=1}^n \sum_{j=1}^n (Te_i, f_j) f_j \otimes e_i$$

Then

$$\begin{aligned} T_n e_k &= \sum_{i=1}^n \sum_{j=1}^n (Te_i, f_j) (e_k, e_i) f_j \\ &= \sum_{j=1}^n (Te_k, f_j) f_j \end{aligned}$$

It follows

$$\|T_n e_k\| \leq \|Te_k\|$$

and

$$\lim_{n \rightarrow \infty} T_n e_k = T e_k.$$

Therefore, from the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \|T - T_n\|_{\mathcal{L}_2}^2 \equiv \lim_{n \rightarrow \infty} \sum_k \|(T - T_n) e_k\|^2 = 0.$$

Therefore, the linear combinations of the $f_j \otimes e_i$ are dense in $\mathcal{L}_2(H, G)$ and this proves completeness of the orthonormal basis.

This also shows $\mathcal{L}_2(H, G)$ is separable. From 15.7.46 it also shows that every $T \in \mathcal{L}_2(H, G)$ is the limit in the operator norm of a sequence of compact operators. This follows because each of the $f_j \otimes e_i$ is easily seen to be a compact operator because if $x_m \rightarrow x$ weakly, then

$$f_j \otimes e_i(x_m) = (x_m, e_i) f_j \rightarrow (x, e_i) f_j = f_j \otimes e_i(x)$$

and since if $\{x_m\}$ is any bounded sequence, there exists a subsequence, $\{x_{n_k}\}$ which converges weakly and by the above, $f_j \otimes e_i(x_{n_k}) \rightarrow f_j \otimes e_i(x)$ showing bounded sets are mapped to precompact sets. Therefore, each $T \in \mathcal{L}_2(H, G)$ must also be a compact operator. Here is why.

Let B be a bounded set in which $\|x\| < M$ for all $x \in B$ and consider TB . I need to show TB is totally bounded. Let $\varepsilon > 0$ be given. Then let $\|T_m - T\| < \frac{\varepsilon}{3M}$ where T_m is a compact operator like those described above and let $\{T_m x_j\}_{j=1}^N$ be an $\varepsilon/3$ net for $T_m(B)$. Then

$$\|Tx_j - T_m x_j\| < \frac{\varepsilon}{3}$$

and so letting $x \in B$, pick x_j such that $\|T_m x - T_m x_j\| < \varepsilon/3$. Then

$$\begin{aligned} \|Tx - Tx_j\| &\leq \|Tx - T_m x\| + \|T_m x - T_m x_j\| + \|T_m x_j - Tx_j\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

showing $\{Tx_j\}_{j=1}^N$ is an ε net for TB .

Finally, consider 15.7.47. Let $\{e_k\}$ be an orthonormal basis for H and consider the following computation which establishes this equation.

$$\begin{aligned} \|Y \otimes X\|_{\mathcal{L}_2}^2 &\equiv \sum_{k=1}^{\infty} \|Y \otimes X(e_k)\|^2 \\ &= \sum_{k=1}^{\infty} \|(e_k, X)Y\|^2 \\ &= \|Y\|_G^2 \sum_{k=1}^{\infty} |(e_k, X)|^2 \\ &= \|Y\|_G^2 \|X\|_H^2 < \infty. \end{aligned} \tag{15.7.48}$$

This proves the theorem.

15.8 Compact Operators In Banach Space

In general for $A \in \mathcal{L}(X, Y)$ the following definition holds.

Definition 15.8.1 *Let $A \in \mathcal{L}(X, Y)$. Then A is compact if whenever $B \subseteq X$ is a bounded set, AB is precompact. Equivalently, if $\{x_n\}$ is a bounded sequence in X , then $\{Ax_n\}$ has a subsequence which converges in Y .*

An important result is the following theorem about the adjoint of a compact operator.

Theorem 15.8.2 *Let $A \in \mathcal{L}(X, Y)$ be compact. Then the adjoint operator, $A^* \in \mathcal{L}(Y', X')$ is also compact.*

Proof: Let $\{y_n^*\}$ be a bounded sequence in Y' . Let B be the closure of the unit ball in X . Then AB is precompact. Then it is clear that the functions $\{y_n^*\}$ are equicontinuous and uniformly bounded on the compact set, $\overline{A(B)}$. By the Ascoli Arzela theorem, there is a subsequence $\{y_{n_k}^*\}$ which converges uniformly to a continuous function, f on $\overline{A(B)}$. Now define g on AX by

$$g(Ax) = \|x\| f\left(A\left(\frac{x}{\|x\|}\right)\right), g(A0) = 0.$$

Thus for $x_1, x_2 \neq 0$, and a, b scalars,

$$\begin{aligned} g(aAx_1 + bAx_2) &\equiv \|ax_1 + bx_2\| f\left(\frac{A(ax_1 + bx_2)}{\|ax_1 + bx_2\|}\right) \\ &\equiv \lim_{k \rightarrow \infty} \|ax_1 + bx_2\| y_{n_k}^*\left(\frac{A(ax_1 + bx_2)}{\|ax_1 + bx_2\|}\right) \\ &= \lim_{k \rightarrow \infty} ay_{n_k}^*(Ax_1) + by_{n_k}^*(Ax_2) \\ &= a \lim_{k \rightarrow \infty} \|x_1\| y_{n_k}^*\left(\frac{Ax_1}{\|x_1\|}\right) + b \lim_{k \rightarrow \infty} \|x_2\| y_{n_k}^*\left(\frac{Ax_2}{\|x_2\|}\right) \\ &= a \|x_1\| f\left(\frac{Ax_1}{\|x_1\|}\right) + b \|x_2\| f\left(\frac{Ax_2}{\|x_2\|}\right) \\ &\equiv ag(Ax_1) + bg(Ax_2) \end{aligned}$$

showing that g is linear on AX . Also

$$|g(Ax)| = \lim_{k \rightarrow \infty} \left| \|x\| y_{n_k}^*\left(A\left(\frac{x}{\|x\|}\right)\right) \right| \leq C \|x\| \left\| A\left(\frac{x}{\|x\|}\right) \right\| = C \|Ax\|$$

and so by the Hahn Banach theorem, there exists y^* extending g to all of Y having the same operator norm.

$$y^*(Ax) = \lim_{k \rightarrow \infty} \|x\| y_{n_k}^*\left(A\left(\frac{x}{\|x\|}\right)\right) = \lim_{k \rightarrow \infty} y_{n_k}^*(Ax)$$

Thus $A^*y_{n_k}^*(x) \rightarrow A^*y^*(x)$ for every x . In addition to this, for $x \in B$,

$$\begin{aligned} \|A^*y^*(x) - A^*y_{n_k}^*(x)\| &= \|y^*(Ax) - y_{n_k}^*(Ax)\| \\ &= \|g(Ax) - g_{n_k}^*(Ax)\| \\ &= \left\| \|x\| f\left(A\left(\frac{x}{\|x\|}\right)\right) - \|x\| y_{n_k}^*\left(\frac{Ax}{\|x\|}\right) \right\| \\ &\leq \left\| f\left(A\left(\frac{x}{\|x\|}\right)\right) - y_{n_k}^*\left(\frac{Ax}{\|x\|}\right) \right\| \end{aligned}$$

and this is uniformly small for large k due to the uniform convergence of $y_{n_k}^*$ to f on $\overline{A(B)}$. Therefore, $\|A^*y^* - A^*y_{n_k}^*\| \rightarrow 0$.

15.9 The Fredholm Alternative

Recall that if A is an $n \times n$ matrix and if the only solution to the system, $A\mathbf{x} = 0$ is $\mathbf{x} = 0$ then for any $\mathbf{y} \in \mathbb{R}^n$ it follows that there exists a unique solution to the system $A\mathbf{x} = \mathbf{y}$. This holds because the first condition implies A is one to one and therefore, A^{-1} exists. Of course things are much harder in a general Banach space. Here is a simple example for a Hilbert space.

Example 15.9.1 Let $L^2(\mathbb{N}; \mu) = H$ where μ is counting measure. Thus an element of H is a sequence, $\mathbf{a} = \{a_i\}_{i=1}^\infty$ having the property that

$$\|\mathbf{a}\|_H \equiv \left(\sum_{k=1}^\infty |a_k|^2 \right)^{1/2} < \infty.$$

Define $A: H \rightarrow H$ by

$$A\mathbf{a} \equiv \mathbf{b} \equiv \{0, a_1, a_2, \dots\}.$$

Thus A slides the sequence to the right and puts a zero in the first slot. Clearly A is one to one and linear but it cannot be onto because it fails to yield $\mathbf{e}_1 \equiv \{1, 0, 0, \dots\}$.

Notwithstanding the above example, there are theorems which are like the linear algebra theorem mentioned above which hold in an arbitrary Banach spaces in the case where the operator is compact. To begin with here is an interesting lemma.

Lemma 15.9.2 Suppose $A \in \mathcal{L}(X, X)$ is compact for X a Banach space. Then $(I - A)(X)$ is a closed subspace of X .

Proof: Suppose $(I - A)x_n \rightarrow y$. Let

$$\alpha_n \equiv \text{dist}(x_n, \ker(I - A))$$

and let $z_n \in \ker(I - A)$ be such that

$$\alpha_n \leq \|x_n - z_n\| \leq \left(1 + \frac{1}{n}\right) \alpha_n.$$

Thus $(I - A)(x_n - z_n) \rightarrow y$ because $(I - A)z_n = 0$.

Case 1: $\{x_n - z_n\}$ has a bounded subsequence.

If this is so, the compactness of A implies there exists a subsequence, still denoted by n such that $\{A(x_n - z_n)\}_{n=1}^\infty$ is a Cauchy sequence. Since $(I - A)(x_n - z_n) \rightarrow y$, this implies $\{(x_n - z_n)\}$ is also a Cauchy sequence converging to a point, $x \in X$. Then, taking the limit as $n \rightarrow \infty$, $(I - A)x = y$ and so $y \in (I - A)(X)$.

Case 2: $\lim_{n \rightarrow \infty} \|x_n - z_n\| = \infty$. I will show this case cannot occur.

In this case, let $w_n \equiv \frac{x_n - z_n}{\|x_n - z_n\|}$. Thus $(I - A)w_n \rightarrow 0$ and w_n is bounded. Therefore, there exists a subsequence, still denoted by n such that $\{Aw_n\}$ is a Cauchy sequence. Now it follows

$$Aw_n - Aw_m + e_n - e_m = w_n - w_m$$

where $e_k \rightarrow 0$ as $k \rightarrow \infty$. This implies $\{w_n\}$ is a Cauchy sequence which must converge to some $w_\infty \in X$. Therefore, $(I - A)w_\infty = 0$ and so $w_\infty \in \ker(I - A)$. However, this is impossible because of the following argument. If $z \in \ker(I - A)$,

$$\begin{aligned} \|w_n - z\| &= \frac{1}{\|x_n - z_n\|} \|x_n - z_n - \|x_n - z_n\| z\| \\ &\geq \frac{1}{\|x_n - z_n\|} \alpha_n \geq \frac{\alpha_n}{(1 + \frac{1}{n}) \alpha_n} = \frac{n}{n+1}. \end{aligned}$$

Taking the limit, $\|w_\infty - z\| \geq 1$. Since $z \in \ker(I - A)$ is arbitrary, this shows $\text{dist}(w_\infty, \ker(I - A)) \geq 1$.

Since Case 2 does not occur, this proves the lemma.

Theorem 15.9.3 *Let $A \in \mathcal{L}(X, X)$ be a compact operator and let $f \in X$. Then there exists a solution, x , to*

$$x - Ax = f \tag{15.9.49}$$

if and only if

$$x^*(f) = 0 \tag{15.9.50}$$

for all $x^* \in \ker(I - A^*)$.

Proof: Suppose x is a solution to 15.9.49 and let $x^* \in \ker(I - A^*)$. Then

$$x^*(f) = x^*((I - A)(x)) = ((I - A^*)x^*)(x) = 0.$$

Next suppose $x^*(f) = 0$ for all $x^* \in \ker(I - A^*)$. I will show there exists x solving 15.9.49. By Lemma 15.9.2, $(I - A)(X)$ is a closed subspace of X . Is $f \in (I - A)(X)$? If not, then by the Hahn Banach theorem, there exists $x^* \in X'$ such that $x^*(f) \neq 0$ but $x^*((I - A)(x)) = 0$ for all $x \in X$. However last statement says nothing more nor less than $(I - A^*)x^* = 0$. This is a contradiction because for such x^* , it is given that $x^*(f) = 0$. This proves the theorem.

The following corollary is called the Fredholm alternative.

Corollary 15.9.4 *Let $A \in \mathcal{L}(X, X)$ be a compact operator. Then there exists a solution to the equation*

$$x - Ax = f \quad (15.9.51)$$

for all $f \in X$ if and only if $(I - A^*)$ is one to one on X' .

Proof: Suppose $(I - A^*)$ is one to one first. Then if $x^* - A^*x^* = 0$ it follows $x^* = 0$ and so for any $f \in X$, $x^*(f) = 0$ for all $x^* \in \ker(I - A^*)$. By 15.9.3 there exists a solution to $(I - A)x = f$.

Now suppose there exists a solution, x , to $(I - A)x = f$ for every $f \in X$. If $(I - A^*)x^* = 0$, then for every $x \in X$,

$$(I - A^*)x^*(x) = x^*((I - A)(x)) = 0$$

Since $(I - A)$ is onto, this shows $x^* = 0$ and so $(I - A^*)$ is one to one as claimed. This proves the corollary.

The following is just an easier version of the above.

Corollary 15.9.5 *In the case where X is a Hilbert space, the conclusions of Corollary 15.9.4, Theorem 15.9.3, and Lemma 15.9.2 remain true if H' is replaced by H and the adjoint is understood in the usual manner for Hilbert space. That is*

$$(Ax, y)_H = (x, A^*y)_H$$

15.10 Square Roots

In this section, H will be a Hilbert space, real or complex, and T will denote an operator which satisfies the following definition. A useful theorem about the existence of square roots of certain operators is presented. This proof is very elementary. I found it in [42].

Definition 15.10.1 *Let $T \in \mathcal{L}(H, H)$ satisfy $T = T^*$ (Hermitian) and for all $x \in H$,*

$$(Tx, x) \geq 0 \quad (15.10.52)$$

Such an operator is referred to as positive and self adjoint. It is probably better to refer to such an operator as “nonnegative” since the possibility that $Tx = 0$ for some $x \neq 0$ is not being excluded. Instead of “self adjoint” you can also use the term, Hermitian. To save on notation, write

$$T \geq 0$$

to mean T is positive, satisfying 15.10.52.

With the above definition here is a fundamental result about positive self adjoint operators.

Proposition 15.10.2 *Let S, T be positive and self adjoint such that $ST = TS$. Then ST is also positive and self adjoint.*

Proof: First suppose $S \leq I$. Let $S_0 \equiv S$ and define

$$S_{n+1} \equiv S_n - S_n^2.$$

Claim: $I \geq S_n \geq 0$.

Proof of the claim: This is true if $n = 0$. Assume true for n . Then from the definition,

$$S_{n+1} = S_n^2(I - S_n) + (I - S_n)^2 S_n$$

and it is obvious from the definition that the sum of positive operators is positive. Therefore, it suffices to show the two terms in the above are both positive. It is clear from the definition that each S_n is Hermitian because they are just polynomials in S . Also each must commute with T for the same reason. Therefore,

$$(S_n^2(I - S_n)x, x) = ((I - S_n)S_nx, S_nx) \geq 0$$

and also

$$((I - S_n)^2 S_nx, x) = (S_n(I - S_n)x, (I - S_n)x) \geq 0$$

This proves the claim.

Now $S_1 = S - S^2$ and so

$$S = S_1 + S^2 = S^2 + S_1^2 + S_2 = S^2 + S_1^2 + S_2^2 + S_3$$

etc. You replace the S_n at the end with $S_n^2 + S_{n+1}$ because this is the definition of S_{n+1} . Thus

$$S - S_{n+1} = \sum_{k=0}^n S_k^2$$

which shows $S - S_{n+1} \geq 0$ and since each S_{n+1} was just shown to be positive, it follows also that $S \geq S - S_{n+1}$. Hence

$$(Sx, x) \geq ((S - S_{n+1})x, x) = \sum_{k=0}^n (S_k^2x, x) = \sum_{k=0}^n |S_kx|^2$$

which proves $\lim_{k \rightarrow \infty} S_kx = 0$. Now

$$\begin{aligned} (STx, x) &= \left((S_{n+1}Tx, x) + \left(\sum_{k=0}^n S_k^2Tx, x \right) \right) \\ &= \left((S_{n+1}Tx, x) + \sum_{k=0}^n (S_k^2Tx, x) \right) \\ &= \left((Tx, S_{n+1}x) + \sum_{k=0}^n (S_kTx, S_kx) \right) \\ &= \left((Tx, S_{n+1}x) + \sum_{k=0}^n (TS_kx, S_kx) \right) \end{aligned}$$

Since $\lim_{n \rightarrow \infty} S_{n+1}x = 0$ The $\liminf_{n \rightarrow \infty}$ of the above expression is nonnegative. Thus $ST \geq 0$. All this was based on the assumption that $S \leq I$. The next task is to remove this assumption. Let $ST = TS$ where T and S are positive self adjoint operators. Then consider $S/\|S\|$. This is still a positive self adjoint operator and it commutes with T just like S does. Therefore, from the first part,

$$0 \leq \left(\frac{S}{\|S\|}Tx, x \right) = \frac{1}{\|S\|} (STx, x)$$

and this proves the proposition.

The proposition is like the familiar statement about real numbers which says that when you multiply two nonnegative real numbers the result is a nonnegative real number. The next lemma is a generalization of the familiar fact that if you have an increasing sequence of real numbers which is bounded above, then the sequence converges.

Lemma 15.10.3 *Let $\{T_n\}$ be a sequence of self adjoint operators on a Hilbert space, H and let $T_n \leq T_{n+1}$ for all n . Also suppose there exists K , a self adjoint operator such that for all n , $T_n \leq K$. Suppose also that each operator commutes with all the others and that K commutes with all the T_n . Then there exists a self adjoint continuous operator, T such that for all $x \in H$,*

$$T_n x \rightarrow Tx,$$

$T \leq K$, and T commutes with all the T_n and with K .

Proof: Consider $K - T_n \equiv S_n$. Then the $\{S_n\}$ are decreasing, that is,

$$\{(S_n x, x)\}$$

is a decreasing sequence and from the hypotheses, $S_n \geq 0$ so the above sequence is bounded below by 0. Therefore, $\lim_{n \rightarrow \infty} (S_n x, x)$ exists. By Proposition 15.10.2, if $n > m$,

$$S_m^2 - S_n S_m = S_m (S_m - S_n) \geq 0$$

and similarly,

$$S_n S_m - S_n^2 = S_n (S_m - S_n) \geq 0.$$

Therefore, since S_n is self adjoint,

$$\begin{aligned} |T_n x - T_m x|^2 &= |S_n x - S_m x|^2 = ((S_n - S_m)^2 x, x) \\ &= ((S_n^2 - 2S_n S_m + S_m^2) x, x) = ((S_m^2 - S_m S_n) x, x) + ((S_n^2 - S_n S_m) x, x) \\ &\leq ((S_m^2 - S_m S_n) x, x) + ((S_m^2 - S_n^2) x, x) \\ &= ((S_m - S_n)(S_m + S_n) x, x) \leq 2((S_m - S_n) K x, x) \\ &\leq 2((S_m - S_n) K x, K x)^{1/2} ((S_m - S_n) x, x)^{1/2} \end{aligned}$$

The last step follows from an application of the Cauchy Schwarz inequality along with the fact $S_m - S_n \geq 0$. The last expression converges to 0 because $\lim_{n \rightarrow \infty} (S_n x, x)$ exists for each x . It follows $\{T_n x\}$ is a Cauchy sequence. Let Tx be the thing to which it converges. T is obviously linear and

$$(Tx, x) = \lim_{n \rightarrow \infty} (T_n x, x) \leq (Kx, x).$$

Also

$$(KTx, y) = \lim_{n \rightarrow \infty} (KT_n x, y) = \lim_{n \rightarrow \infty} (T_n Kx, y) = (TKx, y)$$

and so $TK = KT$. Similarly, T commutes with all T_n .

In order to show T is continuous, apply the uniform boundedness principle, Theorem 13.1.8. The convergence of $\{T_n x\}$ implies there exists a uniform bound on the norms, $\|T_n\|$ and so

$$|(T_n x, y)| \leq C |x| |y|.$$

Now take the limit as $n \rightarrow \infty$ to conclude

$$|(Tx, y)| \leq C |x| |y|$$

which shows $\|T\| \leq C$. This proves the lemma.

With this preparation, here is the theorem about square roots.

Theorem 15.10.4 *Let $T \in \mathcal{L}(H, H)$ be a positive self adjoint linear operator. Then there exists a unique square root, A with the following properties. $A^2 = T$, A is positive and self adjoint, A commutes with every operator which commutes with T .*

Proof: First suppose $T \leq I$. Then define

$$A_0 \equiv 0, \quad A_{n+1} = A_n + \frac{1}{2} (T - A_n^2).$$

From this it follows that every A_n is a polynomial in T . Therefore, A_n commutes with T and with every operator which commutes with T .

Claim 1: $A_n \leq I$.

Proof of Claim 1: This is true if $n = 0$. Suppose it is true for n . Then by the assumption that $T \leq I$,

$$\begin{aligned} I - A_{n+1} &= I - A_n + \frac{1}{2} (A_n^2 - T) \\ &\geq I - A_n + \frac{1}{2} (A_n^2 - I) \\ &= I - A_n - \frac{1}{2} (I - A_n) (I + A_n) \\ &= (I - A_n) \left(I - \frac{1}{2} (I + A_n) \right) \\ &= (I - A_n) (I - A_n) \frac{1}{2} \geq 0. \end{aligned}$$

Claim 2: $A_n \leq A_{n+1}$

Proof of Claim 2: From the definition of A_n , this is true if $n = 0$ because

$$A_1 = T \geq 0 = A_0.$$

Suppose true for n . Then from Claim 1,

$$\begin{aligned} A_{n+2} - A_{n+1} &= A_{n+1} + \frac{1}{2}(T - A_{n+1}^2) - \left[A_n + \frac{1}{2}(T - A_n^2) \right] \\ &= A_{n+1} - A_n + \frac{1}{2}(A_n^2 - A_{n+1}^2) \\ &= (A_{n+1} - A_n) \left(I - \frac{1}{2}(A_n + A_{n+1}) \right) \\ &\geq (A_{n+1} - A_n) \left(I - \frac{1}{2}(2I) \right) = 0. \end{aligned}$$

Claim 3: $A_n \geq 0$

Proof of Claim 3: This is true if $n = 0$. Suppose it is true for n .

$$\begin{aligned} (A_{n+1}x, x) &= (A_nx, x) + \frac{1}{2}(Tx, x) - \frac{1}{2}(A_n^2x, x) \\ &\geq (A_nx, x) + \frac{1}{2}(Tx, x) - \frac{1}{2}(A_nx, x) \geq 0 \end{aligned}$$

because $A_n - A_n^2 = A_n(I - A_n) \geq 0$ by Proposition 15.10.2.

Now $\{A_n\}$ is a sequence of positive self adjoint operators which are bounded above by I such that each of these operators commutes with every operator which commutes with T . By Lemma 15.10.3, there exists a bounded linear operator, A such that for all x ,

$$A_nx \rightarrow Ax$$

Then A commutes with every operator which commutes with T because each A_n has this property. Also A is a positive operator because each A_n is. From passing to the limit in the definition of A_n ,

$$Ax = Ax + \frac{1}{2}(Tx - A^2x)$$

and so $Tx = A^2x$. This proves the theorem in the case that $T \leq I$.

In the general case, consider $T/\|T\|$. Then

$$\left(\frac{T}{\|T\|}x, x \right) = \frac{1}{\|T\|}(Tx, x) \leq |x|^2 = (Ix, x)$$

and so $T/\|T\| \leq I$. Therefore, it has a square root, B . Let $A = \sqrt{\|T\|}B$. Then A has all the right properties and $A^2 = \|T\|B^2 = \|T\|(T/\|T\|) = T$. This proves the existence part of the theorem.

Next suppose both A and B are square roots of T having all the properties stated in the theorem. Then $AB = BA$ because both A and B commute with every operator which commutes with T .

$$(A(A-B)x, (A-B)x), (B(A-B)x, (A-B)x) \geq 0 \quad (15.10.53)$$

Therefore, on adding these,

$$\begin{aligned} & ((A^2 - AB + BA - B^2)x, (A-B)x) \\ &= ((A^2 - B^2)x, (A-B)x) \\ &= ((T - T)x, (A-B)x) = 0. \end{aligned}$$

It follows both expressions in 15.10.53 equal 0 since both are nonnegative and when they are added the result is 0. Now applying the existence part of the theorem to A , there exists a positive square root of A which is self adjoint. Thus

$$(\sqrt{A}(A-B)x, \sqrt{A}(A-B)x) = 0$$

so $\sqrt{A}(A-B)x = 0$ which implies $A(A-B)x = 0$. Similarly, $B(A-B)x = 0$. Subtracting these and taking the inner product with x ,

$$0 = ((A(A-B) - B(A-B))x, x) = ((A-B)^2x, x) = |(A-B)x|^2$$

and so $Ax = Bx$ which shows $A = B$ since x was arbitrary. This proves the theorem.

15.11 Quotient Spaces

A useful idea is that of a quotient space. It is a way to create another Banach space from a given Banach space and a closed subspace. It generalizes similar concepts which are routine in linear algebra.

Definition 15.11.1 *Let X be a Banach space and let V be a closed subspace of X . Then X/V denotes the set of equivalence classes determined by the equivalence relation which says $x \sim y$ means $x - y \in V$. An individual equivalence class will be denoted by any of the following symbols. $x + V$, $[x]$, or $[x]_V$. Vector space operations are defined as follows:*

$$(x + V) + y + V \equiv x + y + V$$

or in other symbols,

$$[x] + [y] \equiv [x + y]$$

and for $\alpha \in \mathbb{F}$,

$$\alpha[x] \equiv [\alpha x].$$

Also a norm is defined by

$$|[x]| \equiv \inf \{ \|x + v\| : v \in V \}.$$

It is left as an exercise to verify the above algebraic operations are well defined. With the above definition, here is the major theorem about quotient spaces.

Theorem 15.11.2 *Let X be a Banach space and let V be a closed subspace of X . Then with the above definitions of vector space operations, X/V is a Banach space. In the case where $V = \ker(A)$ for $A \in \mathcal{L}(X, Y)$ for Y another Banach space, define $\hat{A} : X/V \rightarrow A(X) \subseteq Y$ by $\hat{A}([x]) \equiv Ax$. Then $\hat{A}$ is continuous and $\|\hat{A}\| = \|A\|$. In fact, $\|\hat{A}\| \leq \|A\|$.*

Proof: First of all, consider the claim that the given norm really is a norm. First note that $\|x + V\| \geq 0$ and $\|x + V\| = 0$ only if $x \in V$ because V is closed. Therefore, $x + V = 0 + V$. Next,

$$\begin{aligned} \|\alpha[x]\| &\equiv \|[\alpha x]\| \equiv \inf \{\|\alpha x + v\| : v \in V\} \\ &= \inf \{\|\alpha x + \alpha v\| : v \in V\} \\ &= |\alpha| \inf \{\|x + v\| : v \in V\} = |\alpha| \|x + V\|. \end{aligned}$$

Consider the triangle inequality.

$$\begin{aligned} \|[x + y]\| &= \inf \{\|x + y + v\| : v \in V\} \\ &\leq \|x + v_1\| + \|y + v_2\| \end{aligned}$$

for any choice of v_1 and v_2 . Therefore, taking the infimum of both sides over v_2 yields

$$\|[x + y]\| \leq \|x + v_1\| + \|[y]\|$$

and then taking the infimum over all v_1 yields

$$\|[x + y]\| \leq \|[x]\| + \|[y]\|.$$

Next consider the claim that X/V is a Banach space. Letting $\{[x_n]\}$ be a Cauchy sequence in X/V , I will show a subsequence of this converges to a point in X/V . This is done by defining a suitable sequence in X and then using completeness of X . By choosing a subsequence, it can be assumed that $\|[x_n] - [x_{n+1}]\| < 2^{-n}$. Let $z_1 \equiv x_1$. Then choose $v_2 \in V$ such that $\|x_2 + v_2 - z_1\| < 2^{-1}$. Let $z_2 = x_2 + v_2$. Suppose $\{z_1, \dots, z_n\}$ have been chosen, each having the property that $[z_k] = [x_k]$ and such that $\|z_k - z_{k+1}\| < 2^{-k}$. Then let v_{n+1} be chosen such that $\|x_{n+1} + v_{n+1} - z_n\| < 2^{-n}$ and let $z_{n+1} \equiv x_{n+1} + v_{n+1}$. Thus $\{z_n\}$ is a Cauchy sequence in X and so it converges to $x \in X$. Then

$$\|[x] - [x_n]\| \leq \|x - (x_n + v_n)\| = \|x - z_n\|$$

and so $\lim_{n \rightarrow \infty} [x_n] = [x]$.

Next consider the claim about $\hat{A}$. This is well defined and linear because if $[x] = [x_1]$, then $x - x_1 \in \ker(A)$ and so $Ax = Ax_1$. Thus $\hat{A}([x]) = \hat{A}([x_1])$. It is linear because

$$\begin{aligned} \hat{A}(\alpha[x] + \beta[y]) &= \hat{A}([\alpha x + \beta y]) = A(\alpha x + \beta y) \\ &= \alpha Ax + \beta Ay = \alpha \hat{A}([x]) + \beta \hat{A}([y]) \end{aligned}$$

Next consider the claim that $\widehat{A}$ is continuous. Letting $v \in V$,

$$\left\| \widehat{A}([x]) \right\| \equiv \|Ax\| = \|A(x+v)\| \leq \|A\| \|x+v\|$$

and so, taking the infimum over all $v \in V$,

$$\left\| \widehat{A}([x]) \right\| \leq \|A\| \|x\|$$

and this shows $\left\| \widehat{A} \right\| \leq \|A\|$. This proves the theorem.

Now with this theorem, here is an interesting application.

Theorem 15.11.3 *Let X_1 and X_2 be Banach spaces which are either reflexive or dual spaces for a separable Banach space and let $A_i \in \mathcal{L}(X_i, Y)$ for Y a reflexive Banach space. The following are equivalent.*

For some $k > 0$

$$A_1 \left(\overline{B(0, 1)} \right) \subseteq A_2 \left(\overline{B(0, k)} \right) \quad (15.11.54)$$

$$\|A_1^* y^*\| \leq k \|A_2^* y^*\| \quad (15.11.55)$$

for all $y \in Y^*$. If either of the above hold, then

$$A_1 X_1 \subseteq A_2 X_2. \quad (15.11.56)$$

Proof: Suppose 15.11.54 first. I show this implies 15.11.56. There are two cases. First suppose A_2 is one to one. Then in this case, $A_2^{-1} A_1 \left(\overline{B(0, 1)} \right) \subseteq \overline{B(0, k)}$. Therefore, if $x \in X_1$,

$$A_2^{-1} A_1 (x / \|x\|) = y \in \overline{B(0, k)}$$

and so

$$A_1 (x) = \|x\| A_2 (y) = A_2 (\|x\| y) \in A_2 (X_2).$$

Next suppose A_2 is not one to one. In this case, letting $\widehat{A}_2$ be the continuous linear map given by

$$\widehat{A}_2([x]) \equiv A_2 x,$$

it follows $\widehat{A}_2$ is 1-1 on $X_2 / \ker(A_2)$. Now note that if $\|x\| \leq k$, then it is also the case that $\|[x]\| \leq k$ and so

$$A_2 \left(\overline{B(0, k)} \right) \subseteq \widehat{A}_2 \left(\overline{B_2(0, k)} \right)$$

where in the second set, $\overline{B_2(0, k)}$ is the unit ball in $X / \ker(A_2)$. It follows from 15.11.54

$$A_1 \left(\overline{B(0, 1)} \right) \subseteq \widehat{A}_2 \left(\overline{B_2(0, k)} \right)$$

and so $\widehat{A}_2^{-1} A_1 \left(\overline{B(0, 1)} \right) \subseteq \overline{B_2(0, k)}$ which implies

$$A_1 (x / \|x\|) = \widehat{A}_2 [y] \in \widehat{A}_2 \left(\overline{B_2(0, k)} \right)$$

Therefore, letting $[y_1] = [y]$ be such that $\|y_1\| < 2k$, it follows

$$A_1(x/\|x\|) = \widehat{A}_2[y_1] = A_2(y_1)$$

and so

$$A_1(x) = A_2(\|x\|y_1) \in A_2(X_2).$$

Next I show the equivalence of 15.11.55 and 15.11.54. First I want to show $A_i(\overline{B(0, r)})$ is closed. Suppose then that for $A = A_1$ or A_2 , $A(x_n) \rightarrow y$ where $x_n \in \overline{B(0, r)}$. In the case the X_i are reflexive, it follows from the Eberlein Smulian theorem there exists a subsequence, still denoted as $\{x_n\}$ which converges weakly to $x \in \overline{B(0, r)}$. Then $Ax_n \rightarrow y$ and $x_n \rightarrow x$ weakly. Thus (x, y) is in the weak closure of the graph of A ,

$$\{(x, Ax) : x \in X_i\}$$

This set is strongly closed and convex and hence it is weakly closed by Theorem 14.3.3 so $y = Ax$ and this shows $A(\overline{B(0, r)})$ is closed. In the other case where X_i is the dual space of a separable Banach space, it follows from Corollary 13.4.6 there exists a subsequence still denoted as $\{x_n\}$ such that $x_n \rightarrow x$ weak $*$ and similarly, (x, y) is in the weak $*$ closure of the graph of A which shows again by Theorem 14.3.3 that (x, y) is in the graph of A , showing again that $A(\overline{B(0, r)})$ is closed.

Suppose 15.11.54. Then letting $y^* \in Y'$,

$$\begin{aligned} \|A_1^* y^*\| &= \sup_{\|x_1\|_{X_1} \leq 1} |y^*(A_1 x_1)| \\ &\leq \sup_{\|x_2\|_{X_2} \leq k} |y^*(A_2 x_2)| = k \|A_2^* y^*\| \end{aligned}$$

which shows 15.11.55.

Now suppose 15.11.55. Then if 15.11.54 does not hold, it follows from the first part which gives $A_i(\overline{B(0, r)})$ a closed set, there exists

$$A_1 x_0 \in A_1(\overline{B(0, 1)}) \setminus A_2(\overline{B(0, k)})$$

Now $A_2(\overline{B(0, k)})$ is closed and convex, hence weakly closed, and so by Theorem 14.2.4 there exists $y_0^* \in Y'$ such that

$$\operatorname{Re} y_0^*(A_2(\overline{B(0, k)})) < c < \operatorname{Re} y_0^*(A_1 x_0)$$

and so

$$\begin{aligned} \|A_1^* y_0^*\| &= \sup_{\|x_1\|_{X_1} \leq 1} |y_0^*(A_1 x_1)| \geq \operatorname{Re} y_0^*(A_1 x_0) \\ &> c > \operatorname{Re} y_0^*(A_2(x_2)) = \operatorname{Re} A_2^* y_0^*(x_2) \end{aligned}$$

whenever $x_2 \in \overline{B(0, k)}$ and so, taking the supremum of all such x_2 ,

$$\|A_1^* y_0^*\| > c > k \|A_2^* y_0^*\|,$$

contradicting 15.11.55. This proves the theorem.

15.12 General Theory Of Continuous Semigroups

Much more on semigroups is available in Yosida [67]. This is just an introduction to the subject.

Definition 15.12.1 *A strongly continuous semigroup defined on H , a Banach space is a function $S : [0, \infty) \rightarrow \mathcal{L}(H, H)$ which satisfies the following for all $x_0 \in H$.*

$$\begin{aligned} S(t) &\in \mathcal{L}(H, H), S(t+s) = S(t)S(s), \\ t &\rightarrow S(t)x_0 \text{ is continuous, } \lim_{t \rightarrow 0+} S(t)x_0 = x_0 \end{aligned}$$

Sometimes such a semigroup is said to be C_0 . It is said to have the linear operator A as its generator if

$$D(A) \equiv \left\{ x : \lim_{h \rightarrow 0} \frac{S(h)x - x}{h} \text{ exists} \right\}$$

and for $x \in D(A)$, A is defined by

$$\lim_{h \rightarrow 0} \frac{S(h)x - x}{h} \equiv Ax$$

The assertion that $t \rightarrow S(t)x_0$ is continuous and that $S(t) \in \mathcal{L}(H, H)$ is not sufficient to say there is a bound on $\|S(t)\|$ for all t . Also the assertion that for each x_0 ,

$$\lim_{t \rightarrow 0+} S(t)x_0 = x_0$$

is not the same as saying that $S(t) \rightarrow I$ in $\mathcal{L}(H, H)$. It is a much weaker assertion. The next theorem gives information on the growth of $\|S(t)\|$. It turns out it has exponential growth.

Lemma 15.12.2 *Let $M \equiv \sup \{\|S(t)\| : t \in [0, T]\}$. Then $M < \infty$.*

Proof: If this is not true, then there exists $t_n \in [0, T]$ such that $\|S(t_n)\| \geq n$. That is the operators $S(t_n)$ are not uniformly bounded. By the uniform boundedness principle, Theorem 13.1.8, there exists $x \in H$ such that $\|S(t_n)x\|$ is not bounded. However, this is impossible because it is given that $t \rightarrow S(t)x$ is continuous on $[0, T]$ and so $t \rightarrow \|S(t)x\|$ must achieve its maximum on this compact set.

Now here is the main result for growth of $\|S(t)\|$.

Theorem 15.12.3 *For M described in Lemma 15.12.2, there exists α such that*

$$\|S(t)\| \leq Me^{\alpha t}.$$

In fact, α can be chosen such that $M^{1/T} = e^\alpha$.

Proof: Let t be arbitrary. Then $t = mT + r(t)$ where $0 \leq r(t) < T$. Then by the semigroup property

$$\begin{aligned} \|S(t)\| &= \|S(mT + r(t))\| \\ &= \|S(r(t))S(T)^m\| \leq M^{m+1} \end{aligned}$$

Now $mT \leq t \leq mT + r(t) \leq (m+1)T$ and so

$$m \leq \frac{t}{T} \leq m+1$$

Therefore,

$$\|S(t)\| \leq M^{(t/T)+1} = M \left(M^{1/T}\right)^t.$$

Let $M^{1/T} \equiv e^\alpha$ and then

$$\|S(t)\| \leq Me^{\alpha t}$$

This proves the theorem.

Definition 15.12.4 Let $S(t)$ be a continuous semigroup as described above. It is called a contraction semigroup if for all $t \geq 0$

$$\|S(t)\| \leq 1.$$

It is called a bounded semigroup if there exists M such that for all $t \geq 0$,

$$\|S(t)\| \leq M$$

Note that for $S(t)$ an arbitrary continuous semigroup satisfying

$$\|S(t)\| \leq Me^{\alpha t},$$

It follows that the semigroup,

$$T(t) = e^{-\alpha t} S(t)$$

is a bounded semigroup which satisfies

$$\|T(t)\| \leq M.$$

Proposition 15.12.5 Given a continuous semigroup $S(t)$, its generator A exists and is a closed densely defined operator. Furthermore, for

$$\|S(t)\| \leq Me^{\alpha t}$$

and $\lambda > \alpha$, $\lambda I - A$ is onto and $(\lambda I - A)^{-1}$ maps H onto $D(A)$ and is in $\mathcal{L}(H, H)$. Also for these values of λ ,

$$(\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} S(t) x dt.$$

For $\lambda > \alpha$, the following estimate holds.

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{M}{|\lambda - \alpha|}$$

Proof: First note $D(A) \neq \emptyset$. In fact $0 \in D(A)$. It follows from Theorem 15.12.3 that for all λ large enough, one can define a Laplace transform,

$$R(\lambda)x \equiv \int_0^\infty e^{-\lambda t} S(t)x dt \in H.$$

Here the integral is the ordinary improper Riemann integral. I claim each of these is in $D(A)$.

$$\frac{S(h) \int_0^\infty e^{-\lambda t} S(t)x dt - \int_0^\infty e^{-\lambda t} S(t)x dt}{h}$$

Using the semigroup property and changing the variables in the first of the above integrals, this equals

$$\begin{aligned} &= \frac{1}{h} \left(e^{\lambda h} \int_h^\infty e^{-\lambda t} S(t)x dt - \int_0^\infty e^{-\lambda t} S(t)x dt \right) \\ &= \frac{1}{h} \left((e^{\lambda h} - 1) \int_0^\infty e^{-\lambda t} S(t)x dt - e^{\lambda h} \int_0^h e^{-\lambda t} S(t)x dt \right) \end{aligned}$$

The limit as $h \rightarrow 0$ exists and equals

$$\lambda R(\lambda)x - x$$

Thus $R(\lambda)x \in D(A)$ as claimed and

$$AR(\lambda)x = \lambda R(\lambda)x - x$$

Hence

$$x = (\lambda I - A)R(\lambda)x. \quad (15.12.57)$$

Since x is arbitrary, this shows that for λ large enough, $\lambda I - A$ is onto.

Why is $D(A)$ dense? It was shown above that $R(\lambda)x$ and therefore $\lambda R(\lambda)x \in D(A)$. Then for $\lambda > \alpha$ where $\|S(t)\| \leq Me^{\alpha t}$,

$$\begin{aligned} \|\lambda R(\lambda)x - x\| &= \left\| \int_0^\infty \lambda e^{-\lambda t} S(t)x dt - \int_0^\infty \lambda e^{-\lambda t} x dt \right\| \\ &\leq \int_0^\infty \|\lambda e^{-\lambda t} (S(t)x - x)\| dt \\ &= \int_0^h \|\lambda e^{-\lambda t} (S(t)x - x)\| dt + \int_h^\infty \|\lambda e^{-\lambda t} (S(t)x - x)\| dt \\ &\leq \int_0^h \|\lambda e^{-\lambda t} (S(t)x - x)\| dt + \int_h^\infty \lambda e^{-(\lambda-\alpha)t} dt (M+1) \|x\| \end{aligned}$$

Now since $S(t)x - x \rightarrow 0$, it follows that for h sufficiently small

$$\begin{aligned} &\leq \frac{\varepsilon}{2} \int_0^h \lambda e^{-\lambda t} dt + \frac{\lambda}{\lambda - \alpha} e^{-(\lambda-\alpha)h} (M+1) \|x\| \\ &\leq \frac{\varepsilon}{2} + \frac{\lambda}{\lambda - \alpha} e^{-(\lambda-\alpha)h} (M+1) \|x\| < \varepsilon \end{aligned}$$

whenever λ is large enough. Thus $D(A)$ is dense as claimed.

Let $x \in D(A)$. Then for $y^* \in H'$,

$$y^* \left(\int_0^t S(s) Ax ds \right) = \int_0^t y^* \left(S(s) \lim_{h \rightarrow 0+} \frac{S(h)x - x}{h} \right) ds$$

The difference quotient is given to have a limit and so the difference quotients are bounded. Therefore, one can use the dominated convergence theorem to take the limit outside the integral and write the above equals

$$\begin{aligned} & \lim_{h \rightarrow 0+} \int_0^t y^* \left(S(s) \frac{S(h)x - x}{h} \right) ds \\ &= \lim_{h \rightarrow 0+} y^* \left(\frac{1}{h} \left(\int_h^{t+h} S(s) x ds - \int_0^t S(s) x ds \right) \right) \\ &= \lim_{h \rightarrow 0+} y^* \left(\int_t^{t+h} S(s) x ds - \int_0^h S(s) x ds \right) \\ &= y^* (S(t)x - x). \end{aligned}$$

Thus since y^* is arbitrary, for $x \in D(A)$

$$S(t)x = x + \int_0^t S(s) Ax ds$$

Why is A closed? Suppose $x_n \rightarrow x$ and $x_n \in D(A)$ while $Ax_n \rightarrow z$. From what was just shown

$$S(t)x_n = x_n + \int_0^t S(s) Ax_n ds$$

and so, passing to the limit this yields

$$S(t)x = x + \int_0^t S(s) z ds$$

which implies

$$\lim_{t \rightarrow 0+} \frac{S(t)x - x}{h} = \lim_{h \rightarrow 0+} \frac{1}{t} \int_0^t S(s) z ds = z$$

which shows $Ax = z$ and $x \in D(A)$. Thus A is closed.

Because of 15.12.57 it follows $R(\lambda)x = (\lambda I - A)^{-1}x$. Also

$$\begin{aligned} \|R(\lambda)x\| &\leq \left\| \int_0^\infty e^{-\lambda t} S(t)x dt \right\| \\ &\leq \int_0^\infty e^{-\lambda t} M e^{\alpha t} dt \|x\| \leq \frac{M}{|\lambda - \alpha|} \|x\| \end{aligned}$$

so $R(\lambda) = (\lambda I - A)^{-1} \in \mathcal{L}(H, H)$ and this also proves the last estimate. Also from 15.12.57, $R(\lambda)$ maps H onto $D(A)$. This proves the proposition.

The linear mapping $(\lambda I - A)^{-1}$ is called the resolvent.

The above proof contains an argument which implies the following corollary.

Corollary 15.12.6 *Let $S(t)$ be a continuous semigroup and let A be its generator. Then for $0 < a < b$ and $x \in D(A)$*

$$S(b)x - S(a)x = \int_a^b S(t)Ax dt$$

and also for $t > 0$ you can take the derivative from the left,

$$\lim_{h \rightarrow 0+} \frac{S(t)x - S(t-h)x}{h} = S(t)Ax$$

Proof: Letting $y^* \in H'$,

$$y^* \left(\int_a^b S(t)Ax dt \right) = \int_a^b y^* \left(S(t) \lim_{h \rightarrow 0} \frac{S(h)x - x}{h} \right) dt$$

The difference quotients are bounded because they converge to Ax . Therefore, from the dominated convergence theorem,

$$\begin{aligned} y^* \left(\int_a^b S(t)Ax dt \right) &= \lim_{h \rightarrow 0} \int_a^b y^* \left(S(t) \frac{S(h)x - x}{h} \right) dt \\ &= \lim_{h \rightarrow 0} y^* \left(\int_a^b S(t) \frac{S(h)x - x}{h} dt \right) \\ &= \lim_{h \rightarrow 0} y^* \left(\frac{1}{h} \int_{a+h}^{b+h} S(t)x dt - \frac{1}{h} \int_a^b S(t)x dt \right) \\ &= \lim_{h \rightarrow 0} y^* \left(\frac{1}{h} \int_b^{b+h} S(t)x dt - \frac{1}{h} \int_a^{a+h} S(t)x dt \right) \\ &= y^* (S(b)x - S(a)x) \end{aligned}$$

Since y^* is arbitrary, this proves the first part. Now from what was just shown, if $t > 0$ and h is small enough,

$$\frac{S(t)x - S(t-h)x}{h} = \frac{1}{h} \int_{t-h}^t S(s)Ax ds$$

which converges to $S(t)Ax$ as $h \rightarrow 0+$. This proves the corollary.

Given a closed densely defined operator, when is it the generator of a bounded semigroup? This is answered in the following theorem which is called the Hille Yosida theorem.

Theorem 15.12.7 *Suppose A is a densely defined linear operator which has the property that for all $\lambda > 0$,*

$$(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$$

which means that $\lambda I - A : D(A) \rightarrow H$ is one to one and onto with continuous inverse. Suppose also that for all $n \in \mathbb{N}$,

$$\left\| \left((\lambda I - A)^{-1} \right)^n \right\| \leq \frac{M}{\lambda^n}. \quad (15.12.58)$$

Then there exists a continuous semigroup, $S(t)$ which has A as its generator and satisfies $\|S(t)\| \leq M$ and A is closed. In fact letting

$$S_\lambda(t) \equiv \exp \left(-\lambda + \lambda^2 (\lambda I - A)^{-1} \right)$$

it follows $\lim_{\lambda \rightarrow \infty} S_\lambda(t)x = S(t)x$ uniformly on finite intervals. Conversely, if A is the generator of $S(t)$, a bounded continuous semigroup having $\|S(t)\| \leq M$, then $(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$ for all $\lambda > 0$ and 15.12.58 holds.

Proof: Consider the operator

$$\lambda (\lambda I - A)^{-1} A$$

On $D(A)$, this equals

$$-\lambda + \lambda^2 (\lambda I - A)^{-1} \quad (15.12.59)$$

which makes sense on all of H , not just on $D(A)$. Also this last expression equals

$$\lambda A (\lambda I - A)^{-1}$$

on all of H because $\lambda I - A$ is given to be onto. Denote this as A_λ to save notation. Thus on $D(A)$,

$$\lambda A (\lambda I - A)^{-1} = \lambda (\lambda I - A)^{-1} A$$

For $x \in D(A)$,

$$\begin{aligned} & \left\| \lambda (\lambda I - A)^{-1} x - x \right\| \\ &= \left\| (\lambda I - A)^{-1} (\lambda x - (\lambda I - A)x) \right\| \\ &= \left\| (\lambda I - A)^{-1} Ax \right\| \leq \frac{M}{\lambda} \|Ax\| \end{aligned}$$

which converges to 0. Therefore, for $x \in D(A)$,

$$\|A_\lambda x - Ax\| = \left\| \lambda (\lambda I - A)^{-1} Ax - Ax \right\| \leq \frac{M \|Ax\|}{\lambda} \quad (15.12.60)$$

so it also converges to 0. Because of 15.12.59, the operator $\lambda A (\lambda I - A)^{-1}$ is continuous. Now using 15.12.59 define an approximate semigroup

$$S_\lambda(t) \equiv e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k \left(\lambda^2 (\lambda I - A)^{-1} \right)^k}{k!}$$

The sum converges in $\mathcal{L}(H, H)$ because it converges absolutely and $\mathcal{L}(H, H)$ is complete. Here is why it converges absolutely. It follows from the assumption in the lemma.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{t^k \left\| \left(\lambda^2 (\lambda I - A)^{-1} \right)^k \right\|}{k!} &\leq \sum_{k=0}^{\infty} \frac{t^k \left\| \left(\lambda^2 (\lambda I - A)^{-1} \right) \right\|^k}{k!} \\ &\leq \sum_{k=0}^{\infty} \frac{t^k \lambda^k M}{k!} = M e^{t\lambda} \end{aligned}$$

Thus

$$\|S_{\lambda}(t)\| \leq e^{-\lambda t} M e^{t\lambda} = M$$

The series converges uniformly on any finite interval thanks to the Weierstrass M test. Thus $t \rightarrow S_{\lambda}(t)$ is continuous and it is also routine to verify the semigroup identity. Clearly $\lim_{t \rightarrow 0} S_{\lambda}(t)x = x$. It is also the case that $S_{\lambda}(t)$ is generated by $-\lambda + \lambda^2 (\lambda I - A)^{-1} = A_{\lambda}$. This is easy to show from differentiating the power series which has a continuous derivative. Thus

$$\begin{aligned} &(-\lambda) e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k \left(\lambda^2 (\lambda I - A)^{-1} \right)^k x}{k!} + e^{-\lambda t} \sum_{k=0}^{\infty} \frac{t^k \left(\lambda^2 (\lambda I - A)^{-1} \right)^{k+1} x}{k!} \\ &= \left(-\lambda + \lambda^2 (\lambda I - A)^{-1} \right) S_{\lambda}(t)x = S_{\lambda}(t) \left(-\lambda + \lambda^2 (\lambda I - A)^{-1} \right) x \end{aligned}$$

Now let $t \rightarrow 0+$ to obtain $\left(-\lambda + \lambda^2 (\lambda I - A)^{-1} \right) x = A_{\lambda}x$.

Claim: For $\lambda, \mu > 0$, $(\lambda I - A)^{-1}$ and $(\mu I - A)^{-1}$ commute.

Proof of claim: Suppose

$$y = (\mu I - A)^{-1} (\lambda I - A)^{-1} x \quad (15.12.61)$$

$$z = (\lambda I - A)^{-1} (\mu I - A)^{-1} x \quad (15.12.62)$$

I need to show $y = z$. First note $z \in D(A)$ and

$$(\lambda I - A)z = (\mu I - A)^{-1} x \in D(A).$$

Hence

$$(\mu I - A)z = (\mu - \lambda)z + (\lambda I - A)z \in D(A).$$

Similarly

$$(\lambda I - A)y, (\mu I - A)y \in D(A).$$

From 15.12.61

$$(\lambda I - A)(\mu I - A)y = x$$

and using 15.12.62,

$$\begin{aligned}
 x &= (\mu I - A)(\lambda I - A)z \\
 &= ((\mu - \lambda)I + (\lambda I - A))(\lambda I - A)z \\
 &= (\mu - \lambda)(\lambda I - A)z + (\lambda I - A)^2 z \\
 &= (\lambda I - A)(\mu - \lambda)z + (\lambda I - A)(\lambda I - A)z \\
 &= (\lambda I - A)(\mu - \lambda)z + (\lambda I - A)((\lambda - \mu)I + (\mu I - A))z \\
 &= (\lambda I - A)(\mu - \lambda)z + (\lambda I - A)((\lambda - \mu)z + (\mu I - A)z) \\
 &= (\lambda I - A)(\mu - \lambda)z + (\lambda I - A)(\lambda - \mu)z + (\lambda I - A)(\mu I - A)z \\
 &= (\lambda I - A)(\mu I - A)z
 \end{aligned}$$

Thus

$$x = (\lambda I - A)(\mu I - A)z = (\lambda I - A)(\mu I - A)y$$

and so $z = y$. This proves the claim.

It follows from the description of $S_\lambda(t)$ that $S_\lambda(t)$ and $S_\mu(s)$ commute and also A_λ commutes with $S_\mu(t)$ for any t .

I want to show that for each $x \in D(A)$,

$$\lim_{\lambda \rightarrow \infty} S_\lambda(t)x \equiv S(t)x$$

where $S(t)$ is the desired semigroup. Let $x \in D(A)$

$$\|S_\lambda(t)x - S_\mu(t)x\| = \left\| \int_0^t \frac{d}{dr} (S_\lambda(t-r)S_\mu(r))x dr \right\|$$

Since A_λ commutes with $S_\mu(r)$, the following formula follows from 15.12.60.

$$\begin{aligned}
 &= \left\| \int_0^t (S_\lambda(t-r)S_\mu(r)A_\mu x - S_\lambda(t-r)A_\lambda S_\mu(r)x) dr \right\| \\
 &\leq \int_0^t \|S_\lambda(t-r)S_\mu(r)(A_\mu x - A_\lambda x)\| dr \\
 &\leq M^2 t \|A_\mu x - A_\lambda x\| \leq M^2 t (\|A_\mu x - Ax\| + \|Ax - A_\lambda x\|) \\
 &\leq \left(\frac{\|Ax\|}{\mu} + \frac{\|Ax\|}{\lambda} \right) t M^2
 \end{aligned}$$

Hence whenever μ, λ large enough, $\|S_\lambda(t)x - S_\mu(t)x\|$ is small. Thus $S_\lambda(t)x$ converges uniformly on finite intervals to something denoted by $S(t)x$. Therefore, $t \rightarrow S(t)x$ is continuous for each $x \in D(A)$ and also

$$\|S(t)x\| = \lim_{\lambda \rightarrow \infty} \|S_\lambda(t)x\| \leq M\|x\|$$

so that $S(t)$ can be extended to a continuous linear map, still called $S(t)$ defined on all of H which also satisfies $\|S(t)\| \leq M$ since $D(A)$ is dense in H . If x is arbitrary, let $y \in D(A)$ be close to x . Then

$$\begin{aligned} \|S(t)x - S_\lambda(t)x\| &\leq \|S(t)x - S(t)y\| + \|S(t)y - S_\lambda(t)y\| \\ &\quad + \|S_\lambda(t)y - S_\lambda(t)x\| \\ &\leq 2M\|x - y\| + \|S(t)y - S_\lambda(t)y\| \end{aligned}$$

and so $\lim_{\lambda \rightarrow \infty} S_\lambda(t)x = S(t)x$ for all x , uniformly on finite intervals. Thus $t \rightarrow S(t)x$ is continuous for any $x \in H$.

It remains to verify A generates $S(t)$ and for all x , $S(t)x - x \rightarrow 0$. From the above,

$$S_\lambda(t)x = x + \int_0^t S_\lambda(s) A_\lambda x ds \quad (15.12.63)$$

and so

$$\lim_{t \rightarrow 0+} \|S_\lambda(t)x - x\| = 0$$

By the uniform convergence just shown, there exists λ large enough that for all $t \in [0, \delta]$,

$$\|S(t)x - S_\lambda(t)x\| < \varepsilon.$$

Then

$$\begin{aligned} \limsup_{t \rightarrow 0+} \|S(t)x - x\| &\leq \limsup_{t \rightarrow 0+} (\|S(t)x - S_\lambda(t)x\| + \|S_\lambda(t)x - x\|) \\ &\leq \limsup_{t \rightarrow 0+} (\varepsilon + \|S_\lambda(t)x - x\|) \leq \varepsilon \end{aligned}$$

It follows $\lim_{t \rightarrow 0+} S(t)x = x$ because ε is arbitrary.

Next, $\lim_{\lambda \rightarrow \infty} A_\lambda x = Ax$ for all $x \in D(A)$ by 15.12.60. Therefore, passing to the limit in 15.12.63 yields from the uniform convergence

$$S(t)x = x + \int_0^t S(s) A x ds$$

and by continuity of $s \rightarrow S(s)Ax$, it follows

$$\lim_{h \rightarrow 0+} \frac{S(h)x - x}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h S(s) A x ds = Ax$$

Thus letting B denote the generator of $S(t)$, $D(A) \subseteq D(B)$ and $A = B$ on $D(A)$. It only remains to verify $D(A) = D(B)$.

To do this, let $\lambda > 0$ and consider the following where $y \in H$ is arbitrary.

$$(\lambda I - B)^{-1}y = (\lambda I - B)^{-1} \left((\lambda I - A)(\lambda I - A)^{-1}y \right)$$

Now $(\lambda I - A)^{-1} y \in D(A) \subseteq D(B)$ and $A = B$ on $D(A)$ and so

$$(\lambda I - A)(\lambda I - A)^{-1} y = (\lambda I - B)(\lambda I - A)^{-1} y$$

which implies,

$$\begin{aligned} (\lambda I - B)^{-1} y &= \\ (\lambda I - B)^{-1} \left((\lambda I - B)(\lambda I - A)^{-1} y \right) &= (\lambda I - A)^{-1} y \end{aligned}$$

Recall from Proposition 15.12.5 that an arbitrary element of $D(B)$ is of the form $(\lambda I - B)^{-1} y$ and this has shown every such vector is in $D(A)$, in fact it equals $(\lambda I - A)^{-1} y$. Hence $D(B) \subseteq D(A)$ which shows A generates $S(t)$ and this proves the first half of the theorem.

Next suppose A is the generator of a semigroup $S(t)$ having $\|S(t)\| \leq M$. Then by Proposition 15.12.5 for all $\lambda > 0$, $(\lambda I - A)$ is onto and

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} S(t) dt$$

thus

$$\begin{aligned} & \left\| \left((\lambda I - A)^{-1} \right)^n \right\| \\ &= \left\| \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1 + \cdots + t_n)} S(t_1 + \cdots + t_n) dt_1 \cdots dt_n \right\| \\ &\leq \int_0^\infty \cdots \int_0^\infty e^{-\lambda(t_1 + \cdots + t_n)} M dt_1 \cdots dt_n = \frac{M}{\lambda^n}. \end{aligned}$$

This proves the theorem.

15.12.1 An Evolution Equation

When Λ generates a continuous semigroup, one can consider a very interesting theorem about evolution equations of the form

$$y' - \Lambda y = g(t)$$

provided $t \rightarrow g(t)$ is C^1 .

Theorem 15.12.8 *Let Λ be the generator of $S(t)$, a continuous semigroup on H , a Banach space and let $t \rightarrow g(t)$ be in $C^1(0, \infty; H)$. Then there exists a unique solution to the initial value problem*

$$y' - \Lambda y = g, \quad y(0) = y_0 \in D(\Lambda)$$

and it is given by

$$y(t) = S(t) y_0 + \int_0^t S(t-s) g(s) ds. \quad (15.12.64)$$

This solution is continuous having continuous derivative and has values in $D(\Lambda)$.

Proof: First I show the following claim.

Claim: $\int_0^t S(t-s)g(s)ds \in D(\Lambda)$ and

$$\Lambda \left(\int_0^t S(t-s)g(s)ds \right) = S(t)g(0) - g(t) + \int_0^t S(t-s)g'(s)ds$$

Proof of the claim:

$$\begin{aligned} & \frac{1}{h} \left(S(h) \int_0^t S(t-s)g(s)ds - \int_0^t S(t-s)g(s)ds \right) \\ &= \frac{1}{h} \left(\int_0^t S(t-s+h)g(s)ds - \int_0^t S(t-s)g(s)ds \right) \\ &= \frac{1}{h} \left(\int_{-h}^{t-h} S(t-s)g(s+h)ds - \int_0^t S(t-s)g(s)ds \right) \\ &= \frac{1}{h} \int_{-h}^0 S(t-s)g(s+h)ds + \int_0^{t-h} S(t-s) \frac{g(s+h) - g(s)}{h} ds \\ &\quad - \frac{1}{h} \int_{t-h}^t S(t-s)g(s)ds \end{aligned}$$

Using the estimate in Theorem 15.12.3 on Page 468 and the dominated convergence theorem, the limit as $h \rightarrow 0$ of the above equals

$$S(t)g(0) - g(t) + \int_0^t S(t-s)g'(s)ds$$

which proves the claim.

Since $y_0 \in D(\Lambda)$,

$$\begin{aligned} S(t)\Lambda y_0 &= S(t) \lim_{h \rightarrow 0} \frac{S(h)y_0 - y_0}{h} \\ &= \lim_{h \rightarrow 0} \frac{S(t+h) - S(t)}{h} y_0 \\ &= \lim_{h \rightarrow 0} \frac{S(h)S(t)y_0 - S(t)y_0}{h} \end{aligned} \tag{15.12.65}$$

Since this limit exists, the last limit in the above exists and equals

$$\Lambda S(t)y_0 \tag{15.12.66}$$

and so $S(t)y_0 \in D(\Lambda)$. Now consider 15.12.64.

$$\frac{y(t+h) - y(t)}{h} = \frac{S(t+h) - S(t)}{h} y_0 +$$

$$\begin{aligned}
& \frac{1}{h} \left(\int_0^{t+h} S(t-s+h) g(s) ds - \int_0^t S(t-s) g(s) ds \right) \\
&= \frac{S(t+h) - S(t)}{h} y_0 + \frac{1}{h} \int_t^{t+h} S(t-s+h) g(s) ds \\
& \quad + \frac{1}{h} \left(S(h) \int_0^t S(t-s) g(s) ds - \int_0^t S(t-s) g(s) ds \right)
\end{aligned}$$

From the claim and 15.12.65, 15.12.66 the limit of the right side is

$$\begin{aligned}
& \Lambda S(t) y_0 + g(t) + \Lambda \left(\int_0^t S(t-s) g(s) ds \right) \\
&= \Lambda \left(S(t) y_0 + \int_0^t S(t-s) g(s) ds \right) + g(t)
\end{aligned}$$

Hence

$$y'(t) = \Lambda y(t) + g(t)$$

and from the formula, y' is continuous since by the claim and 15.12.66 it also equals

$$S(t) \Lambda y_0 + g(t) + S(t) g(0) - g(t) + \int_0^t S(t-s) g'(s) ds$$

which is continuous. The claim and 15.12.66 also shows $y(t) \in D(\Lambda)$. This proves the existence part of the lemma.

It remains to prove the uniqueness part. It suffices to show that if

$$y' - \Lambda y = 0, \quad y(0) = 0$$

and y is C^1 having values in $D(\Lambda)$, then $y = 0$. Suppose then that y is this way. Letting $0 < s < t$,

$$\begin{aligned}
& \frac{d}{ds} (S(t-s) y(s)) \\
& \equiv \lim_{h \rightarrow 0} S(t-s-h) \frac{y(s+h) - y(s)}{h} \\
& \quad - \frac{S(t-s) y(s) - S(t-s-h) y(s)}{h}
\end{aligned}$$

provided the limit exists. Since y' exists and $y(s) \in D(\Lambda)$, this equals

$$S(t-s) y'(s) - S(t-s) \Lambda y(s) = 0.$$

Let $y^* \in H'$. This has shown that on the open interval $(0, t)$ the function $s \rightarrow y^*(S(t-s) y(s))$ has a derivative equal to 0. Also from continuity of S and y , this function is continuous on $[0, t]$. Therefore, it is constant on $[0, t]$ by the mean value theorem. At $s = 0$, this function equals 0. Therefore, it equals 0 on $[0, t]$. Thus for fixed $s > 0$ and letting $t > s$, $y^*(S(t-s) y(s)) = 0$. Now let t decrease toward s . Then $y^*(y(s)) = 0$ and since y^* was arbitrary, it follows $y(s) = 0$. This proves uniqueness.

15.12.2 Adjoint, Hilbert Space

In Hilbert space, there are some special things which are true.

Definition 15.12.9 Let A be a densely defined closed operator on H a real Hilbert space. Then A^* is defined as follows.

$$D(A^*) \equiv \{y \in H : |(Ax, y)| \leq C|x|\}$$

Then since $D(A)$ is dense, there exists a unique element of H denoted by A^*y such that

$$(Ax, y) = (x, A^*y)$$

for all $x \in D(A)$.

Lemma 15.12.10 Let A be closed and densely defined on $D(H) \subseteq H$, a Hilbert space. Then A^* is also closed and densely defined. Also $(A^*)^* = A$. In addition to this, if $(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$, then $(\lambda I - A^*)^{-1} \in \mathcal{L}(H, H)$ and

$$\left(\left((\lambda I - A)^{-1} \right)^n \right)^* = \left((\lambda I - A^*)^{-1} \right)^n$$

Proof: Denote by $[x, y]$ an ordered pair in $H \times H$. Define $\tau : H \times H \rightarrow H \times H$ by

$$\tau[x, y] \equiv [-y, x]$$

Then the definition of adjoint implies that for $\mathcal{G}(B)$ equal to the graph of B ,

$$\mathcal{G}(A^*) = (\tau\mathcal{G}(A))^\perp \quad (15.12.67)$$

In this notation the inner product on $H \times H$ with respect to which $\perp$ is defined is given by

$$([x, y], [a, b]) \equiv (x, a) + (y, b).$$

Here is why this is so. For $[x, A^*x] \in \mathcal{G}(A^*)$ it follows that for all $y \in D(A)$

$$([x, A^*x], [-Ay, y]) = -(Ay, x) + (y, A^*x) = 0$$

and so $[x, A^*x] \in (\tau\mathcal{G}(A))^\perp$ which shows

$$\mathcal{G}(A^*) \subseteq (\tau\mathcal{G}(A))^\perp$$

To obtain the other inclusion, let $[a, b] \in (\tau\mathcal{G}(A))^\perp$. This means that for all $x \in D(A)$,

$$([a, b], [-Ax, x]) = 0.$$

In other words, for all $x \in D(A)$,

$$(Ax, a) = (x, b)$$

and so $|(Ax, a)| \leq C|x|$ for all $x \in D(A)$ which shows $a \in D(A^*)$ and

$$(x, A^*a) = (x, b)$$

for all $x \in D(A)$. Therefore, since $D(A)$ is dense, it follows $b = A^*a$ and so $[a, b] \in \mathcal{G}(A^*)$. This shows the other inclusion.

Note that if V is any subspace of the Hilbert space $H \times H$,

$$(V^\perp)^\perp = \overline{V}$$

and $S^\perp$ is always a closed subspace. Also τ and $\perp$ commute. The reason for this is that $[x, y] \in (\tau V)^\perp$ means that

$$(x, -b) + (y, a) = 0$$

for all $[a, b] \in V$ and $[x, y] \in \tau(V^\perp)$ means $[-y, x] \in V^\perp$ so for all $[a, b] \in V$,

$$(-y, a) + (x, b) = 0$$

which says the same thing. It is also clear that $\tau \circ \tau$ has the effect of multiplication by -1 .

It follows from the above description of the graph of A^* that even if $\mathcal{G}(A)$ were not closed it would still be the case that $\mathcal{G}(A^*)$ is closed.

Why is $D(A^*)$ dense? Suppose $z \in D(A^*)^\perp$. Then for all $y \in D(A^*)$ so that $[y, Ay] \in \mathcal{G}(A^*)$, it follows $[z, 0] \in \mathcal{G}(A^*)^\perp = \left((\tau \mathcal{G}(A))^\perp \right)^\perp = \tau \mathcal{G}(A)$ but this implies

$$[0, z] \in -\mathcal{G}(A)$$

and so $z = -A0 = 0$. Thus $D(A^*)$ must be dense since there is no nonzero vector in $D(A^*)^\perp$.

Since A is a closed operator, meaning $\mathcal{G}(A)$ is closed in $H \times H$, it follows from the above formula that

$$\begin{aligned} \mathcal{G}((A^*)^*) &= \left(\tau \left((\tau \mathcal{G}(A))^\perp \right) \right)^\perp = \left(\tau (\tau \mathcal{G}(A))^\perp \right)^\perp \\ &= \left((-\mathcal{G}(A))^\perp \right)^\perp = \left(\mathcal{G}(A)^\perp \right)^\perp = \mathcal{G}(A) \end{aligned}$$

and so $(A^*)^* = A$.

Now consider the final claim. First let $y \in D(A^*) = D(\lambda I - A^*)$. Then letting $x \in H$ be arbitrary,

$$\begin{aligned} &\left(x, \left((\lambda I - A)(\lambda I - A)^{-1} \right)^* y \right) \\ &= \left((\lambda I - A)(\lambda I - A)^{-1} x, y \right) = \left(x, \left((\lambda I - A)^{-1} \right)^* (\lambda I - A^*) y \right) \end{aligned}$$

Thus

$$\left((\lambda I - A)(\lambda I - A)^{-1} \right)^* = I = \left((\lambda I - A)^{-1} \right)^* (\lambda I - A^*) \quad (15.12.68)$$

on $D(A^*)$. Next let $x \in D(A) = D(\lambda I - A)$ and $y \in H$ arbitrary.

$$(x, y) = \left((\lambda I - A)^{-1} (\lambda I - A) x, y \right) = \left((\lambda I - A) x, \left((\lambda I - A)^{-1} \right)^* y \right)$$

Now it follows $\left| \left((\lambda I - A) x, \left((\lambda I - A)^{-1} \right)^* y \right) \right| \leq |y| |x|$ for any $x \in D(A)$ and so

$$\left((\lambda I - A)^{-1} \right)^* y \in D(A^*)$$

Hence

$$(x, y) = \left(x, (\lambda I - A^*) \left((\lambda I - A)^{-1} \right)^* y \right).$$

Since $x \in D(A)$ is arbitrary and $D(A)$ is dense, it follows

$$(\lambda I - A^*) \left((\lambda I - A)^{-1} \right)^* = I \quad (15.12.69)$$

From 15.12.68 and 15.12.69 it follows

$$(\lambda I - A^*)^{-1} = \left((\lambda I - A)^{-1} \right)^*$$

and $(\lambda I - A^*)$ is one to one and onto with continuous inverse. Finally, from the above,

$$\left((\lambda I - A^*)^{-1} \right)^n = \left(\left((\lambda I - A)^{-1} \right)^* \right)^n = \left(\left((\lambda I - A)^{-1} \right)^n \right)^*.$$

This proves the lemma.

With this preparation, here is an interesting result about the adjoint of the generator of a continuous bounded semigroup. I found this in Balakrishnan [9].

Theorem 15.12.11 *Suppose A is a densely defined closed operator which generates a continuous semigroup, $S(t)$. Then A^* is also a closed densely defined operator which generates $S^*(t)$ and $S^*(t)$ is also a continuous semigroup.*

Proof: First suppose $S(t)$ is also a bounded semigroup, $\|S(t)\| \leq M$. From Lemma 15.12.10 A^* is closed and densely defined. It follows from the Hille Yosida theorem, Theorem 15.12.7 that

$$\left| \left((\lambda I - A)^{-1} \right)^n \right| \leq \frac{M}{\lambda^n}$$

From Lemma 15.12.10 and the fact the adjoint of a bounded linear operator preserves the norm,

$$\begin{aligned} \frac{M}{\lambda^n} &\geq \left| \left(\left((\lambda I - A)^{-1} \right)^n \right)^* \right| = \left| \left((\lambda I - A)^{-1} \right)^* \right|^n \\ &= \left| \left((\lambda I - A^*)^{-1} \right)^n \right| \end{aligned}$$

and so by Theorem 15.12.7 again it follows A^* generates a continuous semigroup, $T(t)$ which satisfies $\|T(t)\| \leq M$. I need to identify $T(t)$ with $S^*(t)$. However, from the proof of Theorem 15.12.7 and Lemma 15.12.10, it follows that for $x \in D(A^*)$ and a suitable sequence $\{\lambda_n\}$,

$$\begin{aligned} (T(t)x, y) &= \left(\lim_{n \rightarrow \infty} e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left(\lambda_n^2 (\lambda_n I - A^*)^{-1} \right)^k}{k!} x, y \right) \\ &= \lim_{n \rightarrow \infty} \left(e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left(\left(\lambda_n^2 (\lambda_n I - A)^{-1} \right)^k \right)^*}{k!} x, y \right) \\ &= \lim_{n \rightarrow \infty} \left(x, e^{-\lambda_n t} \left(\sum_{k=0}^{\infty} \frac{t^k \left(\left(\lambda_n^2 (\lambda_n I - A)^{-1} \right)^k \right)}{k!} \right) y \right) \\ &= (x, S(t)y) = (S^*(t)x, y). \end{aligned}$$

Therefore, since y is arbitrary, $S^*(t) = T(t)$ on $x \in D(A^*)$ a dense set and this shows the two are equal. This proves the proposition in the case where $S(t)$ is also bounded.

Next only assume $S(t)$ is a continuous semigroup. Then by Proposition 15.12.5 there exists $\alpha > 0$ such that

$$\|S(t)\| \leq M e^{\alpha t}.$$

Then consider the operator $-\alpha I + A$ and the bounded semigroup $e^{-\alpha t} S(t)$. For $x \in D(A)$

$$\begin{aligned} \lim_{h \rightarrow 0+} \frac{e^{-\alpha h} S(h)x - x}{h} &= \lim_{h \rightarrow 0+} \left(e^{-\alpha h} \frac{S(h)x - x}{h} + \frac{e^{-\alpha h} - 1}{h} x \right) \\ &= -\alpha x + Ax \end{aligned}$$

Thus $-\alpha I + A$ generates $e^{-\alpha t} S(t)$ and it follows from the first part that $-\alpha I + A^*$ generates $e^{-\alpha t} S^*(t)$. Thus

$$\begin{aligned} -\alpha x + A^*x &= \lim_{h \rightarrow 0+} \frac{e^{-\alpha h} S^*(h)x - x}{h} \\ &= \lim_{h \rightarrow 0+} \left(e^{-\alpha h} \frac{S^*(h)x - x}{h} + \frac{e^{-\alpha h} - 1}{h} x \right) \\ &= -\alpha x + \lim_{h \rightarrow 0+} \frac{S^*(h)x - x}{h} \end{aligned}$$

showing that A^* generates $S^*(t)$. It follows from Proposition 15.12.5 that A^* is closed and densely defined. It is obvious $S^*(t)$ is a semigroup. Why is it continuous?

This also follows from the first part of the argument which establishes that

$$e^{-\alpha t} S^*(t)$$

is continuous. This proves the theorem.

15.12.3 Adjoints, Reflexive Banach Space

Here the adjoint of a generator of a semigroup is considered. I will show that the adjoint of the generator generates the adjoint of the semigroup in a reflexive Banach space. This is about as far as you can go although a general but less satisfactory result is given in Yosida [67].

Definition 15.12.12 *Let A be a densely defined closed operator on H a real Banach space. Then A^* is defined as follows.*

$$D(A^*) \equiv \{y^* \in H' : |y^*(Ax)| \leq C \|x\| \text{ for all } x \in D(A)\}$$

Then since $D(A)$ is dense, there exists a unique element of H' denoted by A^*y such that

$$A^*(y^*)(x) = y^*(Ax)$$

for all $x \in D(A)$.

Lemma 15.12.13 *Let A be closed and densely defined on $D(A) \subseteq H$, a Banach space. Then A^* is also closed and densely defined. Also $(A^*)^* = A$. In addition to this, if $(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$, then $(\lambda I - A^*)^{-1} \in \mathcal{L}(H', H')$ and*

$$\left(\left((\lambda I - A)^{-1} \right)^n \right)^* = \left((\lambda I - A^*)^{-1} \right)^n$$

Proof: Denote by $[x, y]$ an ordered pair in $H \times H$. Define $\tau : H \times H \rightarrow H \times H$ by

$$\tau[x, y] \equiv [-y, x]$$

A similar notation will apply to $H' \times H'$. Then the definition of adjoint implies that for $\mathcal{G}(B)$ equal to the graph of B ,

$$\mathcal{G}(A^*) = (\tau \mathcal{G}(A))^\perp \quad (15.12.70)$$

For $S \subseteq H \times H$, define $S^\perp$ by

$$\{[a^*, b^*] \in H' \times H' : a^*(x) + b^*(y) = 0 \text{ for all } [x, y] \in S\}$$

If $S \subseteq H' \times H'$ a similar definition holds.

$$\{[x, y] \in H \times H : a^*(x) + b^*(y) = 0 \text{ for all } [a^*, b^*] \in S\}$$

Here is why 15.12.70 is so. For $[x^*, A^*x^*] \in \mathcal{G}(A^*)$ it follows that for all $y \in D(A)$

$$x^*(Ay) = A^*x^*(y)$$

and so for all $[y, Ay] \in \mathcal{G}(A)$,

$$-x^*(Ay) + A^*x^*(y) = 0$$

which is what it means to say $[x^*, A^*x^*] \in (\tau\mathcal{G}(A))^\perp$. This shows

$$\mathcal{G}(A^*) \subseteq (\tau\mathcal{G}(A))^\perp$$

To obtain the other inclusion, let $[a^*, b^*] \in (\tau\mathcal{G}(A))^\perp$. This means that for all $[x, Ax] \in \mathcal{G}(A)$,

$$-a^*(Ax) + b^*(x) = 0$$

In other words, for all $x \in D(A)$,

$$|a^*(Ax)| \leq \|b^*\| \|x\|$$

which means by definition, $a^* \in D(A^*)$ and $A^*a^* = b^*$. Thus $[a^*, b^*] \in \mathcal{G}(A^*)$. This shows the other inclusion.

Note that if V is any subspace of $H \times H$,

$$(V^\perp)^\perp = \overline{V}$$

and $S^\perp$ is always a closed subspace. Also τ and $\perp$ commute. The reason for this is that $[x^*, y^*] \in (\tau V)^\perp$ means that

$$-x^*(b) + y^*(a) = 0$$

for all $[a, b] \in V$ and $[x^*, y^*] \in \tau(V^\perp)$ means $[-y^*, x^*] \in -(V^\perp) = V^\perp$ so for all $[a, b] \in V$,

$$-y^*(a) + x^*(b) = 0$$

which says the same thing. It is also clear that $\tau \circ \tau$ has the effect of multiplication by -1 . If $V \subseteq H' \times H'$, the argument for commuting $\perp$ and τ is similar.

It follows from the above description of the graph of A^* that even if $\mathcal{G}(A)$ were not closed it would still be the case that $\mathcal{G}(A^*)$ is closed.

Why is $D(A^*)$ dense? If it is not dense, then by a typical application of the Hahn Banach theorem, there exists $y^{**} \in H''$ such that $y^{**}(D(A^*)) = 0$ but $y^{**} \neq 0$. Since H is reflexive, there exists $y \in H$ such that $x^*(y) = 0$ for all $x^* \in D(A^*)$. Thus

$$[y, 0] \in \mathcal{G}(A^*)^\perp = \left((\tau\mathcal{G}(A))^\perp\right)^\perp = \tau\mathcal{G}(A)$$

and so $[0, y] \in \mathcal{G}(A)$ which means $y = A0 = 0$, a contradiction. Thus $D(A^*)$ is indeed dense. Note this is where it was important to assume the space is reflexive. If you consider $C([0, 1])$ it is not dense in $L^\infty([0, 1])$ but if $f \in L^1([0, 1])$ satisfies $\int_0^1 fg dm = 0$ for all $g \in C([0, 1])$, then $f = 0$. Hence there is no nonzero $f \in C([0, 1])^\perp$.

Since A is a closed operator, meaning $\mathcal{G}(A)$ is closed in $H \times H$, it follows from the above formula that

$$\begin{aligned}\mathcal{G}((A^*)^*) &= \left(\tau \left((\tau \mathcal{G}(A))^{\perp} \right) \right)^{\perp} = \left(\tau (\tau \mathcal{G}(A))^{\perp} \right)^{\perp} \\ &= \left((-\mathcal{G}(A))^{\perp} \right)^{\perp} = \left(\mathcal{G}(A)^{\perp} \right)^{\perp} = \mathcal{G}(A)\end{aligned}$$

and so $(A^*)^* = A$.

Now consider the final claim. First let $y^* \in D(A^*) = D(\lambda I - A^*)$. Then letting $x \in H$ be arbitrary,

$$\begin{aligned}y^*(x) &= \left((\lambda I - A)(\lambda I - A)^{-1} \right)^* y^*(x) \\ &= y^* \left((\lambda I - A)(\lambda I - A)^{-1} x \right)\end{aligned}$$

Since $y^* \in D(A^*)$ and $(\lambda I - A)^{-1} x \in D(A)$, this equals

$$(\lambda I - A)^* y^* \left((\lambda I - A)^{-1} x \right)$$

Now by definition, this equals

$$\left((\lambda I - A)^{-1} \right)^* (\lambda I - A)^* y^*(x)$$

It follows that for $y^* \in D(A^*)$,

$$\begin{aligned}&\left((\lambda I - A)^{-1} \right)^* (\lambda I - A)^* y^* \\ &= \left((\lambda I - A)^{-1} \right)^* (\lambda I - A^*) y^* = y^*\end{aligned}\tag{15.12.71}$$

Next let $y^* \in H'$ be arbitrary and $x \in D(A)$

$$\begin{aligned}y^*(x) &= y^* \left((\lambda I - A)^{-1} (\lambda I - A) x \right) \\ &= \left((\lambda I - A)^{-1} \right)^* y^* ((\lambda I - A) x) \\ &= (\lambda I - A)^* \left((\lambda I - A)^{-1} \right)^* y^*(x)\end{aligned}$$

In going from the second to the third line, the first line shows $\left((\lambda I - A)^{-1} \right)^* y^* \in D(A^*)$ and so the third line follows. Since $D(A)$ is dense, it follows

$$(\lambda I - A^*) \left((\lambda I - A)^{-1} \right)^* = I\tag{15.12.72}$$

Then 15.12.71 and 15.12.72 show $\lambda I - A^*$ is one to one and onto from $D(A^*)$ to H' and

$$(\lambda I - A^*)^{-1} = \left((\lambda I - A)^{-1} \right)^*.$$

Finally, from the above,

$$\left((\lambda I - A^*)^{-1}\right)^n = \left(\left((\lambda I - A)^{-1}\right)^*\right)^n = \left(\left((\lambda I - A)^{-1}\right)^n\right)^*.$$

This proves the lemma.

With this preparation, here is an interesting result about the adjoint of the generator of a continuous bounded semigroup.

Theorem 15.12.14 *Suppose A is a densely defined closed operator which generates a continuous semigroup, $S(t)$. Then A^* is also a closed densely defined operator which generates $S^*(t)$ and $S^*(t)$ is also a continuous semigroup.*

Proof: First suppose $S(t)$ is also a bounded semigroup, $\|S(t)\| \leq M$. From Lemma 15.12.13 A^* is closed and densely defined. It follows from the Hille Yosida theorem, Theorem 15.12.7 that

$$\left\|\left((\lambda I - A)^{-1}\right)^n\right\| \leq \frac{M}{\lambda^n}$$

From Lemma 15.12.13 and the fact the adjoint of a bounded linear operator preserves the norm,

$$\begin{aligned} \frac{M}{\lambda^n} &\geq \left\|\left(\left((\lambda I - A)^{-1}\right)^n\right)^*\right\| = \left\|\left(\left((\lambda I - A)^{-1}\right)^*\right)^n\right\| \\ &= \left\|\left((\lambda I - A^*)^{-1}\right)^n\right\| \end{aligned}$$

and so by Theorem 15.12.7 again it follows A^* generates a continuous semigroup, $T(t)$ which satisfies $\|T(t)\| \leq M$. I need to identify $T(t)$ with $S^*(t)$. However, from the proof of Theorem 15.12.7 and Lemma 15.12.13, it follows that for $x^* \in D(A^*)$ and a suitable sequence $\{\lambda_n\}$,

$$\begin{aligned} T(t)x^*(y) &= \lim_{n \rightarrow \infty} e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left(\lambda_n^2 (\lambda_n I - A^*)^{-1}\right)^k}{k!} x^*(y) \\ &= \lim_{n \rightarrow \infty} e^{-\lambda_n t} \sum_{k=0}^{\infty} \frac{t^k \left(\left(\lambda_n^2 (\lambda_n I - A)^{-1}\right)^k\right)^*}{k!} x^*(y) \\ &= \lim_{n \rightarrow \infty} x^* \left(e^{-\lambda_n t} \left(\sum_{k=0}^{\infty} \frac{t^k \left(\left(\lambda_n^2 (\lambda_n I - A)^{-1}\right)^k\right)}{k!} y \right) \right) \\ &= x^*(S(t)y) = S^*(t)x^*(y). \end{aligned}$$

Therefore, since y is arbitrary, $S^*(t) = T(t)$ on $x \in D(A^*)$ a dense set and this shows the two are equal. In particular, $S^*(t)$ is a semigroup because $T(t)$ is. This proves the proposition in the case where $S(t)$ is also bounded.

Next only assume $S(t)$ is a continuous semigroup. Then by Proposition 15.12.5 there exists $\alpha > 0$ such that

$$\|S(t)\| \leq Me^{\alpha t}.$$

Then consider the operator $-\alpha I + A$ and the bounded semigroup $e^{-\alpha t}S(t)$. For $x \in D(A)$

$$\begin{aligned} \lim_{h \rightarrow 0+} \frac{e^{-\alpha h} S(h)x - x}{h} &= \lim_{h \rightarrow 0+} \left(e^{-\alpha h} \frac{S(h)x - x}{h} + \frac{e^{-\alpha h} - 1}{h} x \right) \\ &= -\alpha x + Ax \end{aligned}$$

Thus $-\alpha I + A$ generates $e^{-\alpha t}S(t)$ and it follows from the first part that $-\alpha I + A^*$ generates the semigroup $e^{-\alpha t}S^*(t)$. Thus

$$\begin{aligned} -\alpha x + A^*x &= \lim_{h \rightarrow 0+} \frac{e^{-\alpha h} S^*(h)x - x}{h} \\ &= \lim_{h \rightarrow 0+} \left(e^{-\alpha h} \frac{S^*(h)x - x}{h} + \frac{e^{-\alpha h} - 1}{h} x \right) \\ &= -\alpha x + \lim_{h \rightarrow 0+} \frac{S^*(h)x - x}{h} \end{aligned}$$

showing that A^* generates $S^*(t)$. It follows from Proposition 15.12.5 that A^* is closed and densely defined. It is obvious $S^*(t)$ is a semigroup. Why is it continuous? This also follows from the first part of the argument which establishes that

$$t \rightarrow e^{-\alpha t}S^*(t)x$$

is continuous. This proves the theorem.

Representation Theorems

16.1 Radon Nikodym Theorem

This chapter is on various representation theorems. The first theorem, the Radon Nikodym Theorem, is a representation theorem for one measure in terms of another. The approach given here is due to Von Neumann and depends on the Riesz representation theorem for Hilbert space, Theorem 15.1.14 on Page 426.

Definition 16.1.1 *Let μ and λ be two measures defined on a σ -algebra, $\mathcal{S}$, of subsets of a set, Ω . λ is absolutely continuous with respect to μ , written as $\lambda \ll \mu$, if $\lambda(E) = 0$ whenever $\mu(E) = 0$.*

It is not hard to think of examples which should be like this. For example, suppose one measure is volume and the other is mass. If the volume of something is zero, it is reasonable to expect the mass of it should also be equal to zero. In this case, there is a function called the density which is integrated over volume to obtain mass. The Radon Nikodym theorem is an abstract version of this notion. Essentially, it gives the existence of the density function.

Theorem 16.1.2 (Radon Nikodym) *Let λ and μ be finite measures defined on a σ -algebra, $\mathcal{S}$, of subsets of Ω . Suppose $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega, \mu)$ such that $f(x) \geq 0$ and*

$$\lambda(E) = \int_E f \, d\mu.$$

If it is not necessarily the case that $\lambda \ll \mu$, there are two measures, $\lambda_\perp$ and $\lambda_\parallel$ such that $\lambda = \lambda_\perp + \lambda_\parallel$, $\lambda_\parallel \ll \mu$ and there exists a set of μ measure zero, N such that for all E measurable, $\lambda_\perp(E) = \lambda(E \cap N) = \lambda_\perp(E \cap N)$. In this case the two measures, $\lambda_\perp$ and $\lambda_\parallel$ are unique and the representation of $\lambda = \lambda_\perp + \lambda_\parallel$ is called the Lebesgue decomposition of λ . The measure $\lambda_\parallel$ is the absolutely continuous part of λ and $\lambda_\perp$ is called the singular part of λ .

Proof: Let $\Lambda : L^2(\Omega, \mu + \lambda) \rightarrow \mathbb{C}$ be defined by

$$\Lambda g = \int_\Omega g \, d\lambda.$$

By Holder's inequality,

$$|\Lambda g| \leq \left(\int_{\Omega} 1^2 d\lambda \right)^{1/2} \left(\int_{\Omega} |g|^2 d(\lambda + \mu) \right)^{1/2} = \lambda(\Omega)^{1/2} \|g\|_2$$

where $\|g\|_2$ is the L^2 norm of g taken with respect to $\mu + \lambda$. Therefore, since Λ is bounded, it follows from Theorem 13.1.4 on Page 363 that $\Lambda \in (L^2(\Omega, \mu + \lambda))'$, the dual space $L^2(\Omega, \mu + \lambda)$. By the Riesz representation theorem in Hilbert space, Theorem 15.1.14, there exists a unique $h \in L^2(\Omega, \mu + \lambda)$ with

$$\Lambda g = \int_{\Omega} g d\lambda = \int_{\Omega} h g d(\mu + \lambda). \quad (16.1.1)$$

The plan is to show h is real and nonnegative at least a.e. Therefore, consider the set where $\text{Im } h$ is positive.

$$E = \{x \in \Omega : \text{Im } h(x) > 0\},$$

Now let $g = \chi_E$ and use 16.1.1 to get

$$\lambda(E) = \int_E (\text{Re } h + i \text{Im } h) d(\mu + \lambda). \quad (16.1.2)$$

Since the left side of 16.1.2 is real, this shows

$$\begin{aligned} 0 &= \int_E (\text{Im } h) d(\mu + \lambda) \\ &\geq \int_{E_n} (\text{Im } h) d(\mu + \lambda) \\ &\geq \frac{1}{n} (\mu + \lambda)(E_n) \end{aligned}$$

where

$$E_n \equiv \left\{ x : \text{Im } h(x) \geq \frac{1}{n} \right\}$$

Thus $(\mu + \lambda)(E_n) = 0$ and since $E = \cup_{n=1}^{\infty} E_n$, it follows $(\mu + \lambda)(E) = 0$. A similar argument shows that for

$$E = \{x \in \Omega : \text{Im } h(x) < 0\},$$

$(\mu + \lambda)(E) = 0$. Thus there is no loss of generality in assuming h is real-valued.

The next task is to show h is nonnegative. This is done in the same manner as above. Define the set where it is negative and then show this set has measure zero.

Let $E \equiv \{x : h(x) < 0\}$ and let $E_n \equiv \{x : h(x) < -\frac{1}{n}\}$. Then let $g = \chi_{E_n}$. Since $E = \cup_n E_n$, it follows that if $(\mu + \lambda)(E) > 0$ then this is also true for $(\mu + \lambda)(E_n)$ for all n large enough. Then from 16.1.2

$$\lambda(E_n) = \int_{E_n} h d(\mu + \lambda) \leq -(1/n) (\mu + \lambda)(E_n) < 0,$$

a contradiction. Thus it can be assumed $h \geq 0$.

At this point the argument splits into two cases.

Case Where $\lambda \ll \mu$.

In this case, $h < 1$. Let $E = [h \geq 1]$ and let $g = \mathcal{X}_E$. Then

$$\lambda(E) = \int_E h d(\mu + \lambda) \geq \mu(E) + \lambda(E).$$

Therefore $\mu(E) = 0$. Since $\lambda \ll \mu$, it follows that $\lambda(E) = 0$ also. Thus it can be assumed

$$0 \leq h(x) < 1$$

for all x .

From 16.1.1, whenever $g \in L^2(\Omega, \mu + \lambda)$,

$$\int_{\Omega} g(1-h)d\lambda = \int_{\Omega} hgd\mu. \quad (16.1.3)$$

Now let E be a measurable set and define

$$g(x) \equiv \sum_{i=0}^n h^i(x) \mathcal{X}_E(x)$$

in 16.1.3. This yields

$$\int_E (1 - h^{n+1}(x)) d\lambda = \int_E \sum_{i=1}^{n+1} h^i(x) d\mu. \quad (16.1.4)$$

Let $f(x) = \sum_{i=1}^{\infty} h^i(x)$ and use the Monotone Convergence theorem in 16.1.4 to let $n \rightarrow \infty$ and conclude

$$\lambda(E) = \int_E f d\mu.$$

$f \in L^1(\Omega, \mu)$ because λ is finite.

The function, f is unique μ a.e. because, if g is another function which also serves to represent λ , consider for each $n \in \mathbb{N}$ the set,

$$E_n \equiv \left[f - g > \frac{1}{n} \right]$$

and conclude that

$$0 = \int_{E_n} (f - g) d\mu \geq \frac{1}{n} \mu(E_n).$$

Therefore, $\mu(E_n) = 0$. It follows that

$$\mu([f - g > 0]) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$$

Similarly, the set where g is larger than f has measure zero. This proves the theorem.

Case where it is not necessarily true that $\lambda \ll \mu$.

In this case, let $N = [h \geq 1]$ and let $g = \mathcal{X}_N$. Then

$$\lambda(N) = \int_N h d(\mu + \lambda) \geq \mu(N) + \lambda(N).$$

and so $\mu(N) = 0$ and so $\mu(E) = \mu(E \cap N^C)$. Now define a measure, $\lambda_\perp$ by

$$\lambda_\perp(E) \equiv \lambda(E \cap N)$$

so

$$\lambda_\perp(E \cap N) \equiv \lambda(E \cap N \cap N) = \lambda(E \cap N) \equiv \lambda_\perp(E)$$

and let $\lambda_\parallel \equiv \lambda - \lambda_\perp$. Thus,

$$\lambda_\parallel(E) = \lambda(E) - \lambda_\perp(E) \equiv \lambda(E) - \lambda(E \cap N) = \lambda(E \cap N^C).$$

Suppose now that $\lambda_\parallel(E) > 0$. It follows from the first part of the proof that since $h < 1$ on N^C

$$\begin{aligned} 0 &< \lambda_\parallel(E) = \lambda(E \cap N^C) = \int_{E \cap N^C} h d(\mu + \lambda) \\ &< \mu(E \cap N^C) + \lambda(E \cap N^C) = \mu(E) + \lambda_\parallel(E) \end{aligned}$$

which shows that $\mu(E) > 0$. Thus if $\mu(E) = 0$ it follows $\lambda_\parallel(E) = 0$ and so $\lambda_\parallel \ll \mu$.

It only remains to verify the two measures $\lambda_\perp$ and $\lambda_\parallel$ are unique. Suppose then that ν_1 and ν_2 play the roles of $\lambda_\perp$ and $\lambda_\parallel$ respectively. Let N_1 play the role of N in the definition of ν_1 and let g_1 play the role of g for ν_2 . I will show that $g = g_1$ μ a.e. Let $E_k \equiv [g_1 - g > 1/k]$ for $k \in \mathbb{N}$. Then on observing that $\lambda_\perp - \nu_1 = \nu_2 - \lambda_\parallel$

$$\begin{aligned} 0 &= (\lambda_\perp - \nu_1)(E_n \cap (N_1 \cup N)^C) = \int_{E_n \cap (N_1 \cup N)^C} (g_1 - g) d\mu \\ &\geq \frac{1}{k} \mu(E_k \cap (N_1 \cup N)^C) = \frac{1}{k} \mu(E_k). \end{aligned}$$

and so $\mu(E_k) = 0$. Therefore, $\mu([g_1 - g > 0]) = 0$ because $[g_1 - g > 0] = \cup_{k=1}^\infty E_k$. It follows $g_1 \leq g$ μ a.e. Similarly, $g \geq g_1$ μ a.e. Therefore, $\nu_2 = \lambda_\parallel$ and so $\lambda_\perp = \nu_1$ also. This proves the theorem.

The f in the theorem for the absolutely continuous case is sometimes denoted by $\frac{d\lambda}{d\mu}$ and is called the Radon Nikodym derivative.

The next corollary is a useful generalization to σ finite measure spaces.

Corollary 16.1.3 Suppose $\lambda \ll \mu$ and there exist sets $S_n \in \mathcal{S}$ with

$$S_n \cap S_m = \emptyset, \cup_{n=1}^\infty S_n = \Omega,$$

and $\lambda(S_n), \mu(S_n) < \infty$. Then there exists $f \geq 0$, where f is μ measurable, and

$$\lambda(E) = \int_E f \, d\mu$$

for all $E \in \mathcal{S}$. The function f is $\mu + \lambda$ a.e. unique.

Proof: Define the σ algebra of subsets of S_n ,

$$\mathcal{S}_n \equiv \{E \cap S_n : E \in \mathcal{S}\}.$$

Then both λ , and μ are finite measures on $\mathcal{S}_n$, and $\lambda \ll \mu$. Thus, by Theorem 16.1.2, there exists a nonnegative $\mathcal{S}_n$ measurable function f_n , with $\lambda(E) = \int_E f_n \, d\mu$ for all $E \in \mathcal{S}_n$. Define $f(x) = f_n(x)$ for $x \in S_n$. Since the S_n are disjoint and their union is all of Ω , this defines f on all of Ω . The function, f is measurable because

$$f^{-1}((a, \infty]) = \cup_{n=1}^{\infty} f_n^{-1}((a, \infty]) \in \mathcal{S}.$$

Also, for $E \in \mathcal{S}$,

$$\begin{aligned} \lambda(E) &= \sum_{n=1}^{\infty} \lambda(E \cap S_n) = \sum_{n=1}^{\infty} \int \chi_{E \cap S_n}(x) f_n(x) \, d\mu \\ &= \sum_{n=1}^{\infty} \int \chi_{E \cap S_n}(x) f(x) \, d\mu \end{aligned}$$

By the monotone convergence theorem

$$\begin{aligned} \sum_{n=1}^{\infty} \int \chi_{E \cap S_n}(x) f(x) \, d\mu &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \int \chi_{E \cap S_n}(x) f(x) \, d\mu \\ &= \lim_{N \rightarrow \infty} \int \sum_{n=1}^N \chi_{E \cap S_n}(x) f(x) \, d\mu \\ &= \int \sum_{n=1}^{\infty} \chi_{E \cap S_n}(x) f(x) \, d\mu = \int_E f \, d\mu. \end{aligned}$$

This proves the existence part of the corollary.

To see f is unique, suppose f_1 and f_2 both work and consider for $n \in \mathbb{N}$

$$E_k \equiv \left[f_1 - f_2 > \frac{1}{k} \right].$$

Then

$$0 = \lambda(E_k \cap S_n) - \lambda(E_k \cap S_n) = \int_{E_k \cap S_n} f_1(x) - f_2(x) \, d\mu.$$

Hence $\mu(E_k \cap S_n) = 0$ for all n so

$$\mu(E_k) = \lim_{n \rightarrow \infty} \mu(E_k \cap S_n) = 0.$$

Hence $\mu([f_1 - f_2 > 0]) \leq \sum_{k=1}^{\infty} \mu(E_k) = 0$. Therefore, $\lambda([f_1 - f_2 > 0]) = 0$ also. Similarly

$$(\mu + \lambda)([f_1 - f_2 < 0]) = 0.$$

This version of the Radon Nikodym theorem will suffice for most applications, but more general versions are available. To see one of these, one can read the treatment in Hewitt and Stromberg [31]. This involves the notion of decomposable measure spaces, a generalization of σ finite.

Not surprisingly, there is a simple generalization of the Lebesgue decomposition part of Theorem 16.1.2.

Corollary 16.1.4 *Let $(\Omega, \mathcal{S})$ be a set with a σ algebra of sets. Suppose λ and μ are two measures defined on the sets of $\mathcal{S}$ and suppose there exists a sequence of disjoint sets of $\mathcal{S}$, $\{\Omega_i\}_{i=1}^{\infty}$ such that $\lambda(\Omega_i), \mu(\Omega_i) < \infty$. Then there is a set of μ measure zero, N and measures $\lambda_{\perp}$ and $\lambda_{\parallel}$ such that*

$$\lambda_{\perp} + \lambda_{\parallel} = \lambda, \lambda_{\parallel} \ll \mu, \lambda_{\perp}(E) = \lambda(E \cap N) = \lambda_{\perp}(E \cap N).$$

Proof: Let $\mathcal{S}_i \equiv \{E \cap \Omega_i : E \in \mathcal{S}\}$ and for $E \in \mathcal{S}_i$, let $\lambda^i(E) = \lambda(E)$ and $\mu^i(E) = \mu(E)$. Then by Theorem 16.1.2 there exist unique measures $\lambda_{\perp}^i$ and $\lambda_{\parallel}^i$ such that $\lambda^i = \lambda_{\perp}^i + \lambda_{\parallel}^i$, a set of μ^i measure zero, $N_i \in \mathcal{S}_i$ such that for all $E \in \mathcal{S}_i$, $\lambda_{\perp}^i(E) = \lambda^i(E \cap N_i)$ and $\lambda_{\parallel}^i \ll \mu^i$. Define for $E \in \mathcal{S}$

$$\lambda_{\perp}(E) \equiv \sum_i \lambda_{\perp}^i(E \cap \Omega_i), \lambda_{\parallel}(E) \equiv \sum_i \lambda_{\parallel}^i(E \cap \Omega_i), N \equiv \cup_i N_i.$$

First observe that $\lambda_{\perp}$ and $\lambda_{\parallel}$ are measures.

$$\begin{aligned} \lambda_{\perp}(\cup_{j=1}^{\infty} E_j) &\equiv \sum_i \lambda_{\perp}^i(\cup_{j=1}^{\infty} E_j \cap \Omega_i) = \sum_i \sum_j \lambda_{\perp}^i(E_j \cap \Omega_i) \\ &= \sum_j \sum_i \lambda_{\perp}^i(E_j \cap \Omega_i) = \sum_j \sum_i \lambda(E_j \cap \Omega_i \cap N_i) \\ &= \sum_j \sum_i \lambda_{\perp}^i(E_j \cap \Omega_i) = \sum_j \lambda_{\perp}(E_j). \end{aligned}$$

The argument for $\lambda_{\parallel}$ is similar. Now

$$\mu(N) = \sum_i \mu(N \cap \Omega_i) = \sum_i \mu^i(N_i) = 0$$

and

$$\begin{aligned} \lambda_{\perp}(E) &\equiv \sum_i \lambda_{\perp}^i(E \cap \Omega_i) = \sum_i \lambda^i(E \cap \Omega_i \cap N_i) \\ &= \sum_i \lambda(E \cap \Omega_i \cap N) = \lambda(E \cap N). \end{aligned}$$

Also if $\mu(E) = 0$, then $\mu^i(E \cap \Omega_i) = 0$ and so $\lambda_{||}^i(E \cap \Omega_i) = 0$. Therefore,

$$\lambda_{||}(E) = \sum_i \lambda_{||}^i(E \cap \Omega_i) = 0.$$

The decomposition is unique because of the uniqueness of the $\lambda_{||}^i$ and $\lambda_{\perp}^i$ and the observation that some other decomposition must coincide with the given one on the Ω_i .

16.2 Vector Measures

The next topic will use the Radon Nikodym theorem. It is the topic of vector and complex measures. The main interest is in complex measures although a vector measure can have values in any topological vector space. Whole books have been written on this subject. See for example the book by Diestel and Uhl [19] titled Vector measures.

Definition 16.2.1 *Let $(V, ||\cdot||)$ be a normed linear space and let $(\Omega, \mathcal{S})$ be a measure space. A function $\mu : \mathcal{S} \rightarrow V$ is a vector measure if μ is countably additive. That is, if $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint sets of $\mathcal{S}$,*

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Note that it makes sense to take finite sums because it is given that μ has values in a vector space in which vectors can be summed. In the above, $\mu(E_i)$ is a vector. It might be a point in $\mathbb{R}^n$ or in any other vector space. In many of the most important applications, it is a vector in some sort of function space which may be infinite dimensional. The infinite sum has the usual meaning. That is

$$\sum_{i=1}^{\infty} \mu(E_i) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mu(E_i)$$

where the limit takes place relative to the norm on V .

Definition 16.2.2 *Let $(\Omega, \mathcal{S})$ be a measure space and let μ be a vector measure defined on $\mathcal{S}$. A subset, $\pi(E)$, of $\mathcal{S}$ is called a partition of E if $\pi(E)$ consists of finitely many disjoint sets of $\mathcal{S}$ and $\cup \pi(E) = E$. Let*

$$|\mu|(E) = \sup \left\{ \sum_{F \in \pi(E)} \|\mu(F)\| : \pi(E) \text{ is a partition of } E \right\}.$$

$|\mu|$ is called the total variation of μ .

The next theorem may seem a little surprising. It states that, if finite, the total variation is a nonnegative measure.

Theorem 16.2.3 *If $|\mu|(\Omega) < \infty$, then $|\mu|$ is a measure on $\mathcal{S}$. Even if $|\mu|(\Omega) = \infty$, $|\mu|(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} |\mu|(E_i)$. That is $|\mu|$ is subadditive and $|\mu|(A) \leq |\mu|(B)$ whenever $A, B \in \mathcal{S}$ with $A \subseteq B$.*

Proof: Consider the last claim. Let $a < |\mu|(A)$ and let $\pi(A)$ be a partition of A such that

$$a < \sum_{F \in \pi(A)} ||\mu(F)||.$$

Then $\pi(A) \cup \{B \setminus A\}$ is a partition of B and

$$|\mu|(B) \geq \sum_{F \in \pi(A)} ||\mu(F)|| + ||\mu(B \setminus A)|| > a.$$

Since this is true for all such a , it follows $|\mu|(B) \geq |\mu|(A)$ as claimed.

Let $\{E_j\}_{j=1}^{\infty}$ be a sequence of disjoint sets of $\mathcal{S}$ and let $E_{\infty} = \cup_{j=1}^{\infty} E_j$. Then letting $a < |\mu|(E_{\infty})$, it follows from the definition of total variation there exists a partition of E_{∞} , $\pi(E_{\infty}) = \{A_1, \dots, A_n\}$ such that

$$a < \sum_{i=1}^n ||\mu(A_i)||.$$

Also,

$$A_i = \cup_{j=1}^{\infty} A_i \cap E_j$$

and so by the triangle inequality, $||\mu(A_i)|| \leq \sum_{j=1}^{\infty} ||\mu(A_i \cap E_j)||$. Therefore, by the above, and either Fubini's theorem or Lemma 8.3.3 on Page 190

$$\begin{aligned} a &< \sum_{i=1}^n \overbrace{\sum_{j=1}^{\infty} ||\mu(A_i \cap E_j)||}^{\geq ||\mu(A_i)||} \\ &= \sum_{j=1}^{\infty} \sum_{i=1}^n ||\mu(A_i \cap E_j)|| \\ &\leq \sum_{j=1}^{\infty} |\mu|(E_j) \end{aligned}$$

because $\{A_i \cap E_j\}_{i=1}^n$ is a partition of E_j .

Since a is arbitrary, this shows

$$|\mu|(\cup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} |\mu|(E_j).$$

If the sets, E_j are not disjoint, let $F_1 = E_1$ and if F_n has been chosen, let $F_{n+1} \equiv E_{n+1} \setminus \cup_{i=1}^n E_i$. Thus the sets, F_i are disjoint and $\cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} E_i$. Therefore,

$$|\mu|(\cup_{j=1}^{\infty} E_j) = |\mu|(\cup_{j=1}^{\infty} F_j) \leq \sum_{j=1}^{\infty} |\mu|(F_j) \leq \sum_{j=1}^{\infty} |\mu|(E_j)$$

and proves $|\mu|$ is always subadditive as claimed regardless of whether $|\mu|(\Omega) < \infty$.

Now suppose $|\mu|(\Omega) < \infty$ and let E_1 and E_2 be sets of $\mathcal{S}$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A_1^i \cdots A_{n_i}^i\} = \pi(E_i)$, a partition of E_i which is chosen such that

$$|\mu|(E_i) - \varepsilon < \sum_{j=1}^{n_i} |\mu(A_j^i)| \quad i = 1, 2.$$

Such a partition exists because of the definition of the total variation. Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ denoted by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|\mu|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} |\mu(F)| > |\mu|(E_1) + |\mu|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|\mu|(E_1 \cup E_2) \geq |\mu|(E_1) + |\mu|(E_2). \quad (16.2.5)$$

Then 16.2.5 implies that whenever the E_i are disjoint, $|\mu|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |\mu|(E_j)$. Therefore,

$$\sum_{j=1}^{\infty} |\mu|(E_j) \geq |\mu|(\cup_{j=1}^{\infty} E_j) \geq |\mu|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |\mu|(E_j).$$

Since n is arbitrary,

$$|\mu|(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |\mu|(E_j)$$

which shows that $|\mu|$ is a measure as claimed. This proves the theorem.

In the case that μ is a complex measure, it is always the case that $|\mu|(\Omega) < \infty$.

Theorem 16.2.4 Suppose μ is a complex measure on $(\Omega, \mathcal{S})$ where $\mathcal{S}$ is a σ algebra of subsets of Ω . That is, whenever, $\{E_i\}$ is a sequence of disjoint sets of $\mathcal{S}$,

$$\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i).$$

Then $|\mu|(\Omega) < \infty$.

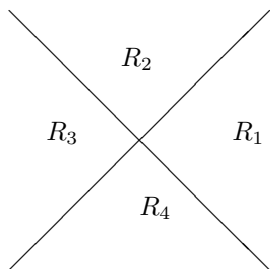
Proof: First here is a claim.

Claim: Suppose $|\mu|(E) = \infty$. Then there are subsets of E , A and B such that $E = A \cup B$, $|\mu(A)|, |\mu(B)| > 1$ and $|\mu|(B) = \infty$.

Proof of the claim: From the definition of $|\mu|$, there exists a partition of E , $\pi(E)$ such that

$$\sum_{F \in \pi(E)} |\mu(F)| > 20(1 + |\mu(E)|). \quad (16.2.6)$$

Here 20 is just a nice sized number. No effort is made to be delicate in this argument. Also note that $\mu(E) \in \mathbb{C}$ because it is given that μ is a complex measure. Consider the following picture consisting of two lines in the complex plane having slopes 1 and -1 which intersect at the origin, dividing the complex plane into four closed sets, R_1, R_2, R_3 , and R_4 as shown.



Let π_i consist of those sets, A of $\pi(E)$ for which $\mu(A) \in R_i$. Thus, some sets, A of $\pi(E)$ could be in two of the π_i if $\mu(A)$ is on one of the intersecting lines. This is not important. The thing which is important is that if $\mu(A) \in R_1$ or R_3 , then $\frac{\sqrt{2}}{2} |\mu(A)| \leq |\operatorname{Re}(\mu(A))|$ and if $\mu(A) \in R_2$ or R_4 then $\frac{\sqrt{2}}{2} |\mu(A)| \leq |\operatorname{Im}(\mu(A))|$ and $\operatorname{Re}(z)$ has the same sign for z in R_1 and R_3 while $\operatorname{Im}(z)$ has the same sign for z in R_2 or R_4 . Then by 16.2.6, it follows that for some i ,

$$\sum_{F \in \pi_i} |\mu(F)| > 5(1 + |\mu(E)|). \quad (16.2.7)$$

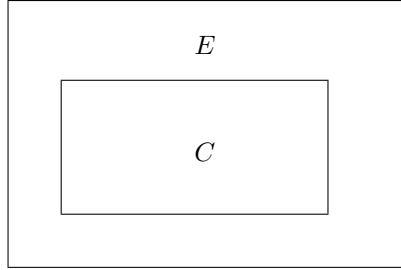
Suppose i equals 1 or 3. A similar argument using the imaginary part applies if i equals 2 or 4. Then,

$$\begin{aligned} \left| \sum_{F \in \pi_i} \mu(F) \right| &\geq \left| \sum_{F \in \pi_i} \operatorname{Re}(\mu(F)) \right| = \sum_{F \in \pi_i} |\operatorname{Re}(\mu(F))| \\ &\geq \frac{\sqrt{2}}{2} \sum_{F \in \pi_i} |\mu(F)| > 5 \frac{\sqrt{2}}{2} (1 + |\mu(E)|). \end{aligned}$$

Now letting C be the union of the sets in π_i ,

$$|\mu(C)| = \left| \sum_{F \in \pi_i} \mu(F) \right| > \frac{5}{2} (1 + |\mu(E)|) > 1. \quad (16.2.8)$$

Define $D \equiv E \setminus C$.



Then $\mu(C) + \mu(E \setminus C) = \mu(E)$ and so

$$\begin{aligned} \frac{5}{2}(1 + |\mu(E)|) &< |\mu(C)| = |\mu(E) - \mu(E \setminus C)| \\ &= |\mu(E) - \mu(D)| \leq |\mu(E)| + |\mu(D)| \end{aligned}$$

and so

$$1 < \frac{5}{2} + \frac{3}{2}|\mu(E)| < |\mu(D)|.$$

Now since $|\mu|(E) = \infty$, it follows from Theorem 16.2.4 that $\infty = |\mu|(E) \leq |\mu|(C) + |\mu|(D)$ and so either $|\mu|(C) = \infty$ or $|\mu|(D) = \infty$. If $|\mu|(C) = \infty$, let $B = C$ and $A = D$. Otherwise, let $B = D$ and $A = C$. This proves the claim.

Now suppose $|\mu|(\Omega) = \infty$. Then from the claim, there exist A_1 and B_1 such that $|\mu|(B_1) = \infty$, $|\mu(B_1)|, |\mu(A_1)| > 1$, and $A_1 \cup B_1 = \Omega$. Let $B_1 \equiv \Omega \setminus A$ play the same role as Ω and obtain $A_2, B_2 \subseteq B_1$ such that $|\mu|(B_2) = \infty$, $|\mu(B_2)|, |\mu(A_2)| > 1$, and $A_2 \cup B_2 = B_1$. Continue in this way to obtain a sequence of disjoint sets, $\{A_i\}$ such that $|\mu(A_i)| > 1$. Then since μ is a measure,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

but this is impossible because $\lim_{i \rightarrow \infty} \mu(A_i) \neq 0$. This proves the theorem.

Theorem 16.2.5 *Let $(\Omega, \mathcal{S})$ be a measure space and let $\lambda : \mathcal{S} \rightarrow \mathbb{C}$ be a complex vector measure. Thus $|\lambda|(\Omega) < \infty$. Let $\mu : \mathcal{S} \rightarrow [0, \mu(\Omega)]$ be a finite measure such that $\lambda \ll \mu$. Then there exists a unique $f \in L^1(\Omega)$ such that for all $E \in \mathcal{S}$,*

$$\int_E f d\mu = \lambda(E).$$

Proof: It is clear that $\operatorname{Re} \lambda$ and $\operatorname{Im} \lambda$ are real-valued vector measures on $\mathcal{S}$. Since $|\lambda|(\Omega) < \infty$, it follows easily that $|\operatorname{Re} \lambda|(\Omega)$ and $|\operatorname{Im} \lambda|(\Omega) < \infty$. This is clear because

$$|\lambda(E)| \geq |\operatorname{Re} \lambda(E)|, |\operatorname{Im} \lambda(E)|.$$

Therefore, each of

$$\frac{|\operatorname{Re} \lambda| + \operatorname{Re} \lambda}{2}, \frac{|\operatorname{Re} \lambda| - \operatorname{Re} \lambda}{2}, \frac{|\operatorname{Im} \lambda| + \operatorname{Im} \lambda}{2}, \text{ and } \frac{|\operatorname{Im} \lambda| - \operatorname{Im} \lambda}{2}$$

are finite measures on $\mathcal{S}$. It is also clear that each of these finite measures are absolutely continuous with respect to μ and so there exist unique nonnegative functions in $L^1(\Omega)$, f_1, f_2, g_1, g_2 such that for all $E \in \mathcal{S}$,

$$\begin{aligned} \frac{1}{2}(|\operatorname{Re} \lambda| + \operatorname{Re} \lambda)(E) &= \int_E f_1 d\mu, \\ \frac{1}{2}(|\operatorname{Re} \lambda| - \operatorname{Re} \lambda)(E) &= \int_E f_2 d\mu, \\ \frac{1}{2}(|\operatorname{Im} \lambda| + \operatorname{Im} \lambda)(E) &= \int_E g_1 d\mu, \\ \frac{1}{2}(|\operatorname{Im} \lambda| - \operatorname{Im} \lambda)(E) &= \int_E g_2 d\mu. \end{aligned}$$

Now let $f = f_1 - f_2 + i(g_1 - g_2)$.

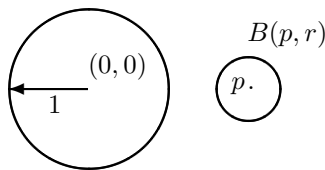
The following corollary is about representing a vector measure in terms of its total variation. It is like representing a complex number in the form $re^{i\theta}$. The proof requires the following lemma.

Lemma 16.2.6 *Suppose $(\Omega, \mathcal{S}, \mu)$ is a measure space and f is a function in $L^1(\Omega, \mu)$ with the property that*

$$\left| \int_E f d\mu \right| \leq \mu(E)$$

for all $E \in \mathcal{S}$. Then $|f| \leq 1$ a.e.

Proof of the lemma: Consider the following picture.



where $B(p, r) \cap B(0, 1) = \emptyset$. Let $E = f^{-1}(B(p, r))$. In fact $\mu(E) = 0$. If $\mu(E) \neq 0$ then

$$\begin{aligned} \left| \frac{1}{\mu(E)} \int_E f d\mu - p \right| &= \left| \frac{1}{\mu(E)} \int_E (f - p) d\mu \right| \\ &\leq \frac{1}{\mu(E)} \int_E |f - p| d\mu < r \end{aligned}$$

because on E , $|f(x) - p| < r$. Hence

$$\left| \frac{1}{\mu(E)} \int_E f d\mu \right| > 1$$

because it is closer to p than r . (Refer to the picture.) However, this contradicts the assumption of the lemma. It follows $\mu(E) = 0$. Since the set of complex numbers, z such that $|z| > 1$ is an open set, it equals the union of countably many balls, $\{B_i\}_{i=1}^\infty$. Therefore,

$$\begin{aligned} \mu(f^{-1}(\{z \in \mathbb{C} : |z| > 1\})) &= \mu(\cup_{k=1}^\infty f^{-1}(B_k)) \\ &\leq \sum_{k=1}^\infty \mu(f^{-1}(B_k)) = 0. \end{aligned}$$

Thus $|f(x)| \leq 1$ a.e. as claimed. This proves the lemma.

Corollary 16.2.7 *Let λ be a complex vector measure with $|\lambda|(\Omega) < \infty^1$. Then there exists a unique $f \in L^1(\Omega)$ such that $\lambda(E) = \int_E f d|\lambda|$. Furthermore, $|f| = 1$ for $|\lambda|$ a.e. This is called the polar decomposition of λ .*

Proof: First note that $\lambda \ll |\lambda|$ and so such an L^1 function exists and is unique. It is required to show $|f| = 1$ a.e. If $|\lambda|(E) \neq 0$,

$$\left| \frac{\lambda(E)}{|\lambda|(E)} \right| = \left| \frac{1}{|\lambda|(E)} \int_E f d|\lambda| \right| \leq 1.$$

Therefore by Lemma 16.2.6, $|f| \leq 1$, $|\lambda|$ a.e. Now let

$$E_n = \left[|f| \leq 1 - \frac{1}{n} \right].$$

Let $\{F_1, \dots, F_m\}$ be a partition of E_n . Then

$$\begin{aligned} \sum_{i=1}^m |\lambda(F_i)| &= \sum_{i=1}^m \left| \int_{F_i} f d|\lambda| \right| \leq \sum_{i=1}^m \int_{F_i} |f| d|\lambda| \\ &\leq \sum_{i=1}^m \int_{F_i} \left(1 - \frac{1}{n} \right) d|\lambda| = \sum_{i=1}^m \left(1 - \frac{1}{n} \right) |\lambda|(F_i) \\ &= |\lambda|(E_n) \left(1 - \frac{1}{n} \right). \end{aligned}$$

Then taking the supremum over all partitions,

$$|\lambda|(E_n) \leq \left(1 - \frac{1}{n} \right) |\lambda|(E_n)$$

which shows $|\lambda|(E_n) = 0$. Hence $|\lambda|([|f| < 1]) = 0$ because $[|f| < 1] = \cup_{n=1}^\infty E_n$. This proves Corollary 16.2.7.

Corollary 16.2.8 *Let λ be a complex vector measure such that $\lambda \ll \mu$ where μ is σ finite. Then there exists a unique $g \in L^1(\Omega, \mu)$ such that $\lambda(E) = \int_E g d\mu$.*

¹As proved above, the assumption that $|\lambda|(\Omega) < \infty$ is redundant.

Proof: By Corollary 16.2.7 and Theorem 16.2.4 which says that $|\lambda|$ is finite, there exists a unique f such that $|f| = 1$ $|\lambda|$ a.e. and

$$\lambda(E) = \int_E f d|\lambda|.$$

Now $|\lambda| \ll \mu$ and so it follows from Corollary 16.1.3 there exists a unique nonnegative measurable function h such that for all E measurable,

$$|\lambda|(E) = \int_E h d\mu$$

where since $|\lambda|$ is finite, $h \in L^1(\Omega, \mu)$. It follows from approximating f with simple functions and using the above formula that

$$\lambda(E) = \int_E f h d\mu.$$

Then let $g = L^1(\Omega, \mu)$. This proves the corollary.

Corollary 16.2.9 Suppose $(\Omega, \mathcal{S})$ is a measure space and μ is a finite nonnegative measure on $\mathcal{S}$. Then for $h \in L^1(\mu)$, define a complex measure, λ by

$$\lambda(E) \equiv \int_E h d\mu.$$

Then

$$|\lambda|(E) = \int_E |h| d\mu.$$

Furthermore, $|h| = \bar{g}h$ where $gd|\lambda|$ is the polar decomposition of λ ,

$$\lambda(E) = \int_E g d|\lambda|$$

Proof: From Corollary 16.2.7 there exists g such that $|g| = 1$, $|\lambda|$ a.e. and for all $E \in \mathcal{S}$

$$\lambda(E) = \int_E g d|\lambda| = \int_E h d\mu.$$

Let s_n be a sequence of simple functions converging pointwise to $\bar{g}$. Then from the above,

$$\int_E g s_n d|\lambda| = \int_E s_n h d\mu.$$

Passing to the limit using the dominated convergence theorem,

$$\int_E d|\lambda| = \int_E \bar{g} h d\mu.$$

It follows $\bar{g}h \geq 0$ a.e. and $|\bar{g}| = 1$. Therefore, $|h| = |\bar{g}h| = \bar{g}h$. It follows from the above, that

$$|\lambda|(E) = \int_E d|\lambda| = \int_E \bar{g} h d\mu = \int_E d|\lambda| = \int_E |h| d\mu$$

and this proves the corollary.

16.3 Representation Theorems For The Dual Space Of L^p

Recall the concept of the dual space of a Banach space in the Chapter on Banach space starting on Page 361. The next topic deals with the dual space of L^p for $p \geq 1$ in the case where the measure space is σ finite or finite. In what follows $q = \infty$ if $p = 1$ and otherwise, $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 16.3.1 (*Riesz representation theorem*) *Let $p > 1$ and let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. If $\Lambda \in (L^p(\Omega))'$, then there exists a unique $h \in L^q(\Omega)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that*

$$\Lambda f = \int_{\Omega} h f d\mu.$$

This function satisfies $\|h\|_q = \|\Lambda\|$ where $\|\Lambda\|$ is the operator norm of Λ .

Proof: (Uniqueness) If h_1 and h_2 both represent Λ , consider

$$f = |h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}),$$

where $\overline{h}$ denotes complex conjugation. By Holder's inequality, it is easy to see that $f \in L^p(\Omega)$. Thus

$$\begin{aligned} 0 &= \Lambda f - \Lambda f = \\ &= \int h_1 |h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}) - h_2 |h_1 - h_2|^{q-2}(\overline{h_1} - \overline{h_2}) d\mu \\ &= \int |h_1 - h_2|^q d\mu. \end{aligned}$$

Therefore $h_1 = h_2$ and this proves uniqueness.

Now let $\lambda(E) = \Lambda(\mathcal{X}_E)$. Since this is a finite measure space $\mathcal{X}_E$ is an element of $L^p(\Omega)$ and so it makes sense to write $\Lambda(\mathcal{X}_E)$. In fact λ is a complex measure having finite total variation. Let $A_1, \dots, A_n$ be a partition of Ω .

$$|\Lambda \mathcal{X}_{A_i}| = w_i(\Lambda \mathcal{X}_{A_i}) = \Lambda(w_i \mathcal{X}_{A_i})$$

for some $w_i \in \mathbb{C}$, $|w_i| = 1$. Thus

$$\begin{aligned} \sum_{i=1}^n |\lambda(A_i)| &= \sum_{i=1}^n |\Lambda(\mathcal{X}_{A_i})| = \Lambda\left(\sum_{i=1}^n w_i \mathcal{X}_{A_i}\right) \\ &\leq \|\Lambda\| \left(\int \left|\sum_{i=1}^n w_i \mathcal{X}_{A_i}\right|^p d\mu\right)^{\frac{1}{p}} = \|\Lambda\| \left(\int_{\Omega} d\mu\right)^{\frac{1}{p}} = \|\Lambda\| \mu(\Omega)^{\frac{1}{p}}. \end{aligned}$$

This is because if $x \in \Omega$, x is contained in exactly one of the A_i and so the absolute value of the sum in the first integral above is equal to 1. Therefore $|\lambda|(\Omega) < \infty$

because this was an arbitrary partition. Also, if $\{E_i\}_{i=1}^\infty$ is a sequence of disjoint sets of $\mathcal{S}$, let

$$F_n = \cup_{i=1}^n E_i, \quad F = \cup_{i=1}^\infty E_i.$$

Then by the Dominated Convergence theorem,

$$\|\mathcal{X}_{F_n} - \mathcal{X}_F\|_p \rightarrow 0.$$

Therefore, by continuity of Λ ,

$$\lambda(F) = \Lambda(\mathcal{X}_F) = \lim_{n \rightarrow \infty} \Lambda(\mathcal{X}_{F_n}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \Lambda(\mathcal{X}_{E_k}) = \sum_{k=1}^\infty \lambda(E_k).$$

This shows λ is a complex measure with $|\lambda|$ finite.

It is also clear from the definition of λ that $\lambda \ll \mu$. Therefore, by the Radon Nikodym theorem, there exists $h \in L^1(\Omega)$ with

$$\lambda(E) = \int_E h d\mu = \Lambda(\mathcal{X}_E).$$

Actually $h \in L^q$ and satisfies the other conditions above. Let $s = \sum_{i=1}^m c_i \mathcal{X}_{E_i}$ be a simple function. Then since Λ is linear,

$$\Lambda(s) = \sum_{i=1}^m c_i \Lambda(\mathcal{X}_{E_i}) = \sum_{i=1}^m c_i \int_{E_i} h d\mu = \int h s d\mu. \quad (16.3.9)$$

Claim: If f is uniformly bounded and measurable, then

$$\Lambda(f) = \int h f d\mu.$$

Proof of claim: Since f is bounded and measurable, there exists a sequence of simple functions, $\{s_n\}$ which converges to f pointwise and in $L^p(\Omega)$. This follows from Theorem 8.3.9 on Page 197 upon breaking f up into positive and negative parts of real and complex parts. In fact this theorem gives uniform convergence. Then

$$\Lambda(f) = \lim_{n \rightarrow \infty} \Lambda(s_n) = \lim_{n \rightarrow \infty} \int h s_n d\mu = \int h f d\mu,$$

the first equality holding because of continuity of Λ , the second following from 16.3.9 and the third holding by the dominated convergence theorem.

This is a very nice formula but it still has not been shown that $h \in L^q(\Omega)$.

Let $E_n = \{x : |h(x)| \leq n\}$. Thus $|h \mathcal{X}_{E_n}| \leq n$. Then

$$|h \mathcal{X}_{E_n}|^{q-2} (\bar{h} \mathcal{X}_{E_n}) \in L^p(\Omega).$$

By the claim, it follows that

$$\|h \mathcal{X}_{E_n}\|_q^q = \int h |h \mathcal{X}_{E_n}|^{q-2} (\bar{h} \mathcal{X}_{E_n}) d\mu = \Lambda(|h \mathcal{X}_{E_n}|^{q-2} (\bar{h} \mathcal{X}_{E_n}))$$

$$\leq \|\Lambda\| \left\| |h\mathcal{X}_{E_n}|^{q-2}(\bar{h}\mathcal{X}_{E_n}) \right\|_p = \|\Lambda\| \|h\mathcal{X}_{E_n}\|_{\frac{q}{q-1}}^{\frac{q}{q-1}},$$

the last equality holding because $q-1 = q/p$ and so

$$\begin{aligned} \left(\int |h\mathcal{X}_{E_n}|^{q-2}(\bar{h}\mathcal{X}_{E_n})^p d\mu \right)^{1/p} &= \left(\int (|h\mathcal{X}_{E_n}|^{q/p})^p d\mu \right)^{1/p} \\ &= \|h\mathcal{X}_{E_n}\|_{\frac{q}{p}}^{\frac{q}{p}} \end{aligned}$$

Therefore, since $q - \frac{q}{p} = 1$, it follows that

$$\|h\mathcal{X}_{E_n}\|_q \leq \|\Lambda\|.$$

Letting $n \rightarrow \infty$, the Monotone Convergence theorem implies

$$\|h\|_q \leq \|\Lambda\|. \quad (16.3.10)$$

Now that h has been shown to be in $L^q(\Omega)$, it follows from 16.3.9 and the density of the simple functions, Theorem 12.2.1 on Page 345, that

$$\Lambda f = \int h f d\mu$$

for all $f \in L^p(\Omega)$.

It only remains to verify the last claim.

$$\|\Lambda\| = \sup \left\{ \int h f : \|f\|_p \leq 1 \right\} \leq \|h\|_q \leq \|\Lambda\|$$

by 16.3.10, and Holder's inequality. This proves the theorem.

To represent elements of the dual space of $L^1(\Omega)$, another Banach space is needed.

Definition 16.3.2 Let $(\Omega, \mathcal{S}, \mu)$ be a measure space. $L^\infty(\Omega)$ is the vector space of measurable functions such that for some $M > 0$, $|f(x)| \leq M$ for all x outside of some set of measure zero ($|f(x)| \leq M$ a.e.). Define $f = g$ when $f(x) = g(x)$ a.e. and $\|f\|_\infty \equiv \inf \{M : |f(x)| \leq M \text{ a.e.}\}.$

Theorem 16.3.3 $L^\infty(\Omega)$ is a Banach space.

Proof: It is clear that $L^\infty(\Omega)$ is a vector space. Is $\|\cdot\|_\infty$ a norm?

Claim: If $f \in L^\infty(\Omega)$, then $|f(x)| \leq \|f\|_\infty$ a.e.

Proof of the claim: $\{x : |f(x)| \geq \|f\|_\infty + n^{-1}\} \equiv E_n$ is a set of measure zero according to the definition of $\|f\|_\infty$. Furthermore, $\{x : |f(x)| > \|f\|_\infty\} = \cup_n E_n$ and so it is also a set of measure zero. This verifies the claim.

Now if $\|f\|_\infty = 0$ it follows that $f(x) = 0$ a.e. Also if $f, g \in L^\infty(\Omega)$,

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\|_\infty + \|g\|_\infty$$

a.e. and so $\|f\|_\infty + \|g\|_\infty$ serves as one of the constants, M in the definition of $\|f + g\|_\infty$. Therefore,

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

Next let c be a number. Then $|cf(x)| = |c||f(x)| \leq |c|\|f\|_\infty$ and so $\|cf\|_\infty \leq |c|\|f\|_\infty$. Therefore since c is arbitrary, $\|f\|_\infty = \|c(1/c)f\|_\infty \leq |\frac{1}{c}|\|cf\|_\infty$ which implies $|c|\|f\|_\infty \leq \|cf\|_\infty$. Thus $\|\cdot\|_\infty$ is a norm as claimed.

To verify completeness, let $\{f_n\}$ be a Cauchy sequence in $L^\infty(\Omega)$ and use the above claim to get the existence of a set of measure zero, E_{nm} such that for all $x \notin E_{nm}$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$$

Let $E = \cup_{n,m} E_{nm}$. Thus $\mu(E) = 0$ and for each $x \notin E$, $\{f_n(x)\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$. Let

$$f(x) = \begin{cases} 0 & \text{if } x \in E \\ \lim_{n \rightarrow \infty} f_n(x) & \text{if } x \notin E \end{cases} = \lim_{n \rightarrow \infty} \chi_{E^c}(x) f_n(x).$$

Then f is clearly measurable because it is the limit of measurable functions. If

$$F_n = \{x : |f_n(x)| > \|f_n\|_\infty\}$$

and $F = \cup_{n=1}^\infty F_n$, it follows $\mu(F) = 0$ and that for $x \notin F \cup E$,

$$|f(x)| \leq \liminf_{n \rightarrow \infty} |f_n(x)| \leq \liminf_{n \rightarrow \infty} \|f_n\|_\infty < \infty$$

because $\{\|f_n\|_\infty\}$ is a Cauchy sequence. ($|\|f_n\|_\infty - \|f_m\|_\infty| \leq \|f_n - f_m\|_\infty$ by the triangle inequality.) Thus $f \in L^\infty(\Omega)$. Let n be large enough that whenever $m > n$,

$$\|f_m - f_n\|_\infty < \varepsilon.$$

Then, if $x \notin E$,

$$\begin{aligned} |f(x) - f_n(x)| &= \lim_{m \rightarrow \infty} |f_m(x) - f_n(x)| \\ &\leq \lim_{m \rightarrow \infty} \inf \|f_m - f_n\|_\infty < \varepsilon. \end{aligned}$$

Hence $\|f - f_n\|_\infty < \varepsilon$ for all n large enough. This proves the theorem.

The next theorem is the Riesz representation theorem for $(L^1(\Omega))'$.

Theorem 16.3.4 (*Riesz representation theorem*) Let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. If $\Lambda \in (L^1(\Omega))'$, then there exists a unique $h \in L^\infty(\Omega)$ such that

$$\Lambda(f) = \int_\Omega h f \, d\mu$$

for all $f \in L^1(\Omega)$. If h is the function in $L^\infty(\Omega)$ representing $\Lambda \in (L^1(\Omega))'$, then $\|h\|_\infty = \|\Lambda\|$.

Proof: Just as in the proof of Theorem 16.3.1, there exists a unique $h \in L^1(\Omega)$ such that for all simple functions, s ,

$$\Lambda(s) = \int hs \, d\mu. \quad (16.3.11)$$

To show $h \in L^\infty(\Omega)$, let $\varepsilon > 0$ be given and let

$$E = \{x : |h(x)| \geq \|\Lambda\| + \varepsilon\}.$$

Let $|k| = 1$ and $hk = |h|$. Since the measure space is finite, $k \in L^1(\Omega)$. As in Theorem 16.3.1 let $\{s_n\}$ be a sequence of simple functions converging to k in $L^1(\Omega)$, and pointwise. It follows from the construction in Theorem 8.3.9 on Page 197 that it can be assumed $|s_n| \leq 1$. Therefore

$$\Lambda(k\mathcal{X}_E) = \lim_{n \rightarrow \infty} \Lambda(s_n\mathcal{X}_E) = \lim_{n \rightarrow \infty} \int_E hs_n \, d\mu = \int_E hk \, d\mu$$

where the last equality holds by the Dominated Convergence theorem. Therefore,

$$\begin{aligned} \|\Lambda\|\mu(E) &\geq |\Lambda(k\mathcal{X}_E)| = \left| \int_E hk\mathcal{X}_E \, d\mu \right| = \int_E |h| \, d\mu \\ &\geq (\|\Lambda\| + \varepsilon)\mu(E). \end{aligned}$$

It follows that $\mu(E) = 0$. Since $\varepsilon > 0$ was arbitrary, $\|\Lambda\| \geq \|h\|_\infty$. It was shown that $h \in L^\infty(\Omega)$, the density of the simple functions in $L^1(\Omega)$ and 16.3.11 imply

$$\Lambda f = \int hf \, d\mu, \quad \|\Lambda\| \geq \|h\|_\infty. \quad (16.3.12)$$

This proves the existence part of the theorem. To verify uniqueness, suppose h_1 and h_2 both represent Λ and let $f \in L^1(\Omega)$ be such that $|f| \leq 1$ and $f(h_1 - h_2) = |h_1 - h_2|$. Then

$$0 = \Lambda f - \Lambda f = \int (h_1 - h_2)f \, d\mu = \int |h_1 - h_2| \, d\mu.$$

Thus $h_1 = h_2$. Finally,

$$\|\Lambda\| = \sup\left\{ \left| \int hf \, d\mu \right| : \|f\|_1 \leq 1 \right\} \leq \|h\|_\infty \leq \|\Lambda\|$$

by 16.3.12.

Next these results are extended to the σ finite case.

Lemma 16.3.5 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space and suppose there exists a measurable function, r such that $r(x) > 0$ for all x , there exists M such that $|r(x)| < M$ for all x , and $\int r \, d\mu < \infty$. Then for*

$$\Lambda \in (L^p(\Omega, \mu))', \quad p \geq 1,$$

there exists a unique $h \in L^{p'}(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int h f d\mu.$$

Also $\|h\| = \|\Lambda\|$. ($\|h\| = \|h\|_{p'}$ if $p > 1$, $\|h\|_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proof: Define a new measure $\tilde{\mu}$, according to the rule

$$\tilde{\mu}(E) \equiv \int_E r d\mu. \quad (16.3.13)$$

Thus $\tilde{\mu}$ is a finite measure on $\mathcal{S}$. Now define a mapping, $\eta : L^p(\Omega, \mu) \rightarrow L^p(\Omega, \tilde{\mu})$ by

$$\eta f = r^{-\frac{1}{p}} f.$$

Then

$$\|\eta f\|_{L^p(\tilde{\mu})}^p = \int \left| r^{-\frac{1}{p}} f \right|^p r d\mu = \|f\|_{L^p(\mu)}^p$$

and so η is one to one and in fact preserves norms. I claim that also η is onto. To see this, let $g \in L^p(\Omega, \tilde{\mu})$ and consider the function, $r^{\frac{1}{p}} g$. Then

$$\int \left| r^{\frac{1}{p}} g \right|^p d\mu = \int |g|^p r d\mu = \int |g|^p d\tilde{\mu} < \infty$$

Thus $r^{\frac{1}{p}} g \in L^p(\Omega, \mu)$ and $\eta \left(r^{\frac{1}{p}} g \right) = g$ showing that η is onto as claimed. Thus η is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which η^* must be one to one and onto.

$$\begin{array}{ccccc} h, L^{p'}(\tilde{\mu}) & L^p(\tilde{\mu})', \tilde{\Lambda} & \xrightarrow{\eta^*} & L^p(\mu)', \Lambda \\ & & \eta & & \\ & L^p(\tilde{\mu}) & \xleftarrow{\quad} & L^p(\mu) \end{array}$$

Then for $\Lambda \in L^p(\mu)'$, there exists a unique $\tilde{\Lambda} \in L^p(\tilde{\mu})'$ such that $\eta^* \tilde{\Lambda} = \Lambda$, $\|\tilde{\Lambda}\| = \|\Lambda\|$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in L^{p'}(\tilde{\mu})$ which represents $\tilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $\|h\|_{L^{p'}(\tilde{\mu})} = \|\tilde{\Lambda}\| = \|\Lambda\|$ and for all $f \in L^p(\mu)$,

$$\begin{aligned} \Lambda(f) &= \eta^* \tilde{\Lambda}(f) \equiv \tilde{\Lambda}(\eta f) = \int h(\eta f) d\tilde{\mu} = \int r h \left(f^{-\frac{1}{p}} f \right) d\mu \\ &= \int r^{\frac{1}{p'}} h f d\mu. \end{aligned}$$

Now

$$\int \left| r^{\frac{1}{p'}} h \right|^{p'} d\mu = \int |h|^{p'} r d\mu = \|h\|_{L^{p'}(\tilde{\mu})}^{p'} < \infty.$$

Thus $\left\| r^{\frac{1}{p'}} h \right\|_{L^{p'}(\mu)} = \|h\|_{L^{p'}(\tilde{\mu})} = \|\tilde{\Lambda}\| = \|\Lambda\|$ and represents Λ in the appropriate way. If $p = 1$, then $1/p' \equiv 0$. This proves the Lemma.

A situation in which the conditions of the lemma are satisfied is the case where the measure space is σ finite. In fact, you should show this is the only case in which the conditions of the above lemma hold.

Theorem 16.3.6 (*Riesz representation theorem*) Let $(\Omega, \mathcal{S}, \mu)$ be σ finite and let

$$\Lambda \in (L^p(\Omega, \mu))', \quad p \geq 1.$$

Then there exists a unique $h \in L^q(\Omega, \mu)$, $L^\infty(\Omega, \mu)$ if $p = 1$ such that

$$\Lambda f = \int h f d\mu.$$

Also $\|h\| = \|\Lambda\|$. ($\|h\| = \|h\|_q$ if $p > 1$, $\|h\|_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proof: Let $\{\Omega_n\}$ be a sequence of disjoint elements of $\mathcal{S}$ having the property that

$$0 < \mu(\Omega_n) < \infty, \quad \cup_{n=1}^\infty \Omega_n = \Omega.$$

Define

$$r(x) = \sum_{n=1}^\infty \frac{1}{n^2} \chi_{\Omega_n}(x) \mu(\Omega_n)^{-1}, \quad \tilde{\mu}(E) = \int_E r d\mu.$$

Thus

$$\int_\Omega r d\mu = \tilde{\mu}(\Omega) = \sum_{n=1}^\infty \frac{1}{n^2} < \infty$$

so $\tilde{\mu}$ is a finite measure. The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

With the Riesz representation theorem, it is easy to show that

$$L^p(\Omega), \quad p > 1$$

is a reflexive Banach space. Recall Definition 13.2.14 on Page 378 for the definition.

Theorem 16.3.7 For $(\Omega, \mathcal{S}, \mu)$ a σ finite measure space and $p > 1$, $L^p(\Omega)$ is reflexive.

Proof: Let $\delta_r : (L^r(\Omega))' \rightarrow L^{r'}(\Omega)$ be defined for $\frac{1}{r} + \frac{1}{r'} = 1$ by

$$\int (\delta_r \Lambda) g \, d\mu = \Lambda g$$

for all $g \in L^r(\Omega)$. From Theorem 16.3.6 δ_r is one to one, onto, continuous and linear. By the open map theorem, δ_r^{-1} is also one to one, onto, and continuous ($\delta_r \Lambda$ equals the representer of Λ). Thus δ_r^* is also one to one, onto, and continuous by Corollary 13.2.11. Now observe that $J = \delta_p^* \circ \delta_q^{-1}$. To see this, let $z^* \in (L^q)'$, $y^* \in (L^p)'$,

$$\begin{aligned} \delta_p^* \circ \delta_q^{-1}(\delta_q z^*)(y^*) &= (\delta_p^* z^*)(y^*) \\ &= z^*(\delta_p y^*) \\ &= \int (\delta_q z^*)(\delta_p y^*) d\mu, \end{aligned}$$

$$\begin{aligned} J(\delta_q z^*)(y^*) &= y^*(\delta_q z^*) \\ &= \int (\delta_p y^*)(\delta_q z^*) d\mu. \end{aligned}$$

Therefore $\delta_p^* \circ \delta_q^{-1} = J$ on $\delta_q(L^q)' = L^p$. But the two δ maps are onto and so J is also onto.

16.4 The Dual Space Of $C(X)$

Consider the dual space of $C(X)$ where X is a compact Hausdorff space. It will turn out to be a space of measures. To show this, the following lemma will be convenient.

Lemma 16.4.1 *Suppose λ is a mapping which is defined on the positive continuous functions defined on X , some topological space which satisfies*

$$\lambda(af + bg) = a\lambda(f) + b\lambda(g) \quad (16.4.14)$$

whenever $a, b \geq 0$ and $f, g \geq 0$. Then there exists a unique extension of λ to all of $C(X)$, Λ such that whenever $f, g \in C(X)$ and $a, b \in \mathbb{C}$, it follows

$$\Lambda(af + bg) = a\Lambda(f) + b\Lambda(g).$$

Proof: Let $C(X; \mathbb{R})$ be the real-valued functions in $C(X)$ and define

$$\Lambda_R(f) = \lambda f^+ - \lambda f^-$$

for $f \in C(X; \mathbb{R})$. Use the identity

$$(f_1 + f_2)^+ + f_1^- + f_2^- = f_1^+ + f_2^+ + (f_1 + f_2)^-$$

and 16.4.14 to write

$$\lambda(f_1 + f_2)^+ - \lambda(f_1 + f_2)^- = \lambda f_1^+ - \lambda f_1^- + \lambda f_2^+ - \lambda f_2^-,$$

it follows that $\Lambda_R(f_1 + f_2) = \Lambda_R(f_1) + \Lambda_R(f_2)$. To show that Λ_R is linear, it is necessary to verify that $\Lambda_R(cf) = c\Lambda_R(f)$ for all $c \in \mathbb{R}$. But

$$(cf)^\pm = cf^\pm,$$

if $c \geq 0$ while

$$(cf)^+ = -c(f)^-,$$

if $c < 0$ and

$$(cf)^- = (-c)f^+,$$

if $c < 0$. Thus, if $c < 0$,

$$\begin{aligned} \Lambda_R(cf) &= \lambda(cf)^+ - \lambda(cf)^- = \lambda((-c)f^-) - \lambda((-c)f^+) \\ &= -c\lambda(f^-) + c\lambda(f^+) = c(\lambda(f^+) - \lambda(f^-)) = c\Lambda_R(f). \end{aligned}$$

A similar formula holds more easily if $c \geq 0$. Now let

$$\Lambda f = \Lambda_R(\operatorname{Re} f) + i\Lambda_R(\operatorname{Im} f)$$

for arbitrary $f \in C(X)$. This is linear as desired. It is obvious that $\Lambda(f + g) = \Lambda(f) + \Lambda(g)$ from the fact that taking the real and imaginary parts are linear operations. The only thing to check is whether you can factor out a complex scalar.

$$\Lambda((a + ib)f) = \Lambda(af) + \Lambda(ibf)$$

$$\equiv \Lambda_R(a \operatorname{Re} f) + i\Lambda_R(a \operatorname{Im} f) + \Lambda_R(-b \operatorname{Im} f) + i\Lambda_R(b \operatorname{Re} f)$$

because $ibf = ib \operatorname{Re} f - b \operatorname{Im} f$ and so $\operatorname{Re}(ibf) = -b \operatorname{Im} f$ and $\operatorname{Im}(ibf) = b \operatorname{Re} f$. Therefore, the above equals

$$\begin{aligned} &= (a + ib)\Lambda_R(\operatorname{Re} f) + i(a + ib)\Lambda_R(\operatorname{Im} f) \\ &= (a + ib)(\Lambda_R(\operatorname{Re} f) + i\Lambda_R(\operatorname{Im} f)) = (a + ib)\Lambda f \end{aligned}$$

The extension is obviously unique. This proves the lemma.

Let $L \in C(X)'$. Also denote by $C^+(X)$ the set of nonnegative continuous functions defined on X . Define for $f \in C^+(X)$

$$\lambda(f) = \sup\{|Lg| : |g| \leq f\}.$$

Note that $\lambda(f) < \infty$ because $|Lg| \leq \|L\| \|g\| \leq \|L\| \|f\|$ for $|g| \leq f$. Then the following lemma is important.

Lemma 16.4.2 *If $c \geq 0$, $\lambda(cf) = c\lambda(f)$, $f_1 \leq f_2$ implies $\lambda f_1 \leq \lambda f_2$, and*

$$\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2).$$

Proof: The first two assertions are easy to see so consider the third. Let $|g_j| \leq f_j$ and let $\tilde{g}_j = e^{i\theta_j} g_j$ where θ_j is chosen such that $e^{i\theta_j} Lg_j = |Lg_j|$. Thus $L\tilde{g}_j = |Lg_j|$. Then

$$|\tilde{g}_1 + \tilde{g}_2| \leq f_1 + f_2.$$

Hence

$$\begin{aligned} |Lg_1| + |Lg_2| &= L\tilde{g}_1 + L\tilde{g}_2 = \\ L(\tilde{g}_1 + \tilde{g}_2) &= |L(\tilde{g}_1 + \tilde{g}_2)| \leq \lambda(f_1 + f_2). \end{aligned} \quad (16.4.15)$$

Choose g_1 and g_2 such that $|Lg_i| + \varepsilon > \lambda(f_i)$. Then 16.4.15 shows

$$\lambda(f_1) + \lambda(f_2) - 2\varepsilon \leq \lambda(f_1 + f_2).$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2). \quad (16.4.16)$$

Now let $|g| \leq f_1 + f_2$, $|Lg| \geq \lambda(f_1 + f_2) - \varepsilon$. Let

$$h_i(x) = \begin{cases} \frac{f_i(x)g(x)}{f_1(x)+f_2(x)} & \text{if } f_1(x) + f_2(x) > 0, \\ 0 & \text{if } f_1(x) + f_2(x) = 0. \end{cases}$$

Then h_i is continuous and $h_1(x) + h_2(x) = g(x)$, $|h_i| \leq f_i$. Therefore,

$$\begin{aligned} -\varepsilon + \lambda(f_1 + f_2) &\leq |Lg| \leq |Lh_1 + Lh_2| \leq |Lh_1| + |Lh_2| \\ &\leq \lambda(f_1) + \lambda(f_2). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows with 16.4.16 that

$$\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2) \leq \lambda(f_1 + f_2)$$

which proves the lemma.

Let Λ be defined in Lemma 16.4.1. Then Λ is linear by this lemma. Also, if $f \geq 0$,

$$\Lambda f = \Lambda_R f = \lambda(f) \geq 0.$$

Therefore, Λ is a positive linear functional on $C(X)$ ($= C_c(X)$ since X is compact). By Theorem 9.4.2 on Page 234, there exists a unique Radon measure μ such that

$$\Lambda f = \int_X f \, d\mu$$

for all $f \in C(X)$. Thus $\Lambda(1) = \mu(X)$. What follows is the Riesz representation theorem for $C(X)'$.

Theorem 16.4.3 *Let $L \in (C(X))'$. Then there exists a Radon measure μ and a function $\sigma \in L^\infty(X, \mu)$ such that*

$$L(f) = \int_X f \, \sigma \, d\mu.$$

Proof: Let $f \in C(X)$. Then there exists a unique Radon measure μ such that

$$|Lf| \leq \Lambda(|f|) = \int_X |f| d\mu = \|f\|_1.$$

Since μ is a Radon measure, $C(X)$ is dense in $L^1(X, \mu)$. Therefore L extends uniquely to an element of $(L^1(X, \mu))'$. By the Riesz representation theorem for L^1 , there exists a unique $\sigma \in L^\infty(X, \mu)$ such that

$$Lf = \int_X f \sigma d\mu$$

for all $f \in C(X)$.

16.5 The Dual Space Of $C_0(X)$

It is possible to give a simple generalization of the above theorem. For X a locally compact Hausdorff space, $\tilde{X}$ denotes the one point compactification of X . Thus, $\tilde{X} = X \cup \{\infty\}$ and the topology of $\tilde{X}$ consists of the usual topology of X along with all complements of compact sets which are defined as the open sets containing ∞ . Also $C_0(X)$ will denote the space of continuous functions, f , defined on X such that in the topology of $\tilde{X}$, $\lim_{x \rightarrow \infty} f(x) = 0$. For this space of functions, $\|f\|_0 \equiv \sup \{|f(x)| : x \in X\}$ is a norm which makes this into a Banach space. Then the generalization is the following corollary.

Corollary 16.5.1 *Let $L \in (C_0(X))'$ where X is a locally compact Hausdorff space. Then there exists $\sigma \in L^\infty(X, \mu)$ for μ a finite Radon measure such that for all $f \in C_0(X)$,*

$$L(f) = \int_X f \sigma d\mu.$$

Proof: Let

$$\tilde{D} \equiv \left\{ f \in C(\tilde{X}) : f(\infty) = 0 \right\}.$$

Thus $\tilde{D}$ is a closed subspace of the Banach space $C(\tilde{X})$. Let $\theta : C_0(X) \rightarrow \tilde{D}$ be defined by

$$\theta f(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x = \infty. \end{cases}$$

Then θ is an isometry of $C_0(X)$ and $\tilde{D}$. ($\|\theta u\| = \|u\|$.) The following diagram is obtained.

$$\begin{array}{ccccc} C_0(X)' & \xleftarrow{\theta^*} & (\tilde{D})' & \xleftarrow{i^*} & C(\tilde{X})' \\ C_0(X) & \xrightarrow{\theta} & \tilde{D} & \xrightarrow{i} & C(\tilde{X}) \end{array}$$

By the Hahn Banach theorem, there exists $L_1 \in C(\tilde{X})'$ such that $\theta^* i^* L_1 = L$. Now apply Theorem 16.4.3 to get the existence of a finite Radon measure, μ_1 , on $\tilde{X}$ and a function $\sigma \in L^\infty(\tilde{X}, \mu_1)$, such that

$$L_1 g = \int_{\tilde{X}} g \sigma d\mu_1.$$

Letting the σ algebra of μ_1 measurable sets be denoted by $\mathcal{S}_1$, define

$$\mathcal{S} \equiv \{E \setminus \{\infty\} : E \in \mathcal{S}_1\}$$

and let μ be the restriction of μ_1 to $\mathcal{S}$. If $f \in C_0(X)$,

$$L f = \theta^* i^* L_1 f \equiv L_1 i \theta f = L_1 \theta f = \int_{\tilde{X}} \theta f \sigma d\mu_1 = \int_X f \sigma d\mu.$$

This proves the corollary.

16.6 More Attractive Formulations

In this section, Corollary 16.5.1 will be refined and placed in an arguably more attractive form. The measures involved will always be complex Borel measures defined on a σ algebra of subsets of X , a locally compact Hausdorff space.

Definition 16.6.1 *Let λ be a complex measure. Then $\int f d\lambda \equiv \int f h d|\lambda|$ where $h d|\lambda|$ is the polar decomposition of λ described above. The complex measure, λ is called regular if $|\lambda|$ is regular.*

The following lemma says that the difference of regular complex measures is also regular.

Lemma 16.6.2 *Suppose $\lambda_i, i = 1, 2$ is a complex Borel measure with total variation finite² defined on X , a locally compact Hausdorff space. Then $\lambda_1 - \lambda_2$ is also a regular measure on the Borel sets.*

Proof: Let E be a Borel set. That way it is in the σ algebras associated with both λ_i . Then by regularity of λ_i , there exist K and V compact and open respectively such that $K \subseteq E \subseteq V$ and $|\lambda_i|(V \setminus K) < \varepsilon/2$. Therefore,

$$\begin{aligned} \sum_{A \in \pi(V \setminus K)} |(\lambda_1 - \lambda_2)(A)| &= \sum_{A \in \pi(V \setminus K)} |\lambda_1(A) - \lambda_2(A)| \\ &\leq \sum_{A \in \pi(V \setminus K)} |\lambda_1(A)| + |\lambda_2(A)| \\ &\leq |\lambda_1|(V \setminus K) + |\lambda_2|(V \setminus K) < \varepsilon. \end{aligned}$$

Therefore, $|\lambda_1 - \lambda_2|(V \setminus K) \leq \varepsilon$ and this shows $\lambda_1 - \lambda_2$ is regular as claimed.

²Recall this is automatic for a complex measure.

Theorem 16.6.3 *Let $L \in C_0(X)'$. Then there exists a unique complex measure, λ with $|\lambda|$ regular and Borel, such that for all $f \in C_0(X)$,*

$$L(f) = \int_X f d\lambda.$$

Furthermore, $\|L\| = |\lambda|(X)$.

Proof: By Corollary 16.5.1 there exists $\sigma \in L^\infty(X, \mu)$ where μ is a Radon measure such that for all $f \in C_0(X)$,

$$L(f) = \int_X f \sigma d\mu.$$

Let a complex Borel measure, λ be given by

$$\lambda(E) \equiv \int_E \sigma d\mu.$$

This is a well defined complex measure because μ is a finite measure. By Corollary 16.2.9

$$|\lambda|(E) = \int_E |\sigma| d\mu \quad (16.6.17)$$

and $\sigma = g|\sigma|$ where $g d|\lambda|$ is the polar decomposition for λ . Therefore, for $f \in C_0(X)$,

$$L(f) = \int_X f \sigma d\mu = \int_X f g |\sigma| d\mu = \int_X f g d|\lambda| \equiv \int_X f d\lambda. \quad (16.6.18)$$

From 16.6.17 and the regularity of μ , it follows that $|\lambda|$ is also regular.

What of the claim about $\|L\|$? By the regularity of $|\lambda|$, it follows that $C_0(X)$ (In fact, $C_c(X)$) is dense in $L^1(X, |\lambda|)$. Since $|\lambda|$ is finite, $g \in L^1(X, |\lambda|)$. Therefore, there exists a sequence of functions in $C_0(X)$, $\{f_n\}$ such that $f_n \rightarrow \bar{g}$ in $L^1(X, |\lambda|)$. Therefore, there exists a subsequence, still denoted by $\{f_n\}$ such that $f_n(x) \rightarrow \bar{g}(x)$ $|\lambda|$ a.e. also. But since $|\bar{g}(x)| = 1$ a.e. it follows that $h_n(x) \equiv \frac{f_n(x)}{|f_n(x)| + \frac{1}{n}}$ also converges pointwise $|\lambda|$ a.e. Then from the dominated convergence theorem and 16.6.18

$$\|L\| \geq \lim_{n \rightarrow \infty} \int_X h_n g d|\lambda| = |\lambda|(X).$$

Also, if $\|f\|_{C_0(X)} \leq 1$, then

$$|L(f)| = \left| \int_X f g d|\lambda| \right| \leq \int_X |f| d|\lambda| \leq |\lambda|(X) \|f\|_{C_0(X)}$$

and so $\|L\| \leq |\lambda|(X)$. This proves everything but uniqueness.

Suppose λ and λ_1 both work. Then for all $f \in C_0(X)$,

$$0 = \int_X f d(\lambda - \lambda_1) = \int_X f h d|\lambda - \lambda_1|$$

where $h d|\lambda - \lambda_1|$ is the polar decomposition for $\lambda - \lambda_1$. By Lemma 16.6.2 $\lambda - \lambda_1$ is regular and so, as above, there exists $\{f_n\}$ such that $|f_n| \leq 1$ and $f_n \rightarrow \bar{h}$ pointwise. Therefore, $\int_X d|\lambda - \lambda_1| = 0$ so $\lambda = \lambda_1$. This proves the theorem.

16.7 Sequential Compactness In L^1

Lemma 16.7.1 *Let $\mathcal{C} \equiv \{E_i\}_{i=1}^\infty$ be a countable collection of sets and let $\Omega_1 \equiv \bigcup_{i=1}^\infty E_i$. Then there exists an algebra of sets, $\mathcal{A}$, such that $\mathcal{A} \supseteq \mathcal{C}$ and $\mathcal{A}$ is countable.*

Proof: Let $\mathcal{C}_1$ denote all finite unions of sets of $\mathcal{C}$ and also include Ω_1 and $\emptyset$. Thus $\mathcal{C}_1$ is countable. Next let $\mathcal{B}_1$ denote all sets of the form $\Omega_1 \setminus A$ such that $A \in \mathcal{C}_1$. Next let $\mathcal{C}_2$ denote all finite unions of sets of $\mathcal{B}_1 \cup \mathcal{C}_1$. Then let $\mathcal{B}_2$ denote all sets of the form $\Omega_1 \setminus A$ such that $A \in \mathcal{C}_2$ and let $\mathcal{C}_3 = \mathcal{B}_2 \cup \mathcal{C}_2$. Continuing this way yields an increasing sequence, $\{\mathcal{C}_n\}$ each of which is countable. Let

$$\mathcal{A} \equiv \bigcup_{i=1}^\infty \mathcal{C}_i.$$

Then $\mathcal{A}$ is countable. Also $\mathcal{A}$ is an algebra. Here is why. Suppose $A, B \in \mathcal{A}$. Then there exists n such that both $A, B \in \mathcal{C}_{n-1}$. It follows $A \cup B \in \mathcal{C}_n \subseteq \mathcal{A}$ from the construction. It only remains to show that $A \setminus B \in \mathcal{A}$. Taking complements with respect to Ω_1 , it follows from the construction that A^C, B^C are both in $\mathcal{B}_{n-1} \subseteq \mathcal{C}_n$. Thus,

$$A^C \cup B^C \in \mathcal{C}_n$$

and so

$$A \setminus B = (A^C \cup B^C)^C \in \mathcal{B}_n \subseteq \mathcal{C}_{n+1} \subseteq \mathcal{A}.$$

This shows $\mathcal{A}$ is an algebra of sets of Ω_1 which is also countable and contains $\mathcal{C}$.

Lemma 16.7.2 *Let $\{f_n\}$ be a sequence of functions in $L^1(\Omega, \mathcal{S}, \mu)$. Then there exists a σ finite set of $\mathcal{S}$, Ω_1 , and a σ algebra of subsets of Ω_1 , $\mathcal{S}_1$, such that $\mathcal{S}_1 \subseteq \mathcal{S}$, $f_n = 0$ off Ω_1 , $f_n \in L^1(\Omega_1, \mathcal{S}_1, \mu)$, and $\mathcal{S}_1 = \sigma(\mathcal{A})$, the σ algebra generated by $\mathcal{A}$, for some $\mathcal{A}$ a countable algebra.*

Proof: Let $\mathcal{E}_n$ denote the sets which are of the form

$$\left\{ f_n^{-1}(B(z, r)) : z \in \mathbb{Q} + i\mathbb{Q}, r > 0, r \in \mathbb{Q}, \text{ and } 0 \notin \overline{B(z, r)} \right\}$$

Since each $\mathcal{E}_n$ is countable, so is

$$\mathcal{E} \equiv \bigcup_{n=1}^\infty \mathcal{E}_n$$

Now let $\Omega_1 \equiv \bigcup \mathcal{E}$. I claim Ω_1 is σ finite. To see this, let

$$W_n = \left\{ \omega \in \Omega : |f_k(\omega)| > \frac{1}{n} \text{ for some } k = 1, 2, \dots, n \right\}$$

Thus if $\omega \in W_n$, it follows that for some $r \in \mathbb{Q}$, $z \in \mathbb{Q} + i\mathbb{Q}$ sufficiently close to $f_k(\omega)$

$$\omega \in f_k^{-1}(B(z, r)) \in \mathcal{E}_k$$

and so $\omega \in \bigcup_{k=1}^n \mathcal{E}_k$ and consequently, $W_n \in \bigcup_{k=1}^n \mathcal{E}_k$. Also

$$\mu(W_n) \frac{1}{n} \leq \int_{W_n} \sum_{k=1}^n |f_k(\omega)| d\mu < \infty.$$

Now if $\omega \in \Omega_1$, then for some k , ω is contained in a set of $\mathcal{E}_k$. Therefore, for that k ,

$$f_k(\omega) \in B(z, r)$$

where r is a positive rational number and $z \in \mathbb{Q} + i\mathbb{Q}$ and $\overline{B(z, r)}$ does not contain 0. Therefore, $f_k(\omega)$ is at a positive distance from 0 and so for large enough n , $\omega \in W_n$. Take n so large that $1/n$ is less than the distance from $\overline{B(z, r)}$ to 0 and also larger than k .

By Lemma 16.7.1 there exists a countable algebra of sets $\mathcal{A}$ which contains $\mathcal{E}$. Let $\mathcal{S}_1 \equiv \sigma(\mathcal{A})$. It remains to show $f_n(\omega) = 0$ off Ω_1 for all n . Let $\omega \notin \Omega_1$. Then ω is not contained in any set of $\mathcal{E}_k$ and so $f_k(\omega)$ cannot be nonzero. Hence $f_k(\omega) = 0$. This proves the lemma.

The following Theorem is the main result on sequential compactness in $L^1(\Omega, \mathcal{S}, \mu)$.

Theorem 16.7.3 *Let $K \subseteq L^1(\Omega, \mathcal{S}, \mu)$ be such that for some $C > 0$ and all $f \in K$,*

$$\|f\|_{L^1} \leq C \quad (16.7.19)$$

and K also satisfies the property that if $\{E_n\}$ is a decreasing sequence of measurable sets such that $\cap_{n=1}^{\infty} E_n = \emptyset$, then for all $\varepsilon > 0$ there exists n_ε such that if $n \geq n_\varepsilon$, then

$$\left| \int_{E_n} f d\mu \right| < \varepsilon \quad (16.7.20)$$

for all $f \in K$. Then every sequence of functions of K has an $L^1(\Omega, \mu)$ weakly convergent subsequence.

Proof: Take $\{f_n\}$ a sequence in K and let $\mathcal{A}, \mathcal{S}_1, \Omega_1$ be as in Lemma 16.7.2. Thus $\mathcal{A}$ is a countable algebra and by assumption, for each $E \in \mathcal{A}$,

$$\left\{ \int_E f_n d\mu \right\}$$

is a bounded sequence and so there exists a convergent subsequence. Therefore, from a Cantor diagonalization argument, there exists a subsequence, denoted by $\{g_n\}$ such that

$$\left\{ \int_E g_n d\mu \right\}$$

converges for every $E \in \mathcal{A}$.

Let

$$\mathcal{M} \equiv \left\{ E \in \mathcal{S}_1 = \sigma(\mathcal{A}) \text{ such that } \lim_{n \rightarrow \infty} \int_E g_n d\mu \text{ exists} \right\}.$$

Then it has been shown that $\mathcal{A} \subseteq \mathcal{M}$. Suppose $E_k \uparrow E$ where $E_k \in \mathcal{M}$. Then letting $\varepsilon > 0$ be given, the assumption shows that for k large enough,

$$\left| \int_{E \setminus E_k} g_n d\mu \right| < \varepsilon$$

for all g_n . Therefore, picking such a k ,

$$\left| \int_E g_n d\mu - \int_E g_m d\mu \right| \leq 2\varepsilon + \left| \int_{E_k} g_n d\mu - \int_{E_k} g_m d\mu \right| < 3\varepsilon$$

provided m, n are large enough. Therefore, $\{\int_E g_n d\mu\}$ is a Cauchy sequence and so it converges.

In the case that $E_k \downarrow E$ use the assumption to conclude there exists a k large enough that

$$\left| \int_{E_k \setminus E} g_n d\mu \right| < \varepsilon$$

for all g_n . Then

$$\begin{aligned} \left| \int_E g_n d\mu - \int_E g_m d\mu \right| &= \left| \int_{E_k} g_n d\mu - \int_{E_k} g_m d\mu \right| \\ &\quad + \left| \int_{E_k \setminus E} g_n d\mu \right| + \left| \int_{E_k \setminus E} g_m d\mu \right| \\ &\leq \left| \int_{E_k} g_n d\mu - \int_{E_k} g_m d\mu \right| + 2\varepsilon < 3\varepsilon \end{aligned}$$

provided m, n large enough. Again $\{\int_E g_n d\mu\}$ is a Cauchy sequence. This shows $\mathcal{M}$ is a monotone class and so by the monotone class theorem, Theorem 9.9.5 on Page 266 it follows $\mathcal{M} = \mathcal{S}_1 \equiv \sigma(\mathcal{A})$.

Therefore, picking $E \in \mathcal{S}_1$, you can define a complex measure,

$$\lambda(E) \equiv \lim_{n \rightarrow \infty} \int_E g_n d\mu$$

Then $\lambda \ll \mu$ and so by Corollary 16.2.8 on Page 501 and the fact shown above that Ω_1 is σ finite there exists a unique $\mathcal{S}_1$ measurable $g \in L^1(\Omega_1, \mu)$ such that

$$\lambda(E) = \int_E g d\mu \equiv \lim_{n \rightarrow \infty} \int_E g_n d\mu.$$

Extend g to equal 0 outside Ω_1 .

It remains to show $\{g_n\}$ converges weakly. It has just been shown that for every s a simple function measurable with respect to $\mathcal{S}_1$

$$\int_{\Omega} g_n s d\mu = \int_{\Omega_1} g_n s d\mu \rightarrow \int_{\Omega_1} g s d\mu = \int_{\Omega} g s d\mu$$

Now let $f \in L^\infty(\Omega_1, \mathcal{S}_1, \mu)$ and pick a uniformly bounded representative of this function. Then by Theorem 8.3.9 on Page 197 there exists a sequence of simple functions converging uniformly to f and so

$$\left\{ \int_{\Omega} g_n f d\mu \right\}$$

converges because

$$\begin{aligned} \left| \int_{\Omega} g_n f d\mu - \int_{\Omega} g f d\mu \right| &\leq \left| \int_{\Omega} g_n s d\mu - \int_{\Omega} g s d\mu \right| \\ &\quad + \int_{\Omega} |g_n| \varepsilon d\mu + \int_{\Omega} |g| \varepsilon d\mu \\ &\leq C\varepsilon + \|g\|_1 \varepsilon + \left| \int_{\Omega} g_n s d\mu - \int_{\Omega} g s d\mu \right| \end{aligned}$$

for suitable simple s satisfying $\sup_{\omega \in \Omega_1} |s(\omega) - f(\omega)| < \varepsilon$ and the last term converges to 0 as $n \rightarrow \infty$.

$(L^1(\Omega, \mathcal{S}, \mu))'$ is a space I don't know much about due to a possible lack of σ finiteness of Ω . However, it does follow that for i the inclusion map of $L^1(\Omega_1, \mathcal{S}_1, \mu)$ into $L^1(\Omega, \mathcal{S}, \mu)$ which merely extends the function as 0 off Ω_1 and $f \in (L^1(\Omega, \mathcal{S}, \mu))'$, there exists $h \in L^\infty(\Omega_1)$ such that for all $g \in L^1(\Omega_1, \mathcal{S}_1, \mu)$

$$i^* f(g) = \int_{\Omega_1} h g d\mu.$$

This is because $i^* f \in (L^1(\Omega_1, \mathcal{S}_1, \mu))'$ and Ω_1 is σ finite and so the Riesz representation theorem applies to get a unique such $h \in L^\infty(\Omega_1)$. Then since all the g_n equal 0 off Ω_1 ,

$$f(g_n) = i^* f(g_n) = \int_{\Omega_1} h g_n d\mu$$

for a unique $h \in L^\infty(\Omega_1, \mathcal{S}_1, \mu)$ due to the Riesz representation theorem which holds here because Ω_1 was shown to be σ finite. Therefore,

$$\lim_{n \rightarrow \infty} f(g_n) = \lim_{n \rightarrow \infty} \int_{\Omega_1} h g_n d\mu = \int_{\Omega_1} h g d\mu = i^* f(g) = f(g).$$

This proves the theorem.

For more on this theorem see [22]. I have only discussed the sufficiency of the conditions to give sequential compactness. They also discuss the necessity of these conditions.

There is another nice condition which implies the above results which is seen in books on probability. It is the concept of equi integrability.

Definition 16.7.4 Let $(\Omega, \mathcal{S}, \mu)$ be a measure space in which $\mu(\Omega) < \infty$. Then $K \subseteq L^1(\Omega, \mathcal{S}, \mu)$ is said to be equi integrable if

$$\lim_{\lambda \rightarrow \infty} \sup_{f \in K} \int_{\{|f| \geq \lambda\}} |f| d\mu = 0$$

Lemma 16.7.5 Let K be an equi integrable set. Then there exists $C > 0$ such that for all $f \in K$,

$$\|f\|_{L^1} \leq C \quad (16.7.21)$$

and K also satisfies the property that if $\{E_n\}$ is a decreasing sequence of measurable sets such that $\cap_{n=1}^{\infty} E_n = \emptyset$, then for all $\varepsilon > 0$ there exists n_ε such that if $n \geq n_\varepsilon$, then

$$\left| \int_{E_n} f d\mu \right| < \varepsilon \quad (16.7.22)$$

for all $f \in K$.

Proof: Choose λ_0 such that

$$\sup_{f \in K} \int_{[|f| \geq \lambda_0]} |f| d\mu \leq 1.$$

Then for $f \in K$,

$$\begin{aligned} \int_{\Omega} |f| d\mu &= \int_{[|f| \geq \lambda_0]} |f| d\mu + \int_{[|f| < \lambda_0]} |f| d\mu \\ &\leq 1 + \lambda_0 \mu(\Omega) \equiv C \end{aligned}$$

and this proves 16.7.21.

Next suppose $\{E_n\}$ is a decreasing sequence which has empty intersection and let $\varepsilon > 0$ and choose λ_ε such that

$$\sup_{f \in K} \int_{[|f| \geq \lambda_\varepsilon]} |f| d\mu \leq \varepsilon/2.$$

Then since μ is finite, there exists n_ε such that if $n \geq n_\varepsilon$, then $\mu(E_n) \leq \varepsilon/2(1 + \lambda_\varepsilon)$. Then letting $f \in K$,

$$\begin{aligned} \int_{E_n} |f| d\mu &= \int_{E_n \cap [|f| \geq \lambda_\varepsilon]} |f| d\mu + \int_{E_n \cap [|f| < \lambda_\varepsilon]} |f| d\mu \\ &\leq \varepsilon/2 + \int_{E_n} \lambda_\varepsilon d\mu < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

This proves 16.7.22 and proves the lemma.

Corollary 16.7.6 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space in which $\mu(\Omega) < \infty$ and let $K \subseteq L^1(\Omega, \mathcal{S}, \mu)$ be equi integrable. Then every sequence from K has a weakly convergent subsequence.*

Proof: From Lemma 16.7.5 the hypotheses of Theorem 16.7.3 are satisfied.

It is also convenient to consider the following proposition.

Proposition 16.7.7 *Let $(\Omega, \mathcal{S}, \mu)$ be a measure space in which $\mu(\Omega) < \infty$. Then $K \subseteq L^1(\Omega, \mathcal{S}, \mu)$ is equi integrable if and only if K is uniformly integrable and there exists a constant, M such that for all $f \in K$, $\|f\|_{L^1} \leq M$.*

Proof: First suppose K is equi integrable. Then pick λ such that for all $f \in K$,

$$\int_{[|f| \geq \lambda]} |f| d\mu < 1.$$

Then for $f \in K$

$$\begin{aligned} \int_{\Omega} |f| d\mu &= \int_{[|f| \geq \lambda]} |f| d\mu + \int_{[|f| < \lambda]} |f| d\mu \\ &\leq 1 + \lambda \mu(\Omega) \equiv M. \end{aligned}$$

Also, if $\varepsilon > 0$, pick λ so large that for all $f \in K$

$$\int_{[|f| \geq \lambda]} |f| d\mu < \frac{\varepsilon}{2}.$$

Then letting $A \in \mathcal{S}$,

$$\begin{aligned} \int_A |f| d\mu &= \int_{A \cap [|f| \geq \lambda]} |f| d\mu + \int_{A \cap [|f| < \lambda]} |f| d\mu \\ &< \frac{\varepsilon}{2} + \lambda P(A) \end{aligned}$$

and so if $P(A)$ is sufficiently small, this is less than ε . Thus K is uniformly integrable.

Now suppose $\|f\|_1 \leq M$ for all $f \in K$ and K is uniformly integrable. Then

$$\int_{[|f| \geq \lambda]} |f| d\mu \geq \lambda P([|f| \geq \lambda])$$

and so

$$P([|f| \geq \lambda]) \leq \frac{1}{\lambda} \int_{[|f| \geq \lambda]} |f| d\mu \leq \frac{1}{\lambda} \int_{\Omega} |f| d\mu \leq \frac{M}{\lambda}$$

and so, by the assumption of uniform integrability,

$$\int_{[|f| \geq \lambda]} |f| d\mu < \varepsilon$$

for all $f \in K$ provided λ is large enough. This proves the proposition.

16.8 Exercises

1. Suppose μ is a vector measure having values in $\mathbb{R}^n$ or $\mathbb{C}^n$. Can you show that $|\mu|$ must be finite? **Hint:** You might define for each $\mathbf{e}_i$, one of the standard basis vectors, the real or complex measure, $\mu_{\mathbf{e}_i}$ given by $\mu_{\mathbf{e}_i}(E) \equiv \mathbf{e}_i \cdot \mu(E)$. Why would this approach not yield anything for an infinite dimensional normed linear space in place of $\mathbb{R}^n$?

2. The Riesz representation theorem of the L^p spaces can be used to prove a very interesting inequality. Let $r, p, q \in (1, \infty)$ satisfy

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1.$$

Then

$$\frac{1}{q} = 1 + \frac{1}{r} - \frac{1}{p} > \frac{1}{r}$$

and so $r > q$. Let $\theta \in (0, 1)$ be chosen so that $\theta r = q$. Then also we have

$$\frac{1}{r} = \left(\overbrace{1 - \frac{1}{p'}}^{1/p+1/p'=1} \right) + \frac{1}{q} - 1 = \frac{1}{q} - \frac{1}{p'}$$

and so

$$\frac{\theta}{q} = \frac{1}{q} - \frac{1}{p'}$$

which implies $p'(1 - \theta) = q$. Now let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $f, g \geq 0$. Justify the steps in the following argument using what was just shown that $\theta r = q$ and $p'(1 - \theta) = q$. Let

$$\begin{aligned} & h \in L^{r'}(\mathbb{R}^n) \cdot \left(\frac{1}{r} + \frac{1}{r'} = 1 \right) \\ & \int f * g(\mathbf{x}) |h(\mathbf{x})| dx = \int \int f(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) |h(\mathbf{x})| dx dy \\ & \leq \int \int |f(\mathbf{y})| |g(\mathbf{x} - \mathbf{y})|^\theta |g(\mathbf{x} - \mathbf{y})|^{1-\theta} |h(\mathbf{x})| dy dx \\ & \leq \int \left(\int \left(|g(\mathbf{x} - \mathbf{y})|^{1-\theta} |h(\mathbf{x})| \right)^{r'} dx \right)^{1/r'} \\ & \quad \left(\int \left(|f(\mathbf{y})| |g(\mathbf{x} - \mathbf{y})|^\theta \right)^r dx \right)^{1/r} dy \\ & \leq \left[\int \left(\int \left(|g(\mathbf{x} - \mathbf{y})|^{1-\theta} |h(\mathbf{x})| \right)^{r'} dx \right)^{p'/r'} dy \right]^{1/p'} \\ & \quad \left[\int \left(\int \left(|f(\mathbf{y})| |g(\mathbf{x} - \mathbf{y})|^\theta \right)^r dx \right)^{p/r} dy \right]^{1/p} \\ & \leq \left[\int \left(\int \left(|g(\mathbf{x} - \mathbf{y})|^{1-\theta} |h(\mathbf{x})| \right)^{p'} dy \right)^{r'/p'} dx \right]^{1/r'} \end{aligned}$$

$$\begin{aligned}
& \left[\int |f(\mathbf{y})|^p \left(\int |g(\mathbf{x} - \mathbf{y})|^{\theta r} dx \right)^{p/r} dy \right]^{1/p} \\
&= \left[\int |h(\mathbf{x})|^{r'} \left(\int |g(\mathbf{x} - \mathbf{y})|^{(1-\theta)p'} dy \right)^{r'/p'} dx \right]^{1/r'} \|g\|_q^{q/r} \|f\|_p \\
&= \|g\|_q^{q/r} \|g\|_q^{q/p'} \|f\|_p \|h\|_{r'} = \|g\|_q \|f\|_p \|h\|_{r'}. \tag{16.8.23}
\end{aligned}$$

Young's inequality says that

$$\|f * g\|_r \leq \|g\|_q \|f\|_p. \tag{16.8.24}$$

Therefore $\|f * g\|_r \leq \|g\|_q \|f\|_p$. How does this inequality follow from the above computation? Does 16.8.23 continue to hold if r, p, q are only assumed to be in $[1, \infty]$? Explain. Does 16.8.24 hold even if r, p , and q are only assumed to lie in $[1, \infty]$?

3. Show that in a reflexive Banach space, weak and weak $*$ convergence are the same.
4. Suppose $(\Omega, \mu, \mathcal{S})$ is a finite measure space and that $\{f_n\}$ is a sequence of functions which converge weakly to 0 in $L^p(\Omega)$. Suppose also that $f_n(x) \rightarrow 0$ a.e. Show that then $f_n \rightarrow 0$ in $L^{p-\varepsilon}(\Omega)$ for every $p > \varepsilon > 0$.
5. Give an example of a sequence of functions in $L^\infty(-\pi, \pi)$ which converges weak $*$ to zero but which does not converge pointwise a.e. to zero.

Integrals And Derivatives

17.1 The Fundamental Theorem Of Calculus

The version of the fundamental theorem of calculus found in Calculus has already been referred to frequently. It says that if f is a Riemann integrable function, the function

$$x \rightarrow \int_a^x f(t) dt,$$

has a derivative at every point where f is continuous. It is natural to ask what occurs for f in L^1 . It is an amazing fact that the same result is obtained asside from a set of measure zero even though f , being only in L^1 may fail to be continuous anywhere. Proofs of this result are based on some form of the Vitali covering theorem presented above. In what follows, the measure space is $(\mathbb{R}^n, \mathcal{S}, m)$ where m is n -dimensional Lebesgue measure although the same theorems can be proved for arbitrary Radon measures [46]. To save notation, m is written in place of m_n .

By Lemma 9.1.7 on Page 224 and the completeness of m , the Lebesgue measurable sets are exactly those measurable in the sense of Caratheodory. Also, to save on notation m is also the name of the outer measure defined on all of $\mathcal{P}(\mathbb{R}^n)$ which is determined by m_n . Recall

$$B(\mathbf{p}, r) = \{\mathbf{x} : |\mathbf{x} - \mathbf{p}| < r\}. \quad (17.1.1)$$

Also define the following.

$$\text{If } B = B(\mathbf{p}, r), \text{ then } \widehat{B} = B(\mathbf{p}, 5r). \quad (17.1.2)$$

The first version of the Vitali covering theorem presented above will now be used to establish the fundamental theorem of calculus. The space of locally integrable functions is the most general one for which the maximal function defined below makes sense.

Definition 17.1.1 $f \in L^1_{loc}(\mathbb{R}^n)$ means $f\chi_{B(0,R)} \in L^1(\mathbb{R}^n)$ for all $R > 0$. For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy Littlewood Maximal Function, Mf , is defined by

$$Mf(\mathbf{x}) \equiv \sup_{r>0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y})| dy.$$

Theorem 17.1.2 *If $f \in L^1(\mathbb{R}^n)$, then for $\alpha > 0$,*

$$\overline{m}([Mf > \alpha]) \leq \frac{5^n}{\alpha} \|f\|_1.$$

(Here and elsewhere, $[Mf > \alpha] \equiv \{\mathbf{x} \in \mathbb{R}^n : Mf(\mathbf{x}) > \alpha\}$ with other occurrences of $[\]$ being defined similarly.)

Proof: Let $S \equiv [Mf > \alpha]$. For $\mathbf{x} \in S$, choose $r_{\mathbf{x}} > 0$ with

$$\frac{1}{m(B(\mathbf{x}, r_{\mathbf{x}}))} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f| \, dm > \alpha.$$

The $r_{\mathbf{x}}$ are all bounded because

$$m(B(\mathbf{x}, r_{\mathbf{x}})) < \frac{1}{\alpha} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f| \, dm < \frac{1}{\alpha} \|f\|_1.$$

By the Vitali covering theorem, there are disjoint balls $B(\mathbf{x}_i, r_i)$ such that

$$S \subseteq \bigcup_{i=1}^{\infty} B(\mathbf{x}_i, 5r_i)$$

and

$$\frac{1}{m(B(\mathbf{x}_i, r_i))} \int_{B(\mathbf{x}_i, r_i)} |f| \, dm > \alpha.$$

Therefore

$$\begin{aligned} \overline{m}(S) &\leq \sum_{i=1}^{\infty} m(B(\mathbf{x}_i, 5r_i)) = 5^n \sum_{i=1}^{\infty} m(B(\mathbf{x}_i, r_i)) \\ &\leq \frac{5^n}{\alpha} \sum_{i=1}^{\infty} \int_{B(\mathbf{x}_i, r_i)} |f| \, dm \\ &\leq \frac{5^n}{\alpha} \int_{\mathbb{R}^n} |f| \, dm, \end{aligned}$$

the last inequality being valid because the balls $B(\mathbf{x}_i, r_i)$ are disjoint. This proves the theorem.

Note that at this point it is unknown whether S is measurable. This is why $\overline{m}(S)$ and not $m(S)$ is written.

The following is the fundamental theorem of calculus from elementary calculus.

Lemma 17.1.3 *Suppose g is a continuous function. Then for all $\mathbf{x}$,*

$$\lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) \, d\mathbf{y} = g(\mathbf{x}).$$

Proof: Note that

$$g(\mathbf{x}) = \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) \, dy$$

and so

$$\begin{aligned} & \left| g(\mathbf{x}) - \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) \, dy \right| \\ &= \left| \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} (g(\mathbf{y}) - g(\mathbf{x})) \, dy \right| \\ &\leq \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |g(\mathbf{y}) - g(\mathbf{x})| \, dy. \end{aligned}$$

Now by continuity of g at $\mathbf{x}$, there exists $r > 0$ such that if $|\mathbf{x} - \mathbf{y}| < r$, $|g(\mathbf{y}) - g(\mathbf{x})| < \varepsilon$. For such r , the last expression is less than

$$\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \varepsilon \, dy < \varepsilon.$$

This proves the lemma.

Definition 17.1.4 Let $f \in L^1(\mathbb{R}^k, m)$. A point, $\mathbf{x} \in \mathbb{R}^k$ is said to be a Lebesgue point if

$$\limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm = 0.$$

Note that if $\mathbf{x}$ is a Lebesgue point, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) \, dm = f(\mathbf{x}).$$

and so the symmetric derivative exists at all Lebesgue points.

Theorem 17.1.5 (Fundamental Theorem of Calculus) Let $f \in L^1(\mathbb{R}^k)$. Then there exists a set of measure 0, N , such that if $\mathbf{x} \notin N$, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dy = 0.$$

Proof: Let $\lambda > 0$ and let $\varepsilon > 0$. By density of $C_c(\mathbb{R}^k)$ in $L^1(\mathbb{R}^k, m)$ there exists $g \in C_c(\mathbb{R}^k)$ such that $\|g - f\|_{L^1(\mathbb{R}^k)} < \varepsilon$. Now since g is continuous,

$$\begin{aligned} & \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm \\ &= \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm \\ &\quad - \lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |g(\mathbf{y}) - g(\mathbf{x})| \, dm \end{aligned}$$

$$\begin{aligned}
&= \limsup_{r \rightarrow 0} \left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| - |g(\mathbf{y}) - g(\mathbf{x})| \, dm \right) \\
&\leq \limsup_{r \rightarrow 0} \left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} ||f(\mathbf{y}) - f(\mathbf{x})| - |g(\mathbf{y}) - g(\mathbf{x})|| \, dm \right) \\
&\leq \limsup_{r \rightarrow 0} \left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - g(\mathbf{y}) - (f(\mathbf{x}) - g(\mathbf{x}))| \, dm \right) \\
&\leq \limsup_{r \rightarrow 0} \left(\frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - g(\mathbf{y})| \, dm \right) + |f(\mathbf{x}) - g(\mathbf{x})| \\
&\leq M([f - g])(\mathbf{x}) + |f(\mathbf{x}) - g(\mathbf{x})|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[\mathbf{x} : \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm > \lambda \right] \\
&\subseteq \left[M([f - g]) > \frac{\lambda}{2} \right] \cup \left[|f - g| > \frac{\lambda}{2} \right]
\end{aligned}$$

Now

$$\begin{aligned}
\varepsilon &> \int |f - g| \, dm \geq \int_{[|f - g| > \frac{\lambda}{2}]} |f - g| \, dm \\
&\geq \frac{\lambda}{2} m \left(\left[|f - g| > \frac{\lambda}{2} \right] \right)
\end{aligned}$$

This along with the weak estimate of Theorem 17.1.2 implies

$$\begin{aligned}
&m \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm > \lambda \right] \right) \\
&< \left(\frac{2}{\lambda} 5^k + \frac{2}{\lambda} \right) \|f - g\|_{L^1(\mathbb{R}^k)} \\
&< \left(\frac{2}{\lambda} 5^k + \frac{2}{\lambda} \right) \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows

$$m_n \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm > \lambda \right] \right) = 0.$$

Now let

$$N = \left[\mathbf{x} : \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| \, dm > 0 \right]$$

and

$$N_n = \left[\mathbf{x} : \limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| dm > \frac{1}{n} \right]$$

It was just shown that $m(N_n) = 0$. Also, $N = \cup_{n=1}^{\infty} N_n$. Therefore, $m(N) = 0$ also. It follows that for $\mathbf{x} \notin N$,

$$\limsup_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| dm = 0$$

and this proves a.e. point is a Lebesgue point.

Of course it is sufficient to assume f is only in $L^1_{loc}(\mathbb{R}^k)$.

Corollary 17.1.6 (*Fundamental Theorem of Calculus*) Let $f \in L^1_{loc}(\mathbb{R}^k)$. Then there exists a set of measure 0, N , such that if $\mathbf{x} \notin N$, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0.$$

Proof: Consider $B(\mathbf{0}, n)$ where n is a positive integer. Then $f_n \equiv f \chi_{B(\mathbf{0}, n)} \in L^1(\mathbb{R}^k)$ and so there exists a set of measure 0, N_n such that if $\mathbf{x} \in B(\mathbf{0}, n) \setminus N_n$, then

$$\begin{aligned} & \lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f_n(\mathbf{y}) - f_n(\mathbf{x})| dy \\ &= \lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| dy = 0. \end{aligned}$$

Let $N = \cup_{n=1}^{\infty} N_n$. Then if $\mathbf{x} \notin N$, the above equation holds.

Corollary 17.1.7 If $f \in L^1_{loc}(\mathbb{R}^n)$, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) dy = f(\mathbf{x}) \quad \text{a.e. } \mathbf{x}. \quad (17.1.3)$$

Proof:

$$\begin{aligned} & \left| \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) dy - f(\mathbf{x}) \right| \\ & \leq \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| dy \end{aligned}$$

and the last integral converges to 0 a.e. $\mathbf{x}$.

Definition 17.1.8 For N the set of Theorem 17.1.5 or Corollary 17.1.6, N^C is called the Lebesgue set or the set of Lebesgue points.

The next corollary is a one dimensional version of what was just presented.

Corollary 17.1.9 *Let $f \in L^1(\mathbb{R})$ and let*

$$F(x) = \int_{-\infty}^x f(t) dt.$$

Then for a.e. x , $F'(x) = f(x)$.

Proof: For $h > 0$

$$\frac{1}{h} \int_x^{x+h} |f(y) - f(x)| dy \leq 2 \left(\frac{1}{2h} \right) \int_{x-h}^{x+h} |f(y) - f(x)| dy$$

By Theorem 17.1.5, this converges to 0 a.e. Similarly

$$\frac{1}{h} \int_{x-h}^x |f(y) - f(x)| dy$$

converges to 0 a.e. x .

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{h} \int_x^{x+h} |f(y) - f(x)| dy \quad (17.1.4)$$

and

$$\left| \frac{F(x) - F(x-h)}{h} - f(x) \right| \leq \frac{1}{h} \int_{x-h}^x |f(y) - f(x)| dy. \quad (17.1.5)$$

Now the expression on the right in 17.1.4 and 17.1.5 converges to zero for a.e. x . Therefore, by 17.1.4, for a.e. x the derivative from the right exists and equals $f(x)$ while from 17.1.5 the derivative from the left exists and equals $f(x)$ a.e. It follows

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x) \text{ a.e. } x$$

This proves the corollary.

17.2 Absolutely Continuous Functions

Definition 17.2.1 *Let $[a, b]$ be a closed and bounded interval and let $f : [a, b] \rightarrow \mathbb{R}$. Then f is said to be absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\sum_{i=1}^m |y_i - x_i| < \delta$, then $\sum_{i=1}^m |f(y_i) - f(x_i)| < \varepsilon$.*

Definition 17.2.2 *A finite subset, P of $[a, b]$ is called a partition of $[x, y] \subseteq [a, b]$ if $P = \{x_0, x_1, \dots, x_n\}$ where*

$$x = x_0 < x_1 < \dots < x_n = y.$$

For $f : [a, b] \rightarrow \mathbb{R}$ and $P = \{x_0, x_1, \dots, x_n\}$ define

$$V_P[x, y] \equiv \sum_{i=1}^n |f(x_i) - f(x_{i-1})|.$$

Denoting by $\mathcal{P}[x, y]$ the set of all partitions of $[x, y]$ define

$$V[x, y] \equiv \sup_{P \in \mathcal{P}[x, y]} V_P[x, y].$$

For simplicity, $V[a, x]$ will be denoted by $V(x)$. It is called the total variation of the function, f .

There are some simple facts about the total variation of an absolutely continuous function, f which are contained in the next lemma.

Lemma 17.2.3 *Let f be an absolutely continuous function defined on $[a, b]$ and let V be its total variation function as described above. Then V is an increasing bounded function. Also if P and Q are two partitions of $[x, y]$ with $P \subseteq Q$, then $V_P[x, y] \leq V_Q[x, y]$ and if $[x, y] \subseteq [z, w]$,*

$$V[x, y] \leq V[z, w] \quad (17.2.6)$$

If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[x, y]$, then

$$V[x, y] = \sum_{i=1}^n V[x_i, x_{i-1}]. \quad (17.2.7)$$

Also if $y > x$,

$$V(y) - V(x) \geq |f(y) - f(x)| \quad (17.2.8)$$

and the function, $x \rightarrow V(x) - f(x)$ is increasing. The total variation function, V is absolutely continuous.

Proof: The claim that V is increasing is obvious as is the next claim about $P \subseteq Q$ leading to $V_P[x, y] \leq V_Q[x, y]$. To verify this, simply add in one point at a time and verify that from the triangle inequality, the sum involved gets no smaller. The claim that V is increasing consistent with set inclusion of intervals is also clearly true and follows directly from the definition.

Now let $t < V[x, y]$ where $P_0 = \{x_0, x_1, \dots, x_n\}$ is a partition of $[x, y]$. There exists a partition, P of $[x, y]$ such that $t < V_P[x, y]$. Without loss of generality it can be assumed that $\{x_0, x_1, \dots, x_n\} \subseteq P$ since if not, you can simply add in the points of P_0 and the resulting sum for the total variation will get no smaller. Let P_i be those points of P which are contained in $[x_{i-1}, x_i]$. Then

$$t < V_P[x, y] = \sum_{i=1}^n V_{P_i}[x_{i-1}, x_i] \leq \sum_{i=1}^n V[x_{i-1}, x_i].$$

Since $t < V[x, y]$ is arbitrary,

$$V[x, y] \leq \sum_{i=1}^n V[x_i, x_{i-1}] \quad (17.2.9)$$

Note that 17.2.9 does not depend on f being absolutely continuous. Suppose now that f is absolutely continuous. Let δ correspond to $\varepsilon = 1$. Then if $[x, y]$ is an interval of length no larger than δ , the definition of absolute continuity implies

$$V[x, y] < 1.$$

Then from 17.2.9

$$V[a, n\delta] \leq \sum_{i=1}^n V[a + (i-1)\delta, a + i\delta] < \sum_{i=1}^n 1 = n.$$

Thus V is bounded on $[a, b]$. Now let P_i be a partition of $[x_{i-1}, x_i]$ such that

$$V_{P_i}[x_{i-1}, x_i] > V[x_{i-1}, x_i] - \frac{\varepsilon}{n}$$

Then letting $P = \cup P_i$,

$$-\varepsilon + \sum_{i=1}^n V[x_{i-1}, x_i] < \sum_{i=1}^n V_{P_i}[x_{i-1}, x_i] = V_P[x, y] \leq V[x, y].$$

Since ε is arbitrary, 17.2.7 follows from this and 17.2.9.

Now let $x < y$

$$\begin{aligned} V(y) - f(y) - (V(x) - f(x)) &= V(y) - V(x) - (f(y) - f(x)) \\ &\geq V(y) - V(x) - |f(y) - f(x)| \geq 0. \end{aligned}$$

It only remains to verify that V is absolutely continuous.

Let $\varepsilon > 0$ be given and let δ correspond to $\varepsilon/2$ in the definition of absolute continuity applied to f . Suppose $\sum_{i=1}^n |y_i - x_i| < \delta$ and consider $\sum_{i=1}^n |V(y_i) - V(x_i)|$. By 17.2.9 this last equals $\sum_{i=1}^n V[x_i, y_i]$. Now let P_i be a partition of $[x_i, y_i]$ such that $V_{P_i}[x_i, y_i] + \frac{\varepsilon}{2n} > V[x_i, y_i]$. Then by the definition of absolute continuity,

$$\begin{aligned} \sum_{i=1}^n |V(y_i) - V(x_i)| &= \sum_{i=1}^n V[x_i, y_i] \\ &\leq \sum_{i=1}^n V_{P_i}[x_i, y_i] + \eta < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

and shows V is absolutely continuous as claimed.

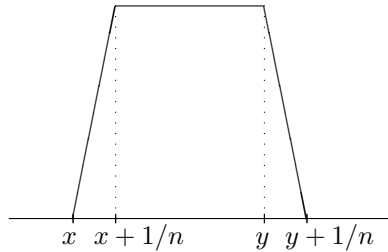
Lemma 17.2.4 Suppose $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous and increasing. Then f' exists a.e., is in $L^1([a, b])$, and

$$f(x) = f(a) + \int_a^x f'(t) dt.$$

Proof: Define L , a positive linear functional on $C([a, b])$ by

$$Lg \equiv \int_a^b g df$$

where this integral is the Riemann Stieltjes integral with respect to the integrating function, f . By the Riesz representation theorem for positive linear functionals, there exists a unique Radon measure, μ such that $Lg = \int g d\mu$. Now consider the following picture for $g_n \in C([a, b])$ in which g_n equals 1 for x between $x + 1/n$ and y .



Then $g_n(t) \rightarrow \chi_{(x, y]}(t)$ pointwise. Therefore, by the dominated convergence theorem,

$$\mu((x, y]) = \lim_{n \rightarrow \infty} \int g_n d\mu.$$

However,

$$\begin{aligned} & \left(f(y) - f\left(x + \frac{1}{n}\right) \right) \\ & \leq \int g_n d\mu = \int_a^b g_n df \leq \left(f\left(y + \frac{1}{n}\right) - f(y) \right) \\ & \quad + \left(f(y) - f\left(x + \frac{1}{n}\right) \right) + \left(f\left(x + \frac{1}{n}\right) - f(x) \right) \end{aligned}$$

and so as $n \rightarrow \infty$ the continuity of f implies

$$\mu((x, y]) = f(y) - f(x).$$

Similarly, $\mu(x, y) = f(y) - f(y)$ and $\mu([x, y]) = f(y) - f(x)$, the argument used to establish this being very similar to the above. It follows in particular that

$$f(x) - f(a) = \int_{[a, x]} d\mu.$$

Note that up till now, no reference has been made to the absolute continuity of f . Any increasing continuous function would be fine.

Now if E is a Borel set such that $m(E) = 0$, Then the outer regularity of m implies there exists an open set, V containing E such that $m(V) < \delta$ where δ corresponds to ε in the definition of absolute continuity of f . Then letting $\{I_k\}$ be the connected components of V it follows $E \subseteq \cup_{k=1}^{\infty} I_k$ with $\sum_k m(I_k) = m(V) < \delta$. Therefore, from absolute continuity of f , it follows that for $I_k = (a_k, b_k)$ and each n

$$\mu(\cup_{k=1}^n I_k) = \sum_{k=1}^n \mu(I_k) = \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$$

and so letting $n \rightarrow \infty$,

$$\mu(E) \leq \mu(V) = \sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \leq \varepsilon.$$

Since ε is arbitrary, it follows $\mu(E) = 0$. Therefore, $\mu \ll m$ and so by the Radon Nikodym theorem there exists a unique $h \in L^1([a, b])$ such that

$$\mu(E) = \int_E h dm.$$

In particular,

$$\mu([a, x]) = f(x) - f(a) = \int_{[a, x]} h dm.$$

From the fundamental theorem of calculus $f'(x) = h(x)$ at every Lebesgue point of h . Therefore, writing in usual notation,

$$f(x) = f(a) + \int_a^x f'(t) dt$$

as claimed. This proves the lemma.

With the above lemmas, the following is the main theorem about absolutely continuous functions.

Theorem 17.2.5 *Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous if and only if $f'(x)$ exists a.e., $f' \in L^1([a, b])$ and*

$$f(x) = f(a) + \int_a^x f'(t) dt.$$

Proof: Suppose first that f is absolutely continuous. By Lemma 17.2.3 the total variation function, V is absolutely continuous and $f(x) = V(x) - (V(x) - f(x))$ where both V and $V - f$ are increasing and absolutely continuous. By Lemma 17.2.4

$$\begin{aligned} f(x) - f(a) &= V(x) - V(a) - [(V(x) - f(x)) - (V(a) - f(a))] \\ &= \int_a^x V'(t) dt - \int_a^x (V - f)'(t) dt. \end{aligned}$$

Now f' exists and is in L^1 because $f = V - (V - f)$ and V and $V - f$ have derivatives in L^1 . Therefore, $(V - f)' = V' - f'$ and so the above reduces to

$$f(x) - f(a) = \int_a^x f'(t) dt.$$

This proves one half of the theorem.

Now suppose $f' \in L^1$ and $f(x) = f(a) + \int_a^x f'(t) dt$. It is necessary to verify that f is absolutely continuous. But this follows easily from Lemma 8.5.2 on Page 212 which implies that a single function, f' is uniformly integrable. This lemma implies that if $\sum_i |y_i - x_i|$ is sufficiently small then

$$\sum_i \left| \int_{x_i}^{y_i} f'(t) dt \right| = \sum_i |f(y_i) - f(x_i)| < \varepsilon.$$

The following simple corollary is a case of Rademacher's theorem.

Corollary 17.2.6 *Suppose $f : [a, b] \rightarrow \mathbb{R}$ is Lipschitz continuous,*

$$|f(x) - f(y)| \leq K|x - y|.$$

Then $f'(x)$ exists a.e. and

$$f(x) = f(a) + \int_a^x f'(t) dt.$$

Proof: It is easy to see that f is absolutely continuous. Therefore, Theorem 17.2.5 applies.

17.3 Differentiation Of Measures With Respect To Lebesgue Measure

Recall the Vitali covering theorem in Corollary 10.4.5 on Page 297.

Corollary 17.3.1 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of open balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}(E \setminus \cup_{j=1}^\infty B_j) = 0$.*

Definition 17.3.2 *Let μ be a Radon measure defined on $\mathbb{R}^n$. Then*

$$\frac{d\mu}{dm}(\mathbf{x}) \equiv \lim_{r \rightarrow 0} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))}$$

whenever this limit exists.

It turns out this limit exists for m a.e. $\mathbf{x}$. To verify this here is another definition.

Definition 17.3.3 Let $f(r)$ be a function having values in $[-\infty, \infty]$. Then

$$\begin{aligned}\limsup_{r \rightarrow 0+} f(r) &\equiv \lim_{r \rightarrow 0} (\sup \{f(t) : t \in [0, r]\}) \\ \liminf_{r \rightarrow 0+} f(r) &\equiv \lim_{r \rightarrow 0} (\inf \{f(t) : t \in [0, r]\})\end{aligned}$$

This is well defined because the function $r \rightarrow \inf \{f(t) : t \in [0, r]\}$ is increasing and $r \rightarrow \sup \{f(t) : t \in [0, r]\}$ is decreasing. Also note that $\lim_{r \rightarrow 0+} f(r)$ exists if and only if

$$\limsup_{r \rightarrow 0+} f(r) = \liminf_{r \rightarrow 0+} f(r)$$

and if this happens

$$\lim_{r \rightarrow 0+} f(r) = \liminf_{r \rightarrow 0+} f(r) = \limsup_{r \rightarrow 0+} f(r).$$

The claims made in the above definition follow immediately from the definition of what is meant by a limit in $[-\infty, \infty]$ and are left for the reader.

Theorem 17.3.4 Let μ be a Borel measure on $\mathbb{R}^n$ then $\frac{d\mu}{dm}(\mathbf{x})$ exists in $[-\infty, \infty]$ *m a.e.*

Proof: Let $p < q$ and let p, q be rational numbers. Define

$$\begin{aligned}N_{pq}(M) &\equiv \left\{ \mathbf{x} \in \mathbb{R}^n \text{ such that } \limsup_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > q \right. \\ &\quad \left. > p > \liminf_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} \right\} \cap B(\mathbf{0}, M), \\ N_{pq} &\equiv \left\{ \mathbf{x} \in \mathbb{R}^n \text{ such that } \limsup_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > q \right. \\ &\quad \left. > p > \liminf_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} \right\}, \\ N &\equiv \left\{ \mathbf{x} \in \mathbb{R}^n \text{ such that } \limsup_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \right. \\ &\quad \left. \liminf_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} \right\}.\end{aligned}$$

I will show $\overline{m}(N_{pq}(M)) = 0$. Use outer regularity to obtain an open set, V containing $N_{pq}(M)$ such that

$$\overline{m}(N_{pq}(M)) + \varepsilon > m(V).$$

From the definition of $N_{pq}(M)$, it follows that for each $\mathbf{x} \in N_{pq}(M)$ there exist arbitrarily small $r > 0$ such that

$$\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} < p.$$

Only consider those r which are small enough to be contained in $B(\mathbf{0}, M)$ so that the collection of such balls has bounded radii. This is a Vitali cover of $N_{pq}(M)$ and so by Corollary 17.3.1 there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^\infty$ such that

$$\mu(B_i) < pm(B_i), \quad \overline{m}(N_{pq}(M) \setminus \cup_{i=1}^\infty B_i) = 0. \quad (17.3.10)$$

Now for $\mathbf{x} \in N_{pq}(M) \cap (\cup_{i=1}^\infty B_i)$ (most of $N_{pq}(M)$), there exist arbitrarily small balls, $B(\mathbf{x}, r)$, such that $B(\mathbf{x}, r)$ is contained in some set of $\{B_i\}_{i=1}^\infty$ and

$$\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > q.$$

This is a Vitali cover of $N_{pq}(M) \cap (\cup_{i=1}^\infty B_i)$ and so there exists a sequence of disjoint balls of this sort, $\{B'_j\}_{j=1}^\infty$ such that

$$\overline{m}((N_{pq}(M) \cap (\cup_{i=1}^\infty B_i)) \setminus \cup_{j=1}^\infty B'_j) = 0, \quad \mu(B'_j) > qm(B'_j). \quad (17.3.11)$$

It follows from 17.3.10 and 17.3.11 that

$$\overline{m}(N_{pq}(M)) \leq \overline{m}((N_{pq}(M) \cap (\cup_{i=1}^\infty B_i))) \leq m(\cup_{j=1}^\infty B'_j) \quad (17.3.12)$$

Therefore,

$$\begin{aligned} \sum_j \mu(B'_j) &> q \sum_j m(B'_j) \geq q \overline{m}(N_{pq}(M) \cap (\cup_i B_i)) = q \overline{m}(N_{pq}(M)) \\ &\geq p \overline{m}(N_{pq}(M)) \geq p(m(V) - \varepsilon) \geq p \sum_i m(B_i) - p\varepsilon \\ &\geq \sum_i \mu(B_i) - p\varepsilon \geq \sum_j \mu(B'_j) - p\varepsilon. \end{aligned}$$

It follows

$$p\varepsilon \geq (q - p) \overline{m}(N_{pq}(M))$$

Since ε is arbitrary, $m(N_{pq}(M)) = 0$. Now $N_{pq} \subseteq \cup_{M=1}^\infty N_{pq}(M)$ and so $m(N_{pq}) = 0$. Now

$$N = \cup_{p,q \in \mathbb{Q}} N_{pq}$$

and since this is a countable union of sets of measure zero, $m(N) = 0$ also. This proves the theorem.

From Theorem 16.2.4 on Page 497 it follows that if μ is a complex measure then $|\mu|$ is a finite measure. This makes possible the following definition.

Definition 17.3.5 *Let μ be a real measure. Define the following measures. For E a measurable set,*

$$\begin{aligned} \mu^+(E) &\equiv \frac{1}{2} (|\mu| + \mu)(E), \\ \mu^-(E) &\equiv \frac{1}{2} (|\mu| - \mu)(E). \end{aligned}$$

These are measures thanks to Theorem 16.2.3 on Page 496 and $\mu^+ - \mu^- = \mu$. These measures have values in $[0, \infty)$. They are called the positive and negative parts of μ respectively. For μ a complex measure, define $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ by

$$\begin{aligned}\operatorname{Re} \mu(E) &\equiv \frac{1}{2} \left(\mu(E) + \overline{\mu(E)} \right) \\ \operatorname{Im} \mu(E) &\equiv \frac{1}{2i} \left(\mu(E) - \overline{\mu(E)} \right)\end{aligned}$$

Then $\operatorname{Re} \mu$ and $\operatorname{Im} \mu$ are both real measures. Thus for μ a complex measure,

$$\begin{aligned}\mu &= \operatorname{Re} \mu^+ - \operatorname{Re} \mu^- + i (\operatorname{Im} \mu^+ - \operatorname{Im} \mu^-) \\ &= \nu_1 - \nu_2 + i (\nu_3 - \nu_4)\end{aligned}$$

where each ν_i is a real measure having values in $[0, \infty)$.

Then there is an obvious corollary to Theorem 17.3.4.

Corollary 17.3.6 *Let μ be a complex Borel measure on $\mathbb{R}^n$. Then $\frac{d\mu}{dm}(\mathbf{x})$ exists a.e.*

Proof: Letting ν_i be defined in Definition 17.3.5. By Theorem 17.3.4, for m a.e. $\mathbf{x}$, $\frac{d\nu_i}{dm}(\mathbf{x})$ exists. This proves the corollary because μ is just a finite sum of these ν_i .

Theorem 16.1.2 on Page 489, the Radon Nikodym theorem, implies that if you have two finite measures, μ and λ , you can write λ as the sum of a measure absolutely continuous with respect to μ and one which is singular to μ in a unique way. The next topic is related to this. It has to do with the differentiation of a measure which is singular with respect to Lebesgue measure.

Theorem 17.3.7 *Let μ be a Radon measure on $\mathbb{R}^n$ and suppose there exists a μ measurable set, N such that for all Borel sets, E , $\mu(E) = \mu(E \cap N)$ where $\overline{m}(N) = 0$. Then*

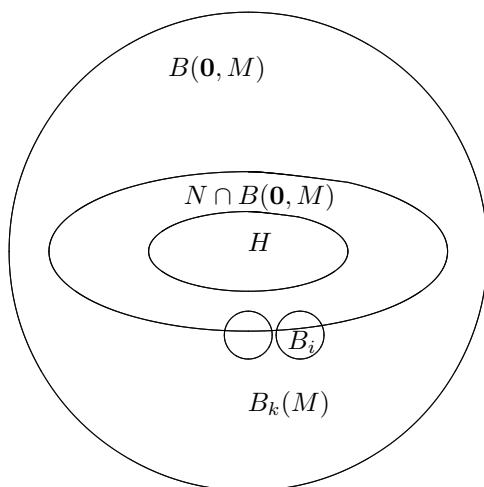
$$\frac{d\mu}{dm}(\mathbf{x}) = 0 \text{ } m \text{ a.e.}$$

Proof: For $k \in \mathbb{N}$, let

$$\begin{aligned}B_k(M) &\equiv \left\{ \mathbf{x} \in N^C : \limsup_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k} \right\} \cap B(\mathbf{0}, M), \\ B_k &\equiv \left\{ \mathbf{x} \in N^C : \limsup_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k} \right\}, \\ B &\equiv \left\{ \mathbf{x} \in N^C : \limsup_{r \rightarrow 0+} \frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > 0 \right\}.\end{aligned}$$

Let $\varepsilon > 0$. Since μ is regular, there exists H , a compact set such that $H \subseteq N \cap B(\mathbf{0}, M)$ and

$$\mu(N \cap B(\mathbf{0}, M) \setminus H) < \varepsilon.$$



For each $\mathbf{x} \in B_k(M)$, there exist arbitrarily small $r > 0$ such that $B(\mathbf{x}, r) \subseteq B(\mathbf{0}, M) \setminus H$ and

$$\frac{\mu(B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} > \frac{1}{k}. \quad (17.3.13)$$

Two such balls are illustrated in the above picture. This is a Vitali cover of $B_k(M)$ and so there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^{\infty}$ such that $\overline{m}(B_k(M) \setminus \cup_i B_i) = 0$. Therefore,

$$\begin{aligned} \overline{m}(B_k(M)) &\leq \overline{m}(B_k(M) \cap (\cup_i B_i)) \leq \sum_i \overline{m}(B_i) \leq k \sum_i \mu(B_i) \\ &= k \sum_i \mu(B_i \cap N) = k \sum_i \mu(B_i \cap N \cap B(\mathbf{0}, M)) \\ &\leq k \mu(N \cap B(\mathbf{0}, M) \setminus H) < \varepsilon k \end{aligned}$$

Since ε was arbitrary, this shows $\overline{m}(B_k(M)) = 0$.

Therefore,

$$\overline{m}(B_k) \leq \sum_{M=1}^{\infty} \overline{m}(B_k(M)) = 0$$

and $\overline{m}(B) \leq \sum_k \overline{m}(B_k) = 0$. Since $\overline{m}(N) = 0$, this proves the theorem.

It is easy to obtain a different version of the above theorem. This is done with the aid of the following lemma.

Lemma 17.3.8 *Suppose μ is a Borel measure on $\mathbb{R}^n$ having values in $[0, \infty)$. Then there exists a Radon measure, μ_1 such that $\mu_1 = \mu$ on all Borel sets.*

Proof: By assumption, $\mu(\mathbb{R}^n) < \infty$ and so it is possible to define a positive linear functional, L on $C_c(\mathbb{R}^n)$ by

$$Lf \equiv \int f d\mu.$$

By the Riesz representation theorem for positive linear functionals of this sort, there exists a unique Radon measure, μ_1 such that for all $f \in C_c(\mathbb{R}^n)$,

$$\int f d\mu_1 = Lf = \int f d\mu.$$

Now let V be an open set and let $K_k \equiv \{\mathbf{x} \in V : \text{dist}(\mathbf{x}, V^c) \leq 1/k\} \cap \overline{B(\mathbf{0}, k)}$. Then $\{K_k\}$ is an increasing sequence of compact sets whose union is V . Let $K_k \prec f_k \prec V$. Then $f_k(\mathbf{x}) \rightarrow \chi_V(\mathbf{x})$ for every $\mathbf{x}$. Therefore,

$$\mu_1(V) = \lim_{k \rightarrow \infty} \int f_k d\mu_1 = \lim_{k \rightarrow \infty} \int f_k d\mu = \mu(V)$$

and so $\mu = \mu_1$ on open sets. Now if K is a compact set, let

$$V_k \equiv \{\mathbf{x} \in \mathbb{R}^n : \text{dist}(\mathbf{x}, K) < 1/k\}.$$

Then V_k is an open set and $\cap_k V_k = K$. Letting $K \prec f_k \prec V_k$, it follows that $f_k(\mathbf{x}) \rightarrow \chi_K(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Therefore, by the dominated convergence theorem with a dominating function, $\chi_{\mathbb{R}^n}$

$$\mu_1(K) = \lim_{k \rightarrow \infty} \int f_k d\mu_1 = \lim_{k \rightarrow \infty} \int f_k d\mu = \mu(K)$$

and so μ and μ_1 are equal on all compact sets. It follows $\mu = \mu_1$ on all countable unions of compact sets and countable intersections of open sets.

Now let E be a Borel set. By regularity of μ_1 , there exist sets, H and G such that H is the countable union of an increasing sequence of compact sets, G is the countable intersection of a decreasing sequence of open sets, $H \subseteq E \subseteq G$, and $\mu_1(H) = \mu_1(G) = \mu_1(E)$. Therefore,

$$\mu_1(H) = \mu(H) \leq \mu(E) \leq \mu(G) = \mu_1(G) = \mu_1(E) = \mu_1(H).$$

therefore, $\mu(E) = \mu_1(E)$ and this proves the lemma.

Corollary 17.3.9 *Suppose μ is a complex Borel measure defined on $\mathbb{R}^n$ for which there exists a μ measurable set, N such that for all Borel sets, E , $\mu(E) = \mu(E \cap N)$ where $\bar{m}(N) = 0$. Then*

$$\frac{d\mu}{dm}(\mathbf{x}) = 0 \text{ m a.e.}$$

Proof: Each of $\text{Re } \mu^+$, $\text{Re } \mu^-$, $\text{Im } \mu^+$, and $\text{Im } \mu^-$ are real measures having values in $[0, \infty)$ and so by Lemma 17.3.8 each is a Radon measure having the same property that μ has in terms of being supported on a set of m measure zero. Therefore, for ν equal to any of these, $\frac{d\nu}{dm}(\mathbf{x}) = 0$ m a.e. This proves the corollary.

17.4 Exercises

1. Suppose A and B are sets of positive Lebesgue measure in $\mathbb{R}^n$. Show that $A - B$ must contain $B(\mathbf{c}, \varepsilon)$ for some $\mathbf{c} \in \mathbb{R}^n$ and $\varepsilon > 0$.

$$A - B \equiv \{\mathbf{a} - \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B\}.$$

Hint: First assume both sets are bounded. This creates no loss of generality. Next there exist $\mathbf{a}_0 \in A$, $\mathbf{b}_0 \in B$ and $\delta > 0$ such that

$$\int_{B(\mathbf{a}_0, \delta)} \chi_A(t) dt > \frac{3}{4}m(B(\mathbf{a}_0, \delta)), \quad \int_{B(\mathbf{b}_0, \delta)} \chi_B(t) dt > \frac{3}{4}m(B(\mathbf{b}_0, \delta)).$$

Now explain why this implies

$$m(A - \mathbf{a}_0 \cap B(\mathbf{0}, \delta)) > \frac{3}{4}m(B(\mathbf{0}, \delta))$$

and

$$m(B - \mathbf{b}_0 \cap B(\mathbf{0}, \delta)) > \frac{3}{4}m(B(\mathbf{0}, \delta)).$$

Explain why

$$m((A - \mathbf{a}_0) \cap (B - \mathbf{b}_0)) > \frac{1}{2}m(B(\mathbf{0}, \delta)) > 0.$$

Let

$$f(\mathbf{x}) \equiv \int \chi_{A-\mathbf{a}_0}(\mathbf{x} + \mathbf{t}) \chi_{B-\mathbf{b}_0}(\mathbf{t}) dt.$$

Explain why $f(\mathbf{0}) > 0$. Next explain why f is continuous and why $f(\mathbf{x}) > 0$ for all $\mathbf{x} \in B(\mathbf{0}, \varepsilon)$ for some $\varepsilon > 0$. Thus if $|\mathbf{x}| < \varepsilon$, there exists $\mathbf{t}$ such that $\mathbf{x} + \mathbf{t} \in A - \mathbf{a}_0$ and $\mathbf{t} \in B - \mathbf{b}_0$. Subtract these.

2. Show Mf is Borel measurable by verifying that $[Mf > \lambda] \equiv E_\lambda$ is actually an open set. **Hint:** If $\mathbf{x} \in E_\lambda$ then for some r , $\int_{B(\mathbf{x}, r)} |f| dm > \lambda m(B(\mathbf{x}, r))$. Then for δ a small enough positive number, $\int_{B(\mathbf{x}, r)} |f| dm > \lambda m(B(\mathbf{x}, r + 2\delta))$. Now pick $\mathbf{y} \in B(\mathbf{x}, \delta)$ and argue that $B(\mathbf{y}, \delta + r) \supseteq B(\mathbf{x}, r)$. Therefore show that,

$$\int_{B(\mathbf{y}, \delta+r)} |f| dm > \int_{B(\mathbf{x}, r)} |f| dm > \lambda m(B(\mathbf{x}, r + 2\delta)) \geq \lambda m(B(\mathbf{y}, r + \delta)).$$

Thus $B(\mathbf{x}, \delta) \subseteq E_\lambda$.

3. Consider the following nested sequence of compact sets, $\{P_n\}$. Let $P_1 = [0, 1]$, $P_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, etc. To go from P_n to P_{n+1} , delete the open interval which is the middle third of each closed interval in P_n . Let $P = \bigcap_{n=1}^\infty P_n$. By the finite intersection property of compact sets, $P \neq \emptyset$. Show $m(P) = 0$. If you feel ambitious also show there is a one to one onto mapping of $[0, 1]$

to P . The set P is called the Cantor set. Thus, although P has measure zero, it has the same number of points in it as $[0, 1]$ in the sense that there is a one to one and onto mapping from one to the other. **Hint:** There are various ways of doing this last part but the most enlightenment is obtained by exploiting the topological properties of the Cantor set rather than some silly representation in terms of sums of powers of two and three. All you need to do is use the Schroder Bernstein theorem and show there is an onto map from the Cantor set to $[0, 1]$. If you do this right and remember the theorems about characterizations of compact metric spaces, Proposition 6.2.5 on Page 142, you may get a pretty good idea why every compact metric space is the continuous image of the Cantor set.

4. Consider the sequence of functions defined in the following way. Let $f_1(x) = x$ on $[0, 1]$. To get from f_n to f_{n+1} , let $f_{n+1} = f_n$ on all intervals where f_n is constant. If f_n is nonconstant on $[a, b]$, let $f_{n+1}(a) = f_n(a)$, $f_{n+1}(b) = f_n(b)$, f_{n+1} is piecewise linear and equal to $\frac{1}{2}(f_n(a) + f_n(b))$ on the middle third of $[a, b]$. Sketch a few of these and you will see the pattern. The process of modifying a nonconstant section of the graph of this function is illustrated in the following picture.



Show $\{f_n\}$ converges uniformly on $[0, 1]$. If $f(x) = \lim_{n \rightarrow \infty} f_n(x)$, show that $f(0) = 0$, $f(1) = 1$, f is continuous, and $f'(x) = 0$ for all $x \notin P$ where P is the Cantor set of Problem 3. This function is called the Cantor function. It is a very important example to remember. Note it has derivative equal to zero a.e. and yet it succeeds in climbing from 0 to 1. Explain why this interesting function is not absolutely continuous although it is continuous. **Hint:** This isn't too hard if you focus on getting a careful estimate on the difference between two successive functions in the list considering only a typical small interval in which the change takes place. The above picture should be helpful.

5. A function, $f : [a, b] \rightarrow \mathbb{R}$ is Lipschitz if $|f(x) - f(y)| \leq K|x - y|$. Show that every Lipschitz function is absolutely continuous. Thus every Lipschitz function is differentiable a.e., $f' \in L^1$, and $f(y) - f(x) = \int_x^y f'(t) dt$.
6. Suppose f, g are both absolutely continuous on $[a, b]$. Show the product of these functions is also absolutely continuous. Explain why $(fg)' = f'g + g'f$ and show the usual integration by parts formula

$$f(b)g(b) - f(a)g(a) - \int_a^b fg' dt = \int_a^b f'g dt.$$

7. In Problem 4 f' failed to give the expected result for $\int_a^b f' dx$ ¹ but at least

¹In this example, you only know that f' exists a.e.

$f' \in L^1$. Suppose f' exists for f a continuous function defined on $[a, b]$. Does it follow that f' is measurable? Can you conclude $f' \in L^1([a, b])$?

8. A sequence of sets, $\{E_i\}$ containing the point $\mathbf{x}$ is said to shrink to $\mathbf{x}$ nicely if there exists a sequence of positive numbers, $\{r_i\}$ and a positive constant, α such that $r_i \rightarrow 0$ and

$$m(E_i) \geq \alpha m(B(\mathbf{x}, r_i)), \quad E_i \subseteq B(\mathbf{x}, r_i).$$

Show the above theorems about differentiation of measures with respect to Lebesgue measure all have a version valid for E_i replacing $B(\mathbf{x}, r)$.

9. Suppose $F(x) = \int_a^x f(t) dt$. Using the concept of nicely shrinking sets in Problem 8 show $F'(x) = f(x)$ a.e.
10. A random variable, X is a measurable real valued function defined on a measure space, $(\Omega, \mathcal{S}, P)$ where P is just a measure with $P(\Omega) = 1$ called a probability measure. The distribution function for X is the function, $F(x) \equiv P([X \leq x])$ in words, $F(x)$ is the probability that X has values no larger than x . Show that F is a right continuous increasing function with the property that $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.
11. Suppose F is an increasing right continuous function.

- (a) Show that $Lf \equiv \int_a^b f dF$ is a well defined positive linear functional on $C_c(\mathbb{R})$ where here $[a, b]$ is a closed interval containing the support of $f \in C_c(\mathbb{R})$.
- (b) Using the Riesz representation theorem for positive linear functionals on $C_c(\mathbb{R})$, let μ denote the Radon measure determined by L . Show that $\mu((a, b]) = F(b) - F(a)$ and $\mu(\{b\}) = F(b) - F(b-)$ where $F(b-) \equiv \lim_{x \rightarrow b-} F(x)$.
- (c) Review Corollary 16.1.4 on Page 494 at this point. Show that the conditions of this corollary hold for μ and m . Consider $\mu_\perp + \mu_\parallel$, the Lebesgue decomposition of μ where $\mu_\parallel \ll m$ and there exists a set of m measure zero, N such that $\mu_\perp(E) = \mu_\perp(E \cap N)$. Show $\mu((0, x]) = \mu_\perp((0, x]) + \int_0^x h(t) dt$ for some $h \in L^1(m)$. Using Theorem 17.3.7 show $h(x) = F'(x)$ m a.e. Explain why $F(x) = F(0) + S(x) + \int_0^x F'(t) dt$ for some function, $S(x)$ which is increasing but has $S'(x) = 0$ a.e. Note this shows in particular that a right continuous increasing function has a derivative a.e.
12. Suppose now that G is just an increasing function defined on $\mathbb{R}$. Show that $G'(x)$ exists a.e. **Hint:** You can mimic the proof of Theorem 17.3.4. The Dini derivatives are defined as

$$D_+ G(x) \equiv \liminf_{h \rightarrow 0+} \frac{G(x+h) - G(x)}{h},$$

$$D^+ G(x) \equiv \limsup_{h \rightarrow 0+} \frac{G(x+h) - G(x)}{h}$$

$$D_-G(x) \equiv \liminf_{h \rightarrow 0+} \frac{G(x) - G(x-h)}{h},$$

$$D^-G(x) \equiv \limsup_{h \rightarrow 0+} \frac{G(x) - G(x-h)}{h}.$$

When $D_+G(x) = D^+G(x)$ the derivative from the right exists and when $D^-G(x) = D_-G(x)$, then the derivative from the left exists. Let (a, b) be an open interval and let

$$N_{pq} \equiv \{x \in (a, b) : D^+G(x) > q > p > D_+G(x)\}.$$

Let $V \subseteq (a, b)$ be an open set containing N_{pq} such that $n(V) < m(N_{pq}) + \varepsilon$. Show using a Vitali covering theorem there is a disjoint sequence of intervals contained in V , $\{(x_i, x_i + h_i)\}_{i=1}^\infty$ such that

$$\frac{G(x_i + h_i) - G(x_i)}{h_i} < p.$$

Next show there is a disjoint sequence of intervals $\{(x'_i, x'_j + h'_j)\}_{j=1}^\infty$ such that each of these is contained in one of the former intervals and

$$\frac{G(x'_j + h'_j) - G(x'_j)}{h'_j} > q, \quad \sum_j h'_j \geq m(N_{pq}).$$

Then

$$\begin{aligned} qm(N_{pq}) &\leq q \sum_j h'_j \leq \sum_j G(x'_j + h'_j) - G(x'_j) \leq \sum_i G(x_i + h_i) - G(x_i) \\ &\leq p \sum_i h_i \leq pm(V) \leq p(m(N_{pq}) + \varepsilon). \end{aligned}$$

Since ε was arbitrary, this shows $m(N_{pq}) = 0$. Taking a union of all N_{pq} for p, q rational, shows the derivative from the right exists a.e. Do a similar argument to show the derivative from the left exists a.e. and then show the derivative from the left equals the derivative from the right a.e. using a similar argument. Thus $G'(x)$ exists on (a, b) a.e. and so it exists a.e. on $\mathbb{R}$ because (a, b) was arbitrary.

Orlitz Spaces

18.1 Basic Theory

All the theorems about the L^p spaces have generalizations to something called an Orlitz space. [1], [49] Instead of the convex function, $A(t) = t^p/p$, one considers a more general convex increasing function called an N function.

Definition 18.1.1 $A : [0, \infty) \rightarrow [0, \infty)$ is an N function if the following two conditions hold.

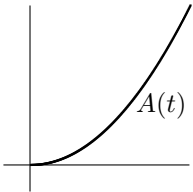
$$A \text{ is convex and strictly increasing} \tag{18.1.1}$$

$$\lim_{t \rightarrow 0+} \frac{A(t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{A(t)}{t} = \infty. \tag{18.1.2}$$

For A an N function,

$$\tilde{A}(s) \equiv \max \{st - A(t) : t \geq 0\}. \tag{18.1.3}$$

As an example see the following picture of a typical N function.



Note that from the assumption, 18.1.2 the maximum in the definition of $\tilde{A}$ must exist. This is because for $t \neq 0$

$$(s - A(t)/t)t$$

is negative for all t large enough. On the other hand, it equals 0 when $t = 0$ and so it suffices to consider only t in a compact set.

Lemma 18.1.2 *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then ϕ is Lipschitz continuous on $[a, b]$.*

Proof: Since it is convex, the difference quotients,

$$\frac{\phi(t) - \phi(a)}{t - a}$$

are increasing because by convexity, if $a < t < x$

$$\frac{t-a}{x-a}\phi(x) + \left(1 - \frac{t-a}{x-a}\right)\phi(a) \geq \phi(t)$$

and this reduces to

$$\frac{\phi(t) - \phi(a)}{t - a} \leq \frac{\phi(x) - \phi(a)}{x - a}.$$

Also these difference quotients are bounded below by

$$\frac{\phi(a) - \phi(a-1)}{1} = \phi(a) - \phi(a-1).$$

Let

$$A \equiv \inf \left\{ \frac{\phi(t) - \phi(a)}{t - a} : t \in (a, b) \right\}.$$

Then A is some finite real number. Similarly there exists a real number B such that for all $t \in (a, b)$,

$$B \geq \frac{\phi(b) - \phi(t)}{b - t}.$$

Now let $a \leq s < t \leq b$. Then

$$\frac{\phi(t) - \phi(s)}{t - s} \geq \frac{\phi(t) - \theta\phi(a) - (1 - \theta)\phi(t)}{t - s}$$

where θ is such that $\theta a + (1 - \theta)t = s$. Thus

$$\theta = \frac{t - s}{t - t_1}$$

and so the above implies

$$\frac{\phi(t) - \phi(s)}{t - s} \geq \frac{t - s}{t - t_1} \frac{\phi(t) - \phi(a)}{t - s} = \frac{\phi(t) - \phi(a)}{t - t_1} \geq A.$$

Similarly,

$$\begin{aligned} \frac{\phi(t) - \phi(s)}{t - s} &\leq \frac{\theta\phi(b) + (1 - \theta)\phi(s) - \phi(s)}{t - s} \\ &= \frac{t - s}{b - s} \frac{\phi(b) - \phi(s)}{t - s} \leq B. \end{aligned}$$

It follows

$$|\phi(t) - \phi(s)| \leq (|A| + |B|) |t - s|$$

and this proves the lemma.

The following is like the inequality, $st \leq t^p/p + s^q/q$, important in the study of L^p spaces.

Proposition 18.1.3 *If A is an N function, then so is $\tilde{A}$ and*

$$A(t) = \max \left\{ ts - \tilde{A}(s) : s \geq 0 \right\}, \quad (18.1.4)$$

so $\tilde{\tilde{A}} = A$. Also

$$st \leq A(t) + \tilde{A}(s) \text{ for all } s, t \geq 0 \quad (18.1.5)$$

and for all $s > 0$,

$$A\left(\frac{\tilde{A}(s)}{s}\right) \leq \tilde{A}(s). \quad (18.1.6)$$

Proof: First consider the claim $\tilde{A}$ is convex. Let $\lambda \in [0, 1]$.

$$\begin{aligned} \tilde{A}(\lambda s_1 + (1 - \lambda) s_2) &\equiv \max \{ [s_1 \lambda + (1 - \lambda) s_2] t - A(t) : t \geq 0 \} \\ &\leq \lambda \max \{ s_1 t - A(t) : t \geq 0 \} + (1 - \lambda) \max \{ s_2 t - A(t) : t \geq 0 \} \\ &= \lambda \tilde{A}(s_1) + (1 - \lambda) \tilde{A}(s_2). \end{aligned}$$

It is obvious $\tilde{A}$ is strictly increasing because st is strictly increasing in s . Next consider 18.1.2.

For $s > 0$ let t_s denote the number where the maximum is achieved. That is,

$$\tilde{A}(s) \equiv st_s - A(t_s).$$

Thus

$$\frac{\tilde{A}(s)}{s} = t_s - \frac{A(t_s)}{s} \geq 0. \quad (18.1.7)$$

It follows from this that

$$\lim_{s \rightarrow 0+} t_s = 0$$

since otherwise, a contradiction results to 18.1.7, the expression becoming negative for small enough s . Thus

$$t_s \geq \frac{\tilde{A}(s)}{s} \geq 0$$

and this shows

$$\lim_{s \rightarrow 0+} \frac{\tilde{A}(s)}{s} = 0.$$

which shows 18.1.2.

To verify the second part of 18.1.2, let t_s be as just described. Then for any $t > 0$

$$\frac{\tilde{A}(s)}{s} = t_s - \frac{A(t_s)}{s} \geq t - \frac{A(t)}{s}$$

It follows

$$\liminf_{s \rightarrow \infty} \frac{\tilde{A}(s)}{s} \geq t.$$

Since t is arbitrary, this proves the second part of 18.1.2.

The inequality 18.1.5 follows from the definition of $\tilde{A}(s)$.

Next consider 18.1.4. It must be shown that

$$A(t_0) = \max \left\{ t_0 s - \tilde{A}(s) : s \geq 0 \right\}.$$

To do so, first note

$$\tilde{A}(s) = \max \{ st - A(t) : t \geq 0 \} \geq st_0 - A(t_0).$$

Hence

$$\max \left\{ t_0 s - \tilde{A}(s) : s \geq 0 \right\} \leq \max \{ t_0 s - [st_0 - A(t_0)] \} = A(t_0).$$

Now let

$$s_0 \equiv \inf \left\{ \frac{A(t) - A(t_0)}{t - t_0} : t > t_0 \right\}.$$

By convexity, the above difference quotients are nondecreasing in t and so

$$s_0(t - t_0) \leq A(t) - A(t_0)$$

for all $t \neq t_0$. Hence for all t ,

$$s_0 t - A(t) \leq s_0 t_0 - A(t_0)$$

and so

$$\tilde{A}(s_0) = s_0 t_0 - A(t_0)$$

implying

$$A(t_0) = s_0 t_0 - \tilde{A}(s_0) \leq \max \left\{ st_0 - \tilde{A}(s) : s \geq 0 \right\} \leq A(t_0).$$

Therefore, 18.1.4 holds.

Consider 18.1.6 next. To do so, let $a = A'$ so that

$$A(t) = \int_0^t a(r) dr, \quad a \text{ increasing.}$$

This is possible by Rademacher's theorem, Corollary 17.2.6 and the fact that since A is convex, it is locally Lipschitz found in Lemma 18.1.2 above. That a is increasing follows from convexity of A . Here is why. For a.e. $s, t \geq 0$, and letting $\lambda \in [0, 1]$,

$$\begin{aligned} \frac{A(s + \lambda(t - s)) - A(s)}{\lambda} &\leq \frac{(1 - \lambda)A(s) + \lambda A(t) - A(s)}{\lambda} \\ &= A(t) - A(s) \end{aligned}$$

Then passing to a limit as $\lambda \rightarrow 0+$,

$$a(s)(t - s) \leq A(t) - A(s).$$

Similarly

$$a(t)(s - t) \leq A(s) - A(t)$$

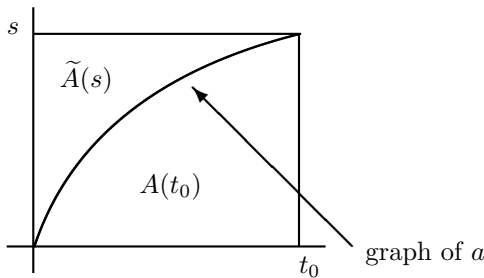
and so

$$(a(t) - a(s))(t - s) \geq 0.$$

(If you like, you can simply assume from the beginning that $A(t)$ is given this way as an integral of a positive increasing function, a , and verify directly that such an A is convex and satisfies the properties of an N function. There is no loss of generality in doing so.) Thus geometrically, $A(t)$ equals the area under the curve defined by a and above the x axis from $x = 0$ to $x = t$. In the definition of $\tilde{A}(s)$ let t_s be the point where the maximum is achieved. Then

$$\tilde{A}(s) = st_s - A(t_s)$$

and so at this point, $\tilde{A}(s) + A(t_s) = st_s$. This means that $\tilde{A}(s)$ is the area to the left of the graph of a which is to the right of the y axis for y between 0 and $a(t_s)$ and that in fact $a(t_s) = s$. The following picture illustrates the reasoning which follows.



Therefore,

$$\begin{aligned} \frac{\tilde{A}(s)}{s} &= t_s - \frac{A(t_s)}{s} = t_s - \frac{1}{s} \int_0^{t_s} a(r) dr \\ &= t_s - \frac{1}{a(t_s)} \int_0^{t_s} a(r) dr = \frac{1}{a(t_s)} \left(t_s s - \int_0^{t_s} a(r) dr \right) \end{aligned}$$

and so

$$\begin{aligned} A\left(\frac{\tilde{A}(s)}{s}\right) &= \int_0^{\tilde{A}(s)/s} a(r) dr = \int_0^{\frac{1}{a(t_s)} \int_0^{t_s} (s-a(r)) dr} a(\tau) d\tau \\ &\leq \int_0^{t_s} s - a(r) dr = st_s - A(t_s) = \tilde{A}(s). \end{aligned}$$

The inequality results from replacing $a(\tau)$ with $a(t_s)$ in the last integral on the top line.

An example of an N function is $A(t) = \frac{t^p}{p}$ for $t \geq 0$ and $p > 1$. For this example, $\tilde{A}(s) = \frac{s^{p'}}{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 18.1.4 Let A be an N function and let $(\Omega, \mathcal{S}, \mu)$ be a measure space. Define

$$K_A(\Omega) \equiv \left\{ u \text{ measurable such that } \int_{\Omega} A(|u|) d\mu < \infty \right\}. \quad (18.1.8)$$

This is called the Orlicz class. Also define

$$L_A(\Omega) \equiv \{ \lambda u : u \in K_A(\Omega) \text{ and } \lambda \in \mathbb{F} \} \quad (18.1.9)$$

where $\mathbb{F}$ is the field of scalars, assumed to be either $\mathbb{R}$ or $\mathbb{C}$.

The pair (A, Ω) is called Δ regular if either of the following conditions hold.

$$A(rx) \leq K_r A(x) \text{ for all } x \in [0, \infty) \quad (18.1.10)$$

or $\mu(\Omega) < \infty$ and for all $r > 0$, there exists M_r and $K_r > 0$ such that

$$A(rx) \leq K_r A(x) \text{ for all } x \geq M_r. \quad (18.1.11)$$

Note there are N functions which are not Δ regular. For example, consider

$$A(x) \equiv e^{x^2} - 1.$$

It can't be Δ regular because

$$\lim_{r \rightarrow \infty} \frac{e^{r^2 x^2} - 1}{e^{x^2} - 1} = \infty.$$

However, functions like x^p/p for $p > 1$ are Δ regular.

Then the following proposition is important.

Proposition 18.1.5 If (A, Ω) is Δ regular, then $K_A(\Omega) = L_A(\Omega)$. In any case, $L_A(\Omega)$ is a vector space and $K_A(\Omega) \subseteq L_A(\Omega)$.

Proof: Suppose (A, Ω) is Δ regular. Then I claim $K_A(\Omega)$ is a vector space. This will verify $K_A(\Omega) = L_A(\Omega)$. Let $f, g \in K_A(\Omega)$ and suppose 18.1.10. Then

$$A(|f+g|) = A\left(2\left(\frac{|f+g|}{2}\right)\right) \leq K_2 A\left(\frac{|f+g|}{2}\right) \leq K_2 \frac{1}{2} [A(|f|) + A(|g|)]$$

so $f+g \in K_A(\Omega)$ in this case. Now suppose 18.1.11

$$\begin{aligned} \int_{\Omega} A(|f+g|) d\mu &= \int_{[|f+g| \leq M_2]} A(|f+g|) d\mu + \int_{[|f+g| > M_2]} A(|f+g|) d\mu \\ &\leq A(M_2) \mu(\Omega) + \int_{\Omega} \frac{K_2}{2} (A(|f|) + A(|g|)) d\mu < \infty. \end{aligned}$$

Thus $f+g \in K_A(\Omega)$ in this case also.

Next consider scalar multiplication. First consider the case of 18.1.10. If $f \in K_A(\Omega)$ and $\alpha \in \mathbb{F}$,

$$\int_{\Omega} A(|\alpha| |f|) d\mu \leq K_{|\alpha|} \int_{\Omega} A(f) d\mu$$

so in the case of 18.1.10 $\alpha f \in K_A(\Omega)$ whenever $f \in K_A(\Omega)$. In the case of 18.1.11,

$$\begin{aligned} \int_{\Omega} A(|\alpha| |f|) d\mu &= \int_{[|\alpha| |f| \leq M_{|\alpha|}]} A(|\alpha| |f|) d\mu + \int_{[|\alpha| |f| > M_{|\alpha|}]} A(|\alpha| |f|) d\mu \\ &\leq A(M_{|\alpha|}) \mu(\Omega) + \int_{\Omega} K_{|\alpha|} A(|f|) d\mu < \infty. \end{aligned}$$

This establishes the first part of the proposition.

Next consider the claim that $L_A(\Omega)$ is always a vector space. First note $K_A(\Omega)$ is always convex due to convexity of A . Let $\lambda u, \alpha v \in L_A(\Omega)$ where $u, v \in K_A(\Omega)$ and let a, b be scalars in $\mathbb{F}$. Then

$$a\lambda u + b\alpha v = |a\lambda| \omega u + |b\alpha| \theta v$$

where $|\omega| = |\theta| = 1$. Then

$$= (|a\lambda| + |b\alpha|) \left(\frac{|a\lambda| \omega u + |b\alpha| \theta v}{|a\lambda| + |b\alpha|} \right)$$

which exhibits $a\lambda u + b\alpha v$ as a multiple of a convex combination of two elements of $K_A(\Omega)$, ωu and θv . Thus $L_A(\Omega)$ is closed with respect to linear combinations. This shows it is a vector space. This proves the proposition.

The following norm for $L_A(\Omega)$ is due to Luxemburg [49]. You might compare this to the definition of a Minkowski functional. The definition of $L_A(\Omega)$ above was cooked up so that the following norm does make sense.

Definition 18.1.6 Define

$$\|u\|_A = \|u\|_{A, \Omega} \equiv \inf \left\{ t > 0 : \int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d\mu \leq 1 \right\}.$$

If two functions of $L_A(\Omega)$ are equal a.e. they are considered to be the same in the usual way.

Proposition 18.1.7 *The number defined in Definition 18.1.6 is a norm on $L_A(\Omega)$. Also, if $\Omega_1 \subseteq \Omega$, then*

$$\|u\|_{A, \Omega_1} \leq \|u\|_{A, \Omega}.$$

Proof: Clearly $\|u\|_A \geq 0$. Is $\|u\|_A$ finite for $u \in L_A(\Omega)$? Let $u \in L_A(\Omega)$ so $u = \lambda v$ where $v \in K_A(\Omega)$. Then for $s > 0$

$$\int_{\Omega} A\left(\frac{|u|}{s|\lambda|}\right) d\mu = \int_{\Omega} A\left(\frac{|v|}{s}\right) d\mu < \infty$$

whenever $s > 1$. Therefore, from the dominated convergence theorem, if s is large enough,

$$\int_{\Omega} A\left(\frac{|u|}{s|\lambda|}\right) d\mu \leq 1$$

and this shows there are values of $t > 0$ such that

$$\int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d\mu \leq 1.$$

Thus $\|u\|_A$ is finite as hoped.

Now suppose $\|u\|_A = 0$ and let

$$E_n \equiv \left\{x : |u(x)| \geq \frac{1}{n}\right\}.$$

Then for arbitrarily small values of t ,

$$\int_{E_n} A\left(\frac{(1/n)}{t}\right) d\mu \leq \int_{\Omega} A\left(\frac{|u(x)|}{t}\right) d\mu \leq 1$$

and so for arbitrarily small values of t ,

$$A\left(\frac{(1/n)}{t}\right) \mu(E_n) \leq 1.$$

Letting $t \rightarrow 0+$ yields a contradiction unless $\mu(E_n) = 0$. Now

$$\mu([|u(x)| > 0]) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0.$$

Thus $u = 0$ as claimed.

Consider the other axioms of a norm. Let $u, v \in L_A(\Omega)$ and let α, β be scalars. Then

$$\|\alpha u + \beta v\|_A \equiv \inf \left\{ t > 0 : \int_{\Omega} A\left(\frac{|u(x) + v(x)|}{t}\right) d\mu \leq 1 \right\}$$

Without loss of generality $\|u\|_A, \|v\|_A < \infty$ since otherwise there is nothing to prove.

$$\begin{aligned}
 \|u + v\|_A &\equiv \inf \left\{ t > 0 : \int_{\Omega} A \left(\frac{|\alpha u(x) + \beta v(x)|}{t} \right) d\mu \leq 1 \right\} \\
 &\leq \inf \left\{ t > 0 : \int_{\Omega} A \left(\frac{|\alpha| |u| + |\beta| |v|}{t} \right) d\mu \leq 1 \right\} \\
 &= \inf \left\{ t > 0 : \int_{\Omega} A \left(\frac{|\alpha| \frac{(|\alpha| + |\beta|)|u|}{t} + |\beta| \frac{(|\alpha| + |\beta|)|v|}{t}}{(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\} \\
 &\leq \inf \left\{ t > 0 : \frac{|\alpha|}{(|\alpha| + |\beta|)} \int_{\Omega} A \left(\frac{|u|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\} \\
 &\quad + \inf \left\{ t > 0 : \frac{|\beta|}{(|\alpha| + |\beta|)} \int_{\Omega} A \left(\frac{|v|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\} \\
 &= |\alpha| \inf \left\{ t/(|\alpha| + |\beta|) > 0 : \int_{\Omega} A \left(\frac{|u|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\} \\
 &\quad + |\beta| \inf \left\{ t/(|\alpha| + |\beta|) > 0 : \int_{\Omega} A \left(\frac{|v|}{t/(|\alpha| + |\beta|)} \right) d\mu \leq 1 \right\} \\
 &= |\alpha| \|u\|_A + |\beta| \|v\|_A.
 \end{aligned}$$

Now let $\Omega_1 \subseteq \Omega$.

$$\begin{aligned}
 \|u\|_{A, \Omega_1} &\equiv \inf \left\{ t > 0 : \int_{\Omega_1} A \left(\frac{|u(x)|}{t} \right) d\mu \leq 1 \right\} \\
 &\leq \inf \left\{ t > 0 : \int_{\Omega} A \left(\frac{|u(x)|}{t} \right) d\mu \leq 1 \right\} \equiv \|u\|_{A, \Omega}.
 \end{aligned}$$

This occurs because if t is in the second set, then it is in the first so the infimum of the second is no smaller than that of the first. This proves the proposition.

Next it is shown that $L_A(\Omega)$ is a Banach space.

Theorem 18.1.8 $L_A(\Omega)$ is a Banach space and every Cauchy sequence has a subsequence which also converges pointwise a.e.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $L_A(\Omega)$ and select a subsequence $\{f_{n_k}\}$ such that

$$\|f_{n_{k+1}} - f_{n_k}\|_A \leq 2^{-k}.$$

Thus

$$f_{n_m}(x) = f_{n_1}(x) + \sum_{k=1}^{m-1} f_{n_{k+1}}(x) - f_{n_k}(x).$$

Let

$$g_m(x) \equiv |f_{n_1}(x)| + \sum_{k=1}^{m-1} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Then

$$\|g_m\|_A \leq \|f_{n_1}\|_A + \sum_{k=1}^{\infty} 2^{-k} \equiv K < \infty.$$

Let

$$g(x) \equiv \lim_{m \rightarrow \infty} g_m(x) \equiv |f_{n_1}(x)| + \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)|.$$

Now $K > \|g_m\|_A$ so

$$1 \geq \int_{\Omega} A\left(\frac{|g_m(x)|}{K}\right) d\mu.$$

By the monotone convergence theorem,

$$1 \geq \int_{\Omega} A\left(\frac{|g(x)|}{K}\right) d\mu$$

showing $g(x) < \infty$ a.e., say for all $x \notin E$ where E is a measurable set having measure zero. Let

$$\begin{aligned} f(x) &\equiv \mathcal{K}_{E^C}(x) \left(f_{n_1}(x) + \sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)) \right) \\ &= \lim_{m \rightarrow \infty} \mathcal{K}_{E^C}(x) f_{n_m}(x). \end{aligned}$$

Thus f is measurable and $f_{n_m}(x) \rightarrow f(x)$ a.e. as $m \rightarrow \infty$.

For $l > k$,

$$\|f_{n_k} - f_{n_l}\|_A < \frac{1}{2^{k-2}}$$

and so

$$1 \geq \int_{\Omega} A\left(\frac{|f_{n_l}(x) - f_{n_k}(x)|}{\left(\frac{1}{2^{k-2}}\right)}\right) d\mu.$$

By Fatou's lemma, let $l \rightarrow \infty$ and obtain

$$1 \geq \int_{\Omega} A\left(\frac{|f(x) - f_{n_k}(x)|}{\left(\frac{1}{2^{k-2}}\right)}\right) d\mu$$

and so $(f - f_{n_k}) 2^{k-2} \in K_A(\Omega)$ and so $f - f_{n_k} \in L_A(\Omega)$, $f_{n_k} \in L_A(\Omega)$. Since $L_A(\Omega)$ is a vector space, this shows $f \in L_A(\Omega)$. Also

$$\|f - f_{n_k}\|_A \leq \frac{1}{2^{k-2}},$$

showing that $f_{n_k} \rightarrow f$ in $L_A(\Omega)$. Since a subsequence converges in $L_A(\Omega)$, it follows the original Cauchy sequence also converges to f in $L_A(\Omega)$. This proves the theorem.

Next consider the space, $E_A(\Omega)$ which will be a subspace of the Orlicz class, $K_A(\Omega)$ just as $L_A(\Omega)$ is a vector space containing the Orlicz class.

Definition 18.1.9 Let $\mathbb{S}$ denote the set of simple functions, s , such that

$$\mu(\{x : s(x) \neq 0\}) < \infty.$$

Then define

$$E_A(\Omega) \equiv \text{the closure in } L_A(\Omega) \text{ of } \mathbb{S}.$$

Proposition 18.1.10 $E_A(\Omega) \subseteq K_A(\Omega) \subseteq L_A(\Omega)$ and they are all equal if (A, Ω) is Δ regular.

Proof: First note that $\mathbb{S} \subseteq K_A(\Omega) \cap E_A(\Omega)$. Let $f \in E_A(\Omega)$. Then by the definition of $E_A(\Omega)$, there exists $s_n \in \mathbb{S}$ such that

$$\|s_n - f\|_A \rightarrow 0.$$

Therefore, for n large enough,

$$\|s_n - f\|_A < \frac{1}{2}$$

and so

$$\int_{\Omega} A\left(\frac{|f - s_n|}{(\frac{1}{2})}\right) d\mu = \int_{\Omega} A(|2f - 2s_n|) d\mu < \infty.$$

Since $\mathbb{S} \subseteq K_A(\Omega)$,

$$\int_{\Omega} A(2|s_n|) d\mu < \infty.$$

Therefore, $2f - 2s_n \in K_A(\Omega)$ and $2s_n \in K_A(\Omega)$ and so, since $K_A(\Omega)$ is convex,

$$\frac{2f - 2s_n}{2} + \frac{2s_n}{2} = f \in K_A(\Omega).$$

This shows $E_A(\Omega) \subseteq K_A(\Omega)$.

Next consider the claim these spaces are all equal in the case that (A, Ω) is Δ regular. It was already shown in Proposition 18.1.5 that in this case,

$$K_A(\Omega) = L_A(\Omega)$$

so it remains to show $E_A(\Omega) = K_A(\Omega)$. Is every $f \in K_A(\Omega)$ the limit in $L_A(\Omega)$ of functions from $\mathbb{S}$? First suppose $\mu(\Omega) = \infty$. Then $A(r|f|) \leq K_r A(|f|)$ and so $A(r|f|) \in L^1(\Omega)$ for any r . Let $\varepsilon > 0$ be given and let

$$\Omega_{\delta} \equiv \{x : |f(x)| \geq \delta\}$$

Then by the dominated convergence theorem,

$$\lim_{\delta \rightarrow 0+} \int_{\Omega \setminus \Omega_\delta} A\left(\frac{|f|}{\varepsilon}\right) d\mu = 0.$$

Choose δ such that

$$\int_{\Omega \setminus \Omega_\delta} A\left(\frac{|f|}{\varepsilon}\right) d\mu < \frac{1}{2}$$

and let $s_n \rightarrow f|_{\Omega_\delta}$ pointwise with $|s_n| \leq |f|_{\Omega_\delta}|$. Then $s_n = 0$ on $\Omega \setminus \Omega_\delta$ and so

$$\begin{aligned} \int_{\Omega} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu &= \int_{\Omega_\delta} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu \\ &+ \int_{\Omega \setminus \Omega_\delta} A\left(\frac{|f|}{\varepsilon}\right) d\mu \leq \int_{\Omega_\delta} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu + \frac{1}{2}. \end{aligned}$$

By the dominated convergence theorem,

$$\int_{\Omega_\delta} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu < \frac{1}{2}$$

for all n large enough. Therefore, for such n ,

$$\int_{\Omega} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu < 1$$

and so $\|f - s_n\|_A \leq \varepsilon$ showing that in this case $E_A(\Omega) \supseteq K_A(\Omega)$ since $\varepsilon > 0$ is arbitrary.

Now suppose $\mu(\Omega) < \infty$. In this case, only assume $A(rt) \leq K_r A(t)$ for t large enough, say for $t \geq M_r$. However, this is enough to conclude $A(r|f|) \in L^1(\Omega)$ for any $r > 0$ because $\mu(\Omega) < \infty$ and $f \in K_A(\Omega)$. Let $s_n \rightarrow f$ pointwise with $|s_n| \leq |f|$, and s simple. Then

$$A\left(\frac{|f - s_n|}{\varepsilon}\right) \leq A\left(\frac{2}{\varepsilon}|f|\right) \in L^1(\Omega)$$

and so the dominated convergence theorem implies

$$\lim_{n \rightarrow \infty} \int_{\Omega} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu = 0.$$

Hence

$$\int_{\Omega} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu < 1$$

for all n large enough and so for such n ,

$$\|f - s_n\|_A \leq \varepsilon$$

which proves the proposition.

It turns out $E_A(\Omega)$ is the largest linear subspace of $K_A(\Omega)$.

Proposition 18.1.11 $E_A(\Omega)$ is the maximal linear subspace of $K_A(\Omega)$.

Proof: Let M be a subspace of $K_A(\Omega)$. Is $M \subseteq E_A(\Omega)$? For $f \in M$, $f/\varepsilon \in K_A(\Omega)$ for all $\varepsilon > 0$ because of the fact that M is a subspace and $f \in M$. Thus $A(|f|/\varepsilon)$ is in $L^1(\Omega)$. Let $\varepsilon > 0$ be given, choose $\delta > 0$ and let

$$F_\delta \equiv \{x : |f(x)| \leq \delta\}$$

By the dominated convergence theorem there exists δ small enough that

$$\int_{F_\delta} A\left(\frac{2|f|}{\varepsilon}\right) d\mu < \frac{1}{2}.$$

Let $|s_n| \leq |f| \chi_{F_\delta^C}$ and $s_n \rightarrow f \chi_{F_\delta^C}$ pointwise for s_n a simple function. Thus $s_n = 0$ on F_δ and so $s_n \in \mathbb{S}$ because $\mu(F_\delta^C) < \infty$. Now

$$\begin{aligned} \int_{\Omega} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu &= \int_{F_\delta} A\left(\frac{|f|}{\varepsilon}\right) d\mu + \int_{F_\delta^C} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu \\ &< \frac{1}{2} + \int_{F_\delta^C} A\left(\frac{|f - s_n|}{\varepsilon}\right) d\mu. \end{aligned}$$

The integrand in the last integral is no larger than $\frac{2|f|}{\varepsilon}$ and so by the dominated convergence theorem, this integral converges to 0 as $n \rightarrow \infty$. In particular, it is eventually less than $\frac{1}{2}$. Therefore, for such n ,

$$\|f - s_n\|_A \leq r.$$

Since r is arbitrary, this shows that $f \in E_A(\Omega)$ which proves the proposition.

Next is a comparison of these function spaces for different choices of the N function. The notation $X \hookrightarrow Y$ for two normed linear spaces means X is a subset of Y and the identity map is continuous.

Proposition 18.1.12 $L_B(\Omega) \hookrightarrow L_A(\Omega)$ if either

$$B(t) \geq A(t) \text{ for all } t \geq 0 \quad (18.1.12)$$

or if

$$B(t) \geq A(t) \text{ for all } t > M \quad (18.1.13)$$

and $\mu(\Omega) < \infty$.

Proof: Let $f \in L_B(\Omega)$ and let

$$\int_{\Omega} B\left(\frac{|f|}{t}\right) d\mu \leq 1.$$

Then if 18.1.12 holds, it follows

$$\int_{\Omega} A\left(\frac{|f|}{t}\right) d\mu \leq 1.$$

Thus if $t \geq \|f\|_B$ then $t \geq \|f\|_A$ which implies $\|f\|_B \geq \|f\|_A$.

Now suppose 18.1.13 holds and $\mu(\Omega) < \infty$. Then $\max(A, B)$ is an N function dominating both A and B for all t . By what was just shown $L_{\max(A, B)}(\Omega) \hookrightarrow L_B(\Omega)$. Then let $f \in L_B(\Omega)$ and let

$$\int_{\Omega} B\left(\frac{|f|}{t}\right) d\mu < 1.$$

Then

$$\begin{aligned} \int_{\Omega} \max(A, B)\left(\frac{|f|}{t}\right) d\mu &= \int_{[\frac{|f|}{t} > M]} B\left(\frac{|f|}{t}\right) d\mu \\ &\quad + \int_{[\frac{|f|}{t} \leq M]} \max(A, B)\left(\frac{|f|}{t}\right) d\mu \\ &\leq \int_{\Omega} B\left(\frac{|f|}{t}\right) d\mu + \mu(\Omega) \max(A, B)(M) < \infty. \end{aligned}$$

It follows $\frac{|f|}{t} \in K_{\max(A, B)}(\Omega)$ and so $f \in L_{\max(A, B)}(\Omega)$. Hence $L_B(\Omega) = L_{\max(A, B)}(\Omega)$ and the identity map from $L_{\max(A, B)}(\Omega)$ to $L_B(\Omega)$ is continuous. Therefore, by the open mapping theorem, the norms $\|\cdot\|_B$ and $\|\cdot\|_{\max(A, B)}$ are equivalent. Hence for $f \in L_B(\Omega)$,

$$\|f\|_A \leq \|f\|_{\max(A, B)} \leq C \|f\|_B.$$

This proves the proposition.

Corollary 18.1.13 *Suppose there exists $C > 0$, a constant such that either*

$$CB(t) \geq A(t)$$

for all $t \geq 0$ or

$$CB(t) \geq A(t)$$

for all $t > M$ and $\mu(\Omega) < \infty$. Then

$$L_B(\Omega) \hookrightarrow L_A(\Omega).$$

Proof: If $f \in L_B(\Omega)$ then $f = \lambda u$ where $u \in K_B(\Omega) = K_{CB}(\Omega)$. Hence $L_{CB}(\Omega) = L_B(\Omega)$ and the two norms on $L_B(\Omega)$,

$$\|\cdot\|_{CB}, \text{ and } \|\cdot\|_B,$$

are equivalent norms by the open mapping theorem. Hence by the Proposition 18.1.12, if $f \in L_B(\Omega)$,

$$\|f\|_A \leq C_1 \|f\|_{CB} \leq C_2 \|f\|_B$$

which proves the corollary.

Definition 18.1.14 A increases essentially more slowly than B if for all $a > 0$,

$$\lim_{t \rightarrow \infty} \frac{A(at)}{B(t)} = 0$$

The next theorem gives added information on how these spaces are related in case that one N function increases essentially more slowly than the other.

Theorem 18.1.15 Suppose $\mu(\Omega) < \infty$ and A increases essentially more slowly than B . Then

$$L_B(\Omega) \hookrightarrow E_A(\Omega)$$

Proof: Let $f \in L_B(\Omega)$. Then there exists $\lambda > 0$ such that

$$\int_{\Omega} B\left(\frac{|f|}{\lambda}\right) d\mu \leq 1.$$

Let r be such that for $t \geq r$,

$$A(|\lambda|t) \leq B(t).$$

Then

$$\begin{aligned} \int_{\Omega} A(|f|) d\mu &= \int_{\{|f| \geq r\}} A(|f|) d\mu + \int_{\{|f| < r\}} A(|f|) d\mu \\ &\leq \int_{\Omega} B\left(\frac{|f|}{|\lambda|}\right) d\mu + A(r) \mu(\Omega) \\ &< 1 + A(r) \mu(\Omega). \end{aligned}$$

Therefore, $L_B(\Omega)$ is a linear space contained in $K_A(\Omega)$. It follows from Proposition 18.1.11 that $L_B(\Omega) \subseteq E_A(\Omega)$. This proves the theorem.

The norm of $E_A(\Omega)$ is the same as the norm on $L_A(\Omega)$ so this shows $L_B(\Omega) \hookrightarrow L_A(\Omega)$.

Note that for $1 < p < q$ and $A(t) = t^p/p$, $B(t) = t^q/q$,

$$A(rt) = r^p t^p$$

and

$$\lim_{t \rightarrow \infty} \frac{A(rt)}{B(t)} = 0$$

showing this case is covered by the above theorem.

If A is increasing essentially more slowly than B and $\mu(\Omega) < \infty$, this has shown the following inclusions

$$E_B(\Omega) \subseteq K_B(\Omega) \subseteq L_B(\Omega) \hookrightarrow E_A(\Omega) \subseteq K_A(\Omega) \subseteq L_A(\Omega).$$

In the case of $A(t) = t^p/p$, $B(t) = t^q/q$ both (A, Ω) and (B, Ω) are Δ regular and so in this case or any other case where the N functions are Δ regular, the above sequence of inclusions reduces to

$$E_B(\Omega) = K_B(\Omega) = L_B(\Omega) \hookrightarrow E_A(\Omega) = K_A(\Omega) = L_A(\Omega).$$

18.2 Dual Spaces In Orlicz Space

Recall that for $s, t \geq 0$,

$$st \leq A(t) + \tilde{A}(s).$$

Let $v \in L_{\tilde{A}}(\Omega)$ and $u \in L_A(\Omega)$. Then there is a version of Holder's inequality as follows. For $\varepsilon > 0$,

$$\frac{|v|}{\|v\|_{\tilde{A}} + \varepsilon} \in K_{\tilde{A}}(\Omega), \quad \frac{|u|}{\|u\|_A + \varepsilon} \in K_A(\Omega).$$

Therefore,

$$\begin{aligned} \int_{\Omega} \left(\frac{|u|}{\|u\|_A + \varepsilon} \right) \left(\frac{|v|}{\|v\|_{\tilde{A}} + \varepsilon} \right) d\mu &\leq \int_{\Omega} A \left(\frac{|u|}{\|u\|_A + \varepsilon} \right) d\mu \\ &+ \int_{\Omega} \tilde{A} \left(\frac{|v|}{\|v\|_{\tilde{A}} + \varepsilon} \right) d\mu \leq 2 \end{aligned}$$

and so $uv \in L^1(\Omega)$ and

$$\left| \int_{\Omega} uv d\mu \right| \leq \int_{\Omega} |u| |v| d\mu \leq 2 (\|v\|_{\tilde{A}} + \varepsilon) (\|u\|_A + \varepsilon).$$

Since ε is arbitrary this shows

$$\left| \int_{\Omega} uv d\mu \right| \leq \int_{\Omega} |u| |v| d\mu \leq 2 \|v\|_{\tilde{A}} \|u\|_A. \quad (18.2.14)$$

Defining L_v for $v \in \tilde{A}$ by

$$L_v(u) \equiv \int_{\Omega} uv d\mu,$$

it follows $L_v \in L_A(\Omega)'$. From now on assume the measure space is σ finite. That is, there exist measurable sets, Ω_k satisfying the following:

$$\Omega = \cup_{k=1}^{\infty} \Omega_k, \quad \mu(\Omega_k) < \infty, \quad \Omega_k \subseteq \Omega_{k+1}.$$

Then

Proposition 18.2.1 *For $v \in L_{\tilde{A}}(\Omega)$, the following inequality holds.*

$$\|v\|_{\tilde{A}} \leq \|L_v\| \leq 2 \|v\|_{\tilde{A}}.$$

Here L_v is considered as either an element of $E_A(\Omega)'$ or $L_A(\Omega)'$ and $\|L_v\|$ refers to the operator norm in either dual space.

Proof: The inequality 18.2.14 implies $\|L_v\| \leq 2\|v\|_{\tilde{A}}$. It remains to show the other half of the inequality. If $L_v = 0$ there is nothing to show because this would imply that $v = 0$ so assume $\|L_v\| > 0$. Define a measurable function, u , as follows. Letting $r \in (0, 1)$,

$$u(x) \equiv \begin{cases} \tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) / \frac{v(x)}{\|L_v\|} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0. \end{cases} \quad (18.2.15)$$

Now let

$$F_n \equiv \{x : |u(x)| \leq n\} \cap \Omega_n \cap \{x : v(x) \neq 0\} \quad (18.2.16)$$

and define

$$u_n(x) \equiv u(x) \chi_{F_n}(x). \quad (18.2.17)$$

Thus u_n is bounded and equals zero off a set which has finite measure. It follows that

$$A\left(\frac{|u_n|}{\alpha}\right) \in L^1(\Omega)$$

for all $\alpha > 0$. I claim that $\|u_n\|_A \leq 1$. If not, there exists $\varepsilon > 0$ such that $\|u_n\|_A - \varepsilon > 1$. Then since A is convex,

$$1 < \int_{\Omega} A\left(\frac{|u_n|}{\|u_n\|_A - \varepsilon}\right) d\mu \leq \frac{1}{\|u_n\|_A - \varepsilon} \int_{\Omega} A(|u_n|) d\mu.$$

Taking $\varepsilon \rightarrow 0+$, using 18.1.6, and convexity of A along with 18.2.15 and 18.2.17,

$$\begin{aligned} \|u_n\|_A &\leq \int_{\Omega} A(|u_n|) d\mu = \int_{F_n} A\left(r\tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) / \frac{rv(x)}{\|L_v\|}\right) d\mu \\ &\leq r \int_{F_n} A\left(\tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) / \frac{rv(x)}{\|L_v\|}\right) d\mu \leq r \int_{F_n} \tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) d\mu \\ &= r \frac{1}{\|L_v\|} \int_{\Omega} u_n(x) v(x) d\mu \leq r \frac{1}{\|L_v\|} \|u_n\|_A \|L_v\| = r \|u_n\|_A, \end{aligned}$$

a contradiction since $r < 1$. Therefore, from 18.2.15,

$$\|L_v\| \geq |L_v(u_n)| \equiv \int_{F_n} v(x) u(x) d\mu = \|L_v\| \int_{F_n} \tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) d\mu$$

and so

$$1 \geq \int_{F_n} \tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) d\mu$$

By the monotone convergence theorem, letting $n \rightarrow \infty$,

$$1 \geq \int_{\Omega} \tilde{A}\left(\frac{r|v(x)|}{\|L_v\|}\right) d\mu$$

showing that

$$\|v\|_{\tilde{A}} \leq \frac{\|L_v\|}{r}.$$

Since this holds for all $r \in (0, 1)$, it follows $\|L_v\| \geq \|v\|_{\tilde{A}}$ as claimed. This proves the proposition.

Now what follows is the Riesz representation theorem for the dual space of $E_A(\Omega)$.

Theorem 18.2.2 *Suppose $\mu(\Omega) < \infty$ and suppose $L \in E_A(\Omega)'$. Then the map $v \rightarrow L_v$ from $L_{\tilde{A}}(\Omega)$ to $E_A(\Omega)'$ is one to one continuous, linear, and onto. If (Ω, A) is Δ regular then $v \rightarrow L_v$ is one to one, linear, onto and continuous as a map from $L_{\tilde{A}}(\Omega)$ to $L_A(\Omega)'$.*

Proof: It is obvious this map is linear. From Proposition 18.2.1 it is continuous and one to one. It remains only to verify that it is onto. Let $L \in E_A(\Omega)'$ and define a complex valued function, λ , mapping the measurable sets to $\mathbb{C}$ as follows.

$$\lambda(F) \equiv L(\mathcal{X}_F).$$

In case $\mu(F) \neq 0$,

$$\begin{aligned} \int_{\Omega} A\left(\mathcal{X}_F(x) A^{-1}\left(\frac{1}{\mu(F)}\right)\right) d\mu &= \int_F A\left(A^{-1}\left(\frac{1}{\mu(F)}\right)\right) d\mu \\ &= \int_F \frac{1}{\mu(F)} d\mu = 1 \end{aligned}$$

and so

$$\|\mathcal{X}_F\|_A \leq \frac{1}{A^{-1}\left(\frac{1}{\mu(F)}\right)}. \quad (18.2.18)$$

In fact, λ is actually a complex measure. To see this, suppose $F_i \uparrow F$. Then from the formula just derived,

$$\|\mathcal{X}_{F_i} - \mathcal{X}_F\|_A = \|\mathcal{X}_{F \setminus F_i}\|_A \leq \frac{1}{A^{-1}\left(\frac{1}{\mu(F \setminus F_i)}\right)}$$

which converges to zero as $i \rightarrow \infty$. Therefore, if the F_i are disjoint and $F = \bigcup_{i=1}^{\infty} F_i$, let $S_m \equiv \bigcup_{i=1}^m F_i$ so that $S_m \uparrow F$. Then since $\mathcal{X}_{S_m} \rightarrow \mathcal{X}_F$ in $E_A(\Omega)$ and L is continuous,

$$\begin{aligned} \lambda(F) &\equiv L(\mathcal{X}_F) = \lim_{m \rightarrow \infty} L(\mathcal{X}_{S_m}) \\ &= \lim_{m \rightarrow \infty} \sum_{i=1}^m L(\mathcal{X}_{F_i}) = \sum_{i=1}^{\infty} \lambda(F_i). \end{aligned}$$

Next observe that λ is absolutely continuous with respect to μ . To see this, suppose $\mu(F) = 0$. Then if $t > 0$,

$$\int_{\Omega} A\left(\frac{\mathcal{X}_F(x)}{t}\right) d\mu = 0 < 1$$

for all $t > 0$ and so $\|\mathcal{X}_F\|_A = 0$. Therefore, $\lambda(F) \equiv L(\mathcal{X}_F) = 0$.

It follows by the Radon Nikodym theorem there exists $v \in L^1(\Omega)$ such that

$$L(\mathcal{X}_F) = \lambda(F) = \int_F v d\mu.$$

Therefore, for all $s \in \mathbb{S}$,

$$L(s) = \int_F s v d\mu. \quad (18.2.19)$$

I need to show that v is actually in $L_{\tilde{A}}(\Omega)$. If $v = 0$ a.e., there is nothing to prove so assume this is not so. Let u be defined by.

$$u(x) \equiv \begin{cases} \tilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) / \frac{v(x)}{\|L\|} & \text{if } v(x) \neq 0 \\ 0 & \text{if } v(x) = 0. \end{cases} \quad (18.2.20)$$

for $r \in (0, 1)$. Now let

$$F_n \equiv \{x : |u(x)| \leq n\} \cap \{x : v(x) \neq 0\} \quad (18.2.21)$$

and define

$$u_n(x) \equiv u(x) \mathcal{X}_{F_n}(x). \quad (18.2.22)$$

I claim $\|u_n\|_A \leq 1$. It is clear that since $\mu(\Omega) < \infty$, $u_n \in E_A(\Omega)$. If $\|u_n\|_A > 1$, Then for ε small enough,

$$\|u_n\|_A - \varepsilon > 1$$

and so, by convexity of A and the fact that $A(0) = 0$,

$$1 < \int_{\Omega} A\left(\frac{|u_n(x)|}{\|u_n\|_A - \varepsilon}\right) d\mu \leq \frac{1}{\|u_n\|_A - \varepsilon} \int_{\Omega} A(|u_n(x)|) d\mu$$

and so, letting $\varepsilon \rightarrow 0+$ and using 18.1.6 and convexity of A as in the proof of the preceeding proposition,

$$\begin{aligned} \|u_n\|_A &\leq \int_{\Omega} A(|u_n(x)|) d\mu \leq \int_{F_n} A\left(r\tilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) / \frac{r|v(x)|}{\|L\|}\right) d\mu \\ &\leq r \int_{F_n} A\left(\tilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) / \frac{r|v(x)|}{\|L\|}\right) d\mu \leq r \int_{F_n} \tilde{A}\left(\frac{r|v(x)|}{\|L\|}\right) d\mu \\ &\leq r \frac{1}{\|L\|} \int_{F_n} u(x) v(x) d\mu = \frac{r}{\|L\|} \int_{\Omega} u_n v d\mu. \end{aligned} \quad (18.2.23)$$

Now by Theorem 8.3.9 applied to the positive and negative parts of real and imaginary parts, there exists a uniformly bounded sequence of simple functions, $\{s_k\}$ converging uniformly to u_n , implying convergence in $E_A(\Omega)$, and so

$$Lu_n = \lim_{k \rightarrow \infty} Ls_k = \lim_{k \rightarrow \infty} \int_{\Omega} s_k v d\mu = \int_{\Omega} u_n v d\mu. \quad (18.2.24)$$

therefore, from 18.2.23,

$$\|u_n\|_A \leq \frac{r}{\|L\|} \int_{\Omega} u_n v d\mu = \frac{r}{\|L\|} L(u_n) \leq \frac{r}{\|L\|} \|L\| \|u_n\|_A,$$

which is a contradiction since $r < 1$. Therefore, $\|u_n\|_A \leq 1$ and from 18.2.23,

$$\begin{aligned} \|L\| &\geq \|L\| \|u_n\|_A \geq |Lu_n| = \int_{\Omega} u_n v d\mu \\ &\geq \|L\| \int_{F_n} \tilde{A} \left(\frac{r|v(x)|}{\|L\|} \right) d\mu. \end{aligned}$$

Letting $n \rightarrow \infty$ the monotone convergence theorem and the above imply

$$\int_{\Omega} \tilde{A} \left(\frac{r|v(x)|}{\|L\|} \right) d\mu \leq 1$$

which shows that $v \in L_{\tilde{A}}(\Omega)$ and $\|v\|_{\tilde{A}} \leq \frac{\|L\|}{r}$ for all $r \in (0, 1)$. Therefore, $\|v\|_{\tilde{A}} \leq \|L\|$.

Since $v \in L_{\tilde{A}}(\Omega)$ it follows $L_v = L$ on $\mathbb{S}$ and so $L_v = L$ because $\mathbb{S}$ is dense in $E_A(\Omega)$. The last assertion follows from Proposition 18.1.10. This completes the proof.

Hausdorff Measure

19.1 Definition Of Hausdorff Measures

This chapter is on Hausdorff measures. First I will discuss some outer measures. In all that is done here, $\alpha(n)$ will be the volume of the ball in $\mathbb{R}^n$ which has radius 1.

Definition 19.1.1 For a set, E , denote by $r(E)$ the number which is half the diameter of E . Thus

$$r(E) \equiv \frac{1}{2} \sup \{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in E\} \equiv \frac{1}{2} \text{diam}(E)$$

Let $E \subseteq \mathbb{R}^n$.

$$\mathcal{H}_\delta^s(E) \equiv \inf \left\{ \sum_{j=1}^{\infty} \beta(s)(r(C_j))^s : E \subseteq \cup_{j=1}^{\infty} C_j, \text{diam}(C_j) \leq \delta \right\}$$

$$\mathcal{H}^s(E) \equiv \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E).$$

In the above definition, $\beta(s)$ is an appropriate positive constant depending on s . It will turn out that for n an integer, $\beta(n) = \alpha(n)$ where $\alpha(n)$ is the Lebesgue measure of the unit ball, $B(\mathbf{0}, 1)$ where the usual norm is used to determine this ball.

Lemma 19.1.2 $\mathcal{H}^s$ and $\mathcal{H}_\delta^s$ are outer measures.

Proof: It is clear that $\mathcal{H}^s(\emptyset) = 0$ and if $A \subseteq B$, then $\mathcal{H}^s(A) \leq \mathcal{H}^s(B)$ with similar assertions valid for $\mathcal{H}_\delta^s$. Suppose $E = \cup_{i=1}^{\infty} E_i$ and $\mathcal{H}_\delta^s(E_i) < \infty$ for each i . Let $\{C_j^i\}_{j=1}^{\infty}$ be a covering of E_i with

$$\sum_{j=1}^{\infty} \beta(s)(r(C_j^i))^s - \varepsilon/2^i < \mathcal{H}_\delta^s(E_i)$$

and $\text{diam}(C_j^i) \leq \delta$. Then

$$\begin{aligned} \mathcal{H}_\delta^s(E) &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \beta(s)(r(C_j^i))^s \\ &\leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^s(E_i) + \varepsilon/2^i \\ &\leq \varepsilon + \sum_{i=1}^{\infty} \mathcal{H}_\delta^s(E_i). \end{aligned}$$

It follows that since $\varepsilon > 0$ is arbitrary,

$$\mathcal{H}_\delta^s(E) \leq \sum_{i=1}^{\infty} \mathcal{H}_\delta^s(E_i)$$

which shows $\mathcal{H}_\delta^s$ is an outer measure. Now notice that $\mathcal{H}_\delta^s(E)$ is increasing as $\delta \rightarrow 0$. Picking a sequence δ_k decreasing to 0, the monotone convergence theorem implies

$$\mathcal{H}^s(E) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(E_i).$$

This proves the lemma.

The outer measure $\mathcal{H}^s$ is called s dimensional Hausdorff measure when restricted to the σ algebra of $\mathcal{H}^s$ measurable sets.

Next I will show the σ algebra of $\mathcal{H}^s$ measurable sets includes the Borel sets. This is done by the following very interesting condition known as Caratheodory's criterion.

19.1.1 Properties Of Hausdorff Measure

Definition 19.1.3 For two sets, A, B in a metric space, we define

$$\text{dist}(A, B) \equiv \inf \{d(x, y) : x \in A, y \in B\}.$$

Theorem 19.1.4 Let μ be an outer measure on the subsets of (X, d) , a metric space. If

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

whenever $\text{dist}(A, B) > 0$, then the σ algebra of measurable sets contains the Borel sets.

Proof: It suffices to show that closed sets are in $\mathcal{S}$, the σ -algebra of measurable sets, because then the open sets are also in $\mathcal{S}$ and consequently $\mathcal{S}$ contains the Borel sets. Let K be closed and let S be a subset of Ω . Is $\mu(S) \geq \mu(S \cap K) + \mu(S \setminus K)$? It suffices to assume $\mu(S) < \infty$. Let

$$K_n \equiv \{x : \text{dist}(x, K) \leq \frac{1}{n}\}$$

By Lemma 6.1.7 on Page 141, $x \rightarrow \text{dist}(x, K)$ is continuous and so K_n is closed. By the assumption of the theorem,

$$\mu(S) \geq \mu((S \cap K) \cup (S \setminus K_n)) = \mu(S \cap K) + \mu(S \setminus K_n) \quad (19.1.1)$$

since $S \cap K$ and $S \setminus K_n$ are a positive distance apart. Now

$$\mu(S \setminus K_n) \leq \mu(S \setminus K) \leq \mu(S \setminus K_n) + \mu((K_n \setminus K) \cap S). \quad (19.1.2)$$

If $\lim_{n \rightarrow \infty} \mu((K_n \setminus K) \cap S) = 0$ then the theorem will be proved because this limit along with 19.1.2 implies $\lim_{n \rightarrow \infty} \mu(S \setminus K_n) = \mu(S \setminus K)$ and then taking a limit in 19.1.1, $\mu(S) \geq \mu(S \cap K) + \mu(S \setminus K)$ as desired. Therefore, it suffices to establish this limit.

Since K is closed, a point, $x \notin K$ must be at a positive distance from K and so

$$K_n \setminus K = \bigcup_{k=n}^{\infty} K_k \setminus K_{k+1}.$$

Therefore

$$\mu(S \cap (K_n \setminus K)) \leq \sum_{k=n}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})). \quad (19.1.3)$$

If

$$\sum_{k=1}^{\infty} \mu(S \cap (K_k \setminus K_{k+1})) < \infty, \quad (19.1.4)$$

then $\mu(S \cap (K_n \setminus K)) \rightarrow 0$ because it is dominated by the tail of a convergent series so it suffices to show 19.1.4.

$$\begin{aligned} & \sum_{k=1}^M \mu(S \cap (K_k \setminus K_{k+1})) = \\ & \sum_{k \text{ even}, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd}, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})). \end{aligned} \quad (19.1.5)$$

By the construction, the distance between any pair of sets, $S \cap (K_k \setminus K_{k+1})$ for different even values of k is positive and the distance between any pair of sets, $S \cap (K_k \setminus K_{k+1})$ for different odd values of k is positive. Therefore,

$$\sum_{k \text{ even}, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) + \sum_{k \text{ odd}, k \leq M} \mu(S \cap (K_k \setminus K_{k+1})) \leq$$

$$\mu\left(\bigcup_{k \text{ even}} S \cap (K_k \setminus K_{k+1})\right) + \mu\left(\bigcup_{k \text{ odd}} S \cap (K_k \setminus K_{k+1})\right) \leq 2\mu(S) < \infty$$

and so for all M , $\sum_{k=1}^M \mu(S \cap (K_k \setminus K_{k+1})) \leq 2\mu(S)$ showing 19.1.4 and proving the theorem.

With the above theorem, the following theorem is easy to obtain.

Theorem 19.1.5 *The σ algebra of $\mathcal{H}^s$ measurable sets contains the Borel sets and $\mathcal{H}^s$ has the property that for all $E \subseteq \mathbb{R}^n$, there exists a Borel set $F \supseteq E$ such that $\mathcal{H}^s(F) = \mathcal{H}^s(E)$.*

Proof: Let $\text{dist}(A, B) = 2\delta_0 > 0$. Is it the case that

$$\mathcal{H}^s(A) + \mathcal{H}^s(B) = \mathcal{H}^s(A \cup B)?$$

This is what is needed to use Caratheodory's criterion.

Let $\{C_j\}_{j=1}^\infty$ be a covering of $A \cup B$ such that $\text{diam}(C_j) \leq \delta < \delta_0$ for each j and

$$\mathcal{H}_\delta^s(A \cup B) + \varepsilon > \sum_{j=1}^\infty \beta(s)(r(C_j))^s.$$

Thus

$$\mathcal{H}_\delta^s(A \cup B) + \varepsilon > \sum_{j \in J_1} \beta(s)(r(C_j))^s + \sum_{j \in J_2} \beta(s)(r(C_j))^s$$

where

$$J_1 = \{j : C_j \cap A \neq \emptyset\}, \quad J_2 = \{j : C_j \cap B \neq \emptyset\}.$$

Recall $\text{dist}(A, B) = 2\delta_0$, $J_1 \cap J_2 = \emptyset$. It follows

$$\mathcal{H}_\delta^s(A \cup B) + \varepsilon > \mathcal{H}_\delta^s(A) + \mathcal{H}_\delta^s(B).$$

Letting $\delta \rightarrow 0$, and noting $\varepsilon > 0$ was arbitrary, yields

$$\mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Equality holds because $\mathcal{H}^s$ is an outer measure. By Caratheodory's criterion, $\mathcal{H}^s$ is a Borel measure.

To verify the second assertion, note first there is no loss of generality in letting $\mathcal{H}^s(E) < \infty$. Let

$$E \subseteq \cup_{j=1}^\infty C_j, \quad r(C_j) < \delta,$$

and

$$\mathcal{H}_\delta^s(E) + \delta > \sum_{j=1}^\infty \beta(s)(r(C_j))^s.$$

Let

$$F_\delta = \cup_{j=1}^\infty \overline{C_j}.$$

Thus $F_\delta \supseteq E$ and

$$\begin{aligned} \mathcal{H}_\delta^s(E) &\leq \mathcal{H}_\delta^s(F_\delta) \leq \sum_{j=1}^\infty \beta(s)(r(\overline{C_j}))^s \\ &= \sum_{j=1}^\infty \beta(s)(r(C_j))^s < \delta + \mathcal{H}_\delta^s(E). \end{aligned}$$

Let $\delta_k \rightarrow 0$ and let $F = \bigcap_{k=1}^{\infty} F_{\delta_k}$. Then $F \supseteq E$ and

$$\mathcal{H}_{\delta_k}^s(E) \leq \mathcal{H}_{\delta_k}^s(F) \leq \mathcal{H}_{\delta_k}^s(F_{\delta}) \leq \delta_k + \mathcal{H}_{\delta_k}^s(E).$$

Letting $k \rightarrow \infty$,

$$\mathcal{H}^s(E) \leq \mathcal{H}^s(F) \leq \mathcal{H}^s(E)$$

This proves the theorem.

A measure satisfying the conclusion of Theorem 19.1.5 is called a Borel regular measure.

19.2 $\mathcal{H}^n$ And m_n

Next I will compare $\mathcal{H}^n$ and m_n . To do this, recall the following covering theorem which is a summary of Corollaries 10.4.5 and 10.4.4 found on Page 297.

Theorem 19.2.1 *Let $E \subseteq \mathbb{R}^n$ and let $\mathcal{F}$, be a collection of balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^{\infty}$, such that $\overline{m_n}(E \setminus \bigcup_{j=1}^{\infty} B_j) = 0$.*

In the next lemma, the balls are the usual balls taken with respect to the usual distance in $\mathbb{R}^n$.

Lemma 19.2.2 *If $m_n(S) = 0$ then $\mathcal{H}^n(S) = \mathcal{H}_{\delta}^n(S) = 0$. Also, there exists a constant, k such that $\mathcal{H}^n(E) \leq km_n(E)$ for all E Borel. Also, if $Q_0 \equiv [0, 1]^n$, the unit cube, then $\mathcal{H}^n([0, 1]^n) > 0$.*

Proof: Suppose first $m_n(S) = 0$. Without loss of generality, S is bounded. Then by outer regularity, there exists a bounded open V containing S and $m_n(V) < \varepsilon$. For each $\mathbf{x} \in S$, there exists a ball $B_{\mathbf{x}}$ such that $\widehat{B_{\mathbf{x}}} \subseteq V$ and $\delta > r(\widehat{B_{\mathbf{x}}})$. By the Vitali covering theorem there is a sequence of disjoint balls $\{B_k\}$ such that $\{\widehat{B_k}\}$ covers S . Then letting $\alpha(n)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^n$

$$\begin{aligned} \mathcal{H}_{\delta}^n(S) &\leq \sum_k \beta(n) r(\widehat{B_k})^n = \frac{\beta(n)}{\alpha(n)} 5^n \sum_k \alpha(n) r(B_k)^n \\ &\leq \frac{\beta(n)}{\alpha(n)} 5^n m_n(V) < \frac{\beta(n)}{\alpha(n)} 5^n \varepsilon \end{aligned}$$

Since ε is arbitrary, this shows $\mathcal{H}_{\delta}^n(S) = 0$ and now it follows $\mathcal{H}^n(S) = 0$.

Letting U be an open set and $\delta > 0$, consider all balls, B contained in U which have diameters less than δ . This is a Vitali covering of U and therefore by Theorem 19.2.1, there exists $\{B_i\}$, a sequence of disjoint balls of radii less than δ contained

in U such that $\cup_{i=1}^{\infty} B_i$ differs from U by a set of Lebesgue measure zero. Let $\alpha(n)$ be the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Then from what was just shown,

$$\begin{aligned}\mathcal{H}_{\delta}^n(U) &= \mathcal{H}_{\delta}^n(\cup_i B_i) \leq \sum_{i=1}^{\infty} \beta(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} \alpha(n) r(B_i)^n \\ &= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^{\infty} m_n(B_i) = \frac{\beta(n)}{\alpha(n)} m_n(U) \equiv k m_n(U).\end{aligned}$$

Now letting E be Borel, it follows from the outer regularity of m_n there exists a decreasing sequence of open sets, $\{V_i\}$ containing E such that $m_n(V_i) \rightarrow m_n(E)$. Then from the above,

$$\mathcal{H}_{\delta}^n(E) \leq \lim_{i \rightarrow \infty} \mathcal{H}_{\delta}^n(V_i) \leq \lim_{i \rightarrow \infty} k m_n(V_i) = k m_n(E).$$

Since $\delta > 0$ is arbitrary, it follows that also

$$\mathcal{H}^n(E) \leq k m_n(E).$$

This proves the first part of the lemma.

To verify the second part, note that it is obvious $\mathcal{H}_{\delta}^n$ and $\mathcal{H}^n$ are translation invariant because diameters of sets do not change when translated. Therefore, if $\mathcal{H}^n([0, 1]^n) = 0$, it follows $\mathcal{H}^n(\mathbb{R}^n) = 0$ because $\mathbb{R}^n$ is the countable union of translates of $Q_0 \equiv [0, 1]^n$. Since each $\mathcal{H}_{\delta}^n$ is no larger than $\mathcal{H}^n$, the same must hold for $\mathcal{H}_{\delta}^n$. Therefore, there exists a sequence of sets, $\{C_i\}$ each having diameter less than δ such that the union of these sets equals $\mathbb{R}^n$ but

$$1 > \sum_{i=1}^{\infty} \beta(n) r(C_i)^n.$$

Now let B_i be a ball having radius equal to $\text{diam}(C_i) = 2r(C_i)$ which contains C_i . It follows

$$m_n(B_i) = \alpha(n) 2^n r(C_i)^n = \frac{\alpha(n) 2^n}{\beta(n)} \beta(n) r(C_i)^n$$

which implies

$$1 > \sum_{i=1}^{\infty} \beta(n) r(C_i)^n = \sum_{i=1}^{\infty} \frac{\beta(n)}{\alpha(n) 2^n} m_n(B_i) = \infty,$$

a contradiction. This proves the lemma.

Theorem 19.2.3 *By choosing $\beta(n)$ properly, one can obtain $\mathcal{H}^n = m_n$ on all Lebesgue measurable sets.*

Proof: I will show $\mathcal{H}^n$ is a positive multiple of m_n for any choice of $\beta(n)$. Define

$$k = \frac{m_n(Q_0)}{\mathcal{H}^n(Q_0)}$$

where $Q_0 = [0, 1]^n$ is the half open unit cube in $\mathbb{R}^n$. I will show $k\mathcal{H}^n(E) = m_n(E)$ for any Lebesgue measurable set. When this is done, it will follow that by adjusting $\beta(n)$ the multiple can be taken to be 1.

Let $Q = \prod_{i=1}^n [a_i, a_i + 2^{-k})$ be a half open box where $a_i = l2^{-k}$. Thus Q_0 is the union of $(2^k)^n$ of these identical half open boxes. By translation invariance, of $\mathcal{H}^n$ and m_n

$$(2^k)^n \mathcal{H}^n(Q) = \mathcal{H}^n(Q_0) = \frac{1}{k} m_n(Q_0) = \frac{1}{k} (2^k)^n m_n(Q).$$

Therefore, $k\mathcal{H}^n(Q) = m_n(Q)$ for any such half open box and by translation invariance, for the translation of any such half open box. It follows from Lemma 10.1.2 that $k\mathcal{H}^n(U) = m_n(U)$ for all open sets. It follows immediately, since every compact set is the countable intersection of open sets that $k\mathcal{H}^n = m_n$ on compact sets. Therefore, they are also equal on all closed sets because every closed set is the countable union of compact sets. Now let F be an arbitrary Lebesgue measurable set. I will show that F is $\mathcal{H}^n$ measurable and that $k\mathcal{H}^n(F) = m_n(F)$. Let $F_l = B(\mathbf{0}, l) \cap F$. Then there exists H a countable union of compact sets and G a countable intersection of open sets such that

$$H \subseteq F_l \subseteq G \quad (19.2.6)$$

and $m_n(G \setminus H) = 0$ which implies by Lemma 19.2.2

$$m_n(G \setminus H) = k\mathcal{H}^n(G \setminus H) = 0. \quad (19.2.7)$$

To do this, let $\{G_i\}$ be a decreasing sequence of bounded open sets containing F_l and let $\{H_i\}$ be an increasing sequence of compact sets contained in F_l such that

$$k\mathcal{H}^n(G_i \setminus H_i) = m_n(G_i \setminus H_i) < 2^{-i}$$

Then letting $G = \cap_i G_i$ and $H = \cup_i H_i$ this establishes 19.2.6 and 19.2.7. Then by completeness of $\mathcal{H}^n$ it follows F_l is $\mathcal{H}^n$ measurable and

$$k\mathcal{H}^n(F_l) = k\mathcal{H}^n(H) = m_n(H) = m_n(F_l).$$

Now taking $l \rightarrow \infty$, it follows F is $\mathcal{H}^n$ measurable and $k\mathcal{H}^n(F) = m_n(F)$. Therefore, adjusting $\beta(n)$ it can be assumed the constant, k is 1. This proves the theorem.

The exact determination of $\beta(n)$ is more technical.

19.3 Technical Considerations

Let $\alpha(n)$ be the volume of the unit ball in $\mathbb{R}^n$. Thus the volume of $B(\mathbf{0}, r)$ in $\mathbb{R}^n$ is $\alpha(n)r^n$ from the change of variables formula. There is a very important and interesting inequality known as the isodiametric inequality which says that if A is any set in $\mathbb{R}^n$, then

$$\overline{m}(A) \leq \alpha(n)(2^{-1}\text{diam}(A))^n = \alpha(n)r(A)^n.$$

This inequality may seem obvious at first but it is not really. The reason it is not is that there are sets which are not subsets of any sphere having the same diameter as the set. For example, consider an equilateral triangle.

Lemma 19.3.1 *Let $f : \mathbb{R}^{n-1} \rightarrow [0, \infty)$ be Borel measurable and let*

$$S = \{(\mathbf{x}, y) : |y| < f(\mathbf{x})\}.$$

Then S is a Borel set in $\mathbb{R}^n$.

Proof: Set s_k be an increasing sequence of Borel measurable functions converging pointwise to f .

$$s_k(\mathbf{x}) = \sum_{m=1}^{N_k} c_m^k \chi_{E_m^k}(\mathbf{x}).$$

Let

$$S_k = \cup_{m=1}^{N_k} E_m^k \times (-c_m^k, c_m^k).$$

Then $(\mathbf{x}, y) \in S_k$ if and only if $f(\mathbf{x}) > 0$ and $|y| < s_k(\mathbf{x}) \leq f(\mathbf{x})$. It follows that $S_k \subseteq S_{k+1}$ and

$$S = \cup_{k=1}^{\infty} S_k.$$

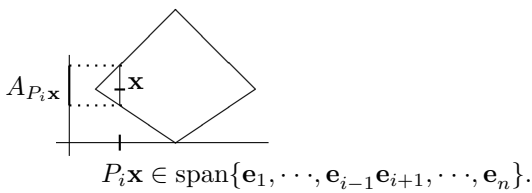
But each S_k is a Borel set and so S is also a Borel set. This proves the lemma.

Let P_i be the projection onto

$$\text{span}(\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n)$$

where the $\mathbf{e}_k$ are the standard basis vectors in $\mathbb{R}^n$, $\mathbf{e}_k$ being the vector having a 1 in the k^{th} slot and a 0 elsewhere. Thus $P_i \mathbf{x} \equiv \sum_{j \neq i} x_j \mathbf{e}_j$. Also let

$$A_{P_i \mathbf{x}} \equiv \{x_i : (x_1, \dots, x_i, \dots, x_n) \in A\}$$



Lemma 19.3.2 *Let $A \subseteq \mathbb{R}^n$ be a Borel set. Then $P_i \mathbf{x} \rightarrow m(A_{P_i \mathbf{x}})$ is a Borel measurable function defined on $P_i(\mathbb{R}^n)$.*

Proof: Let $\mathcal{K}$ be the π system consisting of sets of the form $\prod_{j=1}^n A_j$ where A_i is Borel. Also let $\mathcal{G}$ denote those Borel sets of $\mathbb{R}^n$ such that if $A \in \mathcal{G}$ then

$$P_i \mathbf{x} \rightarrow m((A \cap R_k)_{P_i \mathbf{x}}) \text{ is Borel measurable.}$$

where $R_k = (-k, k)^n$. Thus $\mathcal{K} \in \mathcal{G}$. If $A \in \mathcal{G}$

$$P_i \mathbf{x} \rightarrow m \left((A^C \cap R_k)_{P_i \mathbf{x}} \right)$$

is Borel measurable because it is of the form

$$m \left((R_k)_{P_i \mathbf{x}} \right) - m \left((A \cap R_k)_{P_i \mathbf{x}} \right)$$

and these are Borel measurable functions of $P_i \mathbf{x}$. Also, if $\{A_i\}$ is a disjoint sequence of sets in $\mathcal{G}$ then

$$m \left((\cup_i A_i \cap R_k)_{P_i \mathbf{x}} \right) = \sum_i m \left((A_i \cap R_k)_{P_i \mathbf{x}} \right)$$

and each function of $P_i \mathbf{x}$ is Borel measurable. Thus by the lemma on π systems, Lemma 9.11.3, $\mathcal{G} = \mathcal{B}(\mathbb{R}^n)$ and this proves the lemma.

Now let $A \subseteq \mathbb{R}^n$ be Borel. Let P_i be the projection onto

$$\text{span}(\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n)$$

and as just described,

$$A_{P_i \mathbf{x}} = \{y \in \mathbb{R} : P_i \mathbf{x} + y \mathbf{e}_i \in A\}$$

Thus for $\mathbf{x} = (x_1, \dots, x_n)$,

$$A_{P_i \mathbf{x}} = \{y \in \mathbb{R} : (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_n) \in A\}.$$

Since A is Borel, it follows from Lemma 19.3.1 that

$$P_i \mathbf{x} \rightarrow m(A_{P_i \mathbf{x}})$$

is a Borel measurable function on $P_i \mathbb{R}^n = \mathbb{R}^{n-1}$.

19.3.1 Steiner Symmetrization

Define

$$S(A, \mathbf{e}_i) \equiv \{\mathbf{x} = P_i \mathbf{x} + y \mathbf{e}_i : |y| < 2^{-1} m(A_{P_i \mathbf{x}})\}$$

Lemma 19.3.3 *Let A be a Borel subset of $\mathbb{R}^n$. Then $S(A, \mathbf{e}_i)$ satisfies*

$$P_i \mathbf{x} + y \mathbf{e}_i \in S(A, \mathbf{e}_i) \text{ if and only if } P_i \mathbf{x} - y \mathbf{e}_i \in S(A, \mathbf{e}_i),$$

$$S(A, \mathbf{e}_i) \text{ is a Borel set in } \mathbb{R}^n,$$

$$m_n(S(A, \mathbf{e}_i)) = m_n(A), \tag{19.3.8}$$

$$\text{diam}(S(A, \mathbf{e}_i)) \leq \text{diam}(A). \tag{19.3.9}$$

Proof: The first assertion is obvious from the definition. The Borel measurability of $S(A, \mathbf{e}_i)$ follows from the definition and Lemmas 19.3.2 and 19.3.1. To show Formula 19.3.8,

$$\begin{aligned} m_n(S(A, \mathbf{e}_i)) &= \int_{P_i \mathbb{R}^n} \int_{-2^{-1}m(A_{P_i \mathbf{x}})}^{2^{-1}m(A_{P_i \mathbf{x}})} dx_i dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\ &= \int_{P_i \mathbb{R}^n} m(A_{P_i \mathbf{x}}) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n \\ &= m(A). \end{aligned}$$

Now suppose $\mathbf{x}_1$ and $\mathbf{x}_2 \in S(A, \mathbf{e}_i)$

$$\mathbf{x}_1 = P_i \mathbf{x}_1 + y_1 \mathbf{e}_i, \quad \mathbf{x}_2 = P_i \mathbf{x}_2 + y_2 \mathbf{e}_i.$$

For $\mathbf{x} \in A$ define

$$l(\mathbf{x}) = \sup\{y : P_i \mathbf{x} + y \mathbf{e}_i \in A\}.$$

$$g(\mathbf{x}) = \inf\{y : P_i \mathbf{x} + y \mathbf{e}_i \in A\}.$$

Then it is clear that

$$l(\mathbf{x}_1) - g(\mathbf{x}_1) \geq m(A_{P_i \mathbf{x}_1}) \geq 2|y_1|, \quad (19.3.10)$$

$$l(\mathbf{x}_2) - g(\mathbf{x}_2) \geq m(A_{P_i \mathbf{x}_2}) \geq 2|y_2|. \quad (19.3.11)$$

Claim: $|y_1 - y_2| \leq |l(\mathbf{x}_1) - g(\mathbf{x}_2)|$ or $|y_1 - y_2| \leq |l(\mathbf{x}_2) - g(\mathbf{x}_1)|$.

Proof of Claim: If not,

$$\begin{aligned} 2|y_1 - y_2| &> |l(\mathbf{x}_1) - g(\mathbf{x}_2)| + |l(\mathbf{x}_2) - g(\mathbf{x}_1)| \\ &\geq |l(\mathbf{x}_1) - g(\mathbf{x}_1) + l(\mathbf{x}_2) - g(\mathbf{x}_2)| \\ &= l(\mathbf{x}_1) - g(\mathbf{x}_1) + l(\mathbf{x}_2) - g(\mathbf{x}_2). \\ &\geq 2|y_1| + 2|y_2| \end{aligned}$$

by 19.3.10 and 19.3.11 contradicting the triangle inequality.

Now suppose $|y_1 - y_2| \leq |l(\mathbf{x}_1) - g(\mathbf{x}_2)|$. From the claim,

$$\begin{aligned} |\mathbf{x}_1 - \mathbf{x}_2| &= (|P_i \mathbf{x}_1 - P_i \mathbf{x}_2|^2 + |y_1 - y_2|^2)^{1/2} \\ &\leq (|P_i \mathbf{x}_1 - P_i \mathbf{x}_2|^2 + |l(\mathbf{x}_1) - g(\mathbf{x}_2)|^2)^{1/2} \\ &\leq (|P_i \mathbf{x}_1 - P_i \mathbf{x}_2|^2 + (|z_1 - z_2| + 2\varepsilon)^2)^{1/2} \\ &\leq \text{diam}(A) + O(\sqrt{\varepsilon}) \end{aligned}$$

where z_1 and z_2 are such that $P_i \mathbf{x}_1 + z_1 \mathbf{e}_i \in A$, $P_i \mathbf{x}_2 + z_2 \mathbf{e}_i \in A$, and

$$|z_1 - l(\mathbf{x}_1)| < \varepsilon \text{ and } |z_2 - g(\mathbf{x}_2)| < \varepsilon.$$

If $|y_1 - y_2| \leq |l(\mathbf{x}_2) - g(\mathbf{x}_1)|$, then we use the same argument but let

$$|z_1 - g(\mathbf{x}_1)| < \varepsilon \text{ and } |z_2 - l(\mathbf{x}_2)| < \varepsilon,$$

Since $\mathbf{x}_1, \mathbf{x}_2$ are arbitrary elements of $S(A, \mathbf{e}_i)$ and ε is arbitrary, this proves 19.3.9.

The next lemma says that if A is already symmetric with respect to the j^{th} direction, then this symmetry is not destroyed by taking $S(A, \mathbf{e}_i)$.

Lemma 19.3.4 *Suppose A is a Borel set in $\mathbb{R}^n$ such that $P_j \mathbf{x} + \mathbf{e}_j x_j \in A$ if and only if $P_j \mathbf{x} + (-x_j) \mathbf{e}_j \in A$. Then if $i \neq j$, $P_j \mathbf{x} + \mathbf{e}_j x_j \in S(A, \mathbf{e}_i)$ if and only if $P_j \mathbf{x} + (-x_j) \mathbf{e}_j \in S(A, \mathbf{e}_i)$.*

Proof: By definition,

$$P_j \mathbf{x} + \mathbf{e}_j x_j \in S(A, \mathbf{e}_i)$$

if and only if

$$|x_i| < 2^{-1} m(A_{P_i(P_j \mathbf{x} + \mathbf{e}_j x_j)}).$$

Now

$$x_i \in A_{P_i(P_j \mathbf{x} + \mathbf{e}_j x_j)}$$

if and only if

$$x_i \in A_{P_i(P_j \mathbf{x} + (-x_j) \mathbf{e}_j)}$$

by the assumption on A which says that A is symmetric in the $\mathbf{e}_j$ direction. Hence

$$P_j \mathbf{x} + \mathbf{e}_j x_j \in S(A, \mathbf{e}_i)$$

if and only if

$$|x_i| < 2^{-1} m(A_{P_i(P_j \mathbf{x} + (-x_j) \mathbf{e}_j)})$$

if and only if

$$P_j \mathbf{x} + (-x_j) \mathbf{e}_j \in S(A, \mathbf{e}_i).$$

This proves the lemma.

19.3.2 The Isodiametric Inequality

The next theorem is called the isodiametric inequality. It is the key result used to compare Lebesgue and Hausdorff measures.

Theorem 19.3.5 *Let A be any Lebesgue measurable set in $\mathbb{R}^n$. Then*

$$m_n(A) \leq \alpha(n)(r(A))^n.$$

Proof: Suppose first that A is Borel. Let $A_1 = S(A, \mathbf{e}_1)$ and let $A_k = S(A_{k-1}, \mathbf{e}_k)$. Then by the preceding lemmas, A_n is a Borel set, $\text{diam}(A_n) \leq \text{diam}(A)$, $m_n(A_n) = m_n(A)$, and A_n is symmetric. Thus $\mathbf{x} \in A_n$ if and only if $-\mathbf{x} \in A_n$. It follows that

$$A_n \subseteq \overline{B(\mathbf{0}, r(A_n))}.$$

(If $\mathbf{x} \in A_n \setminus \overline{B(\mathbf{0}, r(A_n))}$, then $-\mathbf{x} \in A_n \setminus \overline{B(\mathbf{0}, r(A_n))}$ and so $\text{diam}(A_n) \geq 2|\mathbf{x}| > \text{diam}(A_n)$.) Therefore,

$$m_n(A_n) \leq \alpha(n)(r(A_n))^n \leq \alpha(n)(r(A))^n.$$

It remains to establish this inequality for arbitrary measurable sets. Letting A be such a set, let $\{K_n\}$ be an increasing sequence of compact subsets of A such that

$$m(A) = \lim_{k \rightarrow \infty} m(K_k).$$

Then

$$\begin{aligned} m(A) &= \lim_{k \rightarrow \infty} m(K_k) \leq \lim_{k \rightarrow \infty} \sup \alpha(n)(r(K_k))^n \\ &\leq \alpha(n)(r(A))^n. \end{aligned}$$

This proves the theorem.

19.4 The Proper Value Of $\beta(n)$

I will show that the proper determination of $\beta(n)$ is $\alpha(n)$, the volume of the unit ball. Since $\beta(n)$ has been adjusted such that $k = 1$, $m_n(B(\mathbf{0}, 1)) = \mathcal{H}^n(B(\mathbf{0}, 1))$. There exists a covering of $B(\mathbf{0}, 1)$ of sets of radii less than δ , $\{C_i\}_{i=1}^\infty$ such that

$$\mathcal{H}_\delta^n(B(\mathbf{0}, 1)) + \varepsilon > \sum_i \beta(n) r(C_i)^n$$

Then by Theorem 19.3.5, the isodiametric inequality,

$$\begin{aligned} \mathcal{H}_\delta^n(B(\mathbf{0}, 1)) + \varepsilon &> \sum_i \beta(n) r(C_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_i \alpha(n) r(\overline{C_i})^n \\ &\geq \frac{\beta(n)}{\alpha(n)} \sum_i m_n(\overline{C_i}) \geq \frac{\beta(n)}{\alpha(n)} m_n(B(\mathbf{0}, 1)) = \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1)) \end{aligned}$$

Now taking the limit as $\delta \rightarrow 0$,

$$\mathcal{H}^n(B(\mathbf{0}, 1)) + \varepsilon \geq \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1))$$

and since $\varepsilon > 0$ is arbitrary, this shows $\alpha(n) \geq \beta(n)$.

By the Vitali covering theorem, there exists a sequence of disjoint balls, $\{B_i\}$ such that $B(\mathbf{0}, 1) = (\cup_{i=1}^\infty B_i) \cup N$ where $m_n(N) = 0$. Then $\mathcal{H}_\delta^n(N) = 0$ can be concluded because $\mathcal{H}_\delta^n \leq \mathcal{H}^n$ and Lemma 19.2.2. Using $m_n(B(\mathbf{0}, 1)) = \mathcal{H}^n(B(\mathbf{0}, 1))$ again,

$$\begin{aligned} \mathcal{H}_\delta^n(B(\mathbf{0}, 1)) &= \mathcal{H}_\delta^n(\cup_i B_i) \leq \sum_{i=1}^\infty \beta(n) r(B_i)^n \\ &= \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^\infty \alpha(n) r(B_i)^n = \frac{\beta(n)}{\alpha(n)} \sum_{i=1}^\infty m_n(B_i) \\ &= \frac{\beta(n)}{\alpha(n)} m_n(\cup_i B_i) = \frac{\beta(n)}{\alpha(n)} m_n(B(\mathbf{0}, 1)) = \frac{\beta(n)}{\alpha(n)} \mathcal{H}^n(B(\mathbf{0}, 1)) \end{aligned}$$

which implies $\alpha(n) \leq \beta(n)$ and so the two are equal. This proves that if $\alpha(n) = \beta(n)$, then the $\mathcal{H}^n = m_n$ on the measurable sets of $\mathbb{R}^n$.

This gives another way to think of Lebesgue measure which is a particularly nice way because it is coordinate free, depending only on the notion of distance.

For $s < n$, note that $\mathcal{H}^s$ is not a Radon measure because it will not generally be finite on compact sets. For example, let $n = 2$ and consider $\mathcal{H}^1(L)$ where L is a line segment joining $(0, 0)$ to $(1, 0)$. Then $\mathcal{H}^1(L)$ is no smaller than $\mathcal{H}^1(L)$ when L is considered a subset of $\mathbb{R}^1$, $n = 1$. Thus by what was just shown, $\mathcal{H}^1(L) \geq 1$. Hence $\mathcal{H}^1([0, 1] \times [0, 1]) = \infty$. The situation is this: L is a one-dimensional object inside $\mathbb{R}^2$ and $\mathcal{H}^1$ is giving a one-dimensional measure of this object. In fact, Hausdorff measures can make such heuristic remarks as these precise. Define the Hausdorff dimension of a set, A , as

$$\dim(A) = \inf\{s : \mathcal{H}^s(A) = 0\}$$

19.4.1 A Formula For $\alpha(n)$

What is $\alpha(n)$? Recall the gamma function which makes sense for all $p > 0$.

$$\Gamma(p) \equiv \int_0^\infty e^{-t} t^{p-1} dt.$$

Lemma 19.4.1 *The following identities hold.*

$$p\Gamma(p) = \Gamma(p+1),$$

$$\Gamma(p)\Gamma(q) = \left(\int_0^1 x^{p-1}(1-x)^{q-1} dx \right) \Gamma(p+q),$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof: Using integration by parts,

$$\begin{aligned} \Gamma(p+1) &= \int_0^\infty e^{-t} t^p dt = -e^{-t} t^p \Big|_0^\infty + p \int_0^\infty e^{-t} t^{p-1} dt \\ &= p\Gamma(p) \end{aligned}$$

Next

$$\begin{aligned}
 \Gamma(p)\Gamma(q) &= \int_0^\infty e^{-t} t^{p-1} dt \int_0^\infty e^{-s} s^{q-1} ds \\
 &= \int_0^\infty \int_0^\infty e^{-(t+s)} t^{p-1} s^{q-1} dt ds \\
 &= \int_0^\infty \int_s^\infty e^{-u} (u-s)^{p-1} s^{q-1} du ds \\
 &= \int_0^\infty \int_0^u e^{-u} (u-s)^{p-1} s^{q-1} ds du \\
 &= \int_0^\infty \int_0^1 e^{-u} (u-ux)^{p-1} (ux)^{q-1} u dx du \\
 &= \int_0^\infty \int_0^1 e^{-u} u^{p+q-1} (1-x)^{p-1} x^{q-1} dx du \\
 &= \Gamma(p+q) \left(\int_0^1 x^{p-1} (1-x)^{q-1} dx \right).
 \end{aligned}$$

It remains to find $\Gamma\left(\frac{1}{2}\right)$.

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} dt = \int_0^\infty e^{-u^2} \frac{1}{u} 2u du = 2 \int_0^\infty e^{-u^2} du$$

Now

$$\begin{aligned}
 \left(\int_0^\infty e^{-x^2} dx \right)^2 &= \int_0^\infty e^{-x^2} dx \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \\
 &= \int_0^\infty \int_0^{\pi/2} e^{-r^2} r d\theta dr = \frac{1}{4} \pi
 \end{aligned}$$

and so

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-u^2} du = \sqrt{\pi}$$

This proves the lemma.

Next let n be a positive integer.

Theorem 19.4.2 $\alpha(n) = \pi^{n/2} (\Gamma(n/2 + 1))^{-1}$ where $\Gamma(s)$ is the gamma function

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt.$$

Proof: First let $n = 1$.

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Thus

$$\pi^{1/2}(\Gamma(1/2 + 1))^{-1} = \frac{2}{\sqrt{\pi}}\sqrt{\pi} = 2 = \alpha(1).$$

and this shows the theorem is true if $n = 1$.

Assume the theorem is true for n and let B_{n+1} be the unit ball in $\mathbb{R}^{n+1}$. Then by the result in $\mathbb{R}^n$,

$$\begin{aligned} m_{n+1}(B_{n+1}) &= \int_{-1}^1 \alpha(n)(1 - x_{n+1}^2)^{n/2} dx_{n+1} \\ &= 2\alpha(n) \int_0^1 (1 - t^2)^{n/2} dt. \end{aligned}$$

Doing an integration by parts and using Lemma 19.4.1

$$\begin{aligned} &= 2\alpha(n)n \int_0^1 t^2(1 - t^2)^{(n-2)/2} dt \\ &= 2\alpha(n)n \frac{1}{2} \int_0^1 u^{1/2}(1 - u)^{n/2-1} du \\ &= n\alpha(n) \int_0^1 u^{3/2-1}(1 - u)^{n/2-1} du \\ &= n\alpha(n)\Gamma(3/2)\Gamma(n/2)(\Gamma((n+3)/2))^{-1} \\ &= n\pi^{n/2}(\Gamma(n/2 + 1))^{-1}(\Gamma((n+3)/2))^{-1}\Gamma(3/2)\Gamma(n/2) \\ &= n\pi^{n/2}(\Gamma(n/2)(n/2))^{-1}(\Gamma((n+1)/2 + 1))^{-1}\Gamma(3/2)\Gamma(n/2) \\ &= 2\pi^{n/2}\Gamma(3/2)(\Gamma((n+1)/2 + 1))^{-1} \\ &= \pi^{(n+1)/2}(\Gamma((n+1)/2 + 1))^{-1}. \end{aligned}$$

This proves the theorem.

From now on, in the definition of Hausdorff measure, it will always be the case that $\beta(s) = \alpha(s)$. As shown above, this is the right thing to have $\beta(s)$ to equal if s is a positive integer because this yields the important result that Hausdorff measure is the same as Lebesgue measure. Note the formula, $\pi^{s/2}(\Gamma(s/2 + 1))^{-1}$ makes sense for any $s \geq 0$.

19.4.2 Hausdorff Measure And Linear Transformations

Hausdorff measure makes possible a unified development of n dimensional area. As in the case of Lebesgue measure, the first step in this is to understand basic considerations related to linear transformations. Recall that for $L \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^l)$, L^* is defined by

$$(L\mathbf{u}, \mathbf{v}) = (\mathbf{u}, L^*\mathbf{v}).$$

Also recall Theorem 4.12.6 on Page 93 which is stated here for convenience. This theorem says you can write a linear transformation as the composition of two linear transformations, one which preserves length and the other which distorts, the right

polar decomposition. The one which distorts is the one which will have a nontrivial interaction with Hausdorff measure while the one which preserves lengths does not change Hausdorff measure. These ideas are behind the following theorems and lemmas.

Theorem 19.4.3 *Let F be an $n \times m$ matrix where $m \geq n$. Then there exists an $m \times n$ matrix R and a $n \times n$ matrix U such that*

$$F = RU, \quad U = U^*,$$

all eigenvalues of U are non negative,

$$U^2 = F^*F, \quad R^*R = I,$$

and $|R\mathbf{x}| = |\mathbf{x}|$.

Lemma 19.4.4 *Let $R \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, and $R^*R = I$. Then if $A \subseteq \mathbb{R}^n$,*

$$\mathcal{H}^n(RA) = \mathcal{H}^n(A).$$

In fact, if $P : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies $|P\mathbf{x} - P\mathbf{y}| = |\mathbf{x} - \mathbf{y}|$, then

$$\mathcal{H}^n(PA) = \mathcal{H}^n(A).$$

Proof: Note that

$$|R(\mathbf{x} - \mathbf{y})|^2 = (R(\mathbf{x} - \mathbf{y}), R(\mathbf{x} - \mathbf{y})) = (R^*R(\mathbf{x} - \mathbf{y}), \mathbf{x} - \mathbf{y}) = |\mathbf{x} - \mathbf{y}|^2$$

Thus R preserves lengths.

Now let P be an arbitrary mapping which preserves lengths and let A be bounded, $P(A) \subseteq \cup_{j=1}^{\infty} C_j$, $r(C_j) < \delta$, and

$$\mathcal{H}_{\delta}^n(PA) + \varepsilon > \sum_{j=1}^{\infty} \alpha(n)(r(C_j))^n.$$

Since P preserves lengths, it follows P is one to one on $P(\mathbb{R}^n)$ and P^{-1} also preserves lengths on $P(\mathbb{R}^n)$. Replacing each C_j with $C_j \cap (PA)$,

$$\begin{aligned} \mathcal{H}_{\delta}^n(PA) + \varepsilon &> \sum_{j=1}^{\infty} \alpha(n)r(C_j \cap (PA))^n \\ &= \sum_{j=1}^{\infty} \alpha(n)r(P^{-1}(C_j \cap (PA)))^n \\ &\geq \mathcal{H}_{\delta}^n(A). \end{aligned}$$

Thus $\mathcal{H}_{\delta}^n(PA) \geq \mathcal{H}_{\delta}^n(A)$.

Now let $A \subseteq \cup_{j=1}^{\infty} C_j$, $\text{diam}(C_j) \leq \delta$, and

$$\mathcal{H}_{\delta}^n(A) + \varepsilon \geq \sum_{j=1}^{\infty} \alpha(n) (r(C_j))^n$$

Then

$$\begin{aligned} \mathcal{H}_{\delta}^n(A) + \varepsilon &\geq \sum_{j=1}^{\infty} \alpha(n) (r(C_j))^n \\ &= \sum_{j=1}^{\infty} \alpha(n) (r(PC_j))^n \\ &\geq \mathcal{H}_{\delta}^n(PA). \end{aligned}$$

Hence $\mathcal{H}_{\delta}^n(PA) = \mathcal{H}_{\delta}^n(A)$. Letting $\delta \rightarrow 0$ yields the desired conclusion in the case where A is bounded. For the general case, let $A_r = A \cap B(0, r)$. Then $\mathcal{H}^n(PA_r) = \mathcal{H}^n(A_r)$. Now let $r \rightarrow \infty$. This proves the lemma.

Lemma 19.4.5 *Let $F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $n \leq m$, and let $F = RU$ where R and U are described in Theorem 4.12.6 on Page 93. Then if $A \subseteq \mathbb{R}^n$ is Lebesgue measurable,*

$$\mathcal{H}^n(FA) = \det(U)m_n(A).$$

Proof: Using Theorem 10.5.7 on Page 301 and Theorem 19.2.3,

$$\begin{aligned} \mathcal{H}^n(FA) &= \mathcal{H}^n(RUA) \\ &= \mathcal{H}^n(UA) = m_n(UA) = \det(U)m_n(A). \end{aligned}$$

Definition 19.4.6 *Define J to equal $\det(U)$. Thus*

$$J = \det((F^*F)^{1/2}) = (\det(F^*F))^{1/2}.$$

19.5 The Area Formula

19.5.1 Preliminary Results

It was shown in Lemma 19.4.5 that

$$\mathcal{H}^n(FA) = \det(U)m_n(A)$$

where $F = RU$ with R preserving distances and U a symmetric matrix having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where F is replaced by a nonlinear mapping, **h**. It contains as a special case the earlier change of variables formula. There are two parts to this development. The first part is to generalize Lemma 19.4.5 to the case of nonlinear maps. When this is done, the area formula can be presented.

In this section, U will be an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ will be a Lebesgue measurable set. Assume $m \geq n$ and

$$\mathbf{h} : U \rightarrow \mathbb{R}^m \text{ is continuous,} \quad (19.5.12)$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \quad (19.5.13)$$

Also assume that for every $\mathbf{x} \in A$, there exists $R_{\mathbf{x}}$ and $L_{\mathbf{x}}$ such that for all $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})$,

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{x} - \mathbf{y}| \quad (19.5.14)$$

This last condition is weaker than saying $\mathbf{h}$ is Lipschitz. Instead, it is an assumption that $\mathbf{h}$ is locally Lipschitz, the Lipschitz constant depending on the point considered. An interesting case in which this would hold would be when $\mathbf{h}$ is differentiable on U and $\|D\mathbf{h}(\mathbf{x})\|$ is uniformly bounded near each point $\mathbf{x}$. Actually, it is the case that 19.5.14 will suffice to obtain 19.5.13 on all but a subset of measure zero of A but this has not been shown yet. Also, the condition 19.5.12 is redundant because you can simply replace U with the union of the sets $B(\mathbf{x}, R_{\mathbf{x}})$ for $\mathbf{x} \in A$. I think it is easiest to retain this condition because differentiability is defined for functions whose domains are open sets. To make this more formal, here is a definition.

Definition 19.5.1 *Let $\mathbf{h}$ be defined in some open set containing a set, A . Then $\mathbf{h}$ is locally Lipschitz on A if for every $\mathbf{x} \in A$ there exists $R_{\mathbf{x}} > 0$ and a constant, $L_{\mathbf{x}}$ such that whenever $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})$,*

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{z} - \mathbf{y}|.$$

Lemma 19.5.2 *If $T \subseteq A$ and $m_n(T) = 0$, then $\mathcal{H}^n(\mathbf{h}(T)) = 0$.*

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : \|D\mathbf{h}(\mathbf{x})\| < k\}.$$

Thus $T = \cup_k T_k$. I will show $\mathbf{h}(T_k)$ has $\mathcal{H}^n$ measure zero and then it will follow that

$$\mathbf{h}(T) = \cup_{k=1}^{\infty} \mathbf{h}(T_k)$$

must also have measure zero.

Let $\varepsilon > 0$ be given. By outer regularity, there exists an open set, V , containing T_k which is contained in U such that $m_n(V) < \frac{\varepsilon}{k^n 6^n}$. For $\mathbf{x} \in T_k$ it follows from differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so whenever $r_{\mathbf{x}}$ is small enough, $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| < 5r_{\mathbf{x}}, |o(\mathbf{v})| < kr_{\mathbf{x}}$. Therefore, if $|\mathbf{v}| < 5r_{\mathbf{x}}$,

$$D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v}) \in B(\mathbf{0}, 5kr_{\mathbf{x}}) + B(\mathbf{0}, kr_{\mathbf{x}}) \subseteq B(\mathbf{0}, 6kr_{\mathbf{x}})$$

and so

$$\mathbf{h}(B(\mathbf{x}, 5r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 6kr_{\mathbf{x}}).$$

Letting $\delta > 0$ be given, the Vitali covering theorem implies there exists a sequence of disjoint balls $\{B_i\}$, $B_i = B(\mathbf{x}_i, r_{\mathbf{x}_i})$, which are contained in V such that the sequence of enlarged balls, $\{\widehat{B}_i\}$, having the same center but 5 times the radius, covers T_k and $6kr_{\mathbf{x}_i} < \delta$. Then

$$\begin{aligned} \mathcal{H}_\delta^n(\mathbf{h}(T_k)) &\leq \mathcal{H}_\delta^n\left(\mathbf{h}\left(\bigcup_{i=1}^\infty \widehat{B}_i\right)\right) \\ &\leq \sum_{i=1}^\infty \mathcal{H}_\delta^n\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^\infty \mathcal{H}_\delta^n(B(\mathbf{h}(\mathbf{x}_i), 6kr_{\mathbf{x}_i})) \\ &\leq \sum_{i=1}^\infty \alpha(n) (6kr_{\mathbf{x}_i})^n = (6k)^n \sum_{i=1}^\infty \alpha(n) r_{\mathbf{x}_i}^n \\ &= (6k)^n \sum_{i=1}^\infty m_n(B(\mathbf{x}_i, r_{\mathbf{x}_i})) \\ &\leq (6k)^n m_n(V) \leq (6k)^n \frac{\varepsilon}{k^n 6^n} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $\mathcal{H}_\delta^n(\mathbf{h}(T_k)) = 0$. Since δ is arbitrary, this implies $\mathcal{H}^n(\mathbf{h}(T_k)) = 0$. Now

$$\mathcal{H}^n(\mathbf{h}(T)) = \lim_{k \rightarrow \infty} \mathcal{H}^n(\mathbf{h}(T_k)) = 0.$$

This proves the lemma.

Lemma 19.5.3 *If S is a Lebesgue measurable subset of A , then $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k)$, $k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F , which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = S_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 19.5.2, $\mathcal{H}^n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is $\mathcal{H}^n$ measurable because of completeness of Hausdorff measure, which comes from $\mathcal{H}^n$ being obtained from an outer measure. Now $\mathbf{h}(S) = \bigcup_{k=1}^\infty \mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable. This proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [58], will be important for the following arguments. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 19.5.4 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \rightarrow \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r \tag{19.5.15}$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(B) \supseteq B(\mathbf{0}, r(1 - \varepsilon)).$$

Proof: Suppose $\mathbf{a} \in B(\mathbf{0}, r(1 - \varepsilon)) \setminus \mathbf{F}(B)$.

I claim that $\mathbf{a} \neq \mathbf{F}(\mathbf{v})$ for all $\mathbf{v} \in \overline{B}$. Here is why. By assumption, if $\mathbf{F}(\mathbf{v}) = \mathbf{a}$, then $|\mathbf{v}| = r$ and so

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| = |\mathbf{a} - \mathbf{v}| \geq |\mathbf{v}| - |\mathbf{a}| > r - r(1 - \varepsilon) = r\varepsilon,$$

a contradiction to 19.5.15.

Now letting $\mathbf{G} : \overline{B} \rightarrow \overline{B}$, be defined by

$$\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a} - \mathbf{F}(\mathbf{v}))}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|},$$

it follows $\mathbf{G}$ is continuous. Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}| = r$. Taking the inner product with $\mathbf{v}$,

$$\begin{aligned} (\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{v} + \mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a} - \mathbf{v}, \mathbf{v}) + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\ &\leq \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [r^2(1 - \varepsilon) - r^2 + r^2\varepsilon] = 0, \end{aligned}$$

a contradiction to $|\mathbf{v}| = r$. Therefore, $B(\mathbf{0}, r(1 - \varepsilon)) \setminus \mathbf{F}(B) = \emptyset$ and this proves the lemma.

By Theorem 4.12.6 on Page 93, when $D\mathbf{h}(\mathbf{x})$ exists,

$$D\mathbf{h}(\mathbf{x}) = R(\mathbf{x})U(\mathbf{x})$$

where $(U(\mathbf{x})\mathbf{u}, \mathbf{v}) = (U(\mathbf{x})\mathbf{v}, \mathbf{u})$, $(U(\mathbf{x})\mathbf{u}, \mathbf{u}) \geq 0$ and $R^*R = I$.

Lemma 19.5.5 *In this situation, $|R^*\mathbf{u}| \leq |\mathbf{u}|$.*

Proof: First note that

$$\begin{aligned} (\mathbf{u} - RR^*\mathbf{u}, RR^*\mathbf{u}) &= (\mathbf{u}, RR^*\mathbf{u}) - |RR^*\mathbf{u}|^2 \\ &= |R^*\mathbf{u}|^2 - |R^*\mathbf{u}|^2 = 0, \end{aligned}$$

and so

$$\begin{aligned} |\mathbf{u}|^2 &= |\mathbf{u} - RR^*\mathbf{u} + RR^*\mathbf{u}|^2 \\ &= |\mathbf{u} - RR^*\mathbf{u}|^2 + |RR^*\mathbf{u}|^2 \\ &= |\mathbf{u} - RR^*\mathbf{u}|^2 + |R^*\mathbf{u}|^2. \end{aligned}$$

This proves the lemma.

Lemma 19.5.6 *If $|P\mathbf{x} - P\mathbf{y}| \leq L|\mathbf{x} - \mathbf{y}|$, then for E a set,*

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E).$$

Proof: Without loss of generality, assume $\mathcal{H}^n(E) < \infty$. Let $\delta > 0$ and let $\{C_i\}_{i=1}^\infty$ be a covering of E such that $r(C_i) \leq \delta$ for each i and

$$\sum_{i=1}^\infty \alpha(n) r(C_i)^n \leq \mathcal{H}_\delta^n(E) + \varepsilon.$$

Then $\{PC_i\}_{i=1}^\infty$ is a covering of PE such that $r(PC_i) \leq L\delta$. Therefore,

$$\begin{aligned} \mathcal{H}_{L\delta}^n(PE) &\leq \sum_{i=1}^\infty \alpha(n) r(PC_i)^n \\ &\leq L^n \sum_{i=1}^\infty \alpha(n) r(C_i)^n \leq L^n \mathcal{H}_\delta^n(E) + L^n \varepsilon \\ &\leq L^n \mathcal{H}^n(E) + L^n \varepsilon. \end{aligned}$$

Letting $\delta \rightarrow 0$,

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E) + L^n \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the Lemma.

Then the following corollary follows from Lemma 19.5.5.

Corollary 19.5.7 *Let $T \subseteq \mathbb{R}^m$. Then*

$$\mathcal{H}^n(T) \geq \mathcal{H}^n(RR^*T) = \mathcal{H}^n(R^*T).$$

Definition 19.5.8 *Let E be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if*

$$\lim_{r \rightarrow 0} \frac{m_n(E \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} = 1.$$

Recall that from the fundamental theorem of calculus applied to $\mathcal{X}_E$ almost every point of E is a point of density.

Lemma 19.5.9 *Let $\mathbf{x} \in A$ be a point where $U(\mathbf{x})^{-1}$ exists. Then if $\varepsilon \in (0, 1)$ the following hold for all r small enough.*

$$\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x}) + R(\mathbf{x})U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon)), \quad (19.5.16)$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \leq m_n(U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))). \quad (19.5.17)$$

$$R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r)) \supseteq R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon)), \quad (19.5.18)$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \geq m_n(U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon))), \quad (19.5.19)$$

If $\mathbf{x}$ is also a point of density of A , then

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} = 1. \quad (19.5.20)$$

Proof: Since $D\mathbf{h}(\mathbf{x})$ exists,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(|\mathbf{v}|). \quad (19.5.21)$$

Consequently, when r is small enough, 19.5.16 holds.

Using the fact $R(\mathbf{x})$ preserves all distances, and Theorem 19.2.3 which says $\mathcal{H}^n = m_n$ on the Borel sets of $\mathbb{R}^n$ implies,

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &\leq \mathcal{H}^n(R(\mathbf{x})U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))) \\ &= \mathcal{H}^n(U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))) = m_n(U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))) \end{aligned}$$

which shows 19.5.17.

From 19.5.21,

$$R^*(\mathbf{x})\mathbf{h}(\mathbf{x} + \mathbf{v}) = R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})(\mathbf{v} + o(|\mathbf{v}|)).$$

Thus, from the assumption that $U(\mathbf{x})^{-1}$ exists and letting $\mathbf{F}(\mathbf{v})$ be given by

$$\mathbf{F}(\mathbf{v}) \equiv U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x} + \mathbf{v}) - U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) \quad (19.5.22)$$

It follows

$$\mathbf{F}(\mathbf{v}) - \mathbf{v} = o(|\mathbf{v}|)$$

and so Lemma 19.5.4 implies that for all r small enough,

$$\begin{aligned} \mathbf{F}(B(\mathbf{0}, r)) &\equiv U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x} + B(\mathbf{0}, r)) - U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) \\ &\supseteq B(\mathbf{0}, (1 - \varepsilon)r). \end{aligned}$$

Therefore,

$$R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r)) \supseteq R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})B(\mathbf{0}, (1 - \varepsilon)r)$$

which proves 19.5.18. Therefore,

$$R(\mathbf{x})R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r)) \supseteq$$

$$R(\mathbf{x})R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + R(\mathbf{x})U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon)).$$

From Lemma 19.5.7, this implies

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &\geq \mathcal{H}^n(R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r))) \\ &= \mathcal{H}^n(R(\mathbf{x})R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r))) \\ &\geq \mathcal{H}^n(R(\mathbf{x})U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon))) \\ &= \mathcal{H}^n(U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon))) = m_n(U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon))) \end{aligned}$$

which shows 19.5.19.

Now suppose that $\mathbf{x}$ is also a point of density of A . Then whenever r is small enough,

$$\frac{m_n(A \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} > 1 - \varepsilon.$$

Consequently, for such r ,

$$\begin{aligned} 1 &= \frac{m_n(A \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} + \frac{m_n(B(\mathbf{x}, r) \setminus A)}{m_n(B(\mathbf{x}, r))} \\ &> 1 - \varepsilon + \frac{m_n(B(\mathbf{x}, r) \setminus A)}{\alpha(n) r^n} \end{aligned}$$

and so

$$m_n(B(\mathbf{x}, r) \setminus A) < \varepsilon \alpha(n) r^n. \quad (19.5.23)$$

Also,

$$\mathbf{h}(B(\mathbf{x}, r) \cap A) \cup \mathbf{h}(B(\mathbf{x}, r) \setminus A) = \mathbf{h}(B(\mathbf{x}, r))$$

and so

$$\begin{aligned} &\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A)) + \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \setminus A)) \\ &\geq \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \end{aligned}$$

Then also letting r also be smaller than $R_{\mathbf{x}}$ mentioned in 19.5.14, it follows from Lemmas 19.5.6, 19.5.3, and 19.5.23, 19.5.18 that

$$\begin{aligned} 1 &\geq \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} \\ &\geq \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) - \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \setminus A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} \\ &\geq 1 - \frac{L_{\mathbf{x}}^n m_n(B(\mathbf{x}, r) \setminus A)}{\mathcal{H}^n(R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x}, r)))} \\ &\geq 1 - \frac{L_{\mathbf{x}}^n m_n(B(\mathbf{x}, r) \setminus A)}{m_n(U(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon)))} \\ &\geq 1 - \frac{L_{\mathbf{x}}^n \varepsilon \alpha(n) r^n}{\det(U(\mathbf{x})) \alpha(n) r^n (1 - \varepsilon)^n} \\ &= 1 - g(\varepsilon) \end{aligned}$$

where $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$. Since ε is arbitrary, this proves 19.5.20.

The next theorem is the generalization of Lemma 19.4.5 to nonlinear maps.

Theorem 19.5.10 *Let $\mathbf{h} : U \rightarrow \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U . Also suppose that for every $\mathbf{x} \in A$, $D\mathbf{h}(\mathbf{x})$ exists. Then for $\mathbf{x} \in A$,*

$$J(\mathbf{x}) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))}, \quad (19.5.24)$$

where $J(\mathbf{x}) \equiv \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

Proof: Suppose first that $U(\mathbf{x})^{-1}$ exists. Using 19.5.19, 19.5.17 and the change of variables formula for linear maps,

$$\begin{aligned} & J(\mathbf{x})(1-\varepsilon)^n \\ &= \frac{m_n(U(\mathbf{x})B(\mathbf{0}, r(1-\varepsilon)))}{m_n(B(\mathbf{x}, r))} \leq \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\ &\leq \frac{m_n(U(\mathbf{x})B(\mathbf{0}, r(1+\varepsilon)))}{m_n(B(\mathbf{x}, r))} = J(\mathbf{x})(1+\varepsilon)^n \end{aligned}$$

whenever r is small enough. It follows that since $\varepsilon > 0$ is arbitrary, 19.5.24 holds.

Now suppose $U(\mathbf{x})^{-1}$ does not exist. The first part shows that the conclusion of the theorem holds when $J(\mathbf{x}) \neq 0$. I will apply this to a modified function. Let

$$\mathbf{k} : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n$$

be defined as

$$\mathbf{k}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \varepsilon \mathbf{x} \end{pmatrix}.$$

Then

$$D\mathbf{k}(\mathbf{x})^* D\mathbf{k}(\mathbf{x}) = D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n$$

and so

$$\begin{aligned} J\mathbf{k}(\mathbf{x})^2 &\equiv \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n) \\ &= \det(Q^* DQ + \varepsilon^2 I_n) \end{aligned}$$

where D is a diagonal matrix having the nonnegative eigenvalues of $D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})$ down the main diagonal. Thus, since one of these eigenvalues equals 0, letting λ_i^2 denote the i^{th} eigenvalue, there exists a constant, C independent of ε such that

$$0 < J\mathbf{k}(\mathbf{x})^2 = \prod_{i=1}^n (\lambda_i^2 + \varepsilon^2) \leq C^2 \varepsilon^2. \quad (19.5.25)$$

Therefore, what was just shown applies to $\mathbf{k}$.

Let

$$T \equiv \{(\mathbf{h}(\mathbf{w}), \mathbf{0})^T : \mathbf{w} \in B(\mathbf{x}, r)\},$$

$$\begin{aligned} T_\varepsilon &\equiv \{(\mathbf{h}(\mathbf{w}), \varepsilon \mathbf{w})^T : \mathbf{w} \in B(\mathbf{x}, r)\} \\ &\equiv \mathbf{k}(B(\mathbf{x}, r)), \end{aligned}$$

then

$$T = PT_\varepsilon$$

where P is the projection map defined by

$$P \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{x} \\ 0 \end{pmatrix}.$$

Since P decreases distances, it follows from Lemma 19.5.6

$$\begin{aligned}\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &= \mathcal{H}^n(T) = \mathcal{H}^n(PT_\varepsilon) \\ &\leq \mathcal{H}^n(T_\varepsilon) = \mathcal{H}^n(\mathbf{k}(B(\mathbf{x}, r))).\end{aligned}$$

It follows from 19.5.25 and the first part of the proof applied to $\mathbf{k}$ that

$$\begin{aligned}C\varepsilon &\geq J\mathbf{k}(\mathbf{x}) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{k}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\ &\geq \limsup_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))}.\end{aligned}$$

Since ε is arbitrary, this establishes 19.5.24 in the case where $U(\mathbf{x})^{-1}$ does not exist and completes the proof of the theorem.

Define the following set for future reference.

$$S \equiv \{\mathbf{x} \in A : U(\mathbf{x})^{-1} \text{ does not exist}\} \quad (19.5.26)$$

19.5.2 The Area Formula

Assume $\mathbf{h} : A \rightarrow \mathbb{R}^m$ is one to one in addition to 19.5.12 - 19.5.14. Since $\mathbf{h}$ is one to one, Lemma 19.5.3 implies one can define a measure, ν , on the σ -algebra of Lebesgue measurable sets as follows.

$$\nu(E) \equiv \mathcal{H}^n(\mathbf{h}(E \cap A)).$$

By Lemma 19.5.3, this is a measure and $\nu \ll m$. Therefore by the corollary to the Radon Nikodym theorem, Corollary 16.1.3 on Page 492, there exists $f \in L^1_{loc}(\mathbb{R}^n)$, $f \geq 0$, $f(\mathbf{x}) = 0$ if $\mathbf{x} \notin A$, and

$$\nu(E) = \int_E f dm = \int_{A \cap E} f dm.$$

What is f ? I will show that $f(\mathbf{x}) = J(\mathbf{x}) = \det(U(\mathbf{x}))$ a.e. Define

$$E \equiv \{\mathbf{x} \in A : \mathbf{x} \text{ is not a point of density of } A\} \cup$$

$$\{\mathbf{x} \in A : \mathbf{x} \text{ is not a Lebesgue point of } f\}.$$

Then E is a set of measure zero and if $\mathbf{x} \in (A \setminus E)$, Lemma 19.5.9 and Theorem 19.5.10 imply

$$\begin{aligned}f(\mathbf{x}) &= \lim_{r \rightarrow 0} \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) dm \\ &= \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_n(B(\mathbf{x}, r))} \\ &= \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\ &= J(\mathbf{x}).\end{aligned}$$

Therefore, $f(\mathbf{x}) = J(\mathbf{x})$ a.e., whenever $\mathbf{x} \in A \setminus E$.

Now let F be a Borel set in $\mathbb{R}^m$. Recall this implies F is $\mathcal{H}^n$ measurable. Then

$$\begin{aligned} \int_{\mathbf{h}(A)} \mathcal{X}_F(\mathbf{y}) d\mathcal{H}^n &= \int \mathcal{X}_{F \cap \mathbf{h}(A)}(\mathbf{y}) d\mathcal{H}^n \\ &= \mathcal{H}^n(\mathbf{h}(\mathbf{h}^{-1}(F) \cap A)) \\ &= \nu(\mathbf{h}^{-1}(F)) = \int \mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm \\ &= \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm. \end{aligned} \quad (19.5.27)$$

Note there are no measurability questions in the above formula because $\mathbf{h}^{-1}(F)$ is a Borel set due to the continuity of $\mathbf{h}$. The Borel measurability of $J(\mathbf{x})$ also follows from the observation that $\mathbf{h}$ is continuous and therefore, the partial derivatives are Borel measurable, being the limit of continuous functions. Then $J(\mathbf{x})$ is just a continuous function of these partial derivatives. However, things are not so clear if E is only assumed $\mathcal{H}^n$ measurable. Is there a similar formula for F only $\mathcal{H}^n$ measurable?

First consider the case where E is only $\mathcal{H}^n$ measurable but

$$\mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

By Theorem 19.1.5 on Page 568, there exists a Borel set $F \supseteq E \cap \mathbf{h}(A)$ such that

$$\mathcal{H}^n(F) = \mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

Then from 19.5.27,

$$\mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) = 0 \text{ a.e.}$$

But

$$0 \leq \mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) \leq \mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) \quad (19.5.28)$$

which shows the two functions in 19.5.28 are equal a.e. Therefore $\mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x})$ is Lebesgue measurable and so from 19.5.27,

$$\begin{aligned} 0 &= \int \mathcal{X}_{E \cap \mathbf{h}(A)}(\mathbf{y}) d\mathcal{H}^n = \int \mathcal{X}_{F \cap \mathbf{h}(A)}(\mathbf{y}) d\mathcal{H}^n \\ &= \int \mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm_n = \int \mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) dm_n, \end{aligned} \quad (19.5.29)$$

which shows 19.5.27 holds in this case where

$$\mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

Now let $A_R \equiv A \cap B(\mathbf{0}, R)$ where R is large enough that $A_R \neq \emptyset$ and let E be $\mathcal{H}^n$ measurable. By Theorem 19.1.5, there exists $F \supseteq E \cap \mathbf{h}(A_R)$ such that F is Borel and

$$\mathcal{H}^n(F \setminus (E \cap \mathbf{h}(A_R))) = 0. \quad (19.5.30)$$

Then

$$(E \cap \mathbf{h}(A_R)) \cup (F \setminus (E \cap \mathbf{h}(A_R))) \cap \mathbf{h}(A_R) = F \cap \mathbf{h}(A_R)$$

and so

$$\mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F)} J = \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(E)} J + \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F \setminus (E \cap \mathbf{h}(A_R)))} J$$

where from 19.5.30 and 19.5.29, the second function on the right of the equal sign is Lebesgue measurable and equals zero a.e. Therefore, the first function on the right of the equal sign is also Lebesgue measurable and equals the function on the left a.e. Thus,

$$\begin{aligned} \int \mathcal{X}_{E \cap \mathbf{h}(A_R)}(\mathbf{y}) d\mathcal{H}^n &= \int \mathcal{X}_{F \cap \mathbf{h}(A_R)}(\mathbf{y}) d\mathcal{H}^n \\ &= \int \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm_n = \int \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) dm_n. \end{aligned} \quad (19.5.31)$$

Letting $R \rightarrow \infty$ yields 19.5.31 with A replacing A_R and the function

$$\mathbf{x} \rightarrow \mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable. Writing this in a more familiar form yields

$$\int_{\mathbf{h}(A)} \mathcal{X}_E(\mathbf{y}) d\mathcal{H}^n = \int_A \mathcal{X}_E(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n. \quad (19.5.32)$$

From this, it follows that if s is a nonnegative $\mathcal{H}^n$ measurable simple function, 19.5.32 continues to be valid with s in place of $\mathcal{X}_E$. Then approximating an arbitrary nonnegative $\mathcal{H}^n$ measurable function, g , by an increasing sequence of simple functions, it follows that 19.5.32 holds with g in place of $\mathcal{X}_E$ and there are no measurability problems because $\mathbf{x} \rightarrow g(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is Lebesgue measurable. This proves the area formula.

Theorem 19.5.11 *Let $g : \mathbf{h}(A) \rightarrow [0, \infty]$ be $\mathcal{H}^n$ measurable where $\mathbf{h}$ is a continuous function and A is a Lebesgue measurable set which satisfies 19.5.12 - 19.5.14. That is, U is an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ is a Lebesgue measurable set, $m \geq n$, and*

$$\mathbf{h} : A \rightarrow \mathbb{R}^m \text{ is continuous,} \quad (19.5.33)$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \quad (19.5.34)$$

Also assume that for every $\mathbf{x} \in A$, there exists $R_{\mathbf{x}}$ and $L_{\mathbf{x}}$ such that for all $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})$,

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{x} - \mathbf{y}| \quad (19.5.35)$$

Then

$$\mathbf{x} \rightarrow (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

19.6 The Area Formula Alternate Version

19.6.1 Preliminary Results

It was shown in Lemma 19.4.5 that

$$\mathcal{H}^n(FA) = \det(U)m_n(A)$$

where $F = RU$ with R preserving distances and U a symmetric matrix having all positive eigenvalues. The area formula gives a generalization of this simple relationship to the case where F is replaced by a nonlinear mapping, $\mathbf{h}$. It contains as a special case the earlier change of variables formula. There are two parts to this development. The first part is to generalize Lemma 19.4.5 to the case of nonlinear maps. When this is done, the area formula can be presented.

In this section, U will be an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ will be a Lebesgue measurable set. Assume $m \geq n$ and

$$\mathbf{h} : U \rightarrow \mathbb{R}^m \text{ is continuous,} \quad (19.6.36)$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \quad (19.6.37)$$

$$\mathcal{H}^n(\mathbf{h}(U \setminus A)) = 0 \quad (19.6.38)$$

These conditions are different than the ones considered earlier. Here no Lipschitz assumption is needed on $\mathbf{h}$. In this sense, these conditions are more general than those considered earlier. However, they are not really more general because of 19.6.38 which says that A is essentially an open set. This was not necessary earlier. The area formula which results from the above conditions is a generalization of the change of variables formula given in Rudin [58] and the proof is essentially the same as the proof given in this book with modifications to account for the Hausdorff measure.

Lemma 19.6.1 *If $T \subseteq A$ and $m_n(T) = 0$, then $\mathcal{H}^n(\mathbf{h}(T)) = 0$.*

Proof: Let

$$T_k \equiv \{\mathbf{x} \in T : \|D\mathbf{h}(\mathbf{x})\| < k\}.$$

Thus $T = \cup_k T_k$. I will show $\mathbf{h}(T_k)$ has $\mathcal{H}^n$ measure zero and then it will follow that

$$\mathbf{h}(T) = \cup_{k=1}^{\infty} \mathbf{h}(T_k)$$

must also have measure zero.

Let $\varepsilon > 0$ be given. By outer regularity, there exists an open set, V , containing T_k which is contained in U such that $m_n(V) < \frac{\varepsilon}{k^n 6^n}$. For $\mathbf{x} \in T_k$ it follows from differentiability,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v})$$

and so whenever $r_{\mathbf{x}}$ is small enough, $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V$ and whenever $|\mathbf{v}| < 5r_{\mathbf{x}}$, $|o(\mathbf{v})| < kr_{\mathbf{x}}$. Therefore, if $|\mathbf{v}| < 5r_{\mathbf{x}}$,

$$D\mathbf{h}(\mathbf{x})\mathbf{v} + o(\mathbf{v}) \in B(\mathbf{0}, 5kr_{\mathbf{x}}) + B(\mathbf{0}, kr_{\mathbf{x}}) \subseteq B(\mathbf{0}, 6kr_{\mathbf{x}})$$

and so

$$\mathbf{h}(B(\mathbf{x}, 5r_{\mathbf{x}})) \subseteq B(\mathbf{h}(\mathbf{x}), 6kr_{\mathbf{x}}).$$

Letting $\delta > 0$ be given, the Vitali covering theorem implies there exists a sequence of disjoint balls $\{B_i\}$, $B_i = B(\mathbf{x}_i, r_{\mathbf{x}_i})$, which are contained in V such that the sequence of enlarged balls, $\{\widehat{B}_i\}$, having the same center but 5 times the radius, covers T_k and $6kr_{\mathbf{x}_i} < \delta$. Then

$$\begin{aligned} \mathcal{H}_\delta^n(\mathbf{h}(T_k)) &\leq \mathcal{H}_\delta^n\left(\mathbf{h}\left(\bigcup_{i=1}^\infty \widehat{B}_i\right)\right) \\ &\leq \sum_{i=1}^\infty \mathcal{H}_\delta^n\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \leq \sum_{i=1}^\infty \mathcal{H}_\delta^n(B(\mathbf{h}(\mathbf{x}_i), 6kr_{\mathbf{x}_i})) \\ &\leq \sum_{i=1}^\infty \alpha(n) (6kr_{\mathbf{x}_i})^n = (6k)^n \sum_{i=1}^\infty \alpha(n) r_{\mathbf{x}_i}^n \\ &= (6k)^n \sum_{i=1}^\infty m_n(B(\mathbf{x}_i, r_{\mathbf{x}_i})) \\ &\leq (6k)^n m_n(V) \leq (6k)^n \frac{\varepsilon}{k^n 6^n} = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $\mathcal{H}_\delta^n(\mathbf{h}(T_k)) = 0$. Since δ is arbitrary, this implies $\mathcal{H}^n(\mathbf{h}(T_k)) = 0$. Now

$$\mathcal{H}^n(\mathbf{h}(T)) = \lim_{k \rightarrow \infty} \mathcal{H}^n(\mathbf{h}(T_k)) = 0.$$

This proves the lemma.

Lemma 19.6.2 *If S is a Lebesgue measurable subset of A , then $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable.*

Proof: Let $S_k = S \cap B(\mathbf{0}, k)$, $k \in \mathbb{N}$. By inner regularity of Lebesgue measure, there exists a set, F , which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = S_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(S_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By Lemma 19.6.1, $\mathcal{H}^n(\mathbf{h}(T)) = 0$ and so $\mathbf{h}(S_k)$ is $\mathcal{H}^n$ measurable because of completeness of Hausdorff measure, which comes from $\mathcal{H}^n$ being obtained from an outer measure. Now $\mathbf{h}(S) = \bigcup_{k=1}^\infty \mathbf{h}(S_k)$ and so it is also true that $\mathbf{h}(S)$ is $\mathcal{H}^n$ measurable. This proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [58], will be important for the following arguments. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 19.6.3 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \rightarrow \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r \quad (19.6.39)$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(B) \supseteq B(\mathbf{0}, r(1 - \varepsilon)).$$

Proof: Suppose $\mathbf{a} \in B(\mathbf{0}, r(1 - \varepsilon)) \setminus \mathbf{F}(B)$.

I claim that $\mathbf{a} \neq \mathbf{F}(\mathbf{v})$ for all $\mathbf{v} \in \overline{B}$. Here is why. By assumption, if $\mathbf{F}(\mathbf{v}) = \mathbf{a}$, then $|\mathbf{v}| = r$ and so

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| = |\mathbf{a} - \mathbf{v}| \geq |\mathbf{v}| - |\mathbf{a}| > r - r(1 - \varepsilon) = r\varepsilon,$$

a contradiction to 19.6.39.

Now letting $\mathbf{G} : \overline{B} \rightarrow \overline{B}$, be defined by

$$\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a} - \mathbf{F}(\mathbf{v}))}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|},$$

it follows $\mathbf{G}$ is continuous. Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}| = r$. Taking the inner product with $\mathbf{v}$,

$$\begin{aligned} (\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{v} + \mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a} - \mathbf{v}, \mathbf{v}) + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\ &\leq \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [r^2(1 - \varepsilon) - r^2 + r^2\varepsilon] = 0, \end{aligned}$$

a contradiction to $|\mathbf{v}| = r$. Therefore, $B(\mathbf{0}, r(1 - \varepsilon)) \setminus \mathbf{F}(B) = \emptyset$ and this proves the lemma.

By Theorem 4.12.6 on Page 93, when $D\mathbf{h}(\mathbf{x})$ exists,

$$D\mathbf{h}(\mathbf{x}) = R(\mathbf{x}) U(\mathbf{x})$$

where $(U(\mathbf{x})\mathbf{u}, \mathbf{v}) = (U(\mathbf{x})\mathbf{v}, \mathbf{u})$, $(U(\mathbf{x})\mathbf{u}, \mathbf{u}) \geq 0$ and $R^*R = I$.

Lemma 19.6.4 *In this situation, $|R^*\mathbf{u}| \leq |\mathbf{u}|$.*

Proof: First note that

$$\begin{aligned} (\mathbf{u} - RR^*\mathbf{u}, RR^*\mathbf{u}) &= (\mathbf{u}, RR^*\mathbf{u}) - |RR^*\mathbf{u}|^2 \\ &= |R^*\mathbf{u}|^2 - |R^*\mathbf{u}|^2 = 0, \end{aligned}$$

and so

$$\begin{aligned} |\mathbf{u}|^2 &= |\mathbf{u} - RR^*\mathbf{u} + RR^*\mathbf{u}|^2 \\ &= |\mathbf{u} - RR^*\mathbf{u}|^2 + |RR^*\mathbf{u}|^2 \\ &= |\mathbf{u} - RR^*\mathbf{u}|^2 + |R^*\mathbf{u}|^2. \end{aligned}$$

This proves the lemma.

Lemma 19.6.5 *If $|P\mathbf{x} - P\mathbf{y}| \leq L|\mathbf{x} - \mathbf{y}|$, then for E a set,*

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E).$$

Proof: Without loss of generality, assume $\mathcal{H}^n(E) < \infty$. Let $\delta > 0$ and let $\{C_i\}_{i=1}^\infty$ be a covering of E such that $\text{diam}(C_i) \leq \delta$ for each i and

$$\sum_{i=1}^\infty \alpha(n) r(C_i)^n \leq \mathcal{H}_\delta^n(E) + \varepsilon.$$

Then $\{PC_i\}_{i=1}^\infty$ is a covering of PE such that $\text{diam}(PC_i) \leq L\delta$. Therefore,

$$\begin{aligned} \mathcal{H}_{L\delta}^n(PE) &\leq \sum_{i=1}^\infty \alpha(n) r(PC_i)^n \\ &\leq L^n \sum_{i=1}^\infty \alpha(n) r(C_i)^n \leq L^n \mathcal{H}_\delta^n(E) + L^n \varepsilon \\ &\leq \mathcal{H}^n(E) + \varepsilon. \end{aligned}$$

Letting $\delta \rightarrow 0$,

$$\mathcal{H}^n(PE) \leq L^n \mathcal{H}^n(E) + L^n \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the Lemma.

Then the following corollary follows from Lemma 19.6.4.

Corollary 19.6.6 *Let $T \subseteq \mathbb{R}^m$. Then*

$$\mathcal{H}^n(T) \geq \mathcal{H}^n(RR^*T) = \mathcal{H}^n(R^*T).$$

Definition 19.6.7 *Let E be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if*

$$\lim_{r \rightarrow 0} \frac{m_n(E \cap B(\mathbf{x}, r))}{m_n(B(\mathbf{x}, r))} = 1.$$

Recall that from the fundamental theorem of calculus applied to $\mathcal{X}_E$ almost every point of E is a point of density.

Lemma 19.6.8 *Let $\mathbf{x} \in A$ be a point where $U(\mathbf{x})^{-1}$ exists. Then if $\varepsilon \in (0, 1)$ the following hold for all r small enough.*

$$\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x}) + R(\mathbf{x})U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon)), \quad (19.6.40)$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \leq m_n(U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))). \quad (19.6.41)$$

$$R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r)) \supseteq R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon)), \quad (19.6.42)$$

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \geq m_n(U(\mathbf{x})B(\mathbf{0}, r(1 - \varepsilon))), \quad (19.6.43)$$

If $\mathbf{x}$ is a point of A , then

$$\lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} = 1. \quad (19.6.44)$$

Proof: Since $D\mathbf{h}(\mathbf{x})$ exists,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(|\mathbf{v}|). \quad (19.6.45)$$

Consequently, when r is small enough, 19.6.40 holds.

Using the fact $R(\mathbf{x})$ preserves all distances, and Theorem 19.2.3 which says $\mathcal{H}^n = m_n$ on the Borel sets of $\mathbb{R}^n$, this implies,

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &\leq \mathcal{H}^n(R(\mathbf{x})U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))) \\ &= \mathcal{H}^n(U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))) = m_n(U(\mathbf{x})B(\mathbf{0}, r(1 + \varepsilon))) \end{aligned}$$

which shows 19.6.41.

From 19.6.45,

$$R^*(\mathbf{x})\mathbf{h}(\mathbf{x} + \mathbf{v}) = R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})(\mathbf{v} + o(|\mathbf{v}|)).$$

Thus, from the assumption that $U(\mathbf{x})^{-1}$ exists and letting $\mathbf{F}(\mathbf{v})$ be given by

$$\mathbf{F}(\mathbf{v}) \equiv U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x} + \mathbf{v}) - U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) \quad (19.6.46)$$

Since $\mathbf{h}$ is continuous near A , it follows

$$\mathbf{F}(\mathbf{v}) - \mathbf{v} = o(|\mathbf{v}|)$$

and so Lemma 19.6.3 implies that for all r small enough,

$$\begin{aligned} \mathbf{F}(B(\mathbf{0}, r)) &\equiv U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x} + B(\mathbf{0}, r)) - U(\mathbf{x})^{-1}R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) \\ &\supseteq B(\mathbf{0}, (1 - \varepsilon)r). \end{aligned}$$

Therefore,

$$R^*(\mathbf{x})\mathbf{h}(B(\mathbf{x}, r)) \supseteq R^*(\mathbf{x})\mathbf{h}(\mathbf{x}) + U(\mathbf{x})B(\mathbf{0}, (1 - \varepsilon)r)$$

which proves 19.6.42. Therefore,

$$\begin{aligned} R(\mathbf{x}) R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x}, r)) &\supseteq \\ R(\mathbf{x}) R^*(\mathbf{x}) \mathbf{h}(\mathbf{x}) + R(\mathbf{x}) U(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon)). \end{aligned}$$

From Lemma 19.6.6, this implies

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &\geq \mathcal{H}^n(R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x}, r))) \\ &= \mathcal{H}^n(R(\mathbf{x}) R^*(\mathbf{x}) \mathbf{h}(B(\mathbf{x}, r))) \\ &\geq \mathcal{H}^n(R(\mathbf{x}) U(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon))) \\ &= \mathcal{H}^n(U(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon))) = m_n(U(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon))) \end{aligned}$$

which shows 19.6.43.

Let $\mathbf{x} \in A$. Choosing r small enough that $B(\mathbf{x}, r) \subseteq U$,

$$\mathbf{h}(B(\mathbf{x}, r) \cap A) \cup \mathbf{h}(B(\mathbf{x}, r) \setminus A) = \mathbf{h}(B(\mathbf{x}, r))$$

and so

$$\begin{aligned} &\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A)) + \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \setminus A)) \\ &\geq \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) \end{aligned}$$

Now by assumption 19.6.38, $\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \setminus A)) = 0$ and so for all r small enough,

$$\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A)) = \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))).$$

This establishes 19.6.44.

The next theorem is the generalization of Lemma 19.4.5 to nonlinear maps.

Theorem 19.6.9 *Let $\mathbf{h} : U \rightarrow \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$, $n \leq m$, $\mathbf{h}$ is continuous on U , $\mathbf{h}$ is differentiable on $A \subseteq U$, and $\mathcal{H}^n(U \setminus A) = 0$. Then for $\mathbf{x} \in A$,*

$$J(\mathbf{x}) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))}, \quad (19.6.47)$$

where $J(\mathbf{x}) \equiv \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

Proof: Suppose first that $U(\mathbf{x})^{-1}$ exists. Using 19.6.43, 19.6.41 and the change of variables formula for linear maps,

$$\begin{aligned} &J(\mathbf{x}) (1 - \varepsilon)^n \\ &= \frac{m_n(U(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon)))}{m_n(B(\mathbf{x}, r))} \leq \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\ &\leq \frac{m_n(U(\mathbf{x}) B(\mathbf{0}, r(1 + \varepsilon)))}{m_n(B(\mathbf{x}, r))} = J(\mathbf{x}) (1 + \varepsilon)^n \end{aligned}$$

whenever r is small enough. It follows that since $\varepsilon > 0$ is arbitrary, 19.6.47 holds.

Now suppose $U(\mathbf{x})^{-1}$ does not exist. The first part shows that the conclusion of the theorem holds when $J(\mathbf{x}) \neq 0$. I will apply this to a modified function. Let

$$\mathbf{k} : \mathbb{R}^n \rightarrow \mathbb{R}^m \times \mathbb{R}^n$$

be defined as

$$\mathbf{k}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \varepsilon \mathbf{x} \end{pmatrix}.$$

Then

$$D\mathbf{k}(\mathbf{x})^* D\mathbf{k}(\mathbf{x}) = D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n$$

and so

$$\begin{aligned} J\mathbf{k}(\mathbf{x})^2 &\equiv \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}) + \varepsilon^2 I_n) \\ &= \det(Q^* DQ + \varepsilon^2 I_n) \end{aligned}$$

where D is a diagonal matrix having the nonnegative eigenvalues of $D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})$ down the main diagonal. Thus, since one of these eigenvalues equals 0, letting λ_i^2 denote the i^{th} eigenvalue, there exists a constant, C independent of ε such that

$$0 < J\mathbf{k}(\mathbf{x})^2 = \prod_{i=1}^n (\lambda_i^2 + \varepsilon^2) \leq C^2 \varepsilon^2. \quad (19.6.48)$$

Therefore, what was just shown applies to $\mathbf{k}$.

Let

$$T \equiv \{(\mathbf{h}(\mathbf{w}), \mathbf{0})^T : \mathbf{w} \in B(\mathbf{x}, r)\},$$

$$\begin{aligned} T_\varepsilon &\equiv \{(\mathbf{h}(\mathbf{w}), \varepsilon \mathbf{w})^T : \mathbf{w} \in B(\mathbf{x}, r)\} \\ &\equiv \mathbf{k}(B(\mathbf{x}, r)), \end{aligned}$$

then

$$T = PT_\varepsilon$$

where P is the projection map defined by

$$P \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{x} \\ \mathbf{0} \end{pmatrix}.$$

Since P decreases distances, it follows from Lemma 19.6.5

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r))) &= \mathcal{H}^n(T) = \mathcal{H}^n(PT_\varepsilon) \\ &\leq \mathcal{H}^n(T_\varepsilon) = \mathcal{H}^n(\mathbf{k}(B(\mathbf{x}, r))). \end{aligned}$$

It follows from 19.6.48 and the first part of the proof applied to $\mathbf{k}$ that

$$\begin{aligned} C\varepsilon &\geq J\mathbf{k}(\mathbf{x}) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{k}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\ &\geq \limsup_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))}. \end{aligned}$$

Since ε is arbitrary, this establishes 19.6.47 in the case where $U(\mathbf{x})^{-1}$ does not exist and completes the proof of the theorem.

19.6.2 The Area Formula

Assume $\mathbf{h} : A \rightarrow \mathbb{R}^m$ is one to one in addition to 19.6.36 - 19.6.38. Since $\mathbf{h}$ is one to one on A , Lemma 19.6.2 implies one can define a measure, ν , on the σ - algebra of Lebesgue measurable sets as follows.

$$\nu(E) \equiv \mathcal{H}^n(\mathbf{h}(E \cap A)).$$

By Lemma 19.6.2, this is a measure and $\nu \ll m$. Therefore by the corollary to the Radon Nikodym theorem, Corollary 16.1.3 on Page 492, there exists $f \in L^1_{loc}(\mathbb{R}^n)$, $f \geq 0$, $f(\mathbf{x}) = 0$ if $\mathbf{x} \notin A$, and

$$\nu(E) = \int_E f dm_n = \int_{A \cap E} f dm_n.$$

What is f ? I will show that $f(\mathbf{x}) = J(\mathbf{x}) = \det(U(\mathbf{x}))$ a.e. Let $\mathbf{x}$ be a Lebesgue point of f . Then by Lemma 19.6.8 and Theorem 19.6.9

$$\begin{aligned} f(\mathbf{x}) &= \lim_{r \rightarrow 0} \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(\mathbf{y}) dm \\ &= \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{m_n(B(\mathbf{x}, r))} \\ &= \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r) \cap A))}{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\ &= J(\mathbf{x}). \end{aligned}$$

Therefore, $f(\mathbf{x}) = J(\mathbf{x})$ a.e., whenever $\mathbf{x}$ is a Lebesgue point of f .

Now let F be a Borel set in $\mathbb{R}^m$. Recall this implies F is $\mathcal{H}^n$ measurable. Then

$$\begin{aligned} \int_{\mathbf{h}(A)} \mathcal{X}_F(\mathbf{y}) d\mathcal{H}^n &= \int \mathcal{X}_{F \cap \mathbf{h}(A)}(\mathbf{y}) d\mathcal{H}^n \\ &= \mathcal{H}^n(\mathbf{h}(\mathbf{h}^{-1}(F) \cap A)) \\ &= \nu(\mathbf{h}^{-1}(F)) = \int \mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm_n \\ &= \int_A \mathcal{X}_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n. \end{aligned} \tag{19.6.49}$$

Note there are no measurability questions in the above formula because $\mathbf{h}^{-1}(F)$ is a Borel set due to the continuity of $\mathbf{h}$. The Borel measurability of $J(\mathbf{x})$ also follows from the observation that $\mathbf{h}$ is continuous and therefore, the partial derivatives are Borel measurable, being the limit of continuous functions. Then $J(\mathbf{x})$ is just a continuous function of these partial derivatives. However, things are not so clear if E is only assumed $\mathcal{H}^n$ measurable. Is there a similar formula for F only $\mathcal{H}^n$ measurable?

First consider the case where E is only $\mathcal{H}^n$ measurable but

$$\mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

By Theorem 19.1.5 on Page 568, there exists a Borel set $F \supseteq E \cap \mathbf{h}(A)$ such that

$$\mathcal{H}^n(F) = \mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

Then from 19.6.49,

$$\mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) = 0 \text{ a.e.}$$

But

$$0 \leq \mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) \leq \mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) \quad (19.6.50)$$

which shows the two functions in 19.6.50 are equal a.e. Therefore $\mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x})$ is Lebesgue measurable and so from 19.6.49,

$$\begin{aligned} 0 &= \int \mathcal{X}_{E \cap \mathbf{h}(A)}(\mathbf{y}) d\mathcal{H}^n = \int \mathcal{X}_{F \cap \mathbf{h}(A)}(\mathbf{y}) d\mathcal{H}^n \\ &= \int \mathcal{X}_{A \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm_n = \int \mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) dm_n, \end{aligned} \quad (19.6.51)$$

which shows 19.6.49 holds in this case where

$$\mathcal{H}^n(E \cap \mathbf{h}(A)) = 0.$$

Now let $A_R \equiv A \cap B(\mathbf{0}, R)$ where R is large enough that $A_R \neq \emptyset$ and let E be $\mathcal{H}^n$ measurable. By Theorem 19.1.5, there exists $F \supseteq E \cap \mathbf{h}(A_R)$ such that F is Borel and

$$\mathcal{H}^n(F \setminus (E \cap \mathbf{h}(A_R))) = 0. \quad (19.6.52)$$

Then

$$(E \cap \mathbf{h}(A_R)) \cup (F \setminus (E \cap \mathbf{h}(A_R))) \cap \mathbf{h}(A_R) = F \cap \mathbf{h}(A_R)$$

and so

$$\mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F)} J = \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(E)} J + \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F \setminus (E \cap \mathbf{h}(A_R)))} J$$

where from 19.6.52 and 19.6.51, the second function on the right of the equal sign is Lebesgue measurable and equals zero a.e. Therefore, the first function on the right of the equal sign is also Lebesgue measurable and equals the function on the left a.e. Thus,

$$\int \mathcal{X}_{E \cap \mathbf{h}(A_R)}(\mathbf{y}) d\mathcal{H}^n = \int \mathcal{X}_{F \cap \mathbf{h}(A_R)}(\mathbf{y}) d\mathcal{H}^n$$

$$= \int \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm_n = \int \mathcal{X}_{A_R \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x}) dm_n. \quad (19.6.53)$$

Letting $R \rightarrow \infty$ yields 19.6.53 with A replacing A_R and the function

$$\mathbf{x} \rightarrow \mathcal{X}_{A \cap \mathbf{h}^{-1}(E)}(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable. Writing this in a more familiar form yields

$$\int_{\mathbf{h}(A)} \mathcal{X}_E(\mathbf{y}) d\mathcal{H}^n = \int_A \mathcal{X}_E(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n. \quad (19.6.54)$$

From this, it follows that if s is a nonnegative $\mathcal{H}^n$ measurable simple function, 19.6.54 continues to be valid with s in place of $\mathcal{X}_E$. Then approximating an arbitrary nonnegative $\mathcal{H}^n$ measurable function, g , by an increasing sequence of simple functions, it follows that 19.6.54 holds with g in place of $\mathcal{X}_E$ and there are no measurability problems because $\mathbf{x} \rightarrow g(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is Lebesgue measurable. This proves the area formula.

Theorem 19.6.10 *Let $g : \mathbf{h}(A) \rightarrow [0, \infty]$ be $\mathcal{H}^n$ measurable where $\mathbf{h}$ is a continuous function and A is a Lebesgue measurable set which satisfies 19.6.36 - 19.6.38. That is, U is an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and continuous and $A \subseteq U$ is a Lebesgue measurable set, $m \geq n$, and*

$$\mathbf{h} : U \rightarrow \mathbb{R}^m \text{ is continuous, } \mathbf{h} \text{ one to one on } A, \quad (19.6.55)$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \quad (19.6.56)$$

$$\mathcal{H}^n(\mathbf{h}(U \setminus A)) = 0, \quad (19.6.57)$$

Then

$$\mathbf{x} \rightarrow (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

19.7 The Divergence Theorem

As an important application of the area formula I will give a general version of the divergence theorem. It will always be assumed $n \geq 2$. Actually it is not necessary to make this assumption but what results in the case where $n = 1$ is nothing more than the fundamental theorem of calculus and the considerations necessary to draw this conclusion seem unnecessarily tedious. You have to consider $\mathcal{H}^0$, zero dimensional Hausdorff measure. It is left as an exercise but I will not present it.

It will be convenient to have some lemmas and theorems in hand before beginning the proof. First recall the Tietze extension theorem on Page 153. It is stated next for convenience.

Theorem 19.7.1 *Let M be a closed nonempty subset of a metric space (X, d) and let $f : M \rightarrow [a, b]$ be continuous at every point of M . Then there exists a function, g continuous on all of X which coincides with f on M such that $g(X) \subseteq [a, b]$.*

The next topic needed is the concept of an infinitely differentiable partition of unity.

Definition 19.7.2 *Let $\mathfrak{C}$ be a set whose elements are subsets of $\mathbb{R}^n$.¹ Then $\mathfrak{C}$ is said to be locally finite if for every $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, $U_{\mathbf{x}}$ containing $\mathbf{x}$ such that $U_{\mathbf{x}}$ has nonempty intersection with only finitely many sets of $\mathfrak{C}$.*

Lemma 19.7.3 *Let $\mathfrak{C}$ be a set whose elements are open subsets of $\mathbb{R}^n$ and suppose $\bigcup \mathfrak{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\{U_i\}_{i=1}^{\infty}$ such that each U_i is bounded, each U_i is a subset of some set of $\mathfrak{C}$, and $\bigcup_{i=1}^{\infty} U_i \supseteq H$.*

Proof: Let $W_k \equiv B(\mathbf{0}, k)$, $W_0 = W_{-1} = \emptyset$. For each $\mathbf{x} \in H \cap \overline{W_k}$ there exists an open set, $U_{\mathbf{x}}$ such that $U_{\mathbf{x}}$ is a subset of some set of $\mathfrak{C}$ and $U_{\mathbf{x}} \subseteq W_{k+1} \setminus \overline{W_{k-1}}$. Then since $H \cap \overline{W_k}$ is compact, there exist finitely many of these sets, $\{U_i^k\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_k}$. If $H \cap \overline{W_k} = \emptyset$, let $m(k) = 0$ and there are no such sets obtained. The desired countable list of open sets is $\bigcup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \in W_k$ where k is the first positive integer with $\mathbf{x} \in W_k$. Then $W_k \setminus \overline{W_{k-1}}$ is an open set containing $\mathbf{x}$ and this open set can have nonempty intersection only with with a set of $\{U_i^k\}_{i=1}^{m(k)} \cup \{U_i^{k-1}\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\bigcup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$ is locally finite.

The set, $\{U_i\}_{i=1}^{\infty}$ is said to be a locally finite cover of H . The following lemma gives some important reasons why a locally finite list of sets is so significant. First of all consider the rational numbers, $\{r_i\}_{i=1}^{\infty}$ each rational number is a closed set.

$$\mathbb{Q} = \{r_i\}_{i=1}^{\infty} = \bigcup_{i=1}^{\infty} \overline{\{r_i\}} \neq \overline{\bigcup_{i=1}^{\infty} \{r_i\}} = \mathbb{R}$$

The set of rational numbers is definitely not locally finite.

Lemma 19.7.4 *Let $\mathfrak{C}$ be locally finite. Then*

$$\overline{\bigcup \mathfrak{C}} = \bigcup \{\overline{H} : H \in \mathfrak{C}\}.$$

Next suppose the elements of $\mathfrak{C}$ are open sets and that for each $U \in \mathfrak{C}$, there exists a differentiable function, ψ_U having $\text{spt}(\psi_U) \subseteq U$. Then you can define the following finite sum for each $\mathbf{x} \in \mathbb{R}^n$

$$f(\mathbf{x}) \equiv \sum \{\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

Furthermore, f is also a differentiable function² and

$$Df(\mathbf{x}) = \sum \{D\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

¹The definition applies with no change to a general topological space in place of $\mathbb{R}^n$.

²If each ψ_U were only continuous, one could conclude f is continuous. Here the main interest is differentiable.

Proof: Let $\mathbf{p}$ be a limit point of $\cup \mathfrak{C}$ and let W be an open set which intersects only finitely many sets of $\mathfrak{C}$. Then $\mathbf{p}$ must be a limit point of one of these sets. It follows $\mathbf{p} \in \cup \{\overline{H} : H \in \mathfrak{C}\}$ and so $\overline{\cup \mathfrak{C}} \subseteq \cup \{\overline{H} : H \in \mathfrak{C}\}$. The inclusion in the other direction is obvious.

Now consider the second assertion. Letting $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, W intersecting only finitely many open sets of $\mathfrak{C}$, $U_1, U_2, \dots, U_m$. Then for all $\mathbf{y} \in W$,

$$f(\mathbf{y}) = \sum_{i=1}^m \psi_{U_i}(\mathbf{y})$$

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

Definition 19.7.5 Let K be a closed subset of an open set, U . $K \prec f \prec U$ if f is continuous, has values in $[0, 1]$, equals 1 on K , and has compact support contained in U .

Lemma 19.7.6 Let U be a bounded open set and let K be a closed subset of U . Then there exist an open set, W , such that $W \subseteq \overline{W} \subseteq U$ and a function, $f \in C_c^\infty(U)$ such that $K \prec f \prec U$.

Proof: The set, K is compact so is at a positive distance from U^C . Let

$$W \equiv \{\mathbf{x} : \text{dist}(\mathbf{x}, K) < 3^{-1} \text{dist}(K, U^C)\}.$$

Also let

$$W_1 \equiv \{\mathbf{x} : \text{dist}(\mathbf{x}, K) < 2^{-1} \text{dist}(K, U^C)\}$$

Then it is clear

$$K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$$

Now consider the function,

$$h(\mathbf{x}) \equiv \frac{\text{dist}(\mathbf{x}, W_1^C)}{\text{dist}(\mathbf{x}, W_1^C) + \text{dist}(\mathbf{x}, \overline{W})}$$

Since $\overline{W}$ is compact it is at a positive distance from W_1^C and so h is a well defined continuous function which has compact support contained in $\overline{W_1}$, equals 1 on W , and has values in $[0, 1]$. Now let ϕ_k be a mollifier. Letting

$$k^{-1} < \min(\text{dist}(K, W^C), 2^{-1} \text{dist}(\overline{W_1}, U^C)),$$

it follows that for such k , the function, $h * \phi_k \in C_c^\infty(U)$, has values in $[0, 1]$, and equals 1 on K . Let $f = h * \phi_k$.

The above lemma is used repeatedly in the following.

Lemma 19.7.7 *Let K be a closed set and let $\{V_i\}_{i=1}^\infty$ be a locally finite list of bounded open sets whose union contains K . Then there exist functions, $\psi_i \in C_c^\infty(V_i)$ such that for all $\mathbf{x} \in K$,*

$$1 = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$$

and the function $f(\mathbf{x})$ given by

$$f(\mathbf{x}) = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$$

is in $C^\infty(\mathbb{R}^n)$.

Proof: Let $K_1 = K \setminus \bigcup_{i=2}^\infty V_i$. Thus K_1 is compact because $K_1 \subseteq V_1$. Let

$$K_1 \subseteq W_1 \subseteq \overline{W_1} \subseteq V_1$$

Thus $W_1, V_2, \dots, V_n$ covers K and $\overline{W_1} \subseteq V_1$. Suppose $W_1, \dots, W_r$ have been defined such that $\overline{W_i} \subseteq V_i$ for each i , and $W_1, \dots, W_r, V_{r+1}, \dots, V_n$ covers K . Then let

$$K_{r+1} \equiv K \setminus ((\bigcup_{i=r+2}^\infty V_i) \cup (\bigcup_{j=1}^r W_j)).$$

It follows K_{r+1} is compact because $K_{r+1} \subseteq V_{r+1}$. Let W_{r+1} satisfy

$$K_{r+1} \subseteq W_{r+1} \subseteq \overline{W_{r+1}} \subseteq V_{r+1}$$

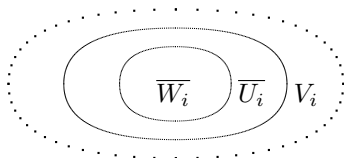
Continuing this way defines a sequence of open sets, $\{W_i\}_{i=1}^\infty$ with the property

$$\overline{W_i} \subseteq V_i, K \subseteq \bigcup_{i=1}^\infty W_i.$$

Note $\{W_i\}_{i=1}^\infty$ is locally finite because the original list, $\{V_i\}_{i=1}^\infty$ was locally finite. Now let U_i be open sets which satisfy

$$\overline{W_i} \subseteq U_i \subseteq \overline{U_i} \subseteq V_i.$$

Similarly, $\{U_i\}_{i=1}^\infty$ is locally finite.



Since the set, $\{W_i\}_{i=1}^\infty$ is locally finite, it follows $\overline{\bigcup_{i=1}^\infty W_i} = \bigcup_{i=1}^\infty \overline{W_i}$ and so it is possible to define ϕ_i and γ , infinitely differentiable functions having compact support such that

$$\overline{U_i} \prec \phi_i \prec V_i, \bigcup_{i=1}^\infty \overline{W_i} \prec \gamma \prec \bigcup_{i=1}^\infty U_i.$$

Now define

$$\psi_i(\mathbf{x}) = \begin{cases} \gamma(\mathbf{x})\phi_i(\mathbf{x}) / \sum_{j=1}^{\infty} \phi_j(\mathbf{x}) & \text{if } \sum_{j=1}^{\infty} \phi_j(\mathbf{x}) \neq 0, \\ 0 & \text{if } \sum_{j=1}^{\infty} \phi_j(\mathbf{x}) = 0. \end{cases}$$

If $\mathbf{x}$ is such that $\sum_{j=1}^{\infty} \phi_j(\mathbf{x}) = 0$, then $\mathbf{x} \notin \cup_{i=1}^{\infty} \overline{U_i}$ because ϕ_i equals one on $\overline{U_i}$. Consequently $\gamma(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$ thanks to the fact that $\cup_{i=1}^{\infty} \overline{U_i}$ is closed and so $\psi_i(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$. Hence ψ_i is infinitely differentiable at such $\mathbf{x}$. If $\sum_{j=1}^{\infty} \phi_j(\mathbf{x}) \neq 0$, this situation persists near $\mathbf{x}$ because each ϕ_j is continuous and so ψ_i is infinitely differentiable at such points also thanks to Lemma 19.7.4. Therefore ψ_i is infinitely differentiable. If $\mathbf{x} \in K$, then $\gamma(\mathbf{x}) = 1$ and so $\sum_{j=1}^{\infty} \psi_j(\mathbf{x}) = 1$. Clearly $0 \leq \psi_i(\mathbf{x}) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The functions, $\{\psi_i\}$ are called a C^∞ partition of unity.

The method of proof of this lemma easily implies the following useful corollary.

Corollary 19.7.8 *If H is a compact subset of V_i for some V_i there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Lemma 37.1.6.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i$, $\phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H .

Lemma 19.7.9 *Let Ω be a metric space with the closed balls compact and suppose μ is a measure defined on the Borel sets of Ω which is finite on compact sets. Then there exists a unique Radon measure, $\bar{\mu}$ which equals μ on the Borel sets. In particular μ must be both inner and outer regular on all Borel sets.*

Proof: Define a positive linear functional, $\Lambda(f) = \int f d\mu$. Let $\bar{\mu}$ be the Radon measure which comes from the Riesz representation theorem for positive linear functionals. Thus for all f continuous,

$$\int f d\mu = \int f d\bar{\mu}.$$

If V is an open set, let $\{f_n\}$ be a sequence of continuous functions which is increasing and converges to χ_V pointwise. Then applying the monotone convergence theorem,

$$\int \chi_V d\mu = \mu(V) = \int \chi_V d\bar{\mu} = \bar{\mu}(V)$$

and so the two measures coincide on all open sets. Every compact set is a countable intersection of open sets and so the two measures coincide on all compact sets. Now let $B(a, n)$ be a ball of radius n and let E be a Borel set contained in this ball. Then by regularity of $\bar{\mu}$ there exist sets F, G such that G is a countable intersection of open sets and F is a countable union of compact sets such that $F \subseteq E \subseteq G$ and $\bar{\mu}(G \setminus F) = 0$. Now $\mu(G) = \bar{\mu}(G)$ and $\mu(F) = \bar{\mu}(F)$. Thus

$$\begin{aligned} \bar{\mu}(G \setminus F) + \bar{\mu}(F) &= \bar{\mu}(G) \\ &= \mu(G) = \mu(G \setminus F) + \mu(F) \end{aligned}$$

and so $\mu(G \setminus F) = \bar{\mu}(G \setminus F)$. Thus

$$\mu(E) = \mu(F) = \bar{\mu}(F) = \bar{\mu}(G) = \bar{\mu}(E).$$

If E is an arbitrary Borel set, then

$$\mu(E \cap B(a, n)) = \bar{\mu}(E \cap B(a, n))$$

and letting $n \rightarrow \infty$, this yields $\mu(E) = \bar{\mu}(E)$.

One more lemma will be useful.

Lemma 19.7.10 *Let V be a bounded open set and let X be the closed subspace of $C(\bar{V})$, the space of continuous functions defined on $\bar{V}$, which is given by the following.*

$$X = \{u \in C(\bar{V}) : u(\mathbf{x}) = 0 \text{ on } \partial V\}.$$

Then $C_c^\infty(V)$ is dense in X with respect to the norm given by

$$\|u\| = \max \{|u(x)| : x \in \bar{V}\}$$

Proof: Let $O \subseteq \bar{O} \subseteq W \subseteq \bar{W} \subseteq V$ be such that $\text{dist}(\bar{O}, V^c) < \eta$ and let $\psi_\delta(\cdot)$ be a mollifier. Let $u \in X$ and consider $\mathcal{X}_W u * \psi_\delta$. Let $\varepsilon > 0$ be given and let η be small enough that $|u(\mathbf{x})| < \varepsilon/2$ whenever $\mathbf{x} \in V \setminus \bar{O}$. Then if δ is small enough $|\mathcal{X}_W u * \psi_\delta(\mathbf{x}) - u(\mathbf{x})| < \varepsilon$ for all $\mathbf{x} \in \bar{O}$ and $\mathcal{X}_W u * \psi_\delta$ is in $C_c^\infty(V)$. For $\mathbf{x} \in V \setminus \bar{O}$, $|\mathcal{X}_W u * \psi_\delta(\mathbf{x})| \leq \varepsilon/2$ and so for such $\mathbf{x}$,

$$|\mathcal{X}_W u * \psi_\delta(\mathbf{x}) - u(\mathbf{x})| \leq \varepsilon.$$

This proves the lemma since ε was arbitrary.

Definition 19.7.11 *A bounded open set, $U \subseteq \mathbb{R}^n$ is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, $Q_1, \dots, Q_N$,*

$$Q_i = \prod_{j=1}^n (a_j^i, b_j^i)$$

such that $\partial U \equiv \bar{U} \setminus U$ is contained in their union. Also, for each Q_i , there exists k and a Lipschitz function, g_i such that $U \cap Q_i$ is of the form

$$\left\{ \mathbf{x} : (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i) \text{ and } a_k^i < x_k < g_i(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \right\} \quad (19.7.58)$$

or else of the form

$$\left\{ \mathbf{x} : (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i) \text{ and } g_i(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) < x_k < b_j^i \right\} \quad (19.7.59)$$

The function, g_i has a derivative on $A_i \subseteq \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i)$ where

$$m_{n-1} \left(\prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i) \setminus A_i \right) = 0.$$

Also, there exists an open set, Q_0 such that $Q_0 \subseteq \overline{Q_0} \subseteq U$ and $\overline{U} \subseteq Q_0 \cup Q_1 \cup \dots \cup Q_N$.

Note that since there are only finitely many Q_i and each g_i is Lipschitz, it follows from an application of Lemma 19.5.6 that $\mathcal{H}^{n-1}(\partial U) < \infty$. Also from Lemma 19.7.9 $\mathcal{H}^{n-1}$ is inner and outer regular on ∂U .

Lemma 19.7.12 Suppose U is a bounded open set as described above. Then there exists a unique function in $L^\infty(\partial U, \mathcal{H}^{n-1})^n$, $\mathbf{n}(\mathbf{y})$ for $\mathbf{y} \in \partial U$ such that $|\mathbf{n}(\mathbf{y})| = 1$, $\mathbf{n}$ is $\mathcal{H}^{n-1}$ measurable, (meaning each component of $\mathbf{n}$ is $\mathcal{H}^{n-1}$ measurable) and for every $\mathbf{w} \in \mathbb{R}^n$ satisfying $|\mathbf{w}| = 1$, and for every $f \in C_c^1(\mathbb{R}^n)$,

$$\lim_{t \rightarrow 0} \int_U \frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} dx = \int_{\partial U} f(\mathbf{n} \cdot \mathbf{w}) d\mathcal{H}^{n-1}$$

Proof: Let $\overline{U} \subseteq V \subseteq \overline{V} \subseteq \cup_{i=0}^N Q_i$ and let $\{\psi_i\}_{i=0}^N$ be a C^∞ partition of unity on $\overline{V}$ such that $\text{spt}(\psi_i) \subseteq Q_i$. Then for all t small enough and $\mathbf{x} \in U$,

$$\frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} = \frac{1}{t} \sum_{i=0}^N \psi_i f(\mathbf{x} + t\mathbf{w}) - \psi_i f(\mathbf{x}).$$

Thus using the dominated convergence theorem,

$$\begin{aligned} & \lim_{t \rightarrow 0} \int_U \frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} dx \\ &= \lim_{t \rightarrow 0} \int_U \left(\frac{1}{t} \sum_{i=0}^N \psi_i f(\mathbf{x} + t\mathbf{w}) - \psi_i f(\mathbf{x}) \right) dx \\ &= \int_U \sum_{i=0}^N \sum_{j=1}^n D_j(\psi_i f)(\mathbf{x}) w_j dx \end{aligned}$$

$$= \int_U \sum_{j=1}^n D_j (\psi_0 f) (\mathbf{x}) w_j dx + \sum_{i=1}^N \int_U \sum_{j=1}^n D_j (\psi_i f) (\mathbf{x}) w_j dx \quad (19.7.60)$$

Since $\text{spt}(\psi_0) \subseteq Q_0$, it follows the first term in the above equals zero. In the second term, fix i . Without loss of generality, suppose the k in the above definition equals n and 19.7.58 holds. This just makes things a little easier to write. Thus g_i is a function of

$$(x_1, \dots, x_{n-1}) \in \prod_{j=1}^{n-1} (a_j^i, b_j^i) \equiv B_i$$

Then

$$\begin{aligned} & \int_U \sum_{j=1}^n D_j (\psi_i f) (\mathbf{x}) w_j dx \\ &= \int_{B_i} \int_{a_n^i}^{g_i(x_1, \dots, x_{n-1})} \sum_{j=1}^n D_j (\psi_i f) (\mathbf{x}) w_j dx_n dx_1 \cdots dx_{n-1} \\ &= \int_{B_i} \int_{-\infty}^{g_i(x_1, \dots, x_{n-1})} \sum_{j=1}^n D_j (\psi_i f) (\mathbf{x}) w_j dx_n dx_1 \cdots dx_{n-1} \end{aligned}$$

Letting $x_n = y + g_i(x_1, \dots, x_{n-1})$ and changing the variable, this equals

$$\begin{aligned} &= \int_{B_i} \int_{-\infty}^0 \sum_{j=1}^n D_j (\psi_i f) (x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1})) \cdot \\ & \quad w_j dy dx_1 \cdots dx_{n-1} \\ &= \int_{A_i} \int_{-\infty}^0 \sum_{j=1}^n D_j (\psi_i f) (x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1})) \cdot \\ & \quad w_j dy dx_1 \cdots dx_{n-1} \end{aligned}$$

Recall D_j refers to the partial derivative taken with respect to the entry in the j^{th} slot. In the n^{th} slot is found not just x_n but $y + g_i(x_1, \dots, x_{n-1})$ so a differentiation with respect to x_j will not be the same as D_j . In fact, it will introduce another term involving $g_{i,j}$. Thus from the chain rule,

$$\begin{aligned} &= \int_{A_i} \int_{-\infty}^0 \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f(x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1}))) w_j - \\ & \quad D_n (\psi_i f) (x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1})) \cdot \\ & \quad g_{i,j}(x_1, \dots, x_{n-1}) w_j dy dx_1 \cdots dx_{n-1} \\ & \quad + \int_{A_i} \int_{-\infty}^0 D_n (\psi_i f) (x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1})) \cdot \quad (19.7.61) \\ & \quad w_n dy dx_1 \cdots dx_{n-1} \end{aligned}$$

Consider the term

$$\int_{A_i} \int_{-\infty}^0 \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f(x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1}))) \cdot w_j dy dx_1 \cdots dx_{n-1}.$$

This equals

$$\int_{B_i} \int_{-\infty}^0 \sum_{j=1}^{n-1} \frac{\partial}{\partial x_j} (\psi_i f(x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1}))) \cdot w_j dy dx_1 \cdots dx_{n-1},$$

and now interchanging the order of integration and using the fact that $\text{spt}(\psi_i) \subseteq Q_i$, it follows this term equals zero. (The reason this is valid is that

$$x_j \rightarrow \psi_i f(x_1, \dots, x_{n-1}, y + g_i(x_1, \dots, x_{n-1}))$$

is the composition of Lipschitz functions and is therefore Lipschitz. Therefore, this function is absolutely continuous and can be recovered by integrating its derivative.)

Then, changing the variable back to x_n it follows 19.7.61 reduces to

$$\begin{aligned} &= - \int_{A_i} \int_{-\infty}^{g_i(x_1, \dots, x_{n-1})} \left(\sum_{j=1}^{n-1} D_n(\psi_i f)(x_1, \dots, x_{n-1}, x_n) \cdot \right. \\ &\quad \left. g_{i,j}(x_1, \dots, x_{n-1}) w_j \right) dx_n dx_1 \cdots dx_{n-1} \\ &+ \int_{A_i} \int_{-\infty}^{g_i(x_1, \dots, x_{n-1})} D_n(\psi_i f)(x_1, \dots, x_{n-1}, x_n) w_n dx_n dx_1 \cdots dx_{n-1} \end{aligned}$$

Doing the integrals using the observation that $g_{i,j}(x_1, \dots, x_{n-1})$ does not depend on x_n , this reduces further to

$$\int_{A_i} (\psi_i f)(x_1, \dots, x_{n-1}, x_n) \mathbf{N}_i(x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1})) \cdot \mathbf{w} dm_{n-1} \quad (19.7.62)$$

where $\mathbf{N}_i(x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1}))$ is given by

$$(-g_{i,1}(x_1, \dots, x_{n-1}), -g_{i,2}(x_1, \dots, x_{n-1}), \dots, -g_{i,n-1}(x_1, \dots, x_{n-1}), 1). \quad (19.7.63)$$

At this point I need a technical lemma which will allow the use of the area formula. The part of the boundary of U which is contained in Q_i is the image of the map, $\mathbf{h}_i(x_1, \dots, x_{n-1})$ given by $(x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1}))$ for $(x_1, \dots, x_{n-1}) \in A_i$. I need a formula for

$$\det(D\mathbf{h}_i(x_1, \dots, x_{n-1})^* D\mathbf{h}_i(x_1, \dots, x_{n-1}))^{1/2}.$$

To avoid interrupting the argument, I will state the lemma here and prove it later.

Lemma 19.7.13

$$\begin{aligned} & \det (D\mathbf{h}_i(x_1, \dots, x_{n-1})^* D\mathbf{h}_i(x_1, \dots, x_{n-1}))^{1/2} \\ &= \sqrt{1 + \sum_{j=1}^{n-1} g_{i,j}(x_1, \dots, x_{n-1})^2} \equiv J_i(x_1, \dots, x_{n-1}). \end{aligned}$$

For

$$\mathbf{y} = (x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1})) \in \partial U \cap Q_i$$

and $\mathbf{n}$ defined by

$$\mathbf{n}_i(\mathbf{y}) = \frac{1}{J_i(x_1, \dots, x_{n-1})} \mathbf{N}_i(\mathbf{y})$$

it follows from the description of $J_i(x_1, \dots, x_{n-1})$ given in the above lemma, that $\mathbf{n}_i$ is a unit vector. All components of $\mathbf{n}_i$ are continuous functions of limits of continuous functions. Therefore, $\mathbf{n}_i$ is Borel measurable and so it is $\mathcal{H}^{n-1}$ measurable. Now 19.7.62 reduces to

$$\begin{aligned} & \int_{A_i} (\psi_i f)(x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1})) \times \\ & \mathbf{n}_i(x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1})) \cdot \mathbf{w} J_i(x_1, \dots, x_{n-1}) dm_{n-1}. \end{aligned}$$

By the area formula this equals

$$\int_{\mathbf{h}(A_i)} \psi_i f(\mathbf{y}) \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1}.$$

Now by Lemma 19.5.6 and the equality of m_{n-1} and $\mathcal{H}^{n-1}$ on $\mathbb{R}^{n-1}$, the above integral equals

$$\int_{\partial U \cap Q_i} \psi_i f(\mathbf{y}) \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} = \int_{\partial U} \psi_i f(\mathbf{y}) \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1}.$$

Returning to 19.7.60 similar arguments apply to the other terms and therefore,

$$\begin{aligned} & \lim_{t \rightarrow 0} \int_U \frac{f(\mathbf{x} + t\mathbf{w}) - f(\mathbf{x})}{t} dm_n \\ &= \sum_{i=1}^N \int_{\partial U} \psi_i f(\mathbf{y}) \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} \\ &= \int_{\partial U} f(\mathbf{y}) \sum_{i=1}^N \psi_i(\mathbf{y}) \mathbf{n}_i(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} \\ &= \int_{\partial U} f(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} \end{aligned} \tag{19.7.64}$$

Then let $\mathbf{n}(\mathbf{y}) \equiv \sum_{i=1}^N \psi_i(\mathbf{y}) \mathbf{n}_i(\mathbf{y})$.

I need to show first there is no other $\mathbf{n}$ which satisfies 19.7.64 and then I need to show that $|\mathbf{n}(\mathbf{y})| = 1$. Note that it is clear $|\mathbf{n}(\mathbf{y})| \leq 1$ because each $\mathbf{n}_i$ is a unit vector and this is just a convex combination of these. Suppose then that $\mathbf{n}_1 \in L^\infty(\partial U, \mathcal{H}^{n-1})$ also works in 19.7.64. Then for all $f \in C_c^1(\mathbb{R}^n)$,

$$\int_{\partial U} f(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} = \int_{\partial U} f(\mathbf{y}) \mathbf{n}_1(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1}.$$

Suppose $h \in C(\partial U)$. Then by the Tietze extension theorem, there exists $f \in C_c(\mathbb{R}^n)$ such that the restriction of f to ∂U equals h . Now by Lemma 19.7.10 applied to a bounded open set containing the support of f , there exists a sequence $\{f_m\}$ of functions in $C_c^1(\mathbb{R}^n)$ converging uniformly to f . Therefore,

$$\begin{aligned} & \int_{\partial U} h(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} \\ &= \lim_{m \rightarrow \infty} \int_{\partial U} f_m(\mathbf{y}) \mathbf{n}(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} \\ &= \lim_{m \rightarrow \infty} \int_{\partial U} f_m(\mathbf{y}) \mathbf{n}_1(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1} \\ &= \int_{\partial U} h(\mathbf{y}) \mathbf{n}_1(\mathbf{y}) \cdot \mathbf{w} d\mathcal{H}^{n-1}. \end{aligned}$$

Now $\mathcal{H}^{n-1}$ is a Radon measure on ∂U and so the continuous functions on ∂U are dense in $L^1(\partial U, \mathcal{H}^{n-1})$. It follows $\mathbf{n} \cdot \mathbf{w} = \mathbf{n}_1 \cdot \mathbf{w}$ a.e. Now let $\{\mathbf{w}_m\}_{m=1}^\infty$ be a countable dense subset of the unit sphere. From what was just shown, $\mathbf{n} \cdot \mathbf{w}_m = \mathbf{n}_1 \cdot \mathbf{w}_m$ except for a set of measure zero, N_m . Letting $N = \cup_m N_m$, it follows that for $\mathbf{y} \notin N$, $\mathbf{n}(\mathbf{y}) \cdot \mathbf{w}_m = \mathbf{n}_1(\mathbf{y}) \cdot \mathbf{w}_m$ for all m . Since the set is dense, it follows $\mathbf{n}(\mathbf{y}) \cdot \mathbf{w} = \mathbf{n}_1(\mathbf{y}) \cdot \mathbf{w}$ for all $\mathbf{y} \notin N$ and for all $\mathbf{w}$ a unit vector. Therefore, $\mathbf{n}(\mathbf{y}) = \mathbf{n}_1(\mathbf{y})$ for all $\mathbf{y} \notin N$ and this shows $\mathbf{n}$ is unique. In particular, although it appears to depend on the partition of unity $\{\psi_i\}$ from its definition, this is not the case.

It only remains to verify $|\mathbf{n}(\mathbf{y})| = 1$ a.e. I will do this by showing how to compute $\mathbf{n}$. In particular, I will show that $\mathbf{n} = \mathbf{n}_i$ a.e. on $\partial U \cap Q_i$. Let $W \subseteq \overline{W} \subseteq Q_i \cap \partial U$ where W is open in ∂U . Let O be an open set such that $O \cap \partial U = W$ and $\overline{O} \subseteq Q_i$. Using Corollary 19.7.8 there exists a C^∞ partition of unity $\{\psi_m\}$ such that $\psi_i = 1$ on $\overline{O}$. Therefore, if $m \neq i$, $\psi_m = 0$ on $\overline{O}$. Then if $f \in C_c^1(O)$,

$$\begin{aligned} & \int_W f \mathbf{w} \cdot \mathbf{n} d\mathcal{H}^{n-1} \\ &= \int_{\partial U} f \mathbf{w} \cdot \mathbf{n} d\mathcal{H}^{n-1} = \int_U \nabla f \cdot \mathbf{w} dm_n \\ &= \int_U \nabla(\psi_i f) \cdot \mathbf{w} dm_n \end{aligned}$$

which by the first part of the argument given above equals

$$\int_W \psi_i f \mathbf{n}_i \cdot \mathbf{w} d\mathcal{H}^{n-1} = \int_W f \mathbf{w} \cdot \mathbf{n}_i d\mathcal{H}^{n-1}.$$

Thus for all $f \in C_c^1(O)$,

$$\int_W f \mathbf{w} \cdot \mathbf{n} d\mathcal{H}^{n-1} = \int_W f \mathbf{w} \cdot \mathbf{n}_i d\mathcal{H}^{n-1} \quad (19.7.65)$$

Since $C_c^1(O)$ is dense in $C_c(O)$, the above equation is also true for all $f \in C_c(O)$. Now letting $h \in C_c(W)$, the Tietze extension theorem implies there exists $f_1 \in C(\bar{O})$ whose restriction to $\bar{W}$ equals h . Let f be defined by

$$f_1(\mathbf{x}) \frac{\text{dist}(\mathbf{x}, O^C)}{\text{dist}(\mathbf{x}, \text{spt}(h)) + \text{dist}(\mathbf{x}, O^C)} = f(\mathbf{x}).$$

Then $f = h$ on W and so this has shown that for all $h \in C_c(W)$, 19.7.65 holds for h in place of f . But as observed earlier, $\mathcal{H}^{n-1}$ is outer and inner regular on ∂U and so $C_c(W)$ is dense in $L^1(W, \mathcal{H}^{n-1})$ which implies $\mathbf{w} \cdot \mathbf{n}(\mathbf{y}) = \mathbf{w} \cdot \mathbf{n}_i(\mathbf{y})$ for a.e. $\mathbf{y}$. Considering a countable dense subset of the unit sphere as above, this implies $\mathbf{n}(\mathbf{y}) = \mathbf{n}_i(\mathbf{y})$ a.e. $\mathbf{y}$. This proves $|\mathbf{n}(\mathbf{y})| = 1$ a.e. and in fact $\mathbf{n}(\mathbf{y})$ can be computed by using the formula for $\mathbf{n}_i(\mathbf{y})$. This proves the lemma.

It remains to prove Lemma 19.7.13.

Proof of Lemma 19.7.13: Let $\mathbf{h}(\mathbf{x}) = (x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1}))^T$

$$D\mathbf{h}(\mathbf{x}) = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \vdots \\ 0 & & 1 \\ g_{,x_1} & \cdots & g_{,x_{n-1}} \end{pmatrix}$$

Therefore,

$$J(\mathbf{x}) = (\det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x})))^{1/2}.$$

Therefore, $J(\mathbf{x})$ is the square root of the determinant of the following $n \times n$ matrix.

$$\begin{pmatrix} 1 + (g_{,x_1})^2 & g_{,x_1}g_{,x_2} & \cdots & g_{,x_1}g_{,x_{n-1}} \\ g_{,x_2}g_{,x_1} & 1 + (g_{,x_2})^2 & \cdots & g_{,x_2}g_{,x_{n-1}} \\ \vdots & & \ddots & \vdots \\ g_{,x_{n-1}}g_{,x_1} & g_{,x_{n-1}}g_{,x_2} & \cdots & 1 + (g_{,x_{n-1}})^2 \end{pmatrix}. \quad (19.7.66)$$

I need to show the determinant of the above matrix equals

$$1 + \sum_{i=1}^{n-1} (g_{,x_i}(\mathbf{x}))^2.$$

This is implied by the following claim. To simplify the notation I will replace $n-1$ with n .

Claim: Let $a_1, \dots, a_n$ be real numbers and let $A(a_1, \dots, a_n)$ be the matrix which has $1 + a_i^2$ in the ii^{th} slot and $a_i a_j$ in the ij^{th} slot when $i \neq j$. Then

$$\det A = 1 + \sum_{i=1}^n a_i^2.$$

Proof of the claim: The matrix, $A(a_1, \dots, a_n)$ is of the form

$$A(a_1, \dots, a_n) = \begin{pmatrix} 1 + a_1^2 & a_1 a_2 & \cdots & a_1 a_n \\ a_1 a_2 & 1 + a_2^2 & & a_2 a_n \\ \vdots & & \ddots & \vdots \\ a_1 a_n & a_2 a_n & \cdots & 1 + a_n^2 \end{pmatrix}$$

Now consider the product of a matrix and its transpose, $B^T B$ below.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & & 0 & a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & a_n \\ -a_1 & -a_2 & \cdots & -a_n & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & -a_n \\ a_1 & a_2 & \cdots & a_n & 1 \end{pmatrix} \quad (19.7.67)$$

This product equals a matrix of the form

$$\begin{pmatrix} A(a_1, \dots, a_n) & \mathbf{0} \\ \mathbf{0} & 1 + \sum_{i=1}^n a_i^2 \end{pmatrix}$$

Therefore, $(1 + \sum_{i=1}^n a_i^2) \det(A(a_1, \dots, a_n)) = \det(B)^2 = \det(B^T)^2$. However, using row operations,

$$\begin{aligned} \det B^T &= \det \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & & 0 & a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & a_n \\ 0 & 0 & \cdots & 0 & 1 + \sum_{i=1}^n a_i^2 \end{pmatrix} \\ &= 1 + \sum_{i=1}^n a_i^2 \end{aligned}$$

and therefore,

$$\left(1 + \sum_{i=1}^n a_i^2\right) \det(A(a_1, \dots, a_n)) = \left(1 + \sum_{i=1}^n a_i^2\right)^2$$

which shows $\det(A(a_1, \dots, a_n)) = (1 + \sum_{i=1}^n a_i^2)$. This proves the claim.

Now the above lemma implies the divergence theorem.

Theorem 19.7.14 *Let U be a bounded open set with a Lipschitz boundary which lies on one side of its boundary. Then if $f \in C_c^1(\mathbb{R}^n)$,*

$$\int_U f_{,k}(\mathbf{x}) \, dm_n = \int_{\partial U} f n_k d\mathcal{H}^{n-1} \quad (19.7.68)$$

where $\mathbf{n} = (n_1, \dots, n_n)$ is the $\mathcal{H}^{n-1}$ measurable unit vector of Lemma 19.7.12. Also, if $\mathbf{F}$ is a vector field such that each component is in $C_c^1(\mathbb{R}^n)$, then

$$\int_U \nabla \cdot \mathbf{F}(\mathbf{x}) dm_n = \int_{\partial U} \mathbf{F} \cdot \mathbf{n} d\mathcal{H}^{n-1}. \quad (19.7.69)$$

Proof: To obtain 19.7.68 apply Lemma 19.7.12 to $\mathbf{w} = \mathbf{e}_k$. Then to obtain 19.7.69 from this,

$$\begin{aligned} & \int_U \nabla \cdot \mathbf{F}(\mathbf{x}) dm_n \\ &= \sum_{j=1}^n \int_U F_{j,j} dm_n = \sum_{j=1}^n \int_{\partial U} F_j n_j d\mathcal{H}^{n-1} \\ &= \int_{\partial U} \sum_{j=1}^n F_j n_j d\mathcal{H}^{n-1} = \int_{\partial U} \mathbf{F} \cdot \mathbf{n} d\mathcal{H}^{n-1}. \end{aligned}$$

This proves the theorem.

What is the geometric significance of the vector, $\mathbf{n}$? Recall that in the part of the boundary contained in Q_i , this vector points in the same direction as the vector

$$\mathbf{N}_i(x_1, \dots, x_{n-1}, g_i(x_1, \dots, x_{n-1}))$$

given by

$$(-g_{i,1}(x_1, \dots, x_{n-1}), -g_{i,2}(x_1, \dots, x_{n-1}), \dots, -g_{i,n-1}(x_1, \dots, x_{n-1}), 1) \quad (19.7.70)$$

in the case where $k = n$. This vector is the gradient of the function,

$$x_n - g_i(x_1, \dots, x_{n-1})$$

and so is perpendicular to the level surface given by

$$x_n - g_i(x_1, \dots, x_{n-1}) = 0$$

in the case where g_i is C^1 . It also points away from U so the vector $\mathbf{n}$ is the unit outer normal. The other cases work similarly.

The divergence theorem is valid in situations more general than for Lipschitz boundaries. What you need is essentially the ability to say that the functions, g_i above can be differentiated a.e. and more importantly that these functions can be recovered by integrating their partial derivatives. In other words, you need absolute continuity in each variable. Later in the chapter on weak derivatives, examples of such functions which are more general than Lipschitz functions will be discussed. However, the Lipschitz functions are pretty general and will suffice for now.

Differentiation With Respect To General Radon Measures

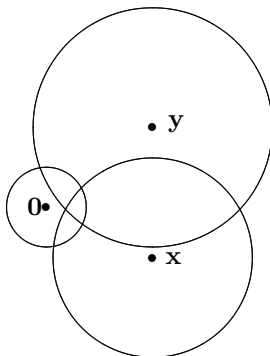
This is a brief chapter on certain important topics on the differentiation theory for general Radon measures. For different proofs and some results which are not discussed here, a good source is [24] which is where I first read some of these things.

20.1 Besicovitch Covering Theorem

The fundamental theorem of calculus presented above for Lebesgue measures can be generalized to arbitrary Radon measures. It turns out that the same approach works if a different covering theorem is employed instead of the Vitali theorem. This covering theorem is the Besicovitch covering theorem of this section. It is necessary because for a general Radon measure μ , it is no longer the case that the measure is translation invariant. This implies there is no way to estimate $\mu(\hat{B})$ in terms of $\mu(B)$ and thus the Vitali covering theorem is of no use. In the Besicovitch covering theorem the balls in the covering are not enlarged as they are in the Vitali theorem. In this theorem they can also be either open or closed or neither open nor closed. The balls can also be taken with respect to any norm on $\mathbb{R}^n$. The notation, $B(\mathbf{x}, r)$ in the above argument will denote any set which satisfies

$$\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < r\} \subseteq B(\mathbf{x}, r) \subseteq \overline{\{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < r\}}$$

and the norm, $\|\cdot\|$ is just some norm on $\mathbb{R}^n$. The following picture is a distorted picture of the situation described in the following lemma.



Lemma 20.1.1 *Let $10 \leq r_x \leq r_y$ and suppose $B(\mathbf{x}, r_x)$ and $B(\mathbf{y}, r_y)$ both have nonempty intersection with $B(\mathbf{0}, 1)$ but neither of these balls contains $\mathbf{0}$. Suppose also that*

$$\|\mathbf{x} - \mathbf{y}\| \geq r_y$$

so that neither ball contains both centers in its interior. Then

$$\left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| \geq \frac{4}{5}.$$

Proof: By hypothesis,

$$\|\mathbf{x}\| \geq r_x \geq \|\mathbf{x}\| - 1, \quad \|\mathbf{y}\| \geq r_y \geq \|\mathbf{y}\| - 1.$$

Then

$$\begin{aligned} \left\| \frac{\mathbf{x}}{\|\mathbf{x}\|} - \frac{\mathbf{y}}{\|\mathbf{y}\|} \right\| &= \left\| \frac{\mathbf{x}\|\mathbf{y}\| - \|\mathbf{x}\|\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \right\| \\ &= \left\| \frac{\mathbf{x}\|\mathbf{y}\| - \mathbf{y}\|\mathbf{y}\| + \mathbf{y}\|\mathbf{y}\| - \|\mathbf{x}\|\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} \right\| \\ &\geq \frac{\|\mathbf{x} - \mathbf{y}\|}{\|\mathbf{x}\|} - \frac{|\|\mathbf{y}\| - \|\mathbf{x}\||}{\|\mathbf{x}\|} \end{aligned} \tag{20.1.1}$$

Now there are two cases.

First suppose $\|\mathbf{y}\| \geq \|\mathbf{x}\|$. Then the above is larger than

$$\begin{aligned} &\geq \frac{r_y}{\|\mathbf{x}\|} - \frac{\|\mathbf{y}\|}{\|\mathbf{x}\|} + 1 \geq \frac{r_y}{\|\mathbf{x}\|} - \frac{(r_y + 1)}{\|\mathbf{x}\|} + 1 \\ &= 1 - \frac{1}{\|\mathbf{x}\|} \geq 1 - \frac{1}{r_x} \geq 1 - \frac{1}{10} = \frac{9}{10}. \end{aligned}$$

Next suppose $\|\mathbf{x}\| \geq \|\mathbf{y}\|$. Then 20.1.1 is at least as large as

$$\begin{aligned}
 & \frac{r_y}{\|\mathbf{x}\|} - \frac{\|\mathbf{x}\| - \|\mathbf{y}\|}{\|\mathbf{x}\|} \\
 = & \frac{r_y}{\|\mathbf{x}\|} - 1 + \frac{\|\mathbf{y}\|}{\|\mathbf{x}\|} \\
 \geq & \frac{2r_y}{\|\mathbf{x}\|} - 1 \geq \frac{2r_y}{r_x + 1} - 1 \\
 \geq & \frac{2r_x}{r_x + 1} - 1 \geq \frac{20}{10 + 1} - 1 \\
 = & .81818
 \end{aligned}$$

This proves the lemma.

Lemma 20.1.2 *There exists L_n depending on dimension, n , such that for $\{\mathbf{x}_k\}_{k=1}^m$ distinct points on $\partial B(\mathbf{0}, 1)$, if $m \geq L_n$, then the distance between some pair of points of $\{\mathbf{x}_k\}_{k=1}^m$ is less than $4/5$.*

Proof: Let $\{\mathbf{z}_j\}_{j=1}^{L_n-1}$ be a $1/3$ net on $\partial B(\mathbf{0}, 1)$. Then for $m \geq L_n$, if $\{\mathbf{x}_k\}_{k=1}^m$ is a set of m distinct points on $\partial B(\mathbf{0}, 1)$, there must exist $\mathbf{x}_i$ and $\mathbf{x}_j$ for $i \neq j$ such that both $\mathbf{x}_i$ and $\mathbf{x}_j$ are contained in some $B(\mathbf{z}_k, 1/3)$. This follows from the pigeon hole principle. There are more $\mathbf{x}_i$ than there are $B(\mathbf{z}_k, 1/3)$ and so one of these must have more than one $\mathbf{x}_k$ in it. But then

$$\|\mathbf{x}_i - \mathbf{x}_j\| \leq \|\mathbf{x}_i - \mathbf{z}_k\| + \|\mathbf{z}_k - \mathbf{x}_j\| \leq \frac{2}{3} < \frac{4}{5}$$

This proves the lemma.

Corollary 20.1.3 *Let $B_0 = B(\mathbf{0}, 1)$ and let $B_j = B(\mathbf{x}_j, r_j)$ for $j = 1, \dots, K$ such that $r_j \geq 10$, $\mathbf{0} \notin B_j$ for all $j > 0$, $B_j \cap B_0 \neq \emptyset$, and for all $i \neq j$,*

$$\|\mathbf{x}_i - \mathbf{x}_j\| \geq \max(r_i, r_j).$$

That is, no B_j contains two centers in its interior. Then $K \leq L_n$, the constant of the above lemma.

Proof: By Lemma 20.1.2, if $K > L_n$, there exist two of the centers, $\mathbf{x}_i$ and $\mathbf{x}_j$ such that $\left\| \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|} - \frac{\mathbf{x}_j}{\|\mathbf{x}_j\|} \right\| < 4/5$. By Lemma 20.1.1, $\|\mathbf{x}_i - \mathbf{x}_j\| < \max(r_i, r_j)$ contrary to the assumptions of the corollary. Hence $K \leq L_n$ as claimed.

Theorem 20.1.4 *There exists a constant N_n , depending only on n with the following property. If $\mathcal{F}$ is any collection of nonempty balls in $\mathbb{R}^n$ with*

$$\sup \{ \text{diam}(B) : B \in \mathcal{F} \} < D < \infty$$

and if A is the set of centers of the balls in $\mathcal{F}$, then there exist subsets of $\mathcal{F}$, $\mathcal{G}_1, \dots, \mathcal{G}_{N_n}$, such that each $\mathcal{G}_i$ is a countable collection of disjoint balls from $\mathcal{F}$ and

$$A \subseteq \bigcup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}.$$

Lemma 20.1.5 *In the situation of Theorem 20.1.4, suppose the set of centers A is bounded. Then there exists a sequence of balls from $\mathcal{F}$, $\{B_j\}_{j=1}^J$ where $J \leq \infty$ such that*

$$r(B_1) \geq \frac{3}{4} \sup \{r(B) : B \in \mathcal{F}\} \quad (20.1.2)$$

and if

$$A_m \equiv A \setminus (\cup_{i=1}^m B_i) \neq \emptyset, \quad (20.1.3)$$

then

$$r(B_{m+1}) \geq \frac{3}{4} \sup \{r : B(\mathbf{a}, r) \in \mathcal{F}, \mathbf{a} \in A_m\}. \quad (20.1.4)$$

Letting $B_j = B(\mathbf{a}_j, r_j)$, this sequence satisfies

$$A \subseteq \cup_{i=1}^J B_i, \quad r(B_k) \leq \frac{4}{3} r(B_j) \text{ for } j < k, \quad \{B(\mathbf{a}_j, r_j/3)\}_{j=1}^J \text{ are disjoint.} \quad (20.1.5)$$

Proof: Pick B_1 satisfying 20.1.2. If $B_1, \dots, B_m$ have been chosen, and A_m is given in 20.1.3, then if it equals $\emptyset$, it follows $A \subseteq \cup_{i=1}^m B_i$. Set $J = m$. If $A_m \neq \emptyset$, pick B_{m+1} to satisfy 20.1.4. This defines the desired sequence. It remains to verify the claims in 20.1.5. Consider the second claim. Letting $A_0 \equiv A$, $A_k \subseteq A_{j-1}$ and so

$$\sup \{r : B(\mathbf{a}, r) \in \mathcal{F}, \mathbf{a} \in A_{j-1}\} \geq r_k.$$

Hence $r_j \geq (3/4)r_k$. This proves the second claim of 20.1.5.

Consider the third claim of 20.1.5. Suppose to the contrary that $\mathbf{x} \in B(\mathbf{a}_j, r_j/3) \cap B(\mathbf{a}_i, r_i/3)$ where $i < j$. Then

$$\begin{aligned} \|\mathbf{a}_i - \mathbf{a}_j\| &\leq \|\mathbf{a}_i - \mathbf{x}\| + \|\mathbf{x} - \mathbf{a}_j\| \\ &< \frac{1}{3}(r_i + r_j) \leq \frac{1}{3}\left(r_i + \frac{4}{3}r_i\right) \\ &= \frac{7}{9}r_i < r_i \end{aligned}$$

contrary to the construction which requires $\mathbf{a}_j \notin B(\mathbf{a}_i, r_i)$.

Finally consider the first claim of 20.1.5. It is true if $J < \infty$. This follows from the construction. If $J = \infty$, then since A is bounded and the balls, $B(\mathbf{a}_j, r_j/3)$ are disjoint, it must be the case that $\lim_{i \rightarrow \infty} r_i = 0$. Suppose $J = \infty$ so that $A_m \neq \emptyset$ for all m . If $\mathbf{a}_0$ fails to be covered, then $\mathbf{a}_0 \in A_k$ for all k . Let $\mathbf{a}_0 \in B(\mathbf{a}_0, r_0) \in \mathcal{F}$ for some ball $B(\mathbf{a}_0, r_0)$. Then for i large enough, $r_i < \frac{1}{10}r_0$ and so since $\mathbf{a}_0 \in A_{i-1}$,

$$\frac{3}{4}r_0 \leq \frac{3}{4} \sup \{r : B(\mathbf{a}, r) \in \mathcal{F}, \mathbf{a} \in A_{i-1}\} \leq r_i < \frac{1}{10}r_0,$$

a contradiction. This proves the lemma.

Lemma 20.1.6 *There exists a constant M_n depending only on n such that for each $1 \leq k \leq J$, M_n exceeds the number of sets B_j for $j < k$ which have nonempty intersection with B_k .*

Proof: These sets B_j which intersect B_k are of two types. Either they have large radius, $r_j > 10r_k$, or they have small radius, $r_j \leq 10r_k$. In this argument let $\text{card}(S)$ denote the number of elements in the set S . Define for fixed k ,

$$I \equiv \{j : 1 \leq j < k, B_j \cap B_k \neq \emptyset, r_j \leq 10r_k\},$$

$$K \equiv \{j : 1 \leq j < k, B_j \cap B_k \neq \emptyset, r_j > 10r_k\}.$$

Claim 1: $B(\mathbf{a}_j, \frac{r_j}{3}) \subseteq B(\mathbf{a}_k, 15r_k)$ for $j \in I$.

Proof: Let $j \in I$. Then $B_j \cap B_k \neq \emptyset$ and $r_j \leq 10r_k$. Now if

$$\mathbf{x} \in B(\mathbf{a}_j, \frac{r_j}{3}),$$

then since $r_j \leq 10r_k$,

$$\begin{aligned} \|\mathbf{x} - \mathbf{a}_k\| &\leq \|\mathbf{x} - \mathbf{a}_j\| + \|\mathbf{a}_j - \mathbf{a}_k\| \leq \frac{r_j}{3} + r_j + r_k = \\ \frac{4}{3}r_j + r_k &\leq \frac{4}{3}(10r_k) + r_k = \frac{43}{3}r_k < 15r_k. \end{aligned}$$

Therefore, $B(\mathbf{a}_j, \frac{r_j}{3}) \subseteq B(\mathbf{a}_k, 15r_k)$.

Claim 2: $\text{card}(I) \leq 60^n$.

Proof: Recall $r(B_k) \leq \frac{4}{3}r(B_j)$. Then letting $\alpha(n)r^n$ be the Lebesgue measure of the n dimensional ball of radius r , (Note this $\alpha(n)$ depends on the norm used.)

$$\begin{aligned} \alpha(n) 15^n r_k^n &\equiv m_n(B(\mathbf{a}_k, 15r_k)) \geq \sum_{j \in I} m_n\left(B\left(\mathbf{a}_j, \frac{r_j}{3}\right)\right) \\ &= \sum_{j \in I} \alpha(n) \left(\frac{r_j}{3}\right)^n \geq \sum_{j \in I} \alpha(n) \left(\frac{r_k}{4}\right)^n \left(\text{since } r_k \leq \frac{4}{3}r_j\right) \\ &= \text{card}(I) \alpha(n) \left(\frac{r_k}{4}\right)^n \end{aligned}$$

and so it follows $\text{card}(I) \leq 60^n$ as claimed.

Claim 3: $\text{card}(K) \leq L_n$ where L_n is the constant of Corollary 20.1.3.

Proof: Consider $\{B_j : j \in K\}$ and B_k . Let $\mathbf{f}(\mathbf{x}) \equiv r_k^{-1}(\mathbf{x} - \mathbf{x}_k)$. Then $\mathbf{f}(B_k) = B(\mathbf{0}, 1)$ and

$$\mathbf{f}(B_j) = r_k^{-1}B(\mathbf{x}_j - \mathbf{x}_k, r_j) = B\left(\frac{\mathbf{x}_j - \mathbf{x}_k}{r_k}, r_j/r_k\right).$$

Then $r_j/r_k \geq 10$ because $j \in K$. None of the balls, $\mathbf{f}(B_j)$ contain $\mathbf{0}$ but all these balls intersect $B(\mathbf{0}, 1)$ and as just noted, each of these balls has radius ≥ 10 and none of them contains two centers on its interior. By Corollary 20.1.3, it follows there are no more than L_n of them. This proves the claim. A constant which will satisfy the desired conditions is

$$M_n \equiv L_n + 60^n + 1.$$

This completes the proof of Lemma 20.1.6.

Next subdivide the balls $\{B_i\}_{i=1}^J$ into M_n subsets $\mathcal{G}_1, \dots, \mathcal{G}_{M_n}$ each of which consists of disjoint balls. This is done in the following way. Let $B_1 \in \mathcal{G}_1$. If $B_1, \dots, B_k$ have each been assigned to one of the sets $\mathcal{G}_1, \dots, \mathcal{G}_{M_n}$, let $B_{k+1} \in \mathcal{G}_r$ where r is the smallest index having the property that B_{k+1} does not intersect any of the balls already in $\mathcal{G}_r$. There must exist such an index $r \in \{1, \dots, M_n\}$ because otherwise $B_{k+1} \cap B_j \neq \emptyset$ for at least M_n values of $j < k+1$ contradicting Lemma 20.1.6. By Lemma 20.1.5

$$A \subseteq \cup_{i=1}^{M_n} \{B : B \in \mathcal{G}_i\} = \cup_{j=1}^J B_j.$$

This proves Theorem 20.1.4 in the case where A is bounded.

To complete the proof of this theorem, the restriction that A is bounded must be removed. Define

$$A_l \equiv A \cap \{\mathbf{x} \in \mathbb{R}^n : 10(l-1)D \leq \|\mathbf{x}\| < 10lD\}, \quad l = 1, 2, \dots$$

and

$$\mathcal{F}_l = \{B(\mathbf{a}, r) : B(\mathbf{a}, r) \in \mathcal{F} \text{ and } \mathbf{a} \in A_l\}.$$

Then since D is an upper bound for all the diameters of these balls,

$$(\cup \mathcal{F}_l) \cap (\cup \mathcal{F}_m) = \emptyset \quad (20.1.6)$$

whenever $m \geq l+2$. Therefore, applying what was just shown to the pair $(A_l, \mathcal{F}_l)$, there exist subsets of $\mathcal{F}_l$, $\mathcal{G}_1^l \cdots \mathcal{G}_{M_n}^l$ such that each $\mathcal{G}_i^l$ is a countable collection of disjoint balls of $\mathcal{F}_l \subseteq \mathcal{F}$ and

$$A_l \subseteq \cup_{i=1}^{M_n} \{B : B \in \mathcal{G}_i^l\}.$$

Now let $\mathcal{G}_j \equiv \cup_{l=1}^{\infty} \mathcal{G}_j^{2l-1}$ for $1 \leq j \leq M_n$ and for $1 \leq j \leq M_n$, let $\mathcal{G}_{j+M_n} \equiv \cup_{l=1}^{\infty} \mathcal{G}_j^{2l}$. Thus, letting $N_n \equiv 2M_n$,

$$A = \cup_{l=1}^{\infty} A_{2l} \cup \cup_{l=1}^{\infty} A_{2l-1} \subseteq \cup_{j=1}^{N_n} \{B : B \in \mathcal{G}_j\}$$

and by 20.1.6, each $\mathcal{G}_j$ is a countable set of disjoint balls of $\mathcal{F}$. This proves the Besicovitch covering theorem.

20.2 Fundamental Theorem Of Calculus For Radon Measures

In this section the Besicovitch covering theorem will be used to give a generalization of the Lebesgue differentiation theorem to general Radon measures. In what follows, μ will be a Radon measure,

$$Z \equiv \{\mathbf{x} \in \mathbb{R}^n : \mu(B(\mathbf{x}, r)) = 0 \text{ for some } r > 0\},$$

$$\int_{B(\mathbf{x},r)} f d\mu \equiv \begin{cases} 0 & \text{if } \mathbf{x} \in Z, \\ \frac{1}{\mu(B(\mathbf{x},r))} \int_{B(\mathbf{x},r)} f d\mu & \text{if } \mathbf{x} \notin Z, \end{cases}$$

and the maximal function $Mf : \mathbb{R}^n \rightarrow [0, \infty]$ is given by

$$Mf(\mathbf{x}) \equiv \sup_{r \leq 1} \int_{B(\mathbf{x},r)} |f| d\mu.$$

Lemma 20.2.1 *Z is measurable and $\mu(Z) = 0$.*

Proof: For each $\mathbf{x} \in Z$, there exists a ball $B(\mathbf{x}, r)$ with $\mu(B(\mathbf{x}, r)) = 0$. Let $\mathcal{C}$ be the collection of these balls. Since $\mathbb{R}^n$ has a countable basis, a countable subset, $\tilde{\mathcal{C}}$, of $\mathcal{C}$ also covers Z . Let

$$\tilde{\mathcal{C}} = \{B_i\}_{i=1}^{\infty}.$$

Then letting $\bar{\mu}$ denote the outer measure determined by μ ,

$$\bar{\mu}(Z) \leq \sum_{i=1}^{\infty} \bar{\mu}(B_i) = \sum_{i=1}^{\infty} \mu(B_i) = 0$$

Therefore, Z is measurable and has measure zero as claimed.

Theorem 20.2.2 *Let μ be a Radon measure and let $f \in L^1(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r \rightarrow 0} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) = 0$$

for μ a.e. $\mathbf{x} \in \mathbb{R}^n$.

Proof: First consider the following claim which is a weak type estimate of the same sort used when differentiating with respect to Lebesgue measure.

Claim 1:

$$\bar{\mu}([Mf > \varepsilon]) \leq N_n \varepsilon^{-1} \|f\|_1$$

Proof: First note $A \cap Z = \emptyset$. For each $\mathbf{x} \in A$ there exists a ball $B_{\mathbf{x}} = B(\mathbf{x}, r_{\mathbf{x}})$ with $r_{\mathbf{x}} \leq 1$ and

$$\mu(B_{\mathbf{x}})^{-1} \int_{B(\mathbf{x}, r_{\mathbf{x}})} |f| d\mu > \varepsilon.$$

Let $\mathcal{F}$ be this collection of balls so that A is the set of centers of balls of $\mathcal{F}$. By the Besicovitch covering theorem,

$$A \subseteq \cup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}$$

where $\mathcal{G}_i$ is a collection of disjoint balls of $\mathcal{F}$. Now for some i ,

$$\bar{\mu}(A)/N_n \leq \mu(\cup \{B : B \in \mathcal{G}_i\})$$

because if this is not so, then

$$\bar{\mu}(A) \leq \sum_{i=1}^{N_n} \mu(\cup \{B : B \in \mathcal{G}_i\}) < \sum_{i=1}^{N_n} \frac{\bar{\mu}(A)}{N_n} = \bar{\mu}(A),$$

a contradiction. Therefore for this i ,

$$\begin{aligned} \frac{\bar{\mu}(A)}{N_n} &\leq \mu(\cup \{B : B \in \mathcal{G}_i\}) = \sum_{B \in \mathcal{G}_i} \mu(B) \leq \sum_{B \in \mathcal{G}_i} \varepsilon^{-1} \int_B |f| d\mu \\ &\leq \varepsilon^{-1} \int_{\mathbb{R}^n} |f| d\mu = \varepsilon^{-1} \|f\|_1. \end{aligned}$$

This shows Claim 1.

Claim 2: If g is any continuous function defined on $\mathbb{R}^n$, then

$$\lim_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |g(\mathbf{y}) - g(\mathbf{x})| d\mu(y) = 0$$

and if $\mathbf{x} \notin Z$,

$$\lim_{r \rightarrow 0} \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} g(\mathbf{y}) d\mu(y) = g(\mathbf{x}). \quad (20.2.7)$$

Proof: If $\mathbf{x} \in Z$ there is nothing to prove. If $\mathbf{x} \notin Z$, then since g is continuous at $\mathbf{x}$, whenever r is small enough,

$$\begin{aligned} &\int_{B(\mathbf{x}, r)} |g(\mathbf{y}) - g(\mathbf{x})| d\mu(y) \\ &= \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} |g(\mathbf{y}) - g(\mathbf{x})| d\mu(y) \\ &\leq \frac{1}{\mu(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \varepsilon d\mu(y) = \varepsilon. \end{aligned}$$

20.2.7 follows from the above and the triangle inequality. This proves the claim.

Now let $g \in C_c(\mathbb{R}^n)$ and $\mathbf{x} \notin Z$. Then from the above observations about continuous functions,

$$\begin{aligned} &\bar{\mu} \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) > \varepsilon \right] \right) \quad (20.2.8) \\ &\leq \bar{\mu} \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - g(\mathbf{y})| d\mu(y) > \frac{\varepsilon}{2} \right] \right) \\ &\quad + \bar{\mu} \left(\left[\mathbf{x} : |g(\mathbf{x}) - f(\mathbf{x})| > \frac{\varepsilon}{2} \right] \right). \end{aligned}$$

$$\leq \bar{\mu} \left(\left[M(f - g) > \frac{\varepsilon}{2} \right] \right) + \bar{\mu} \left(\left[|f - g| > \frac{\varepsilon}{2} \right] \right) \quad (20.2.9)$$

Now

$$\int_{[|f-g|>\frac{\varepsilon}{2}]} |f - g| d\mu \geq \frac{\varepsilon}{2} \bar{\mu} \left(\left[|f - g| > \frac{\varepsilon}{2} \right] \right)$$

and so from Claim 1 20.2.9 and hence 20.2.8 is dominated by

$$\left(\frac{2}{\varepsilon} + \frac{N_n}{\varepsilon} \right) \|f - g\|_{L^1(\mathbb{R}^n, \mu)}.$$

But by regularity of Radon measures, $C_c(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n, \mu)$ and so since g in the above is arbitrary, this shows 20.2.8 equals 0. Now

$$\begin{aligned} & \bar{\mu} \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) > 0 \right] \right) \\ &= \bar{\mu} \left(\bigcup_{m=1}^{\infty} \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) > \frac{1}{m} \right] \right) \right) \\ &\leq \sum_{m=1}^{\infty} \bar{\mu} \left(\left[\mathbf{x} : \limsup_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) > \frac{1}{m} \right] \right) = 0. \end{aligned}$$

By Lemma 20.2.1 the set Z is a set of measure zero and so if

$$\mathbf{x} \notin \left[\limsup_{r \rightarrow 0} \int_{B(\cdot, r)} |f(\mathbf{y}) - f(\cdot)| d\mu(y) > 0 \right] \cup Z$$

the above has shown

$$\begin{aligned} 0 &\leq \liminf_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) \\ &\leq \limsup_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) = 0 \end{aligned}$$

which proves the theorem.

The following corollary is the main result referred to as the Lebesgue Besicovitch Differentiation theorem.

Corollary 20.2.3 *If $f \in L^1_{loc}(\mathbb{R}^n, \mu)$,*

$$\lim_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) = 0 \quad \mu \text{ a.e. } \mathbf{x}. \quad (20.2.10)$$

Proof: If f is replaced by $f\chi_{B(\mathbf{0}, k)}$ then the conclusion 20.2.10 holds for all $\mathbf{x} \notin F_k$ where F_k is a set of μ measure 0. Letting $k = 1, 2, \dots$, and $F \equiv \bigcup_{k=1}^{\infty} F_k$, it follows that F is a set of measure zero and for any $\mathbf{x} \notin F$, and $k \in \{1, 2, \dots\}$, 20.2.10

holds if f is replaced by $f\mathcal{X}_{B(\mathbf{0},k)}$. Picking any such $\mathbf{x}$, and letting $k > |\mathbf{x}| + 1$, this shows

$$\begin{aligned} & \lim_{r \rightarrow 0} \int_{B(\mathbf{x},r)} |f(\mathbf{y}) - f(\mathbf{x})| d\mu(y) \\ &= \lim_{r \rightarrow 0} \int_{B(\mathbf{x},r)} |f\mathcal{X}_{B(\mathbf{0},k)}(\mathbf{y}) - f\mathcal{X}_{B(\mathbf{0},k)}(\mathbf{x})| d\mu(y) = 0. \end{aligned}$$

This proves the corollary.

20.3 Slicing Measures

Let μ be a finite Radon measure. I will show here that a formula of the following form holds.

$$\mu(F) = \int_F d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_F(\mathbf{x}, \mathbf{y}) d\nu_{\mathbf{x}}(y) d\alpha(x)$$

where $\alpha(E) = \mu(E \times \mathbb{R}^m)$. When this is done, the measures, $\nu_{\mathbf{x}}$, are called slicing measures and this shows that an integral with respect to μ can be written as an iterated integral in terms of the measure α and the slicing measures, $\nu_{\mathbf{x}}$. This is like going backwards in the construction of product measure. One starts with a measure, μ , defined on the Cartesian product and produces α and an infinite family of slicing measures from it whereas in the construction of product measure, one starts with two measures and obtains a new measure on a σ algebra of subsets of the Cartesian product of two spaces. First here are two technical lemmas.

Lemma 20.3.1 *The space $C_c(\mathbb{R}^m)$ with the norm*

$$\|f\| \equiv \sup \{|f(\mathbf{y})| : \mathbf{y} \in \mathbb{R}^m\}$$

is separable.

Proof: Let $\mathcal{D}_l$ consist of all functions which are of the form

$$\sum_{|\alpha| \leq N} a_{\alpha} \mathbf{y}^{\alpha} \left(\text{dist} \left(\mathbf{y}, B(\mathbf{0}, l+1)^C \right) \right)^{n_{\alpha}}$$

where $a_{\alpha} \in \mathbb{Q}$, α is a multi-index, and n_{α} is a positive integer. Then $\mathcal{D}_l$ is countable, separates the points of $\overline{B(\mathbf{0}, l)}$ and annihilates no point of $\overline{B(\mathbf{0}, l)}$. By the Stone Weierstrass theorem $\mathcal{D}_l$ is dense in the space $C(\overline{B(\mathbf{0}, l)})$ and so $\cup \{\mathcal{D}_l : l \in \mathbb{N}\}$ is a countable dense subset of $C_c(\mathbb{R}^m)$.

From the regularity of Radon measures, the following lemma follows.

Lemma 20.3.2 *If μ and ν are two Radon measures defined on σ algebras, $\mathcal{S}_{\mu}$ and $\mathcal{S}_{\nu}$, of subsets of $\mathbb{R}^n$ and if $\mu(V) = \nu(V)$ for all V open, then $\mu = \nu$ and $\mathcal{S}_{\mu} = \mathcal{S}_{\nu}$.*

Proof: Every compact set is a countable intersection of open sets so the two measures agree on every compact set. Hence it is routine that the two measures agree on every G_δ and F_σ set. (Recall G_δ sets are countable intersections of open sets and F_σ sets are countable unions of closed sets.) Now suppose $E \in \mathcal{S}_\nu$ is a bounded set. Then by regularity of ν there exists G a G_δ set and F , an F_σ set such that $F \subseteq E \subseteq G$ and $\nu(G \setminus F) = 0$. Then it is also true that $\mu(G \setminus F) = 0$. Hence $E = F \cup (E \setminus F)$ and $E \setminus F$ is a subset of $G \setminus F$, a set of μ measure zero. By completeness of μ , it follows $E \in \mathcal{S}_\mu$ and

$$\mu(E) = \mu(F) = \nu(F) = \nu(E).$$

If $E \in \mathcal{S}_\nu$ not necessarily bounded, let $E_m = E \cap B(0, m)$ and then $E_m \in \mathcal{S}_\mu$ and $\mu(E_m) = \nu(E_m)$. Letting $m \rightarrow \infty$, $E \in \mathcal{S}_\mu$ and $\mu(E) = \nu(E)$. Similarly, $\mathcal{S}_\mu \subseteq \mathcal{S}_\nu$ and the two measures are equal on $\mathcal{S}_\mu$.

The main result in the section is the following theorem.

Theorem 20.3.3 *Let μ be a finite Radon measure on $\mathbb{R}^{n+m}$ defined on a σ algebra, $\mathcal{F}$. Then there exists a unique finite Radon measure, α , defined on a σ algebra, $\mathcal{S}$, of sets of $\mathbb{R}^n$ which satisfies*

$$\alpha(E) = \mu(E \times \mathbb{R}^m) \quad (20.3.11)$$

for all E Borel. There also exists a Borel set of α measure zero, N , such that for each $\mathbf{x} \notin N$, there exists a Radon probability measure $\nu_{\mathbf{x}}$ such that if f is a nonnegative μ measurable function or a μ measurable function in $L^1(\mu)$,

$$\mathbf{y} \rightarrow f(\mathbf{x}, \mathbf{y}) \text{ is } \nu_{\mathbf{x}} \text{ measurable } \alpha \text{ a.e.}$$

$$\mathbf{x} \rightarrow \int_{\mathbb{R}^m} f(\mathbf{x}, \mathbf{y}) d\nu_{\mathbf{x}}(\mathbf{y}) \text{ is } \alpha \text{ measurable} \quad (20.3.12)$$

and

$$\int_{\mathbb{R}^{n+m}} f(\mathbf{x}, \mathbf{y}) d\mu = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(\mathbf{x}, \mathbf{y}) d\nu_{\mathbf{x}}(\mathbf{y}) \right) d\alpha(\mathbf{x}). \quad (20.3.13)$$

If $\widehat{\nu}_{\mathbf{x}}$ is any other collection of Radon measures satisfying 20.3.12 and 20.3.13, then $\widehat{\nu}_{\mathbf{x}} = \nu_{\mathbf{x}}$ for α a.e. $\mathbf{x}$.

Proof: First consider the uniqueness of α . Suppose α_1 is another Radon measure satisfying 20.3.11. Then in particular, α_1 and α agree on open sets and so the two measures are the same by Lemma 20.3.2.

To establish the existence of α , define α_0 on Borel sets by

$$\alpha_0(E) = \mu(E \times \mathbb{R}^m).$$

Thus α_0 is a finite Borel measure and so it is finite on compact sets. Lemma 11.1.3 on Page 11.1.3 implies the existence of the Radon measure α extending α_0 .

Next consider the uniqueness of $\nu_{\mathbf{x}}$. Suppose $\nu_{\mathbf{x}}$ and $\widehat{\nu}_{\mathbf{x}}$ satisfy all conclusions of the theorem with exceptional sets denoted by N and $\widehat{N}$ respectively. Then,

enlarging N and $\hat{N}$, one may also assume, using Lemma 20.2.1, that for $\mathbf{x} \notin N \cup \hat{N}$, $\alpha(B(\mathbf{x}, r)) > 0$ whenever $r > 0$. Now let

$$A = \prod_{i=1}^m (a_i, b_i]$$

where a_i and b_i are rational. Thus there are countably many such sets. Then from the conclusion of the theorem, if $\mathbf{x}_0 \notin N \cup \hat{N}$,

$$\begin{aligned} & \frac{1}{\alpha(B(\mathbf{x}_0, r))} \int_{B(\mathbf{x}_0, r)} \int_{\mathbb{R}^m} \mathcal{X}_A(\mathbf{y}) d\nu_{\mathbf{x}}(\mathbf{y}) d\alpha \\ &= \frac{1}{\alpha(B(\mathbf{x}_0, r))} \int_{B(\mathbf{x}_0, r)} \int_{\mathbb{R}^m} \mathcal{X}_A(\mathbf{y}) d\hat{\nu}_{\mathbf{x}}(\mathbf{y}) d\alpha, \end{aligned}$$

and by the Lebesgue Besicovitch Differentiation theorem, there exists a set of α measure zero, E_A , such that if $\mathbf{x}_0 \notin E_A \cup N \cup \hat{N}$, then the limit in the above exists as $r \rightarrow 0$ and yields

$$\nu_{\mathbf{x}_0}(A) = \hat{\nu}_{\mathbf{x}_0}(A).$$

Letting E denote the union of all the sets E_A for A as described above, it follows that E is a set of measure zero and if $\mathbf{x}_0 \notin E \cup N \cup \hat{N}$ then $\nu_{\mathbf{x}_0}(A) = \hat{\nu}_{\mathbf{x}_0}(A)$ for all such sets A . But every open set can be written as a disjoint union of sets of this form and so for all such $\mathbf{x}_0$, $\nu_{\mathbf{x}_0}(V) = \hat{\nu}_{\mathbf{x}_0}(V)$ for all V open. By Lemma 20.3.2 this shows the two measures are equal and proves the uniqueness assertion for $\nu_{\mathbf{x}}$. It remains to show the existence of the measures $\nu_{\mathbf{x}}$. This will be done with the aid of the following lemma. The idea is to define a positive linear functional which will yield the desired measure, $\nu_{\mathbf{x}}$ and this lemma will help in making this definition.

Lemma 20.3.4 *There exists a set N of α measure 0, independent of $f \in C_c(\mathbb{R}^{n+m})$ such that if $\mathbf{x} \notin N$, $\alpha(B(\mathbf{x}, r)) > 0$ for all $r > 0$ and*

$$\lim_{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \times \mathbb{R}^m} f d\mu = g_f(\mathbf{x})$$

where g_f is a α measurable function with the property that

$$\int_{\mathbb{R}^n} g_f(\mathbf{x}) d\alpha = \int_{\mathbb{R}^n \times \mathbb{R}^m} f d\mu.$$

Proof: Let $f \in C_c(\mathbb{R}^{n+m})$ and let

$$\eta_f(E) \equiv \int_{E \times \mathbb{R}^m} f d\mu.$$

Then η_f is a finite measure defined on the Borel sets with $\eta_f \ll \alpha$. By the Radon Nikodym theorem, there exists a Borel measurable function $\tilde{g}_f$ such that for all Borel E ,

$$\eta_f(E) \equiv \int_{E \times \mathbb{R}^m} f d\mu = \int_E \tilde{g}_f d\alpha. \quad (20.3.14)$$

By the theory of differentiation for Radon measures, there exists a set of α measure zero, N_f such that if $\mathbf{x} \notin N_f$, then $\alpha(B(\mathbf{x}, r)) > 0$ for all $r > 0$ and

$$\lim_{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \times \mathbb{R}^m} f d\mu = \lim_{r \rightarrow 0} \int_{B(\mathbf{x}, r)} \tilde{g}_f d\alpha = \tilde{g}_f(\mathbf{x}).$$

Let $\mathcal{D}$ be a countable dense subset of $C_c(\mathbb{R}^{n+m})$ and let

$$N \equiv \cup \{N_f : f \in \mathcal{D}\}.$$

Then if $f \in C_c(\mathbb{R}^{n+m})$ is arbitrary, and $\mathbf{x} \notin N$, referring to 20.3.14, it follows there exists $g \in \mathcal{D}$ close enough to f such that for all r_1, r_2 ,

$$\left| \frac{1}{\alpha(B(\mathbf{x}, r_1))} \int_{B(\mathbf{x}, r_1) \times \mathbb{R}^m} f d\mu - \frac{1}{\alpha(B(\mathbf{x}, r_2))} \int_{B(\mathbf{x}, r_2) \times \mathbb{R}^m} f d\mu \right| < \frac{\varepsilon}{2} + \left| \frac{1}{\alpha(B(\mathbf{x}, r_1))} \int_{B(\mathbf{x}, r_1) \times \mathbb{R}^m} g d\mu - \frac{1}{\alpha(B(\mathbf{x}, r_2))} \int_{B(\mathbf{x}, r_2) \times \mathbb{R}^m} g d\mu \right|$$

Therefore, taking r_i small enough, the right side is less than ε . Since ε is arbitrary, this shows the limit as $r \rightarrow 0$ exists. Define this limit which exists for all $f \in C_c(\mathbb{R}^{n+m})$ and $\mathbf{x} \notin N$ as $g_f(\mathbf{x})$. By the first part of the argument, $g_f(\mathbf{x}) = \tilde{g}_f(\mathbf{x})$ a.e. Thus, $g_f(\mathbf{x})$ is α measurable because it equals a Borel measurable function α a.e. The final formula follows from

$$\int_{\mathbb{R}^n} g_f(\mathbf{x}) d\alpha = \int_{\mathbb{R}^n} \tilde{g}_f(\mathbf{x}) d\alpha \equiv \eta_f(\mathbb{R}^n) \equiv \int_{\mathbb{R}^n \times \mathbb{R}^m} f d\mu.$$

This proves the lemma.

Continuing with the proof of the theorem, let $\mathbf{x} \notin N$ and let $f(\mathbf{z}, \mathbf{y}) \equiv \psi(\mathbf{z})\phi(\mathbf{y})$ where ψ and ϕ are continuous functions with compact support in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. Suppose first that $\psi(\mathbf{x}) = 1$. Then define a positive linear functional

$$L_{\mathbf{x}}\phi \equiv \lim_{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \times \mathbb{R}^m} \psi(\mathbf{z})\phi(\mathbf{y}) d\mu(z, y).$$

This functional may appear to depend on the choice of ψ satisfying $\psi(\mathbf{x}) = 1$ but this is not the case because all such ψ 's used in the definition of $L_{\mathbf{x}}$ are continuous.

Let $\nu_{\mathbf{x}}$ be the Radon measure representing $L_{\mathbf{x}}$. Thus replacing an arbitrary $\psi \in C_c(\mathbb{R}^n)$ with $\frac{\psi}{\psi(\mathbf{x})}$, in the case when $\psi(\mathbf{x}) \neq 0$,

$$\begin{aligned} \psi(\mathbf{x}) L_{\mathbf{x}}(\phi) &= \psi(\mathbf{x}) \int_{\mathbb{R}^m} \phi(\mathbf{y}) d\nu_{\mathbf{x}}(\mathbf{y}) \\ &= \lim_{r \rightarrow 0} \frac{\psi(\mathbf{x})}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \times \mathbb{R}^m} \frac{\psi}{\psi(\mathbf{x})} \phi d\mu \\ &= \lim_{r \rightarrow 0} \frac{1}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \times \mathbb{R}^m} \psi \phi d\mu \end{aligned}$$

By Lemma 20.3.4,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \psi \phi d\nu_{\mathbf{x}} d\alpha = \int_{\mathbb{R}^n} g_{\psi\phi} d\alpha = \int_{\mathbb{R}^n \times \mathbb{R}^m} \psi \phi d\mu.$$

Letting ψ_k, ϕ_k increase to 1 pointwise, the monotone convergence theorem implies

$$\int_{\mathbb{R}^n} \nu_{\mathbf{x}}(\mathbb{R}^m) d\alpha = \int_{\mathbb{R}^n \times \mathbb{R}^m} d\mu = \mu(\mathbb{R}^n \times \mathbb{R}^m) < \infty \quad (20.3.15)$$

showing that $\mathbf{x} \mapsto \nu_{\mathbf{x}}(\mathbb{R}^m)$ is a function in $L^1(\alpha)$. In particular, this function is finite α a.e. Summarizing, the above has shown that whenever $\psi \in C_c(\mathbb{R}^n)$ and $\phi \in C_c(\mathbb{R}^m)$

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \psi \phi d\nu_{\mathbf{x}} d\alpha = \int_{\mathbb{R}^n \times \mathbb{R}^m} \psi \phi d\mu. \quad (20.3.16)$$

Also $\nu_{\mathbf{x}}$ is a finite measure α a.e. and $\nu_{\mathbf{x}}$ and α are Radon measures.

Next it is shown that $\psi\phi$ can be replaced by $\mathcal{X}_E$ where E is an arbitrary μ measurable set. To do so, let

$$R_1 \equiv \prod_{i=1}^n (a_i, b_i], \quad R_2 \equiv \prod_{i=1}^m (c_i, d_i] \quad (20.3.17)$$

and let ψ_k be a sequence of functions in $C_c(\mathbb{R}^n)$ which is bounded, piecewise linear in each variable, and converging pointwise to $\mathcal{X}_{R_1}$. Also let ϕ_k be a similar sequence converging pointwise to $\mathcal{X}_{R_2}$. Then by the dominated convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_{R_1}(\mathbf{x}) \mathcal{X}_{R_2}(\mathbf{y}) d\mu &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n \times \mathbb{R}^m} \psi_k(\mathbf{x}) \phi_k(\mathbf{y}) d\mu \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \psi_k(\mathbf{x}) \left(\int_{\mathbb{R}^m} \phi_k(\mathbf{y}) d\nu_{\mathbf{x}} \right) d\alpha. \end{aligned} \quad (20.3.18)$$

Since $\nu_{\mathbf{x}}$ is finite α a.e., it follows that for α a.e. $\mathbf{x}$,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^m} \phi_k(\mathbf{y}) d\nu_{\mathbf{x}} = \int_{\mathbb{R}^m} \mathcal{X}_{R_2}(\mathbf{y}) d\nu_{\mathbf{x}}.$$

Since the ϕ_k are uniformly bounded, 20.3.15 implies the existence of a dominating function for the integrand in 20.3.18. Therefore, one can take the limit inside the integrals and obtain

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_{R_1}(\mathbf{x}) \mathcal{X}_{R_2}(\mathbf{y}) d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_{R_1}(\mathbf{x}) \mathcal{X}_{R_2}(\mathbf{y}) d\nu_{\mathbf{x}} d\alpha$$

Every open set, V in $\mathbb{R}^{n+m}$ is a countable disjoint union of such half open rectangles and so the monotone convergence theorem implies for all V open in $\mathbb{R}^{n+m}$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_V d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_V d\nu_{\mathbf{x}} d\alpha. \quad (20.3.19)$$

Since every compact set is the countable intersection of open sets, the above formula holds for V replaced with K where K is compact. Then it follows from the dominated convergence and monotone convergence theorems that whenever H is either a G_δ (countable intersection of open sets) or a F_σ (countable union of closed sets)

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_H d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_H d\nu_{\mathbf{x}} d\alpha.$$

Now let E be μ measurable. Using the regularity of μ there exists F, G such that F is F_σ , G is G_δ , $\mu(G \setminus F) = 0$, and $F \subseteq E \subseteq G$. Also a routine application of the dominated convergence theorem and 20.3.19 shows

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_{(G \setminus F)} d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_{(G \setminus F)} d\nu_{\mathbf{x}} d\alpha$$

and $(G \setminus F)_{\mathbf{x}} \equiv \{\mathbf{y} : (\mathbf{x}, \mathbf{y}) \in G \setminus F\}$ is $\nu_{\mathbf{x}}$ measurable for α a.e. $\mathbf{x}$, wherever $\nu_{\mathbf{x}}$ is a finite measure, and for α a.e. $\mathbf{x}$, $\nu_{\mathbf{x}}(G \setminus F)_{\mathbf{x}} = 0$. Therefore, for α a.e. $\mathbf{x}$, $E_{\mathbf{x}}$ is $\nu_{\mathbf{x}}$ measurable because $E_{\mathbf{x}} = F_{\mathbf{x}} + S_{\mathbf{x}}$ where $S_{\mathbf{x}} \subseteq (G \setminus F)_{\mathbf{x}}$, a set of $\nu_{\mathbf{x}}$ measure zero and $F_{\mathbf{x}}$ is an F_σ set which is measurable because $\nu_{\mathbf{x}}$ is a Radon measure coming as it does from a positive linear functional. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_E d\mu &= \int_{\mathbb{R}^n \times \mathbb{R}^m} \mathcal{X}_F d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_F d\nu_{\mathbf{x}} d\alpha \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \mathcal{X}_E d\nu_{\mathbf{x}} d\alpha. \end{aligned} \quad (20.3.20)$$

It follows from 20.3.20 that one can replace $\mathcal{X}_E$ with an arbitrary nonnegative μ measurable simple function, s . Letting f be a nonnegative μ measurable function, it follows there is an increasing sequence of nonnegative simple functions converging to f pointwise and so by the monotone convergence theorem,

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f d\mu = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f d\nu_{\mathbf{x}} d\alpha$$

where $\mathbf{y} \rightarrow \mathbf{f}(\mathbf{x}, \mathbf{y})$ is $\nu_{\mathbf{x}}$ measurable for α a.e. $\mathbf{x}$ and $\mathbf{x} \rightarrow \int_{\mathbb{R}^m} f d\nu_{\mathbf{x}}$ is α measurable so the iterated integral makes sense.

To see $\nu_{\mathbf{x}}$ is a probability measure for a.e. $\mathbf{x}$,

$$\begin{aligned} &\frac{1}{\alpha(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \int_{\mathbb{R}^m} d\nu_{\mathbf{x}} d\alpha \\ &= \frac{1}{\alpha(B(\mathbf{x}, r))} \mu(B(\mathbf{x}, r) \times \mathbb{R}^m) = 1 \end{aligned}$$

and so, using the fundamental theorem of calculus it follows that upon passing to a limit as $r \rightarrow 0$, it follows that for α a.e. $\mathbf{x}$

$$\nu_{\mathbf{x}}(\mathbb{R}^m) = \int_{\mathbb{R}^m} d\nu_{\mathbf{x}} = 1$$

Due to the regularity of the measures, all sets of measure zero may be taken to be Borel. In the case of $f \in L^1(\mu)$, one applies the above to the positive and negative parts of the real and imaginary parts. This proves the theorem.

20.4 Vitali Coverings

There is another covering theorem which may also be referred to as the Besicovitch covering theorem. As before, the balls can be taken with respect to any norm on $\mathbb{R}^n$. At first, the balls will be closed but this assumption will be removed.

Definition 20.4.1 *A collection of balls, $\mathcal{F}$ covers a set, E in the sense of Vitali if whenever $\mathbf{x} \in E$ and $\varepsilon > 0$, there exists a ball $B \in \mathcal{F}$ whose center is $\mathbf{x}$ having diameter less than ε .*

I will give a proof of the following theorem.

Theorem 20.4.2 *Let μ be a Radon measure on $\mathbb{R}^n$ and let E be a set with $\bar{\mu}(E) < \infty$. Where $\bar{\mu}$ is the outer measure determined by μ . Suppose $\mathcal{F}$ is a collection of closed balls which cover E in the sense of Vitali. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that*

$$\bar{\mu}(E \setminus \cup_{j=1}^{\infty} B_j) = 0.$$

Proof: Let N_n be the constant of the Besicovitch covering theorem. Choose $r > 0$ such that

$$(1-r)^{-1} \left(1 - \frac{1}{2N_n + 2} \right) \equiv \lambda < 1.$$

If $\bar{\mu}(E) = 0$, there is nothing to prove so assume $\bar{\mu}(E) > 0$. Let U_1 be an open set containing E with $(1-r)\mu(U_1) < \bar{\mu}(E)$ and $2\bar{\mu}(E) > \mu(U_1)$, and let $\mathcal{F}_1$ be those sets of $\mathcal{F}$ which are contained in U_1 whose centers are in E . Thus $\mathcal{F}_1$ is also a Vitali cover of E . Now by the Besicovitch covering theorem proved earlier, there exist balls, B , of $\mathcal{F}_1$ such that

$$E \subseteq \cup_{i=1}^{N_n} \{B : B \in \mathcal{G}_i\}$$

where $\mathcal{G}_i$ consists of a collection of disjoint balls of $\mathcal{F}_1$. Therefore,

$$\bar{\mu}(E) \leq \sum_{i=1}^{N_n} \sum_{B \in \mathcal{G}_i} \mu(B)$$

and so, for some $i \leq N_n$,

$$(N_n + 1) \sum_{B \in \mathcal{G}_i} \mu(B) > \bar{\mu}(E).$$

It follows there exists a finite set of balls of $\mathcal{G}_i$, $\{B_1, \dots, B_{m_1}\}$ such that

$$(N_n + 1) \sum_{i=1}^{m_1} \mu(B_i) > \bar{\mu}(E) \quad (20.4.21)$$

and so

$$(2N_n + 2) \sum_{i=1}^{m_1} \mu(B_i) > 2\bar{\mu}(E) > \mu(U_1).$$

Since $2\bar{\mu}(E) \geq \mu(U_1)$, 20.4.21 implies

$$\frac{\mu(U_1)}{2N_2 + 2} \leq \frac{2\bar{\mu}(E)}{2N_2 + 2} = \frac{\bar{\mu}(E)}{N_2 + 1} < \sum_{i=1}^{m_1} \mu(B_i).$$

Also U_1 was chosen such that $(1 - r)\mu(U_1) < \bar{\mu}(E)$, and so

$$\begin{aligned} \lambda\bar{\mu}(E) &\geq \lambda(1 - r)\mu(U_1) = \left(1 - \frac{1}{2N_n + 2}\right)\mu(U_1) \\ &\geq \mu(U_1) - \sum_{i=1}^{m_1} \mu(B_i) = \mu(U_1) - \mu(\cup_{j=1}^{m_1} B_j) \\ &= \mu(U_1 \setminus \cup_{j=1}^{m_1} B_j) \geq \bar{\mu}(E \setminus \cup_{j=1}^{m_1} B_j). \end{aligned}$$

Since the balls are closed, you can consider the sets of $\mathcal{F}$ which have empty intersection with $\cup_{j=1}^{m_1} B_j$ and this new collection of sets will be a Vitali cover of $E \setminus \cup_{j=1}^{m_1} B_j$. Letting this collection of balls play the role of $\mathcal{F}$ in the above argument and letting $E \setminus \cup_{j=1}^{m_1} B_j$ play the role of E , repeat the above argument and obtain disjoint sets of $\mathcal{F}$,

$$\{B_{m_1+1}, \dots, B_{m_2}\},$$

such that

$$\lambda\bar{\mu}(E \setminus \cup_{j=1}^{m_1} B_j) > \bar{\mu}((E \setminus \cup_{j=1}^{m_1} B_j) \setminus \cup_{j=m_1+1}^{m_2} B_j) = \bar{\mu}(E \setminus \cup_{j=1}^{m_2} B_j),$$

and so

$$\lambda^2\bar{\mu}(E) > \bar{\mu}(E \setminus \cup_{j=1}^{m_2} B_j).$$

Continuing in this way, yields a sequence of disjoint balls $\{B_i\}$ contained in $\mathcal{F}$ and

$$\bar{\mu}(E \setminus \cup_{j=1}^{\infty} B_j) \leq \bar{\mu}(E \setminus \cup_{j=1}^{m_k} B_j) < \lambda^k \bar{\mu}(E)$$

for all k . Therefore, $\bar{\mu}(E \setminus \cup_{j=1}^{\infty} B_j) = 0$ and this proves the Theorem.

It is not necessary to assume $\bar{\mu}(E) < \infty$.

Corollary 20.4.3 *Let μ be a Radon measure on $\mathbb{R}^n$. Letting $\bar{\mu}$ be the outer measure determined by μ , suppose $\mathcal{F}$ is a collection of closed balls which cover E in the sense of Vitali. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that*

$$\bar{\mu}(E \setminus \cup_{j=1}^{\infty} B_j) = 0.$$

Proof: Since μ is a Radon measure it is finite on compact sets. Therefore, there are at most countably many numbers, $\{b_i\}_{i=1}^\infty$ such that $\mu(\partial B(\mathbf{0}, b_i)) > 0$. It follows there exists an increasing sequence of positive numbers, $\{r_i\}_{i=1}^\infty$ such that $\lim_{i \rightarrow \infty} r_i = \infty$ and $\mu(\partial B(\mathbf{0}, r_i)) = 0$. Now let

$$\begin{aligned} D_1 &\equiv \{\mathbf{x} : \|\mathbf{x}\| < r_1\}, D_2 \equiv \{\mathbf{x} : r_1 < \|\mathbf{x}\| < r_2\}, \\ \dots, D_m &\equiv \{\mathbf{x} : r_{m-1} < \|\mathbf{x}\| < r_m\}, \dots \end{aligned}$$

Let $\mathcal{F}_m$ denote those closed balls of $\mathcal{F}$ which are contained in D_m . Then letting E_m denote $E \cap D_m$, $\mathcal{F}_m$ is a Vitali cover of E_m , $\bar{\mu}(E_m) < \infty$, and so by Theorem 20.4.2, there exists a countable sequence of balls from $\mathcal{F}_m$ $\{B_j^m\}_{j=1}^\infty$, such that $\bar{\mu}(E_m \setminus \cup_{j=1}^\infty B_j^m) = 0$. Then consider the countable collection of balls, $\{B_j^m\}_{j,m=1}^\infty$.

$$\begin{aligned} \bar{\mu}(E \setminus \cup_{m=1}^\infty \cup_{j=1}^\infty B_j^m) &\leq \bar{\mu}(\cup_{j=1}^\infty \partial B(\mathbf{0}, r_i)) + \\ + \sum_{m=1}^\infty \bar{\mu}(E_m \setminus \cup_{j=1}^\infty B_j^m) &= 0 \end{aligned}$$

This proves the corollary.

You don't need to assume the balls are closed. In fact, the balls can be open closed or anything in between and the same conclusion can be drawn.

Corollary 20.4.4 *Let μ be a Radon measure on $\mathbb{R}^n$. Letting $\bar{\mu}$ be the outer measure determined by μ , suppose $\mathcal{F}$ is a collection of balls which cover E in the sense of Vitali, open closed or neither. Then there exists a sequence of disjoint balls, $\{B_i\} \subseteq \mathcal{F}$ such that*

$$\bar{\mu}(E \setminus \cup_{j=1}^\infty B_j) = 0.$$

Proof: Let $\mathbf{x} \in E$. Thus $\mathbf{x}$ is the center of arbitrarily small balls from $\mathcal{F}$. Since μ is a Radon measure, at most countably many radii, r of these balls can have the property that $\mu(\partial B(\mathbf{0}, r)) > 0$. Let $\mathcal{F}'$ denote the closures of the balls of $\mathcal{F}$, $\bar{B}(\mathbf{x}, r)$ with the property that $\mu(\partial B(\mathbf{x}, r)) > 0$. Since for each $\mathbf{x} \in E$ there are only countably many exceptions, $\mathcal{F}'$ is still a Vitali cover of E . Therefore, by Corollary 20.4.3 there is a disjoint sequence of these balls of $\mathcal{F}'$, $\{\bar{B}_i\}_{i=1}^\infty$ for which

$$\bar{\mu}(E \setminus \cup_{j=1}^\infty \bar{B}_j) = 0$$

However, since their boundaries have μ measure zero, it follows

$$\bar{\mu}(E \setminus \cup_{j=1}^\infty B_j) = 0.$$

This proves the corollary.

20.5 Differentiation Of Radon Measures

This section is a generalization of earlier material in which a measure was differentiated with respect to Lebesgue measure. Here an arbitrary Radon measure will

be differentiated with respect to another arbitrary Radon measure. In this section, $B(\mathbf{x}, r)$ will denote a ball with center $\mathbf{x}$ and radius r . Also, let λ and μ be Radon measures and as above, Z will denote a μ measure zero set off of which $\mu(B(\mathbf{x}, r)) > 0$ for all $r > 0$.

Definition 20.5.1 For $\mathbf{x} \notin Z$, define the upper and lower symmetric derivatives as

$$\overline{D}_\mu \lambda(\mathbf{x}) \equiv \limsup_{r \rightarrow 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}, \quad \underline{D}_\mu \lambda(\mathbf{x}) \equiv \liminf_{r \rightarrow 0} \frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))}.$$

respectively. Also define

$$D_\mu \lambda(\mathbf{x}) \equiv \overline{D}_\mu \lambda(\mathbf{x}) = \underline{D}_\mu \lambda(\mathbf{x})$$

in the case when both the upper and lower derivatives are equal.

Lemma 20.5.2 Let λ and μ be Radon measures. If A is a bounded subset of $\{\mathbf{x} \notin Z : \overline{D}_\mu \lambda(\mathbf{x}) \geq a\}$, then

$$\overline{\lambda}(A) \geq a \overline{\mu}(A)$$

and if A is a bounded subset of $\{\mathbf{x} \notin Z : \underline{D}_\mu \lambda(\mathbf{x}) \leq a\}$, then

$$\overline{\lambda}(A) \leq a \overline{\mu}(A)$$

Proof: Suppose first that A is a bounded subset of $\{\mathbf{x} \notin Z : \overline{D}_\mu \lambda(\mathbf{x}) \geq a\}$, let $\varepsilon > 0$, and let V be a bounded open set with $V \supseteq A$ and $\lambda(V) - \varepsilon < \overline{\lambda}(A)$, $\mu(V) - \varepsilon < \overline{\mu}(A)$. Then if $\mathbf{x} \in A$,

$$\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))} > a - \varepsilon, \quad B(\mathbf{x}, r) \subseteq V,$$

for infinitely many values of r which are arbitrarily small. Thus the collection of such balls constitutes a Vitali cover for A . By Corollary 20.4.4 there is a disjoint sequence of these balls $\{B_i\}$ such that

$$\overline{\mu}(A \setminus \cup_{i=1}^{\infty} B_i) = 0. \quad (20.5.22)$$

Therefore,

$$(a - \varepsilon) \sum_{i=1}^{\infty} \mu(B_i) < \sum_{i=1}^{\infty} \lambda(B_i) \leq \lambda(V) < \varepsilon + \overline{\lambda}(A)$$

and so

$$\begin{aligned} a \sum_{i=1}^{\infty} \mu(B_i) &\leq \varepsilon + \varepsilon \mu(V) + \overline{\lambda}(A) \\ &\leq \varepsilon + \varepsilon (\overline{\mu}(A) + \varepsilon) + \overline{\lambda}(A) \end{aligned} \quad (20.5.23)$$

Now

$$\overline{\mu}(A \setminus \cup_{i=1}^{\infty} B_i) + \overline{\mu}(\cup_{i=1}^{\infty} B_i) \geq \overline{\mu}(A)$$

and so by 20.5.22 and the fact the B_i are disjoint,

$$\begin{aligned} a\bar{\mu}(A) &\leq a\bar{\mu}(\cup_{i=1}^{\infty} B_i) = a \sum_{i=1}^{\infty} \mu(B_i) \\ &\leq \varepsilon + \varepsilon(\bar{\mu}(A) + \varepsilon) + \bar{\lambda}(A). \end{aligned} \quad (20.5.24)$$

Hence $a\bar{\mu}(A) \leq \bar{\lambda}(A)$ since $\varepsilon > 0$ was arbitrary.

Now suppose A is a bounded subset of $\{\mathbf{x} \notin Z : \underline{D}_{\mu}\lambda(\mathbf{x}) \leq a\}$ and let V be a bounded open set containing A with $\mu(V) - \varepsilon < \bar{\mu}(A)$. Then if $\mathbf{x} \in A$,

$$\frac{\lambda(B(\mathbf{x}, r))}{\mu(B(\mathbf{x}, r))} < a + \varepsilon, \quad B(\mathbf{x}, r) \subseteq V$$

for values of r which are arbitrarily small. Therefore, by Corollary 20.4.4 again, there exists a disjoint sequence of these balls, $\{B_i\}$ satisfying this time,

$$\bar{\lambda}(A \setminus \cup_{i=1}^{\infty} B_i) = 0.$$

Then by arguments similar to the above,

$$\bar{\lambda}(A) \leq \sum_{i=1}^{\infty} \lambda(B_i) < (a + \varepsilon)\mu(V) < (a + \varepsilon)(\bar{\mu}(A) + \varepsilon).$$

Since ε was arbitrary, this proves the lemma.

Theorem 20.5.3 *There exists a set of measure zero, N containing Z such that for $\mathbf{x} \notin N$, $D_{\mu}\lambda(\mathbf{x})$ exists and also $\mathcal{X}_{N^c}(\cdot) D_{\mu}\lambda(\cdot)$ is a μ measurable function. Furthermore, $D_{\mu}\lambda(\mathbf{x}) < \infty$ μ a.e.*

Proof: First I show $D_{\mu}\lambda(\mathbf{x})$ exists a.e. Let $0 \leq a < b < \infty$ and let A be any bounded subset of

$$N(a, b) \equiv \{\mathbf{x} \notin Z : \bar{D}_{\mu}\lambda(\mathbf{x}) > b > a > \underline{D}_{\mu}\lambda(\mathbf{x})\}.$$

By Lemma 20.5.2,

$$a\bar{\mu}(A) \geq \bar{\lambda}(A) \geq b\bar{\mu}(A)$$

and so $\mu(A) = 0$ and A is μ measurable. It follows $\mu(N(a, b)) = 0$ because

$$\mu(N(a, b)) \leq \sum_{m=1}^{\infty} \mu(N(a, b) \cap B(\mathbf{0}, m)) = 0.$$

Define

$$N_0 \equiv \{\mathbf{x} \notin Z : \bar{D}_{\mu}\lambda(\mathbf{x}) > \underline{D}_{\mu}\lambda(\mathbf{x})\}.$$

Thus $\mu(N_0) = 0$ because

$$N_0 \subseteq \cup \{N(a, b) : 0 \leq a < b, \text{ and } a, b \in \mathbb{Q}\}$$

Therefore, N_0 is also μ measurable and has μ measure zero. Letting $N \equiv N_0 \cup Z$, it follows $D_\mu \lambda(\mathbf{x})$ exists on N^C . It remains to verify $\mathcal{X}_{N^C}(\cdot) D_\mu \lambda(\cdot)$ is finite a.e. and is μ measurable.

Let

$$I = \{\mathbf{x} : D_\mu \lambda(\mathbf{x}) = \infty\}.$$

Then by Lemma 20.5.2

$$\bar{\lambda}(I \cap B(\mathbf{0}, m)) \geq a\bar{\mu}(I \cap B(\mathbf{0}, m))$$

for all a and since λ is finite on bounded sets, the above implies $\bar{\mu}(I \cap B(\mathbf{0}, m)) = 0$ for each m which implies that I is μ measurable and has μ measure zero since

$$I = \cup_{m=1}^{\infty} I_m.$$

Letting η be an arbitrary Radon measure, let $r > 0$, and suppose $\eta(\partial B(\mathbf{x}, r)) = 0$. (Since η is finite on every ball, there are only countably many r such that $\eta(\partial B(\mathbf{x}, r)) > 0$.) and let V be an open set containing $\overline{B(\mathbf{x}, r)}$. Then whenever $\mathbf{y}$ is close enough to $\mathbf{x}$, it follows that $B(\mathbf{y}, r)$ is also a subset of V . Since V is an arbitrary open set containing $\overline{B(\mathbf{x}, r)}$, it follows

$$\eta(B(\mathbf{x}, r)) = \eta(\overline{B(\mathbf{x}, r)}) \geq \limsup_{\mathbf{y} \rightarrow \mathbf{x}} \eta(B(\mathbf{y}, r))$$

and so $\mathbf{y} \rightarrow \eta(B(\mathbf{y}, r))$ an upper semicontinuous real valued function of $\mathbf{x}$, one which satisfies

$$f(\mathbf{x}) \geq \limsup_{n \rightarrow \infty} f(\mathbf{x}_n)$$

whenever $\mathbf{x}_n \rightarrow \mathbf{x}$. Now it is routine to verify that a function f is upper semicontinuous if and only if $f^{-1}([-\infty, a))$ is open for all $a \in \mathbb{R}$. Therefore, $f^{-1}([-\infty, a))$ is a Borel set for all $a \in \mathbb{R}$ and so f is Borel measurable by Lemma 8.1.6. Now the measurability of $\mathcal{X}_{N^C}(\cdot) D_\mu \lambda(\cdot)$ follows from

$$\mathcal{X}_{N^C}(\mathbf{x}) D_\mu \lambda(\mathbf{x}) = \lim_{r_i \rightarrow 0} \frac{\lambda(B(\mathbf{x}, r_i))}{\mu(B(\mathbf{x}, r_i))} \mathcal{X}_{N^C}(\mathbf{x})$$

where r_i is such that $\partial B(\mathbf{x}, r_i)$ has μ and λ measure zero.

20.6 The Radon Nikodym Theorem For Radon Measures

The above theory can be used to give an alternate treatment of the Radon Nikodym theorem which exhibits the Radon Nikodym derivative as a specific limit.

Theorem 20.6.1 *Let λ and μ be Radon measures and suppose $\lambda \ll \mu$. Then for all E a μ measurable set,*

$$\lambda(E) = \int_E (D_\mu \lambda) d\mu.$$

Proof: Let $t > 1$ and let E be a μ measurable set which is bounded and a subset of N^C where N is the exceptional set of μ measure zero in Theorem 20.5.3 off of which $\mu(B(\mathbf{x}, r)) > 0$ for all $r > 0$ and $D_\mu \lambda(\mathbf{x})$ exists. Consider

$$E_m \equiv E \cap \{\mathbf{x} \in N^C : t^m \leq D_\mu \lambda(\mathbf{x}) < t^{m+1}\}$$

for $m \in \mathbb{Z}$, the integers. First note that

$$E \cap \{\mathbf{x} \in N^C : D_\mu \lambda(\mathbf{x}) = 0\}$$

has λ measure zero because by Lemma 20.5.2,

$$\lambda(E \cap \{\mathbf{x} \in N^C : D_\mu \lambda(\mathbf{x}) = 0\}) \leq a\mu(E)$$

for all $a > 0$ and $\mu(E)$ is finite due to the assumption that E is bounded and μ is a Radon measure. Therefore, by Lemma 20.5.2,

$$\begin{aligned} \lambda(E) &= \sum_{m \in \mathbb{Z}} \lambda(E_m) \leq \sum_{m \in \mathbb{Z}} t^{m+1} \mu(E_m) = t \sum_{m \in \mathbb{Z}} t^m \mu(E_m) \\ &\leq t \sum_{m \in \mathbb{Z}} \int_{E_m} D_\mu \lambda(\mathbf{x}) d\mu = t \int_E D_\mu \lambda(\mathbf{x}) d\mu. \end{aligned}$$

Also by this same lemma,

$$\begin{aligned} \lambda(E) &= \sum_{m \in \mathbb{Z}} \lambda(E_m) \geq \sum_{m \in \mathbb{Z}} t^m \mu(E_m) = t^{-1} \sum_{m \in \mathbb{Z}} t^{m+1} \mu(E_m) \\ &\geq t^{-1} \sum_{m \in \mathbb{Z}} \int_{E_m} D_\mu \lambda(\mathbf{x}) d\mu = t^{-1} \int_E D_\mu \lambda(\mathbf{x}) d\mu. \end{aligned}$$

Thus,

$$t \int_E D_\mu \lambda(\mathbf{x}) d\mu \geq \lambda(E) \geq t^{-1} \int_E D_\mu \lambda(\mathbf{x}) d\mu$$

and letting $t \rightarrow 1$, it follows

$$\lambda(E) = \int_E D_\mu \lambda(\mathbf{x}) d\mu. \quad (20.6.25)$$

Now if E is an arbitrary measurable set, contained in N^C , this formula holds with E replaced with $E \cap B(\mathbf{0}, k)$. Letting $k \rightarrow \infty$ and using the monotone convergence theorem, the above formula holds for all $E \subseteq N^C$. Since N is a set of μ measure zero, it follows N is also a set of λ measure zero due to the assumption of absolute continuity. Therefore 20.6.25 continues to hold for arbitrary μ measurable sets, E . This proves the theorem.

Fourier Transforms

21.1 An Algebra Of Special Functions

First recall the following definition of a polynomial.

Definition 21.1.1 $\alpha = (\alpha_1, \dots, \alpha_n)$ for $\alpha_1 \dots \alpha_n$ nonnegative integers is called a multi-index. For α a multi-index, $|\alpha| \equiv \alpha_1 + \dots + \alpha_n$ and if $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} = (x_1, \dots, x_n),$$

and f a function, define

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}.$$

A polynomial in n variables of degree m is a function of the form

$$p(\mathbf{x}) = \sum_{|\alpha| \leq m} a_\alpha \mathbf{x}^\alpha.$$

Here α is a multi-index as just described and $a_\alpha \in \mathbb{C}$. Also define for $\alpha = (\alpha_1, \dots, \alpha_n)$ a multi-index

$$D^\alpha f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}.$$

Definition 21.1.2 Define $\mathcal{G}_1$ to be the functions of the form $p(\mathbf{x}) e^{-a|\mathbf{x}|^2}$ where $a > 0$ and $p(\mathbf{x})$ is a polynomial. Let $\mathcal{G}$ be all finite sums of functions in $\mathcal{G}_1$. Thus $\mathcal{G}$ is an algebra of functions which has the property that if $f \in \mathcal{G}$ then $\bar{f} \in \mathcal{G}$.

It is always assumed, unless stated otherwise that the measure will be Lebesgue measure.

Lemma 21.1.3 $\mathcal{G}$ is dense in $C_0(\mathbb{R}^n)$ with respect to the norm,

$$\|f\|_\infty \equiv \sup \{|f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\}$$

Proof: By the Weierstrass approximation theorem, it suffices to show $\mathcal{G}$ separates the points and annihilates no point. It was already observed in the above definition that $\bar{f} \in \mathcal{G}$ whenever $f \in \mathcal{G}$. If $\mathbf{y}_1 \neq \mathbf{y}_2$ suppose first that $|\mathbf{y}_1| \neq |\mathbf{y}_2|$. Then in this case, you can let $f(\mathbf{x}) \equiv e^{-|\mathbf{x}|^2}$ and $f \in \mathcal{G}$ and $f(\mathbf{y}_1) \neq f(\mathbf{y}_2)$. If $|\mathbf{y}_1| = |\mathbf{y}_2|$, then suppose $y_{1k} \neq y_{2k}$. This must happen for some k because $\mathbf{y}_1 \neq \mathbf{y}_2$. Then let $f(\mathbf{x}) \equiv x_k e^{-|\mathbf{x}|^2}$. Thus $\mathcal{G}$ separates points. Now $e^{-|\mathbf{x}|^2}$ is never equal to zero and so $\mathcal{G}$ annihilates no point of $\mathbb{R}^n$. This proves the lemma.

These functions are clearly quite specialized. Therefore, the following theorem is somewhat surprising.

Theorem 21.1.4 *For each $p \geq 1, p < \infty, \mathcal{G}$ is dense in $L^p(\mathbb{R}^n)$.*

Proof: Let $f \in L^p(\mathbb{R}^n)$. Then there exists $g \in C_c(\mathbb{R}^n)$ such that $\|f - g\|_p < \varepsilon$. Now let $b > 0$ be large enough that

$$\int_{\mathbb{R}^n} \left(e^{-b|\mathbf{x}|^2} \right)^p dx < \varepsilon^p.$$

Then $\mathbf{x} \rightarrow g(\mathbf{x}) e^{b|\mathbf{x}|^2}$ is in $C_c(\mathbb{R}^n) \subseteq C_0(\mathbb{R}^n)$. Therefore, from Lemma 21.1.3 there exists $\psi \in \mathcal{G}$ such that

$$\left\| g e^{b|\cdot|^2} - \psi \right\|_\infty < 1$$

Therefore, letting $\phi(\mathbf{x}) \equiv e^{-b|\mathbf{x}|^2} \psi(\mathbf{x})$ it follows that $\phi \in \mathcal{G}$ and for all $\mathbf{x} \in \mathbb{R}^n$,

$$|g(\mathbf{x}) - \phi(\mathbf{x})| < e^{-b|\mathbf{x}|^2}$$

Therefore,

$$\left(\int_{\mathbb{R}^n} |g(\mathbf{x}) - \phi(\mathbf{x})|^p dx \right)^{1/p} \leq \left(\int_{\mathbb{R}^n} \left(e^{-b|\mathbf{x}|^2} \right)^p dx \right)^{1/p} < \varepsilon.$$

It follows

$$\|f - \phi\|_p \leq \|f - g\|_p + \|g - \phi\|_p < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves the theorem.

The following lemma is also interesting even if it is obvious.

Lemma 21.1.5 *For $\psi \in \mathcal{G}$, p a polynomial, and α, β multiindices, $D^\alpha \psi \in \mathcal{G}$ and $p\psi \in \mathcal{G}$. Also*

$$\sup\{|\mathbf{x}^\beta D^\alpha \psi(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty$$

21.2 Fourier Transforms Of Functions In $\mathcal{G}$

Definition 21.2.1 *For $\psi \in \mathcal{G}$ Define the Fourier transform, F and the inverse Fourier transform, F^{-1} by*

$$F\psi(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x},$$

$$F^{-1}\psi(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} \psi(\mathbf{x}) d\mathbf{x}.$$

where $\mathbf{t} \cdot \mathbf{x} \equiv \sum_{i=1}^n t_i x_i$. Note there is no problem with this definition because ψ is in $L^1(\mathbb{R}^n)$ and therefore,

$$|e^{i\mathbf{t}\cdot\mathbf{x}} \psi(\mathbf{x})| \leq |\psi(\mathbf{x})|,$$

an integrable function.

One reason for using the functions, $\mathcal{G}$ is that it is very easy to compute the Fourier transform of these functions. The first thing to do is to verify F and F^{-1} map $\mathcal{G}$ to $\mathcal{G}$ and that $F^{-1} \circ F(\psi) = \psi$.

Lemma 21.2.2 *The following formulas are true*

$$\int_{\mathbb{R}} e^{-c(x+it)^2} dx = \int_{\mathbb{R}} e^{-c(x-it)^2} dx = \frac{\sqrt{\pi}}{\sqrt{c}}, \quad (21.2.1)$$

$$\int_{\mathbb{R}^n} e^{-c(\mathbf{x}+it)\cdot(\mathbf{x}+it)} d\mathbf{x} = \int_{\mathbb{R}^n} e^{-c(\mathbf{x}-it)\cdot(\mathbf{x}-it)} d\mathbf{x} = \left(\frac{\sqrt{\pi}}{\sqrt{c}} \right)^n, \quad (21.2.2)$$

$$\int_{\mathbb{R}} e^{-ct^2} e^{-ist} dt = \int_{\mathbb{R}} e^{-ct^2} e^{ist} dt = e^{-\frac{s^2}{4c}} \frac{\sqrt{\pi}}{\sqrt{c}}, \quad (21.2.3)$$

$$\int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2} e^{-is\cdot\mathbf{t}} d\mathbf{t} = \int_{\mathbb{R}^n} e^{-c|\mathbf{t}|^2} e^{is\cdot\mathbf{t}} d\mathbf{t} = e^{-\frac{|s|^2}{4c}} \left(\frac{\sqrt{\pi}}{\sqrt{c}} \right)^n. \quad (21.2.4)$$

Proof: Consider the first one. Simple manipulations yield

$$H(t) \equiv \int_{\mathbb{R}} e^{-c(x+it)^2} dx = e^{ct^2} \int_{\mathbb{R}} e^{-cx^2} \cos(2cxt) dx.$$

Now using the dominated convergence theorem to justify passing derivatives inside the integral where necessary and using integration by parts,

$$\begin{aligned} H'(t) &= 2cte^{ct^2} \int_{\mathbb{R}} e^{-cx^2} \cos(2cxt) dx - e^{ct^2} \int_{\mathbb{R}} e^{-cx^2} \sin(2cxt) 2xc dx \\ &= 2ctH(t) - e^{ct^2} 2ct \int_{\mathbb{R}} e^{-cx^2} \cos(2cxt) dx = 2ct(H(t) - H(t)) = 0 \end{aligned}$$

and so $H(t) = H(0) = \int_{\mathbb{R}} e^{-cx^2} dx \equiv I$. Thus

$$I^2 = \int_{\mathbb{R}^2} e^{-c(x^2+y^2)} dx dy = \int_0^\infty \int_0^{2\pi} e^{-cr^2} r d\theta dr = \frac{\pi}{c}.$$

Therefore, $I = \sqrt{\pi}/\sqrt{c}$. Since the sign of t is unimportant, this proves 21.2.1. This also proves 21.2.2 after writing as iterated integrals.

Consider 21.2.3.

$$\begin{aligned}\int_{\mathbb{R}} e^{-ct^2} e^{ist} dt &= \int_{\mathbb{R}} e^{-c\left(t^2 - \frac{ist}{c} + \left(\frac{is}{2c}\right)^2\right)} e^{-\frac{s^2}{4c}} dt \\ &= e^{-\frac{s^2}{4c}} \int_{\mathbb{R}} e^{-c\left(t - \frac{is}{2c}\right)^2} dt = e^{-\frac{s^2}{4c}} \frac{\sqrt{\pi}}{\sqrt{c}}.\end{aligned}$$

Changing the variable $t \rightarrow -t$ gives the other part of 21.2.3.

Finally 21.2.4 follows from using iterated integrals.

With these formulas, it is easy to verify F, F^{-1} map $\mathcal{G}$ to $\mathcal{G}$ and $F \circ F^{-1} = F^{-1} \circ F = id$.

Theorem 21.2.3 *Each of F and F^{-1} map $\mathcal{G}$ to $\mathcal{G}$. Also $F^{-1} \circ F(\psi) = \psi$ and $F \circ F^{-1}(\psi) = \psi$.*

Proof: The first claim will be shown if it is shown that $F\psi \in \mathcal{G}$ for $\psi(\mathbf{x}) = \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2}$ because an arbitrary function of $\mathcal{G}$ is a finite sum of scalar multiples of functions such as ψ . Using Lemma 21.2.2,

$$\begin{aligned}F\psi(\mathbf{t}) &\equiv \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-it \cdot \mathbf{x}} \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2} d\mathbf{x} \\ &= \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D_t^\alpha \left(\int_{\mathbb{R}^n} e^{-it \cdot \mathbf{x}} e^{-b|\mathbf{x}|^2} d\mathbf{x} \right) \\ &= \left(\frac{1}{2\pi}\right)^{n/2} (i)^{-|\alpha|} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{2b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n \right)\end{aligned}$$

and this is clearly in $\mathcal{G}$ because it equals a polynomial times $e^{-\frac{|\mathbf{t}|^2}{2b}}$. It remains to verify the other assertion. As in the first case, it suffices to consider $\psi(\mathbf{x}) = \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2}$.

$$\begin{aligned}&F^{-1} \circ F(\psi)(\mathbf{s}) \\ &\equiv \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{is \cdot \mathbf{t}} \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-it \cdot \mathbf{x}} \mathbf{x}^\alpha e^{-b|\mathbf{x}|^2} d\mathbf{x} dt \\ &= \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^n} e^{is \cdot \mathbf{t}} (-i)^{-|\alpha|} D_t^\alpha \left(\int_{\mathbb{R}^n} e^{-it \cdot \mathbf{x}} e^{-b|\mathbf{x}|^2} d\mathbf{x} dt \right) \\ &= \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{is \cdot \mathbf{t}} \left(\frac{1}{2\pi}\right)^{n/2} (-i)^{-|\alpha|} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{4b}} \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n \right) dt \\ &= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n (-i)^{-|\alpha|} \int_{\mathbb{R}^n} e^{is \cdot \mathbf{t}} D_t^\alpha \left(e^{-\frac{|\mathbf{t}|^2}{4b}} \right) dt \\ &= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n (-i)^{-|\alpha|} (-1)^{|\alpha|} \mathbf{s}^\alpha (i)^{|\alpha|} \int_{\mathbb{R}^n} e^{is \cdot \mathbf{t}} e^{-\frac{|\mathbf{t}|^2}{4b}} dt \\ &= \left(\frac{1}{2\pi}\right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n \mathbf{s}^\alpha \int_{\mathbb{R}^n} e^{is \cdot \mathbf{t}} e^{-\frac{|\mathbf{t}|^2}{4b}} dt\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2\pi} \right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n \mathbf{s}^\alpha e^{-\frac{|\mathbf{s}|^2}{4(1/(4b))}} \left(\frac{\sqrt{\pi}}{\sqrt{1/(4b)}} \right)^n \\
&= \left(\frac{1}{2\pi} \right)^n \left(\frac{\sqrt{\pi}}{\sqrt{b}} \right)^n \mathbf{s}^\alpha e^{-b|\mathbf{s}|^2} \left(\sqrt{\pi} 2\sqrt{b} \right)^n = \mathbf{s}^\alpha e^{-b|\mathbf{s}|^2} = \psi(\mathbf{s}).
\end{aligned}$$

This little computation proves the theorem. The other case is entirely similar.

21.3 Fourier Transforms Of Just About Anything

21.3.1 Fourier Transforms Of $\mathcal{G}^*$

Definition 21.3.1 Let $\mathcal{G}^*$ denote the vector space of linear functions defined on $\mathcal{G}$ which have values in $\mathbb{C}$. Thus $T \in \mathcal{G}^*$ means $T : \mathcal{G} \rightarrow \mathbb{C}$ and T is linear,

$$T(a\psi + b\phi) = aT(\psi) + bT(\phi) \text{ for all } a, b \in \mathbb{C}, \quad \psi, \phi \in \mathcal{G}$$

Let $\psi \in \mathcal{G}$. Then define $T_\psi \in \mathcal{G}^*$ by

$$T_\psi(\phi) \equiv \int_{\mathbb{R}^n} \psi(\mathbf{x}) \phi(\mathbf{x}) dx$$

Lemma 21.3.2 The following is obtained for all $\phi, \psi \in \mathcal{G}$.

$$T_{F\psi}(\phi) = T_\psi(F\phi), \quad T_{F^{-1}\psi}(\phi) = T_\psi(F^{-1}\phi)$$

Also if $\psi \in \mathcal{G}$ and $T_\psi = 0$, then $\psi = 0$.

Proof:

$$\begin{aligned}
T_{F\psi}(\phi) &\equiv \int_{\mathbb{R}^n} F\psi(\mathbf{t}) \phi(\mathbf{t}) dt \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} \psi(\mathbf{x}) dx \phi(\mathbf{t}) dt \\
&= \int_{\mathbb{R}^n} \psi(\mathbf{x}) \left(\frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} \phi(\mathbf{t}) dt dx \\
&= \int_{\mathbb{R}^n} \psi(\mathbf{x}) F\phi(\mathbf{x}) dx \equiv T_\psi(F\phi)
\end{aligned}$$

The other claim is similar.

Suppose now $T_\psi = 0$. Then

$$\int_{\mathbb{R}^n} \psi \phi dx = 0$$

for all $\phi \in \mathcal{G}$. Therefore, this is true for $\phi = \psi$ and so $\psi = 0$. This proves the lemma.

From now on regard $\mathcal{G} \subseteq \mathcal{G}^*$ and for $\psi \in \mathcal{G}$ write $\psi(\phi)$ instead of $T_\psi(\phi)$. It was just shown that with this interpretation¹,

$$F\psi(\phi) = \psi(F(\phi)), \quad F^{-1}\psi(\phi) = \psi(F^{-1}\phi).$$

This lemma suggests a way to define the Fourier transform of something in $\mathcal{G}^*$.

Definition 21.3.3 For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by

$$FT(\phi) \equiv T(F\phi), \quad F^{-1}T(\phi) \equiv T(F^{-1}\phi)$$

Lemma 21.3.4 F and F^{-1} are both one to one, onto, and are inverses of each other.

Proof: First note F and F^{-1} are both linear. This follows directly from the definition. Suppose now $FT = 0$. Then $FT(\phi) = T(F\phi) = 0$ for all $\phi \in \mathcal{G}$. But F and F^{-1} map $\mathcal{G}$ onto $\mathcal{G}$ because if $\psi \in \mathcal{G}$, then $\psi = F(F^{-1}(\psi))$. Therefore, $T = 0$ and so F is one to one. Similarly F^{-1} is one to one. Now

$$F^{-1}(FT)(\phi) \equiv (FT)(F^{-1}\phi) \equiv T(F(F^{-1}(\phi))) = T\phi.$$

Therefore, $F^{-1} \circ F(T) = T$. Similarly, $F \circ F^{-1}(T) = T$. Thus both F and F^{-1} are one to one and onto and are inverses of each other as suggested by the notation. This proves the lemma.

Probably the most interesting things in $\mathcal{G}^*$ are functions of various kinds. The following lemma has to do with this situation.

Lemma 21.3.5 If $f \in L^1_{loc}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f\phi dx = 0$ for all $\phi \in C_c(\mathbb{R}^n)$, then $f = 0$ a.e.

Proof: First suppose $f \geq 0$. Let

$$E \equiv \{\mathbf{x} : f(\mathbf{x}) \geq r\}, \quad E_R \equiv E \cap B(\mathbf{0}, R).$$

Let K_m be an increasing sequence of compact sets and let V_m be a decreasing sequence of open sets satisfying

$$K_m \subseteq E_R \subseteq V_m, \quad m_n(V_m) \leq m_n(K_m) + 2^{-m}, \quad V_1 \subseteq B(\mathbf{0}, R).$$

Therefore,

$$m_n(V_m \setminus K_m) \leq 2^{-m}.$$

Let

$$\phi_m \in C_c(V_m), \quad K_m \prec \phi_m \prec V_m.$$

¹This is not all that different from what was done with the derivative. Remember when you consider the derivative of a function of one variable, in elementary courses you think of it as a number but thinking of it as a linear transformation acting on $\mathbb{R}$ is better because this leads to the concept of a derivative which generalizes to functions of many variables. So it is here. You can think of $\psi \in \mathcal{G}$ as simply an element of $\mathcal{G}$ but it is better to think of it as an element of $\mathcal{G}^*$ as just described.

Then $\phi_m(\mathbf{x}) \rightarrow \mathcal{X}_{E_R}(\mathbf{x})$ a.e. because the set where $\phi_m(\mathbf{x})$ fails to converge to this set is contained in the set of all $\mathbf{x}$ which are in infinitely many of the sets $V_m \setminus K_m$. This set has measure zero because

$$\sum_{m=1}^{\infty} m_n(V_m \setminus K_m) < \infty$$

and so, by the dominated convergence theorem,

$$0 = \lim_{m \rightarrow \infty} \int_{\mathbb{R}^n} f \phi_m dx = \lim_{m \rightarrow \infty} \int_{V_1} f \phi_m dx = \int_{E_R} f dx \geq r m(E_R).$$

Thus, $m_n(E_R) = 0$ and therefore $m_n(E) = \lim_{R \rightarrow \infty} m_n(E_R) = 0$. Since $r > 0$ is arbitrary, it follows

$$m_n([f > 0]) = \cup_{k=1}^{\infty} m_n([f > k^{-1}]) = 0.$$

Now suppose f has values in $\mathbb{R}$. Let $E_+ = [f \geq 0]$ and $E_- = [f < 0]$. Thus these are two measurable sets. As in the first part, let K_m and V_m be sequences of compact and open sets such that $K_m \subseteq E_+ \cap B(\mathbf{0}, R) \subseteq V_m \subseteq B(\mathbf{0}, R)$ and let $K_m \prec \phi_m \prec V_m$ with $m_n(V_m \setminus K_m) < 2^{-m}$. Thus $\phi_m \in C_c(\mathbb{R}^n)$ and the sequence converges pointwise to $\mathcal{X}_{E_+ \cap B(\mathbf{0}, R)}$. Then by the dominated convergence theorem, if ψ is any function in $C_c(\mathbb{R}^n)$

$$0 = \int f \phi_m \psi dm_n \rightarrow \int f \psi \mathcal{X}_{E_+ \cap B(\mathbf{0}, R)} dm_n.$$

Hence, letting $R \rightarrow \infty$,

$$\int f \psi \mathcal{X}_{E_+} dm_n = \int f_+ \psi dm_n = 0$$

Since ψ is arbitrary, the first part of the argument applies to f_+ and implies $f_+ = 0$. Similarly $f_- = 0$. Finally, if f is complex valued, the assumptions mean

$$\int \operatorname{Re}(f) \phi dm_n = 0, \quad \int \operatorname{Im}(f) \phi dm_n = 0$$

for all $\phi \in C_c(\mathbb{R}^n)$ and so both $\operatorname{Re}(f), \operatorname{Im}(f)$ equal zero a.e. This proves the lemma.

Corollary 21.3.6 *Let $f \in L^1(\mathbb{R}^n)$ and suppose*

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \phi(\mathbf{x}) dx = 0$$

for all $\phi \in \mathcal{G}$. Then $f = 0$ a.e.

Proof: Let $\psi \in C_c(\mathbb{R}^n)$. Then by the Stone Weierstrass approximation theorem, there exists a sequence of functions, $\{\phi_k\} \subseteq \mathcal{G}$ such that $\phi_k \rightarrow \psi$ uniformly. Then by the dominated convergence theorem,

$$\int f\psi dx = \lim_{k \rightarrow \infty} \int f\phi_k dx = 0.$$

By Lemma 21.3.5 $f = 0$.

The next theorem is the main result of this sort.

Theorem 21.3.7 *Let $f \in L^p(\mathbb{R}^n)$, $p \geq 1$, or suppose f is measurable and has polynomial growth,*

$$|f(\mathbf{x})| \leq K(1 + |\mathbf{x}|^2)^m$$

for some $m \in \mathbb{N}$. Then if

$$\int f\psi dx = 0$$

for all $\psi \in \mathcal{G}$ then it follows $f = 0$.

Proof: The case where $f \in L^1(\mathbb{R}^n)$ was dealt with in Corollary 21.3.6. Suppose $f \in L^p(\mathbb{R}^n)$ for $p > 1$. Then by Holder's inequality and the density of $\mathcal{G}$ in $L^{p'}(\mathbb{R}^n)$, it follows that $\int fg dx = 0$ for all $g \in L^{p'}(\mathbb{R}^n)$. By the Riesz representation theorem, $f = 0$.

It remains to consider the case where f has polynomial growth. Thus $\mathbf{x} \rightarrow f(\mathbf{x})e^{-|\mathbf{x}|^2} \in L^1(\mathbb{R}^n)$. Therefore, for all $\psi \in \mathcal{G}$,

$$0 = \int f(\mathbf{x})e^{-|\mathbf{x}|^2}\psi(\mathbf{x}) dx$$

because $e^{-|\mathbf{x}|^2}\psi(\mathbf{x}) \in \mathcal{G}$. Therefore, by the first part, $f(\mathbf{x})e^{-|\mathbf{x}|^2} = 0$ a.e.

The following theorem shows that you can consider most functions you are likely to encounter as elements of $\mathcal{G}^*$.

Theorem 21.3.8 *Let f be a measurable function with polynomial growth,*

$$|f(\mathbf{x})| \leq C(1 + |\mathbf{x}|^2)^N \quad \text{for some } N,$$

or let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1, \infty]$. Then $f \in \mathcal{G}^*$ if

$$f(\phi) \equiv \int f\phi dx.$$

Proof: Let f have polynomial growth first. Then the above integral is clearly well defined and so in this case, $f \in \mathcal{G}^*$.

Next suppose $f \in L^p(\mathbb{R}^n)$ with $\infty > p \geq 1$. Then it is clear again that the above integral is well defined because of the fact that ϕ is a sum of polynomials

times exponentials of the form $e^{-c|\mathbf{x}|^2}$ and these are in $L^{p'}(\mathbb{R}^n)$. Also $\phi \rightarrow f(\phi)$ is clearly linear in both cases. This proves the theorem.

This has shown that for nearly any reasonable function, you can define its Fourier transform as described above. Also you should note that $\mathcal{G}^*$ includes $C_0(\mathbb{R}^n)'$, the space of complex measures whose total variation are Radon measures. It is especially interesting when the Fourier transform yields another function of some sort.

21.3.2 Fourier Transforms Of Functions In $L^1(\mathbb{R}^n)$

First suppose $f \in L^1(\mathbb{R}^n)$.

Theorem 21.3.9 *Let $f \in L^1(\mathbb{R}^n)$. Then $Ff(\phi) = \int_{\mathbb{R}^n} g\phi dt$ where*

$$g(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}$$

and $F^{-1}f(\phi) = \int_{\mathbb{R}^n} g\phi dt$ where $g(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}$. In short,

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x},$$

$$F^{-1}f(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}.$$

Proof: From the definition and Fubini's theorem,

$$\begin{aligned} Ff(\phi) &\equiv \int_{\mathbb{R}^n} f(\mathbf{t}) F\phi(\mathbf{t}) dt = \int_{\mathbb{R}^n} f(\mathbf{t}) \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} \phi(\mathbf{x}) d\mathbf{x} dt \\ &= \int_{\mathbb{R}^n} \left(\left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{t}) e^{-i\mathbf{t} \cdot \mathbf{x}} dt \right) \phi(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Since $\phi \in \mathcal{G}$ is arbitrary, it follows from Theorem 21.3.7 that $Ff(\mathbf{x})$ is given by the claimed formula. The case of F^{-1} is identical.

Here are interesting properties of these Fourier transforms of functions in L^1 .

Theorem 21.3.10 *If $f \in L^1(\mathbb{R}^n)$ and $\|f_k - f\|_1 \rightarrow 0$, then Ff_k and $F^{-1}f_k$ converge uniformly to Ff and $F^{-1}f$ respectively. If $f \in L^1(\mathbb{R}^n)$, then $F^{-1}f$ and Ff are both continuous and bounded. Also,*

$$\lim_{|\mathbf{x}| \rightarrow \infty} F^{-1}f(\mathbf{x}) = \lim_{|\mathbf{x}| \rightarrow \infty} Ff(\mathbf{x}) = 0. \quad (21.3.5)$$

Furthermore, for $f \in L^1(\mathbb{R}^n)$ both Ff and $F^{-1}f$ are uniformly continuous.

Proof: The first claim follows from the following inequality.

$$\begin{aligned} |Ff_k(\mathbf{t}) - Ff(\mathbf{t})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{-i\mathbf{t}\cdot\mathbf{x}} f_k(\mathbf{x}) - e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x})| dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f_k(\mathbf{x}) - f(\mathbf{x})| dx \\ &= (2\pi)^{-n/2} \|f - f_k\|_1. \end{aligned}$$

which a similar argument holding for F^{-1} .

Now consider the second claim of the theorem.

$$|Ff(\mathbf{t}) - Ff(\mathbf{t}')| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{-i\mathbf{t}\cdot\mathbf{x}} - e^{-i\mathbf{t}'\cdot\mathbf{x}}| |f(\mathbf{x})| dx$$

The integrand is bounded by $2|f(\mathbf{x})|$, a function in $L^1(\mathbb{R}^n)$ and converges to 0 as $\mathbf{t}' \rightarrow \mathbf{t}$ and so the dominated convergence theorem implies Ff is continuous. To see $Ff(\mathbf{t})$ is uniformly bounded,

$$|Ff(\mathbf{t})| \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(\mathbf{x})| dx < \infty.$$

A similar argument gives the same conclusions for F^{-1} .

It remains to verify 21.3.5 and the claim that Ff and $F^{-1}f$ are uniformly continuous.

$$|Ff(\mathbf{t})| \leq \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} f(\mathbf{x}) dx \right|$$

Now let $\varepsilon > 0$ be given and let $g \in C_c^\infty(\mathbb{R}^n)$ such that $(2\pi)^{-n/2} \|g - f\|_1 < \varepsilon/2$. Then

$$\begin{aligned} |Ff(\mathbf{t})| &\leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} |f(\mathbf{x}) - g(\mathbf{x})| dx \\ &\quad + \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx \right| \\ &\leq \varepsilon/2 + \left| (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx \right|. \end{aligned}$$

Now integrating by parts, it follows that for $\|\mathbf{t}\|_\infty \equiv \max\{|t_j| : j = 1, \dots, n\} > 0$

$$|Ff(\mathbf{t})| \leq \varepsilon/2 + (2\pi)^{-n/2} \left| \frac{1}{\|\mathbf{t}\|_\infty} \int_{\mathbb{R}^n} \sum_{j=1}^n \left| \frac{\partial g(\mathbf{x})}{\partial x_j} \right| dx \right| \quad (21.3.6)$$

and this last expression converges to zero as $\|\mathbf{t}\|_\infty \rightarrow \infty$. The reason for this is that if $t_j \neq 0$, integration by parts with respect to x_j gives

$$(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} g(\mathbf{x}) dx = (2\pi)^{-n/2} \frac{1}{-it_j} \int_{\mathbb{R}^n} e^{-i\mathbf{t}\cdot\mathbf{x}} \frac{\partial g(\mathbf{x})}{\partial x_j} dx.$$

Therefore, choose the j for which $\|\mathbf{t}\|_\infty = |t_j|$ and the result of 21.3.6 holds. Therefore, from 21.3.6, if $\|\mathbf{t}\|_\infty$ is large enough, $|Ff(\mathbf{t})| < \varepsilon$. Similarly, $\lim_{\|\mathbf{t}\| \rightarrow \infty} F^{-1}(\mathbf{t}) = 0$. Consider the claim about uniform continuity. Let $\varepsilon > 0$ be given. Then there exists R such that if $\|\mathbf{t}\|_\infty > R$, then $|Ff(\mathbf{t})| < \frac{\varepsilon}{2}$. Since Ff is continuous, it is uniformly continuous on the compact set, $[-R-1, R+1]^n$. Therefore, there exists δ_1 such that if $\|\mathbf{t} - \mathbf{t}'\|_\infty < \delta_1$ for $\mathbf{t}', \mathbf{t} \in [-R-1, R+1]^n$, then

$$|Ff(\mathbf{t}) - Ff(\mathbf{t}')| < \varepsilon/2. \quad (21.3.7)$$

Now let $0 < \delta < \min(\delta_1, 1)$ and suppose $\|\mathbf{t} - \mathbf{t}'\|_\infty < \delta$. If both $\mathbf{t}, \mathbf{t}'$ are contained in $[-R, R]^n$, then 21.3.7 holds. If $\mathbf{t} \in [-R, R]^n$ and $\mathbf{t}' \notin [-R, R]^n$, then both are contained in $[-R-1, R+1]^n$ and so this verifies 21.3.7 in this case. The other case is that neither point is in $[-R, R]^n$ and in this case,

$$\begin{aligned} |Ff(\mathbf{t}) - Ff(\mathbf{t}')| &\leq |Ff(\mathbf{t})| + |Ff(\mathbf{t}')| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves the theorem.

There is a very interesting relation between the Fourier transform and convolutions.

Theorem 21.3.11 *Let $f, g \in L^1(\mathbb{R}^n)$. Then $f * g \in L^1$ and $F(f * g) = (2\pi)^{n/2} Ff Fg$.*

Proof: Consider

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})| dy dx.$$

The function, $(\mathbf{x}, \mathbf{y}) \rightarrow |f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})|$ is Lebesgue measurable and so by Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})| dy dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})| dx dy \\ &= \|f\|_1 \|g\|_1 < \infty. \end{aligned}$$

It follows that for a.e. $\mathbf{x}$, $\int_{\mathbb{R}^n} |f(\mathbf{x} - \mathbf{y}) g(\mathbf{y})| dy < \infty$ and for each of these values of $\mathbf{x}$, it follows that $\int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) dy$ exists and equals a function of $\mathbf{x}$ which is in $L^1(\mathbb{R}^n)$, $f * g(\mathbf{x})$. Now

$$\begin{aligned} &F(f * g)(\mathbf{t}) \\ &\equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} f * g(\mathbf{x}) dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{x}} \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) dy dx \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot \mathbf{y}} g(\mathbf{y}) \int_{\mathbb{R}^n} e^{-i\mathbf{t} \cdot (\mathbf{x} - \mathbf{y})} f(\mathbf{x} - \mathbf{y}) dx dy \\ &= (2\pi)^{n/2} Ff(\mathbf{t}) Fg(\mathbf{t}). \end{aligned}$$

There are many other considerations involving Fourier transforms of functions in $L^1(\mathbb{R}^n)$.

21.3.3 Fourier Transforms Of Functions In $L^2(\mathbb{R}^n)$

Consider Ff and $F^{-1}f$ for $f \in L^2(\mathbb{R}^n)$. First note that the formula given for Ff and $F^{-1}f$ when $f \in L^1(\mathbb{R}^n)$ will not work for $f \in L^2(\mathbb{R}^n)$ unless f is also in $L^1(\mathbb{R}^n)$. Recall that $\overline{a + ib} = a - ib$.

Theorem 21.3.12 For $\phi \in \mathcal{G}$, $\|F\phi\|_2 = \|F^{-1}\phi\|_2 = \|\phi\|_2$.

Proof: First note that for $\psi \in \mathcal{G}$,

$$F(\bar{\psi}) = \overline{F^{-1}(\psi)}, \quad F^{-1}(\bar{\psi}) = \overline{F(\psi)}. \quad (21.3.8)$$

This follows from the definition. For example,

$$\begin{aligned} F\bar{\psi}(\mathbf{t}) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot \mathbf{x}} \bar{\psi}(\mathbf{x}) d\mathbf{x} \\ &= \overline{(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x}} \end{aligned}$$

Let $\phi, \psi \in \mathcal{G}$. It was shown above that

$$\int_{\mathbb{R}^n} (F\phi)\psi(\mathbf{t}) d\mathbf{t} = \int_{\mathbb{R}^n} \phi(F\psi) d\mathbf{x}.$$

Similarly,

$$\int_{\mathbb{R}^n} \phi(F^{-1}\psi) d\mathbf{x} = \int_{\mathbb{R}^n} (F^{-1}\phi)\psi d\mathbf{t}. \quad (21.3.9)$$

Now, 21.3.8 - 21.3.9 imply

$$\begin{aligned} \int_{\mathbb{R}^n} |\phi|^2 d\mathbf{x} &= \int_{\mathbb{R}^n} \phi \overline{F^{-1}(F\phi)} d\mathbf{x} \\ &= \int_{\mathbb{R}^n} \phi F(\overline{F\phi}) d\mathbf{x} \\ &= \int_{\mathbb{R}^n} F\phi(\overline{F\phi}) d\mathbf{x} \\ &= \int_{\mathbb{R}^n} |F\phi|^2 d\mathbf{x}. \end{aligned}$$

Similarly

$$\|\phi\|_2 = \|F^{-1}\phi\|_2.$$

This proves the theorem.

Lemma 21.3.13 Let $f \in L^2(\mathbb{R}^n)$ and let $\phi_k \rightarrow f$ in $L^2(\mathbb{R}^n)$ where $\phi_k \in \mathcal{G}$. (Such a sequence exists because of density of $\mathcal{G}$ in $L^2(\mathbb{R}^n)$.) Then Ff and $F^{-1}f$ are both in $L^2(\mathbb{R}^n)$ and the following limits take place in L^2 .

$$\lim_{k \rightarrow \infty} F(\phi_k) = F(f), \quad \lim_{k \rightarrow \infty} F^{-1}(\phi_k) = F^{-1}(f).$$

Proof: Let $\psi \in \mathcal{G}$ be given. Then

$$\begin{aligned} Ff(\psi) &\equiv f(F\psi) \equiv \int_{\mathbb{R}^n} f(\mathbf{x}) F\psi(\mathbf{x}) dx \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \phi_k(\mathbf{x}) F\psi(\mathbf{x}) dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} F\phi_k(\mathbf{x}) \psi(\mathbf{x}) dx. \end{aligned}$$

Also by Theorem 21.3.12 $\{F\phi_k\}_{k=1}^{\infty}$ is Cauchy in $L^2(\mathbb{R}^n)$ and so it converges to some $h \in L^2(\mathbb{R}^n)$. Therefore, from the above,

$$Ff(\psi) = \int_{\mathbb{R}^n} h(\mathbf{x}) \psi(\mathbf{x})$$

which shows that $F(f) \in L^2(\mathbb{R}^n)$ and $h = F(f)$. The case of F^{-1} is entirely similar. This proves the lemma.

Since Ff and $F^{-1}f$ are in $L^2(\mathbb{R}^n)$, this also proves the following theorem.

Theorem 21.3.14 *If $f \in L^2(\mathbb{R}^n)$, Ff and $F^{-1}f$ are the unique elements of $L^2(\mathbb{R}^n)$ such that for all $\phi \in \mathcal{G}$,*

$$\int_{\mathbb{R}^n} Ff(\mathbf{x}) \phi(\mathbf{x}) dx = \int_{\mathbb{R}^n} f(\mathbf{x}) F\phi(\mathbf{x}) dx, \quad (21.3.10)$$

$$\int_{\mathbb{R}^n} F^{-1}f(\mathbf{x}) \phi(\mathbf{x}) dx = \int_{\mathbb{R}^n} f(\mathbf{x}) F^{-1}\phi(\mathbf{x}) dx. \quad (21.3.11)$$

Theorem 21.3.15 (*Plancherel*)

$$\|f\|_2 = \|Ff\|_2 = \|F^{-1}f\|_2. \quad (21.3.12)$$

Proof: Use the density of $\mathcal{G}$ in $L^2(\mathbb{R}^n)$ to obtain a sequence, $\{\phi_k\}$ converging to f in $L^2(\mathbb{R}^n)$. Then by Lemma 21.3.13

$$\|Ff\|_2 = \lim_{k \rightarrow \infty} \|F\phi_k\|_2 = \lim_{k \rightarrow \infty} \|\phi_k\|_2 = \|f\|_2.$$

Similarly,

$$\|f\|_2 = \|F^{-1}f\|_2.$$

This proves the theorem.

The following corollary is a simple generalization of this. To prove this corollary, use the following simple lemma which comes as a consequence of the Cauchy Schwarz inequality.

Lemma 21.3.16 *Suppose $f_k \rightarrow f$ in $L^2(\mathbb{R}^n)$ and $g_k \rightarrow g$ in $L^2(\mathbb{R}^n)$. Then*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} f_k g_k dx = \int_{\mathbb{R}^n} f g dx$$

Proof:

$$\begin{aligned} \left| \int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} f g dx \right| &\leq \left| \int_{\mathbb{R}^n} f_k g_k dx - \int_{\mathbb{R}^n} f_k g dx \right| + \\ &\quad \left| \int_{\mathbb{R}^n} f_k g dx - \int_{\mathbb{R}^n} f g dx \right| \\ &\leq \|f_k\|_2 \|g - g_k\|_2 + \|g\|_2 \|f_k - f\|_2. \end{aligned}$$

Now $\|f_k\|_2$ is a Cauchy sequence and so it is bounded independent of k . Therefore, the above expression is smaller than ε whenever k is large enough. This proves the lemma.

Corollary 21.3.17 For $f, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f \bar{g} dx = \int_{\mathbb{R}^n} Ff \overline{Fg} dx = \int_{\mathbb{R}^n} F^{-1}f \overline{F^{-1}g} dx.$$

Proof: First note the above formula is obvious if $f, g \in \mathcal{G}$. To see this, note

$$\begin{aligned} \int_{\mathbb{R}^n} Ff \overline{Fg} dx &= \int_{\mathbb{R}^n} Ff(\mathbf{x}) \overline{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\mathbf{x} \cdot \mathbf{t}} g(\mathbf{t}) dt} dx \\ &= \int_{\mathbb{R}^n} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \mathbf{t}} Ff(\mathbf{x}) dx \overline{g(\mathbf{t})} dt \\ &= \int_{\mathbb{R}^n} (F^{-1} \circ F) f(\mathbf{t}) \overline{g(\mathbf{t})} dt \\ &= \int_{\mathbb{R}^n} f(\mathbf{t}) \overline{g(\mathbf{t})} dt. \end{aligned}$$

The formula with F^{-1} is exactly similar.

Now to verify the corollary, let $\phi_k \rightarrow f$ in $L^2(\mathbb{R}^n)$ and let $\psi_k \rightarrow g$ in $L^2(\mathbb{R}^n)$. Then by Lemma 21.3.13

$$\begin{aligned} \int_{\mathbb{R}^n} Ff \overline{Fg} dx &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} F\phi_k \overline{F\psi_k} dx \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \phi_k \overline{\psi_k} dx \\ &= \int_{\mathbb{R}^n} f \bar{g} dx \end{aligned}$$

A similar argument holds for F^{-1} . This proves the corollary.

How does one compute Ff and $F^{-1}f$?

Theorem 21.3.18 For $f \in L^2(\mathbb{R}^n)$, let $f_r = f \chi_{E_r}$ where E_r is a bounded measurable set with $E_r \uparrow \mathbb{R}^n$. Then the following limits hold in $L^2(\mathbb{R}^n)$.

$$Ff = \lim_{r \rightarrow \infty} Ff_r, \quad F^{-1}f = \lim_{r \rightarrow \infty} F^{-1}f_r.$$

Proof: $\|f - f_r\|_2 \rightarrow 0$ and so $\|Ff - Ff_r\|_2 \rightarrow 0$ and $\|F^{-1}f - F^{-1}f_r\|_2 \rightarrow 0$ by Plancherel's Theorem. This proves the theorem.

What are Ff_r and $F^{-1}f_r$? Let $\phi \in \mathcal{G}$

$$\begin{aligned} \int_{\mathbb{R}^n} Ff_r \phi dx &= \int_{\mathbb{R}^n} f_r F\phi dx \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_r(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} \phi(\mathbf{y}) dy dx \\ &= \int_{\mathbb{R}^n} [(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} dx] \phi(\mathbf{y}) dy. \end{aligned}$$

Since this holds for all $\phi \in \mathcal{G}$, a dense subset of $L^2(\mathbb{R}^n)$, it follows that

$$Ff_r(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} dx.$$

Similarly

$$F^{-1}f_r(\mathbf{y}) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x}) e^{i\mathbf{x} \cdot \mathbf{y}} dx.$$

This shows that to take the Fourier transform of a function in $L^2(\mathbb{R}^n)$, it suffices to take the limit as $r \rightarrow \infty$ in $L^2(\mathbb{R}^n)$ of $(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f_r(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} dx$. A similar procedure works for the inverse Fourier transform.

Note this reduces to the earlier definition in case $f \in L^1(\mathbb{R}^n)$. Now consider the convolution of a function in L^2 with one in L^1 .

Theorem 21.3.19 *Let $h \in L^2(\mathbb{R}^n)$ and let $f \in L^1(\mathbb{R}^n)$. Then $h * f \in L^2(\mathbb{R}^n)$,*

$$F^{-1}(h * f) = (2\pi)^{n/2} F^{-1}h F^{-1}f,$$

$$F(h * f) = (2\pi)^{n/2} Fh Ff,$$

and

$$\|h * f\|_2 \leq \|h\|_2 \|f\|_1. \quad (21.3.13)$$

Proof: An application of Minkowski's inequality yields

$$\left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |h(\mathbf{x} - \mathbf{y})| |f(\mathbf{y})| dy \right)^2 dx \right)^{1/2} \leq \|f\|_1 \|h\|_2. \quad (21.3.14)$$

Hence $\int |h(\mathbf{x} - \mathbf{y})| |f(\mathbf{y})| dy < \infty$ a.e. $\mathbf{x}$ and

$$\mathbf{x} \rightarrow \int h(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dy$$

is in $L^2(\mathbb{R}^n)$. Let $E_r \uparrow \mathbb{R}^n$, $m(E_r) < \infty$. Thus,

$$h_r \equiv \chi_{E_r} h \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n),$$

and letting $\phi \in \mathcal{G}$,

$$\begin{aligned}
 & \int F(h_r * f)(\phi) dx \\
 & \equiv \int (h_r * f)(F\phi) dx \\
 & = (2\pi)^{-n/2} \int \int \int h_r(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) e^{-i\mathbf{x} \cdot \mathbf{t}} \phi(\mathbf{t}) dt dy dx \\
 & = (2\pi)^{-n/2} \int \int \left(\int h_r(\mathbf{x} - \mathbf{y}) e^{-i(\mathbf{x} - \mathbf{y}) \cdot \mathbf{t}} dx \right) f(\mathbf{y}) e^{-i\mathbf{y} \cdot \mathbf{t}} dy \phi(\mathbf{t}) dt \\
 & = \int (2\pi)^{n/2} Fh_r(\mathbf{t}) Ff(\mathbf{t}) \phi(\mathbf{t}) dt.
 \end{aligned}$$

Since ϕ is arbitrary and $\mathcal{G}$ is dense in $L^2(\mathbb{R}^n)$,

$$F(h_r * f) = (2\pi)^{n/2} Fh_r Ff.$$

Now by Minkowski's Inequality, $h_r * f \rightarrow h * f$ in $L^2(\mathbb{R}^n)$ and also it is clear that $h_r \rightarrow h$ in $L^2(\mathbb{R}^n)$; so, by Plancherel's theorem, you may take the limit in the above and conclude

$$F(h * f) = (2\pi)^{n/2} Fh Ff.$$

The assertion for F^{-1} is similar and 21.3.13 follows from 21.3.14.

21.3.4 The Schwartz Class

The problem with $\mathcal{G}$ is that it does not contain $C_c^\infty(\mathbb{R}^n)$. I have used it in presenting the Fourier transform because the functions in $\mathcal{G}$ have a very specific form which made some technical details work out easier than in any other approach I have seen. The Schwartz class is a larger class of functions which does contain $C_c^\infty(\mathbb{R}^n)$ and also has the same nice properties as $\mathcal{G}$. The functions in the Schwartz class are infinitely differentiable and they vanish very rapidly as $|\mathbf{x}| \rightarrow \infty$ along with all their partial derivatives. This is the description of these functions, not a specific form involving polynomials times $e^{-\alpha|\mathbf{x}|^2}$. To describe this precisely requires some notation.

Definition 21.3.20 $f \in \mathfrak{S}$, the Schwartz class, if $f \in C^\infty(\mathbb{R}^n)$ and for all positive integers N ,

$$\rho_N(f) < \infty$$

where

$$\rho_N(f) = \sup\{(1 + |\mathbf{x}|^2)^N |D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n, |\alpha| \leq N\}.$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and

$$\sup\{|\mathbf{x}^\beta D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty \quad (21.3.15)$$

for all multi indices α and β .

Also note that if $f \in \mathfrak{S}$, then $p(f) \in \mathfrak{S}$ for any polynomial, p with $p(0) = 0$ and that

$$\mathfrak{S} \subseteq L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

for any $p \geq 1$. To see this assertion about the $p(f)$, it suffices to consider the case of the product of two elements of the Schwartz class. If $f, g \in \mathfrak{S}$, then $D^\alpha(fg)$ is a finite sum of derivatives of f times derivatives of g . Therefore, $\rho_N(fg) < \infty$ for all N . You may wonder about examples of things in $\mathfrak{S}$. Clearly any function in $C_c^\infty(\mathbb{R}^n)$ is in $\mathfrak{S}$. However there are other functions in $\mathfrak{S}$. For example $e^{-|\mathbf{x}|^2}$ is in $\mathfrak{S}$ as you can verify for yourself and so is any function from $\mathcal{G}$. Note also that the density of $C_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ shows that $\mathfrak{S}$ is dense in $L^p(\mathbb{R}^n)$ for every p .

Recall the Fourier transform of a function in $L^1(\mathbb{R}^n)$ is given by

$$Ff(\mathbf{t}) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-it \cdot \mathbf{x}} f(\mathbf{x}) d\mathbf{x}.$$

Therefore, this gives the Fourier transform for $f \in \mathfrak{S}$. The nice property which $\mathfrak{S}$ has in common with $\mathcal{G}$ is that the Fourier transform and its inverse map $\mathfrak{S}$ one to one onto $\mathfrak{S}$. This means I could have presented the whole of the above theory as well as what follows in terms of $\mathfrak{S}$ and its algebraic dual, $\mathfrak{S}^*$ rather than in terms of $\mathcal{G}$ and $\mathcal{G}^*$. However, it is more technical. Nevertheless, letting $\mathfrak{S}$ play the role of $\mathcal{G}$ in the above is convenient in certain applications because it is easier to reduce to $\mathfrak{S}$ than $\mathcal{G}$. I will make use of this simple observation whenever it will simplify a presentation. The fundamental result which makes it possible is the following.

Theorem 21.3.21 *If $f \in \mathfrak{S}$, then Ff and $F^{-1}f$ are also in $\mathfrak{S}$.*

Proof: To begin with, let $\alpha = \mathbf{e}_j = (0, 0, \dots, 1, 0, \dots, 0)$, the 1 in the j^{th} slot.

$$\frac{F^{-1}f(\mathbf{t} + h\mathbf{e}_j) - F^{-1}f(\mathbf{t})}{h} = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot \mathbf{x}} f(\mathbf{x}) \left(\frac{e^{ihx_j} - 1}{h} \right) d\mathbf{x}. \quad (21.3.16)$$

Consider the integrand in 21.3.16.

$$\begin{aligned} \left| e^{it \cdot \mathbf{x}} f(\mathbf{x}) \left(\frac{e^{ihx_j} - 1}{h} \right) \right| &= |f(\mathbf{x})| \left| \left(\frac{e^{i(h/2)x_j} - e^{-i(h/2)x_j}}{h} \right) \right| \\ &= |f(\mathbf{x})| \left| \frac{i \sin((h/2)x_j)}{(h/2)} \right| \\ &\leq |f(\mathbf{x})| |x_j| \end{aligned}$$

and this is a function in $L^1(\mathbb{R}^n)$ because $f \in \mathfrak{S}$. Therefore by the Dominated Convergence Theorem,

$$\begin{aligned} \frac{\partial F^{-1}f(\mathbf{t})}{\partial t_j} &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot \mathbf{x}} i x_j f(\mathbf{x}) d\mathbf{x} \\ &= i(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{it \cdot \mathbf{x}} \mathbf{x} \mathbf{e}_j f(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Now $\mathbf{x}^{e_j} f(\mathbf{x}) \in \mathfrak{S}$ and so one can continue in this way and take derivatives indefinitely. Thus $F^{-1}f \in C^\infty(\mathbb{R}^n)$ and from the above argument,

$$D^\alpha F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} (i\mathbf{x})^\alpha f(\mathbf{x}) d\mathbf{x}.$$

To complete showing $F^{-1}f \in \mathfrak{S}$,

$$\mathbf{t}^\beta D^\alpha F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} \mathbf{t}^\beta (i\mathbf{x})^\alpha f(\mathbf{x}) d\mathbf{x}.$$

Integrate this integral by parts to get

$$\mathbf{t}^\beta D^\alpha F^{-1}f(\mathbf{t}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} i^{|\beta|} e^{i\mathbf{t} \cdot \mathbf{x}} D^\beta ((i\mathbf{x})^\alpha f(\mathbf{x})) d\mathbf{x}. \quad (21.3.17)$$

Here is how this is done.

$$\begin{aligned} \int_{\mathbb{R}} e^{it_j x_j} t_j^{\beta_j} (i\mathbf{x})^\alpha f(\mathbf{x}) dx_j &= \frac{e^{it_j x_j}}{it_j} t_j^{\beta_j} (i\mathbf{x})^\alpha f(\mathbf{x}) \Big|_{-\infty}^{\infty} + \\ &\quad i \int_{\mathbb{R}} e^{it_j x_j} t_j^{\beta_j-1} D^{e_j} ((i\mathbf{x})^\alpha f(\mathbf{x})) dx_j \end{aligned}$$

where the boundary term vanishes because $f \in \mathfrak{S}$. Returning to 21.3.17, use the fact that $|e^{ia}| = 1$ to conclude

$$|\mathbf{t}^\beta D^\alpha F^{-1}f(\mathbf{t})| \leq C \int_{\mathbb{R}^n} |D^\beta ((i\mathbf{x})^\alpha f(\mathbf{x}))| d\mathbf{x} < \infty.$$

It follows $F^{-1}f \in \mathfrak{S}$. Similarly $Ff \in \mathfrak{S}$ whenever $f \in \mathfrak{S}$.

Theorem 21.3.22 *Let $\psi \in \mathfrak{S}$. Then $(F \circ F^{-1})(\psi) = \psi$ and $(F^{-1} \circ F)(\psi) = \psi$ whenever $\psi \in \mathfrak{S}$. Also F and F^{-1} map $\mathfrak{S}$ one to one and onto $\mathfrak{S}$.*

Proof: The first claim follows from the fact that F and F^{-1} are inverses of each other which was established above. For the second, let $\psi \in \mathfrak{S}$. Then $\psi = F(F^{-1}\psi)$. Thus F maps $\mathfrak{S}$ onto $\mathfrak{S}$. If $F\psi = 0$, then do F^{-1} to both sides to conclude $\psi = 0$. Thus F is one to one and onto. Similarly, F^{-1} is one to one and onto.

Note the above equations involving F and F^{-1} hold pointwise everywhere because $F\psi$ and $F^{-1}\psi$ are continuous.

21.3.5 Convolution

To begin with it is necessary to discuss the meaning of ϕf where $f \in \mathcal{G}^*$ and $\phi \in \mathcal{G}$. What should it mean? First suppose $f \in L^p(\mathbb{R}^n)$ or measurable with polynomial growth. Then ϕf also has these properties. Hence, it should be the case that $\phi f(\psi) = \int_{\mathbb{R}^n} \phi f \psi dx = \int_{\mathbb{R}^n} f(\phi\psi) dx$. This motivates the following definition.

Definition 21.3.23 Let $T \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then $\phi T \equiv T\phi \in \mathcal{G}^*$ will be defined by

$$\phi T(\psi) \equiv T(\phi\psi).$$

The next topic is that of convolution. It was just shown that

$$F(f * \phi) = (2\pi)^{n/2} F\phi Ff, \quad F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1}f$$

whenever $f \in L^2(\mathbb{R}^n)$ and $\phi \in \mathcal{G}$ so the same definition is retained in the general case because it makes perfect sense and agrees with the earlier definition.

Definition 21.3.24 Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$. Then define the convolution of f with an element of $\mathcal{G}$ as follows.

$$f * \phi \equiv (2\pi)^{n/2} F^{-1}(F\phi Ff) \in \mathcal{G}^*$$

There is an obvious question. With this definition, is it true that $F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1}f$ as it was earlier?

Theorem 21.3.25 Let $f \in \mathcal{G}^*$ and let $\phi \in \mathcal{G}$.

$$F(f * \phi) = (2\pi)^{n/2} F\phi Ff, \quad (21.3.18)$$

$$F^{-1}(f * \phi) = (2\pi)^{n/2} F^{-1}\phi F^{-1}f. \quad (21.3.19)$$

Proof: Note that 21.3.18 follows from Definition 21.3.24 and both assertions hold for $f \in \mathcal{G}$. Consider 21.3.19. Here is a simple formula involving a pair of functions in $\mathcal{G}$.

$$\begin{aligned} & (\psi * F^{-1}F^{-1}\phi)(\mathbf{x}) \\ &= \left(\int \int \int \psi(\mathbf{x} - \mathbf{y}) e^{i\mathbf{y} \cdot \mathbf{y}_1} e^{i\mathbf{y}_1 \cdot \mathbf{z}} \phi(\mathbf{z}) dz dy_1 dy \right) (2\pi)^n \\ &= \left(\int \int \int \psi(\mathbf{x} - \mathbf{y}) e^{-i\mathbf{y} \cdot \tilde{\mathbf{y}}_1} e^{-i\tilde{\mathbf{y}}_1 \cdot \mathbf{z}} \phi(\mathbf{z}) dz d\tilde{\mathbf{y}}_1 dy \right) (2\pi)^n \\ &= (\psi * FF\phi)(\mathbf{x}). \end{aligned}$$

Now for $\psi \in \mathcal{G}$,

$$\begin{aligned} (2\pi)^{n/2} F(F^{-1}\phi F^{-1}f)(\psi) &\equiv (2\pi)^{n/2} (F^{-1}\phi F^{-1}f)(F\psi) \equiv \\ (2\pi)^{n/2} F^{-1}f(F^{-1}\phi F\psi) &\equiv (2\pi)^{n/2} f(F^{-1}(F^{-1}\phi F\psi)) = \\ f\left((2\pi)^{n/2} F^{-1}((FF^{-1}F^{-1}\phi)(F\psi))\right) &\equiv \\ f(\psi * F^{-1}F^{-1}\phi) &= f(\psi * FF\phi) \end{aligned} \quad (21.3.20)$$

Also

$$\begin{aligned}
 (2\pi)^{n/2} F^{-1} (F\phi Ff) (\psi) &\equiv (2\pi)^{n/2} (F\phi Ff) (F^{-1}\psi) \equiv \\
 (2\pi)^{n/2} Ff (F\phi F^{-1}\psi) &\equiv (2\pi)^{n/2} f (F (F\phi F^{-1}\psi)) = \\
 &= f \left(F \left((2\pi)^{n/2} (F\phi F^{-1}\psi) \right) \right) \\
 &= f \left(F \left((2\pi)^{n/2} (F^{-1} F F\phi F^{-1}\psi) \right) \right) = f (F (F^{-1} (F F\phi * \psi))) \\
 f (F F\phi * \psi) &= f (\psi * F F\phi). \tag{21.3.21}
 \end{aligned}$$

The last line follows from the following.

$$\begin{aligned}
 \int F F\phi (\mathbf{x} - \mathbf{y}) \psi (\mathbf{y}) d\mathbf{y} &= \int F\phi (\mathbf{x} - \mathbf{y}) F\psi (\mathbf{y}) d\mathbf{y} \\
 &= \int F\psi (\mathbf{x} - \mathbf{y}) F\phi (\mathbf{y}) d\mathbf{y} \\
 &= \int \psi (\mathbf{x} - \mathbf{y}) F F\phi (\mathbf{y}) d\mathbf{y}.
 \end{aligned}$$

From 21.3.21 and 21.3.20, since ψ was arbitrary,

$$(2\pi)^{n/2} F (F^{-1}\phi F^{-1}f) = (2\pi)^{n/2} F^{-1} (F\phi Ff) \equiv f * \phi$$

which shows 21.3.19.

21.4 Exercises

1. For $f \in L^1(\mathbb{R}^n)$, show that if $F^{-1}f \in L^1$ or $Ff \in L^1$, then f equals a continuous bounded function a.e.
2. Suppose $f, g \in L^1(\mathbb{R})$ and $Ff = Fg$. Show $f = g$ a.e.
3. Show that if $f \in L^1(\mathbb{R}^n)$, then $\lim_{|\mathbf{x}| \rightarrow \infty} Ff(\mathbf{x}) = 0$.
4. $\uparrow$ Suppose $f * f = f$ or $f * f = 0$ and $f \in L^1(\mathbb{R})$. Show $f = 0$.
5. For this problem define $\int_a^\infty f(t) dt \equiv \lim_{r \rightarrow \infty} \int_a^r f(t) dt$. Note this coincides with the Lebesgue integral when $f \in L^1(a, \infty)$. Show

$$(a) \int_0^\infty \frac{\sin(u)}{u} du = \frac{\pi}{2}$$

$$(b) \lim_{r \rightarrow \infty} \int_\delta^\infty \frac{\sin(ru)}{u} du = 0 \text{ whenever } \delta > 0.$$

$$(c) \text{ If } f \in L^1(\mathbb{R}), \text{ then } \lim_{r \rightarrow \infty} \int_{\mathbb{R}} \sin(ru) f(u) du = 0.$$

Hint: For the first two, use $\frac{1}{u} = \int_0^\infty e^{-ut} dt$ and apply Fubini's theorem to $\int_0^R \sin u \int_{\mathbb{R}} e^{-ut} dt du$. For the last part, first establish it for $f \in C_c^\infty(\mathbb{R})$ and then use the density of this set in $L^1(\mathbb{R})$ to obtain the result. This is sometimes called the Riemann Lebesgue lemma.

6. † Suppose that $g \in L^1(\mathbb{R})$ and that at some $x > 0$, g is locally Holder continuous from the right and from the left. This means

$$\lim_{r \rightarrow 0+} g(x+r) \equiv g(x+)$$

exists,

$$\lim_{r \rightarrow 0+} g(x-r) \equiv g(x-)$$

exists and there exist constants $K, \delta > 0$ and $r \in (0, 1]$ such that for $|x-y| < \delta$,

$$|g(x+) - g(y)| < K|x-y|^r$$

for $y > x$ and

$$|g(x-) - g(y)| < K|x-y|^r$$

for $y < x$. Show that under these conditions,

$$\begin{aligned} \lim_{r \rightarrow \infty} \frac{2}{\pi} \int_0^\infty \frac{\sin(ur)}{u} \left(\frac{g(x-u) + g(x+u)}{2} \right) du \\ = \frac{g(x+) + g(x-)}{2}. \end{aligned}$$

7. † Let $g \in L^1(\mathbb{R})$ and suppose g is locally Holder continuous from the right and from the left at x . Show that then

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{ixt} \int_{-\infty}^\infty e^{-ity} g(y) dy dt = \frac{g(x+) + g(x-)}{2}.$$

This is very interesting. If $g \in L^2(\mathbb{R})$, this shows $F^{-1}(Fg)(x) = \frac{g(x+) + g(x-)}{2}$, the midpoint of the jump in g at the point, x . In particular, if $g \in \mathcal{G}$, $F^{-1}(Fg) = g$. **Hint:** Show the left side of the above equation reduces to

$$\frac{2}{\pi} \int_0^\infty \frac{\sin(ur)}{u} \left(\frac{g(x-u) + g(x+u)}{2} \right) du$$

and then use Problem 6 to obtain the result.

8. † A measurable function g defined on $(0, \infty)$ has exponential growth if $|g(t)| \leq Ce^{\eta t}$ for some η . For $\text{Re}(s) > \eta$, define the Laplace Transform by

$$Lg(s) \equiv \int_0^\infty e^{-su} g(u) du.$$

Assume that g has exponential growth as above and is Holder continuous from the right and from the left at t . Pick $\gamma > \eta$. Show that

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi} \int_{-R}^R e^{\gamma t} e^{iyt} Lg(\gamma + iy) dy = \frac{g(t+) + g(t-)}{2}.$$

This formula is sometimes written in the form

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} Lg(s) ds$$

and is called the complex inversion integral for Laplace transforms. It can be used to find inverse Laplace transforms. **Hint:**

$$\begin{aligned} \frac{1}{2\pi} \int_{-R}^R e^{\gamma t} e^{iyt} Lg(\gamma + iy) dy = \\ \frac{1}{2\pi} \int_{-R}^R e^{\gamma t} e^{iyt} \int_0^\infty e^{-(\gamma+iy)u} g(u) du dy. \end{aligned}$$

Now use Fubini's theorem and do the integral from $-R$ to R to get this equal to

$$\frac{e^{\gamma t}}{\pi} \int_{-\infty}^\infty e^{-\gamma u} \bar{g}(u) \frac{\sin(R(t-u))}{t-u} du$$

where $\bar{g}$ is the zero extension of g off $[0, \infty)$. Then this equals

$$\frac{e^{\gamma t}}{\pi} \int_{-\infty}^\infty e^{-\gamma(t-u)} \bar{g}(t-u) \frac{\sin(Ru)}{u} du$$

which equals

$$\frac{2e^{\gamma t}}{\pi} \int_0^\infty \frac{\bar{g}(t-u) e^{-\gamma(t-u)} + \bar{g}(t+u) e^{-\gamma(t+u)}}{2} \frac{\sin(Ru)}{u} du$$

and then apply the result of Problem 6.

9. Suppose $f \in \mathfrak{S}$. Show $F(f_{x_j})(\mathbf{t}) = it_j Ff(\mathbf{t})$.
10. Let $f \in \mathfrak{S}$ and let k be a positive integer.

$$\|f\|_{k,2} \equiv (\|f\|_2^2 + \sum_{|\alpha| \leq k} \|D^\alpha f\|_2^2)^{1/2}.$$

One could also define

$$\|f\|_{k,2} \equiv \left(\int_{\mathbb{R}^n} |Ff(\mathbf{x})|^2 (1 + |\mathbf{x}|^2)^k dx \right)^{1/2}.$$

Show both $\|\cdot\|_{k,2}$ and $\|\cdot\|_{k,2}$ are norms on $\mathfrak{S}$ and that they are equivalent. These are Sobolev space norms. For which values of k does the second norm make sense? How about the first norm?

11. $\uparrow$ Define $H^k(\mathbb{R}^n)$, $k \geq 0$ by $f \in L^2(\mathbb{R}^n)$ such that

$$\left(\int |Ff(\mathbf{x})|^2 (1 + |\mathbf{x}|^2)^k dx \right)^{\frac{1}{2}} < \infty,$$

$$|||f|||_{k,2} \equiv \left(\int |Ff(\mathbf{x})|^2 (1 + |\mathbf{x}|^2)^k dx \right)^{\frac{1}{2}}.$$

Show $H^k(\mathbb{R}^n)$ is a Banach space, and that if k is a positive integer, $H^k(\mathbb{R}^n) = \{ f \in L^2(\mathbb{R}^n) : \text{there exists } \{u_j\} \subseteq \mathcal{G} \text{ with } \|u_j - f\|_2 \rightarrow 0 \text{ and } \{u_j\} \text{ is a Cauchy sequence in } ||| \cdot |||_{k,2} \text{ of Problem 10} \}$. This is one way to define Sobolev Spaces. **Hint:** One way to do the second part of this is to define a new measure, μ by

$$\mu(E) \equiv \int_E (1 + |\mathbf{x}|^2)^k dx.$$

Then show μ is a Radon measure and show there exists $\{g_m\}$ such that $g_m \in \mathcal{G}$ and $g_m \rightarrow Ff$ in $L^2(\mu)$. Thus $g_m = Ff_m$, $f_m \in \mathcal{G}$ because F maps $\mathcal{G}$ onto $\mathcal{G}$. Then by Problem 10, $\{f_m\}$ is Cauchy in the norm $||| \cdot |||_{k,2}$.

12. $\uparrow$ If $2k > n$, show that if $f \in H^k(\mathbb{R}^n)$, then f equals a bounded continuous function a.e. **Hint:** Show that for k this large, $Ff \in L^1(\mathbb{R}^n)$, and then use Problem 1. To do this, write

$$|Ff(\mathbf{x})| = |Ff(\mathbf{x})|(1 + |\mathbf{x}|^2)^{\frac{k}{2}}(1 + |\mathbf{x}|^2)^{-\frac{k}{2}},$$

So

$$\int |Ff(\mathbf{x})| dx = \int |Ff(\mathbf{x})|(1 + |\mathbf{x}|^2)^{\frac{k}{2}}(1 + |\mathbf{x}|^2)^{-\frac{k}{2}} dx.$$

Use the Cauchy Schwarz inequality. This is an example of a Sobolev imbedding Theorem.

13. Let $u \in \mathcal{G}$. Then $Fu \in \mathcal{G}$ and so, in particular, it makes sense to form the integral,

$$\int_{\mathbb{R}} Fu(\mathbf{x}', x_n) dx_n$$

where $(\mathbf{x}', x_n) = \mathbf{x} \in \mathbb{R}^n$. For $u \in \mathcal{G}$, define $\gamma u(\mathbf{x}') \equiv u(\mathbf{x}', 0)$. Find a constant such that $F(\gamma u)(\mathbf{x}')$ equals this constant times the above integral.

Hint: By the dominated convergence theorem

$$\int_{\mathbb{R}} Fu(\mathbf{x}', x_n) dx_n = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} Fu(\mathbf{x}', x_n) dx_n.$$

Now use the definition of the Fourier transform and Fubini's theorem as required in order to obtain the desired relationship.

14. Recall the Fourier series of a function in $L^2(-\pi, \pi)$ converges to the function in $L^2(-\pi, \pi)$. Prove a similar theorem with $L^2(-\pi, \pi)$ replaced by $L^2(-m\pi, m\pi)$ and the functions

$$\left\{ (2\pi)^{-(1/2)} e^{inx} \right\}_{n \in \mathbb{Z}}$$

used in the Fourier series replaced with

$$\left\{ (2m\pi)^{-(1/2)} e^{i \frac{n}{m} x} \right\}_{n \in \mathbb{Z}}$$

Now suppose f is a function in $L^2(\mathbb{R})$ satisfying $Ff(t) = 0$ if $|t| > m\pi$. Show that if this is so, then

$$f(x) = \frac{1}{\pi} \sum_{n \in \mathbb{Z}} f\left(\frac{-n}{m}\right) \frac{\sin(\pi(mx+n))}{mx+n}.$$

Here m is a positive integer. This is sometimes called the Shannon sampling theorem. **Hint:** First note that since $Ff \in L^2$ and is zero off a finite interval, it follows $Ff \in L^1$. Also

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-m\pi}^{m\pi} e^{itx} Ff(x) dx$$

and you can conclude from this that f has all derivatives and they are all bounded. Thus f is a very nice function. You can replace Ff with its Fourier series. Then consider carefully the Fourier coefficient of Ff . Argue it equals $f\left(\frac{-n}{m}\right)$ or at least an appropriate constant times this. When you get this the rest will fall quickly into place if you use Ff is zero off $[-m\pi, m\pi]$.

Fourier Analysis In $\mathbb{R}^n$ An Introduction

The purpose of this chapter is to present some of the most important theorems on Fourier analysis in $\mathbb{R}^n$. These theorems are the Marcinkiewicz interpolation theorem, the Calderon Zygmund decomposition, and Mihlin's theorem. They are all fundamental results whose proofs depend on the methods of real analysis.

22.1 The Marcinkiewicz Interpolation Theorem

Let $(\Omega, \mu, \mathcal{S})$ be a measure space.

Definition 22.1.1 $L^p(\Omega) + L^1(\Omega)$ will denote the space of measurable functions, f , such that f is the sum of a function in $L^p(\Omega)$ and $L^1(\Omega)$. Also, if $T : L^p(\Omega) + L^1(\Omega) \rightarrow$ space of measurable functions, T is subadditive if

$$|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|.$$

T is of type (p, p) if there exists a constant independent of $f \in L^p(\Omega)$ such that

$$\|Tf\|_p \leq A \|f\|_p, \quad f \in L^p(\Omega).$$

T is weak type (p, p) if there exists a constant A independent of f such that

$$\mu([x : |Tf(x)| > \alpha]) \leq \left(\frac{A}{\alpha} \|f\|_p\right)^p, \quad f \in L^p(\Omega).$$

The following lemma involves writing a function as a sum of functions whose values are small and one whose values are large.

Lemma 22.1.2 If $p \in [1, r]$, then $L^p(\Omega) \subseteq L^1(\Omega) + L^r(\Omega)$.

Proof: Let $\lambda > 0$ and let $f \in L^p(\Omega)$

$$f_1(x) \equiv \begin{cases} f(x) & \text{if } |f(x)| \leq \lambda \\ 0 & \text{if } |f(x)| > \lambda \end{cases}, \quad f_2(x) \equiv \begin{cases} f(x) & \text{if } |f(x)| > \lambda \\ 0 & \text{if } |f(x)| \leq \lambda \end{cases}.$$

Thus $f(x) = f_1(x) + f_2(x)$.

$$\int |f_1(x)|^r d\mu = \int_{\{|f| \leq \lambda\}} |f(x)|^r d\mu \leq \lambda^{r-p} \int_{\{|f| \leq \lambda\}} |f(x)|^p d\mu < \infty.$$

Therefore, $f_1 \in L^r(\Omega)$.

$$\int |f_2(x)| d\mu = \int_{\{|f| > \lambda\}} |f(x)| d\mu \leq \mu[|f| > \lambda]^{1/p'} \left(\int |f|^p d\mu \right)^{1/p} < \infty.$$

This proves the lemma since $f = f_1 + f_2$, $f_1 \in L^r$ and $f_2 \in L^1$.

For f a function having nonnegative real values, $\alpha \rightarrow \mu([f > \alpha])$ is called the distribution function.

Lemma 22.1.3 *Let $\phi(0) = 0$, ϕ is strictly increasing, and C^1 . Let $f : \Omega \rightarrow [0, \infty)$ be measurable. Then*

$$\int_{\Omega} (\phi \circ f) d\mu = \int_0^{\infty} \phi'(\alpha) \mu[f > \alpha] d\alpha. \quad (22.1.1)$$

Proof: First suppose

$$f = \sum_{i=1}^m a_i \chi_{E_i}$$

where $a_i > 0$ and the a_i are all distinct nonzero values of f , the sets, E_i being disjoint. Thus,

$$\int_{\Omega} (\phi \circ f) d\mu = \sum_{i=1}^m \phi(a_i) \mu(E_i).$$

Suppose without loss of generality $a_1 < a_2 < \dots < a_m$. Observe

$$\alpha \rightarrow \mu([f > \alpha])$$

is constant on the intervals $[0, a_1), [a_1, a_2), \dots$. For example, on $[a_i, a_{i+1})$, this function has the value

$$\sum_{j=i+1}^m \mu(E_j).$$

The function equals zero on $[a_m, \infty)$. Therefore,

$$\alpha \rightarrow \phi'(\alpha) \mu([|f| > \alpha])$$

is Lebesgue measurable and letting $a_0 = 0$, the second integral in 22.1.1 equals

$$\begin{aligned}
 \int_0^\infty \phi'(\alpha) \mu([f > \alpha]) d\alpha &= \sum_{i=1}^m \int_{a_{i-1}}^{a_i} \phi'(\alpha) \mu([f > \alpha]) d\alpha \\
 &= \sum_{i=1}^m \sum_{j=i}^m \mu(E_j) \int_{a_{i-1}}^{a_i} \phi'(\alpha) d\alpha \\
 &= \sum_{j=1}^m \sum_{i=1}^j \mu(E_j) (\phi(a_i) - \phi(a_{i-1})) \\
 &= \sum_{j=1}^m \mu(E_j) \phi(a_j) = \int_{\Omega} (\phi \circ f) d\mu
 \end{aligned}$$

and so this establishes 22.1.1 in the case when f is a nonnegative simple function. Since every measurable nonnegative function may be written as the pointwise limit of such simple functions, the desired result will follow by the Monotone convergence theorem and the next claim.

Claim: If $f_n \uparrow f$, then for each $\alpha > 0$,

$$\mu([f > \alpha]) = \lim_{n \rightarrow \infty} \mu([f_n > \alpha]).$$

Proof of the claim: $[f_n > \alpha] \uparrow [f > \alpha]$ because if $f(x) > \alpha$ then for large enough n , $f_n(x) > \alpha$ and so

$$\mu([f_n > \alpha]) \uparrow \mu([f > \alpha]).$$

This proves the lemma. (Note the importance of the strict inequality in $[f > \alpha]$ in proving the claim.)

The next theorem is the main result in this section. It is called the Marcinkiewicz interpolation theorem.

Theorem 22.1.4 *Let $(\Omega, \mu, \mathcal{S})$ be a σ finite measure space, $1 < r < \infty$, and let*

$$T : L^1(\Omega) + L^r(\Omega) \rightarrow \text{space of measurable functions}$$

be subadditive, weak (r, r) , and weak $(1, 1)$. Then T is of type (p, p) for every $p \in (1, r)$ and

$$\|Tf\|_p \leq A_p \|f\|_p$$

where the constant A_p depends only on p and the constants in the definition of weak $(1, 1)$ and weak (r, r) .

Proof: Let $\alpha > 0$ and let f_1 and f_2 be defined as in Lemma 22.1.2,

$$f_1(x) \equiv \begin{cases} f(x) & \text{if } |f(x)| \leq \alpha \\ 0 & \text{if } |f(x)| > \alpha \end{cases}, \quad f_2(x) \equiv \begin{cases} f(x) & \text{if } |f(x)| > \alpha \\ 0 & \text{if } |f(x)| \leq \alpha \end{cases}.$$

Thus $f = f_1 + f_2$ where $f_1 \in L^r$ and $f_2 \in L^1$. Since T is subadditive ,

$$[|Tf| > \alpha] \subseteq [|Tf_1| > \alpha/2] \cup [|Tf_2| > \alpha/2] .$$

Let $p \in (1, r)$. By Lemma 22.1.3,

$$\begin{aligned} \int |Tf|^p d\mu &\leq p \int_0^\infty \alpha^{p-1} \mu(|Tf_1| > \alpha/2) d\alpha + \\ &+ p \int_0^\infty \alpha^{p-1} \mu(|Tf_2| > \alpha/2) d\alpha . \end{aligned}$$

Therefore, since T is weak $(1, 1)$ and weak (r, r) ,

$$\int |Tf|^p d\mu \leq p \int_0^\infty \alpha^{p-1} \left(\frac{2A_r}{\alpha} \|f_1\|_r \right)^r d\alpha + p \int_0^\infty \alpha^{p-1} \frac{2A_1}{\alpha} \|f_2\|_1 d\alpha . \quad (22.1.2)$$

Therefore, the right side of 22.1.2 equals

$$\begin{aligned} p(2A_r)^r \int_0^\infty \alpha^{p-1-r} \int_\Omega |f_1|^r d\mu d\alpha + 2A_1 p \int_0^\infty \alpha^{p-2} \int_\Omega |f_2| d\mu d\alpha = \\ p(2A_r)^r \int_\Omega \int_0^\infty \alpha^{p-1-r} |f_1|^r d\alpha d\mu + 2A_1 p \int_\Omega \int_0^\infty \alpha^{p-2} |f_2| d\alpha d\mu . \end{aligned}$$

Now $f_1(x) = 0$ unless $|f_1(x)| \leq \alpha$ and $f_2(x) = 0$ unless $|f_2(x)| > \alpha$ so this equals

$$p(2A_r)^r \int_\Omega |f(x)|^r \int_{|f(x)|}^\infty \alpha^{p-1-r} d\alpha d\mu + 2A_1 p \int_\Omega |f(x)| \int_0^{|f(x)|} \alpha^{p-2} d\alpha d\mu$$

which equals

$$\begin{aligned} \frac{2^r A_r^r p}{r-p} \int_\Omega |f(x)|^p d\mu + \frac{2pA_1}{p-1} \int_\Omega |f(x)|^p d\mu \\ \leq \max \left(\frac{2^r A_r^r p}{r-p}, \frac{2pA_1}{p-1} \right) \|f\|_{L^p(\Omega)}^p \end{aligned}$$

and this proves the theorem.

22.2 The Calderon Zygmund Decomposition

For a given nonnegative integrable function, $\mathbb{R}^n$ can be decomposed into a set where the function is small and a set which is the union of disjoint cubes on which the average of the function is under some control. The measure in this section will always be Lebesgue measure on $\mathbb{R}^n$. This theorem depends on the Lebesgue theory of differentiation.

Theorem 22.2.1 *Let $f \geq 0$, $\int f dx < \infty$, and let α be a positive constant. Then there exist sets F and Ω such that*

$$\mathbb{R}^n = F \cup \Omega, \quad F \cap \Omega = \emptyset \quad (22.2.3)$$

$$f(x) \leq \alpha \text{ a.e. on } F \quad (22.2.4)$$

$\Omega = \cup_{k=1}^{\infty} Q_k$ where the interiors of the cubes are disjoint and for each cube, Q_k ,

$$\alpha < \frac{1}{m(Q_k)} \int_{Q_k} f(x) dx \leq 2^n \alpha. \quad (22.2.5)$$

Proof: Let S_0 be a tiling of $\mathbb{R}^n$ into cubes having sides of length M where M is chosen large enough that if Q is one of these cubes, then

$$\frac{1}{m(Q)} \int_Q f dm \leq \alpha. \quad (22.2.6)$$

Suppose $S_0, \dots, S_m$ have been chosen. To get S_{m+1} , replace each cube of S_m by the 2^n cubes obtained by bisecting the sides. Then S_{m+1} consists of exactly those cubes of S_m for which 22.2.6 holds and let T_{m+1} consist of the bisected cubes from S_m for which 22.2.6 does not hold. Now define

$$F \equiv \{\mathbf{x} : \mathbf{x} \text{ is contained in some cube from } S_m \text{ for all } m\},$$

$$\Omega \equiv \mathbb{R}^n \setminus F = \cup_{m=1}^{\infty} \cup \{Q : Q \in T_m\}$$

Note that the cubes from T_m have pair wise disjoint interiors and also the interiors of cubes from T_m have empty intersections with the interiors of cubes of T_k if $k \neq m$.

Let $\mathbf{x}$ be a point of Ω and let $\mathbf{x}$ be in a cube of T_m such that m is the first index for which this happens. Let Q be the cube in S_{m-1} containing $\mathbf{x}$ and let Q^* be the cube in the bisection of Q which contains $\mathbf{x}$. Therefore 22.2.6 does not hold for Q^* . Thus

$$\alpha < \frac{1}{m(Q^*)} \int_{Q^*} f dx \leq \frac{m(Q)}{m(Q^*)} \overbrace{\frac{1}{m(Q)} \int_Q f dx}^{\leq \alpha} \leq 2^n \alpha$$

which shows Ω is the union of cubes having disjoint interiors for which 22.2.5 holds.

Now a.e. point of F is a Lebesgue point of f . Let $\mathbf{x}$ be such a point of F and suppose $\mathbf{x} \in Q_k$ for $Q_k \in S_k$. Let $d_k \equiv$ diameter of Q_k . Thus $d_k \rightarrow 0$.

$$\begin{aligned} \frac{1}{m(Q_k)} \int_{Q_k} |f(\mathbf{y}) - f(\mathbf{x})| dy &\leq \frac{1}{m(Q_k)} \int_{B(\mathbf{x}, d_k)} |f(\mathbf{y}) - f(\mathbf{x})| dy \\ &= \frac{m(B(\mathbf{x}, d_k))}{m(Q_k)} \frac{1}{m(B(\mathbf{x}, d_k))} \int_{B(\mathbf{x}, d_k)} |f(\mathbf{x}) - f(\mathbf{y})| dy \\ &\leq K_n \frac{1}{m(B(\mathbf{x}, d_k))} \int_{B(\mathbf{x}, d_k)} |f(\mathbf{x}) - f(\mathbf{y})| dy \end{aligned}$$

where K_n is a constant which depends on n and measures the ratio of the volume of a ball with diameter $2d$ and a cube with diameter d . The last expression converges to 0 because $\mathbf{x}$ is a Lebesgue point. Hence

$$f(\mathbf{x}) = \lim_{k \rightarrow \infty} \frac{1}{m(Q_k)} \int_{Q_k} f(\mathbf{y}) \, dy \leq \alpha$$

and this shows $f(\mathbf{x}) \leq \alpha$ a.e. on F . This proves the theorem.

22.3 Mihlin’s Theorem

In this section, the Marcinkiewicz interpolation theorem and Calderon Zygmund decomposition will be used to establish a remarkable theorem of Mihlin, a generalization of Plancherel’s theorem to the L^p spaces. It is of fundamental importance in the study of elliptic partial differential equations and can also be used to give proofs for the theory of singular integrals. Mihlin’s theorem involves a conclusion which is of the form

$$\|F^{-1}\rho * \phi\|_p \leq A_p \|\phi\|_p \tag{22.3.7}$$

for $p > 1$ and $\phi \in \mathcal{G}$. Thus $F^{-1}\rho*$ extends to a continuous linear map defined on L^p because of the density of $\mathcal{G}$. It is proved by showing various weak type estimates and then applying the Marcinkiewicz Interpolation Theorem to get an estimate like the above.

Recall that by Corollary 21.3.19, if $f \in L^2(\mathbb{R}^n)$ and if $\phi \in \mathcal{G}$, then $f*\phi \in L^2(\mathbb{R}^n)$ and

$$F(f*\phi)(\mathbf{x}) = (2\pi)^{n/2} F\phi(\mathbf{x}) Ff(\mathbf{x}).$$

The next lemma is essentially a weak $(1,1)$ estimate. The inequality 22.3.7 is established under the condition, 22.3.8 and then it is shown there exist conditions which are easier to verify which imply condition 22.3.8. I think the approach used here is due to Hormander [34] and is found in Berg and Lofstrom [10]. For many more references and generalizations, you might look in Triebel [64]. A different proof based on singular integrals is in Stein [63]. Functions, ρ which yield an inequality of the sort in 22.3.7 are called L^p multipliers.

Lemma 22.3.1 *Suppose $\rho \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and suppose also there exists a constant C_1 such that*

$$\int_{|\mathbf{x}| \geq 2|\mathbf{y}|} |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x})| \, dx \leq C_1. \tag{22.3.8}$$

Then there exists a constant A depending only on $C_1, \|\rho\|_\infty$, and n such that

$$m\left(\left[\mathbf{x} : |F^{-1}\rho*\phi(\mathbf{x})| > \alpha\right]\right) \leq \frac{A}{\alpha} \|\phi\|_1$$

for all $\phi \in \mathcal{G}$.

Proof: Let $\phi \in \mathcal{G}$ and use the Calderon decomposition to write $\mathbb{R}^n = E \cup \Omega$ where Ω is a union of cubes, $\{Q_i\}$ with disjoint interiors such that

$$\alpha m(Q_i) \leq \int_{Q_i} |\phi(\mathbf{x})| dx \leq 2^n \alpha m(Q_i), \quad |\phi(\mathbf{x})| \leq \alpha \text{ a.e. on } E. \quad (22.3.9)$$

The proof is accomplished by writing ϕ as the sum of a good function and a bad function and establishing a similar weak inequality for these two functions separately. Then this information is used to obtain the desired conclusion.

$$g(\mathbf{x}) = \begin{cases} \phi(\mathbf{x}) & \text{if } \mathbf{x} \in E \\ \frac{1}{m(Q_i)} \int_{Q_i} \phi(\mathbf{x}) dx & \text{if } \mathbf{x} \in Q_i \subseteq \Omega \end{cases}, \quad g(\mathbf{x}) + b(\mathbf{x}) = \phi(\mathbf{x}). \quad (22.3.10)$$

Thus

$$\int_{Q_i} b(\mathbf{x}) dx = \int_{Q_i} (\phi(\mathbf{x}) - g(\mathbf{x})) dx = \int_{Q_i} \phi(\mathbf{x}) dx - \int_{Q_i} \phi(\mathbf{x}) dx = 0. \quad (22.3.11)$$

$$b(\mathbf{x}) = 0 \text{ if } \mathbf{x} \notin \Omega. \quad (22.3.12)$$

Claim:

$$\|g\|_2^2 \leq \alpha(1 + 4^n) \|\phi\|_1, \quad \|g\|_1 \leq \|\phi\|_1. \quad (22.3.13)$$

Proof of claim:

$$\|g\|_2^2 = \|g\|_{L^2(E)}^2 + \|g\|_{L^2(\Omega)}^2.$$

Thus

$$\begin{aligned} \|g\|_{L^2(\Omega)}^2 &= \sum_i \int_{Q_i} |g(x)|^2 dx \\ &\leq \sum_i \int_{Q_i} \left(\frac{1}{m(Q_i)} \int_{Q_i} |\phi(y)| dy \right)^2 dx \\ &\leq \sum_i \int_{Q_i} (2^n \alpha)^2 dx \leq 4^n \alpha^2 \sum_i m(Q_i) \\ &\leq 4^n \alpha^2 \frac{1}{\alpha} \sum_i \int_{Q_i} |\phi(x)| dx \leq 4^n \alpha \|\phi\|_1. \end{aligned}$$

$$\|g\|_{L^2(E)}^2 = \int_E |\phi(x)|^2 dx \leq \alpha \int_E |\phi(x)| dx = \alpha \|\phi\|_1.$$

Now consider the second of the inequalities in 22.3.13.

$$\begin{aligned} \|g\|_1 &= \int_E |g(\mathbf{x})| dx + \int_\Omega |g(\mathbf{x})| dx \\ &= \int_E |\phi(\mathbf{x})| dx + \sum_i \int_{Q_i} |g| dx \\ &\leq \int_E |\phi(\mathbf{x})| dx + \sum_i \int_{Q_i} \frac{1}{m(Q_i)} \int_{Q_i} |\phi(\mathbf{x})| dm(x) dm \\ &= \int_E |\phi(\mathbf{x})| dx + \sum_i \int_{Q_i} |\phi(\mathbf{x})| dm(x) = \|\phi\|_1 \end{aligned}$$

This proves the claim. From the claim, it follows that $b \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

Because of 22.3.13, $g \in L^1(\mathbb{R}^n)$ and so $F^{-1}\rho * g \in L^2(\mathbb{R}^n)$. (Since $\rho \in L^2$, it follows $F^{-1}\rho \in L^2$ and so this convolution is indeed in L^2 .) By Plancherel's theorem,

$$\|F^{-1}\rho * g\|_2 = \|F(F^{-1}\rho * g)\|_2.$$

By Corollary 21.3.19 on Page 651, the expression on the right equals

$$(2\pi)^{n/2} \|\rho Fg\|_2$$

and so

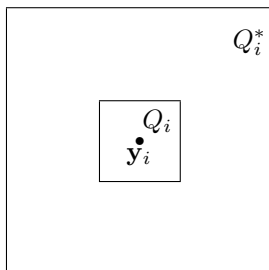
$$\|F^{-1}\rho * g\|_2 = (2\pi)^{n/2} \|\rho Fg\|_2 \leq C_n \|\rho\|_\infty \|g\|_2.$$

From this and 22.3.13

$$\begin{aligned} & m([|F^{-1}\rho * g| \geq \alpha/2]) \\ & \leq \frac{C_n \|\rho\|_\infty^2}{\alpha^2} \alpha(1 + 4^n) \|\phi\|_1 = C_n \alpha^{-1} \|\phi\|_1. \end{aligned} \quad (22.3.14)$$

This is what is wanted so far as g is concerned. Next it is required to estimate $m([|F^{-1}\rho * b| \geq \alpha/2])$.

If Q is one of the cubes whose union is Ω , let Q^* be the cube with the same center as Q but whose sides are $2\sqrt{n}$ times as long.



Let

$$\Omega^* \equiv \bigcup_{i=1}^{\infty} Q_i^*$$

and let

$$E^* \equiv \mathbb{R}^n \setminus \Omega^*.$$

Thus $E^* \subseteq E$. Let $\mathbf{x} \in E^*$. Then because of 22.3.11,

$$\begin{aligned} & \int_{Q_i} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) d\mathbf{y} \\ & = \int_{Q_i} [F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x} - \mathbf{y}_i)] b(\mathbf{y}) d\mathbf{y}, \end{aligned} \quad (22.3.15)$$

where $\mathbf{y}_i$ is the center of Q_i . Consequently if the sides of Q_i have length $2t/\sqrt{n}$, 22.3.15 implies

$$\int_{E^*} \left| \int_{Q_i} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) d\mathbf{y} \right| dx \leq \quad (22.3.16)$$

$$\begin{aligned}
& \int_{E^*} \int_{Q_i} |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x} - \mathbf{y}_i)| |b(\mathbf{y})| dy dx \\
&= \int_{Q_i} \int_{E^*} |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x} - \mathbf{y}_i)| dx |b(\mathbf{y})| dy \quad (22.3.17) \\
&\leq \int_{Q_i} \int_{|\mathbf{x} - \mathbf{y}_i| \geq 2t} |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x} - \mathbf{y}_i)| dx |b(\mathbf{y})| dy \quad (22.3.18)
\end{aligned}$$

since if $\mathbf{x} \in E^*$, then $|\mathbf{x} - \mathbf{y}_i| \geq 2t$. Now for $\mathbf{y} \in Q_i$,

$$|\mathbf{y} - \mathbf{y}_i| \leq \left(\sum_{j=1}^n \left(\frac{t}{\sqrt{n}} \right)^2 \right)^{1/2} = t.$$

From 22.3.8 and the change of variables $\mathbf{u} = \mathbf{x} - \mathbf{y}_i$ 22.3.16 - 22.3.18 imply

$$\int_{E^*} \left| \int_{Q_i} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) dy \right| dx \leq C_1 \int_{Q_i} |b(\mathbf{y})| dy. \quad (22.3.19)$$

Now from 22.3.19, and the fact that $b = 0$ off Ω ,

$$\begin{aligned}
\int_{E^*} |F^{-1}\rho * b(\mathbf{x})| dx &= \int_{E^*} \left| \int_{\mathbb{R}^n} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) dy \right| dx \\
&= \int_{E^*} \left| \sum_{i=1}^{\infty} \int_{Q_i} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) dy \right| dx \\
&\leq \int_{E^*} \sum_{i=1}^{\infty} \left| \int_{Q_i} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) dy \right| dx \\
&= \sum_{i=1}^{\infty} \int_{E^*} \left| \int_{Q_i} F^{-1}\rho(\mathbf{x} - \mathbf{y}) b(\mathbf{y}) dy \right| dx \\
&\leq \sum_{i=1}^{\infty} C_1 \int_{Q_i} |b(\mathbf{y})| dy = C_1 \|b\|_1.
\end{aligned}$$

Thus, by 22.3.13,

$$\begin{aligned}
\int_{E^*} |F^{-1}\rho * b(\mathbf{x})| dx &\leq C_1 \|b\|_1 \\
&\leq C_1 [\|\phi\|_1 + \|g\|_1] \\
&\leq C_1 [\|\phi\|_1 + \|\phi\|_1] \\
&\leq 2C_1 \|\phi\|_1.
\end{aligned}$$

Consequently,

$$m \left(\left[|F^{-1}\rho * b| \geq \frac{\alpha}{2} \right] \cap E^* \right) \leq \frac{4C_1}{\alpha} \|\phi\|_1.$$

From 22.3.10, 22.3.14, and 22.3.9,

$$\begin{aligned} m \left[|F^{-1}\rho * \phi| > \alpha \right] &\leq m \left[|F^{-1}\rho * g| \geq \frac{\alpha}{2} \right] + m \left[|F^{-1}\rho * b| \geq \frac{\alpha}{2} \right] \\ &\leq \frac{C_n}{\alpha} \|\phi\|_1 + m \left(\left[|F^{-1}\rho * b| \geq \frac{\alpha}{2} \right] \cap E^* \right) + m(\Omega^*) \\ &\leq \frac{C_n}{\alpha} \|\phi\|_1 + \frac{4C_1}{\alpha} \|\phi\|_1 + C_n m(\Omega) \leq \frac{A}{\alpha} \|\phi\|_1 \end{aligned}$$

because

$$m(\Omega) \leq \alpha^{-1} \|\phi\|_1$$

by 22.3.9. This proves the lemma.

The next lemma extends this lemma by giving a weak (2, 2) estimate and a (2, 2) estimate.

Lemma 22.3.2 *Suppose $\rho \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and suppose also that there exists a constant C_1 such that*

$$\int_{|\mathbf{x}| > 2|\mathbf{y}|} |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x})| dx \leq C_1. \quad (22.3.20)$$

Then $F^{-1}\rho*$ maps $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ to measurable functions and there exists a constant A depending only on $C_1, n, \|\rho\|_\infty$ such that

$$m \left([|F^{-1}\rho * f| > \alpha] \right) \leq A \frac{\|f\|_1}{\alpha} \text{ if } f \in L^1(\mathbb{R}^n), \quad (22.3.21)$$

$$m \left([|F^{-1}\rho * f| > \alpha] \right) \leq \left(A \frac{\|f\|_2}{\alpha} \right)^2 \text{ if } f \in L^2(\mathbb{R}^n). \quad (22.3.22)$$

Thus, $F^{-1}\rho*$ is weak type (1, 1) and weak type (2, 2). Also

$$\|F^{-1}\rho * f\|_2 \leq A \|f\|_2 \text{ if } f \in L^2(\mathbb{R}^n). \quad (22.3.23)$$

Proof: By Plancherel's theorem $F^{-1}\rho$ is in $L^2(\mathbb{R}^n)$. If $f \in L^1(\mathbb{R}^n)$, then by Minkowski's inequality,

$$F^{-1}\rho * f \in L^2(\mathbb{R}^n).$$

Now let $g \in L^2(\mathbb{R}^n)$. By Holder's inequality,

$$\int |F^{-1}\rho(\mathbf{x} - \mathbf{y})| |g(\mathbf{y})| dy \leq \left(\int |F^{-1}\rho(\mathbf{x} - \mathbf{y})|^2 dy \right)^{1/2} \left(\int |g(\mathbf{y})|^2 dy \right)^{1/2} < \infty$$

and so the following is well defined a.e.

$$F^{-1}\rho * g(\mathbf{x}) \equiv \int F^{-1}\rho(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) dy$$

also,

$$\begin{aligned} |F^{-1}\rho * g(\mathbf{x}) - F^{-1}\rho * g(\mathbf{x}')| &\leq \int |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x}' - \mathbf{y})| |g(\mathbf{y})| dy \\ &\leq \|F^{-1}\rho - F^{-1}\rho_{\mathbf{x}' - \mathbf{x}}\| \|g\|_{l^2} \end{aligned}$$

and by continuity of translation in $L^2(\mathbb{R}^n)$, this shows $\mathbf{x} \rightarrow F^{-1}\rho * g(\mathbf{x})$ is continuous. Therefore, $F^{-1}\rho *$ maps $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ to the space of measurable functions. (Continuous functions are measurable.) It is clear that $F^{-1}\rho *$ is subadditive.

If $\phi \in \mathcal{G}$, Plancherel's theorem implies as before,

$$\begin{aligned} \|F^{-1}\rho * \phi\|_2 &= \|F(F^{-1}\rho * \phi)\|_2 = \\ (2\pi)^{n/2} \|\rho F\phi\|_2 &\leq (2\pi)^{n/2} \|\rho\|_\infty \|\phi\|_2. \end{aligned} \quad (22.3.24)$$

Now let $f \in L^2(\mathbb{R}^n)$ and let $\phi_k \in \mathcal{G}$, with

$$\|\phi_k - f\|_2 \rightarrow 0.$$

Then by Holder's inequality,

$$\int F^{-1}\rho(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dy = \lim_{k \rightarrow \infty} \int F^{-1}\rho(\mathbf{x} - \mathbf{y}) \phi_k(\mathbf{y}) dy$$

and so by Fatou's lemma, Plancherel's theorem, and 22.3.24,

$$\begin{aligned} \|F^{-1}\rho * f\|_2 &= \left(\int \left| \int F^{-1}\rho(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dy \right|^2 dx \right)^{1/2} \leq \\ &\leq \liminf_{k \rightarrow \infty} \left(\int \left| \int F^{-1}\rho(\mathbf{x} - \mathbf{y}) \phi_k(\mathbf{y}) dy \right|^2 dx \right)^{1/2} = \liminf_{k \rightarrow \infty} \|F^{-1}\rho * \phi_k\|_2 \\ &\leq \|\rho\|_\infty (2\pi)^{n/2} \liminf_{k \rightarrow \infty} \|\phi_k\|_2 = \|\rho\|_\infty (2\pi)^{n/2} \|f\|_2. \end{aligned}$$

Thus, 22.3.23 holds with $A = \|\rho\|_\infty (2\pi)^{n/2}$. Consequently,

$$\begin{aligned} A \|f\|_2 &\geq \left(\int_{\{|F^{-1}\rho * f| > \alpha\}} |F^{-1}\rho * f(\mathbf{x})|^2 dx \right)^{1/2} \\ &\geq \alpha m(\{|F^{-1}\rho * f| > \alpha\})^{1/2} \end{aligned}$$

and so 22.3.22 follows.

It remains to prove 22.3.21 which holds for all $f \in \mathcal{G}$ by Lemma 22.3.1. Let $f \in L^1(\mathbb{R}^n)$ and let $\phi_k \rightarrow f$ in $L^1(\mathbb{R}^n)$, $\phi_k \in \mathcal{G}$. Without loss of generality, assume that both f and $F^{-1}\rho$ are Borel measurable. Therefore, by Minkowski's inequality, and Plancherel's theorem,

$$\|F^{-1}\rho * \phi_k - F^{-1}\rho * f\|_2$$

$$\begin{aligned}
&\leq \left(\int \left| \int F^{-1} \rho(\mathbf{x} - \mathbf{y}) (\phi_k(\mathbf{y}) - f(\mathbf{y})) d\mathbf{y} \right|^2 d\mathbf{x} \right)^{1/2} \\
&\leq \|\phi_k - f\|_1 \|\rho\|_2
\end{aligned}$$

which shows that $F^{-1} \rho * \phi_k$ converges to $F^{-1} \rho * f$ in $L^2(\mathbb{R}^n)$. Therefore, there exists a subsequence such that the convergence is pointwise a.e. Then, denoting the subsequence by k ,

$$\mathcal{X}_{[|F^{-1} \rho * f| > \alpha]}(\mathbf{x}) \leq \liminf_{k \rightarrow \infty} \mathcal{X}_{[|F^{-1} \rho * \phi_k| > \alpha]}(\mathbf{x}) \text{ a.e. } \mathbf{x}.$$

Thus by Lemma 22.3.1 and Fatou's lemma, there exists a constant, A , depending on C_1, n , and $\|\rho\|_\infty$ such that

$$\begin{aligned}
m([|F^{-1} \rho * f| > \alpha]) &\leq \liminf_{k \rightarrow \infty} m([|F^{-1} \rho * \phi_k| > \alpha]) \\
&\leq \liminf_{k \rightarrow \infty} A \frac{\|\phi_k\|_1}{\alpha} = A \frac{\|f\|_1}{\alpha}.
\end{aligned}$$

This shows 22.3.21 and proves the lemma.

Theorem 22.3.3 *Let $\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and suppose*

$$\int_{|\mathbf{x}| \geq 2|\mathbf{y}|} |F^{-1} \rho(\mathbf{x} - \mathbf{y}) - F^{-1} \rho(\mathbf{x})| d\mathbf{x} \leq C_1.$$

Then for each $p \in (1, \infty)$, there exists a constant, A_p , depending only on

$$p, n, \|\rho\|_\infty,$$

and C_1 such that for all $\phi \in \mathcal{G}$,

$$\|F^{-1} \rho * \phi\|_p \leq A_p \|\phi\|_p.$$

Proof: From Lemma 22.3.2, $F^{-1} \rho *$ is weak $(1, 1)$, weak $(2, 2)$, and maps

$$L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$$

to measurable functions. Therefore, by the Marcinkiewicz interpolation theorem, there exists a constant A_p depending only on p, C_1, n , and $\|\rho\|_\infty$ for $p \in (1, 2]$, such that for $f \in L^p(\mathbb{R}^n)$, and $p \in (1, 2]$,

$$\|F^{-1} \rho * f\|_p \leq A_p \|f\|_p.$$

Thus the theorem is proved for these values of p . Now suppose $p > 2$. Then $p' < 2$ where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

By Plancherel's theorem and Theorem 21.3.25,

$$\begin{aligned}
 \int F^{-1} \rho * \phi(\mathbf{x}) \psi(\mathbf{x}) dx &= (2\pi)^{n/2} \int \rho(\mathbf{x}) F\phi(\mathbf{x}) F\psi(\mathbf{x}) dx \\
 &= \int F(F^{-1} \rho * \psi) F\phi dx \\
 &= \int (F^{-1} \rho * \psi)(\phi) dx.
 \end{aligned}$$

Thus by the case for $p \in (1, 2)$ and Holder's inequality,

$$\begin{aligned}
 \left| \int F^{-1} \rho * \phi(\mathbf{x}) \psi(\mathbf{x}) dx \right| &= \left| \int (F^{-1} \rho * \psi)(\phi) dx \right| \\
 &\leq \|F^{-1} \rho * \psi\|_{p'} \|\phi\|_p \\
 &\leq A_{p'} \|\psi\|_{p'} \|\phi\|_p.
 \end{aligned}$$

Letting $L\psi \equiv \int F^{-1} \rho * \phi(\mathbf{x}) \psi(\mathbf{x}) dx$, this shows $L \in L^{p'}(\mathbb{R}^n)'$ and $\|L\|_{(L^{p'})'} \leq A_{p'} \|\phi\|_p$ which implies by the Riesz representation theorem that $F^{-1} \rho * \phi$ represents L and

$$\|L\|_{(L^{p'})'} = \|F^{-1} \rho * \phi\|_{L^p} \leq A_{p'} \|\phi\|_p$$

Since $p' = p/(p-1)$, this proves the theorem.

It is possible to give verifiable conditions on ρ which imply 22.3.20. The condition on ρ which is presented here is the existence of a constant, C_0 such that

$$C_0 \geq \sup\{|\mathbf{x}|^{|\alpha|} |D^\alpha \rho(\mathbf{x})| : |\alpha| \leq L, \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}\}, L > n/2. \quad (22.3.25)$$

$$\rho \in C^L(\mathbb{R}^n \setminus \{\mathbf{0}\}) \text{ where } L \text{ is an integer.}$$

Here α is a multi-index and $|\alpha| = \sum_{i=1}^n \alpha_i$. The condition says roughly that ρ is pretty smooth away from $\mathbf{0}$ and all the partial derivatives vanish pretty fast as $|\mathbf{x}| \rightarrow \infty$. Also recall the notation

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $\alpha = (\alpha_1 \cdots \alpha_n)$. For more general conditions, see [34].

Lemma 22.3.4 *Let 22.3.25 hold and suppose $\psi \in C_c^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$. Then for each α , $|\alpha| \leq L$, there exists a constant $C \equiv C(\alpha, n, \psi)$ independent of k such that*

$$\sup_{\mathbf{x} \in \mathbb{R}^n} |\mathbf{x}|^{|\alpha|} |D^\alpha(\rho(\mathbf{x}) \psi(2^k \mathbf{x}))| \leq CC_0.$$

Proof:

$$|\mathbf{x}|^{|\alpha|} |D^\alpha(\rho(\mathbf{x}) \psi(2^k \mathbf{x}))| \leq |\mathbf{x}|^{|\alpha|} \sum_{\beta+\gamma=\alpha} |D^\beta \rho(\mathbf{x})| 2^{k|\gamma|} |D^\gamma \psi(2^k \mathbf{x})|$$

$$\begin{aligned}
&= \sum_{\beta+\gamma=\alpha} |\mathbf{x}|^{|\beta|} |D^\beta \rho(\mathbf{x})| |2^k \mathbf{x}|^{|\gamma|} |D^\gamma \psi(2^k \mathbf{x})| \\
&\leq C_0 C(\alpha, n) \sum_{|\gamma| \leq |\alpha|} \sup\{|z|^{|\gamma|} |D^\gamma \psi(\mathbf{z})| : \mathbf{z} \in \mathbb{R}^n\} = C_0 C(\alpha, n, \psi)
\end{aligned}$$

and this proves the lemma.

Lemma 22.3.5 *There exists*

$$\phi \in C_c^\infty([|\mathbf{x}| : 4^{-1} < |\mathbf{x}| < 4]), \quad \phi(\mathbf{x}) \geq 0$$

and

$$\sum_{k=-\infty}^{\infty} \phi(2^k \mathbf{x}) = 1$$

for each $\mathbf{x} \neq \mathbf{0}$.

Proof: Let

$$\psi \geq 0, \quad \psi = 1 \text{ on } [2^{-1} \leq |\mathbf{x}| \leq 2],$$

$$\text{spt}(\psi) \subseteq [4^{-1} < |\mathbf{x}| < 4].$$

Consider

$$g(\mathbf{x}) = \sum_{k=-\infty}^{\infty} \psi(2^k \mathbf{x}).$$

Then for each $\mathbf{x}$, only finitely many terms are not equal to 0. Also, $g(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. To verify this last claim, note that for some k an integer, $|\mathbf{x}| \in [2^l, 2^{l+2}]$. Therefore, choose k an integer such that $2^k |\mathbf{x}| \in [2^{-1}, 2]$. For example, let $k = -l - 1$. This works because $2^k |\mathbf{x}| \in [2^l 2^k, 2^{l+2} 2^k] = [2^{l-l-1}, 2^{l+2-l-1}] = [2^{-1}, 2]$. Therefore, for this value of k , $\psi(2^k \mathbf{x}) = 1$ so $g(\mathbf{x}) > 0$.

Now notice that

$$g(2^r \mathbf{x}) = \sum_{k=-\infty}^{\infty} \psi(2^k 2^r \mathbf{x}) = \sum_{k=-\infty}^{\infty} \psi(2^k \mathbf{x}) = g(\mathbf{x}).$$

Let $\phi(\mathbf{x}) \equiv \psi(\mathbf{x}) g(\mathbf{x})^{-1}$. Then

$$\sum_{k=-\infty}^{\infty} \phi(2^k \mathbf{x}) = \sum_{k=-\infty}^{\infty} \frac{\psi(2^k \mathbf{x})}{g(2^k \mathbf{x})} = g(\mathbf{x})^{-1} \sum_{k=-\infty}^{\infty} \psi(2^k \mathbf{x}) = 1$$

for each $\mathbf{x} \neq \mathbf{0}$. This proves the lemma.

Now define

$$\rho_m(\mathbf{x}) \equiv \sum_{k=-m}^m \rho(\mathbf{x}) \phi(2^k \mathbf{x}), \quad \gamma_k(\mathbf{x}) \equiv \rho(\mathbf{x}) \phi(2^k \mathbf{x}).$$

Let $t > 0$ and let $|\mathbf{y}| \leq t$. Consider the problem of estimating

$$\int_{|\mathbf{x}| \geq 2t} |F^{-1}\gamma_k(\mathbf{x} - \mathbf{y}) - F^{-1}\gamma_k(\mathbf{x})| dx. \quad (22.3.26)$$

In the following estimates, $C(a, b, \dots, d)$ will denote a generic constant depending only on the indicated objects, $a, b, \dots, d$. For the first estimate, note that since $|\mathbf{y}| \leq t$, 22.3.26 is no larger than

$$\begin{aligned} 2 \int_{|\mathbf{x}| \geq t} |F^{-1}\gamma_k(\mathbf{x})| dx &= 2 \int_{|\mathbf{x}| \geq t} |F^{-1}\gamma_k(\mathbf{x})| |\mathbf{x}|^{-L} |\mathbf{x}|^L dx \\ &\leq 2 \left(\int_{|\mathbf{x}| \geq t} |\mathbf{x}|^{-2L} dx \right)^{1/2} \left(\int_{|\mathbf{x}| \geq t} |\mathbf{x}|^{2L} |F^{-1}\gamma_k(\mathbf{x})|^2 dx \right)^{1/2} \end{aligned}$$

Using spherical coordinates and Plancherel's theorem,

$$\begin{aligned} &\leq C(n, L) t^{n/2-L} \left(\int |\mathbf{x}|^{2L} |F^{-1}\gamma_k(\mathbf{x})|^2 dx \right)^{1/2} \\ &\leq C(n, L) t^{n/2-L} \left(\int \sum_{j=1}^n |x_j|^{2L} |F^{-1}\gamma_k(\mathbf{x})|^2 dx \right)^{1/2} \\ &\leq C(n, L) t^{n/2-L} \left(\sum_{j=1}^n \int |F^{-1}D_j^L \gamma_k(\mathbf{x})|^2 dx \right)^{1/2} \\ &= C(n, L) t^{n/2-L} \left(\sum_{j=1}^n \int_{S_k} |D_j^L \gamma_k(\mathbf{x})|^2 dx \right)^{1/2} \end{aligned} \quad (22.3.27)$$

where

$$S_k \equiv [\mathbf{x} : 2^{-2-k} < |\mathbf{x}| < 2^{2-k}], \quad (22.3.28)$$

a set containing the support of γ_k . Now from the definition of γ_k ,

$$|D_j^L \gamma_k(\mathbf{z})| = |D_j^L(\rho(\mathbf{z})\phi(2^k \mathbf{z}))|.$$

By Lemma 22.3.4, this is no larger than

$$C(L, n, \phi) C_0 |\mathbf{z}|^{-L}. \quad (22.3.29)$$

It follows, using polar coordinates, that the last expression in 22.3.27 is no larger than

$$\begin{aligned} C(n, L, \phi, C_0) t^{n/2-L} \left(\int_{S_k} |\mathbf{z}|^{-2L} dz \right)^{1/2} &\leq C(n, L, \phi, C_0) t^{n/2-L}. \quad (22.3.30) \\ \left(\int_{2^{-2-k}}^{2^{2-k}} \rho^{n-1-2L} d\rho \right)^{1/2} &\leq C(n, L, \phi, C_0) t^{n/2-L} 2^{k(L-n/2)}. \end{aligned}$$

Now estimate 22.3.26 in another way. The support of γ_k is in S_k , a bounded set, and so $F^{-1}\gamma_k$ is differentiable. Therefore,

$$\int_{|\mathbf{x}| \geq 2t} |F^{-1}\gamma_k(\mathbf{x} - \mathbf{y}) - F^{-1}\gamma_k(\mathbf{x})| dx =$$

$$\begin{aligned}
& \int_{|\mathbf{x}| \geq 2t} \left| \int_0^1 \sum_{j=1}^n D_j F^{-1} \gamma_k(\mathbf{x} - s\mathbf{y}) y_j ds \right| dx \\
& \leq t \int_{|\mathbf{x}| \geq 2t} \int_0^1 \sum_{j=1}^n |D_j F^{-1} \gamma_k(\mathbf{x} - s\mathbf{y})| ds dx \\
& \leq t \int \sum_{j=1}^n |D_j F^{-1} \gamma_k(\mathbf{x})| dx \\
& \leq t \sum_{j=1}^n \left(\int (1 + |2^{-k} \mathbf{x}|^2)^{-L} dx \right)^{1/2} \\
& \quad \cdot \left(\int (1 + |2^{-k} \mathbf{x}|^2)^L |D_j F^{-1} \gamma_k(\mathbf{x})|^2 dx \right)^{1/2} \\
& \leq C(n, L) t 2^{kn/2} \sum_{j=1}^n \left(\int (1 + |2^{-k} \mathbf{x}|^2)^L |D_j F^{-1} \gamma_k(\mathbf{x})|^2 dx \right)^{1/2}. \quad (22.3.31)
\end{aligned}$$

Now consider the j^{th} term in the last sum in 22.3.31.

$$\begin{aligned}
& \int (1 + |2^{-k} \mathbf{x}|^2)^L |D_j F^{-1} \gamma_k(\mathbf{x})|^2 dx \leq \\
& C(n, L) \int \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} |\mathbf{x}^{2\alpha}| |D_j F^{-1} \gamma_k(\mathbf{x})|^2 dx \\
& = C(n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int \mathbf{x}^{2\alpha} |F^{-1}(\pi_j \gamma_k)(\mathbf{x})|^2 dx
\end{aligned} \quad (22.3.32)$$

where $\pi_j(\mathbf{z}) \equiv z_j$. This last assertion follows from

$$D_j \int e^{-i\mathbf{x} \cdot \mathbf{y}} \gamma_k(\mathbf{y}) d\mathbf{y} = \int (-i) e^{-i\mathbf{x} \cdot \mathbf{y}} y_j \gamma_k(\mathbf{y}) d\mathbf{y}.$$

Therefore, a similar computation and Plancherel's theorem implies 22.3.32 equals

$$\begin{aligned}
& = C(n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int |F^{-1} D^\alpha (\pi_j \gamma_k)(\mathbf{x})|^2 dx \\
& = C(n, L) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int_{S_k} |D^\alpha (z_j \gamma_k(\mathbf{z}))|^2 dz
\end{aligned} \quad (22.3.33)$$

where S_k is given in 22.3.28. Now

$$\begin{aligned}
|D^\alpha (z_j \gamma_k(\mathbf{z}))| &= 2^{-k} |D^\alpha (\rho(\mathbf{z}) z_j 2^k \phi(2^k \mathbf{z}))| \\
&= 2^{-k} |D^\alpha (\rho(\mathbf{z}) \psi_j(2^k \mathbf{z}))|
\end{aligned}$$

where $\psi_j(\mathbf{z}) \equiv z_j \phi(\mathbf{z})$. By Lemma 22.3.4, this is dominated by

$$2^{-k} C(\alpha, n, \phi, j, C_0) |\mathbf{z}|^{-|\alpha|}.$$

Therefore, 22.3.33 is dominated by

$$\begin{aligned} & C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} \int_{S_k} 2^{-2k} |\mathbf{z}|^{-2|\alpha|} dz \\ & \leq C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-2k|\alpha|} 2^{-2k} (2^{-2-k})^{(-2|\alpha|)} (2^{2-k})^n \\ & \leq C(L, n, \phi, j, C_0) \sum_{|\alpha| \leq L} 2^{-kn-2k} \\ & \leq C(L, n, \phi, j, C_0) 2^{-kn} 2^{-2k}. \end{aligned}$$

It follows that 22.3.31 is no larger than

$$C(L, n, \phi, C_0) t^{kn/2} 2^{-kn/2} 2^{-k} = C(L, n, \phi, C_0) t 2^{-k}. \quad (22.3.34)$$

It follows from 22.3.34 and 22.3.30 that if $|\mathbf{y}| \leq t$,

$$\begin{aligned} & \int_{|\mathbf{x}| \geq 2t} |F^{-1} \gamma_k(\mathbf{x} - \mathbf{y}) - F^{-1} \gamma_k(\mathbf{x})| dx \leq \\ & C(L, n, \phi, C_0) \min(t 2^{-k}, (2^{-k} t)^{n/2-k}). \end{aligned}$$

With this inequality, the next lemma which is the desired result can be obtained.

Lemma 22.3.6 *There exists a constant depending only on the indicated objects, $C_1 = C(L, n, \phi, C_0)$ such that when $|\mathbf{y}| \leq t$,*

$$\begin{aligned} & \int_{|\mathbf{x}| \geq 2t} |F^{-1} \rho(\mathbf{x} - \mathbf{y}) - F^{-1} \rho(\mathbf{x})| dx \leq C_1 \\ & \int_{|\mathbf{x}| \geq 2t} |F^{-1} \rho_m(\mathbf{x} - \mathbf{y}) - F^{-1} \rho_m(\mathbf{x})| dx \leq C_1. \end{aligned} \quad (22.3.35)$$

Proof: $F^{-1} \rho = \lim_{m \rightarrow \infty} F^{-1} \rho_m$ in $L^2(\mathbb{R}^n)$. Let $m_k \rightarrow \infty$ be such that convergence is pointwise a.e. Then if $|\mathbf{y}| \leq t$, Fatou's lemma implies

$$\begin{aligned} & \int_{|\mathbf{x}| \geq 2t} |F^{-1} \rho(\mathbf{x} - \mathbf{y}) - F^{-1} \rho(\mathbf{x})| dx \leq \\ & \liminf_{l \rightarrow \infty} \int_{|\mathbf{x}| \geq 2t} |F^{-1} \rho_{m_l}(\mathbf{x} - \mathbf{y}) - F^{-1} \rho_{m_l}(\mathbf{x})| dx \end{aligned}$$

$$\begin{aligned}
&\leq \liminf_{l \rightarrow \infty} \sum_{k=-m_l}^{m_l} \int_{|\mathbf{x}| \geq 2t} |F^{-1}\gamma_k(\mathbf{x} - \mathbf{y}) - F^{-1}\gamma_k(\mathbf{x})| dx \\
&\leq C(L, n, \phi, C_0) \sum_{k=-\infty}^{\infty} \min(t2^{-k}, (2^{-k}t)^{n/2-L}). \tag{22.3.36}
\end{aligned}$$

Now consider the sum in 22.3.36,

$$\sum_{k=-\infty}^{\infty} \min(t2^{-k}, (2^{-k}t)^{n/2-L}). \tag{22.3.37}$$

$t2^j = \min(t2^j, (2^j t)^{n/2-L})$ exactly when $t2^j \leq 1$. This occurs if and only if $j \leq -\ln(t)/\ln(2)$. Therefore 22.3.37 is no larger than

$$\sum_{j \leq -\ln(t)/\ln(2)} 2^j t + \sum_{j \geq -\ln(t)/\ln(2)} (2^j t)^{n/2-L}.$$

Letting $a = L - n/2$, this equals

$$\begin{aligned}
&t \sum_{k \geq \ln(t)/\ln(2)} 2^{-k} + t^{-a} \sum_{j \geq -\ln(t)/\ln(2)} (2^{-a})^j \\
&\leq 2t \left(\frac{1}{2}\right)^{\ln(t)/\ln(2)} + t^{-a} \left(\frac{1}{2^a}\right)^{-\ln(t)/\ln(2)} \\
&= 2t \left(\frac{1}{2}\right)^{\log_2(t)} + t^{-a} \left(\frac{1}{2^a}\right)^{-\log_2(t)} \\
&= 2 + 1 = 3.
\end{aligned}$$

Similarly, 22.3.35 holds. This proves the lemma.

Now it is possible to prove Mihlin's theorem.

Theorem 22.3.7 (*Mihlin's theorem*) Suppose ρ satisfies

$$C_0 \geq \sup\{|\mathbf{x}|^{|\alpha|} |D^\alpha \rho(\mathbf{x})| : |\alpha| \leq L, \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}\},$$

where L is an integer greater than $n/2$ and $\rho \in C^L(\mathbb{R}^n \setminus \{\mathbf{0}\})$. Then for every $p > 1$, there exists a constant A_p depending only on p, C_0, ϕ, n , and L , such that for all $\psi \in \mathcal{G}$,

$$\|F^{-1}\rho * \psi\|_p \leq A_p \|\psi\|_p.$$

Proof: Since ρ_m satisfies 22.3.35, and is obviously in $L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, Theorem 22.3.3 implies there exists a constant A_p depending only on $p, n, \|\rho_m\|_\infty$, and C_1 such that for all $\psi \in \mathcal{G}$ and $p \in (1, \infty)$,

$$\|F^{-1}\rho_m * \psi\|_p \leq A_p \|\psi\|_p.$$

Now $\|\rho_m\|_\infty \leq \|\rho\|_\infty$ because

$$|\rho_m(\mathbf{x})| \leq |\rho(\mathbf{x})| \sum_{k=-m}^m \phi(2^k \mathbf{x}) \leq |\rho(\mathbf{x})|. \quad (22.3.38)$$

Therefore, since $C_1 = C_1(L, n, \phi, C_0)$ and $C_0 \geq \|\rho\|_\infty$,

$$\|F^{-1}\rho_m * \psi\|_p \leq A_p(L, n, \phi, C_0, p) \|\psi\|_p.$$

In particular, A_p does not depend on m . Now, by 22.3.38, the observation that $\rho \in L^\infty(\mathbb{R}^n)$, $\lim_{m \rightarrow \infty} \rho_m(\mathbf{y}) = \rho(\mathbf{y})$ and the dominated convergence theorem, it follows that for $\theta \in \mathcal{G}$,

$$\begin{aligned} |(F^{-1}\rho * \psi)(\theta)| &\equiv \left| (2\pi)^{n/2} \int \rho(\mathbf{x}) F\psi(\mathbf{x}) F^{-1}\theta(\mathbf{x}) d\mathbf{x} \right| \\ &= \lim_{m \rightarrow \infty} |(F^{-1}\rho_m * \psi)(\theta)| \leq \lim_{m \rightarrow \infty} \sup \|F^{-1}\rho_m * \psi\|_p \|\theta\|_{p'} \\ &\leq A_p(L, n, \phi, C_0, p) \|\psi\|_p \|\theta\|_{p'}. \end{aligned}$$

Hence $F^{-1}\rho * \psi \in L^p(\mathbb{R}^n)$ and $\|F^{-1}\rho * \psi\|_p \leq A_p \|\psi\|_p$. This proves the theorem.

22.4 Singular Integrals

If $K \in L^1(\mathbb{R}^n)$ then when $p > 1$,

$$\|K * f\|_p \leq \|f\|_p.$$

It turns out that some meaning can be assigned to $K * f$ for some functions K which are not in L^1 . This involves assuming a certain form for K and exploiting cancellation. The resulting theory of singular integrals is very useful. To illustrate, an application will be given to the Helmholtz decomposition of vector fields in the next section. Like Mihlin's theorem, the theory presented here rests on Theorem 22.3.3, restated here for convenience.

Theorem 22.4.1 *Let $\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and suppose*

$$\int_{|\mathbf{x}| \geq 2|\mathbf{y}|} |F^{-1}\rho(\mathbf{x} - \mathbf{y}) - F^{-1}\rho(\mathbf{x})| d\mathbf{x} \leq C_1.$$

Then for each $p \in (1, \infty)$, there exists a constant, A_p , depending only on

$$p, n, \|\rho\|_\infty,$$

and C_1 such that for all $\phi \in \mathcal{G}$,

$$\|F^{-1}\rho * \phi\|_p \leq A_p \|\phi\|_p.$$

Lemma 22.4.2 *Suppose*

$$K \in L^2(\mathbb{R}^n), \|FK\|_\infty \leq B < \infty, \quad (22.4.39)$$

and

$$\int_{|\mathbf{x}| > 2|\mathbf{y}|} |K(\mathbf{x} - \mathbf{y}) - K(\mathbf{x})| dx \leq B.$$

Then for all $p > 1$, there exists a constant, $A(p, n, B)$, depending only on the indicated quantities such that

$$\|K * f\|_p \leq A(p, n, B) \|f\|_p$$

for all $f \in \mathcal{G}$.

Proof: Let $FK = \rho$ so $F^{-1}\rho = K$. Then from 22.4.39 $\rho \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $K = F^{-1}\rho$. By Theorem 22.3.3 listed above,

$$\|K * f\|_p = \|F^{-1}\rho * f\|_p \leq A(p, n, B) \|f\|_p$$

for all $f \in \mathcal{G}$. This proves the lemma.

The next lemma provides a situation in which the above conditions hold.

Lemma 22.4.3 *Suppose*

$$|K(\mathbf{x})| \leq B |\mathbf{x}|^{-n}, \quad (22.4.40)$$

$$\int_{a < |\mathbf{x}| < b} K(\mathbf{x}) dx = 0, \quad (22.4.41)$$

$$\int_{|\mathbf{x}| > 2|\mathbf{y}|} |K(\mathbf{x} - \mathbf{y}) - K(\mathbf{x})| dx \leq B. \quad (22.4.42)$$

Define

$$K_\varepsilon(\mathbf{x}) = \begin{cases} K(\mathbf{x}) & \text{if } |\mathbf{x}| \geq \varepsilon, \\ 0 & \text{if } |\mathbf{x}| < \varepsilon. \end{cases} \quad (22.4.43)$$

Then there exists a constant $C(n)$ such that

$$\int_{|\mathbf{x}| > 2|\mathbf{y}|} |K_\varepsilon(\mathbf{x} - \mathbf{y}) - K_\varepsilon(\mathbf{x})| dx \leq C(n) B \quad (22.4.44)$$

and

$$\|FK_\varepsilon\|_\infty \leq C(n) B. \quad (22.4.45)$$

Proof: In the argument, $C(n)$ will denote a generic constant depending only on n . Consider 22.4.44 first. The integral is broken up according to whether $|\mathbf{x}|, |\mathbf{x} - \mathbf{y}| > \varepsilon$.

$ \mathbf{x} $	$> \varepsilon$	$> \varepsilon$	$< \varepsilon$	$< \varepsilon$
$ \mathbf{x} - \mathbf{y} $	$> \varepsilon$	$< \varepsilon$	$< \varepsilon$	$> \varepsilon$

$$\begin{aligned}
& \int_{|\mathbf{x}| \geq 2|\mathbf{y}|} |K_\varepsilon(\mathbf{x} - \mathbf{y}) - K_\varepsilon(\mathbf{x})| dx = \\
& \int_{|\mathbf{x}| \geq 2|\mathbf{y}|, |\mathbf{x} - \mathbf{y}| > \varepsilon, |\mathbf{x}| < \varepsilon} |K_\varepsilon(\mathbf{x} - \mathbf{y}) - K_\varepsilon(\mathbf{x})| dx + \\
& + \int_{|\mathbf{x}| \geq 2|\mathbf{y}|, |\mathbf{x} - \mathbf{y}| < \varepsilon, |\mathbf{x}| \geq \varepsilon} |K_\varepsilon(\mathbf{x} - \mathbf{y}) - K_\varepsilon(\mathbf{x})| dx + \\
& \int_{|\mathbf{x}| \geq 2|\mathbf{y}|, |\mathbf{x} - \mathbf{y}| > \varepsilon, |\mathbf{x}| > \varepsilon} |K_\varepsilon(\mathbf{x} - \mathbf{y}) - K_\varepsilon(\mathbf{x})| dx + \\
& + \int_{|\mathbf{x}| \geq 2|\mathbf{y}|, |\mathbf{x} - \mathbf{y}| < \varepsilon, |\mathbf{x}| < \varepsilon} |K_\varepsilon(\mathbf{x} - \mathbf{y}) - K_\varepsilon(\mathbf{x})| dx.
\end{aligned} \tag{22.4.46}$$

Now consider the terms in the above expression. The last integral in 22.4.46 equals 0 from the definition of K_ε . The third integral on the right is no larger than B by the definition of K_ε and 22.4.42. Consider the second integral on the right. This integral is no larger than

$$\int_{|\mathbf{x}| \geq 2|\mathbf{y}|, |\mathbf{x}| \geq \varepsilon, |\mathbf{x} - \mathbf{y}| < \varepsilon} B |\mathbf{x}|^{-n} dx.$$

Now $|\mathbf{x}| \leq |\mathbf{y}| + \varepsilon \leq |\mathbf{x}|/2 + \varepsilon$ and so $|\mathbf{x}| < 2\varepsilon$. Thus this is no larger than

$$\int_{\varepsilon \leq |\mathbf{x}| \leq 2\varepsilon} B |\mathbf{x}|^{-n} dx = B \int_{S^{n-1}} \int_{\varepsilon}^{2\varepsilon} \rho^{n-1} \frac{1}{\rho^n} d\rho d\sigma \leq BC(n) \ln 2 = C(n) B.$$

It remains to estimate the first integral on the right in 22.4.46. This integral is bounded by

$$\int_{|\mathbf{x}| \geq 2|\mathbf{y}|, |\mathbf{x} - \mathbf{y}| > \varepsilon, |\mathbf{x}| < \varepsilon} B |\mathbf{x} - \mathbf{y}|^{-n} dx$$

In the integral above, $|\mathbf{x}| < \varepsilon$ and so $|\mathbf{x} - \mathbf{y}| - |\mathbf{y}| < \varepsilon$. Therefore, $|\mathbf{x} - \mathbf{y}| < \varepsilon + |\mathbf{y}| < \varepsilon + |\mathbf{x}|/2 < \varepsilon + \varepsilon/2 = (3/2)\varepsilon$. Hence $\varepsilon \leq |\mathbf{x} - \mathbf{y}| \leq (3/2)|\mathbf{x} - \mathbf{y}|$. Therefore, the above integral is no larger than

$$\int_{\varepsilon}^{(3/2)\varepsilon} B |\mathbf{z}|^{-n} dz = B \int_{S^{n-1}} \int_{\varepsilon}^{(3/2)\varepsilon} \rho^{-1} d\rho d\sigma = BC(n) \ln(3/2).$$

This establishes 22.4.44.

Now it remains to show 22.4.45, a statement about the Fourier transforms of K_ε . Fix ε and let $\mathbf{y} \neq \mathbf{0}$ also be given.

$$K_{\varepsilon R}(\mathbf{y}) \equiv \begin{cases} K_\varepsilon(\mathbf{y}) & \text{if } |\mathbf{y}| < R, \\ 0 & \text{if } |\mathbf{y}| \geq R \end{cases}$$

where $R > \frac{3\pi}{|\mathbf{y}|}$. (The 3 here isn't important. It just needs to be larger than 1.) Then

$$|FK_{\varepsilon R}(\mathbf{y})| \leq \left| \int_{0 < |\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \right| + \left| \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \right|$$

$$= \mathbf{A} + \mathbf{B}. \quad (22.4.47)$$

Consider **A**. By 22.4.41

$$\int_{\varepsilon < |\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}) dx = 0$$

and so

$$\mathbf{A} = \left| \int_{\varepsilon < |\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}) (e^{-i\mathbf{x} \cdot \mathbf{y}} - 1) dx \right|$$

Now

$$|e^{-i\mathbf{x} \cdot \mathbf{y}} - 1| = |2 - 2 \cos(\mathbf{x} \cdot \mathbf{y})|^{1/2} \leq 2 |\mathbf{x} \cdot \mathbf{y}| \leq 2 |\mathbf{x}| |\mathbf{y}|$$

so, using polar coordinates, this expression is no larger than

$$2B \int_{\varepsilon < |\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} |\mathbf{x}|^{-n} |\mathbf{x}| |\mathbf{y}| dx \leq C(n) B |\mathbf{y}| \int_{\varepsilon}^{3\pi/|\mathbf{y}|} d\rho \leq BC(n).$$

Next, consider **B**. This estimate is based on the trick which follows. Let

$$\mathbf{z} \equiv \mathbf{y}\pi / |\mathbf{y}|^2$$

so that

$$|\mathbf{z}| = \pi / |\mathbf{y}|, \quad \mathbf{z} \cdot \mathbf{y} = \pi.$$

Then

$$\begin{aligned} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} dx &= \frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} dx \\ &\quad - \frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i(\mathbf{x}+\mathbf{z}) \cdot \mathbf{y}} dx. \end{aligned} \quad (22.4.48)$$

Here is why. Note in the second of these integrals,

$$\begin{aligned} & -\frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i(\mathbf{x}+\mathbf{z}) \cdot \mathbf{y}} dx \\ &= -\frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} e^{-i\mathbf{z} \cdot \mathbf{y}} dx \\ &= -\frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} e^{-i\pi} dx \\ &= \frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_{\varepsilon}(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} dx. \end{aligned}$$

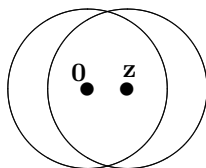
Then changing the variables in 22.4.48,,

$$\begin{aligned}
 & \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\
 = & \frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\
 & - \frac{1}{2} \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}-\mathbf{z}| \leq R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} = \\
 & \frac{1}{2} \int_{|\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} - \frac{1}{2} \int_{|\mathbf{x}-\mathbf{z}| \leq R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\
 & + \frac{1}{2} \int_{|\mathbf{x}-\mathbf{z}| \leq 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} - \frac{1}{2} \int_{|\mathbf{x}| \leq 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x}.
 \end{aligned} \tag{22.4.49}$$

Since $|\mathbf{z}| = \pi/|\mathbf{y}|$, it follows $|\mathbf{z}| = \frac{\pi}{|\mathbf{y}|} < \frac{3\pi}{|\mathbf{y}|} < R$ and so the following picture describes the situation. In this picture, the radius of each ball equals either R or $3\pi|\mathbf{y}|^{-1}$ and each integral above is taken over one of the two balls in the picture, either the one centered at $\mathbf{0}$ or the one centered at $\mathbf{z}$.



To begin with, consider the integrals which involve $K_\varepsilon(\mathbf{x}-\mathbf{z})$.

$$\begin{aligned}
 & \int_{|\mathbf{x}-\mathbf{z}| \leq R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\
 = & \int_{|\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\
 - & \int_{|\mathbf{x}-\mathbf{z}| > R, |\mathbf{x}| < R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \\
 + & \int_{|\mathbf{x}-\mathbf{z}| < R, |\mathbf{x}| > R} K_\varepsilon(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x}.
 \end{aligned} \tag{22.4.50}$$

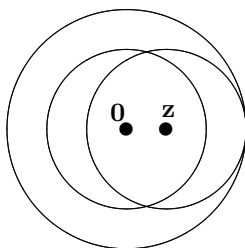
Look at the picture. Similarly,

$$\begin{aligned}
 & \int_{|\mathbf{x}-\mathbf{z}| \leq 3\pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{x} \\
 &= \int_{|\mathbf{x}| \leq 3\pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{x} \\
 - & \int_{|\mathbf{x}-\mathbf{z}| > 3\pi|\mathbf{y}|^{-1}, |\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{x} + \\
 & \int_{|\mathbf{x}-\mathbf{z}| < 3\pi|\mathbf{y}|^{-1}, |\mathbf{x}| > 3\pi|\mathbf{y}|^{-1}} K_{\varepsilon}(\mathbf{x}-\mathbf{z}) e^{-i\mathbf{x}\cdot\mathbf{y}} d\mathbf{x}.
 \end{aligned} \tag{22.4.51}$$

The last integral in 22.4.50 is taken over a set that is contained in

$$B(\mathbf{0}, R + |\mathbf{z}|) \setminus B(\mathbf{0}, R)$$

illustrated in the following picture as the region between the small ball centered at $\mathbf{0}$ and the big ball which surrounds the two small balls



and so this integral is dominated by

$$B\left(\frac{1}{(R - |\mathbf{z}|)^n}\right) \alpha(n) ((R + |\mathbf{z}|)^n - R^n),$$

an expression which converges to 0 as $R \rightarrow \infty$. Similarly, the second integral on the right in 22.4.50 converges to zero as $R \rightarrow \infty$. Now consider the last two integrals in 22.4.51. Letting $3\pi|\mathbf{y}|^{-1}$ play the role of R and using $|\mathbf{z}| = \pi/|\mathbf{y}|$, these are each dominated by an expression of the form

$$\begin{aligned}
 & B\left(\frac{1}{(3\pi|\mathbf{y}|^{-1} - |\mathbf{z}|)^n}\right) \alpha(n) \left((3\pi|\mathbf{y}|^{-1} + |\mathbf{z}|)^n - (3\pi|\mathbf{y}|^{-1})^n \right) \\
 &= B\left(\frac{1}{(3\pi|\mathbf{y}|^{-1} - \pi|\mathbf{y}|^{-1})^n}\right) \alpha(n) \cdot \\
 & \quad \left((3\pi|\mathbf{y}|^{-1} + \pi|\mathbf{y}|^{-1})^n - (3\pi|\mathbf{y}|^{-1})^n \right)
 \end{aligned}$$

$$= \alpha(n) B \frac{|\mathbf{y}|^n}{(2\pi)^n |\mathbf{y}|^n} \frac{1}{((4\pi)^n - (3\pi)^n)} = C(n) B.$$

Returning to 22.4.49, the terms involving $\mathbf{x} - \mathbf{y}$ have now been estimated. Thus, collecting the terms which have not yet been estimated along with those that have,

$$\begin{aligned} \mathbf{B} &= \left| \int_{3\pi|\mathbf{y}|^{-1} < |\mathbf{x}| \leq R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \right| \\ &\leq \frac{1}{2} \left| \int_{|\mathbf{x}| < R} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} - \int_{|\mathbf{x}| < R} K_\varepsilon(\mathbf{x} - \mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \right. \\ &\quad \left. + \int_{|\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x} - \mathbf{z}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} - \int_{|\mathbf{x}| < 3\pi|\mathbf{y}|^{-1}} K_\varepsilon(\mathbf{x}) e^{-i\mathbf{x} \cdot \mathbf{y}} d\mathbf{x} \right| \\ &\quad + C(n) B + g(R) \end{aligned}$$

where $g(R) \rightarrow 0$ as $R \rightarrow \infty$. Using $|\mathbf{z}| = \pi/|\mathbf{y}|$ again,

$$\mathbf{B} \leq \frac{1}{2} \int_{3|\mathbf{z}| < |\mathbf{x}| < R} |K_\varepsilon(\mathbf{x}) - K_\varepsilon(\mathbf{x} - \mathbf{z})| d\mathbf{x} + C(n) B + g(R).$$

But the integral in the above is dominated by $C(n) B$ by 22.4.44 which was established earlier. Therefore, from 22.4.47,

$$|FK_{\varepsilon R}| \leq C(n) B + g(R)$$

where $g(R) \rightarrow 0$.

Now $K_{\varepsilon R} \rightarrow K_\varepsilon$ in $L^2(\mathbb{R}^n)$ because

$$\begin{aligned} \|K_{\varepsilon R} - K_\varepsilon\|_{L^2(\mathbb{R}^n)} &\leq B \int_{|\mathbf{x}| > R} \frac{1}{|\mathbf{x}|^{2n}} d\mathbf{x} \\ &= B \int_{S^{n-1}} \int_R^\infty \frac{1}{\rho^{n+1}} d\rho d\sigma, \end{aligned}$$

which converges to 0 as $R \rightarrow \infty$ and so $FK_{\varepsilon R} \rightarrow FK_\varepsilon$ in $L^2(\mathbb{R}^n)$ by Plancherel's theorem. Therefore, by taking a subsequence, still denoted by R , $FK_{\varepsilon R}(\mathbf{y}) \rightarrow FK_\varepsilon(\mathbf{y})$ a.e. which shows

$$|FK_\varepsilon(\mathbf{y})| \leq C(n) B \text{ a.e.}$$

This proves the lemma.

Corollary 22.4.4 Suppose 22.4.40 - 22.4.42 hold. Then if $g \in C_c^1(\mathbb{R}^n)$, $K_\varepsilon * g$ converges uniformly and in $L^p(\mathbb{R}^n)$ as $\varepsilon \rightarrow 0$.

Proof:

$$K_\varepsilon * g(\mathbf{x}) \equiv \int K_\varepsilon(\mathbf{y}) g(\mathbf{x} - \mathbf{y}) dy.$$

Let $0 < \eta < \varepsilon$. Then since $g \in C_c^1(\mathbb{R}^n)$, there exists a constant, K such that $K|\mathbf{u} - \mathbf{v}| \geq |g(\mathbf{u}) - g(\mathbf{v})|$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

$$\begin{aligned} |K_\varepsilon * g(\mathbf{x}) - K_\eta * g(\mathbf{x})| &\leq BK \int_{\eta < |\mathbf{y}| < \varepsilon} \frac{1}{|\mathbf{y}|^n} |\mathbf{y}| dy \\ &= BK \int_{S^{n-1}} \int_\eta^\varepsilon d\rho d\sigma = C_n |\varepsilon - \eta|. \end{aligned}$$

This proves the corollary.

Theorem 22.4.5 Suppose 22.4.40 - 22.4.42. Then for K_ε given by 22.4.43 and $p > 1$, there exists a constant $A(p, n, B)$ such that for all $f \in L^p(\mathbb{R}^n)$,

$$\|K_\varepsilon * f\|_p \leq A(p, n, B) \|f\|_p. \quad (22.4.52)$$

Also, for each $f \in L^p(\mathbb{R}^n)$,

$$Tf \equiv \lim_{\varepsilon \rightarrow 0} K_\varepsilon * f \quad (22.4.53)$$

exists in $L^p(\mathbb{R}^n)$ and for all $f \in L^p(\mathbb{R}^n)$,

$$\|Tf\|_p \leq A(p, n, B) \|f\|_p. \quad (22.4.54)$$

Thus T is a linear and continuous map defined on $L^p(\mathbb{R}^n)$ for each $p > 1$.

Proof: From 22.4.40 it follows $K_\varepsilon \in L^{p'}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ where, as usual, $1/p + 1/p' = 1$. By continuity of translation in $L^{p'}(\mathbb{R}^n)$, $x \rightarrow K_\varepsilon * f(x)$ is a continuous function. By Lemma 22.4.3, $\|FK_\varepsilon\|_\infty \leq C(n)B$ for all ε . Therefore, by Lemma 22.4.2,

$$\|K_\varepsilon * g\|_p \leq A(p, n, B) \|g\|_p$$

for all $g \in \mathcal{G}$. Now let $f \in L^p(\mathbb{R}^n)$ and $g_k \rightarrow f$ in $L^p(\mathbb{R}^n)$ where $g_k \in \mathcal{G}$. Then

$$\begin{aligned} |K_\varepsilon * f(\mathbf{x}) - K_\varepsilon * g_k(\mathbf{x})| &\leq \int |K_\varepsilon(\mathbf{x} - \mathbf{y})| |g_k(\mathbf{y}) - f(\mathbf{y})| dy \\ &\leq \|K_\varepsilon\|_{p'} \|g_k - f\|_p \end{aligned}$$

which shows that $K_\varepsilon * g_k(\mathbf{x}) \rightarrow K_\varepsilon * f(\mathbf{x})$ pointwise and so by Fatou's lemma,

$$\begin{aligned} \|K_\varepsilon * f\|_p &\leq \liminf_{k \rightarrow \infty} \|K_\varepsilon * g_k\|_p \leq \liminf_{k \rightarrow \infty} A(p, n, B) \|g_k\|_p \\ &= A(p, n, B) \|f\|_p. \end{aligned}$$

This verifies 22.4.52.

To verify 22.4.53, let $\delta > 0$ be given and let

$$f \in L^p(\mathbb{R}^n), g \in C_c^\infty(\mathbb{R}^n).$$

$$\begin{aligned}
\|K_\varepsilon * f - K_\eta * f\|_p &\leq \|K_\varepsilon * (f - g)\|_p + \|K_\varepsilon * g - K_\eta * g\|_p \\
&\quad + \|K_\eta * (f - g)\|_p \\
&\leq 2A(p, n, B) \|f - g\|_p + \|K_\varepsilon * g - K_\eta * g\|_p.
\end{aligned}$$

Choose g such that $2A(p, n, B) \|f - g\|_p \leq \delta/2$. Then if ε, η are small enough, Corollary 22.4.4 implies the last term is also less than $\delta/2$. Thus, $\lim_{\varepsilon \rightarrow 0} K_\varepsilon * f$ exists in $L^p(\mathbb{R}^n)$. Let Tf be the element of $L^p(\mathbb{R}^n)$ to which it converges. Then 22.4.54 follows and T is obviously linear because

$$\begin{aligned}
T(af + bg) &= \lim_{\varepsilon \rightarrow 0} K_\varepsilon * (af + bg) = \lim_{\varepsilon \rightarrow 0} (aK_\varepsilon * f + bK_\varepsilon * g) \\
&= aTf + bTg.
\end{aligned}$$

This proves the theorem.

When do conditions 22.4.40-22.4.42 hold? It turns out this happens for K given by the following.

$$K(\mathbf{x}) \equiv \frac{\Omega(\mathbf{x})}{|\mathbf{x}|^n}, \quad (22.4.55)$$

where

$$\Omega(\lambda \mathbf{x}) = \Omega(\mathbf{x}) \text{ for all } \lambda > 0, \quad (22.4.56)$$

$$\Omega \text{ is Lipschitz on } S^{n-1},$$

$$\int_{S^{n-1}} \Omega(\mathbf{x}) d\sigma = 0. \quad (22.4.57)$$

Theorem 22.4.6 *For K given by 22.4.55 - 22.4.57, it follows there exists a constant B such that*

$$|K(\mathbf{x})| \leq B |\mathbf{x}|^{-n}, \quad (22.4.58)$$

$$\int_{a < |\mathbf{x}| < b} K(\mathbf{x}) dx = 0, \quad (22.4.59)$$

$$\int_{|\mathbf{x}| > 2|\mathbf{y}|} |K(\mathbf{x} - \mathbf{y}) - K(\mathbf{x})| dx \leq B. \quad (22.4.60)$$

Consequently, the conclusions of Theorem 22.4.5 hold also.

Proof: 22.4.58 is obvious. To verify 22.4.59,

$$\begin{aligned}
\int_{a < |\mathbf{x}| < b} K(\mathbf{x}) dx &= \int_a^b \int_{S^{n-1}} \frac{\Omega(\rho \mathbf{w})}{\rho^n} \rho^{n-1} d\sigma d\rho \\
&= \int_a^b \frac{1}{\rho} \int_{S^{n-1}} \Omega(\mathbf{w}) d\sigma d\rho = 0.
\end{aligned}$$

It remains to show 22.4.60.

$$\begin{aligned}
K(\mathbf{x} - \mathbf{y}) - K(\mathbf{x}) &= |\mathbf{x} - \mathbf{y}|^{-n} \left(\Omega\left(\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\right) - \Omega\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \right) \\
&\quad + \Omega(\mathbf{x}) \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^n} - \frac{1}{|\mathbf{x}|^n} \right)
\end{aligned} \quad (22.4.61)$$

where 22.4.56 was used to write $\Omega\left(\frac{\mathbf{z}}{|\mathbf{z}|}\right) = \Omega(\mathbf{z})$. The first group of terms in 22.4.61 is dominated by

$$|\mathbf{x} - \mathbf{y}|^{-n} \text{Lip}(\Omega) \left| \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|} - \frac{\mathbf{x}}{|\mathbf{x}|} \right|$$

and an estimate is required for $|\mathbf{x}| > 2|\mathbf{y}|$. Since $|\mathbf{x}| > 2|\mathbf{y}|$,

$$|\mathbf{x} - \mathbf{y}|^{-n} \leq (|\mathbf{x}| - |\mathbf{y}|)^{-n} \leq \frac{2^n}{|\mathbf{x}|^n}.$$

Also

$$\begin{aligned} \left| \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|} - \frac{\mathbf{x}}{|\mathbf{x}|} \right| &= \left| \frac{(\mathbf{x} - \mathbf{y})|\mathbf{x}| - \mathbf{x}|\mathbf{x} - \mathbf{y}|}{|\mathbf{x}||\mathbf{x} - \mathbf{y}|} \right| \\ &\leq \left| \frac{(\mathbf{x} - \mathbf{y})|\mathbf{x}| - \mathbf{x}|\mathbf{x} - \mathbf{y}|}{|\mathbf{x}|(|\mathbf{x}| - |\mathbf{y}|)} \right| \leq \left| \frac{(\mathbf{x} - \mathbf{y})|\mathbf{x}| - \mathbf{x}|\mathbf{x} - \mathbf{y}|}{|\mathbf{x}|(|\mathbf{x}|/2)} \right| \\ &= \frac{2}{|\mathbf{x}|^2} |\mathbf{x}|\mathbf{x} - \mathbf{y}|\mathbf{x}| - \mathbf{x}|\mathbf{x} - \mathbf{y}| = \frac{2}{|\mathbf{x}|^2} |\mathbf{x}||\mathbf{x}| - |\mathbf{x} - \mathbf{y}| - \mathbf{y}|\mathbf{x}| \\ &\leq \frac{2}{|\mathbf{x}|^2} |\mathbf{x}||\mathbf{x}| - |\mathbf{x} - \mathbf{y}| + |\mathbf{y}||\mathbf{x}| \leq \frac{2}{|\mathbf{x}|^2} (|\mathbf{x}||\mathbf{x}| - |\mathbf{x} - \mathbf{y}|) + |\mathbf{y}||\mathbf{x}| \\ &\leq \frac{4}{|\mathbf{x}|^2} |\mathbf{x}||\mathbf{y}| = 4 \frac{|\mathbf{y}|}{|\mathbf{x}|}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{|\mathbf{x}| > 2|\mathbf{y}|} |\mathbf{x} - \mathbf{y}|^{-n} \left| \Omega\left(\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|}\right) - \Omega\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \right| dx \\ &\leq 4(2^n) \int_{|\mathbf{x}| > 2|\mathbf{y}|} \frac{1}{|\mathbf{x}|^n} \frac{|\mathbf{y}|}{|\mathbf{x}|} dx \text{Lip}(\Omega) \\ &= C(n, \text{Lip} \Omega) \int_{|\mathbf{x}| > 2|\mathbf{y}|} \frac{|\mathbf{y}|}{|\mathbf{x}|^{n+1}} dx \\ &= C(n, \text{Lip} \Omega) \int_{|\mathbf{u}| > 2} \frac{1}{|\mathbf{u}|^{n+1}} du. \end{aligned} \tag{22.4.62}$$

It remains to consider the second group of terms in 22.4.61 when $|\mathbf{x}| > 2|\mathbf{y}|$.

$$\begin{aligned} \left| \frac{1}{|\mathbf{x} - \mathbf{y}|^n} - \frac{1}{|\mathbf{x}|^n} \right| &= \left| \frac{|\mathbf{x}|^n - |\mathbf{x} - \mathbf{y}|^n}{|\mathbf{x} - \mathbf{y}|^n |\mathbf{x}|^n} \right| \\ &\leq \frac{2^n}{|\mathbf{x}|^{2n}} ||\mathbf{x}|^n - |\mathbf{x} - \mathbf{y}|^n| \\ &\leq \frac{2^n}{|\mathbf{x}|^{2n}} |\mathbf{y}| \left[|\mathbf{x}|^{n-1} + |\mathbf{x}|^{n-2} |\mathbf{x} - \mathbf{y}| + \right. \\ &\quad \left. \cdots + |\mathbf{x}||\mathbf{x} - \mathbf{y}|^{n-2} + |\mathbf{x} - \mathbf{y}|^{n-1} \right] \end{aligned}$$

$$\leq \frac{2^n |\mathbf{y}| C(n) |\mathbf{x}|^{n-1}}{|\mathbf{x}|^{2n}} = \frac{C(n) 2^n |\mathbf{y}|}{|\mathbf{x}|^{n+1}}.$$

Thus

$$\begin{aligned} & \int_{|\mathbf{x}| > 2|\mathbf{y}|} \left| \Omega(\mathbf{x}) \left(\frac{1}{|\mathbf{x} - \mathbf{y}|^n} - \frac{1}{|\mathbf{x}|^n} \right) \right| dx \\ & \leq C(n) \int_{|\mathbf{x}| > 2|\mathbf{y}|} \frac{|\mathbf{y}|}{|\mathbf{x}|^{n+1}} dx \\ & \leq C(n) \int_{|u| > 2} \frac{1}{|u|^{n+1}} du. \end{aligned} \quad (22.4.63)$$

From 22.4.62 and 22.4.63,

$$\int_{|\mathbf{x}| > 2|\mathbf{y}|} |K(\mathbf{x} - \mathbf{y}) - K(\mathbf{x})| dx \leq C(n, \text{Lip } \Omega).$$

This proves the theorem.

22.5 Helmholtz Decompositions

It turns out that every vector field which has its components in L^p can be written as a sum of a gradient and a vector field which has zero divergence. This is a very remarkable result, especially when applied to vector fields which are only in L^p . Recall that for u a function of n variables, $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$.

Definition 22.5.1 Define

$$\Phi(\mathbf{y}) \equiv \begin{cases} -\frac{1}{a_1} \ln |\mathbf{y}|, & \text{if } n = 2, \\ \frac{1}{(n-2)a_{n-1}} |\mathbf{y}|^{2-n}, & \text{if } n > 2. \end{cases}$$

where a_k denotes the area of the unit sphere, S^k .

Then it is routine to verify $\Delta \Phi = 0$ away from 0. In fact, if $n > 2$,

$$\Phi_{,ii}(\mathbf{y}) = C_n \left[\frac{1}{|\mathbf{y}|^n} - n \frac{y_i^2}{|\mathbf{y}|^{n+2}} \right], \quad \Phi_{,ij}(\mathbf{y}) = C_n \frac{y_i y_j}{|\mathbf{y}|^{n+2}}, \quad (22.5.64)$$

while if $n = 2$,

$$\begin{aligned} \Phi_{,22}(\mathbf{y}) &= C_2 \frac{y_1^2 - y_2^2}{(y_1^2 + y_2^2)^2}, \quad \Phi_{,11}(\mathbf{y}) = C_2 \frac{y_2^2 - y_1^2}{(y_1^2 + y_2^2)^2}, \\ \Phi_{,ij}(\mathbf{y}) &= C_2 \frac{y_1 y_2}{(y_1^2 + y_2^2)^2}. \end{aligned}$$

Also,

$$\nabla \Phi(\mathbf{y}) = \frac{-\mathbf{y}}{a_{n-1} |\mathbf{y}|^n}. \quad (22.5.65)$$

In the above the subscripts following a comma denote partial derivatives.

Lemma 22.5.2 For $n \geq 2$

$$\Phi_{,ij}(\mathbf{y}) = \frac{\Omega_{ij}(\mathbf{y})}{|\mathbf{y}|^n}$$

where

$$\Omega_{ij} \text{ is Lipschitz continuous on } S^{n-1}, \quad (22.5.66)$$

$$\Omega_{ij}(\lambda \mathbf{y}) = \Omega_{ij}(\mathbf{y}), \quad (22.5.67)$$

for all $\lambda > 0$, and

$$\int_{S^{n-1}} \Omega_{ij}(\mathbf{y}) d\sigma = 0. \quad (22.5.68)$$

Proof:

Proof: The case $n = 2$ is left to the reader. 22.5.66 and 22.5.67 are obvious from the above descriptions. It remains to verify 22.5.68. If $n \geq 3$ and $i \neq j$, then this formula is also clear from 22.5.64. Thus consider the case when $n \geq 3$ and $i = j$. By symmetry,

$$I \equiv \int_{S^{n-1}} 1 - ny_i^2 d\sigma = \int_{S^{n-1}} 1 - ny_j^2 d\sigma.$$

Hence

$$\begin{aligned} nI &= \sum_{i=1}^n \int_{S^{n-1}} 1 - ny_i^2 d\sigma = \int_{S^{n-1}} \left(n - n \sum_i y_i^2 \right) d\sigma \\ &= \int_{S^{n-1}} (n - n) d\sigma = 0. \end{aligned}$$

This proves the lemma.

Let U be a bounded open set locally on one side of its boundary having Lipschitz boundary so the divergence theorem holds and let $B = B(\mathbf{0}, R)$ where

$$B \supseteq U - U \equiv \{\mathbf{x} - \mathbf{y} : \mathbf{x} \in U, \mathbf{y} \in U\}$$

Let $f \in C_c^\infty(U)$ and define for $\mathbf{x} \in U$,

$$u(\mathbf{x}) \equiv \int_B \Phi(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) dy = \int_U \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dy.$$

Let $h(\mathbf{y}) = f(\mathbf{x} - \mathbf{y})$. Then since Φ is in $L^1(B)$,

$$\begin{aligned} \Delta u(\mathbf{x}) &= \int_B \Phi(\mathbf{y}) \Delta f(\mathbf{x} - \mathbf{y}) dy = \int_B \Phi(\mathbf{y}) \Delta h(\mathbf{y}) dy \\ &= \int_{B \setminus B(\mathbf{0}, \varepsilon)} \nabla \cdot (\nabla h(\mathbf{y}) \Phi(\mathbf{y})) - \nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) dy \\ &\quad + \int_{B(\mathbf{0}, \varepsilon)} \Phi(\mathbf{y}) \Delta h(\mathbf{y}) dy. \end{aligned}$$

The last term converges to 0 as $\varepsilon \rightarrow 0$ because Φ is in L^1 and Δh is bounded. Since $\text{spt}(h) \subseteq B$, the divergence theorem implies

$$\Delta u(\mathbf{x}) = - \int_{\partial B(\mathbf{0}, \varepsilon)} \Phi(\mathbf{y}) \nabla h(\mathbf{y}) \cdot \mathbf{n} d\sigma - \int_{B \setminus B(\mathbf{0}, \varepsilon)} \nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) dy + e(\varepsilon) \quad (22.5.69)$$

where here and below, $e(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The first term in 22.5.69 converges to 0 as $\varepsilon \rightarrow 0$ because

$$\left| \int_{\partial B(\mathbf{0}, \varepsilon)} \Phi(\mathbf{y}) \nabla h(\mathbf{y}) \cdot \mathbf{n} d\sigma \right| \leq \begin{cases} C_{nh} \frac{1}{\varepsilon^{n-2}} \varepsilon^{n-1} = C_{nh} \varepsilon & \text{if } n > 2 \\ C_h (\ln \varepsilon) \varepsilon & \text{if } n = 2 \end{cases}$$

and since $\Delta \Phi(\mathbf{y}) = 0$,

$$\nabla \Phi(\mathbf{y}) \cdot \nabla h(\mathbf{y}) = \nabla \cdot (\nabla \Phi(\mathbf{y}) h(\mathbf{y})).$$

Consequently

$$\Delta u(\mathbf{x}) = - \int_{B \setminus B(\mathbf{0}, \varepsilon)} \nabla \cdot (\nabla \Phi(\mathbf{y}) h(\mathbf{y})) dy + e(\varepsilon).$$

Thus, by the divergence theorem, 22.5.65, and the definition of h above,

$$\begin{aligned} \Delta u(\mathbf{x}) &= \int_{\partial B(\mathbf{0}, \varepsilon)} f(\mathbf{x} - \mathbf{y}) \nabla \Phi(\mathbf{y}) \cdot \mathbf{n} d\sigma + e(\varepsilon) \\ &= \int_{\partial B(\mathbf{0}, \varepsilon)} f(\mathbf{x} - \mathbf{y}) \left(-\frac{\mathbf{y}}{a_{n-1} |\mathbf{y}|^n} \right) \cdot \left(-\frac{\mathbf{y}}{|\mathbf{y}|} \right) d\sigma + e(\varepsilon) \\ &= - \left(\int_{\partial B(\mathbf{0}, \varepsilon)} f(\mathbf{x} - \mathbf{y}) d\sigma(y) \right) \frac{1}{a_{n-1} \varepsilon^{n-1}} + e(\varepsilon). \end{aligned}$$

Letting $\varepsilon \rightarrow 0$,

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}).$$

This proves the following lemma.

Lemma 22.5.3 *Let U be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary and let $B \supseteq U - U$ where $B = B(\mathbf{0}, R)$. Let $f \in C_c^\infty(U)$. Then for $\mathbf{x} \in U$,*

$$\int_B \Phi(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) dy = \int_U \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dy,$$

and it follows that if u is given by one of the above formulas, then for all $x \in U$,

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}).$$

Theorem 22.5.4 *Let $f \in L^p(U)$. Then there exists $u \in L^p(U)$ whose weak derivatives are also in $L^p(U)$ such that in the sense of weak derivatives,*

$$-\Delta u = f.$$

It is given by

$$u(\mathbf{x}) = \int_B \Phi(\mathbf{y}) \tilde{f}(\mathbf{x} - \mathbf{y}) dy = \int_U \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) dy \quad (22.5.70)$$

where $\tilde{f}$ denotes the zero extension of f off of U .

Proof: Let $f \in L^p(U)$ and let $f_k \in C_c^\infty(U)$, $\|f_k - f\|_{L^p(U)} \rightarrow 0$, and let u_k be given by 22.5.70 with f_k in place of f . Then by Minkowski's inequality,

$$\begin{aligned} \|u - u_k\|_{L^p(U)} &= \left(\int_U \left(\int_B \Phi(\mathbf{y}) \left| \tilde{f}(\mathbf{x} - \mathbf{y}) - f_k(\mathbf{x} - \mathbf{y}) \right| dy \right)^p dx \right)^{1/p} \\ &\leq \left(\int_B |\Phi(\mathbf{y})| \left(\int_U \left| \tilde{f}(\mathbf{x} - \mathbf{y}) - f_k(\mathbf{x} - \mathbf{y}) \right|^p dx \right)^{1/p} dy \right) \\ &\leq \int_B |\Phi(\mathbf{y})| dy \|f - f_k\|_{L^p(U)} = C(B) \|f - f_k\|_{L^p(U)} \end{aligned}$$

and so $u_k \rightarrow u$ in $L^p(U)$. Also

$$u_{k,i}(\mathbf{x}) = \int_U \Phi_{,i}(\mathbf{x} - \mathbf{y}) f_k(\mathbf{y}) dy = \int_B f_k(\mathbf{x} - \mathbf{y}) \Phi_{,i}(\mathbf{y}) dy.$$

Now let

$$w_i \equiv \int_B \tilde{f}(\mathbf{x} - \mathbf{y}) \Phi_{,i}(\mathbf{y}) dy. \quad (22.5.71)$$

and since $\Phi_{,i} \in L^1(B)$, it follows from Minkowski's inequality that

$$\begin{aligned} \|u_{k,i} - w_i\|_{L^p(U)} &\leq \left(\int_U \left(\int_B \left| f_k(\mathbf{x} - \mathbf{y}) - \tilde{f}(\mathbf{x} - \mathbf{y}) \right| |\Phi_{,i}(\mathbf{y})| dy \right)^p dx \right)^{1/p} \\ &\leq \int_B |\Phi_{,i}(\mathbf{y})| \left(\int_U \left| f_k(\mathbf{x} - \mathbf{y}) - \tilde{f}(\mathbf{x} - \mathbf{y}) \right|^p dx \right)^{1/p} dy \\ &\leq C(B) \|f_k - f\|_{L^p(U)} \end{aligned}$$

and so $u_{k,i} \rightarrow w_i$ in $L^p(U)$.

Now let $\phi \in C_c^\infty(U)$. Then

$$\int_U w_i \phi dx = - \lim_{k \rightarrow \infty} \int_U u_k \phi_{,i} dx = - \int_U u \phi_{,i} dx.$$

Thus $u_{,i} = w_i \in L^p(\mathbb{R}^n)$ and so if $\phi \in C_c^\infty(U)$,

$$\int_U f \phi dx = \lim_{k \rightarrow \infty} \int_U f_k \phi dx = \lim_{k \rightarrow \infty} \int_U \nabla u_k \cdot \nabla \phi dx = \int_U \nabla u \cdot \nabla \phi dx$$

and so $-\Delta u = f$ as claimed. This proves the theorem.

One could also ask whether the second weak partial derivatives of u are in $L^p(U)$. This is where the theory singular integrals is used. Recall from 22.5.70 and 22.5.71 along with the argument of the above lemma, that if u is given by 22.5.70, then u_i is given by 22.5.71 which equals

$$\int_U \Phi_{,i}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}.$$

Lemma 22.5.5 *Let $f \in L^p(U)$ and let*

$$w_i(\mathbf{x}) \equiv \int_U \Phi_{,i}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}.$$

Then $w_{i,j} \in L^p(U)$ for each $j = 1 \cdots n$ and the map $f \rightarrow w_{i,j}$ is continuous and linear on $L^p(U)$.

Proof: First let $f \in C_c^\infty(U)$. For such f ,

$$\begin{aligned} w_i(\mathbf{x}) &= \int_U \Phi_{,i}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^n} \Phi_{,i}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} \\ &= \int_{\mathbb{R}^n} \Phi_{,i}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y} = \int_B \Phi_{,i}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y} \end{aligned}$$

and

$$\begin{aligned} w_{i,j}(\mathbf{x}) &= \int_B \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x} - \mathbf{y}) d\mathbf{y} \\ &= \int_{B \setminus B(\mathbf{0}, \varepsilon)} \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x} - \mathbf{y}) d\mathbf{y} + \int_{B(\mathbf{0}, \varepsilon)} \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x} - \mathbf{y}) d\mathbf{y}. \end{aligned}$$

The second term converges to 0 because $f_{,j}$ is bounded and by 22.5.65, $\Phi_{,i} \in L^1_{loc}$. Thus

$$\begin{aligned} w_{i,j}(\mathbf{x}) &= \int_{B \setminus B(\mathbf{0}, \varepsilon)} \Phi_{,i}(\mathbf{y}) f_{,j}(\mathbf{x} - \mathbf{y}) d\mathbf{y} + e(\varepsilon) \\ &= \int_{B \setminus B(\mathbf{0}, \varepsilon)} -(\Phi_{,i}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}))_{,j} + \Phi_{,ij}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y} + e(\varepsilon) \end{aligned}$$

where $e(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Using the divergence theorem, this yields

$$w_{i,j}(\mathbf{x}) = \int_{\partial B(\mathbf{0}, \varepsilon)} \Phi_{,i}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) n_j d\sigma + \int_{B \setminus B(\mathbf{0}, \varepsilon)} \Phi_{,ij}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y} + e(\varepsilon).$$

Consider the first term on the right. This term equals, after letting $\mathbf{y} = \varepsilon \mathbf{z}$,

$$\begin{aligned} \varepsilon^{n-1} \int_{\partial B(\mathbf{0}, 1)} \Phi_{,i}(\varepsilon \mathbf{z}) f(\mathbf{x} - \varepsilon \mathbf{z}) n_j d\sigma &= C_n \varepsilon^{n-1} \int_{\partial B(\mathbf{0}, 1)} \varepsilon^{1-n} z_i z_j f(\mathbf{x} - \varepsilon \mathbf{z}) d\sigma(z) \\ &= C_n \int_{\partial B(\mathbf{0}, 1)} z_i z_j f(\mathbf{x} - \varepsilon \mathbf{z}) d\sigma(z) \end{aligned}$$

and this converges to 0 if $i \neq j$ and it converges to

$$C_n f(\mathbf{x}) \int_{\partial B(\mathbf{0},1)} z_i^2 d\sigma(z)$$

if $i = j$. Thus

$$w_{i,j}(\mathbf{x}) = C_n \delta_{ij} f(\mathbf{x}) + \int_{B \setminus B(\mathbf{0},\varepsilon)} \Phi_{,ij}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) dy + e(\varepsilon).$$

Letting

$$\Phi_{ij}^\varepsilon \equiv \begin{cases} 0 & \text{if } |\mathbf{y}| < \varepsilon, \\ \Phi_{,ij}(\mathbf{y}) & \text{if } |\mathbf{y}| \geq \varepsilon, \end{cases}$$

it follows

$$w_{i,j}(\mathbf{x}) = C_n \delta_{ij} f(\mathbf{x}) + \Phi_{ij}^\varepsilon * \tilde{f}(\mathbf{x}) + e(\varepsilon).$$

By the theory of singular integrals, there exists a continuous linear map, $K_{ij} \in \mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ such that

$$K_{ij} f \equiv \lim_{\varepsilon \rightarrow 0} \Phi_{ij}^\varepsilon * f.$$

Therefore, letting $\varepsilon \rightarrow 0$,

$$w_{i,j} = C_n \delta_{ij} f + K_{ij} \tilde{f}$$

whenever $f \in C_c^\infty(U)$.

Now let $f \in L^p(U)$, let

$$\|f_k - f\|_{L^p(U)} \rightarrow 0,$$

where $f_k \in C_c^\infty(U)$, and let

$$w_i^k(\mathbf{x}) = \int_U \Phi_{,i}(\mathbf{x} - \mathbf{y}) f_k(\mathbf{y}) dy.$$

Then it follows as before that $w_i^k \rightarrow w_i$ in $L^p(U)$ and

$$w_{i,j}^k = C_n \delta_{ij} f_k + K_{ij} \tilde{f}_k.$$

Now let $\phi \in C_c^\infty(U)$.

$$\begin{aligned} w_{i,j}(\phi) &\equiv - \int_U w_i \phi_{,j} dx = - \lim_{k \rightarrow \infty} \int_U w_i^k \phi_{,j} dx \\ &= \lim_{k \rightarrow \infty} \int_U w_{i,j}^k \phi dx = \lim_{k \rightarrow \infty} \int_U (C_n \delta_{ij} \tilde{f}_k + K_{ij} \tilde{f}_k) \phi dx \\ &= \int_U (C_n \delta_{ij} \tilde{f} + K_{ij} \tilde{f}) \phi dx. \end{aligned}$$

It follows

$$w_{i,j} = C_n \delta_{ij} \tilde{f} + K_{ij} \tilde{f}$$

and this proves the lemma.

Corollary 22.5.6 *In the situation of Theorem 22.5.4, all weak derivatives of u of order 2 are in $L^p(U)$ and also $f \rightarrow u_{,ij}$ is a continuous map.*

Proof:

$$u_{,i}(\mathbf{x}) = \int_U \Phi_{,i}(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y}$$

and so $u_{,ij} \in L^p(U)$ and $f \rightarrow u_{,ij}$ is continuous by Lemma 22.5.5.

With this preparation, it is possible to consider the Helmholtz decomposition. Let $\mathbf{F} \in L^p(U; \mathbb{R}^n)$ and define

$$\phi(\mathbf{x}) \equiv \int_U \nabla \Phi(\mathbf{x} - \mathbf{y}) \cdot \mathbf{F}(\mathbf{y}) d\mathbf{y}. \quad (22.5.72)$$

Then by Lemma 22.5.5,

$$\phi_{,j} = C_n \tilde{F}_j + \sum_i K_{ij} \tilde{F}_i \in L^p(\mathbb{R}^n)$$

and the mapping $\mathbf{F} \rightarrow \nabla \phi$ is continuous from $L^p(U; \mathbb{R}^n)$ to $L^p(U; \mathbb{R}^n)$.

Now suppose $\mathbf{F} \in C_c^\infty(U; \mathbb{R}^n)$. Then

$$\begin{aligned} \phi(\mathbf{x}) &= \int_U \sum_{i=1}^n -\frac{\partial}{\partial y^i} (\Phi(\mathbf{x} - \mathbf{y}) F_i(\mathbf{y})) + \Phi(\mathbf{x} - \mathbf{y}) \nabla \cdot \mathbf{F}(\mathbf{y}) d\mathbf{y} \\ &= \int_U \Phi(\mathbf{x} - \mathbf{y}) \nabla \cdot \mathbf{F}(\mathbf{y}) d\mathbf{y} \end{aligned}$$

and so by Lemma 22.5.3,

$$\nabla \cdot \nabla \phi = \Delta \phi = -\nabla \cdot \mathbf{F}.$$

This continues to hold in the sense of weak derivatives if $\mathbf{F}$ is only in $L^p(U; \mathbb{R}^n)$ because by Minkowski's inequality and 22.5.72 the map $\mathbf{F} \rightarrow \phi$ is continuous. Also note that for $\mathbf{F} \in C_c^\infty(U; \mathbb{R}^n)$,

$$\phi(\mathbf{x}) = \int_B \Phi(\mathbf{y}) \nabla \cdot \mathbf{F}(\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

Next define $\pi : L^p(U; \mathbb{R}^n) \rightarrow L^p(U; \mathbb{R}^n)$ by

$$\pi \mathbf{F} = -\nabla \phi, \quad \phi(\mathbf{x}) = \int_U \nabla \Phi(\mathbf{x} - \mathbf{y}) \cdot \mathbf{F}(\mathbf{y}) d\mathbf{y}.$$

It was already shown that π is continuous, linear, and $\nabla \cdot \pi \mathbf{F} = \nabla \cdot \mathbf{F}$. It is also true that π is a projection. To see this, let $\mathbf{F} \in C_c^\infty(U; \mathbb{R}^n)$. Then for B large enough,

$$\begin{aligned} \pi^2 \mathbf{F}(\mathbf{x}) &= -\nabla \int_B \Phi(\mathbf{z}) \nabla \cdot \pi \mathbf{F}(\mathbf{x} - \mathbf{z}) d\mathbf{z} \\ &= -\nabla \int_B \Phi(\mathbf{z}) \nabla \cdot \nabla \int_B \Phi(\mathbf{w}) \nabla \cdot \mathbf{F}(\mathbf{x} - \mathbf{z} - \mathbf{w}) d\mathbf{w} d\mathbf{z} \\ &= -\nabla \int_B \Phi(\mathbf{z}) \nabla \cdot \mathbf{F}(\mathbf{x} - \mathbf{z}) d\mathbf{z} = \pi \mathbf{F}(\mathbf{x}). \end{aligned}$$

Since π is continuous and $C_c^\infty(U; \mathbb{R}^n)$ is dense in $L^p(U; \mathbb{R}^n)$, $\pi^2 \mathbf{F} = \pi \mathbf{F}$ for all $\mathbf{F} \in L^p(U; \mathbb{R}^n)$. This proves the following theorem which is the Helmholtz decomposition.

Theorem 22.5.7 *There exists a continuous projection*

$$\pi : L^p(U; \mathbb{R}^n) \rightarrow L^p(U; \mathbb{R}^n)$$

such that $\pi \mathbf{F}$ is a gradient and

$$\nabla \cdot (\mathbf{F} - \pi \mathbf{F}) = 0$$

in the sense of weak derivatives.

Note this theorem shows that any L^p vector field is the sum of a gradient and a part which is divergence free. $\mathbf{F} = \mathbf{F} - \pi \mathbf{F} + \pi \mathbf{F}$.

The Bochner Integral

23.1 Strong And Weak Measurability

In this chapter $(\Omega, \mathcal{S}, \mu)$ will be a σ finite measure space and X will be a Banach space which contains the values of either a function or a measure. The Banach space will be either a real or a complex Banach space but the field of scalars does not matter and so it is denoted by $\mathbb{F}$ with the understanding that $\mathbb{F} = \mathbb{C}$ unless otherwise stated. The theory presented here includes the case where $X = \mathbb{R}^n$ or $\mathbb{C}^n$ but it does not include the situation where f could have values in a space like $[0, \infty]$. To begin with here is a definition.

Definition 23.1.1 *A function, $x : \Omega \rightarrow X$, for X a Banach space, is a simple function if it is of the form*

$$x(s) = \sum_{i=1}^n a_i \chi_{B_i}(s)$$

where $B_i \in \mathcal{S}$ and $\mu(B_i) < \infty$ for each i . A function x from Ω to X is said to be strongly measurable if there exists a sequence of simple functions $\{x_n\}$ converging pointwise to x . The function x is said to be weakly measurable if, for each $f \in X'$,

$$f \circ x$$

is a scalar valued measurable function.

Earlier, a function was measurable if inverse images of open sets were measurable. Something similar holds here. The difference is that another condition needs to hold.

Theorem 23.1.2 *x is strongly measurable if and only if $x^{-1}(U)$ is measurable for all U open in X and $x(\Omega)$ is separable.*

Proof: Suppose first $x^{-1}(U)$ is measurable for all U open in X and $x(\Omega)$ is separable. Let $\{a_n\}_{n=1}^\infty$ be the dense subset of $x(\Omega)$. It follows $x^{-1}(B)$ is measurable for all B Borel because

$$\{B : x^{-1}(B) \text{ is measurable}\}$$

is a σ algebra containing the open sets. Let

$$U_k^n \equiv \{z \in X : \|z - a_k\| \leq \min\{\|z - a_l\|_{l=1}^n\}\}.$$

In words, U_k^m is the set of points of X which are as close to a_k as they are to any of the a_l for $l \leq n$.

$$B_k^n \equiv x^{-1}(U_k^n), \quad D_k^n \equiv B_k^n \setminus (\cup_{i=1}^{k-1} B_i^n), \quad D_1^n \equiv B_1^n,$$

and

$$x_n(s) \equiv \sum_{k=1}^n a_k \chi_{D_k^n}(s).$$

Thus $x_n(s)$ is a closest approximation to $x(s)$ from $\{a_k\}_{k=1}^n$ and so $x_n(s) \rightarrow x(s)$ because $\{a_n\}_{n=1}^\infty$ is dense in $x(\Omega)$. Furthermore, x_n is measurable because each D_k^n is measurable.

Since $(\Omega, \mathcal{S}, \mu)$ is σ finite, there exists $\Omega_n \uparrow \Omega$ with $\mu(\Omega_n) < \infty$. Let

$$y_n(s) \equiv \chi_{\Omega_n}(s) x_n(s).$$

Then $y_n(s) \rightarrow x(s)$ for each s because for any $s, s \in \Omega_n$ if n is large enough. Also y_n is a simple function because it equals 0 off a set of finite measure.

Now suppose that x is strongly measurable. Then some sequence of simple functions, $\{x_n\}$, converges pointwise to x . Then $x_n^{-1}(W)$ is measurable for every open set W because it is just a finite union of measurable sets. Thus, $x_n^{-1}(W)$ is measurable for every Borel set W . This follows by considering

$$\{W : x_n^{-1}(W) \text{ is measurable}\}$$

and observing this is a σ algebra which contains the open sets. Since X is a metric space, it follows that if U is an open set in X , there exists a sequence of open sets, $\{V_n\}$ which satisfies

$$\overline{V}_n \subseteq U, \quad \overline{V}_n \subseteq V_{n+1}, \quad U = \cup_{n=1}^\infty V_n.$$

Then

$$x^{-1}(V_m) \subseteq \bigcup_{n < \infty} \bigcap_{k \geq n} x_k^{-1}(V_m) \subseteq x^{-1}(\overline{V}_m).$$

This implies

$$\begin{aligned} x^{-1}(U) &= \bigcup_{m < \infty} x^{-1}(V_m) \\ &\subseteq \bigcup_{m < \infty} \bigcup_{n < \infty} \bigcap_{k \geq n} x_k^{-1}(V_m) \subseteq \bigcup_{m < \infty} x^{-1}(\overline{V}_m) \subseteq x^{-1}(U). \end{aligned}$$

Since

$$x^{-1}(U) = \bigcup_{m < \infty} \bigcup_{n < \infty} \bigcap_{k \geq n} x_k^{-1}(V_m),$$

it follows that $x^{-1}(U)$ is measurable for every open U . It remains to show $x(\Omega)$ is separable. Let

$$D \equiv \text{all values of the simple functions } x_n$$

which converge to x pointwise. Then D is clearly countable and dense in $\overline{D}$, a set which contains $x(\Omega)$.

Claim: $x(\Omega)$ is separable.

Proof of claim: For $n \in \mathbb{N}$, let $\mathcal{B}_n \equiv \{B(d, r) : 0 < r < \frac{1}{n}, r \text{ rational}, d \in D\}$. Thus $\mathcal{B}_n$ is countable. Let $z \in \overline{D}$. Consider $B(z, \frac{1}{n})$. Then there exists $d \in D \cap B(z, \frac{1}{3n})$. Now pick $r \in \mathbb{Q} \cap (\frac{1}{3n}, \frac{1}{n})$ so that $B(d, r) \in \mathcal{B}_n$. Now $z \in B(d, r)$ and so this shows that $x(\Omega) \subseteq \overline{D} \subseteq \cup \mathcal{B}_n$ for each n . Now let $\mathcal{B}'_n$ denote those sets of $\mathcal{B}_n$ which have nonempty intersection with $x(\Omega)$. Say $\mathcal{B}'_n = \{B_k^n\}_{k=1}^\infty$. By the axiom of choice, there exists $x_k^n \in B_k^n \cap x(\Omega)$. Then if $z \in x(\Omega)$, z is contained in some set of $\mathcal{B}'_n$ which also contains a point of $\{x_k^n\}_{k=1}^\infty$. Therefore, z is at least as close as $2/n$ to some point of $\{x_k^n\}_{k=1}^\infty$ which shows $\{x_k^n\}_{n,k=1}^\infty$ is a countable dense subset of $x(\Omega)$. Therefore $x(\Omega)$ is separable. This proves the theorem.

The last part also shows that a subset of a separable metric space is also separable. Therefore, the following simple corollary is obtained.

Corollary 23.1.3 *If X is a separable Banach space then x is strongly measurable if and only if $x^{-1}(U)$ is measurable for all U open in X .*

The next lemma is interesting for its own sake. Roughly it says that if a Banach space is separable, then the unit ball in the dual space is weak * separable. This will be used to prove Pettis's theorem, one of the major theorems in this subject which relates weak measurability to strong measurability.

Lemma 23.1.4 *If X is a separable Banach space with B' the closed unit ball in X' , then there exists a sequence $\{f_n\}_{n=1}^\infty \equiv D' \subseteq B'$ with the property that for every $x \in X$,*

$$\|x\| = \sup_{f \in D'} |f(x)|$$

If H is a dense subset of X' then D' may be chosen to be contained in H .

Proof: Let $\{a_k\}$ be a countable dense set in X and consider the mapping

$$\phi_n : B' \rightarrow \mathbb{F}^n$$

given by

$$\phi_n(f) \equiv (f(a_1), \dots, f(a_n)).$$

Then $\phi_n(B')$ is contained in a compact subset of $\mathbb{F}^n$ because $|f(a_k)| \leq \|a_k\|$. Therefore, there exists a countable dense subset of $\phi_n(B')$, $\{\phi_n(f_k^n)\}_{k=1}^\infty$. Let $D'_n \equiv \{f_k^n\}_{k=1}^\infty$. Let

$$D' \equiv \cup_{k=1}^\infty D'_k.$$

It remains to show this works. Letting $x \in X$ and $\varepsilon > 0$ be given, there exists a_m such that $\|a_m - x\| < \varepsilon$. Then by the usual argument involving the Hahn Banach

theorem, there exists $f_x \in B'$ such that $\|x\| = f_x(x)$. Letting $n > m$, let $g \in B'$ be one of the f_k^n with $\{\phi_n(f_k^n)\}_{k=1}^\infty$ a dense subset of $\phi_n(B')$ such that

$$|g(a_m) - f_x(a_m)| < \varepsilon.$$

Then

$$\begin{aligned} \|x\| &= |f_x(x)| = |f_x(a_m) + f_x(x) - f_x(a_m)| \\ &\leq |f_x(a_m)| + \varepsilon \leq |g(a_m)| + 2\varepsilon \leq |g(x)| + 3\varepsilon \end{aligned}$$

and so since $\varepsilon > 0$ is arbitrary,

$$\|x\| \leq \sup_{g \in D'} |g(x)| \leq \|x\|$$

The last claim is verified by noting that from density of H in X' , each f_k^n can be chosen in H . If f_k^n is just in B' , pick $\{g_{kl}^n\}_{l=1}^\infty \in B' \cap H$ such that $\|f_k^n - g_{kl}^n\| < 2^{-l}$. Then $\{g_{kl}^n\}_{k,l}$ is countable and an enumeration of it will work to show that each f_k^n can be chosen in H . This proves the lemma.

The next theorem is one of the most important results in the subject. It is due to Pettis and appeared in 1938.

Theorem 23.1.5 *If x has values in a separable Banach space, X , and if x is weakly measurable, then x is strongly measurable.*

Proof: It is necessary to show $x^{-1}(U)$ is measurable whenever U is open. Since every open set is a countable union of balls, it suffices to show $x^{-1}(B(a, r))$ is measurable for any ball, $B(a, r)$. Since every open ball is the countable union of closed balls, it suffices to verify $x^{-1}(\overline{B(a, r)})$ is measurable. From Lemma 23.1.4

$$\begin{aligned} x^{-1}(\overline{B(a, r)}) &= \{s : \|x(s) - a\| \leq r\} \\ &= \left\{ s : \sup_{f \in D'} |f(x(s) - a)| \leq r \right\} \\ &= \cap_{f \in D'} \{s : |f(x(s) - a)| \leq r\} \\ &= \cap_{f \in D'} \{s : |f(x(s)) - f(a)| \leq r\} \\ &= \cap_{f \in D'} (f \circ x)^{-1} \overline{B(f(a), r)} \end{aligned}$$

which equals a countable union of measurable sets because it is assumed that $f \circ x$ is measurable for all $f \in X'$. This proves the theorem.

The same method of proof yields the following interesting corollary.

Corollary 23.1.6 *Let X be a separable Banach space and let $\mathcal{B}(X)$ denote the σ algebra of Borel sets. Then $\mathcal{B}(X) = \mathcal{F}$ where $\mathcal{F}$ is the smallest σ algebra of subsets of X which has the property that every function, $x^* \in X'$ is $\mathcal{F}$ measurable.*

Proof: First I need to show $\mathcal{F}$ contains open balls because then $\mathcal{F}$ will contain the open sets, since every open set is a countable union of open balls, which will imply $\mathcal{F} \supseteq \mathcal{B}(X)$. As noted above, it suffices to verify $\mathcal{F}$ contains the closed balls because every open ball is a countable union of closed balls. Let D' be those functionals in B' defined in Lemma 23.1.4. Then

$$\begin{aligned} \{x : \|x - a\| \leq r\} &= \left\{x : \sup_{x^* \in D'} |x^*(x - a)| \leq r\right\} \\ &= \cap_{x^* \in D'} \{x : |x^*(x - a)| \leq r\} \\ &= \cap_{x^* \in D'} \{x : |x^*(x) - x^*(a)| \leq r\} \\ &= \cap_{x^* \in D'} x^{*-1} \left(\overline{B(x^*(a), r)}\right) \end{aligned}$$

which is measurable because this is a countable intersection of measurable sets. Thus $\mathcal{F}$ contains open sets so $\mathcal{F} \supseteq \mathcal{B}(X)$.

To show the other direction for the inclusion, note that each x^* is $\mathcal{B}(X)$ measurable because $x^{*-1}(\text{open set}) = \text{open set}$. Therefore, $\mathcal{B}(X)$ is a σ algebra with respect to which each x^* is measurable and $\mathcal{F}$ is the smallest of these so $\mathcal{B}(X) \subseteq \mathcal{F}$. This proves the corollary.

It is important to verify the limit of strongly measurable functions is itself strongly measurable. This happens under very general conditions. Suppose X is any separable metric space and let τ denote the open sets of X . Then it is routine to see that

$$\tau \text{ has a countable basis, } \mathcal{B}. \quad (23.1.1)$$

Whenever $U \in \mathcal{B}$, there exists a sequence of open sets, $\{V_m\}_{m=1}^\infty$, such that

$$\cdots V_m \subseteq \overline{V_m} \subseteq V_{m+1} \subseteq \cdots, \quad U = \bigcup_{m=1}^\infty V_m. \quad (23.1.2)$$

Theorem 23.1.7 *Let f_n and f be functions mapping Ω to X where $\mathcal{F}$ is a σ algebra of measurable sets of Ω and (X, τ) is a topological space satisfying 23.1.1 - 23.1.2. Then if f_n is measurable, and $f(\omega) = \lim_{n \rightarrow \infty} f_n(\omega)$, it follows that f is also measurable. (Pointwise limits of measurable functions are measurable.)*

Proof: Let $\mathcal{B}$ be the countable basis of 23.1.1 and let $U \in \mathcal{B}$. Let $\{V_m\}$ be the sequence of 23.1.2. Since f is the pointwise limit of f_n ,

$$f^{-1}(V_m) \subseteq \{\omega : f_k(\omega) \in V_m \text{ for all } k \text{ large enough}\} \subseteq f^{-1}(\overline{V_m}).$$

Therefore,

$$\begin{aligned} f^{-1}(U) &= \cup_{m=1}^\infty f^{-1}(V_m) \subseteq \cup_{m=1}^\infty \cup_{n=1}^\infty \cap_{k=n}^\infty f_k^{-1}(V_m) \\ &\subseteq \cup_{m=1}^\infty f^{-1}(\overline{V_m}) = f^{-1}(U). \end{aligned}$$

It follows $f^{-1}(U) \in \mathcal{F}$ because it equals the expression in the middle which is measurable. Now let $W \in \tau$. Since $\mathcal{B}$ is countable, $W = \cup_{n=1}^\infty U_n$ for some sets $U_n \in \mathcal{B}$. Hence

$$f^{-1}(W) = \cup_{n=1}^\infty f^{-1}(U_n) \in \mathcal{F}.$$

This proves the theorem.

Corollary 23.1.8 *x is strongly measurable if and only if $x(\Omega)$ is separable and x is weakly measurable.*

Proof: Strong measurability clearly implies weak measurability. If $x_n(s) \rightarrow x(s)$ where x_n is simple, then $f(x_n(s)) \rightarrow f(x(s))$ for all $f \in X'$. Hence $f \circ x$ is measurable by Theorem 23.1.7 because it is the limit of a sequence of measurable functions. Let D denote the set of all values of x_n . Then $\overline{D}$ is a separable set containing $x(\Omega)$. Thus $\overline{D}$ is a separable metric space. Therefore $x(\Omega)$ is separable also by the last part of the proof of Theorem 23.1.2.

Now suppose D is a countable dense subset of $x(\Omega)$ and x is weakly measurable. Let Z be the subset consisting of all finite linear combinations of D with the scalars coming from the set of rational points of $\mathbb{F}$. Thus, Z is countable. Letting $Y = \overline{Z}$, Y is a separable Banach space containing $x(\Omega)$. If $f \in Y'$, f can be extended to an element of X' by the Hahn Banach theorem. Therefore, x is a weakly measurable Y valued function. Now use Theorem 23.1.5 to conclude x is strongly measurable. This proves the corollary.

Weakly measurable as defined above means $s \rightarrow x^*(x(s))$ is measurable for every $x^* \in X'$. The next lemma ties this to the usual version of measurability in which a function is measurable when inverse images of open sets are measurable.

Lemma 23.1.9 *Let X be a Banach space and let $x : (\Omega, \mathcal{F}) \rightarrow K \subseteq X$ where K is weakly compact and X' is separable. Then x is weakly measurable if and only if $x^{-1}(U) \in \mathcal{F}$ whenever U is a weakly open set.*

Proof: By Corollary 13.4.9 on Page 386, there exists a metric d , such that the metric space topology with respect to d coincides with the weak topology. Since K is compact, it follows that K is also separable. Hence it is completely separable and so there exists a countable basis of open sets, $\mathcal{B}$ for the weak topology on K . It follows that if U is any weakly open set, covered by basic sets of the form $B_A(x, r)$ where A is a finite subset of X' , there exists a countable collection of these sets of the form $B_A(x, r)$ which covers U .

Suppose now that x is weakly measurable. To show $x^{-1}(U) \in \mathcal{F}$ whenever U is weakly open, it suffices to verify $x^{-1}(B_A(z, r)) \in \mathcal{F}$ for any set, $B_A(z, r)$. Let $A = \{x_1^*, \dots, x_m^*\}$. Then

$$\begin{aligned} x^{-1}(B_A(z, r)) &= \{s \in \Omega : \rho_A(x(s) - z) < r\} \\ &\equiv \left\{ s \in \Omega : \max_{x^* \in A} |x^*(x(s) - z)| < r \right\} \\ &= \bigcup_{i=1}^m \{s \in \Omega : |x_i^*(x(s) - z)| < r\} \\ &= \bigcup_{i=1}^m \{s \in \Omega : |x_i^*(x(s)) - x_i^*(z)| < r\} \end{aligned}$$

which is measurable because each $x_i^* \circ x$ is given to be measurable.

Next suppose $x^{-1}(U) \in \mathcal{F}$ whenever U is weakly open. Then in particular this holds when $U = B_{x^*}(z, r)$ for arbitrary x^* . Hence

$$\{s \in \Omega : x(s) \in B_{x^*}(z, r)\} \in \mathcal{F}.$$

But this says the same as

$$\{s \in \Omega : |x^*(x(s)) - x^*(z)| < r\} \in \mathcal{F}$$

Since $x^*(z)$ can be a completely arbitrary element of $\mathbb{F}$, it follows $x^* \circ x$ is an $\mathbb{F}$ valued measurable function. In other words, x is weakly measurable according to the former definition. This proves the lemma.

One can also define weak $*$ measurability and prove a theorem just like the Pettis theorem above. The next lemma is the analogue of Lemma 23.1.4.

Lemma 23.1.10 *Let B be the closed unit ball in X . If X' is separable, there exists a sequence $\{x_m\}_{m=1}^\infty \equiv D \subseteq B$ with the property that for all $y^* \in X'$,*

$$\|y^*\| = \sup_{x \in D} |y^*(x)|.$$

Proof: Let

$$\{x_k^*\}_{k=1}^\infty$$

be the dense subspace of X' . Define $\phi_n : B \rightarrow \mathbb{F}^n$ by

$$\phi_n(x) \equiv (x_1^*(x), \dots, x_n^*(x)).$$

Then $|x_k^*(x)| \leq \|x_k^*\|$ and so $\phi_n(B)$ is contained in a compact subset of $\mathbb{F}^n$. Therefore, there exists a countable set, $D_n \subseteq B$ such that $\phi_n(D_n)$ is dense in $\phi_n(B)$. Let

$$D \equiv \cup_{n=1}^\infty D_n.$$

It remains to verify this works. Let $y^* \in X'$. Then there exists y such that

$$|y^*(y)| > \|y^*\| - \varepsilon.$$

By density, there exists one of the x_k^* from the countable dense subset of X' such that also

$$|x_k^*(y)| > \|y^*\| - \varepsilon, \quad \|x_k^* - y^*\| < \varepsilon.$$

Now $x_k^*(y) \in \phi_k(B)$ and so there exists $x \in D_k \subseteq D$ such that

$$|x_k^*(x)| > \|y^*\| - \varepsilon.$$

Then since $\|x_k^* - y^*\| < \varepsilon$, this implies

$$|y^*(x)| \geq \|y^*\| - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\|y^*\| \leq \sup_{x \in D} |y^*(x)| \leq \|y^*\|$$

and this proves the lemma.

The next theorem is another version of the Pettis theorem. First here is a definition.

Definition 23.1.11 A function y having values in X' is weak $*$ measurable, when for each $x \in X$, $y(\cdot)(x)$ is a measurable scalar valued function.

Theorem 23.1.12 If X' is separable and $y : \Omega \rightarrow X'$ is weak $*$ measurable, then y is strongly measurable.

Proof: It is necessary to show $y^{-1}(B(a^*, r))$ is measurable. This will suffice because the separability of X' implies every open set is the countable union of such balls of the form $B(a^*, r)$. It also suffices to verify inverse images of closed balls are measurable because every open ball is the countable union of closed balls. From Lemma 23.1.10,

$$\begin{aligned} y^{-1}(\overline{B(a^*, r)}) &= \{s : \|y(s) - a^*\| \leq r\} \\ &= \left\{s : \sup_{x \in D} |(y(s) - a^*)(x)| \leq r\right\} \\ &= \left\{s : \sup_{x \in D} |y(s)(x) - a^*(x)| \leq r\right\} \\ &= \bigcap_{x \in D} y(\cdot)(x)^{-1}(\overline{B(a^*(x), r)}) \end{aligned}$$

which is a countable intersection of measurable sets by hypothesis. This proves the theorem.

The following are interesting consequences of the theory developed so far and are of interest independent of the theory of integration of vector valued functions.

Theorem 23.1.13 If X' is separable, then so is X .

Proof: Let $D = \{x_m\} \subseteq B$, the unit ball of X , be the sequence promised by Lemma 23.1.10. Let V be all finite linear combinations of elements of $\{x_m\}$ with rational scalars. Thus $\overline{V}$ is a separable subspace of X . The claim is that $\overline{V} = X$. If not, there exists

$$x_0 \in X \setminus \overline{V}.$$

But by the Hahn Banach theorem there exists $x_0^* \in X'$ satisfying $x_0^*(x_0) \neq 0$, but $x_0^*(v) = 0$ for every $v \in \overline{V}$. Hence

$$\|x_0^*\| = \sup_{x \in D} |x_0^*(x)| = 0,$$

a contradiction. This proves the theorem.

Corollary 23.1.14 If X is reflexive, then X is separable if and only if X' is separable.

Proof: From the above theorem, if X' is separable, then so is X . Now suppose X is separable with a dense subset equal to D . Then since X is reflexive, $J(D)$ is dense in X'' where J is the James map satisfying $Jx(x^*) \equiv x^*(x)$. Then since X'' is separable, it follows from the above theorem that X' is also separable.

23.2 The Bochner Integral

23.2.1 Definition And Basic Properties

Definition 23.2.1 Let $a_k \in X$, a Banach space and let

$$x(s) = \sum_{k=1}^n a_k \chi_{E_k}(s) \quad (23.2.3)$$

where for each k , E_k is measurable and $\mu(E_k) < \infty$. Then define

$$\int_{\Omega} x(s) d\mu \equiv \sum_{k=1}^n a_k \mu(E_k).$$

Proposition 23.2.2 Definition 23.2.1 is well defined.

Proof: It suffices to verify that if

$$\sum_{k=1}^n a_k \chi_{E_k}(s) = 0,$$

then

$$\sum_{k=1}^n a_k \mu(E_k) = 0.$$

Let $f \in X'$. Then

$$f\left(\sum_{k=1}^n a_k \chi_{E_k}(s)\right) = \sum_{k=1}^n f(a_k) \chi_{E_k}(s) = 0$$

and, therefore,

$$0 = \int_{\Omega} \left(\sum_{k=1}^n f(a_k) \chi_{E_k}(s)\right) d\mu = \sum_{k=1}^n f(a_k) \mu(E_k) = f\left(\sum_{k=1}^n a_k \mu(E_k)\right).$$

Since $f \in X'$ is arbitrary, and X' separates the points of X , it follows that

$$\sum_{k=1}^n a_k \mu(E_k) = 0$$

as claimed. This proves the proposition.

It follows easily from this proposition that $\int_{\Omega} d\mu$ is well defined and linear on simple functions.

Definition 23.2.3 A strongly measurable function x is Bochner integrable if there exists a sequence of simple functions x_n converging to x pointwise and satisfying

$$\int_{\Omega} \|x_n(s) - x_m(s)\| d\mu \rightarrow 0 \text{ as } m, n \rightarrow \infty. \quad (23.2.4)$$

If x is Bochner integrable, define

$$\int_{\Omega} x(s) d\mu \equiv \lim_{n \rightarrow \infty} \int_{\Omega} x_n(s) d\mu. \quad (23.2.5)$$

Theorem 23.2.4 The Bochner integral is well defined and if x is Bochner integrable and $f \in X'$,

$$f\left(\int_{\Omega} x(s) d\mu\right) = \int_{\Omega} f(x(s)) d\mu \quad (23.2.6)$$

and

$$\left\| \int_{\Omega} x(s) d\mu \right\| \leq \int_{\Omega} \|x(s)\| d\mu. \quad (23.2.7)$$

Also, the Bochner integral is linear. That is, if a, b are scalars and x, y are two Bochner integrable functions, then

$$\int_{\Omega} (ax(s) + by(s)) d\mu = a \int_{\Omega} x(s) d\mu + b \int_{\Omega} y(s) d\mu \quad (23.2.8)$$

Proof: First it is shown that the triangle inequality holds on simple functions and that the limit in 23.2.5 exists. Thus, if x is given by 23.2.3 with the E_k disjoint,

$$\begin{aligned} & \left\| \int_{\Omega} x(s) d\mu \right\| \\ &= \left\| \int_{\Omega} \sum_{k=1}^n a_k \chi_{E_k}(s) d\mu \right\| = \left\| \sum_{k=1}^n a_k \mu(E_k) \right\| \\ &\leq \sum_{k=1}^n \|a_k\| \mu(E_k) = \int_{\Omega} \sum_{k=1}^n \|a_k\| \chi_{E_k}(s) d\mu = \int_{\Omega} \|x(s)\| d\mu \end{aligned}$$

which shows the triangle inequality holds on simple functions. This implies

$$\begin{aligned} \left\| \int_{\Omega} x_n(s) d\mu - \int_{\Omega} x_m(s) d\mu \right\| &= \left\| \int_{\Omega} (x_n(s) - x_m(s)) d\mu \right\| \\ &\leq \int_{\Omega} \|x_n(s) - x_m(s)\| d\mu \end{aligned}$$

which verifies the existence of the limit in 23.2.5. This completes the first part of the argument.

Next it is shown the integral does not depend on the choice of the sequence satisfying 23.2.4 so that the integral is well defined. Suppose y_n, x_n both satisfy 23.2.4 and converge to x pointwise. By Fatou's lemma,

$$\begin{aligned} \left\| \int_{\Omega} y_n d\mu - \int_{\Omega} x_m d\mu \right\| &\leq \int_{\Omega} \|y_n - x\| d\mu + \int_{\Omega} \|x - x_m\| d\mu \\ &\leq \liminf_{k \rightarrow \infty} \int_{\Omega} \|y_n - y_k\| d\mu + \liminf_{k \rightarrow \infty} \int_{\Omega} \|x_k - x_m\| \\ &\leq \varepsilon/2 + \varepsilon/2 \end{aligned}$$

if m and n are chosen large enough. Since ε is arbitrary, this shows the limit is the same for both sequences and demonstrates the Bochner integral is well defined.

It remains to verify the triangle inequality on Bochner integral functions and the claim about passing a continuous linear functional inside the integral. Let x be Bochner integrable and let x_n be a sequence which satisfies the conditions of the definition. Define

$$y_n(s) \equiv \begin{cases} x_n(s) & \text{if } \|x_n(s)\| \leq 2\|x(s)\|, \\ 0 & \text{if } \|x_n(s)\| > 2\|x(s)\|. \end{cases} \quad (23.2.9)$$

If $x(s) = 0$ then $y_n(s) = 0$ for all n . If $\|x(s)\| > 0$ then for all n large enough,

$$y_n(s) = x_n(s).$$

Thus, $y_n(s) \rightarrow x(s)$ and

$$\|y_n(s)\| \leq 2\|x(s)\|. \quad (23.2.10)$$

By Fatou's lemma,

$$\int_{\Omega} \|x\| d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \|x_n\| d\mu. \quad (23.2.11)$$

Also from 23.2.4 and the triangle inequality on simple functions, $\{\int_{\Omega} \|x_n\| d\mu\}_{n=1}^{\infty}$ is a Cauchy sequence and so it must be bounded. Therefore, by 23.2.10, 23.2.11, and the dominated convergence theorem,

$$0 = \lim_{n, m \rightarrow \infty} \int_{\Omega} \|y_n - y_m\| d\mu \quad (23.2.12)$$

and it follows x_n can be replaced with y_n in Definition 23.2.3.

From Definition 23.2.1,

$$f\left(\int_{\Omega} y_n d\mu\right) = \int_{\Omega} f(y_n) d\mu.$$

Thus,

$$f\left(\int_{\Omega} x d\mu\right) = \lim_{n \rightarrow \infty} f\left(\int_{\Omega} y_n d\mu\right) = \lim_{n \rightarrow \infty} \int_{\Omega} f(y_n) d\mu = \int_{\Omega} f(x) d\mu,$$

the last equation holding from the dominated convergence theorem and 23.2.10 and 23.2.11. This shows 23.2.6. To verify 23.2.7,

$$\begin{aligned} \left\| \int_{\Omega} x(s) d\mu \right\| &= \lim_{n \rightarrow \infty} \left\| \int_{\Omega} y_n(s) d\mu \right\| \\ &\leq \lim_{n \rightarrow \infty} \int_{\Omega} \|y_n(s)\| d\mu = \int_{\Omega} \|x(s)\| d\mu \end{aligned}$$

where the last equation follows from the dominated convergence theorem and 23.2.10, 23.2.11.

It remains to verify 23.2.8. Let $f \in X'$. Then from 23.2.6

$$\begin{aligned} f \left(\int_{\Omega} (ax(s) + by(s)) d\mu \right) &= \int_{\Omega} (af(x(s)) + bf(y(s))) d\mu \\ &= a \int_{\Omega} f(x(s)) d\mu + b \int_{\Omega} f(y(s)) d\mu \\ &= f \left(a \int_{\Omega} x(s) d\mu + b \int_{\Omega} y(s) d\mu \right). \end{aligned}$$

Since X' separates the points of X , it follows

$$\int_{\Omega} (ax(s) + by(s)) d\mu = a \int_{\Omega} x(s) d\mu + b \int_{\Omega} y(s) d\mu$$

and this proves 23.2.8. This proves the theorem.

Theorem 23.2.5 *An X valued function, x , is Bochner integrable if and only if x is strongly measurable and*

$$\int_{\Omega} \|x(s)\| d\mu < \infty. \quad (23.2.13)$$

In this case there exists a sequence of simple functions $\{y_n\}$ satisfying 23.2.4, $y_n(s)$ converging pointwise to $x(s)$,

$$\|y_n(s)\| \leq 2\|x(s)\| \quad (23.2.14)$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|x(s) - y_n(s)\| d\mu = 0. \quad (23.2.15)$$

Proof: Suppose x is strongly measurable and condition 23.2.13 holds. Since x is strongly measurable, there exists a sequence of simple functions, $\{x_n\}$ converging pointwise to x . As before, let

$$y_n(s) = \begin{cases} x_n(s) & \text{if } \|x_n(s)\| \leq 2\|x(s)\|, \\ 0 & \text{if } \|x_n(s)\| > 2\|x(s)\|. \end{cases} \quad (23.2.16)$$

Then 23.2.14 holds for y_n and $y_n(s) \rightarrow x(s)$. Also

$$0 = \lim_{m,n \rightarrow \infty} \int_{\Omega} \|y_n(s) - y_m(s)\| d\mu$$

since otherwise, there would exist $\varepsilon > 0$ and $N_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and $n_\varepsilon, m_\varepsilon > N_\varepsilon$ such that

$$\int_{\Omega} \|y_{n_\varepsilon}(s) - y_{m_\varepsilon}(s)\| d\mu \geq \varepsilon.$$

But then taking a limit as $\varepsilon \rightarrow 0$ and using the dominated convergence theorem and 23.2.14 and 23.2.13, this would imply $0 \geq \varepsilon$. Therefore, x is Bochner integrable. 23.2.15 follows from the dominated convergence theorem and 23.2.14.

Now suppose x is Bochner integrable. Then it is strongly measurable and there exists a sequence of simple functions $\{x_n\}$ such that $x_n(s)$ converges pointwise to x and

$$\lim_{m,n \rightarrow \infty} \int_{\Omega} \|x_n(s) - x_m(s)\| d\mu = 0.$$

Therefore, as before, since $\{\int_{\Omega} x_n d\mu\}_{n=1}^{\infty}$ is a Cauchy sequence, it follows

$$\left\{ \int_{\Omega} \|x_n\| d\mu \right\}_{n=1}^{\infty}$$

is also a Cauchy sequence because

$$\begin{aligned} \left| \int_{\Omega} \|x_n\| d\mu - \int_{\Omega} \|x_m\| d\mu \right| &\leq \int_{\Omega} \left| \|x_n\| - \|x_m\| \right| d\mu \\ &\leq \int_{\Omega} \|x_n - x_m\| d\mu. \end{aligned}$$

Thus

$$\int_{\Omega} \|x\| d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \|x_n\| d\mu < \infty$$

Using 23.2.16 it follows y_n satisfies 23.2.14, converges pointwise to x and then from the dominated convergence theorem 23.2.15 holds. This proves the theorem.

Here is a simple corollary.

Corollary 23.2.6 *Let an X valued function x be Bochner integrable and let $L \in \mathcal{L}(X, Y)$ where Y is another Banach space. Then Lx is a Y valued Bochner integrable function and*

$$L \left(\int_{\Omega} x(s) d\mu \right) = \int_{\Omega} Lx(s) d\mu$$

Proof: From Theorem 23.2.5 there is a sequence of simple functions $\{y_n\}$ having the properties listed in that theorem. Then consider $\{Ly_n\}$ which converges pointwise to Lx . Since L is continuous and linear,

$$\int_{\Omega} \|Ly_n - Lx\|_Y d\mu \leq \|L\| \int_{\Omega} \|y_n - x\|_X d\mu$$

which converges to 0. This implies

$$\lim_{m,n \rightarrow \infty} \int_{\Omega} \|Ly_n - Ly_m\| d\mu = 0$$

and so by definition Lx is Bochner integrable. Also

$$\begin{aligned} \int_{\Omega} x(s) d\mu &= \lim_{n \rightarrow \infty} \int_{\Omega} y_n(s) d\mu \\ \int_{\Omega} Lx(s) d\mu &= \lim_{n \rightarrow \infty} \int_{\Omega} Ly_n(s) d\mu \\ &= \lim_{n \rightarrow \infty} L \int_{\Omega} y_n(s) d\mu \\ \left\| L \left(\int_{\Omega} x(s) d\mu \right) - \int_{\Omega} Lx(s) d\mu \right\|_Y \\ &\leq \left\| L \left(\int_{\Omega} x(s) d\mu \right) - L \int_{\Omega} y_n(s) d\mu \right\|_Y \\ &\quad + \left\| \int_{\Omega} Ly_n(s) d\mu - \int_{\Omega} Lx(s) d\mu \right\|_Y < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

whenever n large enough. This proves the corollary.

23.2.2 Taking A Closed Operator Out Of The Integral

Now let X and Y be separable Banach spaces and suppose $A : D(A) \subseteq X \rightarrow Y$ be a closed operator. Recall this means that the graph of A ,

$$G(A) \equiv \{(x, Ax) : x \in D(A)\}$$

is a closed subset of $X \times Y$ with respect to the product topology obtained from the norm

$$\|(x, y)\| = \max(\|x\|, \|y\|).$$

Thus also $G(A)$ is a separable Banach space with the above norm. You can also consider $D(A)$ as a separable Banach space having the norm

$$\|x\|_{D(A)} \equiv \max(\|x\|, \|Ax\|) \quad (23.2.17)$$

which is isometric to $G(A)$ with the mapping, $\theta x \equiv (x, Ax)$.

Lemma 23.2.7 *A closed subspace of a reflexive Banach space is reflexive.*

Proof: Consider the following diagram in which Y is a closed subspace of the reflexive space, X .

$$\begin{array}{ccc} Y'' & \xrightarrow{i^{**} 1-1} & X'' \\ Y' & \xleftarrow{i^* \text{ onto}} & X' \\ Y & \xrightarrow{i} & X \end{array}$$

This diagram follows from theorems on adjoints presented earlier.

Now let $y^{**} \in Y''$. Then $i^{**}y^{**} = Jy$ because X is reflexive. I want to show that $y \in Y$. If it is not in Y then there exists $x^* \in X'$ such that $x^*(y) \neq 0$ but $x^*(Y) = 0$. Then $i^*x^* = 0$. Hence

$$0 = y^{**}(i^*x^*) = i^{**}y^{**}(x^*) = J(y)(x^*) = x^*(y) \neq 0,$$

a contradiction. Hence $y \in Y$. Since i^* is onto, I want to show that for all $x^* \in X'$,

$$i^*x^*(y) = y^{**}(i^*x^*)$$

because this will imply $y^{**} = J_Y(y)$ where J_Y is the James map from Y to Y' . However, the above is equivalent to the following holding for all $x^* \in X'$.

$$i^*x^*(y) = J_Y(y)(i^*x^*) = i^{**}J_Y(y)(x^*) = i^{**}y^{**}(x^*)$$

Since i^{**} is onto, it follows $J_Y(y) = y^{**}$ and this proves it.

Corollary 23.2.8 *Suppose Y is a reflexive Banach space and X is a Banach space such that there exists a continuous one to one mapping, $g : X \rightarrow Y$ such that $g(X)$ is a closed subset of Y . Then X is reflexive.*

Proof: By the open mapping theorem, $g(X)$ and X are homeomorphic since g^{-1} must also be continuous. Therefore, since $g(X)$ is reflexive, it follows X is also.

Lemma 23.2.9 *Suppose V and W are reflexive Banach spaces and that V is a dense subset of W in the topology of W . Then i^*W' is a dense subset of V' where here i is the inclusion map of V into W .*

Proof: First note that i^* is one to one. If $i^*w^* = 0$ for $w^* \in W'$, then this means that for all $v \in V$,

$$i^*w(v) = w^*(v) = 0$$

and since V is dense in W , this shows $w^* = 0$.

Consider the following diagram

$$\begin{array}{ccc} V'' & \xrightarrow{i^{**}} & W'' \\ V' & \xleftarrow{i^*} & W' \\ V & \xrightarrow{i} & W \end{array}$$

in which i is the inclusion map. Next suppose i^*W' is not dense in V' . Then there exists $v^{**} \in V''$ such that $v^{**} \neq 0$ but $v^{**}(i^*W') = 0$. It follows from V being reflexive, that $v^{**} = Jv_0$ where J is the James map from V to V'' for some $v_0 \in V$. Thus for every $w^* \in W'$,

$$\begin{aligned} 0 &= v^{**}(i^*w^*) = i^{**}v^{**}(w^*) \\ &= i^{**}Jv_0(w^*) = Jv_0(i^*w^*) \\ &= i^*w^*(v_0) = w^*(v_0) \end{aligned}$$

and since W' separates the points of W , it follows $v_0 = 0$ which contradicts $v^{**} \neq 0$. This proves the lemma.

Note that in the proof, only V reflexive was used.

This lemma implies an easy corollary.

Corollary 23.2.10 *Let E and F be reflexive Banach spaces and let A be a closed operator, $A : D(A) \subseteq E \rightarrow F$. Suppose also that $D(A)$ is dense in E . Then making $D(A)$ into a Banach space by using the above graph norm given in 23.2.17, it follows that $D(A)$ is a Banach space and i^*E' is a dense subspace of $D(A)'$.*

Proof: First note that $E \times F$ is a reflexive Banach space and $\mathcal{G}(A)$ is a closed subspace of $E \times F$ so it is also a reflexive Banach space. Now $D(A)$ is isometric to $\mathcal{G}(A)$ and so it follows $D(A)$ is a dense subspace of E which is reflexive. Therefore, from Lemma 23.2.9 the conclusion follows.

With this preparation, here is another interesting theorem. This one is about taking outside the integral a closed linear operator as opposed to a continuous linear operator.

Theorem 23.2.11 *Let X, Y be separable Banach spaces and let $A : D(A) \subseteq X \rightarrow Y$ be a closed operator where $D(A)$ is a dense subset of X . Suppose also that i^*X' is a dense subspace of $D(A)'$ where $D(A)$ is a Banach space having the graph norm described in 23.2.17. Suppose that $(\Omega, \mathcal{F}, \mu)$ is a σ finite measure space and $x : \Omega \rightarrow X$ is strongly measurable and it happens that $x(s) \in D(A)$ for all $s \in \Omega$. Then x is strongly measurable as a mapping into $D(A)$. Also Ax is strongly measurable as a map into Y and if*

$$\int_{\Omega} \|x(s)\| d\mu, \int_{\Omega} \|Ax(s)\| d\mu < \infty, \quad (23.2.18)$$

then

$$\int_{\Omega} x(s) d\mu \in D(A) \quad (23.2.19)$$

and

$$A \int_{\Omega} x(s) d\mu = \int_{\Omega} Ax(s) d\mu. \quad (23.2.20)$$

Proof: First of all, consider the assertion that x is strongly measurable into $D(A)$. Letting $f \in D(A)'$ be given, there exists a sequence, $\{g_n\} \subseteq i^*X'$ such that $g_n \rightarrow f$ in $D(A)'$. Therefore,

$$s \rightarrow g_n(x(s))$$

is measurable by assumption and

$$g_n(x(s)) \rightarrow f(x(s))$$

which shows that $s \rightarrow f(x(s))$ is measurable. By the Pettis theorem, it follows

$$s \rightarrow x(s)$$

is strongly measurable as a map into $D(A)$.

It follows from Theorem 23.2.5 there exists a sequence of simple functions, $\{x_n\}$ of the form

$$x_n(s) = \sum_{k=1}^{m_n} a_k^n \chi_{E_k^n}(s), x_n(s) \in D(A),$$

which converges strongly and pointwise to $x(s)$ in $D(A)$. Thus

$$x_n(s) \rightarrow x(s), Ax_n(s) \rightarrow Ax(s),$$

which shows $s \rightarrow Ax(s)$ is strongly measurable in Y as claimed.

It remains to verify the assertions about the integral. 23.2.18 implies x is Bochner integrable as a function having values in $D(A)$ with the norm on $D(A)$ described above. Therefore, by Theorem 23.2.5 there exists a sequence of simple functions $\{y_n\}$ having values in $D(A)$,

$$\lim_{m,n \rightarrow \infty} \int_{\Omega} \|y_n - y_m\|_{D(A)} d\mu = 0,$$

$y_n(s)$ converging pointwise to $x(s)$,

$$\|y_n(s)\|_{D(A)} \leq 2\|x(s)\|_{D(A)}$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|x(s) - y_n(s)\|_{D(A)} ds = 0.$$

Therefore,

$$\int_{\Omega} y_n(s) d\mu \in D(A), \int_{\Omega} y_n(s) d\mu \rightarrow \int_{\Omega} x(s) d\mu \text{ in } X,$$

and since y_n is a simple function and A is linear,

$$A \int_{\Omega} y_n(s) d\mu = \int_{\Omega} Ay_n(s) d\mu \rightarrow \int_{\Omega} Ax(s) d\mu \text{ in } Y.$$

It follows, since A is a closed operator, that

$$\int_{\Omega} x(s) d\mu \in D(A)$$

and

$$A \int_{\Omega} x(s) d\mu = \int_{\Omega} Ax(s) d\mu.$$

This proves the theorem.

Here is another version of this theorem which has different hypotheses.

Theorem 23.2.12 *Let X and Y be separable Banach spaces and let $A : D(A) \subseteq X \rightarrow Y$ be a closed operator. Also let $(\Omega, \mathcal{F}, \mu)$ be a σ finite measure space and let*

$x : \Omega \rightarrow X$ be Bochner integrable such that $x(s) \in D(A)$ for all s . Also suppose Ax is Bochner integrable. Then

$$\int Axd\mu = A \int xd\mu$$

and $\int xd\mu \in D(A)$.

Proof: Consider the graph of A ,

$$G(A) \equiv \{(x, Ax) : x \in D(A)\} \subseteq X \times Y.$$

Then since A is closed, $G(A)$ is a closed separable Banach space with the norm $\|(x, y)\| \equiv \max(\|x\|, \|y\|)$. Therefore, for $g^* \in G(A)'$, one can apply the Hahn Banach theorem and obtain $(x^*, y^*) \in (X \times Y)'$ such that $g^*(x, Ax) = (x^*(x), y^*(Ax))$. Now it follows from the assumptions that $s \rightarrow (x^*(x(s)), y^*(Ax(s)))$ is measurable with values in $G(A)$. It is also separably valued because this is true of $G(A)$. By the Pettis theorem, $s \rightarrow (x(s), A(x(s)))$ must be strongly measurable. Also $\int \|x(s)\| + \|A(x(s))\| d\mu < \infty$ by assumption and so there exists a sequence of simple functions having values in $G(A)$, $\{(x_n(s), Ax_n(s))\}$ which converges to $(x(s), A(x(s)))$ pointwise such that $\int \|(x_n, Ax_n) - (x, Ax)\| d\mu \rightarrow 0$ in $G(A)$. Now for simple functions is it routine to verify that

$$\begin{aligned} \int (x_n, Ax_n) d\mu &= \left(\int x_n d\mu, \int Ax_n d\mu \right) \\ &= \left(\int x_n d\mu, A \int x_n d\mu \right) \end{aligned}$$

Also

$$\begin{aligned} \left\| \int x_n d\mu - \int x d\mu \right\| &\leq \int \|x_n - x\| d\mu \\ &\leq \int \|(x_n, Ax_n) - (x, Ax)\| d\mu \end{aligned}$$

which converges to 0. Also

$$\begin{aligned} \left\| \int Ax_n d\mu - \int Axd\mu \right\| &= \left\| A \int x_n d\mu - \int Axd\mu \right\| \\ &\leq \int \|Ax_n - Ax\| d\mu \\ &\leq \int \|(x_n, Ax_n) - (x, Ax)\| d\mu \end{aligned}$$

and this converges to 0. Therefore, $\int x_n d\mu \rightarrow \int x d\mu$ and $A \int x_n d\mu \rightarrow \int Axd\mu$. Since each $\int x_n d\mu \in D(A)$, and A is closed, this implies $\int x d\mu \in D(A)$ and $A \int x d\mu = \int Axd\mu$. This proves the theorem.

23.3 Operator Valued Functions

Consider the case where $A(s) \in \mathcal{L}(X, Y)$ for X and Y separable Banach spaces. With the operator norm $\mathcal{L}(X, Y)$ is a Banach space and so if A is strongly measurable, the Bochner integral can be defined as before. However, it is also possible to define the Bochner integral of such operator valued functions for more general situations. In this section, $(\Omega, \mathcal{F}, \mu)$ will be a σ finite measure space as usual.

Lemma 23.3.1 *Let $x \in X$ and suppose A is strongly measurable. Then*

$$s \rightarrow A(s)x$$

is strongly measurable as a map into Y .

Proof: Since A is assumed to be strongly measurable, it is the pointwise limit of simple functions of the form

$$A_n(s) \equiv \sum_{k=1}^{m_n} A_k^n \chi_{E_k^n}(s)$$

where A_k^n is in $\mathcal{L}(X, Y)$. It follows $A_n(s)x \rightarrow A(s)x$ for each s and so, since $s \rightarrow A_n(s)x$ is a simple Y valued function, $s \rightarrow A(s)x$ must be strongly measurable.

Definition 23.3.2 *Suppose $A(s) \in \mathcal{L}(X, Y)$ for each $s \in \Omega$ where X, Y are separable Banach spaces. Suppose also that for each $x \in X$,*

$$s \rightarrow A(s)x \text{ is strongly measurable} \quad (23.3.21)$$

and there exists C such that for each $x \in X$,

$$\int_{\Omega} \|A(s)x\| d\mu < C\|x\| \quad (23.3.22)$$

Then $\int_{\Omega} A(s) d\mu \in \mathcal{L}(X, Y)$ is defined by the following formula.

$$\left(\int_{\Omega} A(s) d\mu \right)(x) \equiv \int_{\Omega} A(s)x d\mu \quad (23.3.23)$$

Lemma 23.3.3 *The above definition is well defined. Furthermore, if 23.3.21 holds then $s \rightarrow \|A(s)\|$ is measurable and if 23.3.22 holds, then*

$$\left\| \int_{\Omega} A(s) d\mu \right\| \leq \int_{\Omega} \|A(s)\| d\mu.$$

Proof: It is clear that in case $s \rightarrow A(s)x$ is measurable for all $x \in X$ there exists a unique $\Psi \in \mathcal{L}(X, Y)$ such that

$$\Psi(x) = \int_{\Omega} A(s)x d\mu.$$

This is because $x \rightarrow \int_{\Omega} A(s)x d\mu$ is linear and continuous. Thus $\Psi = \int_{\Omega} A(s) d\mu$ and the definition is well defined.

Now consider the assertion about $s \rightarrow \|A(s)\|$. Let $D' \subseteq B'$ the closed unit ball in Y' be such that D' is countable and

$$\|y\| = \sup_{y^* \in D'} |y^*(y)|.$$

Also let D be a countable dense subset of B , the unit ball of X . Then

$$\begin{aligned} \{s : \|A(s)\| > \alpha\} &= \left\{s : \sup_{x \in D} \|A(s)x\| > \alpha\right\} \\ &= \cup_{x \in D} \{s : \|A(s)x\| > \alpha\} \\ &= \cup_{x \in D} (\cup_{y^* \in D'} \{y^*(A(s)x) > \alpha\}) \end{aligned}$$

and this is measurable because $s \rightarrow A(s)x$ is strongly, hence weakly measurable.

Now suppose 23.3.22 holds. Then for all x ,

$$\int_{\Omega} \|A(s)x\| d\mu < C \|x\|.$$

It follows that for $\|x\| \leq 1$,

$$\begin{aligned} \left\| \left(\int_{\Omega} A(s) d\mu \right) (x) \right\| &= \left\| \int_{\Omega} A(s)x d\mu \right\| \\ &\leq \int_{\Omega} \|A(s)x\| d\mu \\ &\leq \int_{\Omega} \|A(s)\| d\mu \end{aligned}$$

and so

$$\left\| \int_{\Omega} A(s) d\mu \right\| \leq \int_{\Omega} \|A(s)\| d\mu.$$

This proves the lemma.

Now it is interesting to consider the case where $A(s) \in \mathcal{L}(H, H)$ where $s \rightarrow A(s)x$ is strongly measurable and $A(s)$ is compact and self adjoint. Recall the Kuratowski measurable selection theorem, Theorem 8.1.11 on Page 182 listed here for convenience.

Theorem 23.3.4 *Let E be a compact metric space and let $(\Omega, \mathcal{F})$ be a measure space. Suppose $\psi : E \times \Omega \rightarrow \mathbb{R}$ has the property that $x \rightarrow \psi(x, \omega)$ is continuous and $\omega \rightarrow \psi(x, \omega)$ is measurable. Then there exists a measurable function, f having values in E such that*

$$\psi(f(\omega), \omega) = \sup_{x \in E} \psi(x, \omega).$$

Furthermore, $\omega \rightarrow \psi(f(\omega), \omega)$ is measurable.

23.3.1 Review Of Hilbert Schmidt Theorem

Here I will give a proof of the Hilbert Schmidt theorem which will generalize to a result about measurable operators. Recall the following.

Definition 23.3.5 Define $v \otimes u \in \mathcal{L}(H, H)$ by

$$v \otimes u(x) = (x, u)v.$$

$A \in \mathcal{L}(H, H)$ is a compact operator if whenever $\{x_k\}$ is a bounded sequence, there exists a convergent subsequence of $\{Ax_k\}$. Equivalently, A maps bounded sets to sets whose closures are compact or to use other terminology, A maps bounded sets to sets which are precompact.

Lemma 23.3.6 Let H be a separable Hilbert space and suppose $A \in \mathcal{L}(H, H)$ is a compact operator. Let B denote the closed unit ball in H . Then A is continuous as a map from B with the weak topology into H with the strong topology. For $u, v \in H$, $v \otimes u : H \rightarrow H$ is a compact operator. If A is self adjoint and compact, the function

$$x \rightarrow (Ax, x)$$

is continuous on B with respect to the weak topology on B . The function,

$$x \rightarrow (v \otimes u(x), x)$$

is continuous and the operator $u \otimes u$ is self adjoint.

Proof: Since H is separable, it follows from Corollary 13.4.9 on Page 386 that B can be considered as a metric space. Therefore, showing continuity reduces to showing convergent sequences are taken to convergent sequences. Let $x_n \rightarrow x$ weakly in B . Suppose Ax_n does not converge to Ax . Then there exists a subsequence, still denoted by $\{x_n\}$ such that

$$\|Ax_n - Ax\| \geq \varepsilon > 0 \quad (23.3.24)$$

for all n . Then since A maps bounded sets to compact sets, there is a further subsequence, still denoted by $\{x_n\}$ such that Ax_n converges to some $y \in H$. Therefore,

$$\begin{aligned} (y, w) &= \lim_{n \rightarrow \infty} (Ax_n, w) = \lim_{n \rightarrow \infty} (x_n, A^*w) \\ &= (x, A^*w) = (Ax, w) \end{aligned}$$

which shows $Ax = y$ since w is arbitrary. However, this contradicts 23.3.24.

Next consider the claim about $v \otimes u$. Letting $\{x_n\}$ be a bounded sequence,

$$v \otimes u(x_n) = (x_n, u)v.$$

There exists a weakly convergent subsequence of $\{x_n\}$ say $\{x_{n_k}\}$ converging weakly to $x \in H$. Therefore,

$$\|v \otimes u(x_{n_k}) - v \otimes u(x)\| = \|(x_{n_k}, u) - (x, u)\| \|v\|$$

which converges to 0. Thus $v \otimes u$ is compact as claimed. It takes bounded sets to precompact sets.

To verify the assertion about $x \rightarrow (Ax, x)$, let $x_n \rightarrow x$ weakly. Then

$$\begin{aligned} & |(Ax_n, x_n) - (Ax, x)| \\ & \leq |(Ax_n, x_n) - (Ax, x_n)| + |(Ax, x_n) - (Ax, x)| \\ & \leq |(Ax_n, x_n) - (Ax, x_n)| + |(Ax_n, x) - (Ax, x)| \\ & \leq \|Ax_n - Ax\| \|x_n\| + \|Ax_n - Ax\| \|x\| \leq 2 \|Ax_n - Ax\| \end{aligned}$$

which converges to 0.

$$\begin{aligned} & |(v \otimes u(x_n), x_n) - (v \otimes u(x), x)| \\ & = |(x_n, u)(v, x_n) - (x, u)(v, x)| \end{aligned}$$

and this converges to 0 by weak convergence. It follows from the definition that $u \otimes u$ is self adjoint. This proves the lemma.

Observation 23.3.7 *Note that if A is any self adjoint operator,*

$$\overline{(Ax, x)} = (x, Ax) = (Ax, x).$$

so (Ax, x) is real valued.

Lemma 23.3.8 *Let $A \in \mathcal{L}(H, H)$ and suppose it is self adjoint and compact. Let B denote the closed unit ball in H . Let $e \in B$ be such that*

$$|(Ae, e)| = \max_{x \in B} |(Ax, x)|.$$

Then letting $\lambda = (Ae, e)$, it follows $Ae = \lambda e$. If $\lambda \neq 0$, then $\|e\| = 1$ and if $\lambda = 0$, it can be assumed $e = 0$ so it is still the case $Ae = \lambda e$.

Proof: From the above observation, (Ax, x) is always real and since A is compact, $|(Ax, x)|$ achieves a maximum at e . It remains to verify e is an eigenvector. Note that $\|e\| = 1$ whenever $\lambda \neq 0$ since otherwise $|(Ae, e)|$ could be made larger by replacing e with $e/\|e\|$.

Suppose $\lambda = (Ae, e) > 0$. Then it is easy to verify that $\lambda I - A$ is a nonnegative ($((\lambda I - A)x, x) \geq 0$ for all x .) and self adjoint operator. Therefore, the Cauchy Schwarz inequality can be applied to write

$$((\lambda I - A)e, x) \leq ((\lambda I - A)e, e)^{1/2} ((\lambda I - A)x, x)^{1/2} = 0$$

Since this is true for all x it follows $Ae = \lambda e$.

Next suppose $\lambda = (Ae, e) < 0$. Then $-\lambda = (-Ae, e)$ and the previous result can be applied to $-A$ and $-\lambda$. Thus $-\lambda e = -Ae$ and so $Ae = \lambda e$.

Finally consider the case where $\lambda = 0$. Then $0 = (A0, 0)$ and so it suffices to take $e = 0$ as claimed. This proves the lemma.

With these lemmas here is a major theorem, the Hilbert Schmidt theorem.

Theorem 23.3.9 *Let $A \in \mathcal{L}(H, H)$ be a compact self adjoint operator on a Hilbert space. Then there exist real numbers $\{\lambda_k\}_{k=1}^\infty$ and vectors $\{e_k\}_{k=1}^\infty$ such that*

$$\begin{aligned} \|e_k\| &= 1 \text{ if } \lambda_k \neq 0, \\ \|e_k\| &= 0 \text{ if } \lambda_k = 0, \\ (e_k, e_j)_H &= 0 \text{ if } k \neq j, \\ Ae_k &= \lambda_k e_k, \\ |\lambda_n| &\geq |\lambda_{n+1}| \text{ for all } n, \\ \lim_{n \rightarrow \infty} \lambda_n &= 0, \\ \lim_{n \rightarrow \infty} \left\| A - \sum_{k=1}^n \lambda_k (e_k \otimes e_k) \right\|_{\mathcal{L}(H, H)} &= 0. \end{aligned} \quad (23.3.25)$$

Proof: This is done by considering a sequence of compact self adjoint operators, $A, A_1, A_2, \dots$. Here is how these are defined. Using Lemma 23.3.8 let e_1, λ_1 be given by that lemma such that

$$|(Ae_1, e_1)| = \max_{x \in B} |(Ax, x)|, \quad \lambda_1 = (Ae_1, e_1).$$

Then by that lemma, $Ae_1 = \lambda_1 e_1$ and $\|e_1\| = 1$ if $\lambda_1 \neq 0$ while $e_1 = 0$ if $\lambda_1 = 0$.

If A_n has been obtained, use Lemma 23.3.8 to obtain e_{n+1} and λ_{n+1} such that

$$|(A_n e_{n+1}, e_{n+1})| = \max_{x \in B} |(A_n x, x)|, \quad \lambda_{n+1} = (A_n e_{n+1}, e_{n+1}).$$

By that lemma again, $A_n e_{n+1} = \lambda_{n+1} e_{n+1}$ and $\|e_{n+1}\| = 1$ if $\lambda_{n+1} \neq 0$ while $e_{n+1} = 0$ if $\lambda_{n+1} = 0$. Then

$$A_{n+1} \equiv A_n - \lambda_{n+1} e_{n+1} \otimes e_{n+1}$$

Thus

$$A_n = A - \sum_{k=1}^n \lambda_k e_k \otimes e_k. \quad (23.3.26)$$

Claim 1: If $k < n+1$ then $(e_{n+1}, e_k) = 0$. Also $Ae_k = \lambda_k e_k$ for all k .

Proof of claim: From the above,

$$\lambda_{n+1} e_{n+1} = A_n e_{n+1} = Ae_{n+1} - \sum_{k=1}^n \lambda_k (e_{n+1}, e_k) e_k.$$

If $\lambda_{n+1} = 0$, then $(e_{n+1}, e_k) = 0$ because $e_{n+1} = 0$. If $\lambda_{n+1} \neq 0$, then from the above and an induction hypothesis

$$\begin{aligned} \lambda_{n+1} (e_{n+1}, e_j) &= (Ae_{n+1}, e_j) - \sum_{k=1}^n \lambda_k (e_{n+1}, e_k) (e_k, e_j) \\ &= (e_{n+1}, Ae_j) - \sum_{k=1}^n \lambda_k (e_{n+1}, e_k) (e_k, e_j) \\ &= \lambda_j (e_{n+1}, e_j) - \lambda_j (e_{n+1}, e_j) = 0. \end{aligned}$$

To verify the second part of this claim,

$$\lambda_{n+1}e_{n+1} = A_n e_{n+1} = A e_{n+1} - \sum_{k=1}^n \lambda_k e_k (e_{n+1}, e_k) = A e_{n+1}$$

This proves the claim.

Claim 2: $|\lambda_n| \geq |\lambda_{n+1}|$.

Proof of claim: From 23.3.26 and the definition of A_n and $e_k \otimes e_k$,

$$\begin{aligned} \lambda_{n+1} &= (A_n e_{n+1}, e_{n+1}) \\ &= (A_{n-1} e_{n+1}, e_{n+1}) - \lambda_n |(e_n, e_{n+1})|^2 \\ &= (A_{n-1} e_{n+1}, e_{n+1}) \end{aligned}$$

By the previous claim. Therefore,

$$|\lambda_{n+1}| = |(A_{n-1} e_{n+1}, e_{n+1})| \leq |(A_{n-1} e_n, e_n)| = |\lambda_n|$$

by the definition of $|\lambda_n|$. (e_n makes $|(A_{n-1} x, x)|$ as large as possible, not necessarily e_{n+1} .)

Claim 3: $\lim_{n \rightarrow \infty} \lambda_n = 0$.

Proof of claim: If for some n , $\lambda_n = 0$, then $\lambda_k = 0$ for all $k > n$ by claim 2. Assume then that $\lambda_k \neq 0$ for any k . Then if $\lim_{k \rightarrow \infty} |\lambda_k| = \varepsilon > 0$, contrary to the claim, $\|e_k\| = 1$ for all k and

$$\begin{aligned} \|Ae_n - Ae_m\|^2 &= \|\lambda_n e_n - \lambda_m e_m\|^2 \\ &= \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon^2 \end{aligned}$$

which shows there is no Cauchy subsequence of $\{Ae_n\}_{n=1}^\infty$, which contradicts the compactness of A . This proves the claim.

Claim 4: $\|A_n\| \rightarrow 0$

Proof of claim: Let $x, y \in B$

$$\begin{aligned} |\lambda_{n+1}| &\geq \left| \left(A_n \frac{x+y}{2}, \frac{x+y}{2} \right) \right| \\ &= \left| \frac{1}{4} (A_n x, x) + \frac{1}{4} (A_n y, y) + \frac{1}{2} (A_n x, y) \right| \\ &\geq \frac{1}{2} |(A_n x, y)| - \frac{1}{4} |(A_n x, x) + (A_n y, y)| \\ &\geq \frac{1}{2} |(A_n x, y)| - \frac{1}{4} (|(A_n x, x)| + |(A_n y, y)|) \\ &\geq \frac{1}{2} |(A_n x, y)| - \frac{1}{2} |\lambda_{n+1}| \end{aligned}$$

and so

$$3|\lambda_{n+1}| \geq |(A_n x, y)|.$$

It follows $\|A_n\| \leq 3|\lambda_{n+1}|$. This proves the claim.

By 23.3.26 this proves 23.3.25 and completes the proof.

23.3.2 Measurable Compact Operators

Here the operators will be of the form $A(s)$ where $s \in \Omega$ and $s \rightarrow A(s)x$ is strongly measurable and $A(s)$ is a compact operator in $\mathcal{L}(H, H)$.

Theorem 23.3.10 *Let $A(s) \in \mathcal{L}(H, H)$ be a compact self adjoint operator and H is a separable Hilbert space such that $s \rightarrow A(s)x$ is strongly measurable. Then there exist real numbers $\{\lambda_k(s)\}_{k=1}^\infty$ and vectors $\{e_k(s)\}_{k=1}^\infty$ such that*

$$\begin{aligned} \|e_k(s)\| &= 1 \text{ if } \lambda_k \neq 0, \\ \|e_k(s)\| &= 0 \text{ if } \lambda_k = 0, \\ (e_k(s), e_j(s))_H &= 0 \text{ if } k \neq j, \\ A(s)e_k(s) &= \lambda_k(s)e_k(s), \\ |\lambda_n(s)| &\geq |\lambda_{n+1}(s)| \text{ for all } n, \\ \lim_{n \rightarrow \infty} \lambda_n(s) &= 0, \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left\| A(s) - \sum_{k=1}^n \lambda_k(s) (e_k(s) \otimes e_k(s)) \right\|_{\mathcal{L}(H, H)} = 0.$$

The function $s \rightarrow \lambda_j(s)$ is measurable and $s \rightarrow e_j(s)$ is strongly measurable.

Proof: It is simply a repeat of the above proof of the Hilbert Schmidt theorem except at every step when the e_k and λ_k are defined, you use the Kuratowski measurable selection theorem, Theorem 23.3.4 on Page 716 to obtain $\lambda_k(s)$ is measurable and that $s \rightarrow e_k(s)$ is also measurable.

When you consider $\max_{x \in B} |(A_n(s)x, x)|$, let $\psi(x, s) = |(A_n(s)x, x)|$. Then ψ is continuous in x by Lemma 23.3.6 on Page 717 and it is measurable in s by assumption. Therefore, by the Kuratowski theorem, $e_k(s)$ is measurable in the sense that inverse images of weakly open sets in B are measurable. However, by Lemma 23.1.9 on Page 702 this is the same as weakly measurable. Since H is separable, this implies $s \rightarrow e_k(s)$ is also strongly measurable. The measurability of λ_k and e_k is the only new thing here and so this completes the proof.

23.4 Fubini's Theorem For Bochner Integrals

Now suppose $(\Omega_1, \mathcal{F}, \mu)$ and $(\Omega_2, \mathcal{S}, \lambda)$ are two σ finite measure spaces. Recall the notion of product measure. There was a σ algebra, denoted by $\mathcal{F} \times \mathcal{S}$ which is the smallest σ algebra containing the elementary sets, (finite disjoint unions of measurable rectangles) and a measure, denoted by $\mu \times \lambda$ defined on this σ algebra such that for $E \in \mathcal{F} \times \mathcal{S}$,

$$s_1 \rightarrow \lambda(E_{s_1}), (E_{s_1} \equiv \{s_2 : (s_1, s_2) \in E\})$$

is μ measurable and

$$s_2 \rightarrow \mu(E_{s_2}), \quad (E_{s_2} \equiv \{s_1 : (s_1, s_2) \in E\})$$

is λ measurable. In terms of nonnegative functions which are $\mathcal{F} \times \mathcal{S}$ measurable,

$$\begin{aligned} s_1 &\rightarrow f(s_1, s_2) \text{ is } \mu \text{ measurable,} \\ s_2 &\rightarrow f(s_1, s_2) \text{ is } \lambda \text{ measurable,} \\ s_1 &\rightarrow \int_{\Omega_2} f(s_1, s_2) d\lambda \text{ is } \mu \text{ measurable,} \\ s_2 &\rightarrow \int_{\Omega_1} f(s_1, s_2) d\mu \text{ is } \lambda \text{ measurable,} \end{aligned}$$

and the conclusion of Fubini's theorem holds.

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f d(\mu \times \lambda) &= \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) d\lambda d\mu \\ &= \int_{\Omega_2} \int_{\Omega_1} f(s_1, s_2) d\mu d\lambda. \end{aligned}$$

The following theorem is the version of Fubini's theorem valid for Bochner integrable functions.

Theorem 23.4.1 *Let $f : \Omega_1 \times \Omega_2 \rightarrow X$ be strongly measurable with respect to $\mu \times \lambda$ and suppose*

$$\int_{\Omega_1 \times \Omega_2} \|f(s_1, s_2)\| d(\mu \times \lambda) < \infty. \quad (23.4.27)$$

Then there exist a set of μ measure zero, N and a set of λ measure zero, M such that the following formula holds with all integrals making sense.

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f(s_1, s_2) d(\mu \times \lambda) &= \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_N(s_1) d\lambda d\mu \\ &= \int_{\Omega_2} \int_{\Omega_1} f(s_1, s_2) \mathcal{X}_M(s_2) d\mu d\lambda. \end{aligned}$$

Proof: First note that from 23.4.27 and the usual Fubini theorem for nonnegative valued functions,

$$\int_{\Omega_1 \times \Omega_2} \|f(s_1, s_2)\| d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} \|f(s_1, s_2)\| d\lambda d\mu$$

and so

$$\int_{\Omega_2} \|f(s_1, s_2)\| d\lambda < \infty \quad (23.4.28)$$

for μ a.e. s_1 . Say for all $s_1 \notin N$ where $\mu(N) = 0$.

Let $\phi \in X'$. Then $\phi \circ f$ is $\mathcal{F} \times \mathcal{S}$ measurable and

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2} |\phi \circ f(s_1, s_2)| d(\mu \times \lambda) \\ & \leq \int_{\Omega_1 \times \Omega_2} \|\phi\| \|f(s_1, s_2)\| d(\mu \times \lambda) < \infty \end{aligned}$$

and so from the usual Fubini theorem for complex valued functions,

$$\int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) d\lambda d\mu. \quad (23.4.29)$$

Now also if you fix s_2 , it follows from the definition of strongly measurable and the properties of product measure mentioned above that

$$s_1 \rightarrow f(s_1, s_2)$$

is strongly measurable. Also, by 23.4.28

$$\int_{\Omega_2} \|f(s_1, s_2)\| d\lambda < \infty$$

for $s_1 \notin N$. Therefore, by Theorem 23.2.5 $s_2 \rightarrow f(s_1, s_2) \mathcal{X}_{N^c}(s_1)$ is Bochner integrable. By 23.4.29 and 23.2.6

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) d(\mu \times \lambda) \\ &= \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) d\lambda d\mu \\ &= \int_{\Omega_1} \int_{\Omega_2} \phi(f(s_1, s_2) \mathcal{X}_{N^c}(s_1)) d\lambda d\mu \\ &= \int_{\Omega_1} \phi \left(\int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^c}(s_1) d\lambda \right) d\mu. \end{aligned} \quad (23.4.30)$$

Each iterated integral makes sense and

$$\begin{aligned} s_1 & \rightarrow \int_{\Omega_2} \phi(f(s_1, s_2) \mathcal{X}_{N^c}(s_1)) d\lambda \\ &= \phi \left(\int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^c}(s_1) d\lambda \right) \end{aligned} \quad (23.4.31)$$

is μ measurable because

$$\begin{aligned} (s_1, s_2) & \rightarrow \phi(f(s_1, s_2) \mathcal{X}_{N^c}(s_1)) \\ &= \phi(f(s_1, s_2)) \mathcal{X}_{N^c}(s_1) \end{aligned}$$

is product measurable. Now consider the function,

$$s_1 \rightarrow \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) d\lambda. \quad (23.4.32)$$

I want to show this is also Bochner integrable with respect to μ so I can factor out ϕ once again. It's measurability follows from the Pettis theorem and the above observation 23.4.31. Also,

$$\begin{aligned} & \int_{\Omega_1} \left\| \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) d\lambda \right\| d\mu \\ & \leq \int_{\Omega_1} \int_{\Omega_2} \|f(s_1, s_2)\| d\lambda d\mu \\ & = \int_{\Omega_1 \times \Omega_2} \|f(s_1, s_2)\| d(\mu \times \lambda) < \infty. \end{aligned}$$

Therefore, the function in 23.4.32 is indeed Bochner integrable and so in 23.4.30 the ϕ can be taken outside the last integral. Thus,

$$\begin{aligned} & \phi \left(\int_{\Omega_1 \times \Omega_2} f(s_1, s_2) d(\mu \times \lambda) \right) \\ & = \int_{\Omega_1 \times \Omega_2} \phi \circ f(s_1, s_2) d(\mu \times \lambda) \\ & = \int_{\Omega_1} \int_{\Omega_2} \phi \circ f(s_1, s_2) d\lambda d\mu \\ & = \int_{\Omega_1} \phi \left(\int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) d\lambda \right) d\mu \\ & = \phi \left(\int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) d\lambda d\mu \right). \end{aligned}$$

Since X' separates the points,

$$\int_{\Omega_1 \times \Omega_2} f(s_1, s_2) d(\mu \times \lambda) = \int_{\Omega_1} \int_{\Omega_2} f(s_1, s_2) \mathcal{X}_{N^C}(s_1) d\lambda d\mu.$$

The other formula follows from similar reasoning. This proves the theorem.

23.5 The Spaces $L^p(\Omega; X)$

Definition 23.5.1 $x \in L^p(\Omega; X)$ for $p \in [1, \infty)$ if x is strongly measurable and

$$\int_{\Omega} \|x(s)\|^p d\mu < \infty$$

Also

$$\|x\|_{L^p(\Omega; X)} \equiv \|x\|_p \equiv \left(\int_{\Omega} \|x(s)\|^p d\mu \right)^{1/p}. \quad (23.5.33)$$

As in the case of scalar valued functions, two functions in $L^p(\Omega; X)$ are considered equal if they are equal a.e. With this convention, and using the same arguments found in the presentation of scalar valued functions it is clear that $L^p(\Omega; X)$ is a normed linear space with the norm given by 23.5.33. In fact, $L^p(\Omega; X)$ is a Banach space. This is the main contribution of the next theorem.

Lemma 23.5.2 *If x_n is a Cauchy sequence in $L^p(\Omega; X)$ satisfying*

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|_p < \infty,$$

then there exists $x \in L^p(\Omega; X)$ such that $x_n(s) \rightarrow x(s)$ a.e. and

$$\|x - x_n\|_p \rightarrow 0.$$

Proof: Let

$$g_N(s) \equiv \sum_{n=1}^N \|x_{n+1}(s) - x_n(s)\|_X$$

Then by the triangle inequality,

$$\begin{aligned} \left(\int_{\Omega} g_N(s)^p d\mu \right)^{1/p} &\leq \sum_{n=1}^N \left(\int_{\Omega} \|x_{n+1}(s) - x_n(s)\|^p d\mu \right)^{1/p} \\ &\leq \sum_{n=1}^{\infty} \|x_{n+1} - x_n\|_p < \infty. \end{aligned}$$

Let

$$g(s) = \lim_{N \rightarrow \infty} g_N(s) = \sum_{n=1}^{\infty} \|x_{n+1}(s) - x_n(s)\|_X.$$

By the monotone convergence theorem,

$$\left(\int_{\Omega} g(s)^p d\mu \right)^{1/p} = \lim_{N \rightarrow \infty} \left(\int_{\Omega} g_N(s)^p d\mu \right)^{1/p} < \infty.$$

Therefore, there exists a set of measure 0, E , such that for $s \notin E$, $g(s) < \infty$. Hence, for $s \notin E$,

$$\lim_{N \rightarrow \infty} x_{N+1}(s)$$

exists because

$$x_{N+1}(s) = x_{N+1}(s) - x_1(s) + x_1(s) = \sum_{n=1}^N (x_{n+1}(s) - x_n(s)) + x_1(s).$$

Thus, if $N > M$, and s is a point where $g(s) < \infty$,

$$\begin{aligned} \|x_{N+1}(s) - x_{M+1}(s)\|_X &\leq \sum_{n=M+1}^N \|x_{n+1}(s) - x_n(s)\|_X \\ &\leq \sum_{n=M+1}^{\infty} \|x_{n+1}(s) - x_n(s)\|_X \end{aligned}$$

which shows that $\{x_{N+1}(s)\}_{N=1}^{\infty}$ is a Cauchy sequence. Now let

$$x(s) \equiv \begin{cases} \lim_{N \rightarrow \infty} x_N(s) & \text{if } s \notin E, \\ 0 & \text{if } s \in E. \end{cases}$$

By Theorem 23.1.2, $x_n(\Omega)$ is separable for each n . Therefore, $x(\Omega)$ is also separable. Also, if $f \in X'$, then

$$f(x(s)) = \lim_{N \rightarrow \infty} f(x_N(s))$$

if $s \notin E$ and $f(x(s)) = 0$ if $s \in E$. Therefore, $f \circ x$ is measurable because it is the limit of the measurable functions,

$$f \circ x_N \chi_{EC}.$$

Since x is weakly measurable and $x(\Omega)$ is separable, Corollary 23.1.8 shows that x is strongly measurable. By Fatou's lemma,

$$\int_{\Omega} \|x(s) - x_N(s)\|^p d\mu \leq \liminf_{M \rightarrow \infty} \int_{\Omega} \|x_M(s) - x_N(s)\|^p d\mu.$$

But if N and M are large enough with $M > N$,

$$\begin{aligned} \left(\int_{\Omega} \|x_M(s) - x_N(s)\|^p d\mu \right)^{1/p} &\leq \sum_{n=N}^M \|x_{n+1} - x_n\|_p \\ &\leq \sum_{n=N}^{\infty} \|x_{n+1} - x_n\|_p < \varepsilon \end{aligned}$$

and this shows, since ε is arbitrary, that

$$\lim_{N \rightarrow \infty} \int_{\Omega} \|x(s) - x_N(s)\|^p d\mu = 0.$$

It remains to show $x \in L^p(\Omega; X)$. This follows from the above and the triangle inequality. Thus, for N large enough,

$$\left(\int_{\Omega} \|x(s)\|^p d\mu \right)^{1/p}$$

$$\begin{aligned}
&\leq \left(\int_{\Omega} \|x_N(s)\|^p d\mu \right)^{1/p} + \left(\int_{\Omega} \|x(s) - x_N(s)\|^p d\mu \right)^{1/p} \\
&\leq \left(\int_{\Omega} \|x_N(s)\|^p d\mu \right)^{1/p} + \varepsilon < \infty.
\end{aligned}$$

This proves the lemma.

Theorem 23.5.3 $L^p(\Omega; X)$ is complete. Also every Cauchy sequence has a subsequence which converges pointwise.

Proof: If $\{x_n\}$ is Cauchy in $L^p(\Omega; X)$, extract a subsequence $\{x_{n_k}\}$ satisfying

$$\|x_{n_{k+1}} - x_{n_k}\|_p \leq 2^{-k}$$

and apply Lemma 23.5.2. The pointwise convergence of this subsequence was established in the proof of this lemma. This proves the theorem because if a subsequence of a Cauchy sequence converges, then the Cauchy sequence must also converge.

Observation 23.5.4 If the measure space is Lebesgue measure then you have continuity of translation in $L^p(\mathbb{R}^n; X)$ in the usual way. More generally, for μ a Radon measure on Ω a locally compact Hausdorff space, $C_c(\Omega; X)$ is dense in $L^p(\Omega; X)$. Here $C_c(\Omega; X)$ is the space of continuous X valued functions which have compact support in Ω . The proof of this little observation follows immediately from approximating with simple functions and then applying the appropriate considerations to the simple functions.

Clearly Fatou's lemma and the monotone convergence theorem make no sense for functions with values in a Banach space but the dominated convergence theorem holds in this setting.

Theorem 23.5.5 If x is strongly measurable and $x_n(s) \rightarrow x(s)$ a.e. with

$$\|x_n(s)\| \leq g(s) \text{ a.e.}$$

where $g \in L^1(\Omega)$, then x is Bochner integrable and

$$\int_{\Omega} x(s) d\mu = \lim_{n \rightarrow \infty} \int_{\Omega} x_n(s) d\mu.$$

Proof: $\|x_n(s) - x(s)\| \leq 2g(s)$ a.e. so by the usual dominated convergence theorem,

$$0 = \lim_{n \rightarrow \infty} \int_{\Omega} \|x_n(s) - x(s)\| d\mu.$$

Also,

$$\int_{\Omega} \|x_n(s) - x_m(s)\| d\mu$$

$$\leq \int_{\Omega} \|x_n(s) - x(s)\| d\mu + \int_{\Omega} \|x_m(s) - x(s)\| d\mu,$$

and so $\{x_n\}$ is a Cauchy sequence in $L^1(\Omega; X)$. Therefore, by Theorem 23.5.3, there exists $y \in L^1(\Omega; X)$ and a subsequence $x_{n'}$ satisfying

$$x_{n'}(s) \rightarrow y(s) \text{ a.e. and in } L^1(\Omega; X).$$

But $x(s) = \lim_{n' \rightarrow \infty} x_{n'}(s)$ a.e. and so $x(s) = y(s)$ a.e. Hence

$$\int_{\Omega} \|x(s)\| d\mu = \int_{\Omega} \|y(s)\| d\mu < \infty$$

which shows that x is Bochner integrable. Finally, since the integral is linear,

$$\begin{aligned} \left\| \int_{\Omega} x(s) d\mu - \int_{\Omega} x_n(s) d\mu \right\| &= \left\| \int_{\Omega} (x(s) - x_n(s)) d\mu \right\| \\ &\leq \int_{\Omega} \|x_n(s) - x(s)\| d\mu, \end{aligned}$$

and this last integral converges to 0. This proves the theorem.

One can also give a version of the Vitali convergence theorem.

Definition 23.5.6 Let $\mathcal{A} \subseteq L^1(\Omega; X)$. Then $\mathcal{A}$ is said to be uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\mu(E) < \delta$, it follows

$$\int_E \|f\|_X d\mu < \varepsilon$$

for all $f \in \mathcal{A}$. It is bounded if

$$\sup_{f \in \mathcal{A}} \int_{\Omega} \|f\|_X d\mu < \infty.$$

Theorem 23.5.7 Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let X be a separable Banach space. Let $\{f_n\} \subseteq L^1(\Omega; X)$ be uniformly integrable and bounded such that $f_n(\omega) \rightarrow f(\omega)$ for each $\omega \in \Omega$. Then $f \in L^1(\Omega; X)$ and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|f_n - f\|_X d\mu = 0.$$

Proof: Let $\varepsilon > 0$ be given. Then by uniform integrability there exists $\delta > 0$ such that if $\mu(E) < \delta$ then

$$\int_E \|f_n\| d\mu < \varepsilon/3.$$

By Fatou's lemma the same inequality holds for f . Also Fatou's lemma shows $f \in L^1(\Omega; X)$, f being measurable because of Theorem 8.1.9.

By Egoroff's theorem, Theorem 8.3.11, there exists a set of measure less than δ , E such that the convergence of $\{f_n\}$ to f is uniform off E . Therefore,

$$\begin{aligned} \int_{\Omega} \|f - f_n\| d\mu &\leq \int_E (\|f\|_X + \|f_n\|_X) d\mu + \int_{E^c} \|f - f_n\|_X d\mu \\ &< \frac{2\varepsilon}{3} + \int_{E^c} \frac{\varepsilon}{(\mu(\Omega) + 1)3} d\mu < \varepsilon \end{aligned}$$

if n is large enough. This proves the theorem.

Note that a convenient way to achieve uniform integrability is to simply say $\{f_n\}$ is bounded in $L^p(\Omega; X)$ for some $p > 1$. This follows from Holder's inequality.

$$\int_E \|f_n\| d\mu \leq \left(\int_E d\mu \right)^{1/p'} \left(\int_{\Omega} \|f_n\|^p d\mu \right)^{1/p}.$$

The following theorem is interesting.

Theorem 23.5.8 *Let $1 \leq p < \infty$ and let $p < r \leq \infty$. Then $L^r([0, T], X)$ is a Borel subset of $L^p([0, T]; X)$. Letting $C([0, T]; X)$ denote the functions having values in X which are continuous, $C([0, T]; X)$ is also a Borel subset of $L^p([0, T]; X)$. Here the measure is ordinary one dimensional Lebesgue measure on $[0, T]$.*

Proof: First consider the claim about $L^r([0, T]; X)$. Let

$$B_M \equiv \left\{ x \in L^p([0, T]; X) : \|x\|_{L^r([0, T]; X)} \leq M \right\}.$$

Then B_M is a closed subset of $L^p([0, T]; X)$. Here is why. If $\{x_n\}$ is a sequence of elements of B_M and $x_n \rightarrow x$ in $L^p([0, T]; X)$, then passing to a subsequence, still denoted by x_n , it can be assumed $x_n(s) \rightarrow x(s)$ a.e. Hence Fatou's lemma can be applied to conclude

$$\int_0^T \|x(s)\|^r ds \leq \liminf_{n \rightarrow \infty} \int_0^T \|x_n(s)\|^r ds \leq M^r < \infty.$$

Now $\cup_{M=1}^{\infty} B_M = L^r([0, T]; X)$. Note this did not depend on the measure space used. It would have been equally valid on any measure space.

Consider now $C([0, T]; X)$. The norm on this space is the usual norm, $\|\cdot\|_{\infty}$. The argument above shows $\|\cdot\|_{\infty}$ is a Borel measurable function on $L^p([0, T]; X)$. This is because $B_M \equiv \{x \in L^p([0, T]; X) : \|x\|_{\infty} \leq M\}$ is a closed, hence Borel subset of $L^p([0, T]; X)$. Now let $\theta \in \mathcal{L}(L^p([0, T]; X), L^p(\mathbb{R}; X))$ such that $\theta(x(t)) = x(t)$ for all $t \in [0, T]$ and also $\theta \in \mathcal{L}(C([0, T]; X), BC(\mathbb{R}; X))$ where $BC(\mathbb{R}; X)$ denotes the bounded continuous functions with a norm given by

$$\|x\| \equiv \sup_{t \in \mathbb{R}} \|x(t)\|,$$

and θx has compact support.

For example, you could define

$$\tilde{x}(t) \equiv \begin{cases} x(t) & \text{if } t \in [0, T] \\ x(2T - t) & \text{if } t \in [T, 2T] \\ x(-t) & \text{if } t \in [-T, 0] \\ 0 & \text{if } t \notin [-T, 2T] \end{cases}$$

and let $\Phi \in C_c^\infty(-T, 2T)$ such that $\Phi(t) = 1$ for $t \in [0, T]$. Then you could let

$$\theta x(t) \equiv \Phi(t) \tilde{x}(t).$$

Then let $\{\phi_n\}$ be a mollifier and define

$$\psi_n x(t) \equiv \phi_n * \theta x(t).$$

It follows $\psi_n x$ is uniformly continuous because

$$\begin{aligned} & \|\psi_n x(t) - \psi_n x(t')\|_X \\ & \leq \int_{\mathbb{R}} |\phi_n(t' - s) - \phi_n(t - s)| \|\theta x(s)\|_X ds \\ & \leq C \|x\|_p \left(\int_{\mathbb{R}} |\phi_n(t' - s) - \phi_n(t - s)|^{p'} ds \right)^{1/p'} \end{aligned}$$

Also for $x \in C([0, T]; X)$, it follows from usual mollifier arguments that

$$\|\psi_n x - x\|_{L^\infty([0, T]; X)} \rightarrow 0.$$

Here is why. For $t \in [0, T]$,

$$\begin{aligned} \|\psi_n x(t) - x(t)\|_X & \leq \int_{\mathbb{R}} \phi_n(s) \|\theta x(t - s) - \theta x(t)\| ds \\ & \leq C_\theta \int_{-1/n}^{1/n} \phi_n(s) ds \varepsilon = C_\theta \varepsilon \end{aligned}$$

provided n is large enough due to the compact support and consequent uniform continuity of θx .

If $\|\psi_n x - x\|_{L^\infty([0, T]; X)} \rightarrow 0$, then $\{\psi_n x\}$ must be a Cauchy sequence in $C([0, T]; X)$ and this requires that x equals a continuous function a.e. Thus $C([0, T]; X)$ consists exactly of those functions, x of $L^p([0, T]; X)$ such that $\|\psi_n x - x\|_\infty \rightarrow 0$. It follows

$$C([0, T]; X) = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty \left\{ x \in L^p([0, T]; X) : \|\psi_k x - x\|_\infty \leq \frac{1}{n} \right\}. \quad (23.5.34)$$

It only remains to show

$$S \equiv \{x \in L^p([0, T]; X) : \|\psi_k x - x\|_\infty \leq \alpha\}$$

is a Borel set. Suppose then that $x_n \in S$ and $x_n \rightarrow x$ in $L^p([0, T]; X)$. Then there exists a subsequence, still denoted by n such that $x_n \rightarrow x$ pointwise a.e. as well as in L^p . There exists a set of measure 0 such that for all n , and t not in this set,

$$\begin{aligned} \|\psi_k x_n(t) - x_n(t)\| &\equiv \left\| \int_{-1/k}^{1/k} \phi_k(s) (\theta x_n(t-s)) ds - x_n(t) \right\| \leq \alpha \\ x_n(t) &\rightarrow x(t). \end{aligned}$$

Then

$$\begin{aligned} &\|\psi_k x_n(t) - x_n(t) - (\psi_k x(t) - x(t))\| \\ &\leq \|x_n(t) - x(t)\|_X + \left\| \int_{-1/k}^{1/k} \phi_k(s) (\theta x_n(t-s) - \theta x(t-s)) ds \right\| \\ &\leq \|x_n(t) - x(t)\|_X + C_{k,\theta} \|x_n - x\|_{L^p(0,T;X)} \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$. It follows that for a.e. t ,

$$\|\psi_k x(t) - x(t)\| \leq \alpha.$$

Thus S is closed and so the set in 23.5.34 is a Borel set. This proves the theorem.

As in the scalar case, the following lemma holds in this more general context.

Lemma 23.5.9 *Let (Ω, μ) be a regular measure space where Ω is a locally compact Hausdorff space. Then $C_c(\Omega; X)$ the space of continuous functions having compact support and values in X is dense in $L^p(0, T; X)$ for all $p \in [0, \infty)$. For any σ finite measure space, the simple functions are dense in $L^p(0, T; X)$.*

Proof: First is it shown the simple functions are dense in $L^p(0, T; X)$. Let $f \in L^p(0, T; X)$ and let $\{x_n\}$ denote a sequence of simple functions which converge to f pointwise which also have the property that

$$\|x_n(s)\| \leq 2\|f(s)\|$$

Then

$$\int_{\Omega} \|x_n(s) - f(s)\|^p d\mu \rightarrow 0$$

from the dominated convergence theorem. Therefore, the simple functions are indeed dense in $L^p(0, T; X)$.

Next suppose (Ω, μ) is a regular measure space. If $x(s) \equiv \sum_i a_i \chi_{E_i}(s)$ is a simple function, then by regularity, there exist compact sets, K_i and open sets, V_i such that $K_i \subseteq E_i \subseteq V_i$ and $\mu(V_i \setminus K_i)^{1/p} < \varepsilon / \sum_i \|a_i\|$. Let $K_i \prec h_i \prec V_i$. Then consider

$$\sum_i a_i h_i \in C_c(\Omega).$$

By the triangle inequality,

$$\begin{aligned}
 & \left(\int_{\Omega} \left\| \sum_i a_i h_i(s) - a_i \mathcal{X}_{E_i}(s) \right\|^p d\mu \right)^{1/p} \\
 & \leq \sum_i \left(\int_{\Omega} \|a_i\| |h_i(s) - \mathcal{X}_{E_i}(s)|^p d\mu \right)^{1/p} \\
 & \leq \sum_i \left(\int_{\Omega} \|a_i\|^p |h_i(s) - \mathcal{X}_{E_i}(s)|^p d\mu \right)^{1/p} \\
 & \leq \sum_i \|a_i\| \left(\int_{V_i \setminus K_i} d\mu \right)^{1/p} \\
 & \leq \sum_i \|a_i\| \mu(V_i \setminus K_i)^{1/p} < \varepsilon
 \end{aligned}$$

Since ε is arbitrary, this and the first part of the lemma shows $C_c(\Omega; X)$ is dense in $L^p(\Omega; X)$.

23.6 Measurable Representatives

In this section consider the special case where $X = L^1(B, \nu)$ where $(B, \mathcal{F}, \nu)$ is a σ finite measure space and $x \in L^1(\Omega; X)$. Thus for each $s \in \Omega$, $x(s) \in L^1(B, \nu)$. In general, the map

$$(s, t) \rightarrow x(s)(t)$$

will not be measurable, but one can obtain a measurable representative. This is important because it allows the use of Fubini's theorem on the measurable representative.

By Theorem 23.2.5, there exists a sequence of simple functions, $\{x_n\}$, of the form

$$x_n(s) = \sum_{k=1}^m a_k \mathcal{X}_{E_k}(s) \quad (23.6.35)$$

where $a_k \in L^1(B, \nu)$ which satisfy the conditions of Definition 23.2.3 and

$$\|x_n - x\|_1 \rightarrow 0. \quad (23.6.36)$$

Because of the form of x_n given in 23.6.35, if

$$x_n(s, t) \equiv x_n(s)(t),$$

then x_n is measurable.

$$\int_{\Omega} \int_B |x_n(s, t) - x_m(s, t)| d\nu d\mu \leq \int_{\Omega} \int_B |x_n(s, t) - x(s)(t)| d\nu d\mu$$

$$+ \int_{\Omega} \int_B |x_m(s, t) - x(s)(t)| d\nu d\mu. \quad (23.6.37)$$

It follows from 23.6.37 and 23.6.36 that $\{x_n\}$ is a Cauchy sequence in $L^1(\Omega \times B)$. Therefore, there exists $y \in L^1(\Omega \times B)$ and a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$, such that

$$\lim_{n \rightarrow \infty} x_n(s, t) = y(s, t) \text{ a.e.}$$

and

$$\lim_{n \rightarrow \infty} \|x_n - y\|_1 = 0.$$

It follows that

$$\begin{aligned} \int_{\Omega} \int_B |y(s, t) - x(s)(t)| d\nu d\mu &\leq \int_{\Omega} \int_B |y(s, t) - x_n(s, t)| d\nu d\mu \\ &+ \int_{\Omega} \int_B |x(s)(t) - x_n(s, t)| d\nu d\mu. \end{aligned} \quad (23.6.38)$$

Since $\lim_{n \rightarrow \infty} \|x_n - x\|_1 = 0$, it follows from 23.6.38 that $y = x$ in $L^1(\Omega; X)$. Thus, for a.e. s ,

$$y(s, \cdot) = x(s) \text{ in } X = L^1(B).$$

Now $\int_{\Omega} x(s) d\mu \in X = L^1(B, \nu)$ so it makes sense to ask for $(\int_{\Omega} x(s) d\mu)(t)$, at least a.e. To find what this is, note

$$\left\| \int_{\Omega} x_n(s) d\mu - \int_{\Omega} x(s) d\mu \right\|_X \leq \int_{\Omega} \|x_n(s) - x(s)\|_X d\mu.$$

Therefore, since the right side converges to 0,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \int_{\Omega} x_n(s) d\mu - \int_{\Omega} x(s) d\mu \right\|_X &= \\ \lim_{n \rightarrow \infty} \int_B \left| \left(\int_{\Omega} x_n(s) d\mu \right)(t) - \left(\int_{\Omega} x(s) d\mu \right)(t) \right| d\nu &= 0. \end{aligned}$$

But

$$\left(\int_{\Omega} x_n(s) d\mu \right)(t) = \int_{\Omega} x_n(s, t) d\mu \text{ a.e. } t.$$

Therefore

$$\lim_{n \rightarrow \infty} \int_B \left| \int_{\Omega} x_n(s, t) d\mu - \left(\int_{\Omega} x(s) d\mu \right)(t) \right| d\nu = 0. \quad (23.6.39)$$

Also, since $x_n \rightarrow y$ in $L^1(\Omega \times B)$,

$$0 = \lim_{n \rightarrow \infty} \int_B \int_{\Omega} |x_n(s, t) - y(s, t)| d\mu d\nu \geq$$

$$\lim_{n \rightarrow \infty} \int_B \left| \int_{\Omega} x_n(s, t) d\mu - \int_{\Omega} y(s, t) d\mu \right| d\nu. \quad (23.6.40)$$

From 23.6.39 and 23.6.40

$$\int_{\Omega} y(s, t) d\mu = \left(\int_{\Omega} x(s) d\mu \right)(t) \text{ a.e. } t.$$

This proves the following theorem.

Theorem 23.6.1 *Let $X = L^1(B)$ where $(B, \mathcal{F}, \nu)$ is a σ finite measure space and let $x \in L^1(\Omega; X)$. Then there exists a measurable representative, $y \in L^1(\Omega \times B)$, such that*

$$x(s) = y(s, \cdot) \text{ a.e. } s \text{ in } \Omega,$$

and

$$\int_{\Omega} y(s, t) d\mu = \left(\int_{\Omega} x(s) d\mu \right)(t) \text{ a.e. } t.$$

23.7 Vector Measures

There is also a concept of vector measures.

Definition 23.7.1 *Let $(\Omega, \mathcal{S})$ be a set and a σ algebra of subsets of Ω . A mapping*

$$F : \mathcal{S} \rightarrow X$$

is said to be a vector measure if

$$F(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} F(E_i)$$

whenever $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint elements of $\mathcal{S}$. For F a vector measure,

$$|F|(A) \equiv \sup \left\{ \sum_{F \in \pi(A)} \|\mu(F)\| : \pi(A) \text{ is a partition of } A \right\}.$$

This is the same definition that was given in the case where F would have values in $\mathbb{C}$, the only difference being the fact that now F has values in a general Banach space X as the vector space of values of the vector measure. Recall that a partition of A is a finite set, $\{F_1, \dots, F_m\} \subseteq \mathcal{S}$ such that $\cup_{i=1}^m F_i = A$. The same theorem about $|F|$ proved in the case of complex valued measures holds in this context with the same proof. For completeness, it is included here.

Theorem 23.7.2 *If $|F|(\Omega) < \infty$, then $|F|$ is a measure on $\mathcal{S}$.*

Proof: Let E_1 and E_2 be sets of $\mathcal{S}$ such that $E_1 \cap E_2 = \emptyset$ and let $\{A_1^i \cdots A_{n_i}^i\} = \pi(E_i)$, a partition of E_i which is chosen such that

$$|F|(E_i) - \varepsilon < \sum_{j=1}^{n_i} \|F(A_j^i)\| \quad i = 1, 2.$$

Consider the sets which are contained in either of $\pi(E_1)$ or $\pi(E_2)$, it follows this collection of sets is a partition of $E_1 \cup E_2$ which is denoted here by $\pi(E_1 \cup E_2)$. Then by the above inequality and the definition of total variation,

$$|F|(E_1 \cup E_2) \geq \sum_{F \in \pi(E_1 \cup E_2)} \|F(F)\| > |F|(E_1) + |F|(E_2) - 2\varepsilon,$$

which shows that since $\varepsilon > 0$ was arbitrary,

$$|F|(E_1 \cup E_2) \geq |F|(E_1) + |F|(E_2). \quad (23.7.41)$$

Let $\{E_j\}_{j=1}^\infty$ be a sequence of disjoint sets of $\mathcal{S}$ and let $E_\infty = \cup_{j=1}^\infty E_j$. Then by the definition of total variation there exists a partition of E_∞ , $\pi(E_\infty) = \{A_1, \cdots, A_n\}$ such that

$$|F|(E_\infty) - \varepsilon < \sum_{i=1}^n \|F(A_i)\|.$$

Also,

$$A_i = \cup_{j=1}^\infty A_i \cap E_j$$

and so by the triangle inequality, $\|F(A_i)\| \leq \sum_{j=1}^\infty \|F(A_i \cap E_j)\|$. Therefore, by the above,

$$\begin{aligned} |F|(E_\infty) - \varepsilon &< \sum_{i=1}^n \overbrace{\sum_{j=1}^\infty \|F(A_i \cap E_j)\|}^{\geq \|F(A_i)\|} \\ &= \sum_{j=1}^\infty \sum_{i=1}^n \|F(A_i \cap E_j)\| \\ &\leq \sum_{j=1}^\infty |F|(E_j) \end{aligned}$$

because $\{A_i \cap E_j\}_{i=1}^n$ is a partition of E_j .

Since $\varepsilon > 0$ is arbitrary, this shows

$$|F|(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty |F|(E_j).$$

Also, 23.7.41 implies that whenever the E_i are disjoint, $|F|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |F|(E_j)$. Therefore,

$$\sum_{j=1}^{\infty} |F|(E_j) \geq |F|(\cup_{j=1}^{\infty} E_j) \geq |F|(\cup_{j=1}^n E_j) \geq \sum_{j=1}^n |F|(E_j).$$

Since n is arbitrary,

$$|F|(\cup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} |F|(E_j)$$

which shows that $|F|$ is a measure as claimed. This proves the theorem.

Definition 23.7.3 A Banach space is said to have the Radon Nikodym property if whenever

$(\Omega, \mathcal{S}, \mu)$ is a finite measure space

$F : \mathcal{S} \rightarrow X$ is a vector measure with $|F|(\Omega) < \infty$

$$F \ll \mu$$

then one may conclude there exists $g \in L^1(\Omega; X)$ such that

$$F(E) = \int_E g(s) d\mu$$

for all $E \in \mathcal{S}$.

Some Banach spaces have the Radon Nikodym property and some don't. No attempt is made to give a complete answer to the question of which Banach spaces have this property but the next theorem gives examples of many spaces which do.

Theorem 23.7.4 Suppose X' is a separable dual space. Then X' has the Radon Nikodym property.

Proof: Let $F \ll \mu$ and let $|F|(\Omega) < \infty$ for $F : \mathcal{S} \rightarrow X'$, a vector measure. Pick $x \in X$ and consider the map

$$E \rightarrow F(E)(x)$$

for $E \in \mathcal{S}$. This defines a complex measure which is absolutely continuous with respect to $|F|$. Therefore, by the Radon Nikodym theorem, there exists $f_x \in L^1(\Omega, |F|)$ such that

$$F(E)(x) = \int_E f_x(s) d|F|. \quad (23.7.42)$$

Claim: $|f_x(s)| \leq \|x\|$ for $|F|$ a.e. s .

Proof of claim: Consider the closed ball in $\mathbb{F}$, $\overline{B(0, \|x\|)}$ and let $B \equiv B(p, r)$ be an open ball contained in its complement. Let $f_x^{-1}(B) \equiv E \in \mathcal{S}$. I want to argue that $|F|(E) = 0$ so suppose $|F|(E) > 0$. then

$$|F|(E) \|x\| \geq \|F(E)\| \|x\| \geq |F(E)(x)|$$

and so from 23.7.42,

$$\frac{1}{|F|(E)} \left| \int_E f_x(s) d|F| \right| \leq \|x\|. \quad (23.7.43)$$

But on E , $|f_x(s) - p| < r$ and so

$$\left| \frac{1}{|F|(E)} \int_E f_x(s) d|F| - p \right| < r$$

which contradicts 23.7.43 because $B(p, r)$ was given to have empty intersection with $\overline{B(0, \|x\|)}$. Therefore, $|F|(E) = 0$ as hoped. Now $\mathbb{F} \setminus \overline{B(0, \|x\|)}$ can be covered by countably many such balls and so $|F|(\mathbb{F} \setminus \overline{B(0, \|x\|)}) = 0$.

Denote the exceptional set of measure zero by N_x . By Theorem 23.1.13, X is separable. Letting D be a dense, countable subset of X , define

$$N_1 \equiv \cup_{x \in D} N_x.$$

Thus

$$|F|(N_1) = 0.$$

For any $E \in \mathcal{S}$, $x, y \in D$, and $a, b \in \mathbb{F}$,

$$\begin{aligned} \int_E f_{ax+by}(s) d|F| &= F(E)(ax+by) = aF(E)(x) + bF(E)(y) \\ &= \int_E (af_x(s) + bf_y(s)) d|F|. \end{aligned} \quad (23.7.44)$$

Since 23.7.44 holds for all $E \in \mathcal{S}$, it follows

$$f_{ax+by}(s) = af_x(s) + bf_y(s)$$

for $|F|$ a.e. s and $x, y \in D$. Let $\tilde{D}$ consist of all finite linear combinations of the form $\sum_{i=1}^m a_i x_i$ where a_i is a rational point of $\mathbb{F}$ and $x_i \in D$. If

$$\sum_{i=1}^m a_i x_i \in \tilde{D},$$

the above argument implies

$$f_{\sum_{i=1}^m a_i x_i}(s) = \sum_{i=1}^m a_i f_{x_i}(s) \text{ a.e.}$$

Since $\tilde{D}$ is countable, there exists a set, N_2 , with

$$|F|(N_2) = 0$$

such that for $s \notin N_2$,

$$f_{\sum_{i=1}^m a_i x_i}(s) = \sum_{i=1}^m a_i f_{x_i}(s) \quad (23.7.45)$$

whenever $\sum_{i=1}^m a_i x_i \in \tilde{D}$. Let

$$N = N_1 \cup N_2$$

and let

$$\tilde{h}_x(s) \equiv \mathcal{X}_{N^c}(s) f_x(s)$$

for all $x \in \tilde{D}$. Now for $x \in X$ define

$$h_x(s) \equiv \lim_{x' \rightarrow x} \{\tilde{h}_{x'}(s) : x' \in \tilde{D}\}.$$

This is well defined because if x' and y' are elements of $\tilde{D}$, the above claim and 23.7.45 imply

$$\left| \tilde{h}_{x'}(s) - \tilde{h}_{y'}(s) \right| = \left| \tilde{h}_{(x'-y')}(s) \right| \leq \|x' - y'\|.$$

Using 23.7.45, the dominated convergence theorem may be applied to conclude that for $x_n \rightarrow x$, with $x_n \in \tilde{D}$,

$$\int_E h_x(s) d|F| = \lim_{n \rightarrow \infty} \int_E \tilde{h}_{x_n}(s) d|F| = \lim_{n \rightarrow \infty} F(E)(x_n) = F(E)(x). \quad (23.7.46)$$

It follows from the density of $\tilde{D}$ that for all $x, y \in X$ and $a, b \in \mathbb{F}$,

$$|h_x(s)| \leq \|x\|, \quad h_{ax+by}(s) = ah_x(s) + bh_y(s), \quad (23.7.47)$$

for all s because if $s \in N$, both sides of the equation in 23.7.47 equal 0.

Let $\theta(s)$ be given by

$$\theta(s)(x) = h_x(s).$$

By 23.7.47 it follows that $\theta(s) \in X'$ for each s . Also

$$\theta(s)(x) = h_x(s) \in L^1(\Omega)$$

so $\theta(\cdot)$ is weak $*$ measurable. Since X' is separable, Theorem 23.1.12 implies that θ is strongly measurable. Furthermore, by 23.7.47,

$$\|\theta(s)\| \equiv \sup_{\|x\| \leq 1} |\theta(s)(x)| \leq \sup_{\|x\| \leq 1} |h_x(s)| \leq 1.$$

Therefore,

$$\int \|\theta(s)\| d|F| < \infty$$

so $\theta \in L^1(\Omega; X')$. By 23.2.6, if $E \in \mathcal{S}$,

$$\int_E h_x(s) d|F| = \int_E \theta(s)(x) d|F| = \left(\int_E \theta(s) d|F| \right)(x). \quad (23.7.48)$$

From 23.7.46 and 23.7.48,

$$\left(\int_E \theta(s) d|F| \right)(x) = F(E)(x)$$

for all $x \in X$ and therefore,

$$\int_E \theta(s) d|F| = F(E).$$

Finally, since $F \ll \mu$, $|F| \ll \mu$ also and so there exists $k \in L^1(\Omega)$ such that

$$|F|(E) = \int_E k(s) d\mu$$

for all $E \in \mathcal{S}$, by the Radon Nikodym Theorem. It follows

$$F(E) = \int_E \theta(s) d|F| = \int_E \theta(s) k(s) d\mu.$$

Letting $g(s) = \theta(s) k(s)$, this has proved the theorem.

Corollary 23.7.5 *Any separable reflexive Banach space has the Radon Nikodym property.*

It is not necessary to assume separability in the above corollary. For the proof of a more general result, consult *Vector Measures* by Diestel and Uhl, [19].

23.8 The Riesz Representation Theorem

The Riesz representation theorem for the spaces $L^p(\Omega; X)$ holds under certain conditions. The proof follows the proofs given earlier for scalar valued functions.

Definition 23.8.1 *If X and Y are two Banach spaces, X is isometric to Y if there exists $\theta \in \mathcal{L}(X, Y)$ such that*

$$\|\theta x\|_Y = \|x\|_X.$$

This will be written as $X \cong Y$. The map θ is called an isometry.

The next theorem says that $L^{p'}(\Omega; X')$ is always isometric to a subspace of $(L^p(\Omega; X))'$ for any Banach space, X .

Theorem 23.8.2 *Let X be any Banach space and let $(\Omega, \mathcal{S}, \mu)$ be a finite measure space. Let $p \geq 1$ and let $1/p + 1/p' = 1$. (If $p = 1$, $p' \equiv \infty$.) Then $L^{p'}(\Omega; X')$ is isometric to a subspace of $(L^p(\Omega; X))'$. Also, for $g \in L^{p'}(\Omega; X')$,*

$$\sup_{\|f\|_p \leq 1} \left| \int_{\Omega} g(s) (f(s)) d\mu \right| = \|g\|_{p'}.$$

Proof: First observe that for $f \in L^p(\Omega; X)$ and $g \in L^{p'}(\Omega; X')$,

$$s \rightarrow g(s)(f(s))$$

is a function in $L^1(\Omega)$. (To obtain measurability, write f as a limit of simple functions. Holder's inequality then yields the function is in $L^1(\Omega)$.) Define

$$\theta : L^{p'}(\Omega; X') \rightarrow (L^p(\Omega; X))'$$

by

$$\theta g(f) \equiv \int_{\Omega} g(s)(f(s)) d\mu.$$

Holder's inequality implies

$$\|\theta g\| \leq \|g\|_{p'} \quad (23.8.49)$$

and it is also clear that θ is linear. Next it is required to show

$$\|\theta g\| = \|g\|.$$

This will first be verified for simple functions. Let

$$g(s) = \sum_{i=1}^m c_i \chi_{E_i}(s)$$

where $c_i \in X'$, the E_i are disjoint and

$$\cup_{i=1}^m E_i = \Omega.$$

Then $\|g\| \in L^{p'}(\Omega)$. Let $\varepsilon > 0$ be given. By the scalar Riesz representation theorem, there exists $h \in L^p(\Omega)$ such that $\|h\|_p = 1$ and

$$\int_{\Omega} \|g(s)\|_{X'} h(s) d\mu \geq \|g\|_{L^{p'}(\Omega; X')} - \varepsilon.$$

Now let d_i be chosen such that

$$c_i(d_i) \geq \|c_i\|_{X'} - \varepsilon / \|h\|_{L^1(\Omega)}$$

and $\|d_i\|_X \leq 1$. Let

$$f(s) \equiv \sum_{i=1}^m d_i h(s) \chi_{E_i}(s).$$

Thus $f \in L^p(\Omega; X)$ and $\|f\|_{L^p(\Omega; X)} \leq 1$. This follows from

$$\begin{aligned} \|f\|_p^p &= \int_{\Omega} \sum_{i=1}^m \|d_i\|_X^p |h(s)|^p \chi_{E_i}(s) d\mu \\ &= \sum_{i=1}^m \left(\int_{E_i} |h(s)|^p d\mu \right) \|d_i\|_X^p \leq \int_{\Omega} |h|^p d\mu = 1. \end{aligned}$$

Also

$$\begin{aligned}
 \|\theta g\| &\geq |\theta g(f)| = \left| \int_{\Omega} g(s) (f(s)) d\mu \right| \geq \\
 &\left| \int_{\Omega} \sum_{i=1}^m \left(\|c_i\|_{X'} - \varepsilon / \|h\|_{L^1(\Omega)} \right) h(s) \chi_{E_i}(s) d\mu \right| \\
 &\geq \left| \int_{\Omega} \|g(s)\|_{X'} h(s) d\mu \right| - \varepsilon \left| \int_{\Omega} h(s) / \|h\|_{L^1(\Omega)} d\mu \right| \\
 &\geq \|g\|_{L^{p'}(\Omega; X')} - 2\varepsilon.
 \end{aligned}$$

Since ε was arbitrary,

$$\|\theta g\| \geq \|g\| \quad (23.8.50)$$

and from 23.8.49 this shows equality holds in 23.8.50 whenever g is a simple function.

In general, let $g \in L^{p'}(\Omega; X')$ and let g_n be a sequence of simple functions converging to g in $L^{p'}(\Omega; X')$. Then

$$\|\theta g\| = \lim_{n \rightarrow \infty} \|\theta g_n\| = \lim_{n \rightarrow \infty} \|g_n\| = \|g\|.$$

This proves the theorem and shows θ is the desired isometry.

Theorem 23.8.3 *If X is a Banach space and X' has the Radon Nikodym property, then if $(\Omega, \mathcal{S}, \mu)$ is a finite measure space,*

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X')$$

and in fact the mapping θ of Theorem 23.8.2 is onto.

Proof: Let $l \in (L^p(\Omega; X))'$ and define $F(E) \in X'$ by

$$F(E)(x) \equiv l(\chi_E(\cdot)x).$$

Lemma 23.8.4 *F defined above is a vector measure with values in X' and $|F|(\Omega) < \infty$.*

Proof of the lemma: Clearly $F(E)$ is linear. Also

$$\begin{aligned}
 \|F(E)\| &= \sup_{\|x\| \leq 1} \|F(E)(x)\| \\
 &\leq \|l\| \sup_{\|x\| \leq 1} \|\chi_E(\cdot)x\|_{L^p(\Omega; X)} \leq \|l\| \mu(E)^{1/p}.
 \end{aligned}$$

Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of disjoint elements of $\mathcal{S}$ and let $E = \cup_{n < \infty} E_n$.

$$\begin{aligned}
 \left| F(E)(x) - \sum_{k=1}^n F(E_k)(x) \right| &= \left| l(\chi_E(\cdot)x) - \sum_{i=1}^n l(\chi_{E_i}(\cdot)x) \right| \quad (23.8.51) \\
 &\leq \|l\| \left\| \chi_E(\cdot)x - \sum_{i=1}^n \chi_{E_i}(\cdot)x \right\|_{L^p(\Omega; X)} \\
 &\leq \|l\| \mu \left(\bigcup_{k > n} E_k \right)^{1/p} \|x\|.
 \end{aligned}$$

Since $\mu(\Omega) < \infty$,

$$\lim_{n \rightarrow \infty} \mu \left(\bigcup_{k > n} E_k \right)^{1/p} = 0$$

and so inequality 23.8.51 shows that

$$\lim_{n \rightarrow \infty} \left\| F(E) - \sum_{k=1}^n F(E_k) \right\|_{X'} = 0.$$

To show $|F|(\Omega) < \infty$, let $\varepsilon > 0$ be given, let $\{H_1, \dots, H_n\}$ be a partition of Ω , and let $\|x_i\| \leq 1$ be chosen in such a way that

$$F(H_i)(x_i) > \|F(H_i)\| - \varepsilon/n.$$

Thus

$$\begin{aligned} -\varepsilon + \sum_{i=1}^n \|F(H_i)\| &< \sum_{i=1}^n l(\mathcal{X}_{H_i}(\cdot)x_i) \leq \|l\| \left\| \sum_{i=1}^n \mathcal{X}_{H_i}(\cdot)x_i \right\|_{L^p(\Omega; X)} \\ &\leq \|l\| \left(\int_{\Omega} \sum_{i=1}^n \mathcal{X}_{H_i}(s) d\mu \right)^{1/p} = \|l\| \mu(\Omega)^{1/p}. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\sum_{i=1}^n \|F(H_i)\| < \|l\| \mu(\Omega)^{1/p}.$$

Since the partition was arbitrary, this shows $|F|(\Omega) \leq \|l\| \mu(\Omega)^{1/p}$ and this proves the lemma.

Continuing with the proof of Theorem 23.8.3, note that

$$F \ll \mu.$$

Since X' has the Radon Nikodym property, there exists $g \in L^1(\Omega; X')$ such that

$$F(E) = \int_E g(s) d\mu.$$

Also, from the definition of $F(E)$,

$$\begin{aligned} l \left(\sum_{i=1}^n x_i \mathcal{X}_{E_i}(\cdot) \right) &= \sum_{i=1}^n l(\mathcal{X}_{E_i}(\cdot)x_i) \\ &= \sum_{i=1}^n F(E_i)(x_i) = \sum_{i=1}^n \int_{E_i} g(s)(x_i) d\mu. \end{aligned} \tag{23.8.52}$$

It follows from 23.8.52 that whenever h is a simple function,

$$l(h) = \int_{\Omega} g(s)(h(s)) d\mu. \quad (23.8.53)$$

Let

$$G_n \equiv \{s : \|g(s)\|_{X'} \leq n\}$$

and let

$$j : L^p(G_n; X) \rightarrow L^p(\Omega; X)$$

be given by

$$jh(s) = \begin{cases} h(s) & \text{if } s \in G_n, \\ 0 & \text{if } s \notin G_n. \end{cases}$$

Letting h be a simple function in $L^p(G_n; X)$,

$$j^*l(h) = l(jh) = \int_{G_n} g(s)(h(s)) d\mu. \quad (23.8.54)$$

Since the simple functions are dense in $L^p(G_n; X)$, and $g \in L^{p'}(G_n; X')$, it follows 23.8.54 holds for all $h \in L^p(G_n; X)$. By Theorem 23.8.2,

$$\|g\|_{L^{p'}(G_n; X')} = \|j^*l\|_{(L^p(G_n; X))'} \leq \|l\|_{(L^p(\Omega; X))'}.$$

By the monotone convergence theorem,

$$\|g\|_{L^{p'}(\Omega; X')} = \lim_{n \rightarrow \infty} \|g\|_{L^{p'}(G_n; X')} \leq \|l\|_{(L^p(\Omega; X))'}.$$

Therefore $g \in L^{p'}(\Omega; X')$ and since simple functions are dense in $L^p(\Omega; X)$, 23.8.53 holds for all $h \in L^p(\Omega; X)$. Thus $l = \theta g$ and the theorem is proved because, by Theorem 23.8.2, $\|l\| = \|g\|$ and the mapping θ is onto because l was arbitrary.

As in the scalar case, everything generalizes to the case of σ finite measure spaces. The proof is almost identical.

Lemma 23.8.5 *Let $(\Omega, \mathcal{S}, \mu)$ be a σ finite measure space and let X be a Banach space such that X' has the Radon Nikodym property. Then there exists a measurable function, r such that $r(x) > 0$ for all x , such that $|r(x)| < M$ for all x , and $\int r d\mu < \infty$. For*

$$\Lambda \in (L^p(\Omega; X))', \quad p \geq 1,$$

there exists a unique $h \in L^{p'}(\Omega; X')$, $L^\infty(\Omega; X')$ if $p = 1$ such that

$$\Lambda f = \int h(f) d\mu.$$

Also $\|h\| = \|\Lambda\|$. ($\|h\| = \|h\|_{p'}$ if $p > 1$, $\|h\|_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proof: First suppose r exists as described. Also, to save on notation and to emphasize the similarity with the scalar case, denote the norm in the various spaces by $|\cdot|$. Define a new measure $\tilde{\mu}$, according to the rule

$$\tilde{\mu}(E) \equiv \int_E r d\mu. \quad (23.8.55)$$

Thus $\tilde{\mu}$ is a finite measure on $\mathcal{S}$. Now define a mapping, $\eta : L^p(\Omega; X, \mu) \rightarrow L^p(\Omega; X, \tilde{\mu})$ by

$$\eta f = r^{-\frac{1}{p}} f.$$

Then

$$\|\eta f\|_{L^p(\tilde{\mu})}^p = \int \left| r^{-\frac{1}{p}} f \right|^p r d\mu = \|f\|_{L^p(\mu)}^p$$

and so η is one to one and in fact preserves norms. I claim that also η is onto. To see this, let $g \in L^p(\Omega; X, \tilde{\mu})$ and consider the function, $r^{\frac{1}{p}} g$. Then

$$\int \left| r^{\frac{1}{p}} g \right|^p d\mu = \int |g|^p r d\mu = \int |g|^p d\tilde{\mu} < \infty$$

Thus $r^{\frac{1}{p}} g \in L^p(\Omega; X, \mu)$ and $\eta \left(r^{\frac{1}{p}} g \right) = g$ showing that η is onto as claimed. Thus η is one to one, onto, and preserves norms. Consider the diagram below which is descriptive of the situation in which η^* must be one to one and onto.

$$\begin{array}{ccccc} h, L^{p'}(\tilde{\mu}) & L^p(\tilde{\mu})', \tilde{\Lambda} & \xrightarrow{\eta^*} & L^p(\mu)', \Lambda \\ & L^p(\tilde{\mu}) & \xleftarrow{\eta} & L^p(\mu) \end{array}$$

Then for $\Lambda \in L^p(\mu)'$, there exists a unique $\tilde{\Lambda} \in L^p(\tilde{\mu})'$ such that $\eta^* \tilde{\Lambda} = \Lambda$, $\|\tilde{\Lambda}\| = \|\Lambda\|$. By the Riesz representation theorem for finite measure spaces, there exists a unique $h \in L^{p'}(\tilde{\mu}) \equiv L^{p'}(\Omega; X', \tilde{\mu})$ which represents $\tilde{\Lambda}$ in the manner described in the Riesz representation theorem. Thus $\|h\|_{L^{p'}(\tilde{\mu})} = \|\tilde{\Lambda}\| = \|\Lambda\|$ and for all $f \in L^p(\mu)$,

$$\begin{aligned} \Lambda(f) &= \eta^* \tilde{\Lambda}(f) \equiv \tilde{\Lambda}(\eta f) = \int h(\eta f) d\tilde{\mu} = \int r h \left(r^{-\frac{1}{p}} f \right) d\mu \\ &= \int r^{\frac{1}{p'}} h f d\mu. \end{aligned}$$

Now

$$\int \left| r^{\frac{1}{p'}} h \right|^{p'} d\mu = \int |h|^{p'} r d\mu = \|h\|_{L^{p'}(\tilde{\mu})}^{p'} < \infty.$$

Thus $\left\| r^{\frac{1}{p'}} h \right\|_{L^{p'}(\mu)} = \|h\|_{L^{p'}(\tilde{\mu})} = \|\tilde{\Lambda}\| = \|\Lambda\|$ and represents Λ in the appropriate way. If $p = 1$, then $1/p' \equiv 0$. Now consider the existence of r . Since the measure

space is σ finite, there exist $\{\Omega_n\}$ disjoint, each having positive measure and their union equals Ω . Then define

$$r(\omega) \equiv \sum_{n=1}^{\infty} \frac{1}{n^2} \mu(\Omega_n)^{-1} \chi_{\Omega_n}(\omega)$$

This proves the Lemma.

Theorem 23.8.6 (*Riesz representation theorem*) Let $(\Omega, \mathcal{S}, \mu)$ be σ finite and let X' have the Radon Nikodym property. Then for

$$\Lambda \in (L^p(\Omega; X, \mu))', \quad p \geq 1$$

there exists a unique $h \in L^q(\Omega, X', \mu)$, $L^\infty(\Omega, X', \mu)$ if $p = 1$ such that

$$\Lambda f = \int h(f) d\mu.$$

Also $\|h\| = \|\Lambda\|$. ($\|h\| = \|h\|_q$ if $p > 1$, $\|h\|_\infty$ if $p = 1$). Here

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proof: The above lemma gives the existence part of the conclusion of the theorem. Uniqueness is done as before.

Corollary 23.8.7 If X' is separable, then for $(\Omega, \mathcal{S}, \mu)$ a σ finite measure space,

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X').$$

Corollary 23.8.8 If X is separable and reflexive, then for $(\Omega, \mathcal{S}, \mu)$ a σ finite measure space,

$$(L^p(\Omega; X))' \cong L^{p'}(\Omega; X').$$

Corollary 23.8.9 If X is separable and reflexive and $(\Omega, \mathcal{S}, \mu)$ a σ finite measure space, then if $p \in (1, \infty)$, then $L^p(\Omega; X)$ is reflexive.

Proof: This is just like the scalar valued case.

23.9 Maximal Monotone Operators

23.9.1 Basic Theory

Here is provided a short introduction to some of the most important properties of maximal monotone operators. The following definition describes them.

Definition 23.9.1 Let H be a real Hilbert space and let $A : D(A) \rightarrow \mathcal{P}(H)$ have the following properties.

1. For each $y \in H$ there exists $x \in D(A)$ such that $y \in x + Ax$.
2. A is monotone. That is, if $z \in Ax$ and $w \in Ay$ then

$$(z - w, x - y) \geq 0$$

Such an operator is called a maximal monotone operator.

It turns out that whenever A is maximal monotone, so is λA for all $\lambda > 0$.

Lemma 23.9.2 *Suppose A is maximal monotone. Then so is λA . Also $J_\lambda \equiv (I + \lambda A)^{-1}$ makes sense for each $\lambda > 0$ and is Lipschitz continuous.*

Proof: To begin with consider $(I + A)^{-1}$. Suppose

$$x_1, x_2 \in (I + A)^{-1}(y)$$

Then $y \in (I + A)x_i$ and so $y - x_i \in Ax_i$. By monotonicity

$$(y - x_1 - (y - x_2), x_1 - x_2) \geq 0$$

and so

$$0 \geq |x_1 - x_2|^2$$

which shows $J_1 \equiv (I + A)^{-1}$ makes sense. In fact this is Lipschitz with Lipschitz constant 1. Here is why. $x \in (I + A)J_1x$ and $y \in (I + A)J_1y$. Then

$$x - J_1x \in AJ_1x, \quad y - J_1y \in AJ_1y$$

and so by monotonicity

$$0 \leq (x - J_1x - (y - J_1y), J_1x - J_1y)$$

which yields

$$\begin{aligned} |J_1x - J_1y|^2 &\leq (x - y, J_1x - J_1y) \\ &\leq |x - y| |J_1x - J_1y| \end{aligned}$$

which yields the result.

Next consider the claim that λA is maximal monotone. The monotone part is immediate. The only thing in question is whether $I + \lambda A$ is onto. Let $r \in (-1, 1)$ and pick $f \in H$. Consider solving the equation for u

$$(1 + r)u + Au \ni (1 + r)f \tag{23.9.56}$$

This is equivalent to finding u such that

$$(I + A)u \ni (1 + r)f - ru$$

or in other words finding u such that

$$u = J_1((1+r)f - ru)$$

However, if

$$Tu \equiv J_1((1+r)f - ru),$$

then since $|r| < 1$, T is a contraction mapping and so there exists a unique solution to 23.9.56. Thus

$$u + \frac{1}{1+r}Au \ni f$$

It follows for any $|r| < 1$, $(1+r)^{-1}A$ is maximal monotone. This takes care of all $\lambda \in (\frac{1}{2}, \infty)$. Now do the same thing for $(2/3)A$ to get the result for all $\lambda \in ((\frac{2}{3})(\frac{1}{2}), \infty)$. Now apply the same argument to $(2/3)^2 A$ to get the result for all $\lambda \in ((\frac{2}{3})^2(\frac{1}{2}), \infty)$. Next consider the same argument to $(2/3)^3 A$ to get the desired result for all $\lambda \in ((\frac{2}{3})^3(\frac{1}{2}), \infty)$. Continuing this way shows λA is maximal monotone for all $\lambda > 0$. Also from the first part of the proof $(I + \lambda A)^{-1}$ is Lipschitz continuous with Lipschitz constant 1. This proves the lemma.

A maximal monotone operator can be approximated with a Lipschitz continuous operator which is also monotone and has certain salubrious properties. This operator is called the Yosida approximation and as in the case of linear operators it is obtained by formally considering

$$\frac{A}{1 + \lambda A}$$

If you do the division formally you get the definition for A_λ ,

$$A_\lambda x \equiv \frac{1}{\lambda}x - \frac{1}{\lambda}J_\lambda x \quad (23.9.57)$$

where $J_\lambda = (I + \lambda A)^{-1}$ as above. It is obvious that A_λ is Lipschitz continuous with Lipschitz constant no more than $2/\lambda$. Actually you can show $1/\lambda$ also works but this is not important here.

Lemma 23.9.3 $A_\lambda x \in AJ_\lambda x$ and $|A_\lambda x| \leq |y|$ for all $y \in Ax$ whenever $x \in D(A)$. Also A_λ is monotone.

Proof: Consider the first claim. From the definition,

$$A_\lambda x \equiv \frac{1}{\lambda}x - \frac{1}{\lambda}J_\lambda x$$

Is

$$\frac{1}{\lambda}x - \frac{1}{\lambda}J_\lambda x \in AJ_\lambda x?$$

Is

$$x - J_\lambda x \in \lambda AJ_\lambda x?$$

Is

$$x \in J_\lambda x + \lambda A J_\lambda x?$$

Is

$$x \in (I + \lambda A) J_\lambda x?$$

Certainly so. This is how J_λ is defined.

Now consider the second claim. Let $y \in Ax$ for some $x \in D(A)$. Then by monotonicity and what was just shown

$$0 \leq (A_\lambda x - y, J_\lambda x - x) = -\lambda (A_\lambda x - y, A_\lambda x)$$

and so

$$|A_\lambda x|^2 \leq (y, A_\lambda x) \leq |y| |A_\lambda x|$$

Finally, to show A_λ is monotone,

$$\begin{aligned} & (A_\lambda x - A_\lambda y, x - y) = \\ & \left(\frac{1}{\lambda} x - \frac{1}{\lambda} J_\lambda x - \left(\frac{1}{\lambda} y - \frac{1}{\lambda} J_\lambda y \right), x - y \right) \\ &= \frac{1}{\lambda} |x - y|^2 - \frac{1}{\lambda} (J_\lambda x - J_\lambda y, x - y) \\ &\geq \frac{1}{\lambda} |x - y|^2 - \frac{1}{\lambda} |x - y| |J_\lambda x - J_\lambda y| \\ &\geq \frac{1}{\lambda} |x - y|^2 - \frac{1}{\lambda} |x - y|^2 = 0 \end{aligned}$$

and this proves the lemma.

The next lemma gives a way to determine whether a pair $[x, y]$ is in the graph of A defined as

$$\{[x, y] : y \in Ax\} \equiv \mathcal{G}(A)$$

Here I am writing $[\cdot, \cdot]$ rather than $(\cdot, \cdot)$ to avoid confusion with the inner product. It is the conclusion of this lemma which accounts for the use of the term “maximal”. It essentially says there is no larger monotone graph which includes the one for A .

Lemma 23.9.4 *Suppose $(y_1 - y, x_1 - x) \geq 0$ for all $[x, y] \in \mathcal{G}(A)$ where A is maximal monotone. Then $x_1 \in D(A)$ and $y_1 \in Ax_1$. Also if $[x_k, y_k] \in \mathcal{G}(A)$ and $x_k \rightarrow x, y_k \rightharpoonup y$ where the half arrow denotes weak convergence, then $[x, y] \in \mathcal{G}(A)$.*

Proof: I want to show $y_1 \in Ax_1$ or in other words I want to show

$$x_1 + \lambda y_1 \in x_1 + \lambda Ax_1$$

or in other words

$$J_\lambda(x_1 + \lambda y_1) = x_1.$$

This is the motivation for the following argument.

From Lemma 23.9.3 $A_\lambda(x_1 + \lambda y_1) \in AJ_\lambda(x_1 + \lambda y_1)$ and so by the above assumption

$$\begin{aligned} 0 &\leq (y_1 - A_\lambda(x_1 + \lambda y_1), x_1 - J_\lambda(x_1 + \lambda y_1)) \\ &= \left(y_1 - \left(\frac{1}{\lambda}(x_1 + \lambda y_1) - \frac{1}{\lambda}J_\lambda(x_1 + \lambda y_1) \right), x_1 - J_\lambda(x_1 + \lambda y_1) \right) \\ &= \left(\left(-\frac{1}{\lambda}x_1 + \frac{1}{\lambda}J_\lambda(x_1 + \lambda y_1) \right), x_1 - J_\lambda(x_1 + \lambda y_1) \right) \\ &= -\frac{1}{\lambda}(x_1 - J_\lambda(x_1 + \lambda y_1), x_1 - J_\lambda(x_1 + \lambda y_1)) \end{aligned}$$

which requires

$$x_1 = J_\lambda(x_1 + \lambda y_1)$$

and this says $x_1 \in D(A)$ because J_λ maps into $D(A)$. Also it says

$$x_1 + Ax_1 \ni x_1 + \lambda y_1$$

and so $y_1 \in Ax_1$.

This makes the last claim pretty easy. Suppose $x_k \rightarrow x$ where $x_k \in D(A)$ and that $y_k \in Ax_k$ and $y_k \rightharpoonup y$. I need to verify $y = Ax$ and $x \in D(A)$. Let $[u, v] \in \mathcal{G}(A)$. Then

$$(y - v, x - u) = \lim_{k \rightarrow \infty} (y_k - v, x_k - u) \geq 0$$

and so, by the first part, $x \in D(A)$ and $y \in Ax$. Why does that limit hold? It is because

$$\begin{aligned} &|(y - v, x - u) - (y_k - v, x_k - u)| \\ &\leq |(y - v, x - u) - (y_k - v, x - u)| + |(y_k - v, x_k - x)| \end{aligned}$$

The second term is no larger than

$$|y_k - v| |x_k - x|$$

which converges to 0 since y_k is weakly convergent, hence bounded. The first term converges to 0 because of the assumption that y_k converges weakly to y . This proves the lemma.

What about the sum of maximal monotone operators? This might not be maximal monotone but what you can say is the following.

Proposition 23.9.5 *Let A be maximal monotone and let B be Lipschitz and monotone. Then $A + B$ is maximal monotone.*

Proof: First suppose B has a Lipschitz constant less than 1. The monotonicity is obvious. I need to show that for any y there exists $x \in D(A)$ such that

$$y \in x + Bx + Ax$$

This happens if and only if

$$y - Bx \in (I + A)x$$

if and only if $x = (I + A)^{-1}(y - Bx)$. Let

$$Tx \equiv (I + A)^{-1}(y - Bx)$$

Then T is clearly a contraction mapping because $(I + A)^{-1}$ is Lipschitz with Lipschitz constant 1. Therefore, there exists a unique fixed point and this shows $A + B$ is maximal monotone. Now the same argument applied to $A + B$ shows that $A + 2B$ is maximal monotone. Continuing this way $A + nB$ is maximal monotone. Now for arbitrary B let n be large enough that $n^{-1}B$ has Lipschitz constant less than 1. Then as just explained, $A + n(n^{-1}B) = A + B$ is maximal monotone. This proves the proposition.

The following is a useful result for determining conditions under which $A + B$ is maximal monotone or more particularly whether a given y is in $(I + A + B)(H)$ where A, B are both maximal monotone.

Theorem 23.9.6 *Let A and B be maximal monotone, let*

$$y \in x_\lambda + B_\lambda x_\lambda + Ax_\lambda,$$

and suppose $B_\lambda x_\lambda$ is bounded independent of λ . Then there exists $x \in D(A) \cap D(B)$ such that $y = x + Ax + Bx$.

Proof: First of all, it follows from Proposition 23.9.5 that there exists a unique x_λ . Note

$$\begin{aligned} y - x_\lambda - B_\lambda x_\lambda &\in Ax_\lambda \\ y - x_\mu - B_\mu x_\mu &\in Ax_\mu \end{aligned}$$

and so by monotonicity of A ,

$$(x_\mu - x_\lambda + B_\mu x_\mu - B_\lambda x_\lambda, x_\lambda - x_\mu) \geq 0$$

and so

$$\begin{aligned} |x_\lambda - x_\mu|^2 &\leq (B_\mu x_\mu - B_\lambda x_\lambda, x_\lambda - x_\mu) \\ &= -(B_\lambda x_\lambda - B_\mu x_\mu, x_\lambda - x_\mu) \end{aligned} \quad (23.9.58)$$

I want to write as many things as possible in terms of the B_λ and B_μ . Denote as $J_\lambda(B)$ the operator $(I + \lambda B)^{-1}$. Then

$$B_\lambda x_\lambda = \frac{1}{\lambda} (x_\lambda - J_\lambda(B)x_\lambda)$$

and so

$$x_\lambda = \lambda B_\lambda x_\lambda + J_\lambda(B)x_\lambda$$

Thus 23.9.58 becomes

$$\begin{aligned}
 |x_\lambda - x_\mu|^2 &= \\
 &- (B_\lambda x_\lambda - B_\mu x_\mu, \lambda B_\lambda x_\lambda + J_\lambda(B) x_\lambda - (\mu B_\mu x_\mu + J_\mu(B) x_\mu)) \\
 &= - (B_\lambda x_\lambda - B_\mu x_\mu, \lambda B_\lambda x_\lambda - \mu B_\mu x_\mu) \\
 &\quad + (B_\mu x_\mu - B_\lambda x_\lambda, J_\lambda(B) x_\lambda - J_\mu(B) x_\mu) \\
 &= - (B_\lambda x_\lambda - B_\mu x_\mu, \lambda B_\lambda x_\lambda - \lambda B_\mu x_\mu) - (B_\lambda x_\lambda - B_\mu x_\mu, (\lambda - \mu) B_\mu x_\mu) \\
 &\quad - (B_\lambda x_\lambda - B_\mu x_\mu, J_\lambda(B) x_\lambda - J_\mu(B) x_\mu)
 \end{aligned}$$

Now recall $B_\mu x \in B J_\mu(B) x$. Then by monotonicity the first and last terms to the right of the equal sign in the above are negative. Therefore,

$$|x_\lambda - x_\mu|^2 \leq |(B_\lambda x_\lambda - B_\mu x_\mu, (\lambda - \mu) B_\mu x_\mu)| \leq C |\lambda - \mu|$$

where C is some constant which comes from the assumption the $B_\lambda x_\lambda$ are bounded.

Therefore, letting λ denote a sequence converging to 0 it follows

$$\lim_{\lambda \rightarrow 0} x_\lambda = x_1 \in H$$

for some x , the convergence being strong convergence. Also taking a further subsequence and using weak compactness it can be assumed

$$B_\lambda x_\lambda \rightharpoonup z_1$$

where this time the convergence is weak. Taking another subsequence, it can also be assumed

$$y - x_\lambda - B_\lambda x_\lambda \rightharpoonup z_2 \quad (23.9.59)$$

the convergence being weak convergence. Recall $B_\lambda x_\lambda \in B J_\lambda(B) x_\lambda$ and also note that by assumption there is a constant C independent of λ such that

$$C \geq |B_\lambda x_\lambda| \geq \frac{1}{\lambda} (x_\lambda - J_\lambda(B) x)$$

which shows

$$J_\lambda(B) x_\lambda \rightarrow x_1$$

also. Now it follows from Lemma 23.9.4 that $x_1 \in D(B)$ and $z_1 \in Bx_1$. Recall

$$y - x_\lambda - B_\lambda x_\lambda \in Ax_\lambda$$

and so by the same lemma again,

$$x_1 \in D(A), \quad z_2 \in Ax_1$$

By 23.9.59 it follows

$$y - x_1 - z_1 = z_2 \in Ax_1$$

Thus

$$y = x_1 + z_1 + z_2 \in x_1 + Bx_1 + Ax_1$$

and this proves the theorem.

23.9.2 Evolution Inclusions

One of the interesting things about maximal monotone operators is the concept of evolution inclusions. To facilitate this, here is a little lemma.

Lemma 23.9.7 *Let $f : [0, T] \rightarrow \mathbb{R}$ be continuous and suppose*

$$D^+ f(t) \equiv \limsup_{h \rightarrow 0+} \frac{f(t+h) - f(t)}{h} < g(t)$$

where g is a continuous function. Then

$$f(t) - f(0) \leq \int_0^t g(s) ds.$$

Proof: Suppose this is not so. Then let

$$S \equiv \left\{ t \in [0, T] : f(t) - f(0) > \int_0^t g(s) ds \right\}$$

and it would follow that $S \neq \emptyset$. Let $a = \inf S$. Then there exists a decreasing sequence $h_n \rightarrow 0$ such that

$$f(a + h_n) - f(0) > \int_0^{a+h_n} g(s) ds \quad (23.9.60)$$

First suppose $a = 0$. Then dividing by h_n and taking the limit,

$$g(0) > D^+ f(0) \geq g(0),$$

a contradiction. Therefore, assume $a > 0$. Then by continuity

$$f(a) - f(0) \geq \int_0^a g(s) ds$$

If strict inequality holds, then $a \neq \inf S$. It follows

$$f(a) - f(0) = \int_0^a g(s) ds$$

and so from 23.9.60

$$\frac{f(a + h_n) - f(a)}{h_n} > \frac{1}{h_n} \int_a^{a+h_n} g(s) ds.$$

Then doing $\limsup_{n \rightarrow \infty}$ to both sides,

$$g(a) > D^+ f(a) \geq g(a)$$

the same sort of contradiction obtained earlier. Thus $S = \emptyset$ and this proves the lemma.

The following is the main result.

Theorem 23.9.8 *Let H be a Hilbert space and let A be a maximal monotone operator as described above. Let $f : [0, T] \rightarrow H$ be continuous such that $f' \in L^2(0, T; H)$. Then there exists a unique solution to the evolution inclusion*

$$y' + Ay \ni f, \quad y(0) = y_0 \in D(A)$$

Here y' exists a.e., $y(t) \in D(A)$ a.e., y is continuous.

Proof: Let y_λ be the solution to

$$y'_\lambda + A_\lambda y_\lambda = f, \quad y_\lambda(0) = y_0$$

I will base the entire proof on estimating the solutions to the corresponding integral equation

$$y_\lambda(t) - y_0 + \int_0^t A_\lambda y_\lambda(s) ds = \int_0^t f(s) ds \quad (23.9.61)$$

Let h, k be small positive numbers. Then

$$y_\lambda(t+h) - y_\lambda(t) + \int_t^{t+h} A_\lambda y_\lambda(s) ds = \int_t^{t+h} f(s) ds \quad (23.9.62)$$

Next consider the difference operator

$$D_k g(t) \equiv \frac{g(t+k) - g(t)}{k}$$

Do this D_k to both sides of 23.9.62 where $k < h$. This gives

$$\begin{aligned} D_k(y_\lambda(t+h) - y_\lambda(t)) + \frac{1}{k} \left(\int_{t+h}^{t+h+k} A_\lambda y_\lambda(s) ds - \int_t^{t+k} A_\lambda y_\lambda(s) ds \right) \\ = \frac{1}{k} \left(\int_{t+h}^{t+h+k} f(s) ds - \int_t^{t+k} f(s) ds \right) \end{aligned} \quad (23.9.63)$$

Now multiply both sides by $y_\lambda(t+h+k) - y_\lambda(t+k)$. Consider the first term. To simplify the ideas consider instead

$$\begin{aligned} (D_k g(t), g(t+k)) &= \frac{1}{k} \left(|g(t+k)|^2 - (g(t), g(t+k)) \right) \\ &\geq \frac{1}{k} \left(|g(t+k)|^2 - |g(t)| |g(t+h)| \right) \\ &\geq \frac{1}{k} \left(\frac{1}{2} |g(t+k)|^2 - \frac{1}{2} |g(t)|^2 \right) \end{aligned} \quad (23.9.64)$$

Then applying this simple observation to 23.9.63,

$$\frac{1}{2} \frac{1}{k} \left(|y_\lambda(t+h+k) - y_\lambda(t+k)|^2 - |y_\lambda(t+h) - y_\lambda(t)|^2 \right) +$$

$$\begin{aligned} & \left(\frac{1}{k} \left(\int_{t+h}^{t+h+k} A_\lambda y_\lambda(s) ds - \int_t^{t+k} A_\lambda y_\lambda(s) ds \right), y_\lambda(t+h+k) - y_\lambda(t+k) \right) + \\ & \leq \left(\frac{1}{k} \left(\int_{t+h}^{t+h+k} f(s) ds - \int_t^{t+k} f(s) ds \right), y_\lambda(t+h+k) - y_\lambda(t+k) \right) \end{aligned}$$

Taking $\limsup_{k \rightarrow 0}$ of both sides yields

$$\begin{aligned} & \frac{1}{2} D^+ \left(|y_\lambda(t+h) - y_\lambda(t)|^2 \right) + (A_\lambda y_\lambda(t+h) - A_\lambda y_\lambda(t), y_\lambda(t+h) - y_\lambda(t)) \\ & \leq (f(t+h) - f(t), y_\lambda(t+h) - y_\lambda(t)) \end{aligned}$$

Now recall that A_λ is monotone. Therefore,

$$D^+ \left(|y_\lambda(t+h) - y_\lambda(t)|^2 \right) \leq |f(t+h) - f(t)|^2 + |y_\lambda(t+h) - y_\lambda(t)|^2$$

From Lemma 23.9.7 it follows that for all $\varepsilon > 0$,

$$\begin{aligned} & |y_\lambda(t+h) - y_\lambda(t)|^2 - |y_\lambda(h) - y_0|^2 \\ & \leq \int_0^t |f(s+h) - f(s)|^2 ds + \int_0^t |y_\lambda(s+h) - y_\lambda(s)|^2 ds + \varepsilon t \end{aligned}$$

and so since ε is arbitrary, the term εt can be eliminated. By Gronwall's inequality,

$$|y_\lambda(t+h) - y_\lambda(t)|^2 \leq e^t \left(|y_\lambda(h) - y_0|^2 + \int_0^t |f(s+h) - f(s)|^2 ds \right). \quad (23.9.65)$$

The last integral equals

$$\begin{aligned} & \int_0^t \left| \int_s^{s+h} f'(r) dr \right|^2 ds \leq \int_0^t h \int_s^{s+h} |f'(r)|^2 dr ds \\ & = h \left[\int_0^h \int_0^r |f'(r)|^2 ds dr + \int_h^t \int_{r-h}^r |f'(r)|^2 ds dr + \int_t^{t+h} \int_{r-h}^t |f'(r)|^2 ds dr \right] \\ & \leq h^2 \int_0^{t+h} |f'(r)|^2 dr \end{aligned}$$

and now it follows that for all $t+h < T$,

$$\left| \frac{y_\lambda(t+h) - y_\lambda(t)}{h} \right|^2 \leq e^T \left(\left| \frac{y_\lambda(h) - y_0}{h} \right|^2 + \|f'\|_{L^2(0,T;H)}^2 \right). \quad (23.9.66)$$

Now return to 23.9.62.

$$\left| \frac{y_\lambda(h) - y_0}{h} \right| \leq \left| \frac{1}{h} \int_0^h A_\lambda y_\lambda(s) ds \right| + \left| \frac{1}{h} \int_0^h f(s) ds \right|$$

Then taking $\limsup_{h \rightarrow 0}$ of both sides

$$\limsup_{h \rightarrow 0} \left| \frac{y_\lambda(h) - y_0}{h} \right| \leq |A_\lambda y_0| + |f(0)|$$

From Lemma 23.9.3, $|A_\lambda y_0| \leq |a|$ for all $a \in Ay_0$. This is where $y_0 \in D(A)$ is used. Thus from 23.9.66, there exists a constant C independent of t and h and λ such that

$$\left| \frac{y_\lambda(t+h) - y_\lambda(t)}{h} \right|^2 \leq C$$

From the estimate just obtained and 23.9.62, this implies

$$\frac{y_\lambda(t+h) - y_\lambda(t)}{h} + \frac{1}{h} \int_t^{t+h} A_\lambda y_\lambda(s) ds = \frac{1}{h} \int_t^{t+h} f(s) ds \quad (23.9.67)$$

Now letting $h \rightarrow 0$, it follows that for all $t \in [0, T]$, there exists a constant C independent of t, λ such that

$$|A_\lambda y_\lambda(t)| \leq C. \quad (23.9.68)$$

This is a very nice estimate. The next task is to show uniform convergence of the y_λ as $\lambda \rightarrow 0$. From 23.9.62

$$\begin{aligned} & (D_h(y_\lambda(t) - y_\mu(t)), y_\lambda(t+h) - y_\mu(t+h)) + \\ & \left(\frac{1}{h} \int_t^{t+h} (A_\lambda y_\lambda(s) - A_\mu y_\mu(s)) ds, y_\lambda(t+h) - y_\mu(t+h) \right) = 0 \end{aligned}$$

Then from the argument in 23.9.64,

$$\begin{aligned} & \frac{1}{h} \frac{1}{2} \left(|y_\lambda(t+h) - y_\mu(t+h)|^2 - |y_\lambda(t) - y_\mu(t)|^2 \right) \\ & + \left(\frac{1}{h} \int_t^{t+h} (A_\lambda y_\lambda(s) - A_\mu y_\mu(s)) ds, y_\lambda(t+h) - y_\mu(t+h) \right) \leq 0 \end{aligned}$$

Now take $\limsup_{h \rightarrow 0}$ to obtain

$$\frac{1}{2} D^+ |y_\lambda(t) - y_\mu(t)|^2 + (A_\lambda y_\lambda(t) - A_\mu y_\mu(t), y_\lambda(t) - y_\mu(t)) \leq 0$$

Using the definition of A_λ this equals

$$\begin{aligned} & \frac{1}{2} D^+ |y_\lambda(t) - y_\mu(t)|^2 + \\ & (A_\lambda y_\lambda(t) - A_\mu y_\mu(t), \lambda A_\lambda y_\lambda(t) + J_\lambda y_\lambda(t) - (\mu A_\mu y_\mu(t) + J_\mu y_\mu(t))) \leq 0 \end{aligned}$$

Now this last term splits into the following sum

$$\begin{aligned} & (A_\lambda y_\lambda(t) - A_\mu y_\mu(t), \lambda A_\lambda y_\lambda(t) - \mu A_\mu y_\mu(t)) \\ & + (A_\lambda y_\lambda(t) - A_\mu y_\mu(t), J_\lambda y_\lambda(t) - J_\mu y_\mu(t)) \end{aligned}$$

By Lemma 23.9.3 the second of these terms is nonnegative. Also from the estimate 23.9.68, the first term converges to 0 uniformly in t as $\lambda, \mu \rightarrow 0$. Then by Lemma 23.9.7 it follows that if λ is any sequence converging to 0, $y_\lambda(t)$ is uniformly Cauchy. Let

$$y(t) \equiv \lim_{\lambda \rightarrow 0} y_\lambda(t).$$

Thus y is continuous because it is the uniform limit of continuous functions. Since $A_\lambda y_\lambda(t)$ is uniformly bounded, it also follows

$$y(t) = \lim_{\lambda \rightarrow 0} J_\lambda y_\lambda(t) \text{ uniformly in } t. \quad (23.9.69)$$

Taking a further subsequence, you can assume

$$A_\lambda y_\lambda \rightharpoonup z \text{ weak } * \text{ in } L^\infty(0, T; H). \quad (23.9.70)$$

Thus $z \in L^\infty(0, T; H)$. Recall $A_\lambda y_\lambda \in AJ_\lambda y_\lambda$.

Now A can be considered a maximal monotone operator on $L^2(0, T; H)$ according to the rule

$$Ay(t) \equiv A(y(t))$$

where

$$D(A) \equiv \{f \in L^2(0, T; H) : f(t) \in D(A) \text{ a.e. } t\}$$

By Lemma 23.9.4 applied to A considered as a maximal monotone operator on $L^2(0, T; H)$ and using 23.9.69 and 23.9.70, it follows $y(t) \in D(A)$ a.e. t and $z(t) \in Ay(t)$ a.e. t . Then passing to the limit in 23.9.61 yields

$$y(t) - y_0 + \int_0^t z(s) ds = \int_0^t f(s) ds. \quad (23.9.71)$$

Then by fundamental theorem of calculus, $y'(t)$ exists a.e. t and

$$y' + z = f, \quad y(0) = y_0$$

where $z(t) \in Ay(t)$ a.e.

It remains to verify uniqueness. Suppose $[y_1, z_1]$ is another pair which works. Then from 23.9.71,

$$\begin{aligned} y(t) - y_1(t) + \int_0^t (z(r) - z_1(r)) dr &= 0 \\ y(s) - y_1(s) + \int_0^s (z(r) - z_1(r)) dr &= 0 \end{aligned}$$

Therefore for $s < t$,

$$y(t) - y_1(t) - (y(s) - y_1(s)) = \int_s^t (z(r) - z_1(r)) dr$$

and so

$$||y(t) - y_1(t)| - |y(s) - y_1(s)|| \leq K|s - t|$$

for some K depending on $||z||_{L^\infty}, ||z_1||_{L^\infty}$. Since y, y_1 are bounded, it follows that $t \rightarrow |y(t) - y_1(t)|^2$ is also Lipschitz. Therefore by Corollary 17.2.6, it is the integral of its derivative which exists a.e. So what is this derivative? As before,

$$\begin{aligned} & (D_h(y(t) - y_1(t)), y(t+h) - y_1(t+h)) \\ & + \left(\frac{1}{h} \int_t^{t+h} (z(s) - z_1(s)) ds, y(t+h) - y_1(t+h) \right) = 0 \end{aligned}$$

and so

$$\begin{aligned} & \frac{1}{h} \left(\frac{|y(t+h) - y_1(t+h)|^2}{2} - \frac{|y(t) - y_1(t)|^2}{2} \right) \\ & + \left(\frac{1}{h} \int_t^{t+h} (z(s) - z_1(s)) ds, y(t+h) - y_1(t+h) \right) \leq 0 \end{aligned}$$

Then taking $\lim_{h \rightarrow 0}$ it follows that for a.e. t (Lebesgue points of $z - z_1$ intersected with the points where $|y - y_1|^2$ has a derivative)

$$\frac{1}{2} \frac{d}{dt} |y(t) - y_1(t)|^2 + (z(t) - z_1(t), y(t) - y_1(t)) \leq 0$$

Thus for a.e. t ,

$$\frac{d}{dt} |y(t) - y_1(t)|^2 \leq 0$$

and so

$$|y(t) - y_1(t)|^2 - |y_0 - y_0|^2 = \int_0^t \frac{d}{dt} |y(s) - y_1(s)|^2 ds \leq 0.$$

This proves the theorem.

23.10 Exercises

1. Show $L^1(\mathbb{R})$ is not reflexive. **Hint:** $L^1(\mathbb{R})$ is separable. What about $L^\infty(\mathbb{R})$?
2. If $f \in L^1(\mathbb{R}^n; X)$ for X a Banach space, does the usual fundamental theorem of calculus work? That is, can you say $\lim_{r \rightarrow 0} \frac{1}{m(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} f(t) dm = f(x)$ a.e.?
3. Does the Vitali convergence theorem hold for Bochner integrable functions? If so, give a statement of the appropriate theorem and a proof.

The Yankov von Neumann Aumann theorem

The Yankov von Neumann Aumann theorem deals with the projection of a product measurable set. It is a very difficult but interesting theorem. The material of this chapter is taken from [14], [15], [7], and [35]. We use the standard notation that for $\mathcal{S}$ and $\mathcal{F}$ σ algebras, $\mathcal{S} \times \mathcal{F}$ is the σ algebra generated by the measurable rectangles, the product measure σ algebra. The next result is fairly easy and the proof is left for the reader.

Lemma 24.0.1 *Let (X, d) be a metric space. Then if $d_1(x, y) = \frac{d(x, y)}{1+d(x, y)}$, it follows that d_1 is a metric on X and the basis of open balls taken with respect to d_1 yields the same topology as the basis of open balls taken with respect to d .*

Theorem 24.0.2 *Let (X_i, d_i) denote a complete metric space and let $X \equiv \prod_{i=1}^{\infty} X_i$. Then X is also a complete metric space with the metric*

$$\rho(\mathbf{x}, \mathbf{y}) \equiv \sum_{i=1}^{\infty} 2^{-i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}.$$

Also, if X_i is separable for each i then so is X .

Proof: It is clear from the above lemma that ρ is a metric on X . We need to verify X is complete with this metric. Let $\{\mathbf{x}^n\}$ be a Cauchy sequence in X . Then it is clear from the definition that $\{x_i^n\}$ is a Cauchy sequence for each i and converges to $x_i \in X_i$. Therefore, letting $\varepsilon > 0$ be given, we choose N such that

$$\sum_{k=N}^{\infty} 2^{-k} < \frac{\varepsilon}{2},$$

we choose M large enough that for $n > M$,

$$2^{-i} \frac{d_i(x_i^n, x_i)}{1 + d_i(x_i^n, x_i)} < \frac{\varepsilon}{2(N+1)}$$

for all $i = 1, 2, \dots, N$. Then letting $\mathbf{x} = \{x_i\}$,

$$\rho(\mathbf{x}, \mathbf{x}^n) \leq \frac{\varepsilon N}{2(N+1)} + \sum_{k=N}^{\infty} 2^{-k} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We need to verify that X is separable. Let D_i denote a countable dense set in X_i , $D_i \equiv \{r_k^i\}_{k=1}^{\infty}$. Then let

$$\mathcal{D}_k \equiv D_1 \times \dots \times D_k \times \{r_1^{k+1}\} \times \{r_1^{k+2}\} \times \dots$$

Thus $\mathcal{D}_k$ is a countable subset of X . Let $\mathcal{D} \equiv \cup_{k=1}^{\infty} \mathcal{D}_k$. Then $\mathcal{D}$ is countable and we can see $\mathcal{D}$ is dense in X as follows. The projection of $\mathcal{D}_k$ onto the first k entries is dense in $\prod_{i=1}^k X_i$ and for k large enough the remaining component's contribution to the metric, ρ is very small. Therefore, obtaining $\mathbf{d} \in \mathcal{D}$ close to $\mathbf{x} \in X$ may be accomplished by finding $\mathbf{d} \in \mathcal{D}$ such that $\mathbf{d}$ is close to $\mathbf{x}$ in the first k components for k large enough. Note that we do not use $\prod_{k=1}^{\infty} D_k$!

Definition 24.0.3 *A complete separable metric space is called a polish space.*

Theorem 24.0.4 *Let X be a polish space. Then there exists $f: \mathbb{N}^{\mathbb{N}} \rightarrow X$ which is onto and continuous. Here $\mathbb{N}^{\mathbb{N}} \equiv \prod_{i=1}^{\infty} \mathbb{N}$ and a metric is given according to the above theorem. Thus for $\mathbf{n}, \mathbf{m} \in \mathbb{N}^{\mathbb{N}}$,*

$$\rho(\mathbf{n}, \mathbf{m}) \equiv \sum_{i=1}^{\infty} 2^{-i} \frac{|n_i - m_i|}{1 + |n_i - m_i|}.$$

Proof: Since X is polish, there exists a countable covering of X by closed sets having diameters no larger than 2^{-1} , $\{B(i)\}_{i=1}^{\infty}$. Each of these closed sets is also a polish space and so there exists a countable covering of $B(i)$ by a countable collection of closed sets, $\{B(i, j)\}_{j=1}^{\infty}$ each having diameter no larger than 2^{-2} where $B(i, j) \subseteq B(i) \neq \emptyset$ for all j . Continue this way. Thus

$$B(n_1, n_2, \dots, n_m) = \cup_{i=1}^{\infty} B(n_1, n_2, \dots, n_m, i)$$

and each of $B(n_1, n_2, \dots, n_m, i)$ is a closed set contained in $B(n_1, n_2, \dots, n_m)$ whose diameter is at most half of the diameter of $B(n_1, n_2, \dots, n_m)$. Now we define our mapping from $\mathbb{N}^{\mathbb{N}}$ to X . If $\mathbf{n} = \{n_k\}_{k=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$, we let $f(\mathbf{n}) \equiv \cap_{m=1}^{\infty} B(n_1, n_2, \dots, n_m)$. Since the diameters of these sets converge to 0, there exists a unique point in this countable intersection and this is $f(\mathbf{n})$.

We need to verify f is continuous. Let $\mathbf{n} \in \mathbb{N}^{\mathbb{N}}$ be given and suppose $\mathbf{m}$ is very close to $\mathbf{n}$. The only way this can occur is for n_k to coincide with m_k for many k . Therefore, both $f(\mathbf{n})$ and $f(\mathbf{m})$ must be contained in $B(n_1, n_2, \dots, n_m)$ for some fairly large m . This implies, from the above construction that $f(\mathbf{m})$ is as close to $f(\mathbf{n})$ as 2^{-m} , proving f is continuous. To see that f is onto, note that from the construction, if $x \in X$, then $x \in B(n_1, n_2, \dots, n_m)$ for some choice of $n_1, \dots, n_m$ for each m . Note nothing is said about f being one to one. It probably is not one to one.

Definition 24.0.5 We call a topological space, X a Suslin space if X is a Hausdorff space and there exists a polish space, Z and a continuous function f which maps Z onto X . These are also called analytic sets in some contexts but we will use the term Suslin space in referring to them.

Corollary 24.0.6 X is a Suslin space, if and only if there exists a continuous mapping from $\mathbb{N}^{\mathbb{N}}$ onto X .

Proof: We know there exists a polish space Z and a continuous function, $h : Z \rightarrow X$ which is onto. By the above theorem there exists a continuous map, $g : \mathbb{N}^{\mathbb{N}} \rightarrow Z$ which is onto. Then $h \circ g$ is a continuous map from $\mathbb{N}^{\mathbb{N}}$ onto X . The “if” part of this theorem is accomplished by noting that $\mathbb{N}^{\mathbb{N}}$ is a polish space.

Lemma 24.0.7 Let X be a Suslin space and suppose X_i is a subspace of X which is also a Suslin space. Then $\cup_{i=1}^{\infty} X_i$ and $\cap_{i=1}^{\infty} X_i$ are also Suslin spaces. Also every Borel set in X is a Suslin space.

Proof: Let $f_i : Z_i \rightarrow X_i$ where Z_i is a polish space and f_i is continuous and onto. Without loss of generality we may assume the spaces Z_i are disjoint because if not, we could replace Z_i with $Z_i \times \{i\}$. Now we define a metric, ρ , for $Z \equiv \cup_{i=1}^{\infty} Z_i$ as follows.

$$\rho(x, y) \equiv 1 \text{ if } x \in Z_i, y \in Z_k, i \neq k$$

$$\rho(x, y) \equiv \frac{d_i(x, y)}{1 + d_i(x, y)} \text{ if } x, y \in Z_i.$$

Here d_i is the metric on Z_i . It is easy to verify ρ is a metric and that (Z, ρ) is a polish space. Now we define $f : Z \rightarrow \cup_{i=1}^{\infty} X_i$ as follows. For $x \in Z_i$, $f(x) \equiv f_i(x)$. This is well defined because the Z_i are disjoint. If y is very close to x it must be that x and y are in the same Z_i otherwise this could not happen. Therefore, continuity of f follows from continuity of f_i . This shows countable unions of Suslin subspaces of a Suslin space are Suslin spaces.

If $H \subseteq X$ is a closed subset, then, letting $f : Z \rightarrow X$ be onto and continuous, it follows $f : f^{-1}(H) \rightarrow H$ is onto and continuous. Since $f^{-1}(H)$ is closed, it follows $f^{-1}(H)$ is a polish space. Therefore, H is a Suslin space.

Now we show countable intersections of Suslin spaces are Suslin. It is clear that $\theta : \prod_{i=1}^{\infty} Z_i \rightarrow \prod_{i=1}^{\infty} X_i$ given by $\theta(\mathbf{z}) \equiv \mathbf{x} = \{x_i\}$ where $x_i = f_i(z_i)$ is continuous and onto. Therefore, $\prod_{i=1}^{\infty} X_i$ is a Suslin space. Now let $P \equiv \{\mathbf{y} \in \prod_{i=1}^{\infty} f_i(Z_i) : y_i = y_j \text{ for all } i, j\}$. Then P is a closed subspace of a Suslin space and so it is Suslin. Then we define $h : P \rightarrow \cap_{i=1}^{\infty} X_i$ by $h(\mathbf{y}) \equiv f_i(y_i)$. This shows $\cap_{i=1}^{\infty} X_i$ is Suslin because h is continuous and onto. ($h \circ \theta : \theta^{-1}(P) \rightarrow \cap_{i=1}^{\infty} X_i$ is continuous and $\theta^{-1}(P)$ being a closed subset of a polish space is polish.)

Next let U be an open subset of X . Then $f^{-1}(U)$, being an open subset of a polish space, can be obtained as an increasing limit of closed sets, K_n . Therefore, $U = \cup_{n=1}^{\infty} f(K_n)$. Each $f(K_n)$ is a Suslin space because it is the continuous image of a polish space, K_n . Therefore, by the first part of the lemma, U is a Suslin space. Now let

$$\mathcal{F} \equiv \{E \subseteq X : \text{both } E^C \text{ and } E \text{ are Suslin}\}.$$

We see that $\mathcal{F}$ is closed with respect to taking complements. The first part of this lemma shows $\mathcal{F}$ is closed with respect to countable unions. Therefore, $\mathcal{F}$ is a σ algebra and so, since it contains the open sets, must contain the Borel sets.

It turns out that Suslin spaces tend to be measurable sets. In order to develop this idea, we need a technical lemma.

Lemma 24.0.8 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and denote by μ^* the outer measure generated by μ . Thus*

$$\mu^*(S) \equiv \inf \{ \mu(E) : E \supseteq S, E \in \mathcal{F} \}.$$

Then μ^* is regular, meaning that for every S , there exists $E \in \mathcal{F}$ such that $E \supseteq S$ and $\mu(E) = \mu^*(S)$. If $S_n \uparrow S$, it follows that $\mu^*(S_n) \uparrow \mu^*(S)$. Also if $\mu(\Omega) < \infty$, then a set, E is measurable if and only if

$$\mu^*(\Omega) \geq \mu^*(E) + \mu^*(\Omega \setminus E).$$

Proof: First we verify that μ^* is regular. If $\mu^*(S) = \infty$, let $E = \Omega$. Then $\mu^*(S) = \mu(E)$ and $E \supseteq S$. On the other hand, if $\mu^*(S) < \infty$, then we can obtain $E_n \in \mathcal{F}$ such that $\mu^*(S) + \frac{1}{n} \geq \mu(E_n)$ and $E_n \supseteq S$. Now let $F_n = \bigcap_{i=1}^n E_i$. Then $F_n \supseteq S$ and so $\mu^*(S) + \frac{1}{n} \geq \mu(F_n) \geq \mu^*(S)$. Therefore, letting $F = \bigcap_{k=1}^{\infty} F_k \in \mathcal{F}$ it follows $\mu(F) = \lim_{n \rightarrow \infty} \mu(F_n) = \mu^*(S)$.

Let $E_n \supseteq S_n$ be such that $E_n \in \mathcal{F}$ and $\mu(E_n) = \mu^*(S_n)$. Also let $E_{\infty} \supseteq S$ such that $\mu(E_{\infty}) = \mu^*(S)$ and $E_{\infty} \in \mathcal{F}$. Now consider $B_n \equiv \bigcup_{k=1}^n E_k$. We claim

$$\mu(B_n) = \mu(S_n). \quad (24.0.1)$$

Here is why:

$$\mu(E_1 \setminus E_2) = \mu(E_1) - \mu(E_1 \cap E_2) = \mu^*(S_1) - \mu^*(S_1) = 0.$$

Therefore,

$$\mu(B_2) = \mu(E_1 \cup E_2) = \mu(E_1 \setminus E_2) + \mu(E_2) = \mu(E_2) = \mu^*(S_2).$$

Continuing in this way we see that 24.0.1 holds. Now let $B_n \cap E_{\infty} \equiv C_n$. Then $C_n \uparrow C \equiv \bigcup_{k=1}^{\infty} C_k \in \mathcal{F}$ and $\mu(C_n) = \mu^*(S_n)$. Since $S_n \uparrow S$ and each $C_n \supseteq S_n$, it follows $C \supseteq S$ and therefore,

$$\mu^*(S) \leq \mu(C) = \lim_{n \rightarrow \infty} \mu(C_n) = \lim_{n \rightarrow \infty} \mu^*(S_n) \leq \mu^*(S).$$

Now we verify the second claim of the lemma. It is clear the formula holds whenever E is measurable. Suppose now that the formula holds. Let S be an arbitrary set. We need to verify that

$$\mu^*(S) \geq \mu^*(S \cap E) + \mu^*(S \setminus E).$$

Let $F \supseteq S$, $F \in \mathcal{F}$, and $\mu(F) = \mu^*(S)$. Then since μ^* is subadditive,

$$\mu^*(\Omega \setminus F) \leq \mu^*(E \setminus F) + \mu^*(\Omega \cap E^C \cap F^C). \quad (24.0.2)$$

Since F is measurable,

$$\mu^*(E) = \mu^*(E \cap F) + \mu^*(E \setminus F) \quad (24.0.3)$$

and

$$\mu^*(\Omega \setminus E) = \mu^*(F \setminus E) + \mu^*(\Omega \cap E^C \cap F^C) \quad (24.0.4)$$

and by the hypothesis,

$$\mu^*(\Omega) \geq \mu^*(E) + \mu^*(\Omega \setminus E). \quad (24.0.5)$$

Therefore,

$$\begin{aligned} \mu(\Omega) &\geq \mu^*(E) + \mu^*(\Omega \setminus E) \\ &= \mu^*(E \cap F) + \mu^*(E \setminus F) + \mu^*(\Omega \setminus E) \\ &= \mu^*(E \cap F) + \mu^*(E \setminus F) + \mu^*(F \setminus E) + \mu^*(\Omega \cap E^C \cap F^C) \\ &\geq \mu^*(\Omega \setminus F) + \mu^*(F \setminus E) + \mu^*(E \cap F) \\ &\geq \mu^*(\Omega \setminus F) + \mu^*(F) = \mu(\Omega) \end{aligned}$$

showing that all the inequalities must be equal signs. Hence, referring to the top and fourth lines above,

$$\mu(\Omega) = \mu^*(\Omega \setminus F) + \mu^*(F \setminus E) + \mu^*(E \cap F).$$

Subtracting $\mu^*(\Omega \setminus F) = \mu(\Omega \setminus F)$ from both sides gives

$$\mu^*(S) = \mu(F) = \mu^*(F \setminus E) + \mu^*(E \cap F) \geq \mu^*(S \setminus E) + \mu^*(E \cap S),$$

This proves the lemma.

The next theorem is a major result. It states that the Suslin subsets are measurable under appropriate conditions.

Theorem 24.0.9 *Let Ω be a metric space and let $(\Omega, \mathcal{F}, \mu)$ be a complete Borel measure space with $\mu(\Omega) < \infty$. Denote by μ^* the outer measure generated by μ . Then if A is a Suslin subset of Ω , it follows that A is μ^* measurable.*

Proof: We need to verify that

$$\mu^*(\Omega) \geq \mu^*(A) + \mu^*(\Omega \setminus A).$$

We know from Corollary 24.0.6, there exists a continuous map, $f : \mathbb{N}^{\mathbb{N}} \rightarrow A$ which is onto. Let

$$E(k) \equiv \{\mathbf{n} \in \mathbb{N}^{\mathbb{N}} : n_1 \leq k\}.$$

Then $E(k) \uparrow \mathbb{N}^{\mathbb{N}}$ and so from Lemma 24.0.8 we know $\mu^*(f(E(k))) \uparrow \mu^*(A)$. Therefore, there exists m_1 such that

$$\mu^*(f(E(m_1))) > \mu^*(A) - \frac{\varepsilon}{2}.$$

Now $E(k)$ is clearly not compact but it is trying to be as far as the first component is concerned. Now we let

$$E(m_1, k) \equiv \{\mathbf{n} \in \mathbb{N}^{\mathbb{N}} : n_1 \leq m_1 \text{ and } n_2 \leq k\}.$$

Thus $E(m_1, k) \uparrow E(m_1)$ and so we can pick m_2 such that

$$\mu^*(f(E(m_1, m_2))) > \mu^*(f(E(m_1))) - \frac{\varepsilon}{2^2}.$$

We continue in this way obtaining a decreasing list of sets, $f(E(m_1, m_2, \dots, m_{k-1}, m_k))$, such that

$$\mu^*(f(E(m_1, m_2, \dots, m_{k-1}, m_k))) > \mu^*(f(E(m_1, m_2, \dots, m_{k-1}))) - \frac{\varepsilon}{2^k}.$$

Therefore,

$$\mu^*(f(E(m_1, m_2, \dots, m_{k-1}, m_k))) - \mu^*(A) > \sum_{l=1}^k -\left(\frac{\varepsilon}{2^l}\right) > -\varepsilon.$$

Now define a closed set,

$$C \equiv \bigcap_{k=1}^{\infty} \overline{f(E(m_1, m_2, \dots, m_{k-1}, m_k))}.$$

The sets $f(E(m_1, m_2, \dots, m_{k-1}, m_k))$ are decreasing as $k \rightarrow \infty$ and so

$$\mu^*(C) = \lim_{k \rightarrow \infty} \mu^*\left(\overline{f(E(m_1, m_2, \dots, m_{k-1}, m_k))}\right) \geq \mu^*(A) - \varepsilon.$$

We wish to verify that $C \subseteq A$. If we can do this we will be done because C , being a closed set, is measurable and so

$$\mu^*(\Omega) = \mu^*(C) + \mu^*(\Omega \setminus C) \geq \mu^*(A) - \varepsilon + \mu^*(\Omega \setminus A).$$

Since ε is arbitrary, this will conclude the proof. Therefore, we only need to verify that $C \subseteq A$.

What we know is that each $f(E(m_1, m_2, \dots, m_{k-1}, m_k))$ is contained in A . We do not know their closures are contained in A . We let $\mathbf{m} \equiv \{m_i\}_{i=1}^{\infty}$ where the m_i are defined above. Then letting

$$K \equiv \{\mathbf{n} \in \mathbb{N}^{\mathbb{N}} : n_i \leq m_i \text{ for all } i\},$$

we see that K is a closed, hence complete subset of $\mathbb{N}^{\mathbb{N}}$ which is also totally bounded due to the definition of the distance. Therefore, K is compact and so $f(K)$ is also compact, hence closed due to the assumption that Ω is a Hausdorff space and we know that $f(K) \subseteq A$. We verify that $C = f(K)$. We know $f(K) \subseteq C$. Suppose therefore, $p \in C$. From the definition of C , we know there exists $\mathbf{r}^k \in E(m_1, m_2, \dots, m_{k-1}, m_k)$ such that $d(f(\mathbf{r}^k), p) < \frac{1}{k}$. Denote by $\mathbf{r}^k$ the element of $\mathbb{N}^{\mathbb{N}}$ which consists of modifying $\mathbf{r}^k$ by taking all components after the k^{th} equal to

one. Thus $\tilde{\mathbf{r}}^k \in K$. Now $\{\tilde{\mathbf{r}}^k\}$ is in a compact set and so taking a subsequence we can have $\tilde{\mathbf{r}}^k \rightarrow \mathbf{r} \in K$. But from the metric on $\mathbb{N}^{\mathbb{N}}$, it follows that $\rho(\tilde{\mathbf{r}}^k, \mathbf{r}^k) < \frac{1}{2^{k-2}}$. Therefore, $\mathbf{r}^k \rightarrow \mathbf{r}$ also and so $f(\mathbf{r}^k) \rightarrow f(\mathbf{r}) = p$. Therefore, $p \in f(K)$ and this proves the theorem.

Note we could have proved this under weaker assumptions. If we had assumed only that every point has a countable basis (first axiom of countability) and Ω is Hausdorff, the same argument would work. We will need the following definition.

Definition 24.0.10 Let $\mathcal{F}$ be a σ algebra of sets from Ω and let μ denote a finite measure defined on $\mathcal{F}$. We let $\mathcal{F}_\mu$ denote the completion of $\mathcal{F}$ with respect to μ . Thus we let μ^* be the outer measure determined by μ and $\mathcal{F}_\mu$ will be the σ algebra of μ^* measurable subsets of Ω . We also define $\widehat{\mathcal{F}}$ by

$$\widehat{\mathcal{F}} \equiv \cap \{\mathcal{F}_\mu : \mu \text{ is a finite measure defined on } \mathcal{F}\}.$$

Also, if X is a topological space, we will denote by $B(X)$ the Borel sets of X .

With this notation, we can give the following simple corollary of Theorem 24.0.9.

Corollary 24.0.11 Let Ω be a compact metric space and let A be a Suslin subset of Ω . Then $A \in \widehat{B(\Omega)}$.

Proof: Let μ be a finite measure defined on $B(\Omega)$. By Theorem 24.0.9 $A \in B(\Omega)_\mu$. Since this is true for every finite measure, μ , it follows $A \in \widehat{B(\Omega)}$ as claimed. This proves the corollary.

We give another technical lemma about the completion of measure spaces.

Lemma 24.0.12 Let μ be a finite measure on a σ algebra, Σ . Then $A \in \Sigma_\mu$ if and only if there exists $A_1 \in \Sigma$ and N_1 such that $A = A_1 \cup N_1$ where there exists $N \in \Sigma$ such that $\mu(N) = 0$ and $N_1 \subseteq N$.

Proof: Suppose first $A = A_1 \cup N_1$ where these sets are as described. Let $S \in \mathcal{P}(\Omega)$ and let μ^* denote the outer measure determined by μ . Then since $A_1 \in \Sigma \subseteq \Sigma_\mu$

$$\begin{aligned} \mu^*(S) &\leq \mu^*(S \setminus A) + \mu^*(S \cap A) \\ &\leq \mu^*(S \setminus A_1) + \mu^*(S \cap A_1) + \mu^*(N_1) \\ &= \mu^*(S \setminus A_1) + \mu^*(S \cap A_1) = \mu^*(S) \end{aligned}$$

showing that $A \in \Sigma_\mu$.

Now suppose $A \in \Sigma_\mu$. Then there exists $B_1 \supseteq A$ such that $\mu^*(B_1) = \mu^*(A)$, and $B_1 \in \Sigma$. Also there exists $A_1^C \in \Sigma$ with $A_1^C \supseteq A^C$ and $\mu(A_1^C) = \mu^*(A^C)$. Then $A_1 \subseteq A \subseteq B_1$

$$A \subseteq A_1 \cup (B_1 \setminus A_1).$$

Now

$$\mu(A_1) + \mu^*(A^C) = \mu(A_1) + \mu(A_1^C) = \mu(\Omega)$$

and so

$$\begin{aligned} \mu(B_1 \setminus A_1) &= \mu^*(B_1 \setminus A_1) \\ &= \mu^*(B_1 \setminus A) + \mu^*(A \setminus A_1) \\ &= \mu^*(B_1) - \mu^*(A) + \mu^*(A) - \mu^*(A_1) \\ &= \mu^*(A) - (\mu(\Omega) - \mu^*(A^C)) = 0 \end{aligned}$$

because $A \in \Sigma_\mu$ implying $A = A_1 \cup \overbrace{(B_1 \setminus A_1) \cap A}^{N_1}$ and $N_1 \subseteq N \equiv (B_1 \setminus A_1) \in \Sigma$ with $\mu(N) = 0$. This proves the lemma.

Next we need another definition.

Definition 24.0.13 We say (Ω, Σ) , where Σ is a σ algebra of subsets of Ω , is separable if there exists a sequence $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ such that $\sigma(\{A_n\}) = \Sigma$ and if $w \neq w'$, then there exists $A \in \Sigma$ such that $\mathcal{X}_A(w) \neq \mathcal{X}_A(w')$. This last condition is referred to by saying $\{A_n\}$ separates the points of Ω . Given two measure spaces, (Ω, Σ) and (Ω', Σ') , we say they are isomorphic if there exists a function, $f: \Omega \rightarrow \Omega'$ which is one to one and $f(E) \in \Sigma'$ whenever $E \in \Sigma$ and $f^{-1}(F) \in \Sigma$ whenever $F \in \Sigma'$.

The interesting thing about separable measure spaces is that they are isomorphic to a very simple sort of measure space in which topology plays a significant role.

Lemma 24.0.14 Let (Ω, Σ) be separable. Then there exists $E \in \{0, 1\}^\mathbb{N}$ such that (Ω, Σ) and $(E, B(E))$ are isomorphic.

Proof: First we show $\{A_n\}$ separates the points. We already know Σ separates the points. If this is not so, there exists $\omega, \omega_1 \in \Omega$ such that for all n , $\mathcal{X}_{A_n}(\omega) = \mathcal{X}_{A_n}(\omega_1)$. Then let

$$\mathcal{F} \equiv \{F \in \Sigma : \mathcal{X}_F(\omega) = \mathcal{X}_F(\omega_1)\}$$

Thus $A_n \in \mathcal{F}$ for all n . It is also clear that $\mathcal{F}$ is a σ algebra and so $\mathcal{F} = \Sigma$ contradicting the assumption that Σ separates points. Now we define a function from Ω to $\{0, 1\}^\mathbb{N}$ as follows.

$$f(\omega) \equiv \{\mathcal{X}_{A_n}(\omega)\}_{n=1}^\infty$$

We also let $E \equiv f(\Omega)$. Since the $\{A_n\}$ separate the points, we see that f is one to one. A subbasis for the topology of $\{0, 1\}^\mathbb{N}$ consists of sets of the form $\prod_{i=1}^\infty H_i$ where $H_i = \{0, 1\}$ for all i except one, when $i = j$ and H_j equals either $\{0\}$ or $\{1\}$. Therefore, $f^{-1}(\text{subbasic open set}) \in \Sigma$ because if H_j is the exceptional set then this equals A_j if $H_j = \{1\}$ and A_j^C if $H_j = \{0\}$. Intersections of these subbasic sets with E gives a countable subbasis for E and so the inverse image of all sets

in a countable subbasis for E are in Σ , showing that $f^{-1}(\text{open set}) \in \Sigma$. Now we consider $f(A_n)$.

$$f(A_n) \equiv \{\{\lambda_k\}_{k=1}^\infty : \lambda_n = 1\} \cap E,$$

an open set in E . Hence $f(A_n) \in B(E)$. Now letting

$$\mathcal{F} \equiv \{G \subseteq \Omega : f(G) \in B(E)\},$$

we see that $\mathcal{F}$ is a σ algebra which contains $\{A_n\}_{n=1}^\infty$ and so $\mathcal{F} \supseteq \sigma(\{A_n\}) \equiv \Sigma$. Thus $f(F) \in B(E)$ for all $A \in \Sigma$. This proves the lemma.

Lemma 24.0.15 *Let $\phi : (\Omega_1, \Sigma_1) \rightarrow (\Omega_2, \Sigma_2)$ where $\phi^{-1}(U) \in \Sigma_1$ for all $U \in \Sigma_2$. Then if $F \in \widehat{\Sigma}_2$, it follows $\phi^{-1}(F) \in \widehat{\Sigma}_1$.*

Proof: Let μ be a finite measure on Σ_1 and define a measure $\phi(\mu)$ on Σ_2 by the rule

$$\phi(\mu)(F) \equiv \mu(\phi^{-1}(F)).$$

Now let $A \in \Sigma_{2\phi(\mu)}$. Then by Lemma 24.0.12, $A = A_1 \cup N_1$ where there exists $N \in \Sigma_2$ with $\phi(\mu)(N) = 0$ and $A_1 \in \Sigma_2$. Therefore, from the definition of $\phi(\mu)$, we have $\mu(\phi^{-1}(N)) = 0$ and therefore, $\phi^{-1}(A) = \phi^{-1}(A_1) \cup \phi^{-1}(N_1)$ where $\phi^{-1}(N_1) \subseteq \phi^{-1}(N) \in \Sigma_1$ and $\mu(\phi^{-1}(N)) = 0$. Therefore, $\phi^{-1}(A) \in \Sigma_{1\mu}$ and so if $F \in \widehat{\Sigma}_2$, then

$$F \in \cap \{\Sigma_{2\nu} : \nu \text{ is a finite measure on } \Sigma_2\} \subseteq \cap \{\Sigma_{2\phi(\mu)} : \mu \text{ is a finite measure on } \Sigma_1\},$$

and so $\phi^{-1}(F) \in \Sigma_{1\mu}$. Since μ is arbitrary, this shows $\phi^{-1}(F) \in \widehat{\Sigma}_1$.

The next lemma is a special case of the Yankov von Neumann Aumann projection theorem. It contains the main idea of the proof of the more general theorem.

Lemma 24.0.16 *Let (Ω, Σ) be separable and let X be a Suslin space. Let $G \in \Sigma \times B(X)$. (Recall $\Sigma \times B(X)$ is the σ algebra of product measurable sets, the smallest σ algebra containing the measurable rectangles.) Then*

$$\text{proj}_\Omega(G) \in \widehat{\Sigma}.$$

Proof: Let $f : (\Omega, \Sigma) \rightarrow (E, B(E))$ be the isomorphism of Lemma 24.0.14. We have the following claim.

Claim: $f \times \text{id}_X$ maps $\Sigma \times B(X)$ to $B(E) \times B(X)$.

Proof of the claim: First of all, assume $A \times B$ is a measurable rectangle where $A \in \Sigma$ and $B \in B(X)$. Then by the assumption that f is an isomorphism, $f(A) \in B(E)$ and so

$$f \times \text{id}_X(A \times B) \in B(E) \times B(X).$$

Now let

$$\mathcal{F} \equiv \{P \in \Sigma \times B(X) : f \times \text{id}_X(P) \in B(E) \times B(X)\}.$$

Then we see that $\mathcal{F}$ is a σ algebra and contains the elementary sets. ($\mathcal{F}$ is closed with respect to complements because f is one to one.) Therefore, $\mathcal{F} = \Sigma \times B(X)$ and this proves the claim.

Therefore, since $G \in \Sigma \times B(X)$, we see

$$f \times \text{id}_X(G) \in B(E) \times B(X) \subseteq B(E \times X).$$

The set inclusion follows from the observation that if $A \in B(E)$ and $B \in B(X)$ then $A \times B$ is in $B(E \times X)$ and the collection of sets in $B(E) \times B(X)$ which are in $B(E \times X)$ is a σ algebra.

Therefore, there exists D , a Borel set in $E \times X$ such that $f \times \text{id}_X(G) = D \cap (E \times X)$. Now from this it follows from Lemma 24.0.7 that D is a Suslin space. Letting Y be $\{0, 1\}^{\mathbb{N}}$, it follows that $\text{proj}_Y(D)$ is a Suslin space in Y . By Corollary 24.0.11, we see that $\text{proj}_Y(D) \in \widehat{B(Y)}$. Now

$$\begin{aligned} \text{proj}_\Omega(G) &= \{\omega \in \Omega : \text{there exists } x \in X \text{ with } (\omega, x) \in G\} \\ &= \{\omega \in \Omega : \text{there exists } x \in X \text{ with } (f(\omega), x) \in f \times \text{id}_X(G)\} \\ &= f^{-1}(\{y \in Y : \text{there exists } x \in X \text{ with } (y, x) \in D\}) \\ &= f^{-1}(\text{proj}_Y(D)). \end{aligned}$$

Now $\text{proj}_Y(D) \in \widehat{B(Y)}$ and so Lemma 24.0.15 shows $f^{-1}(\text{proj}_Y(D)) \in \widehat{\Sigma}$. This proves the lemma.

Now we are ready to prove the Yankov von Neumann Aumann projection theorem. First we must present another technical lemma.

Lemma 24.0.17 *Let X be a Hausdorff space and let $G \in \Sigma \times B(X)$ where Σ is a σ algebra of sets of Ω . Then there exists $\Sigma_0 \subseteq \Sigma$ a countably generated σ algebra such that $G \in \Sigma_0 \times B(X)$.*

Proof: First suppose G is a measurable rectangle, $G = A \times B$ where $A \in \Sigma$ and $B \in B(X)$. Letting Σ_0 be the finite σ algebra, $\{\emptyset, A, A^C, \Omega\}$, we see that $G \in \Sigma_0 \times B(X)$. Similarly, if G equals an elementary set, then the conclusion of the lemma holds for G . Let

$$\mathcal{F} \equiv \{H \in \Sigma \times B(X) : H \in \Sigma_0 \times B(X)\}$$

for some countably generated σ algebra, Σ_0 . We just saw that $\mathcal{F}$ contains the elementary sets. If $H \in \mathcal{F}$, then $H^C \in \Sigma_0 \times B(X)$ for the same Σ_0 and so $\mathcal{F}$ is closed with respect to complements. Now suppose $H_n \in \mathcal{F}$. Then for each n , there exists a countably generated σ algebra, Σ_{0n} such that $H_n \in \Sigma_{0n} \times B(X)$. Then $\bigcup_{n=1}^{\infty} H_n \in \sigma(\{\Sigma_{0n} \times B(X)\})$. We will be done when we show

$$\sigma(\{\Sigma_{0n} \times B(X)\}_{n=1}^{\infty}) \subseteq \sigma(\{\Sigma_{0n}\}_{n=1}^{\infty}) \times B(X)$$

because it is clear that $\sigma(\{\Sigma_{0n}\}_{n=1}^{\infty})$ is countably generated. We see that $\sigma(\{\Sigma_{0n} \times B(X)\}_{n=1}^{\infty})$ is generated by sets of the form $A \times B$ where $A \in \Sigma_{0n}$ and $B \in B(X)$. But each

such set is also contained in $\sigma(\{\Sigma_{0n}\}_{n=1}^\infty) \times B(X)$ and so the desired inclusion is obtained. Therefore, $\mathcal{F}$ is a σ algebra and so since $\mathcal{F}$ was shown to contain the measurable rectangles, this verifies $\mathcal{F} = \Sigma \times B(X)$ and this proves the lemma.

Theorem 24.0.18 *Let (Ω, Σ) be a measure space and let $G \in \widehat{\Sigma} \times B(X)$ where X is a Suslin space. Then*

$$\text{proj}_\Omega(G) \in \widehat{\Sigma}.$$

Proof: By the previous lemma, $G \in \Sigma_0 \times B(X)$ where Σ_0 is countably generated. If (Ω, Σ_0) were separable, we could then apply Lemma 24.0.16 and be done. Unfortunately, we don't know Σ_0 separates the points of Ω . Therefore, we define an equivalence class on the points of Ω as follows. We say $\omega \sim \omega_1$ if and only if $\mathcal{X}_A(\omega) = \mathcal{X}_A(\omega_1)$ for all $A \in \Sigma_0$. Now the nice thing to notice about this equivalence relation is that if $\omega \in A \in \Sigma_0$, and if $\omega \sim \omega_1$, then $1 = \mathcal{X}_A(\omega) = \mathcal{X}_A(\omega_1)$ implying $\omega_1 \in A$ also. Therefore, every set of Σ_0 is the union of equivalence classes. It follows that for $A \in \Sigma_0$, and π the map given by $\pi\omega \equiv [\omega]$ where $[\omega]$ is the equivalence class determined by ω ,

$$\pi(A) \cap \pi(\Omega \setminus A) = \emptyset.$$

Suppose now that $H_n \in \Sigma_0 \times B(X)$. If $([\omega], x) \in \cap_{n=1}^\infty \pi \times \text{id}_X(H_n)$, then for each n ,

$$([\omega], x) = (\pi\omega_n, x)$$

for some $(\omega_n, x) \in H_n$. But this implies $\omega \sim \omega_n$ and so from the above observation that the sets of Σ_0 are unions of equivalence classes, it follows that $(\omega, x) \in H_n$. Therefore, $(\omega, x) \in \cap_{n=1}^\infty H_n$ and so $([\omega], x) = \pi \times \text{id}_X(\omega, x)$ where $(\omega, x) \in \cap_{n=1}^\infty H_n$. This shows that

$$\pi \times \text{id}_X(\cap_{n=1}^\infty H_n) \supseteq \cap_{n=1}^\infty \pi \times \text{id}_X(H_n).$$

In fact these two sets are equal because the other inclusion is obvious. We will denote by Ω_1 the set of equivalence classes and Σ_1 will be the subsets, S_1 , of Ω_1 such that $S_1 = \{[\omega] : \omega \in S \in \Sigma_0\}$. Then (Ω_1, Σ_1) is clearly a measure space which is separable. Let

$$\mathcal{F} \equiv \{H \in \Sigma_0 \times B(X) : \pi \times \text{id}_X(H), \pi \times \text{id}_X(H^C) \in \Sigma_1 \times B(X)\}.$$

We see that the measurable rectangles, $A \times B$ where $A \in \Sigma_0$ and $B \in B(X)$ are in $\mathcal{F}$, that from the above observation on countable intersections, $\mathcal{F}$ is closed with respect to countable unions and closed with respect to complements. Therefore, $\mathcal{F}$ is a σ algebra and so $\mathcal{F} = \Sigma_0 \times B(X)$. By Lemma 24.0.14 (Ω_1, Σ_1) is isomorphic to $(E, B(E))$ where E is a subspace of $\{0, 1\}^\mathbb{N}$. Denoting the isomorphism by h , it follows as in Lemma 24.0.16 that $h \times \text{id}_X$ maps $\Sigma_1 \times B(X)$ to $B(E) \times B(X)$. Therefore, we see $f \equiv h \circ \pi$ is a mapping from Ω to E which has the property that $f \times \text{id}_X$ maps $\Sigma_0 \times B(X)$ to $B(E) \times B(X)$. Now from the proof of Lemma 24.0.16 starting with the claim, we see that $G \in \widehat{\Sigma}_0$. However, if μ is a finite measure on $\widehat{\Sigma}$, then $(\widehat{\Sigma})_\mu = \Sigma_\mu$ and so $\widehat{\Sigma}_0 \subseteq (\widehat{\Sigma})_\mu \subseteq \widehat{\Sigma}$. This proves the theorem.

24.1 Multifunctions

Let X be a separable complete metric space and let $(T, \mathcal{C}, \mu)$ be a set, a σ algebra of subsets of T , and a measure μ such that this is a complete σ finite measure space. Also let $\Gamma : T \rightarrow \mathcal{P}_F(X)$, the closed subsets of X .

Definition 24.1.1 We define $\Gamma^-(S) \equiv \{t \in T : \Gamma(t) \cap S \neq \emptyset\}$

We will consider a theory of measurability of set valued functions. The following theorem is the main result in the subject. In this theorem the third condition is what we will refer to as measurable.

Theorem 24.1.2 *The following are equivalent.*

1. For all B a Borel set in X , $\Gamma^-(B) \in \mathcal{C}$.
2. For all F closed in X , $\Gamma^-(F) \in \mathcal{C}$
3. For all U open in X , $\Gamma^-(U) \in \mathcal{C}$
4. There exists a sequence, $\{\sigma_n\}$ of measurable functions satisfying $\sigma_n(t) \in \Gamma(t)$ such that for all $t \in T$,

$$\Gamma(t) = \overline{\{\sigma_n(t) : n \in \mathbb{N}\}}$$

These functions are called measurable selections.

5. For all $x \in X$, $t \mapsto \text{dist}(x, \Gamma(t))$ is a measurable real valued function.
6. $\mathcal{G}(\Gamma) \equiv \{(t, x) : x \in \Gamma(t)\} \subseteq \mathcal{C} \times B(X)$.

Proof: It is obvious that 1.) $\Rightarrow$ 2.). To see that 2.) $\Rightarrow$ 3.) note that $\Gamma^-(\cup_{i=1}^{\infty} F_i) = \cup_{i=1}^{\infty} \Gamma^-(F_i)$. Since any open set in X can be obtained as a countable union of closed sets, this implies 2.) $\Rightarrow$ 3.).

Now we verify that 3.) $\Rightarrow$ 4.). Let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X . For $t \in T$, let $\psi_1(t) = x_n$ where n is the smallest integer such that $\Gamma(t) \cap B(x_n, 1) \neq \emptyset$. Therefore, $\psi_1(t)$ has countably many values, $x_{n_1}, x_{n_2}, \dots$ where $n_1 < n_2 < \dots$. Now

$$\{t : \psi_1(t) = x_n\} =$$

$$\{t : \Gamma(t) \cap B(x_n, 1) \neq \emptyset\} \cap [T \setminus \cup_{k < n} \{t : \Gamma(t) \cap B(x_k, 1) \neq \emptyset\}] \in \mathcal{C}.$$

Thus we see that ψ_1 is measurable and $\text{dist}(\psi_1(t), \Gamma(t)) < 1$. Let

$$T_n \equiv \{t \in T : \psi_1(t) = x_n\}.$$

Then $T_n \in \mathcal{C}$ and $T_n \cap T_m = \emptyset$ for $n \neq m$ and $\cup_{n=1}^{\infty} T_n = T$. Let $D_n \equiv \{x_k : x_k \in B(x_n, 1)\}$. Now for each n , and $t \in T_n$, let $\psi_2(t) = x_k$ where k is the smallest index such that

$x_k \in D_n$ and $B(x_k, \frac{1}{2}) \cap \Gamma(t) \neq \emptyset$. Thus $\text{dist}(\psi_2(t), \Gamma(t)) < \frac{1}{2}$ and $d(\psi_2(t), \psi_1(t)) < 1$. Continue this way obtaining ψ_k a measurable function such that

$$\text{dist}(\psi_k(t), \Gamma(t)) < \frac{1}{2^{k-1}}, \quad d(\psi_k(t), \psi_{k+1}(t)) < \frac{1}{2^{k-2}}.$$

Then for each t , $\{\psi_k(t)\}$ is a Cauchy sequence converging to a point, $\sigma(t) \in \Gamma(t)$. This has shown that if Γ is measurable there exists a measurable selection, $\sigma(t) \in \Gamma(t)$. It remains to show there exists a sequence of these measurable selections, σ_n such that the conclusion of 4.) holds. To do this we define

$$\Gamma_{ni}(t) \equiv \begin{cases} \Gamma(t) \cap B(x_n, 2^{-i}) & \text{if } \Gamma(t) \cap B(x_n, 2^{-i}) \neq \emptyset \\ \Gamma(t) & \text{otherwise.} \end{cases}$$

First we show that Γ_{ni} is measurable. Let U be open. Then

$$\begin{aligned} \{t : \Gamma_{ni}(t) \cap U \neq \emptyset\} &= \{t : \Gamma(t) \cap B(x_n, 2^{-i}) \cap U \neq \emptyset\} \cup \\ &[\{t : \Gamma(t) \cap B(x_n, 2^{-i}) = \emptyset\} \cap \{t : \Gamma(t) \cap U \neq \emptyset\}] \\ &= \{t : \Gamma(t) \cap B(x_n, 2^{-i}) \cap U \neq \emptyset\} \cup \\ &[(T \setminus \{t : \Gamma(t) \cap B(x_n, 2^{-i}) \neq \emptyset\}) \cap \{t : \Gamma(t) \cap U \neq \emptyset\}], \end{aligned}$$

a measurable set. By what was just shown there exists σ_{ni} , a measurable function such that $\sigma_{ni}(t) \in \Gamma_{ni}(t) \subseteq \Gamma(t)$ for all $t \in T$. If $x \in \Gamma(t)$, then $x \in B(x_n, 2^{-i})$ whenever x_n is close enough to x . Therefore, $|\sigma_{ni}(t) - x| < 2^{-i}$. And it follows that condition 4.) holds.

Now we verify that 4.) $\Rightarrow$ 3.). Suppose there exist measurable selections $\sigma_n(t) \in \Gamma(t)$ satisfying condition 4.). Let U be open. Then

$$\{t : \Gamma(t) \cap U \neq \emptyset\} = \cup_{n=1}^{\infty} \sigma_n^{-1}(U) \in \mathcal{C}.$$

Now we verify that 4.) $\Rightarrow$ 5.). Let $F(t) \equiv \text{dist}(x, \Gamma(t))$. Then letting U be an open set in $[0, \infty)$, $F(t) \in U$ if and only if $d(x, \sigma_n(t)) \in U$ for some $\sigma_n(t)$. Let $h_n(t) \equiv d(x, \sigma_n(t))$. Then h_n is measurable and $F^{-1}(U) = \cup_{n=1}^{\infty} h_n^{-1}(U) \in \mathcal{C}$. This shows that for all $x \in X$, $t \rightarrow \text{dist}(x, \Gamma(t))$ is measurable and this proves 5.).

Now we verify that 5.) $\Rightarrow$ 4.). We know $\text{dist}(x, \Gamma(\cdot))$ is measurable and we show $\{t : \Gamma(t) \cap U \neq \emptyset\} \in \mathcal{C}$ whenever U is open and then use 3.) $\Rightarrow$ 4.). Since X is separable, there exists $B(x_i, r_i)$ such that $U = \cup_{i=1}^{\infty} B(x_i, r_i)$. Then

$$\begin{aligned} \{t : \Gamma(t) \cap U \neq \emptyset\} &= \cup_{i=1}^{\infty} \{t : \Gamma(t) \cap B(x_i, r_i) \neq \emptyset\} \\ &= \cup_{i=1}^{\infty} \{t : \text{dist}(x_i, \Gamma(t)) < r_i\} \in \mathcal{C}. \end{aligned}$$

Therefore, 5.) $\Rightarrow$ 4.) as claimed.

Now we must prove 5.) $\Rightarrow$ 6.). We note that $t \rightarrow \text{dist}(x, \Gamma(t))$ is measurable and $x \rightarrow \text{dist}(x, \Gamma(t))$ is continuous. Also, the graph of Γ , $\mathcal{G}(\Gamma)$ is given by

$$\mathcal{G}(\Gamma) = \{(t, x) : \text{dist}(x, \Gamma(t)) = 0\}.$$

We wish to show that $(t, x) \rightarrow \text{dist}(x, \Gamma(t))$ is product measurable because then $\mathcal{G}(\Gamma)$, being the inverse image of $\{0\}$ will be product measurable. Let $\{x_k\}$ be a countable dense set in X and let

$$\phi_k(t, x) \equiv \text{dist}(x_n, \Gamma(t))$$

where n is the first index such that $x \in B(x_n, 2^{-k})$. Then $\phi_k(t, x) \rightarrow \text{dist}(x, \Gamma(t))$ due to the continuity of $x \rightarrow \text{dist}(x, \Gamma(t))$ and so we must argue that ϕ_k is product measurable. On

$$E_n \equiv T \times (B(x_n, 2^{-k}) \setminus \cup_{m < n} B(x_m, 2^{-k})),$$

$\phi_k(t, x) = \text{dist}(x_n, \Gamma(t))$. Thus, on this set, ϕ_k equals a measurable function of t and does not depend on x on E_n . It follows that there are measurable simple $\mathcal{C}$ measurable functions, $s_m(t)$ which increase pointwise to $\text{dist}(x_n, \Gamma(t))$ on E_n . Thus $s_m(t) \chi_{E_n}(x)$ increases to $\phi_k(t, x)$ on E_n showing that $\phi_k \chi_{E_n}$ is product measurable with respect to $\mathcal{C} \times \sigma(\tau)$ since E_n is a measurable rectangle with respect to $\mathcal{C}$ and $\sigma(\tau)$. Therefore, ϕ_k is product measurable and so $(t, x) \rightarrow \text{dist}(x, \Gamma(t))$ is also product measurable.

It remains to prove 6.) $\Rightarrow$ 1.). This follows from Theorem 24.0.18.

$$\Gamma^-(B) \equiv \{t : \Gamma(t) \cap B \neq \emptyset\}$$

$$= \text{proj}_T(\mathcal{G}(\Gamma) \cap (T \cap B)).$$

But from Theorem 24.0.18, $\text{proj}_T(\mathcal{G}(\Gamma) \cap (T \cap B)) \in \widehat{\mathcal{C}} \subseteq \mathcal{C}_\mu = \mathcal{C}$. The last part results from $(T, \mathcal{C}, \mu)$ being a complete measure space. Note that without this assumption we could not draw the conclusion desired.

For much more on multifunctions, you should see the book by Hu and Papageorgiou. The above proof follows the presentation in this book.

Part III

Complex Analysis

The Complex Numbers

The reader is presumed familiar with the algebraic properties of complex numbers, including the operation of conjugation. Here a short review of the distance in $\mathbb{C}$ is presented.

The length of a complex number, referred to as the modulus of z and denoted by $|z|$ is given by

$$|z| \equiv (x^2 + y^2)^{1/2} = (z\bar{z})^{1/2},$$

Then $\mathbb{C}$ is a metric space with the distance between two complex numbers, z and w defined as

$$d(z, w) \equiv |z - w|.$$

This metric on $\mathbb{C}$ is the same as the usual metric of $\mathbb{R}^2$. A sequence, $z_n \rightarrow z$ if and only if $x_n \rightarrow x$ in $\mathbb{R}$ and $y_n \rightarrow y$ in $\mathbb{R}$ where $z = x + iy$ and $z_n = x_n + iy_n$. For example if $z_n = \frac{n}{n+1} + i\frac{1}{n}$, then $z_n \rightarrow 1 + 0i = 1$.

Definition 25.0.3 *A sequence of complex numbers, $\{z_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists N such that $n, m > N$ implies $|z_n - z_m| < \varepsilon$.*

This is the usual definition of Cauchy sequence. There are no new ideas here.

Proposition 25.0.4 *The complex numbers with the norm just mentioned forms a complete normed linear space.*

Proof: Let $\{z_n\}$ be a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of real numbers and so they converge to real numbers, x and y respectively. Thus $z_n = x_n + iy_n \rightarrow x + iy$. $\mathbb{C}$ is a linear space with the field of scalars equal to $\mathbb{C}$. It only remains to verify that $|\cdot|$ satisfies the axioms of a norm which are:

$$|z + w| \leq |z| + |w|$$

$$|z| \geq 0 \text{ for all } z$$

$$|z| = 0 \text{ if and only if } z = 0$$

$$|\alpha z| = |\alpha| |z|.$$

The only one of these axioms of a norm which is not completely obvious is the first one, the triangle inequality. Let $z = x + iy$ and $w = u + iv$

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = |z|^2 + |w|^2 + 2 \operatorname{Re}(z\bar{w}) \\ &\leq |z|^2 + |w|^2 + 2|(z\bar{w})| = (|z| + |w|)^2 \end{aligned}$$

and this verifies the triangle inequality.

Definition 25.0.5 *An infinite sum of complex numbers is defined as the limit of the sequence of partial sums. Thus,*

$$\sum_{k=1}^{\infty} a_k \equiv \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

Just as in the case of sums of real numbers, an infinite sum converges if and only if the sequence of partial sums is a Cauchy sequence.

From now on, when f is a function of a complex variable, it will be assumed that f has values in X , a complex Banach space. Usually in complex analysis courses, f has values in $\mathbb{C}$ but there are many important theorems which don't require this so I will leave it fairly general for a while. Later the functions will have values in $\mathbb{C}$. If you are only interested in this case, think $\mathbb{C}$ whenever you see X .

Definition 25.0.6 *A sequence of functions of a complex variable, $\{f_n\}$ converges uniformly to a function, g for $z \in S$ if for every $\varepsilon > 0$ there exists N_ε such that if $n > N_\varepsilon$, then*

$$\|f_n(z) - g(z)\| < \varepsilon$$

for all $z \in S$. The infinite sum $\sum_{k=1}^{\infty} f_k$ converges uniformly on S if the partial sums converge uniformly on S . Here $\|\cdot\|$ refers to the norm in X , the Banach space in which f has its values.

The following proposition is also a routine application of the above definition. Neither the definition nor this proposition say anything new.

Proposition 25.0.7 *A sequence of functions, $\{f_n\}$ defined on a set S , converges uniformly to some function, g if and only if for all $\varepsilon > 0$ there exists N_ε such that whenever $m, n > N_\varepsilon$,*

$$\|f_n - f_m\|_\infty < \varepsilon.$$

Here $\|f\|_\infty \equiv \sup \{\|f(z)\| : z \in S\}$.

Just as in the case of functions of a real variable, one of the important theorems is the Weierstrass M test. Again, there is nothing new here. It is just a review of earlier material.

Theorem 25.0.8 *Let $\{f_n\}$ be a sequence of complex valued functions defined on $S \subseteq \mathbb{C}$. Suppose there exists M_n such that $\|f_n\|_\infty < M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly on S .*

Proof: Let $z \in S$. Then letting $m < n$

$$\left\| \sum_{k=1}^n f_k(z) - \sum_{k=1}^m f_k(z) \right\| \leq \sum_{k=m+1}^n \|f_k(z)\| \leq \sum_{k=m+1}^{\infty} M_k < \varepsilon$$

whenever m is large enough. Therefore, the sequence of partial sums is uniformly Cauchy on S and therefore, converges uniformly to $\sum_{k=1}^{\infty} f_k(z)$ on S .

25.1 The Extended Complex Plane

The set of complex numbers has already been considered along with the topology of $\mathbb{C}$ which is nothing but the topology of $\mathbb{R}^2$. Thus, for $z_n = x_n + iy_n$, $z_n \rightarrow z \equiv x + iy$ if and only if $x_n \rightarrow x$ and $y_n \rightarrow y$. The norm in $\mathbb{C}$ is given by

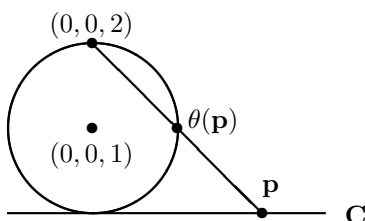
$$|x + iy| \equiv ((x + iy)(x - iy))^{1/2} = (x^2 + y^2)^{1/2}$$

which is just the usual norm in $\mathbb{R}^2$ identifying (x, y) with $x + iy$. Therefore, $\mathbb{C}$ is a complete metric space topologically like $\mathbb{R}^2$ and so the Heine Borel theorem that compact sets are those which are closed and bounded is valid. Thus, as far as topology is concerned, there is nothing new about $\mathbb{C}$.

The extended complex plane, denoted by $\hat{\mathbb{C}}$, consists of the complex plane, $\mathbb{C}$ along with another point not in $\mathbb{C}$ known as ∞ . For example, ∞ could be any point in $\mathbb{R}^3$. A sequence of complex numbers, z_n , converges to ∞ if, whenever K is a compact set in $\mathbb{C}$, there exists a number, N such that for all $n > N$, $z_n \notin K$. Since compact sets in $\mathbb{C}$ are closed and bounded, this is equivalent to saying that for all $R > 0$, there exists N such that if $n > N$, then $z_n \notin B(0, R)$ which is the same as saying $\lim_{n \rightarrow \infty} |z_n| = \infty$ where this last symbol has the same meaning as it does in calculus.

A geometric way of understanding this in terms of more familiar objects involves a concept known as the Riemann sphere.

Consider the unit sphere, S^2 given by $(z - 1)^2 + y^2 + x^2 = 1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, p in the complex plane to the point $(0, 0, 2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty) \equiv (0, 0, 2)$.



Then θ^{-1} is sometimes called stereographic projection. The mapping θ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, z_n converges to ∞ if and only if θz_n converges to $(0, 0, 2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta(z_n) \rightarrow \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 25.1.1 Let $z, w \in \widehat{\mathbb{C}}$ including possibly $w = \infty$. Then let $d(x, w) \equiv |\theta(z) - \theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^3$.

Theorem 25.1.2 $(\widehat{\mathbb{C}}, d)$ is a compact, hence complete metric space.

Proof: Suppose $\{z_n\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\{\theta(z_n)\}$ is a sequence in S^2 which is compact. Therefore, there exists a subsequence, $\{\theta z_{n_k}\}$ and a point, $z \in S^2$ such that $\theta z_{n_k} \rightarrow \theta z$ in S^2 which implies immediately that $d(z_{n_k}, z) \rightarrow 0$. A compact metric space must be complete.

25.2 Exercises

1. Prove the root test for series of complex numbers. If $a_k \in \mathbb{C}$ and $r \equiv \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ then

$$\sum_{k=0}^{\infty} a_k \begin{cases} \text{converges absolutely if } r < 1 \\ \text{diverges if } r > 1 \\ \text{test fails if } r = 1. \end{cases}$$

2. Does $\lim_{n \rightarrow \infty} n \left(\frac{2+i}{3}\right)^n$ exist? Tell why and find the limit if it does exist.
3. Let $A_0 = 0$ and let $A_n \equiv \sum_{k=1}^n a_k$ if $n > 0$. Prove the partial summation formula,

$$\sum_{k=p}^q a_k b_k = A_q b_q - A_{p-1} b_p + \sum_{k=p}^{q-1} A_k (b_k - b_{k+1}).$$

Now using this formula, suppose $\{b_n\}$ is a sequence of real numbers which converges to 0 and is decreasing. Determine those values of ω such that $|\omega| = 1$ and $\sum_{k=1}^{\infty} b_k \omega^k$ converges.

4. Let $f : U \subseteq \mathbb{C} \rightarrow \mathbb{C}$ be given by $f(x + iy) = u(x, y) + iv(x, y)$. Show f is continuous on U if and only if $u : U \rightarrow \mathbb{R}$ and $v : U \rightarrow \mathbb{R}$ are both continuous.

Riemann Stieltjes Integrals

In the theory of functions of a complex variable, the most important results are those involving contour integration. I will base this on the notion of Riemann Stieltjes integrals as in [16], [50], and [32]. The Riemann Stieltjes integral is a generalization of the usual Riemann integral and requires the concept of a function of bounded variation.

Definition 26.0.1 *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be a function. Then γ is of bounded variation if*

$$\sup \left\{ \sum_{i=1}^n |\gamma(t_i) - \gamma(t_{i-1})| : a = t_0 < \cdots < t_n = b \right\} \equiv V(\gamma, [a, b]) < \infty$$

where the sums are taken over all possible lists, $\{a = t_0 < \cdots < t_n = b\}$. The set of points $\gamma([a, b])$ will also be denoted by γ^* .

The idea is that it makes sense to talk of the length of the curve $\gamma([a, b])$, defined as $V(\gamma, [a, b])$. For this reason, in the case that γ is continuous, such an image of a bounded variation function is called a rectifiable curve.

Definition 26.0.2 *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation and let $f : \gamma^* \rightarrow X$. Letting $\mathcal{P} \equiv \{t_0, \dots, t_n\}$ where $a = t_0 < t_1 < \cdots < t_n = b$, define*

$$\|\mathcal{P}\| \equiv \max \{|t_j - t_{j-1}| : j = 1, \dots, n\}$$

and the Riemann Stieltjes sum by

$$S(\mathcal{P}) \equiv \sum_{j=1}^n f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1}))$$

where $\tau_j \in [t_{j-1}, t_j]$. (Note this notation is a little sloppy because it does not identify the specific point, τ_j used. It is understood that this point is arbitrary.) Define $\int_{\gamma} f d\gamma$ as the unique number which satisfies the following condition. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|\mathcal{P}\| \leq \delta$, then

$$\left| \int_{\gamma} f d\gamma - S(\mathcal{P}) \right| < \varepsilon.$$

Sometimes this is written as

$$\int_{\gamma} f d\gamma \equiv \lim_{\|\mathcal{P}\| \rightarrow 0} S(\mathcal{P}).$$

The set of points in the curve, $\gamma([a, b])$ will be denoted sometimes by γ^* .

Then γ^* is a set of points in $\mathbb{C}$ and as t moves from a to b , $\gamma(t)$ moves from $\gamma(a)$ to $\gamma(b)$. Thus γ^* has a first point and a last point. If $\phi : [c, d] \rightarrow [a, b]$ is a continuous nondecreasing function, then $\gamma \circ \phi : [c, d] \rightarrow \mathbb{C}$ is also of bounded variation and yields the same set of points in $\mathbb{C}$ with the same first and last points.

Theorem 26.0.3 *Let ϕ and γ be as just described. Then assuming that*

$$\int_{\gamma} f d\gamma$$

exists, so does

$$\int_{\gamma \circ \phi} f d(\gamma \circ \phi)$$

and

$$\int_{\gamma} f d\gamma = \int_{\gamma \circ \phi} f d(\gamma \circ \phi). \quad (26.0.1)$$

Proof: There exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ such that $\|\mathcal{P}\| < \delta$, then

$$\left| \int_{\gamma} f d\gamma - S(\mathcal{P}) \right| < \varepsilon.$$

By continuity of ϕ , there exists $\sigma > 0$ such that if $\mathcal{Q}$ is a partition of $[c, d]$ with $\|\mathcal{Q}\| < \sigma$, $\mathcal{Q} = \{s_0, \dots, s_n\}$, then $|\phi(s_j) - \phi(s_{j-1})| < \delta$. Thus letting $\mathcal{P}$ denote the points in $[a, b]$ given by $\phi(s_j)$ for $s_j \in \mathcal{Q}$, it follows that $\|\mathcal{P}\| < \delta$ and so

$$\left| \int_{\gamma} f d\gamma - \sum_{j=1}^n f(\gamma(\phi(\tau_j))) (\gamma(\phi(s_j)) - \gamma(\phi(s_{j-1}))) \right| < \varepsilon$$

where $\tau_j \in [s_{j-1}, s_j]$. Therefore, from the definition 26.0.1 holds and

$$\int_{\gamma \circ \phi} f d(\gamma \circ \phi)$$

exists.

This theorem shows that $\int_{\gamma} f d\gamma$ is independent of the particular γ used in its computation to the extent that if ϕ is any nondecreasing continuous function from another interval, $[c, d]$, mapping to $[a, b]$, then the same value is obtained by replacing γ with $\gamma \circ \phi$.

The fundamental result in this subject is the following theorem.

Theorem 26.0.4 Let $f : \gamma^* \rightarrow X$ be continuous and let $\gamma : [a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Then $\int_{\gamma} f d\gamma$ exists. Also letting $\delta_m > 0$ be such that $|t - s| < \delta_m$ implies $\|f(\gamma(t)) - f(\gamma(s))\| < \frac{1}{m}$,

$$\left| \int_{\gamma} f d\gamma - S(\mathcal{P}) \right| \leq \frac{2V(\gamma, [a, b])}{m}$$

whenever $\|\mathcal{P}\| < \delta_m$.

Proof: The function, $f \circ \gamma$, is uniformly continuous because it is defined on a compact set. Therefore, there exists a decreasing sequence of positive numbers, $\{\delta_m\}$ such that if $|s - t| < \delta_m$, then

$$|f(\gamma(t)) - f(\gamma(s))| < \frac{1}{m}.$$

Let

$$F_m \equiv \overline{\{S(\mathcal{P}) : \|\mathcal{P}\| < \delta_m\}}.$$

Thus F_m is a closed set. (The symbol, $S(\mathcal{P})$ in the above definition, means to include all sums corresponding to $\mathcal{P}$ for any choice of τ_j .) It is shown that

$$\text{diam}(F_m) \leq \frac{2V(\gamma, [a, b])}{m} \quad (26.0.2)$$

and then it will follow there exists a unique point, $I \in \cap_{m=1}^{\infty} F_m$. This is because X is complete. It will then follow $I = \int_{\gamma} f(t) d\gamma(t)$. To verify 26.0.2, it suffices to verify that whenever $\mathcal{P}$ and $\mathcal{Q}$ are partitions satisfying $\|\mathcal{P}\| < \delta_m$ and $\|\mathcal{Q}\| < \delta_m$,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq \frac{2}{m} V(\gamma, [a, b]). \quad (26.0.3)$$

Suppose $\|\mathcal{P}\| < \delta_m$ and $\mathcal{Q} \supseteq \mathcal{P}$. Then also $\|\mathcal{Q}\| < \delta_m$. To begin with, suppose that $\mathcal{P} \equiv \{t_0, \dots, t_p, \dots, t_n\}$ and $\mathcal{Q} \equiv \{t_0, \dots, t_{p-1}, t^*, t_p, \dots, t_n\}$. Thus $\mathcal{Q}$ contains only one more point than $\mathcal{P}$. Letting $S(\mathcal{Q})$ and $S(\mathcal{P})$ be Riemann Steiltjes sums,

$$\begin{aligned} S(\mathcal{Q}) &\equiv \sum_{j=1}^{p-1} f(\gamma(\sigma_j))(\gamma(t_j) - \gamma(t_{j-1})) + f(\gamma(\sigma_*))(\gamma(t^*) - \gamma(t_{p-1})) \\ &\quad + f(\gamma(\sigma^*))(\gamma(t_p) - \gamma(t^*)) + \sum_{j=p+1}^n f(\gamma(\sigma_j))(\gamma(t_j) - \gamma(t_{j-1})), \\ S(\mathcal{P}) &\equiv \sum_{j=1}^{p-1} f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1})) + \\ &\quad \underbrace{= f(\gamma(\tau_p))(\gamma(t_p) - \gamma(t_{p-1}))}_{f(\gamma(\tau_p))(\gamma(t^*) - \gamma(t_{p-1})) + f(\gamma(\tau_p))(\gamma(t_p) - \gamma(t^*))} \end{aligned}$$

$$+ \sum_{j=p+1}^n f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})).$$

Therefore,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq \sum_{j=1}^{p-1} \frac{1}{m} |\gamma(t_j) - \gamma(t_{j-1})| + \frac{1}{m} |\gamma(t^*) - \gamma(t_{p-1})| + \frac{1}{m} |\gamma(t_p) - \gamma(t^*)| + \sum_{j=p+1}^n \frac{1}{m} |\gamma(t_j) - \gamma(t_{j-1})| \leq \frac{1}{m} V(\gamma, [a, b]). \quad (26.0.4)$$

Clearly the extreme inequalities would be valid in 26.0.4 if $\mathcal{Q}$ had more than one extra point. You simply do the above trick more than one time. Let $S(\mathcal{P})$ and $S(\mathcal{Q})$ be Riemann Steiltjes sums for which $\|\mathcal{P}\|$ and $\|\mathcal{Q}\|$ are less than δ_m and let $\mathcal{R} \equiv \mathcal{P} \cup \mathcal{Q}$. Then from what was just observed,

$$|S(\mathcal{P}) - S(\mathcal{Q})| \leq |S(\mathcal{P}) - S(\mathcal{R})| + |S(\mathcal{R}) - S(\mathcal{Q})| \leq \frac{2}{m} V(\gamma, [a, b]).$$

and this shows 26.0.3 which proves 26.0.2. Therefore, there exists a unique complex number, $I \in \cap_{m=1}^{\infty} F_m$ which satisfies the definition of $\int_{\gamma} f d\gamma$. This proves the theorem.

The following theorem follows easily from the above definitions and theorem.

Theorem 26.0.5 *Let $f \in C(\gamma^*)$ and let $\gamma : [a, b] \rightarrow \mathbb{C}$ be of bounded variation and continuous. Let*

$$M \geq \max \{ \|f \circ \gamma(t)\| : t \in [a, b] \}. \quad (26.0.5)$$

Then

$$\left\| \int_{\gamma} f d\gamma \right\| \leq MV(\gamma, [a, b]). \quad (26.0.6)$$

Also if $\{f_n\}$ is a sequence of functions of $C(\gamma^*)$ which is converging uniformly to the function, f on γ^* , then

$$\lim_{n \rightarrow \infty} \int_{\gamma} f_n d\gamma = \int_{\gamma} f d\gamma. \quad (26.0.7)$$

Proof: Let 26.0.5 hold. From the proof of the above theorem, when $\|\mathcal{P}\| < \delta_m$,

$$\left\| \int_{\gamma} f d\gamma - S(\mathcal{P}) \right\| \leq \frac{2}{m} V(\gamma, [a, b])$$

and so

$$\left\| \int_{\gamma} f d\gamma \right\| \leq \|S(\mathcal{P})\| + \frac{2}{m} V(\gamma, [a, b])$$

$$\begin{aligned}
&\leq \sum_{j=1}^n M |\gamma(t_j) - \gamma(t_{j-1})| + \frac{2}{m} V(\gamma, [a, b]) \\
&\leq MV(\gamma, [a, b]) + \frac{2}{m} V(\gamma, [a, b]).
\end{aligned}$$

This proves 26.0.6 since m is arbitrary. To verify 26.0.7 use the above inequality to write

$$\begin{aligned}
&\left\| \int_{\gamma} f d\gamma - \int_{\gamma} f_n d\gamma \right\| = \left\| \int_{\gamma} (f - f_n) d\gamma(t) \right\| \\
&\leq \max \{ \|f \circ \gamma(t) - f_n \circ \gamma(t)\| : t \in [a, b] \} V(\gamma, [a, b]).
\end{aligned}$$

Since the convergence is assumed to be uniform, this proves 26.0.7.

It turns out to be much easier to evaluate such integrals in the case where γ is also $C^1([a, b])$. The following theorem about approximation will be very useful but first here is an easy lemma.

Lemma 26.0.6 *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be in $C^1([a, b])$. Then $V(\gamma, [a, b]) < \infty$ so γ is of bounded variation.*

Proof: This follows from the following

$$\begin{aligned}
\sum_{j=1}^n |\gamma(t_j) - \gamma(t_{j-1})| &= \sum_{j=1}^n \left| \int_{t_{j-1}}^{t_j} \gamma'(s) ds \right| \\
&\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} |\gamma'(s)| ds \\
&\leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|\gamma'\|_{\infty} ds \\
&= \|\gamma'\|_{\infty} (b - a).
\end{aligned}$$

Therefore it follows $V(\gamma, [a, b]) \leq \|\gamma'\|_{\infty} (b - a)$. Here $\|\gamma\|_{\infty} = \max \{ |\gamma(t)| : t \in [a, b] \}$.

Theorem 26.0.7 *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Let Ω be an open set containing γ^* and let $f : \Omega \times K \rightarrow X$ be continuous for K a compact set in $\mathbb{C}$, and let $\varepsilon > 0$ be given. Then there exists $\eta : [a, b] \rightarrow \mathbb{C}$ such that $\eta(a) = \gamma(a)$, $\gamma(b) = \eta(b)$, $\eta \in C^1([a, b])$, and*

$$\|\gamma - \eta\| < \varepsilon, \quad (26.0.8)$$

$$\left| \int_{\gamma} f(\cdot, z) d\gamma - \int_{\eta} f(\cdot, z) d\eta \right| < \varepsilon, \quad (26.0.9)$$

$$V(\eta, [a, b]) \leq V(\gamma, [a, b]), \quad (26.0.10)$$

where $\|\gamma - \eta\| \equiv \max \{ |\gamma(t) - \eta(t)| : t \in [a, b] \}$.

Proof: Extend γ to be defined on all $\mathbb{R}$ according to $\gamma(t) = \gamma(a)$ if $t < a$ and $\gamma(t) = \gamma(b)$ if $t > b$. Now define

$$\gamma_h(t) \equiv \frac{1}{2h} \int_{-2h+t+\frac{2h}{(b-a)}(t-a)}^{t+\frac{2h}{(b-a)}(t-a)} \gamma(s) ds.$$

where the integral is defined in the obvious way. That is,

$$\int_a^b \alpha(t) + i\beta(t) dt \equiv \int_a^b \alpha(t) dt + i \int_a^b \beta(t) dt.$$

Therefore,

$$\gamma_h(b) = \frac{1}{2h} \int_b^{b+2h} \gamma(s) ds = \gamma(b),$$

$$\gamma_h(a) = \frac{1}{2h} \int_{a-2h}^a \gamma(s) ds = \gamma(a).$$

Also, because of continuity of γ and the fundamental theorem of calculus,

$$\begin{aligned} \gamma'_h(t) &= \frac{1}{2h} \left\{ \gamma \left(t + \frac{2h}{b-a}(t-a) \right) \left(1 + \frac{2h}{b-a} \right) - \right. \\ &\quad \left. \gamma \left(-2h + t + \frac{2h}{b-a}(t-a) \right) \left(1 + \frac{2h}{b-a} \right) \right\} \end{aligned}$$

and so $\gamma_h \in C^1([a, b])$. The following lemma is significant.

Lemma 26.0.8 $V(\gamma_h, [a, b]) \leq V(\gamma, [a, b])$.

Proof: Let $a = t_0 < t_1 < \dots < t_n = b$. Then using the definition of γ_h and changing the variables to make all integrals over $[0, 2h]$,

$$\begin{aligned} &\sum_{j=1}^n |\gamma_h(t_j) - \gamma_h(t_{j-1})| = \\ &\sum_{j=1}^n \left| \frac{1}{2h} \int_0^{2h} \left[\gamma \left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a) \right) - \right. \right. \\ &\quad \left. \left. \gamma \left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a) \right) \right] ds \right| \\ &\leq \frac{1}{2h} \int_0^{2h} \sum_{j=1}^n \left| \gamma \left(s - 2h + t_j + \frac{2h}{b-a}(t_j - a) \right) - \right. \\ &\quad \left. \gamma \left(s - 2h + t_{j-1} + \frac{2h}{b-a}(t_{j-1} - a) \right) \right| ds. \end{aligned}$$

For a given $s \in [0, 2h]$, the points, $s - 2h + t_j + \frac{2h}{b-a}(t_j - a)$ for $j = 1, \dots, n$ form an increasing list of points in the interval $[a - 2h, b + 2h]$ and so the integrand is bounded above by $V(\gamma, [a - 2h, b + 2h]) = V(\gamma, [a, b])$. It follows

$$\sum_{j=1}^n |\gamma_h(t_j) - \gamma_h(t_{j-1})| \leq V(\gamma, [a, b])$$

which proves the lemma.

With this lemma the proof of the theorem can be completed without too much trouble. Let H be an open set containing γ^* such that $\overline{H}$ is a compact subset of Ω . Let $0 < \varepsilon < \text{dist}(\gamma^*, H^C)$. Then there exists δ_1 such that if $h < \delta_1$, then for all t ,

$$\begin{aligned} |\gamma(t) - \gamma_h(t)| &\leq \frac{1}{2h} \int_{-2h+t+\frac{2h}{b-a}(t-a)}^{t+\frac{2h}{b-a}(t-a)} |\gamma(s) - \gamma(t)| ds \\ &< \frac{1}{2h} \int_{-2h+t+\frac{2h}{b-a}(t-a)}^{t+\frac{2h}{b-a}(t-a)} \varepsilon ds = \varepsilon \end{aligned} \quad (26.0.11)$$

due to the uniform continuity of γ . This proves 26.0.8.

From 26.0.2 and the above lemma, there exists δ_2 such that if $\|\mathcal{P}\| < \delta_2$, then for all $z \in K$,

$$\left\| \int_{\gamma} f(\cdot, z) d\gamma(t) - S(\mathcal{P}) \right\| < \frac{\varepsilon}{3}, \quad \left\| \int_{\gamma_h} f(\cdot, z) d\gamma_h(t) - S_h(\mathcal{P}) \right\| < \frac{\varepsilon}{3}$$

for all h . Here $S(\mathcal{P})$ is a Riemann Steiltjes sum of the form

$$\sum_{i=1}^n f(\gamma(\tau_i), z) (\gamma(t_i) - \gamma(t_{i-1}))$$

and $S_h(\mathcal{P})$ is a similar Riemann Steiltjes sum taken with respect to γ_h instead of γ . Because of 26.0.11 $\gamma_h(t)$ has values in $H \subseteq \Omega$. Therefore, fix the partition, $\mathcal{P}$, and choose h small enough that in addition to this, the following inequality is valid for all $z \in K$.

$$|S(\mathcal{P}) - S_h(\mathcal{P})| < \frac{\varepsilon}{3}$$

This is possible because of 26.0.11 and the uniform continuity of f on $\overline{H} \times K$. It follows

$$\begin{aligned} &\left\| \int_{\gamma} f(\cdot, z) d\gamma(t) - \int_{\gamma_h} f(\cdot, z) d\gamma_h(t) \right\| \leq \\ &\left\| \int_{\gamma} f(\cdot, z) d\gamma(t) - S(\mathcal{P}) \right\| + \|S(\mathcal{P}) - S_h(\mathcal{P})\| \\ &\quad + \left\| S_h(\mathcal{P}) - \int_{\gamma_h} f(\cdot, z) d\gamma_h(t) \right\| < \varepsilon. \end{aligned}$$

Formula 26.0.10 follows from the lemma. This proves the theorem.

Of course the same result is obtained without the explicit dependence of f on z .

This is a very useful theorem because if γ is $C^1([a, b])$, it is easy to calculate $\int_{\gamma} f d\gamma$ and the above theorem allows a reduction to the case where γ is C^1 . The next theorem shows how easy it is to compute these integrals in the case where γ is C^1 . First note that if f is continuous and $\gamma \in C^1([a, b])$, then by Lemma 26.0.6 and the fundamental existence theorem, Theorem 26.0.4, $\int_{\gamma} f d\gamma$ exists.

Theorem 26.0.9 *If $f : \gamma^* \rightarrow X$ is continuous and $\gamma : [a, b] \rightarrow \mathbb{C}$ is in $C^1([a, b])$, then*

$$\int_{\gamma} f d\gamma = \int_a^b f(\gamma(t)) \gamma'(t) dt. \quad (26.0.12)$$

Proof: Let $\mathcal{P}$ be a partition of $[a, b]$, $\mathcal{P} = \{t_0, \dots, t_n\}$ and $\|\mathcal{P}\|$ is small enough that whenever $|t - s| < \|\mathcal{P}\|$,

$$|f(\gamma(t)) - f(\gamma(s))| < \varepsilon \quad (26.0.13)$$

and

$$\left\| \int_{\gamma} f d\gamma - \sum_{j=1}^n f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) \right\| < \varepsilon.$$

Now

$$\sum_{j=1}^n f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) = \int_a^b \sum_{j=1}^n f(\gamma(\tau_j)) \mathcal{X}_{[t_{j-1}, t_j]}(s) \gamma'(s) ds$$

where here

$$\mathcal{X}_{[a, b]}(s) \equiv \begin{cases} 1 & \text{if } s \in [p, q] \\ 0 & \text{if } s \notin [p, q] \end{cases}.$$

Also,

$$\int_a^b f(\gamma(s)) \gamma'(s) ds = \int_a^b \sum_{j=1}^n f(\gamma(s)) \mathcal{X}_{[t_{j-1}, t_j]}(s) \gamma'(s) ds$$

and thanks to 26.0.13,

$$\begin{aligned} & \left\| \overbrace{\int_a^b \sum_{j=1}^n f(\gamma(\tau_j)) \mathcal{X}_{[t_{j-1}, t_j]}(s) \gamma'(s) ds}^{=\sum_{j=1}^n f(\gamma(\tau_j))(\gamma(t_j) - \gamma(t_{j-1}))} - \overbrace{\int_a^b \sum_{j=1}^n f(\gamma(s)) \mathcal{X}_{[t_{j-1}, t_j]}(s) \gamma'(s) ds}^{=\int_a^b f(\gamma(s)) \gamma'(s) ds} \right\| \\ & \leq \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \|f(\gamma(\tau_j)) - f(\gamma(s))\| |\gamma'(s)| ds \leq \|\gamma'\|_{\infty} \sum_j \varepsilon (t_j - t_{j-1}) \\ & = \varepsilon \|\gamma'\|_{\infty} (b - a). \end{aligned}$$

It follows that

$$\begin{aligned} \left\| \int_{\gamma} f d\gamma - \int_a^b f(\gamma(s)) \gamma'(s) ds \right\| &\leq \left\| \int_{\gamma} f d\gamma - \sum_{j=1}^n f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) \right\| \\ &+ \left\| \sum_{j=1}^n f(\gamma(\tau_j)) (\gamma(t_j) - \gamma(t_{j-1})) - \int_a^b f(\gamma(s)) \gamma'(s) ds \right\| \leq \varepsilon \|\gamma'\|_{\infty} (b-a) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, this verifies 26.0.12.

Definition 26.0.10 Let Ω be an open subset of $\mathbb{C}$ and let $\gamma : [a, b] \rightarrow \Omega$ be a continuous function with bounded variation $f : \Omega \rightarrow X$ be a continuous function. Then the following notation is more customary.

$$\int_{\gamma} f(z) dz \equiv \int_{\gamma} f d\gamma.$$

The expression, $\int_{\gamma} f(z) dz$, is called a contour integral and γ is referred to as the contour. A function $f : \Omega \rightarrow X$ for Ω an open set in $\mathbb{C}$ has a primitive if there exists a function, F , the primitive, such that $F'(z) = f(z)$. Thus F is just an antiderivative. Also if $\gamma_k : [a_k, b_k] \rightarrow \mathbb{C}$ is continuous and of bounded variation, for $k = 1, \dots, m$ and $\gamma_k(b_k) = \gamma_{k+1}(a_k)$, define

$$\int_{\sum_{k=1}^m \gamma_k} f(z) dz \equiv \sum_{k=1}^m \int_{\gamma_k} f(z) dz. \quad (26.0.14)$$

In addition to this, for $\gamma : [a, b] \rightarrow \mathbb{C}$, define $-\gamma : [a, b] \rightarrow \mathbb{C}$ by $-\gamma(t) \equiv \gamma(b + a - t)$. Thus γ simply traces out the points of γ^* in the opposite order.

The following lemma is useful and follows quickly from Theorem 26.0.3.

Lemma 26.0.11 In the above definition, there exists a continuous bounded variation function, γ defined on some closed interval, $[c, d]$, such that $\gamma([c, d]) = \cup_{k=1}^m \gamma_k([a_k, b_k])$ and $\gamma(c) = \gamma_1(a_1)$ while $\gamma(d) = \gamma_m(b_m)$. Furthermore,

$$\int_{\gamma} f(z) dz = \sum_{k=1}^m \int_{\gamma_k} f(z) dz.$$

If $\gamma : [a, b] \rightarrow \mathbb{C}$ is of bounded variation and continuous, then

$$\int_{\gamma} f(z) dz = - \int_{-\gamma} f(z) dz.$$

Re stating Theorem 26.0.7 with the new notation in the above definition,

Theorem 26.0.12 *Let K be a compact set in $\mathbb{C}$ and let $f : \Omega \times K \rightarrow X$ be continuous for Ω an open set in $\mathbb{C}$. Also let $\gamma : [a, b] \rightarrow \Omega$ be continuous with bounded variation. Then if $r > 0$ is given, there exists $\eta : [a, b] \rightarrow \Omega$ such that $\eta(a) = \gamma(a)$, $\eta(b) = \gamma(b)$, η is $C^1([a, b])$, and*

$$\left| \int_{\gamma} f(z, w) dz - \int_{\eta} f(z, w) dz \right| < r, \quad \|\eta - \gamma\| < r.$$

It will be very important to consider which functions have primitives. It turns out, it is not enough for f to be continuous in order to possess a primitive. This is in stark contrast to the situation for functions of a real variable in which the fundamental theorem of calculus will deliver a primitive for any continuous function. The reason for the interest in such functions is the following theorem and its corollary.

Theorem 26.0.13 *Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be continuous and of bounded variation. Also suppose $F'(z) = f(z)$ for all $z \in \Omega$, an open set containing γ^* and f is continuous on Ω . Then*

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$

Proof: By Theorem 26.0.12 there exists $\eta \in C^1([a, b])$ such that $\gamma(a) = \eta(a)$, and $\gamma(b) = \eta(b)$ such that

$$\left\| \int_{\gamma} f(z) dz - \int_{\eta} f(z) dz \right\| < \varepsilon.$$

Then since η is in $C^1([a, b])$,

$$\begin{aligned} \int_{\eta} f(z) dz &= \int_a^b f(\eta(t)) \eta'(t) dt = \int_a^b \frac{dF(\eta(t))}{dt} dt \\ &= F(\eta(b)) - F(\eta(a)) = F(\gamma(b)) - F(\gamma(a)). \end{aligned}$$

Therefore,

$$\left\| (F(\gamma(b)) - F(\gamma(a))) - \int_{\gamma} f(z) dz \right\| < \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

Corollary 26.0.14 *If $\gamma : [a, b] \rightarrow \mathbb{C}$ is continuous, has bounded variation, is a closed curve, $\gamma(a) = \gamma(b)$, and $\gamma^* \subseteq \Omega$ where Ω is an open set on which $F'(z) = f(z)$, then*

$$\int_{\gamma} f(z) dz = 0.$$

26.1 Exercises

1. Let $\gamma : [a, b] \rightarrow \mathbb{R}$ be increasing. Show $V(\gamma, [a, b]) = \gamma(b) - \gamma(a)$.
2. Suppose $\gamma : [a, b] \rightarrow \mathbb{C}$ satisfies a Lipschitz condition, $|\gamma(t) - \gamma(s)| \leq K|s - t|$. Show γ is of bounded variation and that $V(\gamma, [a, b]) \leq K|b - a|$.
3. $\gamma : [c_0, c_m] \rightarrow \mathbb{C}$ is piecewise smooth if there exist numbers, $c_k, k = 1, \dots, m$ such that $c_0 < c_1 < \dots < c_{m-1} < c_m$ such that γ is continuous and $\gamma : [c_k, c_{k+1}] \rightarrow \mathbb{C}$ is C^1 . Show that such piecewise smooth functions are of bounded variation and give an estimate for $V(\gamma, [c_0, c_m])$.
4. Let $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ be given by $\gamma(t) = r(\cos mt + i \sin mt)$ for m an integer. Find $\int_{\gamma} \frac{dz}{z}$.
5. Show that if $\gamma : [a, b] \rightarrow \mathbb{C}$ then there exists an increasing function $h : [0, 1] \rightarrow [a, b]$ such that $\gamma \circ h([0, 1]) = \gamma^*$.
6. Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be an arbitrary continuous curve having bounded variation and let f, g have continuous derivatives on some open set containing γ^* . Prove the usual integration by parts formula.

$$\int_{\gamma} f g' dz = f(\gamma(b)) g(\gamma(b)) - f(\gamma(a)) g(\gamma(a)) - \int_{\gamma} f' g dz.$$

7. Let $f(z) \equiv |z|^{-(1/2)} e^{-i\frac{\theta}{2}}$ where $z = |z| e^{i\theta}$. This function is called the principle branch of $z^{-(1/2)}$. Find $\int_{\gamma} f(z) dz$ where γ is the semicircle in the upper half plane which goes from $(1, 0)$ to $(-1, 0)$ in the counter clockwise direction. Next do the integral in which γ goes in the clockwise direction along the semicircle in the lower half plane.
8. Prove an open set, U is connected if and only if for every two points in U , there exists a C^1 curve having values in U which joins them.
9. Let $\mathcal{P}, \mathcal{Q}$ be two partitions of $[a, b]$ with $\mathcal{P} \subseteq \mathcal{Q}$. Each of these partitions can be used to form an approximation to $V(\gamma, [a, b])$ as described above. Recall the total variation was the supremum of sums of a certain form determined by a partition. How is the sum associated with $\mathcal{P}$ related to the sum associated with $\mathcal{Q}$? Explain.
10. Consider the curve,

$$\gamma(t) = \begin{cases} t + it^2 \sin\left(\frac{1}{t}\right) & \text{if } t \in (0, 1] \\ 0 & \text{if } t = 0 \end{cases}.$$

Is γ a continuous curve having bounded variation? What if the t^2 is replaced with t ? Is the resulting curve continuous? Is it a bounded variation curve?

11. Suppose $\gamma : [a, b] \rightarrow \mathbb{R}$ is given by $\gamma(t) = t$. What is $\int_{\gamma} f(t) d\gamma$? Explain.

Fundamentals Of Complex Analysis

27.1 Analytic Functions

Definition 27.1.1 Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \rightarrow X$. Then f is analytic on Ω if for every $z \in \Omega$,

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h} \equiv f'(z)$$

exists and is a continuous function of $z \in \Omega$. Here $h \in \mathbb{C}$.

Note that if f is analytic, it must be the case that f is continuous. It is more common to not include the requirement that f' is continuous but it is shown later that the continuity of f' follows.

What are some examples of analytic functions? In the case where $X = \mathbb{C}$, the simplest example is any polynomial. Thus

$$p(z) \equiv \sum_{k=0}^n a_k z^k$$

is an analytic function and

$$p'(z) = \sum_{k=1}^n a_k k z^{k-1}.$$

More generally, power series are analytic. This will be shown soon but first here is an important definition and a convergence theorem called the root test.

Definition 27.1.2 Let $\{a_k\}$ be a sequence in X . Then $\sum_{k=1}^{\infty} a_k \equiv \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$ whenever this limit exists. When the limit exists, the series is said to converge.

Theorem 27.1.3 Consider $\sum_{k=1}^{\infty} a_k$ and let $\rho \equiv \limsup_{k \rightarrow \infty} \|a_k\|^{1/k}$. Then if $\rho < 1$, the series converges absolutely and if $\rho > 1$ the series diverges spectacularly in the sense that $\lim_{k \rightarrow \infty} a_k \neq 0$. If $\rho = 1$ the test fails. Also $\sum_{k=1}^{\infty} a_k (z-a)^k$ converges on some disk $B(a, R)$. It converges absolutely if $|z-a| < R$ and uniformly on $B(a, r_1)$ whenever $r_1 < R$. The function $f(z) = \sum_{k=1}^{\infty} a_k (z-a)^k$ is continuous on $B(a, R)$.

Proof: Suppose $\rho < 1$. Then there exists $r \in (\rho, 1)$. Therefore, $\|a_k\| \leq r^k$ for all k large enough and so by a comparison test, $\sum_k \|a_k\|$ converges because the partial sums are bounded above. Therefore, the partial sums of the original series form a Cauchy sequence in X and so they also converge due to completeness of X .

Now suppose $\rho > 1$. Then letting $\rho > r > 1$, it follows $\|a_k\|^{1/k} \geq r$ infinitely often. Thus $\|a_k\| \geq r^k$ infinitely often. Thus there exists a subsequence for which $\|a_{n_k}\|$ converges to ∞ . Therefore, the series cannot converge.

Now consider $\sum_{k=1}^{\infty} a_k (z-a)^k$. This series converges absolutely if

$$\limsup_{k \rightarrow \infty} \|a_k\|^{1/k} |z-a| < 1$$

which is the same as saying $|z-a| < 1/\rho$ where $\rho \equiv \limsup_{k \rightarrow \infty} \|a_k\|^{1/k}$. Let $R = 1/\rho$.

Now suppose $r_1 < R$. Consider $|z-a| \leq r_1$. Then for such z ,

$$\|a_k\| |z-a|^k \leq \|a_k\| r_1^k$$

and

$$\limsup_{k \rightarrow \infty} (\|a_k\| r_1^k)^{1/k} = \limsup_{k \rightarrow \infty} \|a_k\|^{1/k} r_1 = \frac{r_1}{R} < 1$$

so $\sum_k \|a_k\| r_1^k$ converges. By the Weierstrass M test, $\sum_{k=1}^{\infty} a_k (z-a)^k$ converges uniformly for $|z-a| \leq r_1$. Therefore, f is continuous on $B(a, R)$ as claimed because it is the uniform limit of continuous functions, the partial sums of the infinite series.

What if $\rho = 0$? In this case,

$$\limsup_{k \rightarrow \infty} \|a_k\|^{1/k} |z-a| = 0 \cdot |z-a| = 0$$

and so $R = \infty$ and the series, $\sum \|a_k\| |z-a|^k$ converges everywhere.

What if $\rho = \infty$? Then in this case, the series converges only at $z = a$ because if $z \neq a$,

$$\limsup_{k \rightarrow \infty} \|a_k\|^{1/k} |z-a| = \infty.$$

Theorem 27.1.4 Let $f(z) \equiv \sum_{k=1}^{\infty} a_k (z-a)^k$ be given in Theorem 27.1.3 where $R > 0$. Then f is analytic on $B(a, R)$. So are all its derivatives.

Proof: Consider $g(z) = \sum_{k=2}^{\infty} a_k k (z-a)^{k-1}$ on $B(a, R)$ where $R = \rho^{-1}$ as above. Let $r_1 < r < R$. Then letting $|z-a| < r_1$ and $h < r - r_1$,

$$\begin{aligned}
 & \left\| \frac{f(z+h) - f(z)}{h} - g(z) \right\| \\
 & \leq \sum_{k=2}^{\infty} \|a_k\| \left| \frac{(z+h-a)^k - (z-a)^k}{h} - k(z-a)^{k-1} \right| \\
 & \leq \sum_{k=2}^{\infty} \|a_k\| \left| \frac{1}{h} \left(\sum_{i=0}^k \binom{k}{i} (z-a)^{k-i} h^i - (z-a)^k \right) - k(z-a)^{k-1} \right| \\
 & = \sum_{k=2}^{\infty} \|a_k\| \left| \frac{1}{h} \left(\sum_{i=1}^k \binom{k}{i} (z-a)^{k-i} h^i \right) - k(z-a)^{k-1} \right| \\
 & \leq \sum_{k=2}^{\infty} \|a_k\| \left| \left(\sum_{i=2}^k \binom{k}{i} (z-a)^{k-i} h^{i-1} \right) \right| \\
 & \leq |h| \sum_{k=2}^{\infty} \|a_k\| \left(\sum_{i=0}^{k-2} \binom{k}{i+2} |z-a|^{k-2-i} |h|^i \right) \\
 & = |h| \sum_{k=2}^{\infty} \|a_k\| \left(\sum_{i=0}^{k-2} \binom{k-2}{i} \frac{k(k-1)}{(i+2)(i+1)} |z-a|^{k-2-i} |h|^i \right) \\
 & \leq |h| \sum_{k=2}^{\infty} \|a_k\| \frac{k(k-1)}{2} \left(\sum_{i=0}^{k-2} \binom{k-2}{i} |z-a|^{k-2-i} |h|^i \right) \\
 & = |h| \sum_{k=2}^{\infty} \|a_k\| \frac{k(k-1)}{2} (|z-a| + |h|)^{k-2} < |h| \sum_{k=2}^{\infty} \|a_k\| \frac{k(k-1)}{2} r^{k-2}.
 \end{aligned}$$

Then

$$\limsup_{k \rightarrow \infty} \left(\|a_k\| \frac{k(k-1)}{2} r^{k-2} \right)^{1/k} = \rho r < 1$$

and so

$$\left\| \frac{f(z+h) - f(z)}{h} - g(z) \right\| \leq C|h|.$$

therefore, $g(z) = f'(z)$. Now by Theorem 27.1.3 it also follows that f' is continuous. Since $r_1 < R$ was arbitrary, this shows that $f'(z)$ is given by the differentiated series above for $|z-a| < R$. Now a repeat of the argument shows all the derivatives of f exist and are continuous on $B(a, R)$.

27.1.1 Cauchy Riemann Equations

Next consider the very important Cauchy Riemann equations which give conditions under which complex valued functions of a complex variable are analytic.

Theorem 27.1.5 Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \rightarrow \mathbb{C}$ be a function, such that for $z = x + iy \in \Omega$,

$$f(z) = u(x, y) + iv(x, y).$$

Then f is analytic if and only if u, v are $C^1(\Omega)$ and

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Furthermore,

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i \frac{\partial v}{\partial x}(x, y).$$

Proof: Suppose f is analytic first. Then letting $t \in \mathbb{R}$,

$$\begin{aligned} f'(z) &= \lim_{t \rightarrow 0} \frac{f(z+t) - f(z)}{t} = \\ \lim_{t \rightarrow 0} \left(\frac{u(x+t, y) + iv(x+t, y)}{t} - \frac{u(x, y) + iv(x, y)}{t} \right) \\ &= \frac{\partial u(x, y)}{\partial x} + i \frac{\partial v(x, y)}{\partial x}. \end{aligned}$$

But also

$$\begin{aligned} f'(z) &= \lim_{t \rightarrow 0} \frac{f(z+it) - f(z)}{it} = \\ \lim_{t \rightarrow 0} \left(\frac{u(x, y+t) + iv(x, y+t)}{it} - \frac{u(x, y) + iv(x, y)}{it} \right) \\ &= \frac{1}{i} \left(\frac{\partial u(x, y)}{\partial y} + i \frac{\partial v(x, y)}{\partial y} \right) \\ &= \frac{\partial v(x, y)}{\partial y} - i \frac{\partial u(x, y)}{\partial y}. \end{aligned}$$

This verifies the Cauchy Riemann equations. We are assuming that $z \rightarrow f'(z)$ is continuous. Therefore, the partial derivatives of u and v are also continuous. To see this, note that from the formulas for $f'(z)$ given above, and letting $z_1 = x_1 + iy_1$

$$\left| \frac{\partial v(x, y)}{\partial y} - \frac{\partial v(x_1, y_1)}{\partial y} \right| \leq |f'(z) - f'(z_1)|,$$

showing that $(x, y) \rightarrow \frac{\partial v(x, y)}{\partial y}$ is continuous since $(x_1, y_1) \rightarrow (x, y)$ if and only if $z_1 \rightarrow z$. The other cases are similar.

Now suppose the Cauchy Riemann equations hold and the functions, u and v are $C^1(\Omega)$. Then letting $h = h_1 + ih_2$,

$$f(z+h) - f(z) = u(x+h_1, y+h_2)$$

$$+iv(x+h_1, y+h_2) - (u(x, y) + iv(x, y))$$

We know u and v are both differentiable and so

$$f(z+h) - f(z) = \frac{\partial u}{\partial x}(x, y)h_1 + \frac{\partial u}{\partial y}(x, y)h_2 +$$

$$i\left(\frac{\partial v}{\partial x}(x, y)h_1 + \frac{\partial v}{\partial y}(x, y)h_2\right) + o(h).$$

Dividing by h and using the Cauchy Riemann equations,

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{\frac{\partial u}{\partial x}(x, y)h_1 + i\frac{\partial v}{\partial y}(x, y)h_2}{h} + \\ &\quad \frac{i\frac{\partial v}{\partial x}(x, y)h_1 + \frac{\partial u}{\partial y}(x, y)h_2}{h} + \frac{o(h)}{h} \\ &= \frac{\partial u}{\partial x}(x, y)\frac{h_1 + ih_2}{h} + i\frac{\partial v}{\partial x}(x, y)\frac{h_1 + ih_2}{h} + \frac{o(h)}{h} \end{aligned}$$

Taking the limit as $h \rightarrow 0$,

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}(x, y).$$

It follows from this formula and the assumption that u, v are $C^1(\Omega)$ that f' is continuous.

It is routine to verify that all the usual rules of derivatives hold for analytic functions. In particular, the product rule, the chain rule, and quotient rule.

27.1.2 An Important Example

An important example of an analytic function is $e^z \equiv \exp(z) \equiv e^x(\cos y + i \sin y)$ where $z = x + iy$. You can verify that this function satisfies the Cauchy Riemann equations and that all the partial derivatives are continuous. Also from the above discussion, $(e^z)' = e^x \cos(y) + ie^x \sin y = e^z$. Later I will show that e^z is given by the usual power series. An important property of this function is that it can be used to parameterize the circle centered at z_0 having radius r .

Lemma 27.1.6 *Let γ denote the closed curve which is a circle of radius r centered at z_0 . Then a parameterization this curve is $\gamma(t) = z_0 + re^{it}$ where $t \in [0, 2\pi]$.*

Proof: $|\gamma(t) - z_0|^2 = |re^{it}re^{-it}| = r^2$. Also, you can see from the definition of the sine and cosine that the point described in this way moves counter clockwise over this circle.

27.2 Exercises

1. Verify all the usual rules of differentiation including the product and chain rules.
2. Suppose f and $f' : U \rightarrow \mathbb{C}$ are analytic and $f(z) = u(x, y) + iv(x, y)$. Verify $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. This partial differential equation satisfied by the real and imaginary parts of an analytic function is called Laplace's equation. We say these functions satisfying Laplace's equation are harmonic functions. If u is a harmonic function defined on $B(0, r)$ show that $v(x, y) \equiv \int_0^y u_x(x, t) dt - \int_0^x u_y(t, 0) dt$ is such that $u + iv$ is analytic.
3. Let $f : U \rightarrow \mathbb{C}$ be analytic and $f(z) = u(x, y) + iv(x, y)$. Show u, v and uv are all harmonic although it can happen that u^2 is not. Recall that a function, w is harmonic if $w_{xx} + w_{yy} = 0$.
4. Define a function $f(z) \equiv \bar{z} \equiv x - iy$ where $z = x + iy$. Is f analytic?
5. If $f(z) = u(x, y) + iv(x, y)$ and f is analytic, verify that

$$\det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = |f'(z)|^2.$$

6. Show that if $u(x, y) + iv(x, y) = f(z)$ is analytic, then $\nabla u \cdot \nabla v = 0$. Recall

$$\nabla u(x, y) = \langle u_x(x, y), u_y(x, y) \rangle.$$

7. Show that every polynomial is analytic.
8. If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U , an open set of $\mathbb{C}$, and if $f : U \rightarrow \mathbb{C}$ is analytic, we can consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, $\langle x, y \rangle$. Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \rightarrow \mathbb{C}$ is analytic. Show that the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$ is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal. **Hint:** To make this easy to show, first observe that $\langle x, y \rangle \cdot \langle a, b \rangle = \frac{1}{2}(z\bar{w} + \bar{z}w)$ where $z = x + iy$ and $w = a + ib$.
9. Analytic functions are even better than what is described in Problem 8. In addition to preserving angles, they also preserve orientation. To verify this show that if $z = x + iy$ and $w = a + ib$ are two complex numbers, then $\langle x, y, 0 \rangle$ and $\langle a, b, 0 \rangle$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $\langle x, y, 0 \rangle \times \langle a, b, 0 \rangle$, yields a vector normal to the two given vectors such that the triple, $\langle x, y, 0 \rangle, \langle a, b, 0 \rangle$, and $\langle x, y, 0 \rangle \times \langle a, b, 0 \rangle$ satisfies the right hand rule

and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product either points in the direction of the positive z axis or in the direction of the negative z axis. Thus, either the vectors $\langle x, y, 0 \rangle, \langle a, b, 0 \rangle, \mathbf{k}$ form a right handed system or the vectors $\langle a, b, 0 \rangle, \langle x, y, 0 \rangle, \mathbf{k}$ form a right handed system. These are the two possible orientations. Show that in the situation of Problem 8 the orientation of $\gamma'(t_0), \eta'(s_0), \mathbf{k}$ is the same as the orientation of the vectors $(f \circ \gamma)'(t_0), (f \circ \eta)'(s_0), \mathbf{k}$. Such mappings are called conformal. If f is analytic and $f'(z) \neq 0$, then we know from this problem and the above that f is a conformal map. **Hint:** You can do this by verifying that $(f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) = |f'(\gamma(t_0))|^2 \gamma'(t_0) \times \eta'(s_0)$. To make the verification easier, you might first establish the following simple formula for the cross product where here $x + iy = z$ and $a + ib = w$.

$$(x, y, 0) \times (a, b, 0) = \operatorname{Re}(z i \bar{w}) \mathbf{k}.$$

10. Write the Cauchy Riemann equations in terms of polar coordinates. Recall the polar coordinates are given by

$$x = r \cos \theta, \quad y = r \sin \theta.$$

This means, letting $u(x, y) = u(r, \theta), v(x, y) = v(r, \theta)$, write the Cauchy Riemann equations in terms of r and θ . You should eventually show the Cauchy Riemann equations are equivalent to

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

11. Show that a real valued analytic function must be constant.

27.3 Cauchy's Formula For A Disk

The Cauchy integral formula is the most important theorem in complex analysis. It will be established for a disk in this chapter and later will be generalized to much more general situations but the version given here will suffice to prove many interesting theorems needed in the later development of the theory. The following are some advanced calculus results.

Lemma 27.3.1 *Let $f : [a, b] \rightarrow \mathbb{C}$. Then $f'(t)$ exists if and only if $\operatorname{Re} f'(t)$ and $\operatorname{Im} f'(t)$ exist. Furthermore,*

$$f'(t) = \operatorname{Re} f'(t) + i \operatorname{Im} f'(t).$$

Proof: The if part of the equivalence is obvious.

Now suppose $f'(t)$ exists. Let both t and $t + h$ be contained in $[a, b]$

$$\left| \frac{\operatorname{Re} f(t+h) - \operatorname{Re} f(t)}{h} - \operatorname{Re}(f'(t)) \right| \leq \left| \frac{f(t+h) - f(t)}{h} - f'(t) \right|$$

and this converges to zero as $h \rightarrow 0$. Therefore, $\operatorname{Re} f'(t) = \operatorname{Re}(f'(t))$. Similarly, $\operatorname{Im} f'(t) = \operatorname{Im}(f'(t))$.

Lemma 27.3.2 *If $g : [a, b] \rightarrow \mathbb{C}$ and g is continuous on $[a, b]$ and differentiable on (a, b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: From the above lemma, you can apply the mean value theorem to the real and imaginary parts of g .

Applying the above lemma to the components yields the following lemma.

Lemma 27.3.3 *If $g : [a, b] \rightarrow \mathbb{C}^n = X$ and g is continuous on $[a, b]$ and differentiable on (a, b) with $g'(t) = 0$, then $g(t)$ is a constant.*

If you want to have X be a complex Banach space, the result is still true.

Lemma 27.3.4 *If $g : [a, b] \rightarrow X$ and g is continuous on $[a, b]$ and differentiable on (a, b) with $g'(t) = 0$, then $g(t)$ is a constant.*

Proof: Let $\Lambda \in X'$. Then $\Lambda g : [a, b] \rightarrow \mathbb{C}$. Therefore, from Lemma 27.3.2, for each $\Lambda \in X'$, $\Lambda g(s) = \Lambda g(t)$ and since X' separates the points, it follows $g(s) = g(t)$ so g is constant.

Lemma 27.3.5 *Let $\phi : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s, t) ds. \quad (27.3.1)$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times [c, d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} ds. \quad (27.3.2)$$

Proof: The first claim follows from the uniform continuity of ϕ on $[a, b] \times [c, d]$, which uniform continuity results from the set being compact. To establish 27.3.2, let t and $t + h$ be contained in $[c, d]$ and form, using the mean value theorem,

$$\begin{aligned} \frac{g(t+h) - g(t)}{h} &= \frac{1}{h} \int_a^b [\phi(s, t+h) - \phi(s, t)] ds \\ &= \frac{1}{h} \int_a^b \frac{\partial \phi(s, t + \theta h)}{\partial t} h ds \\ &= \int_a^b \frac{\partial \phi(s, t + \theta h)}{\partial t} ds, \end{aligned}$$

where θ may depend on s but is some number between 0 and 1. Then by the uniform continuity of $\frac{\partial \phi}{\partial t}$, it follows that 27.3.2 holds.

Corollary 27.3.6 *Let $\phi : [a, b] \times [c, d] \rightarrow \mathbb{C}$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s, t) ds. \quad (27.3.3)$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times [c, d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} ds. \quad (27.3.4)$$

Proof: Apply Lemma 27.3.5 to the real and imaginary parts of ϕ .

Applying the above corollary to the components, you can also have the same result for ϕ having values in $\mathbb{C}^n$.

Corollary 27.3.7 *Let $\phi : [a, b] \times [c, d] \rightarrow \mathbb{C}^n$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s, t) ds. \quad (27.3.5)$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times [c, d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} ds. \quad (27.3.6)$$

If you want to consider ϕ having values in X , a complex Banach space a similar result holds.

Corollary 27.3.8 *Let $\phi : [a, b] \times [c, d] \rightarrow X$ be continuous and let*

$$g(t) \equiv \int_a^b \phi(s, t) ds. \quad (27.3.7)$$

Then g is continuous. If $\frac{\partial \phi}{\partial t}$ exists and is continuous on $[a, b] \times [c, d]$, then

$$g'(t) = \int_a^b \frac{\partial \phi(s, t)}{\partial t} ds. \quad (27.3.8)$$

Proof: Let $\Lambda \in X'$. Then $\Lambda\phi : [a, b] \times [c, d] \rightarrow \mathbb{C}$ is continuous and $\frac{\partial \Lambda\phi}{\partial t}$ exists and is continuous on $[a, b] \times [c, d]$. Therefore, from 27.3.8,

$$\Lambda(g'(t)) = (\Lambda g)'(t) = \int_a^b \frac{\partial \Lambda\phi(s, t)}{\partial t} ds = \Lambda \int_a^b \frac{\partial \phi(s, t)}{\partial t} ds$$

and since X' separates the points, it follows 27.3.8 holds.

The following is Cauchy's integral formula for a disk.

Theorem 27.3.9 Let $f : \Omega \rightarrow X$ be analytic on the open set, Ω and let

$$\overline{B(z_0, r)} \subseteq \Omega.$$

Let $\gamma(t) \equiv z_0 + re^{it}$ for $t \in [0, 2\pi]$. Then if $z \in B(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw. \quad (27.3.9)$$

Proof: Consider for $\alpha \in [0, 1]$,

$$g(\alpha) \equiv \int_0^{2\pi} \frac{f(z + \alpha(z_0 + re^{it} - z))}{re^{it} + z_0 - z} rie^{it} dt.$$

If α equals one, this reduces to the integral in 27.3.9. The idea is to show g is a constant and that $g(0) = f(z) 2\pi i$. First consider the claim about $g(0)$.

$$\begin{aligned} g(0) &= \left(\int_0^{2\pi} \frac{re^{it}}{re^{it} + z_0 - z} dt \right) if(z) \\ &= if(z) \left(\int_0^{2\pi} \frac{1}{1 - \frac{z-z_0}{re^{it}}} dt \right) \\ &= if(z) \int_0^{2\pi} \sum_{n=0}^{\infty} r^{-n} e^{-int} (z - z_0)^n dt \end{aligned}$$

because $\left| \frac{z-z_0}{re^{it}} \right| < 1$. Since this sum converges uniformly you can interchange the sum and the integral to obtain

$$\begin{aligned} g(0) &= if(z) \sum_{n=0}^{\infty} r^{-n} (z - z_0)^n \int_0^{2\pi} e^{-int} dt \\ &= 2\pi if(z) \end{aligned}$$

because $\int_0^{2\pi} e^{-int} dt = 0$ if $n > 0$.

Next consider the claim that g is constant. By Corollary 27.3.7, for $\alpha \in (0, 1)$,

$$\begin{aligned} g'(\alpha) &= \int_0^{2\pi} \frac{f'(z + \alpha(z_0 + re^{it} - z)) (re^{it} + z_0 - z)}{re^{it} + z_0 - z} rie^{it} dt \\ &= \int_0^{2\pi} f'(z + \alpha(z_0 + re^{it} - z)) rie^{it} dt \\ &= \int_0^{2\pi} \frac{d}{dt} \left(f(z + \alpha(z_0 + re^{it} - z)) \frac{1}{\alpha} \right) dt \\ &= f(z + \alpha(z_0 + re^{i2\pi} - z)) \frac{1}{\alpha} - f(z + \alpha(z_0 + re^0 - z)) \frac{1}{\alpha} = 0. \end{aligned}$$

Now g is continuous on $[0, 1]$ and $g'(t) = 0$ on $(0, 1)$ so by Lemma 27.3.3, g equals a constant. This constant can only be $g(0) = 2\pi if(z)$. Thus,

$$g(1) = \int_{\gamma} \frac{f(w)}{w-z} dw = g(0) = 2\pi if(z).$$

This proves the theorem.

This is a very significant theorem. A few applications are given next.

Theorem 27.3.10 *Let $f : \Omega \rightarrow X$ be analytic where Ω is an open set in $\mathbb{C}$. Then f has infinitely many derivatives on Ω . Furthermore, for all $z \in B(z_0, r)$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}} dw \quad (27.3.10)$$

where $\gamma(t) \equiv z_0 + re^{it}$, $t \in [0, 2\pi]$ for r small enough that $B(z_0, r) \subseteq \Omega$.

Proof: Let $z \in B(z_0, r) \subseteq \Omega$ and let $\overline{B(z_0, r)} \subseteq \Omega$. Then, letting $\gamma(t) \equiv z_0 + re^{it}$, $t \in [0, 2\pi]$, and h small enough,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw, \quad f(z+h) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z-h} dw$$

Now

$$\frac{1}{w-z-h} - \frac{1}{w-z} = \frac{h}{(-w+z+h)(-w+z)}$$

and so

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi i} \int_{\gamma} \frac{hf(w)}{(-w+z+h)(-w+z)} dw \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(-w+z+h)(-w+z)} dw. \end{aligned}$$

Now for all h sufficiently small, there exists a constant C independent of such h such that

$$\begin{aligned} &\left| \frac{1}{(-w+z+h)(-w+z)} - \frac{1}{(-w+z)(-w+z)} \right| \\ &= \left| \frac{h}{(w-z-h)(w-z)^2} \right| \leq C|h| \end{aligned}$$

and so, the integrand converges uniformly as $h \rightarrow 0$ to

$$= \frac{f(w)}{(w-z)^2}$$

Therefore, the limit as $h \rightarrow 0$ may be taken inside the integral to obtain

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^2} dw.$$

Continuing in this way, yields 27.3.10.

This is a very remarkable result. It shows the existence of one continuous derivative implies the existence of all derivatives, in contrast to the theory of functions of a real variable. Actually, more than what is stated in the theorem was shown. The above proof establishes the following corollary.

Corollary 27.3.11 Suppose f is continuous on $\partial B(z_0, r)$ and suppose that for all $z \in B(z_0, r)$,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw,$$

where $\gamma(t) \equiv z_0 + re^{it}$, $t \in [0, 2\pi]$. Then f is analytic on $B(z_0, r)$ and in fact has infinitely many derivatives on $B(z_0, r)$.

Another application is the following lemma.

Lemma 27.3.12 Let $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$, suppose $f_n \rightarrow f$ uniformly on $\overline{B(z_0, r)}$, and suppose

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w - z} dw \quad (27.3.11)$$

for $z \in B(z_0, r)$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw, \quad (27.3.12)$$

implying that f is analytic on $B(z_0, r)$.

Proof: From 27.3.11 and the uniform convergence of f_n to f on $\gamma([0, 2\pi])$, the integrals in 27.3.11 converge to

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw.$$

Therefore, the formula 27.3.12 follows.

Uniform convergence on a closed disk of the analytic functions implies the target function is also analytic. This is amazing. Think of the Weierstrass approximation theorem for polynomials. You can obtain a continuous nowhere differentiable function as the uniform limit of polynomials.

The conclusions of the following proposition have all been obtained earlier in Theorem 27.1.4 but they can be obtained more easily if you use the above theorem and lemmas.

Proposition 27.3.13 Let $\{a_n\}$ denote a sequence in X . Then there exists $R \in [0, \infty]$ such that

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges absolutely if $|z - z_0| < R$, diverges if $|z - z_0| > R$ and converges uniformly on $B(z_0, r)$ for all $r < R$. Furthermore, if $R > 0$, the function,

$$f(z) \equiv \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is analytic on $B(z_0, R)$.

Proof: The assertions about absolute convergence are routine from the root test if

$$R \equiv \left(\limsup_{n \rightarrow \infty} |a_n|^{1/n} \right)^{-1}$$

with $R = \infty$ if the quantity in parenthesis equals zero. The root test can be used to verify absolute convergence which then implies convergence by completeness of X .

The assertion about uniform convergence follows from the Weierstrass M test and $M_n \equiv |a_n| r^n$. ($\sum_{n=0}^{\infty} |a_n| r^n < \infty$ by the root test). It only remains to verify the assertion about $f(z)$ being analytic in the case where $R > 0$.

Let $0 < r < R$ and define $f_n(z) \equiv \sum_{k=0}^n a_k (z - z_0)^k$. Then f_n is a polynomial and so it is analytic. Thus, by the Cauchy integral formula above,

$$f_n(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f_n(w)}{w - z} dw$$

where $\gamma(t) = z_0 + re^{it}$, for $t \in [0, 2\pi]$. By Lemma 27.3.12 and the first part of this proposition involving uniform convergence,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw.$$

Therefore, f is analytic on $B(z_0, r)$ by Corollary 27.3.11. Since $r < R$ is arbitrary, this shows f is analytic on $B(z_0, R)$.

This proposition shows that all functions having values in X which are given as power series are analytic on their circle of convergence, the set of complex numbers, z , such that $|z - z_0| < R$. In fact, every analytic function can be realized as a power series.

Theorem 27.3.14 *If $f : \Omega \rightarrow X$ is analytic and if $B(z_0, r) \subseteq \Omega$, then*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad (27.3.13)$$

for all $|z - z_0| < r$. Furthermore,

$$a_n = \frac{f^{(n)}(z_0)}{n!}. \quad (27.3.14)$$

Proof: Consider $|z - z_0| < r$ and let $\gamma(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$. Then for $w \in \gamma([0, 2\pi])$,

$$\left| \frac{z - z_0}{w - z_0} \right| < 1$$

and so, by the Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0) \left(1 - \frac{z-z_0}{w-z_0}\right)} dw \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^n dw. \end{aligned}$$

Since the series converges uniformly, you can interchange the integral and the sum to obtain

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z_0)^{n+1}} \right) (z-z_0)^n \\ &\equiv \sum_{n=0}^{\infty} a_n (z-z_0)^n \end{aligned}$$

By Theorem 27.3.10, 27.3.14 holds.

Note that this also implies that if a function is analytic on an open set, then all of its derivatives are also analytic. This follows from Theorem 27.1.4 which says that a function given by a power series has all derivatives on the disk of convergence.

27.4 Exercises

1. Show that if $|e_k| \leq \varepsilon$, then $|\sum_{k=m}^{\infty} e_k (r^k - r^{k+1})| < \varepsilon$ if $0 \leq r < 1$. **Hint:** Let $|\theta| = 1$ and verify that

$$\theta \sum_{k=m}^{\infty} e_k (r^k - r^{k+1}) = \left| \sum_{k=m}^{\infty} e_k (r^k - r^{k+1}) \right| = \sum_{k=m}^{\infty} \operatorname{Re}(\theta e_k) (r^k - r^{k+1})$$

where $-\varepsilon < \operatorname{Re}(\theta e_k) < \varepsilon$.

2. Abel's theorem says that if $\sum_{n=0}^{\infty} a_n (z-a)^n$ has radius of convergence equal to 1 and if $A = \sum_{n=0}^{\infty} a_n$, then $\lim_{r \rightarrow 1^-} \sum_{n=0}^{\infty} a_n r^n = A$. **Hint:** Show $\sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} A_k (r^k - r^{k+1})$ where A_k denotes the k^{th} partial sum of $\sum a_j$. Thus

$$\sum_{k=0}^{\infty} a_k r^k = \sum_{k=m+1}^{\infty} A_k (r^k - r^{k+1}) + \sum_{k=0}^m A_k (r^k - r^{k+1}),$$

where $|A_k - A| < \varepsilon$ for all $k \geq m$. In the first sum, write $A_k = A + e_k$ and use Problem 1. Use this theorem to verify that $\arctan(1) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1}$.

3. Find the integrals using the Cauchy integral formula.

- (a) $\int_{\gamma} \frac{\sin z}{z-i} dz$ where $\gamma(t) = 2e^{it} : t \in [0, 2\pi]$.
 (b) $\int_{\gamma} \frac{1}{z-a} dz$ where $\gamma(t) = a + re^{it} : t \in [0, 2\pi]$
 (c) $\int_{\gamma} \frac{\cos z}{z^2} dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$
 (d) $\int_{\gamma} \frac{\log(z)}{z^n} dz$ where $\gamma(t) = 1 + \frac{1}{2}e^{it} : t \in [0, 2\pi]$ and $n = 0, 1, 2$. In this problem, $\log(z) \equiv \ln|z| + i \arg(z)$ where $\arg(z) \in (-\pi, \pi)$ and $z = |z|e^{i \arg(z)}$. Thus $e^{\log(z)} = z$ and $\log(z)' = \frac{1}{z}$.

4. Let $\gamma(t) = 4e^{it} : t \in [0, 2\pi]$ and find $\int_{\gamma} \frac{z^2+4}{z(z^2+1)} dz$.

5. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $|z| < R$. Show that then

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

for all $r \in [0, R)$. **Hint:** Let

$$f_n(z) \equiv \sum_{k=0}^n a_k z^k,$$

show

$$\frac{1}{2\pi} \int_0^{2\pi} |f_n(re^{i\theta})|^2 d\theta = \sum_{k=0}^n |a_k|^2 r^{2k}$$

and then take limits as $n \rightarrow \infty$ using uniform convergence.

6. The Cauchy integral formula, marvelous as it is, can actually be improved upon. The Cauchy integral formula involves representing f by the values of f on the boundary of the disk, $B(a, r)$. It is possible to represent f by using only the values of $\operatorname{Re} f$ on the boundary. This leads to the Schwarz formula. Supply the details in the following outline.

Suppose f is analytic on $|z| < R$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (27.4.15)$$

with the series converging uniformly on $|z| = R$. Then letting $|w| = R$,

$$2u(w) = f(w) + \overline{f(\overline{w})}$$

and so

$$2u(w) = \sum_{k=0}^{\infty} a_k w^k + \sum_{k=0}^{\infty} \overline{a_k} (\overline{w})^k. \quad (27.4.16)$$

Now letting $\gamma(t) = Re^{it}$, $t \in [0, 2\pi]$

$$\begin{aligned} \int_{\gamma} \frac{2u(w)}{w} dw &= (a_0 + \overline{a_0}) \int_{\gamma} \frac{1}{w} dw \\ &= 2\pi i (a_0 + \overline{a_0}). \end{aligned}$$

Thus, multiplying 27.4.16 by w^{-1} ,

$$\frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w} dw = a_0 + \overline{a_0}.$$

Now multiply 27.4.16 by $w^{-(n+1)}$ and integrate again to obtain

$$a_n = \frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w^{n+1}} dw.$$

Using these formulas for a_n in 27.4.15, we can interchange the sum and the integral (Why can we do this?) to write the following for $|z| < R$.

$$\begin{aligned} f(z) &= \frac{1}{\pi i} \int_{\gamma} \frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^{k+1} u(w) dw - \overline{a_0} \\ &= \frac{1}{\pi i} \int_{\gamma} \frac{u(w)}{w-z} dw - \overline{a_0}, \end{aligned}$$

which is the Schwarz formula. Now $\operatorname{Re} a_0 = \frac{1}{2\pi i} \int_{\gamma} \frac{u(w)}{w} dw$ and $\overline{a_0} = \operatorname{Re} a_0 - i \operatorname{Im} a_0$. Therefore, we can also write the Schwarz formula as

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{u(w)(w+z)}{(w-z)w} dw + i \operatorname{Im} a_0. \quad (27.4.17)$$

7. Take the real parts of the second form of the Schwarz formula to derive the Poisson formula for a disk,

$$u(re^{i\alpha}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(Re^{i\theta})(R^2 - r^2)}{R^2 + r^2 - 2Rr \cos(\theta - \alpha)} d\theta. \quad (27.4.18)$$

8. Suppose that $u(w)$ is a given real continuous function defined on $\partial B(0, R)$ and define $f(z)$ for $|z| < R$ by 27.4.17. Show that f , so defined is analytic. Explain why u given in 27.4.18 is harmonic. Show that

$$\lim_{r \rightarrow R^-} u(re^{i\alpha}) = u(Re^{i\alpha}).$$

Thus u is a harmonic function which approaches a given function on the boundary and is therefore, a solution to the Dirichlet problem.

9. Suppose $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ for all $|z - z_0| < R$. Show that $f'(z) = \sum_{k=0}^{\infty} a_k k (z - z_0)^{k-1}$ for all $|z - z_0| < R$. **Hint:** Let $f_n(z)$ be a partial sum of f . Show that f'_n converges uniformly to some function, g on $|z - z_0| \leq r$ for any $r < R$. Now use the Cauchy integral formula for a function and its derivative to identify g with f' .
10. Use Problem 9 to find the exact value of $\sum_{k=0}^{\infty} k^2 \left(\frac{1}{3}\right)^k$.
11. Prove the binomial formula,

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$

where

$$\binom{\alpha}{n} \equiv \frac{\alpha \cdots (\alpha - n + 1)}{n!}.$$

Can this be used to give a proof of the binomial formula,

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k?$$

Explain.

12. Suppose f is analytic on $B(z_0, r)$ and continuous on $\overline{B(z_0, r)}$ and $|f(z)| \leq M$ on $\overline{B(z_0, r)}$. Show that then $|f^{(n)}(a)| \leq \frac{Mn!}{r^n}$.

27.5 Zeros Of An Analytic Function

In this section we give a very surprising property of analytic functions which is in stark contrast to what takes place for functions of a real variable.

Definition 27.5.1 *A region is a connected open set.*

It turns out the zeros of an analytic function which is not constant on some region cannot have a limit point. This is also a good time to define the order of a zero.

Definition 27.5.2 *Suppose f is an analytic function defined near a point, α where $f(\alpha) = 0$. Thus α is a zero of the function, f . The zero is of order m if $f(z) = (z - \alpha)^m g(z)$ where g is an analytic function which is not equal to zero at α .*

Theorem 27.5.3 *Let Ω be a connected open set (region) and let $f : \Omega \rightarrow X$ be analytic. Then the following are equivalent.*

1. $f(z) = 0$ for all $z \in \Omega$
2. There exists $z_0 \in \Omega$ such that $f^{(n)}(z_0) = 0$ for all n .

3. There exists $z_0 \in \Omega$ which is a limit point of the set,

$$Z \equiv \{z \in \Omega : f(z) = 0\}.$$

Proof: It is clear the first condition implies the second two. Suppose the third holds. Then for z near z_0

$$f(z) = \sum_{n=k}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

where $k \geq 1$ since z_0 is a zero of f . Suppose $k < \infty$. Then,

$$f(z) = (z - z_0)^k g(z)$$

where $g(z_0) \neq 0$. Letting $z_n \rightarrow z_0$ where $z_n \in Z, z_n \neq z_0$, it follows

$$0 = (z_n - z_0)^k g(z_n)$$

which implies $g(z_n) = 0$. Then by continuity of g , we see that $g(z_0) = 0$ also, contrary to the choice of k . Therefore, k cannot be less than ∞ and so z_0 is a point satisfying the second condition.

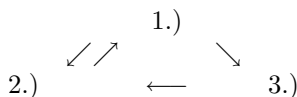
Now suppose the second condition and let

$$S \equiv \left\{ z \in \Omega : f^{(n)}(z) = 0 \text{ for all } n \right\}.$$

It is clear that S is a closed set which by assumption is nonempty. However, this set is also open. To see this, let $z \in S$. Then for all w close enough to z ,

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z)}{k!} (w - z)^k = 0.$$

Thus f is identically equal to zero near $z \in S$. Therefore, all points near z are contained in S also, showing that S is an open set. Now $\Omega = S \cup (\Omega \setminus S)$, the union of two disjoint open sets, S being nonempty. It follows the other open set, $\Omega \setminus S$, must be empty because Ω is connected. Therefore, the first condition is verified. This proves the theorem. (See the following diagram.)



Note how radically different this is from the theory of functions of a real variable. Consider, for example the function

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

which has a derivative for all $x \in \mathbb{R}$ and for which 0 is a limit point of the set, Z , even though f is not identically equal to zero.

Here is a very important application called Euler's formula. Recall that

$$e^z \equiv e^x (\cos(y) + i \sin(y)) \quad (27.5.19)$$

Is it also true that $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$?

Theorem 27.5.4 (*Euler's Formula*) *Let $z = x + iy$. Then*

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Proof: It was already observed that e^z given by 27.5.19 is analytic. So is $\exp(z) \equiv \sum_{k=0}^{\infty} \frac{z^k}{k!}$. In fact the power series converges for all $z \in \mathbb{C}$. Furthermore the two functions, e^z and $\exp(z)$ agree on the real line which is a set which contains a limit point. Therefore, they agree for all values of $z \in \mathbb{C}$.

This formula shows the famous two identities,

$$e^{i\pi} = -1 \text{ and } e^{2\pi i} = 1.$$

27.6 Liouville's Theorem

The following theorem pertains to functions which are analytic on all of $\mathbb{C}$, "entire" functions.

Definition 27.6.1 *A function, $f : \mathbb{C} \rightarrow \mathbb{C}$ or more generally, $f : \mathbb{C} \rightarrow X$ is entire means it is analytic on $\mathbb{C}$.*

Theorem 27.6.2 (*Liouville's theorem*) *If f is a bounded entire function having values in X , then f is a constant.*

Proof: Since f is entire, pick any $z \in \mathbb{C}$ and write

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)}{(w-z)^2} dw$$

where $\gamma_R(t) = z + Re^{it}$ for $t \in [0, 2\pi]$. Therefore,

$$\|f'(z)\| \leq C \frac{1}{R}$$

where C is some constant depending on the assumed bound on f . Since R is arbitrary, let $R \rightarrow \infty$ to obtain $f'(z) = 0$ for any $z \in \mathbb{C}$. It follows from this that f is constant for if z_j $j = 1, 2$ are two complex numbers, let $h(t) = f(z_1 + t(z_2 - z_1))$ for $t \in [0, 1]$. Then $h'(t) = f'(z_1 + t(z_2 - z_1))(z_2 - z_1) = 0$. By Lemmas 27.3.2 - 27.3.4 h is a constant on $[0, 1]$ which implies $f(z_1) = f(z_2)$.

With Liouville's theorem it becomes possible to give an easy proof of the fundamental theorem of algebra. It is ironic that all the best proofs of this theorem in algebra come from the subjects of analysis or topology. Out of all the proofs that have been given of this very important theorem, the following one based on Liouville's theorem is the easiest.

Theorem 27.6.3 (*Fundamental theorem of Algebra*) *Let*

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$$

be a polynomial where $n \geq 1$ and each coefficient is a complex number. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof: Suppose not. Then $p(z)^{-1}$ is an entire function. Also

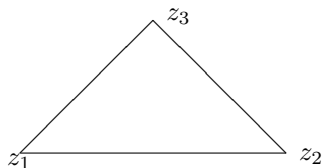
$$|p(z)| \geq |z|^n - (|a_{n-1}||z|^{n-1} + \cdots + |a_1||z| + |a_0|)$$

and so $\lim_{|z| \rightarrow \infty} |p(z)| = \infty$ which implies $\lim_{|z| \rightarrow \infty} |p(z)^{-1}| = 0$. It follows that, since $p(z)^{-1}$ is bounded for z in any bounded set, we must have that $p(z)^{-1}$ is a bounded entire function. But then it must be constant. However since $p(z)^{-1} \rightarrow 0$ as $|z| \rightarrow \infty$, this constant can only be 0. However, $\frac{1}{p(z)}$ is never equal to zero. This proves the theorem.

27.7 The General Cauchy Integral Formula

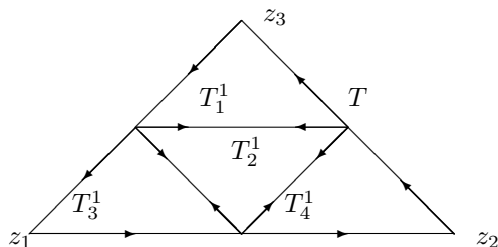
27.7.1 The Cauchy Goursat Theorem

This section gives a fundamental theorem which is essential to the development which follows and is closely related to the question of when a function has a primitive. First of all, if you have two points in $\mathbb{C}$, z_1 and z_2 , you can consider $\gamma(t) \equiv z_1 + t(z_2 - z_1)$ for $t \in [0, 1]$ to obtain a continuous bounded variation curve from z_1 to z_2 . More generally, if $z_1, \dots, z_m$ are points in $\mathbb{C}$ you can obtain a continuous bounded variation curve from z_1 to z_m which consists of first going from z_1 to z_2 and then from z_2 to z_3 and so on, till in the end one goes from z_{m-1} to z_m . We denote this piecewise linear curve as $\gamma(z_1, \dots, z_m)$. Now let T be a triangle with vertices z_1, z_2 and z_3 encountered in the counter clockwise direction as shown.



Denote by $\int_{\partial T} f(z) dz$, the expression, $\int_{\gamma(z_1, z_2, z_3, z_1)} f(z) dz$. Consider the fol-

lowing picture.



By Lemma 26.0.11

$$\int_{\partial T} f(z) dz = \sum_{k=1}^4 \int_{\partial T_k^1} f(z) dz. \quad (27.7.20)$$

On the “inside lines” the integrals cancel as claimed in Lemma 26.0.11 because there are two integrals going in opposite directions for each of these inside lines.

Theorem 27.7.1 (*Cauchy Goursat*) *Let $f : \Omega \rightarrow X$ have the property that $f'(z)$ exists for all $z \in \Omega$ and let T be a triangle contained in Ω . Then*

$$\int_{\partial T} f(w) dw = 0.$$

Proof: Suppose not. Then

$$\left\| \int_{\partial T} f(w) dw \right\| = \alpha \neq 0.$$

From 27.7.20 it follows

$$\alpha \leq \sum_{k=1}^4 \left\| \int_{\partial T_k^1} f(w) dw \right\|$$

and so for at least one of these T_k^1 , denoted from now on as T_1 ,

$$\left\| \int_{\partial T_1} f(w) dw \right\| \geq \frac{\alpha}{4}.$$

Now let T_1 play the same role as T , subdivide as in the above picture, and obtain T_2 such that

$$\left\| \int_{\partial T_2} f(w) dw \right\| \geq \frac{\alpha}{4^2}.$$

Continue in this way, obtaining a sequence of triangles,

$$T_k \supseteq T_{k+1}, \text{diam}(T_k) \leq \text{diam}(T) 2^{-k},$$

and

$$\left\| \int_{\partial T_k} f(w) dw \right\| \geq \frac{\alpha}{4^k}.$$

Then let $z \in \cap_{k=1}^{\infty} T_k$ and note that by assumption, $f'(z)$ exists. Therefore, for all k large enough,

$$\int_{\partial T_k} f(w) dw = \int_{\partial T_k} f(z) + f'(z)(w-z) + g(w) dw$$

where $\|g(w)\| < \varepsilon |w-z|$. Now observe that $w \rightarrow f(z) + f'(z)(w-z)$ has a primitive, namely,

$$F(w) = f(z)w + f'(z)(w-z)^2/2.$$

Therefore, by Corollary 26.0.14.

$$\int_{\partial T_k} f(w) dw = \int_{\partial T_k} g(w) dw.$$

From the definition, of the integral,

$$\begin{aligned} \frac{\alpha}{4^k} &\leq \left\| \int_{\partial T_k} g(w) dw \right\| \leq \varepsilon \text{diam}(T_k) (\text{length of } \partial T_k) \\ &\leq \varepsilon 2^{-k} (\text{length of } T) \text{diam}(T) 2^{-k}, \end{aligned}$$

and so

$$\alpha \leq \varepsilon (\text{length of } T) \text{diam}(T).$$

Since ε is arbitrary, this shows $\alpha = 0$, a contradiction. Thus $\int_{\partial T} f(w) dw = 0$ as claimed.

This fundamental result yields the following important theorem.

Theorem 27.7.2 (Morera¹) *Let Ω be an open set and let $f'(z)$ exist for all $z \in \Omega$. Let $D \equiv \overline{B}(z_0, r) \subseteq \Omega$. Then there exists $\varepsilon > 0$ such that f has a primitive on $B(z_0, r + \varepsilon)$.*

Proof: Choose $\varepsilon > 0$ small enough that $B(z_0, r + \varepsilon) \subseteq \Omega$. Then for $w \in B(z_0, r + \varepsilon)$, define

$$F(w) \equiv \int_{\gamma(z_0, w)} f(u) du.$$

Then by the Cauchy Goursat theorem, and $w \in B(z_0, r + \varepsilon)$, it follows that for $|h|$ small enough,

$$\begin{aligned} \frac{F(w+h) - F(w)}{h} &= \frac{1}{h} \int_{\gamma(w, w+h)} f(u) du \\ &= \frac{1}{h} \int_0^1 f(w+th) h dt = \int_0^1 f(w+th) dt \end{aligned}$$

¹Giancinto Morera 1856-1909. This theorem or one like it dates from around 1886

which converges to $f(w)$ due to the continuity of f at w . This proves the theorem.

The following is a slight generalization of the above theorem which is also referred to as Morera's theorem.

Corollary 27.7.3 *Let Ω be an open set and suppose that whenever*

$$\gamma(z_1, z_2, z_3, z_1)$$

is a closed curve bounding a triangle T , which is contained in Ω , and f is a continuous function defined on Ω , it follows that

$$\int_{\gamma(z_1, z_2, z_3, z_1)} f(z) dz = 0,$$

then f is analytic on Ω .

Proof: As in the proof of Morera's theorem, let $\overline{B(z_0, r)} \subseteq \Omega$ and use the given condition to construct a primitive, F for f on $B(z_0, r)$. Then F is analytic and so by Theorem 27.3.10, it follows that F and hence f have infinitely many derivatives, implying that f is analytic on $B(z_0, r)$. Since z_0 is arbitrary, this shows f is analytic on Ω .

27.7.2 A Redundant Assumption

Earlier in the definition of analytic, it was assumed the derivative is continuous. This assumption is **redundant**.

Theorem 27.7.4 *Let Ω be an open set in $\mathbb{C}$ and suppose $f : \Omega \rightarrow X$ has the property that $f'(z)$ exists for each $z \in \Omega$. Then f is analytic on Ω .*

Proof: Let $z_0 \in \Omega$ and let $B(z_0, r) \subseteq \Omega$. By Morera's theorem f has a primitive, F on $B(z_0, r)$. It follows that F is analytic because it has a derivative, f , and this derivative is continuous. Therefore, by Theorem 27.3.10 F has infinitely many derivatives on $B(z_0, r)$ implying that f also has infinitely many derivatives on $B(z_0, r)$. Thus f is analytic as claimed.

It follows a function is analytic on an open set, Ω if and only if $f'(z)$ exists for $z \in \Omega$. This is because it was just shown the derivative, if it exists, is automatically continuous.

The same proof used to prove Theorem 27.7.2 implies the following corollary.

Corollary 27.7.5 *Let Ω be a convex open set and suppose that $f'(z)$ exists for all $z \in \Omega$. Then f has a primitive on Ω .*

Note that this implies that if Ω is a convex open set on which $f'(z)$ exists and if $\gamma : [a, b] \rightarrow \Omega$ is a closed, continuous curve having bounded variation, then letting F be a primitive of f Theorem 26.0.13 implies

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)) = 0.$$

Notice how different this is from the situation of a function of a real variable! It is possible for a function of a real variable to have a derivative everywhere and yet the derivative can be discontinuous. A simple example is the following.

$$f(x) \equiv \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then $f'(x)$ exists for all $x \in \mathbb{R}$. Indeed, if $x \neq 0$, the derivative equals $2x \sin \frac{1}{x} - \cos \frac{1}{x}$ which has no limit as $x \rightarrow 0$. However, from the definition of the derivative of a function of one variable, $f'(0) = 0$.

27.7.3 Classification Of Isolated Singularities

First some notation.

Definition 27.7.6 Let $B'(a, r) \equiv \{z \in \mathbb{C} \text{ such that } 0 < |z - a| < r\}$. Thus this is the usual ball without the center. A function is said to have an isolated singularity at the point $a \in \mathbb{C}$ if f is analytic on $B'(a, r)$ for some $r > 0$.

It turns out isolated singularities can be neatly classified into three types, removable singularities, poles, and essential singularities. The next theorem deals with the case of a removable singularity.

Definition 27.7.7 An isolated singularity of f is said to be removable if there exists an analytic function, g analytic at a and near a such that $f = g$ at all points near a .

Theorem 27.7.8 Let $f : B'(a, r) \rightarrow X$ be analytic. Thus f has an isolated singularity at a . Suppose also that

$$\lim_{z \rightarrow a} f(z)(z - a) = 0.$$

Then there exists a unique analytic function, $g : B(a, r) \rightarrow X$ such that $g = f$ on $B'(a, r)$. Thus the singularity at a is removable.

Proof: Let $h(z) \equiv (z - a)^2 f(z)$, $h(a) \equiv 0$. Then h is analytic on $B(a, r)$ because it is easy to see that $h'(a) = 0$. It follows h is given by a power series,

$$h(z) = \sum_{k=2}^{\infty} a_k (z - a)^k$$

where $a_0 = a_1 = 0$ because of the observation above that $h'(a) = h(a) = 0$. It follows that for $|z - a| > 0$

$$f(z) = \sum_{k=2}^{\infty} a_k (z - a)^{k-2} \equiv g(z).$$

This proves the theorem.

What of the other case where the singularity is not removable? This situation is dealt with by the amazing Casorati Weierstrass theorem.

Theorem 27.7.9 (*Casorati Weierstrass*) *Let a be an isolated singularity and suppose for some $r > 0$, $f(B'(a, r))$ is not dense in $\mathbb{C}$. Then either a is a removable singularity or there exist finitely many $b_1, \dots, b_M$ for some finite number, M such that for z near a ,*

$$f(z) = g(z) + \sum_{k=1}^M \frac{b_k}{(z-a)^k} \quad (27.7.21)$$

where $g(z)$ is analytic near a .

Proof: Suppose $B(z_0, \delta)$ has no points of $f(B'(a, r))$. Such a ball must exist if $f(B'(a, r))$ is not dense. Then for $z \in B'(a, r)$, $|f(z) - z_0| \geq \delta > 0$. It follows from Theorem 27.7.8 that $\frac{1}{f(z) - z_0}$ has a removable singularity at a . Hence, there exists h an analytic function such that for z near a ,

$$h(z) = \frac{1}{f(z) - z_0}. \quad (27.7.22)$$

There are two cases. First suppose $h(a) = 0$. Then $\sum_{k=1}^{\infty} a_k (z-a)^k = \frac{1}{f(z) - z_0}$ for z near a . If all the $a_k = 0$, this would be a contradiction because then the left side would equal zero for z near a but the right side could not equal zero. Therefore, there is a first m such that $a_m \neq 0$. Hence there exists an analytic function, $k(z)$ which is not equal to zero in some ball, $B(a, \varepsilon)$ such that

$$k(z)(z-a)^m = \frac{1}{f(z) - z_0}.$$

Hence, taking both sides to the -1 power,

$$f(z) - z_0 = \frac{1}{(z-a)^m} \sum_{k=0}^{\infty} b_k (z-a)^k$$

and so 27.7.21 holds.

The other case is that $h(a) \neq 0$. In this case, raise both sides of 27.7.22 to the -1 power and obtain

$$f(z) - z_0 = h(z)^{-1},$$

a function analytic near a . Therefore, the singularity is removable. This proves the theorem.

This theorem is the basis for the following definition which classifies isolated singularities.

Definition 27.7.10 *Let a be an isolated singularity of a complex valued function, f . When 27.7.21 holds for z near a , then a is called a pole. The order of the pole in 27.7.21 is M . If for every $r > 0$, $f(B'(a, r))$ is dense in $\mathbb{C}$ then a is called an essential singularity.*

In terms of the above definition, isolated singularities are either removable, a pole, or essential. There are no other possibilities.

Theorem 27.7.11 Suppose $f : \Omega \rightarrow \mathbb{C}$ has an isolated singularity at $a \in \Omega$. Then a is a pole if and only if

$$\lim_{z \rightarrow a} d(f(z), \infty) = 0$$

in $\widehat{\mathbb{C}}$.

Proof: Suppose first f has a pole at a . Then by definition, $f(z) = g(z) + \sum_{k=1}^M \frac{b_k}{(z-a)^k}$ for z near a where g is analytic. Then

$$\begin{aligned} |f(z)| &\geq \frac{|b_M|}{|z-a|^M} - |g(z)| - \sum_{k=1}^{M-1} \frac{|b_k|}{|z-a|^k} \\ &= \frac{1}{|z-a|^M} \left(|b_M| - \left(|g(z)| |z-a|^M + \sum_{k=1}^{M-1} |b_k| |z-a|^{M-k} \right) \right). \end{aligned}$$

Now $\lim_{z \rightarrow a} \left(|g(z)| |z-a|^M + \sum_{k=1}^{M-1} |b_k| |z-a|^{M-k} \right) = 0$ and so the above inequality proves $\lim_{z \rightarrow a} |f(z)| = \infty$. Referring to the diagram on Page 778, you see this is the same as saying

$$\lim_{z \rightarrow a} |\theta f(z) - (0, 0, 2)| = \lim_{z \rightarrow a} |\theta f(z) - \theta(\infty)| = \lim_{z \rightarrow a} d(f(z), \infty) = 0$$

Conversely, suppose $\lim_{z \rightarrow a} d(f(z), \infty) = 0$. Then from the diagram on Page 778, it follows $\lim_{z \rightarrow a} |f(z)| = \infty$ and in particular, a cannot be either removable or an essential singularity by the Casorati Weierstrass theorem, Theorem 27.7.9. The only case remaining is that a is a pole. This proves the theorem.

Definition 27.7.12 Let $f : \Omega \rightarrow \mathbb{C}$ where Ω is an open subset of $\mathbb{C}$. Then f is called meromorphic if all singularities are isolated and are either poles or removable and this set of singularities has no limit point. It is convenient to regard meromorphic functions as having values in $\widehat{\mathbb{C}}$ where if a is a pole, $f(a) \equiv \infty$. From now on, this will be assumed when a meromorphic function is being considered.

The usefulness of the above convention about $f(a) \equiv \infty$ at a pole is made clear in the following theorem.

Theorem 27.7.13 Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \rightarrow \widehat{\mathbb{C}}$ be meromorphic. Then f is continuous with respect to the metric, d on $\widehat{\mathbb{C}}$.

Proof: Let $z_n \rightarrow z$ where $z \in \Omega$. Then if z is a pole, it follows from Theorem 27.7.11 that

$$d(f(z_n), \infty) \equiv d(f(z_n), f(z)) \rightarrow 0.$$

If z is not a pole, then $f(z_n) \rightarrow f(z)$ in $\mathbb{C}$ which implies $|\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \rightarrow 0$. Recall that θ is continuous on $\mathbb{C}$.

27.7.4 The Cauchy Integral Formula

This section presents the general version of the Cauchy integral formula valid for arbitrary closed rectifiable curves. The key idea in this development is the notion of the winding number. This is the number also called the index, defined in the following theorem. This winding number, along with the earlier results, especially Liouville's theorem, yields an extremely general Cauchy integral formula.

Definition 27.7.14 Let $\gamma : [a, b] \rightarrow \mathbb{C}$ and suppose $z \notin \gamma^*$. The winding number, $n(\gamma, z)$ is defined by

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}.$$

The main interest is in the case where γ is closed curve. However, the same notation will be used for any such curve.

Theorem 27.7.15 Let $\gamma : [a, b] \rightarrow \mathbb{C}$ be continuous and have bounded variation with $\gamma(a) = \gamma(b)$. Also suppose that $z \notin \gamma^*$. Define

$$n(\gamma, z) \equiv \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}. \quad (27.7.23)$$

Then $n(\gamma, \cdot)$ is continuous and integer valued. Furthermore, there exists a sequence, $\eta_k : [a, b] \rightarrow \mathbb{C}$ such that η_k is $C^1([a, b])$,

$$\|\eta_k - \gamma\| < \frac{1}{k}, \eta_k(a) = \eta_k(b) = \gamma(a) = \gamma(b),$$

and $n(\eta_k, z) = n(\gamma, z)$ for all k large enough. Also $n(\gamma, \cdot)$ is constant on every connected component of $\mathbb{C} \setminus \gamma^*$ and equals zero on the unbounded component of $\mathbb{C} \setminus \gamma^*$.

Proof: First consider the assertion about continuity.

$$\begin{aligned} |n(\gamma, z) - n(\gamma, z_1)| &\leq C \left| \int_{\gamma} \left(\frac{1}{w - z} - \frac{1}{w - z_1} \right) dw \right| \\ &\leq \tilde{C} (\text{Length of } \gamma) |z_1 - z| \end{aligned}$$

whenever z_1 is close enough to z . This proves the continuity assertion. Note this did not depend on γ being closed.

Next it is shown that for a closed curve the winding number equals an integer. To do so, use Theorem 26.0.12 to obtain η_k , a function in $C^1([a, b])$ such that $z \notin \eta_k([a, b])$ for all k large enough, $\eta_k(x) = \gamma(x)$ for $x = a, b$, and

$$\left| \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z} - \frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z} \right| < \frac{1}{k}, \|\eta_k - \gamma\| < \frac{1}{k}.$$

It is shown that each of $\frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z}$ is an integer. To simplify the notation, write η instead of η_k .

$$\int_{\eta} \frac{dw}{w - z} = \int_a^b \frac{\eta'(s) ds}{\eta(s) - z}.$$

Define

$$g(t) \equiv \int_a^t \frac{\eta'(s) ds}{\eta(s) - z}. \quad (27.7.24)$$

Then

$$\begin{aligned} \left(e^{-g(t)} (\eta(t) - z) \right)' &= e^{-g(t)} \eta'(t) - e^{-g(t)} g'(t) (\eta(t) - z) \\ &= e^{-g(t)} \eta'(t) - e^{-g(t)} \eta'(t) = 0. \end{aligned}$$

It follows that $e^{-g(t)} (\eta(t) - z)$ equals a constant. In particular, using the fact that $\eta(a) = \eta(b)$,

$$e^{-g(b)} (\eta(b) - z) = e^{-g(a)} (\eta(a) - z) = (\eta(a) - z) = (\eta(b) - z)$$

and so $e^{-g(b)} = 1$. This happens if and only if $-g(b) = 2m\pi i$ for some integer m . Therefore, 27.7.24 implies

$$2m\pi i = \int_a^b \frac{\eta'(s) ds}{\eta(s) - z} = \int_\gamma \frac{dw}{w - z}.$$

Therefore, $\frac{1}{2\pi i} \int_{\eta_k} \frac{dw}{w - z}$ is a sequence of integers converging to $\frac{1}{2\pi i} \int_\gamma \frac{dw}{w - z} \equiv n(\gamma, z)$ and so $n(\gamma, z)$ must also be an integer and $n(\eta_k, z) = n(\gamma, z)$ for all k large enough.

Since $n(\gamma, \cdot)$ is continuous and integer valued, it follows from Corollary 6.7.11 on Page 162 that it must be constant on every connected component of $\mathbb{C} \setminus \gamma^*$. It is clear that $n(\gamma, z)$ equals zero on the unbounded component because from the formula,

$$\lim_{z \rightarrow \infty} |n(\gamma, z)| \leq \lim_{z \rightarrow \infty} V(\gamma, [a, b]) \left(\frac{1}{|z| - c} \right)$$

where $c \geq \max\{|w| : w \in \gamma^*\}$. This proves the theorem.

Corollary 27.7.16 *Suppose $\gamma : [a, b] \rightarrow \mathbb{C}$ is a continuous bounded variation curve and $n(\gamma, z)$ is an integer where $z \notin \gamma^*$. Then $\gamma(a) = \gamma(b)$. Also $z \rightarrow n(\gamma, z)$ for $z \notin \gamma^*$ is continuous.*

Proof: Letting η be a C^1 curve for which $\eta(a) = \gamma(a)$ and $\eta(b) = \gamma(b)$ and which is close enough to γ that $n(\eta, z) = n(\gamma, z)$, the argument is similar to the above. Let

$$g(t) \equiv \int_a^t \frac{\eta'(s) ds}{\eta(s) - z}. \quad (27.7.25)$$

Then

$$\begin{aligned} \left(e^{-g(t)} (\eta(t) - z) \right)' &= e^{-g(t)} \eta'(t) - e^{-g(t)} g'(t) (\eta(t) - z) \\ &= e^{-g(t)} \eta'(t) - e^{-g(t)} \eta'(t) = 0. \end{aligned}$$

Hence

$$e^{-g(t)} (\eta(t) - z) = c \neq 0. \quad (27.7.26)$$

By assumption

$$g(b) = \int_{\eta} \frac{1}{w-z} dw = 2\pi im$$

for some integer, m . Therefore, from 27.7.26

$$1 = e^{2\pi mi} = \frac{\eta(b) - z}{c}.$$

Thus $c = \eta(b) - z$ and letting $t = a$ in 27.7.26,

$$1 = \frac{\eta(a) - z}{\eta(b) - z}$$

which shows $\eta(a) = \eta(b)$. This proves the corollary since the assertion about continuity was already observed.

It is a good idea to consider a simple case to get an idea of what the winding number is measuring. To do so, consider $\gamma : [a, b] \rightarrow \mathbb{C}$ such that γ is continuous, closed and bounded variation. Suppose also that γ is one to one on (a, b) . Such a curve is called a simple closed curve. It can be shown that such a simple closed curve divides the plane into exactly two components, an “inside” bounded component and an “outside” unbounded component. This is called the Jordan Curve theorem or the Jordan separation theorem. This is a difficult theorem which requires some very hard topology such as homology theory or degree theory. It won’t be used here beyond making reference to it. For now, it suffices to simply assume that γ is such that this result holds. This will usually be obvious anyway. Also suppose that it is possible to change the parameter to be in $[0, 2\pi]$, in such a way that $\gamma(t) + \lambda(z + re^{it} - \gamma(t)) - z \neq 0$ for all $t \in [0, 2\pi]$ and $\lambda \in [0, 1]$. (As t goes from 0 to 2π the point $\gamma(t)$ traces the curve $\gamma([0, 2\pi])$ in the counter clockwise direction.) Suppose $z \in D$, the inside of the simple closed curve and consider the curve $\delta(t) = z + re^{it}$ for $t \in [0, 2\pi]$ where r is chosen small enough that $\overline{B}(z, r) \subseteq D$. Then it happens that $n(\delta, z) = n(\gamma, z)$.

Proposition 27.7.17 *Under the above conditions,*

$$n(\delta, z) = n(\gamma, z)$$

and $n(\delta, z) = 1$.

Proof: By changing the parameter, assume that $[a, b] = [0, 2\pi]$. From Theorem 27.7.15 it suffices to assume also that γ is C^1 . Define $h_{\lambda}(t) \equiv \gamma(t) + \lambda(z + re^{it} - \gamma(t))$ for $\lambda \in [0, 1]$. (This function is called a homotopy of the curves γ and δ .) Note that for each $\lambda \in [0, 1]$, $t \mapsto h_{\lambda}(t)$ is a closed C^1 curve. Also,

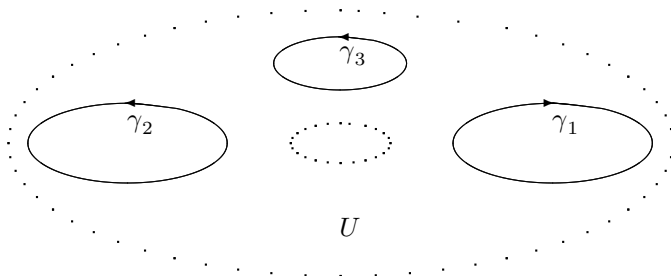
$$\frac{1}{2\pi i} \int_{h_{\lambda}} \frac{1}{w-z} dw = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\gamma'(t) + \lambda(re^{it} - \gamma'(t))}{\gamma(t) + \lambda(z + re^{it} - \gamma(t)) - z} dt.$$

This number is an integer and it is routine to verify that it is a continuous function of λ . When $\lambda = 0$ it equals $n(\gamma, z)$ and when $\lambda = 1$ it equals $n(\delta, z)$. Therefore, $n(\delta, z) = n(\gamma, z)$. It only remains to compute $n(\delta, z)$.

$$n(\delta, z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{rie^{it}}{re^{it}} dt = 1.$$

This proves the proposition.

Now if γ was not one to one but caused the point, $\gamma(t)$ to travel around γ^* twice, you could modify the above argument to have the parameter interval, $[0, 4\pi]$ and still find $n(\delta, z) = n(\gamma, z)$ only this time, $n(\delta, z) = 2$. Thus the winding number is just what its name suggests. It measures the number of times the curve winds around the point. One might ask why bother with the winding number if this is all it does. The reason is that the notion of counting the number of times a curve winds around a point is rather vague. The winding number is precise. It is also the natural thing to consider in the general Cauchy integral formula presented below. Consider a situation typified by the following picture in which Ω is the open set between the dotted curves and γ_j are closed rectifiable curves in Ω .



The following theorem is the general Cauchy integral formula.

Definition 27.7.18 Let $\{\gamma_k\}_{k=1}^n$ be continuous oriented curves having bounded variation. Then this is called a cycle if whenever, $z \notin \cup_{k=1}^n \gamma_k^*$, $\sum_{k=1}^n n(\gamma_k, z)$ is an integer.

By Theorem 27.7.15 if each γ_k is a closed curve, then $\{\gamma_k\}_{k=1}^n$ is a cycle.

Theorem 27.7.19 Let Ω be an open subset of the plane and let $f : \Omega \rightarrow X$ be analytic. If $\gamma_k : [a_k, b_k] \rightarrow \Omega$, $k = 1, \dots, m$ are continuous curves having bounded variation such that for all $z \notin \cup_{k=1}^m \gamma_k([a_k, b_k])$

$$\sum_{k=1}^m n(\gamma_k, z) \text{ equals an integer}$$

and for all $z \notin \Omega$,

$$\sum_{k=1}^m n(\gamma_k, z) = 0.$$

Then for all $z \in \Omega \setminus \cup_{k=1}^m \gamma_k([a_k, b_k])$,

$$f(z) \sum_{k=1}^m n(\gamma_k, z) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw.$$

Proof: Let ϕ be defined on $\Omega \times \Omega$ by

$$\phi(z, w) \equiv \begin{cases} \frac{f(w)-f(z)}{w-z} & \text{if } w \neq z \\ f'(z) & \text{if } w = z \end{cases}.$$

Then ϕ is analytic as a function of both z and w and is continuous in $\Omega \times \Omega$. This is easily seen using Theorem 27.7.8. Consider the case of $w \rightarrow \phi(z, w)$.

$$\lim_{w \rightarrow z} (w-z)(\phi(z, w) - \phi(z, z)) = \lim_{w \rightarrow z} \left(\frac{f(w) - f(z)}{w-z} - f'(z) \right) = 0.$$

Thus $w \rightarrow \phi(z, w)$ has a removable singularity at z . The case of $z \rightarrow \phi(z, w)$ is similar.

Define

$$h(z) \equiv \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \phi(z, w) dw.$$

Is h analytic on Ω ? To show this is the case, verify

$$\int_{\partial T} h(z) dz = 0$$

for every triangle, T , contained in Ω and apply Corollary 27.7.3. To do this, use Theorem 26.0.12 to obtain for each k , a sequence of functions, $\eta_{kn} \in C^1([a_k, b_k])$ such that

$$\eta_{kn}(x) = \gamma_k(x) \text{ for } x \in \{a_k, b_k\}$$

and

$$\eta_{kn}([a_k, b_k]) \subseteq \Omega, \quad \|\eta_{kn} - \gamma_k\| < \frac{1}{n},$$

$$\left\| \int_{\eta_{kn}} \phi(z, w) dw - \int_{\gamma_k} \phi(z, w) dw \right\| < \frac{1}{n}, \quad (27.7.27)$$

for all $z \in T$. Then applying Fubini's theorem,

$$\int_{\partial T} \int_{\eta_{kn}} \phi(z, w) dw dz = \int_{\eta_{kn}} \int_{\partial T} \phi(z, w) dz dw = 0$$

because ϕ is given to be analytic. By 27.7.27,

$$\int_{\partial T} \int_{\gamma_k} \phi(z, w) dw dz = \lim_{n \rightarrow \infty} \int_{\partial T} \int_{\eta_{kn}} \phi(z, w) dw dz = 0$$

and so h is analytic on Ω as claimed.

Now let H denote the set,

$$H \equiv \left\{ z \in \mathbb{C} \setminus \bigcup_{k=1}^m \gamma_k([a_k, b_k]) : \sum_{k=1}^m n(\gamma_k, z) = 0 \right\}.$$

H is an open set because $z \rightarrow \sum_{k=1}^m n(\gamma_k, z)$ is integer valued by assumption and continuous. Define

$$g(z) \equiv \begin{cases} h(z) & \text{if } z \in \Omega \\ \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw & \text{if } z \in H \end{cases} \quad (27.7.28)$$

Why is $g(z)$ well defined? For $z \in \Omega \cap H$, $z \notin \bigcup_{k=1}^m \gamma_k([a_k, b_k])$ and so

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \phi(z, w) dw = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} dw \\ &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{w-z} dw \\ &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw \end{aligned}$$

because $z \in H$. This shows $g(z)$ is well defined. Also, g is analytic on Ω because it equals h there. It is routine to verify that g is analytic on H also because of the second line of 27.7.28. By assumption, $\Omega^C \subseteq H$ because it is assumed that $\sum_k n(\gamma_k, z) = 0$ for $z \notin \Omega$ and so $\Omega \cup H = \mathbb{C}$ showing that g is an entire function.

Now note that $\sum_{k=1}^m n(\gamma_k, z) = 0$ for all z contained in the unbounded component of $\mathbb{C} \setminus \bigcup_{k=1}^m \gamma_k([a_k, b_k])$ which component contains $B(0, r)^C$ for r large enough. It follows that for $|z| > r$, it must be the case that $z \in H$ and so for such z , the bottom description of $g(z)$ found in 27.7.28 is valid. Therefore, it follows

$$\lim_{|z| \rightarrow \infty} \|g(z)\| = 0$$

and so g is bounded and entire. By Liouville's theorem, g is a constant. Hence, from the above equation, the constant can only equal zero.

For $z \in \Omega \setminus \bigcup_{k=1}^m \gamma_k([a_k, b_k])$,

$$\begin{aligned} 0 = h(z) &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \phi(z, w) dw = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w) - f(z)}{w-z} dw = \\ &= \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(w)}{w-z} dw - f(z) \sum_{k=1}^m n(\gamma_k, z). \end{aligned}$$

This proves the theorem.

Corollary 27.7.20 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \rightarrow \Omega$, $k = 1, \dots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^m n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \rightarrow \mathbb{C}$ is analytic,

$$\sum_{k=1}^m \int_{\gamma_k} f(w) dw = 0.$$

Proof: This follows from Theorem 27.7.19 as follows. Let

$$g(w) = f(w)(w - z)$$

where $z \in \Omega \setminus \cup_{k=1}^m \gamma_k([a_k, b_k])$. Then by this theorem,

$$\begin{aligned} 0 &= 0 \sum_{k=1}^m n(\gamma_k, z) = g(z) \sum_{k=1}^m n(\gamma_k, z) = \\ &= \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} f(w) dw. \end{aligned}$$

Another simple corollary to the above theorem is Cauchy's theorem for a simply connected region.

Definition 27.7.21 *An open set, $\Omega \subseteq \mathbb{C}$ is a region if it is open and connected. A region, Ω is simply connected if $\widehat{\mathbb{C}} \setminus \Omega$ is connected where $\widehat{\mathbb{C}}$ is the extended complex plane. In the future, the term simply connected open set will be an open set which is connected and $\widehat{\mathbb{C}} \setminus \Omega$ is connected.*

Corollary 27.7.22 *Let $\gamma : [a, b] \rightarrow \Omega$ be a continuous closed curve of bounded variation where Ω is a simply connected region in $\mathbb{C}$ and let $f : \Omega \rightarrow X$ be analytic. Then*

$$\int_{\gamma} f(w) dw = 0.$$

Proof: Let D denote the unbounded component of $\widehat{\mathbb{C}} \setminus \gamma^*$. Thus $\infty \in \widehat{\mathbb{C}} \setminus \gamma^*$. Then the connected set, $\widehat{\mathbb{C}} \setminus \Omega$ is contained in D since every point of $\widehat{\mathbb{C}} \setminus \Omega$ must be in some component of $\widehat{\mathbb{C}} \setminus \gamma^*$ and ∞ is contained in both $\widehat{\mathbb{C}} \setminus \Omega$ and D . Thus D must be the component that contains $\widehat{\mathbb{C}} \setminus \Omega$. It follows that $n(\gamma, \cdot)$ must be constant on $\widehat{\mathbb{C}} \setminus \Omega$, its value being its value on D . However, for $z \in D$,

$$n(\gamma, z) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} dw$$

and so $\lim_{|z| \rightarrow \infty} n(\gamma, z) = 0$ showing $n(\gamma, z) = 0$ on D . Therefore this verifies the hypothesis of Theorem 27.7.19. Let $z \in \Omega \cap D$ and define

$$g(w) \equiv f(w)(w - z).$$

Thus g is analytic on Ω and by Theorem 27.7.19,

$$0 = n(z, \gamma) g(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(w)}{w - z} dw = \frac{1}{2\pi i} \int_{\gamma} f(w) dw.$$

This proves the corollary.

The following is a very significant result which will be used later.

Corollary 27.7.23 *Suppose Ω is a simply connected open set and $f : \Omega \rightarrow X$ is analytic. Then f has a primitive, F , on Ω . Recall this means there exists F such that $F'(z) = f(z)$ for all $z \in \Omega$.*

Proof: Pick a point, $z_0 \in \Omega$ and let V denote those points, z of Ω for which there exists a curve, $\gamma : [a, b] \rightarrow \Omega$ such that γ is continuous, of bounded variation, $\gamma(a) = z_0$, and $\gamma(b) = z$. Then it is easy to verify that V is both open and closed in Ω and therefore, $V = \Omega$ because Ω is connected. Denote by $\gamma_{z_0, z}$ such a curve from z_0 to z and define

$$F(z) \equiv \int_{\gamma_{z_0, z}} f(w) dw.$$

Then F is well defined because if $\gamma_j, j = 1, 2$ are two such curves, it follows from Corollary 27.7.22 that

$$\int_{\gamma_1} f(w) dw + \int_{-\gamma_2} f(w) dw = 0,$$

implying that

$$\int_{\gamma_1} f(w) dw = \int_{\gamma_2} f(w) dw.$$

Now this function, F is a primitive because, thanks to Corollary 27.7.22

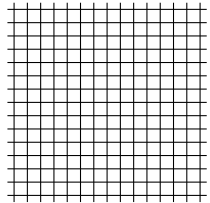
$$\begin{aligned} (F(z+h) - F(z))h^{-1} &= \frac{1}{h} \int_{\gamma_{z, z+h}} f(w) dw \\ &= \frac{1}{h} \int_0^1 f(z+th) h dt \end{aligned}$$

and so, taking the limit as $h \rightarrow 0$, $F'(z) = f(z)$.

27.7.5 An Example Of A Cycle

The next theorem deals with the existence of a cycle with nice properties. Basically, you go around the compact subset of an open set with suitable contours while staying in the open set. The method involves the following simple concept.

Definition 27.7.24 A tiling of $\mathbb{R}^2 = \mathbb{C}$ is the union of infinitely many equally spaced vertical and horizontal lines. You can think of the small squares which result as tiles. To tile the plane or $\mathbb{R}^2 = \mathbb{C}$ means to consider such a union of horizontal and vertical lines. It is like graph paper. See the picture below for a representation of part of a tiling of $\mathbb{C}$.



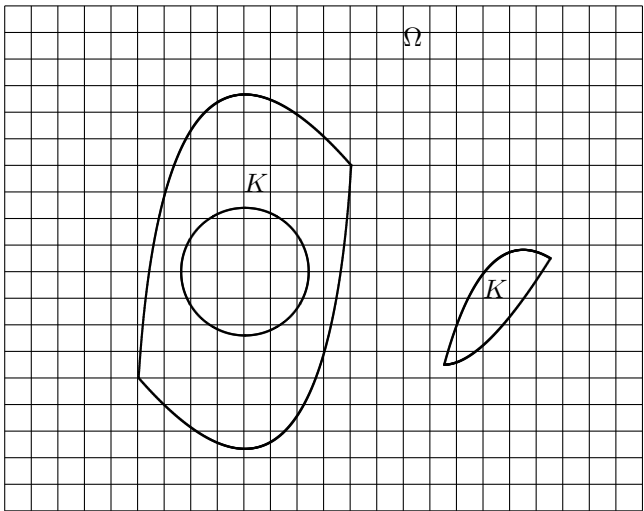
Theorem 27.7.25 Let K be a compact subset of an open set, Ω . Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each j , $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,

$$\sum_{k=1}^m n(\Gamma_k, p) = 1.$$

while for all $z \notin \Omega$

$$\sum_{k=1}^m n(\Gamma_k, z) = 0.$$

Proof: Let $\delta = \text{dist}(K, \Omega^C)$. Since K is compact, $\delta > 0$. Now tile the plane with squares, each of which has diameter less than $\delta/2$.

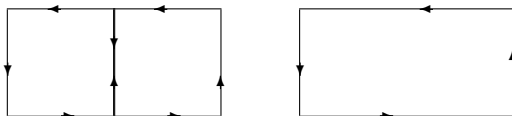


Let S denote the set of all the closed squares in this tiling which have nonempty intersection with K . Thus, all the squares of S are contained in Ω . First suppose p is a point of K which is in the interior of one of these squares in the tiling. Denote by ∂S_k the boundary of S_k one of the squares in S , oriented in the counter clockwise direction and S_m denote the square of S which contains the point, p in its interior. Let the edges of the square, S_j be $\{\gamma_k^j\}_{k=1}^4$. Thus a short computation shows $n(\partial S_m, p) = 1$ but $n(\partial S_j, p) = 0$ for all $j \neq m$. The reason for this is that for z in S_j , the values $\{z - p : z \in S_j\}$ lie in an open square, Q which is located at a positive distance from 0. Then $\widehat{\mathbb{C}} \setminus Q$ is connected and $1/(z - p)$ is analytic on Q . It follows from Corollary 27.7.23 that this function has a primitive on Q and so

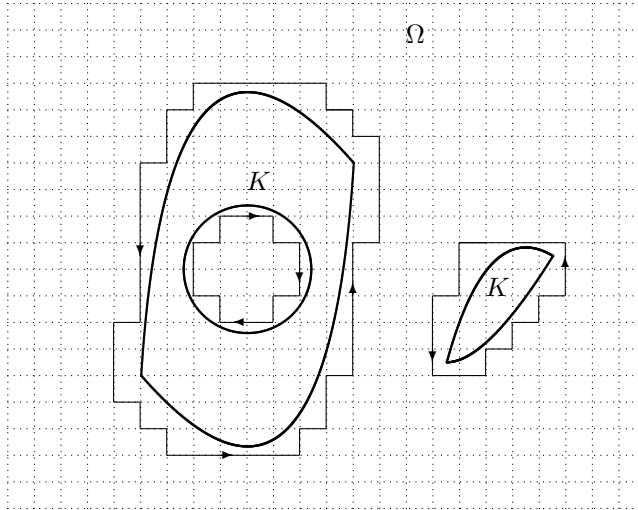
$$\int_{\partial S_j} \frac{1}{z - p} dz = 0.$$

Similarly, if $z \notin \Omega$, $n(\partial S_j, z) = 0$. On the other hand, a direct computation will verify that $n(p, \partial S_m) = 1$. Thus $1 = \sum_{j,k} n(p, \gamma_k^j) = \sum_{S_j \in S} n(p, \partial S_j)$ and if $z \notin \Omega$, $0 = \sum_{j,k} n(z, \gamma_k^j) = \sum_{S_j \in S} n(z, \partial S_j)$.

If γ_k^{j*} coincides with γ_l^{l*} , then the contour integrals taken over this edge are taken in opposite directions and so the edge the two squares have in common can be deleted without changing $\sum_{j,k} n(z, \gamma_k^j)$ for any z not on any of the lines in the tiling. For example, see the picture,



From the construction, if any of the γ_k^{j*} contains a point of K then this point is on one of the four edges of S_j and at this point, there is at least one edge of some S_l which also contains this point. As just discussed, this shared edge can be deleted without changing $\sum_{i,j} n(z, \gamma_k^j)$. Delete the edges of the S_k which intersect K but not the endpoints of these edges. That is, delete the open edges. When this is done, delete all isolated points. Let the resulting oriented curves be denoted by $\{\gamma_k\}_{k=1}^m$. Note that you might have $\gamma_k^* = \gamma_l^*$. The construction is illustrated in the following picture.



Then as explained above, $\sum_{k=1}^m n(p, \gamma_k) = 1$. It remains to prove the claim about the closed curves.

Each orientation on an edge corresponds to a direction of motion over that edge. Call such a motion over the edge a route. Initially, every vertex, (corner of a square in S) has the property there are the same number of routes to and from that vertex. When an open edge whose closure contains a point of K is deleted, every vertex either remains unchanged as to the number of routes to and from that vertex or it loses both a route away and a route to. Thus the property of having the same number of routes to and from each vertex is preserved by deleting these open edges. The isolated points which result lose all routes to and from. It follows that upon removing the isolated points you can begin at any of the remaining vertices and follow the routes leading out from this and successive vertices according to orientation and eventually return to that end. Otherwise, there would be a vertex which would have only one route leading to it which does not happen. Now if you have used all the routes out of this vertex, pick another vertex and do the same process. Otherwise, pick an unused route out of the vertex and follow it to return. Continue this way till all routes are used exactly once, resulting in closed oriented curves, Γ_k . Then

$$\sum_k n(\Gamma_k, p) = \sum_j n(\gamma_j, p) = 1.$$

In case $p \in K$ is on some line of the tiling, it is not on any of the Γ_k because $\Gamma_k^* \cap K = \emptyset$ and so the continuity of $z \rightarrow n(\Gamma_k, z)$ yields the desired result in this case also. This proves the lemma.

27.8 Exercises

1. If U is simply connected, f is analytic on U and f has no zeros in U , show there exists an analytic function, F , defined on U such that $e^F = f$.
2. Let f be defined and analytic near the point $a \in \mathbb{C}$. Show that then $f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k$ whenever $|z-a| < R$ where R is the distance between a and the nearest point where f fails to have a derivative. The number R , is called the radius of convergence and the power series is said to be expanded about a .
3. Find the radius of convergence of the function $\frac{1}{1+z^2}$ expanded about $a = 2$. Note there is nothing wrong with the function, $\frac{1}{1+x^2}$ when considered as a function of a real variable, x for any value of x . However, if you insist on using power series, you find there is a limitation on the values of x for which the power series converges due to the presence in the complex plane of a point, i , where the function fails to have a derivative.
4. Suppose f is analytic on all of $\mathbb{C}$ and satisfies $|f(z)| < A + B|z|^{1/2}$. Show f is constant.
5. What if you defined an open set, U to be simply connected if $\mathbb{C} \setminus U$ is connected. Would it amount to the same thing? **Hint:** Consider the outside of $B(0, 1)$.
6. Let $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Find $\int_{\gamma} \frac{1}{z^n} dz$ for $n = 1, 2, \dots$.
7. Show $i \int_0^{2\pi} (2 \cos \theta)^{2n} d\theta = \int_{\gamma} \left(z + \frac{1}{z}\right)^{2n} \left(\frac{1}{z}\right) dz$ where $\gamma(t) = e^{it} : t \in [0, 2\pi]$. Then evaluate this integral using the binomial theorem and the previous problem.
8. Suppose that for some constants $a, b \neq 0$, $a, b \in \mathbb{R}$, $f(z+ib) = f(z)$ for all $z \in \mathbb{C}$ and $f(z+a) = f(z)$ for all $z \in \mathbb{C}$. If f is analytic, show that f must be constant. Can you generalize this? **Hint:** This uses Liouville's theorem.
9. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic for $z \in U$, an open set. Let $g(z) = u^*(x, y) + iv^*(x, y)$ where

$$\begin{pmatrix} u^* \\ v^* \end{pmatrix} = Q \begin{pmatrix} u \\ v \end{pmatrix}$$

where Q is a unitary matrix. That is $QQ^* = Q^*Q = I$. When will g be analytic?

10. Suppose f is analytic on an open set, U , except for $\gamma^* \subset U$ where γ is a one to one continuous function having bounded variation, but it is known that f is continuous on γ^* . Show that in fact f is analytic on γ^* also. **Hint:** Pick a point on γ^* , say $\gamma(t_0)$ and suppose for now that $t_0 \in (a, b)$. Pick $r > 0$ such that $B = B(\gamma(t_0), r) \subseteq U$. Then show there exists $t_1 < t_0$ and $t_2 > t_0$ such

that $\gamma([t_1, t_2]) \subseteq \overline{B}$ and $\gamma(t_i) \notin B$. Thus $\gamma([t_1, t_2])$ is a path across B going through the center of B which divides B into two open sets, B_1 , and B_2 along with γ^* . Let the boundary of B_k consist of $\gamma([t_1, t_2])$ and a circular arc, C_k . Now letting $z \in B_k$, the line integral of $\frac{f(w)}{w-z}$ over γ^* in two different directions cancels. Therefore, if $z \in B_k$, you can argue that $f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$. By continuity, this continues to hold for $z \in \gamma((t_1, t_2))$. Therefore, f must be analytic on $\gamma((t_1, t_1))$ also. This shows that f must be analytic on $\gamma((a, b))$. To get the endpoints, simply extend γ to have the same properties but defined on $[a - \varepsilon, b + \varepsilon]$ and repeat the above argument or else do this at the beginning and note that you get $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$.

11. Let U be an open set contained in the upper half plane and suppose that there are finitely many line segments on the x axis which are contained in the boundary of U . Now suppose that f is defined, real, and continuous on these line segments and is defined and analytic on U . Now let $\tilde{U}$ denote the reflection of U across the x axis. Show that it is possible to extend f to a function, g defined on all of

$$W \equiv \tilde{U} \cup U \cup \{\text{the line segments mentioned earlier}\}$$

such that g is analytic in W . **Hint:** For $z \in \tilde{U}$, the reflection of U across the x axis, let $g(z) \equiv \overline{f(\bar{z})}$. Show that g is analytic on $\tilde{U} \cup U$ and continuous on the line segments. Then use Problem 10 or Morera's theorem to argue that g is analytic on the line segments also. The result of this problem is known as the Schwarz reflection principle.

12. Show that rotations and translations of analytic functions yield analytic functions and use this observation to generalize the Schwarz reflection principle to situations in which the line segments are part of a line which is not the x axis. Thus, give a version which involves reflection about an arbitrary line.

The Open Mapping Theorem

28.1 A Local Representation

The open mapping theorem, is an even more surprising result than the theorem about the zeros of an analytic function. The following proof of this important theorem uses an interesting local representation of the analytic function.

Theorem 28.1.1 (*Open mapping theorem*) Let Ω be a region in $\mathbb{C}$ and suppose $f : \Omega \rightarrow \mathbb{C}$ is analytic. Then $f(\Omega)$ is either a point or a region. In the case where $f(\Omega)$ is a region, it follows that for each $z_0 \in \Omega$, there exists an open set, V containing z_0 and $m \in \mathbb{N}$ such that for all $z \in V$,

$$f(z) = f(z_0) + \phi(z)^m \quad (28.1.1)$$

where $\phi : V \rightarrow B(0, \delta)$ is one to one, analytic and onto, $\phi(z_0) = 0$, $\phi'(z) \neq 0$ on V and ϕ^{-1} analytic on $B(0, \delta)$. If f is one to one then $m = 1$ for each z_0 and $f^{-1} : f(\Omega) \rightarrow \Omega$ is analytic.

Proof: Suppose $f(\Omega)$ is not a point. Then if $z_0 \in \Omega$ it follows there exists $r > 0$ such that $f(z) \neq f(z_0)$ for all $z \in B(z_0, r) \setminus \{z_0\}$. Otherwise, z_0 would be a limit point of the set,

$$\{z \in \Omega : f(z) - f(z_0) = 0\}$$

which would imply from Theorem 27.5.3 that $f(z) = f(z_0)$ for all $z \in \Omega$. Therefore, making r smaller if necessary and using the power series of f ,

$$f(z) = f(z_0) + (z - z_0)^m g(z) \quad (\stackrel{?}{=} \left((z - z_0) g(z)^{1/m} \right)^m)$$

for all $z \in B(z_0, r)$, where $g(z) \neq 0$ on $B(z_0, r)$. As implied in the above formula, one wonders if you can take the m^{th} root of $g(z)$.

$\frac{g'}{g}$ is an analytic function on $B(z_0, r)$ and so by Corollary 27.7.5 it has a primitive on $B(z_0, r)$, h . Therefore by the product rule and the chain rule, $(ge^{-h})' = 0$ and so there exists a constant, $C = e^{a+ib}$ such that on $B(z_0, r)$,

$$ge^{-h} = e^{a+ib}.$$

Therefore,

$$g(z) = e^{h(z)+a+ib}$$

and so, modifying h by adding in the constant, $a + ib$, $g(z) = e^{h(z)}$ where $h'(z) = \frac{g'(z)}{g(z)}$ on $B(z_0, r)$. Letting

$$\phi(z) = (z - z_0) e^{\frac{h(z)}{m}}$$

implies formula 28.1.1 is valid on $B(z_0, r)$. Now

$$\phi'(z_0) = e^{\frac{h(z_0)}{m}} \neq 0.$$

Shrinking r if necessary you can assume $\phi'(z) \neq 0$ on $B(z_0, r)$. Is there an open set, V contained in $B(z_0, r)$ such that ϕ maps V onto $B(0, \delta)$ for some $\delta > 0$?

Let $\phi(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Consider the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

where u, v are C^1 because ϕ is given to be analytic. The Jacobian of this map at $(x, y) \in B(z_0, r)$ is

$$\begin{vmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{vmatrix} = \begin{vmatrix} u_x(x, y) & -v_x(x, y) \\ v_x(x, y) & u_x(x, y) \end{vmatrix} \\ = u_x(x, y)^2 + v_x(x, y)^2 = |\phi'(z)|^2 \neq 0.$$

This follows from a use of the Cauchy Riemann equations. Also

$$\begin{pmatrix} u(x_0, y_0) \\ v(x_0, y_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore, by the inverse function theorem there exists an open set, V , containing z_0 and $\delta > 0$ such that $(u, v)^T$ maps V one to one onto $B(0, \delta)$. Thus ϕ is one to one onto $B(0, \delta)$ as claimed. Applying the same argument to other points, z of V and using the fact that $\phi'(z) \neq 0$ at these points, it follows ϕ maps open sets to open sets. In other words, ϕ^{-1} is continuous.

It also follows that ϕ^m maps V onto $B(0, \delta^m)$. Therefore, the formula 28.1.1 implies that f maps the open set, V , containing z_0 to an open set. This shows $f(\Omega)$ is an open set because z_0 was arbitrary. It is connected because f is continuous and Ω is connected. Thus $f(\Omega)$ is a region. It remains to verify that ϕ^{-1} is analytic on $B(0, \delta)$. Since ϕ^{-1} is continuous,

$$\lim_{\phi(z_1) \rightarrow \phi(z)} \frac{\phi^{-1}(\phi(z_1)) - \phi^{-1}(\phi(z))}{\phi(z_1) - \phi(z)} = \lim_{z_1 \rightarrow z} \frac{z_1 - z}{\phi(z_1) - \phi(z)} = \frac{1}{\phi'(z)}.$$

Therefore, ϕ^{-1} is analytic as claimed.

It only remains to verify the assertion about the case where f is one to one. If $m > 1$, then $e^{\frac{2\pi i}{m}} \neq 1$ and so for $z_1 \in V$,

$$e^{\frac{2\pi i}{m}} \phi(z_1) \neq \phi(z_1). \quad (28.1.2)$$

But $e^{\frac{2\pi i}{m}} \phi(z_1) \in B(0, \delta)$ and so there exists $z_2 \neq z_1$ (since ϕ is one to one) such that $\phi(z_2) = e^{\frac{2\pi i}{m}} \phi(z_1)$. But then

$$\phi(z_2)^m = \left(e^{\frac{2\pi i}{m}} \phi(z_1) \right)^m = \phi(z_1)^m$$

implying $f(z_2) = f(z_1)$ contradicting the assumption that f is one to one. Thus $m = 1$ and $f'(z) = \phi'(z) \neq 0$ on V . Since f maps open sets to open sets, it follows that f^{-1} is continuous and so

$$\begin{aligned} (f^{-1})'(f(z)) &= \lim_{f(z_1) \rightarrow f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\ &= \lim_{z_1 \rightarrow z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}. \end{aligned}$$

This proves the theorem.

One does not have to look very far to find that this sort of thing does not hold for functions mapping $\mathbb{R}$ to $\mathbb{R}$. Take for example, the function $f(x) = x^2$. Then $f(\mathbb{R})$ is neither a point nor a region. In fact $f(\mathbb{R})$ fails to be open.

Corollary 28.1.2 *Suppose in the situation of Theorem 28.1.1 $m > 1$ for the local representation of f given in this theorem. Then there exists $\delta > 0$ such that if $w \in B(f(z_0), \delta) = f(V)$ for V an open set containing z_0 , then $f^{-1}(w)$ consists of m distinct points in V . (f is m to one on V)*

Proof: Let $w \in B(f(z_0), \delta)$. Then $w = f(\hat{z})$ where $\hat{z} \in V$. Thus $f(\hat{z}) = f(z_0) + \phi(\hat{z})^m$. Consider the m distinct numbers, $\left\{ e^{\frac{2k\pi i}{m}} \phi(\hat{z}) \right\}_{k=1}^m$. Then each of these numbers is in $B(0, \delta)$ and so since ϕ maps V one to one onto $B(0, \delta)$, there are m distinct numbers in V , $\{z_k\}_{k=1}^m$ such that $\phi(z_k) = e^{\frac{2k\pi i}{m}} \phi(\hat{z})$. Then

$$\begin{aligned} f(z_k) &= f(z_0) + \phi(z_k)^m = f(z_0) + \left(e^{\frac{2k\pi i}{m}} \phi(\hat{z}) \right)^m \\ &= f(z_0) + e^{2k\pi i} \phi(\hat{z})^m = f(z_0) + \phi(\hat{z})^m = f(\hat{z}) = w \end{aligned}$$

This proves the corollary.

28.2 Branches Of The Logarithm

The argument used in to prove the next theorem was used in the proof of the open mapping theorem. It is a very important result and deserves to be stated as a theorem.

Theorem 28.2.1 *Let Ω be a simply connected region and suppose $f : \Omega \rightarrow \mathbb{C}$ is analytic and nonzero on Ω . Then there exists an analytic function, g such that $e^{g(z)} = f(z)$ for all $z \in \Omega$.*

Proof: The function, f'/f is analytic on Ω and so by Corollary 27.7.23 there is a primitive for f'/f , denoted as g_1 . Then

$$(e^{-g_1} f)' = -\frac{f'}{f} e^{-g_1} f + e^{-g_1} f' = 0$$

and so since Ω is connected, it follows $e^{-g_1} f$ equals a constant, e^{a+ib} . Therefore, $f(z) = e^{g_1(z)+a+ib}$. Define $g(z) \equiv g_1(z) + a + ib$.

The function, g in the above theorem is called a branch of the logarithm of f and is written as $\log(f(z))$.

Definition 28.2.2 *Let ρ be a ray starting at 0. Thus ρ is a straight line of infinite length extending in one direction with its initial point at 0.*

A special case of the above theorem is the following.

Theorem 28.2.3 *Let ρ be a ray starting at 0. Then there exists an analytic function, $L(z)$ defined on $\mathbb{C} \setminus \rho$ such that*

$$e^{L(z)} = z.$$

This function, L is called a branch of the logarithm. This branch of the logarithm satisfies the usual formula for logarithms, $L(zw) = L(z) + L(w)$ provided $zw \notin \rho$.

Proof: $\mathbb{C} \setminus \rho$ is a simply connected region because its complement with respect to $\hat{\mathbb{C}}$ is connected. Furthermore, the function, $f(z) = z$ is not equal to zero on $\mathbb{C} \setminus \rho$. Therefore, by Theorem 28.2.1 there exists an analytic function $L(z)$ such that $e^{L(z)} = f(z) = z$. Now consider the problem of finding a description of $L(z)$. Each $z \in \mathbb{C} \setminus \rho$ can be written in a unique way in the form

$$z = |z| e^{i \arg_{\theta}(z)}$$

where $\arg_{\theta}(z)$ is the angle in $(\theta, \theta + 2\pi)$ associated with z . (You could of course have considered this to be the angle in $(\theta - 2\pi, \theta)$ associated with z or in infinitely many other open intervals of length 2π . The description of the log is not unique.) Then letting $L(z) = a + ib$

$$z = |z| e^{i \arg_{\theta}(z)} = e^{L(z)} = e^a e^{ib}$$

and so you can let $L(z) = \ln |z| + i \arg_{\theta}(z)$.

Does $L(z)$ satisfy the usual properties of the logarithm? That is, for $z, w \in \mathbb{C} \setminus \rho$, is $L(zw) = L(z) + L(w)$? This follows from the usual rules of exponents. You know $e^{z+w} = e^z e^w$. (You can verify this directly or you can reduce to the case where z, w are real. If z is a fixed real number, then the equation holds for all real w . Therefore,

it must also hold for all complex w because the real line contains a limit point. Now for this fixed w , the equation holds for all z real. Therefore, by similar reasoning, it holds for all complex z .)

Now suppose $z, w \in \mathbb{C} \setminus \rho$ and $zw \notin \rho$. Then

$$e^{L(zw)} = zw, \quad e^{L(z)+L(w)} = e^{L(z)}e^{L(w)} = zw$$

and so $L(zw) = L(z) + L(w)$ as claimed. This proves the theorem.

In the case where the ray is the negative real axis, it is called the principal branch of the logarithm. Thus $\arg(z)$ is a number between $-\pi$ and π .

Definition 28.2.4 Let $\log$ denote the branch of the logarithm which corresponds to the ray for $\theta = \pi$. That is, the ray is the negative real axis. Sometimes this is called the principal branch of the logarithm.

28.3 Maximum Modulus Theorem

Here is another very significant theorem known as the maximum modulus theorem which follows immediately from the open mapping theorem.

Theorem 28.3.1 (maximum modulus theorem) Let Ω be a bounded region and let $f : \Omega \rightarrow \mathbb{C}$ be analytic and $f : \bar{\Omega} \rightarrow \mathbb{C}$ continuous. Then if $z \in \Omega$,

$$|f(z)| \leq \max \{|f(w)| : w \in \partial\Omega\}. \quad (28.3.3)$$

If equality is achieved for any $z \in \Omega$, then f is a constant.

Proof: Suppose f is not a constant. Then $f(\Omega)$ is a region and so if $z \in \Omega$, there exists $r > 0$ such that $B(f(z), r) \subseteq f(\Omega)$. It follows there exists $z_1 \in \Omega$ with $|f(z_1)| > |f(z)|$. Hence $\max \{|f(w)| : w \in \bar{\Omega}\}$ is not achieved at any interior point of Ω . Therefore, the point at which the maximum is achieved must lie on the boundary of Ω and so

$$\max \{|f(w)| : w \in \partial\Omega\} = \max \{|f(w)| : w \in \bar{\Omega}\} > |f(z)|$$

for all $z \in \Omega$ or else f is a constant. This proves the theorem.

You can remove the assumption that Ω is bounded and give a slightly different version.

Theorem 28.3.2 Let $f : \Omega \rightarrow \mathbb{C}$ be analytic on a region, Ω and suppose $\overline{B(a, r)} \subseteq \Omega$. Then

$$|f(a)| \leq \max \{|f(a + re^{i\theta})| : \theta \in [0, 2\pi]\}.$$

Equality occurs for some $r > 0$ and $a \in \Omega$ if and only if f is constant in Ω hence equality occurs for all such a, r .

Proof: The claimed inequality holds by Theorem 28.3.1. Suppose equality in the above is achieved for some $\overline{B(a, r)} \subseteq \Omega$. Then by Theorem 28.3.1 f is equal to a constant, w on $B(a, r)$. Therefore, the function, $f(\cdot) - w$ has a zero set which has a limit point in Ω and so by Theorem 27.5.3 $f(z) = w$ for all $z \in \Omega$.

Conversely, if f is constant, then the equality in the above inequality is achieved for all $\overline{B(a, r)} \subseteq \Omega$.

Next is yet another version of the maximum modulus principle which is in Conway [16]. Let Ω be an open set.

Definition 28.3.3 Define $\partial_\infty \Omega$ to equal $\partial \Omega$ in the case where Ω is bounded and $\partial \Omega \cup \{\infty\}$ in the case where Ω is not bounded.

Definition 28.3.4 Let f be a complex valued function defined on a set $S \subseteq \mathbb{C}$ and let a be a limit point of S .

$$\limsup_{z \rightarrow a} |f(z)| \equiv \lim_{r \rightarrow 0} \{\sup |f(w)| : w \in B'(a, r) \cap S\}.$$

The limit exists because $\{\sup |f(w)| : w \in B'(a, r) \cap S\}$ is decreasing in r . In case $a = \infty$,

$$\limsup_{z \rightarrow \infty} |f(z)| \equiv \lim_{r \rightarrow \infty} \{\sup |f(w)| : |w| > r, w \in S\}$$

Note that if $\limsup_{z \rightarrow a} |f(z)| \leq M$ and $\delta > 0$, then there exists $r > 0$ such that if $z \in B'(a, r) \cap S$, then $|f(z)| < M + \delta$. If $a = \infty$, there exists $r > 0$ such that if $|z| > r$ and $z \in S$, then $|f(z)| < M + \delta$.

Theorem 28.3.5 Let Ω be an open set in $\mathbb{C}$ and let $f : \Omega \rightarrow \mathbb{C}$ be analytic. Suppose also that for every $a \in \partial_\infty \Omega$,

$$\limsup_{z \rightarrow a} |f(z)| \leq M < \infty.$$

Then in fact $|f(z)| \leq M$ for all $z \in \Omega$.

Proof: Let $\delta > 0$ and let $H \equiv \{z \in \Omega : |f(z)| > M + \delta\}$. Suppose $H \neq \emptyset$. Then H is an open subset of Ω . I claim that H is actually bounded. If Ω is bounded, there is nothing to show so assume Ω is unbounded. Then the condition involving the limsup implies there exists $r > 0$ such that if $|z| > r$ and $z \in \Omega$, then $|f(z)| \leq M + \delta/2$. It follows H is contained in $\overline{B(0, r)}$ and so it is bounded. Now consider the components of Ω . One of these components contains points from H . Let this component be denoted as V and let $H_V \equiv H \cap V$. Thus H_V is a bounded open subset of V . Let U be a component of H_V . First suppose $\overline{U} \subseteq V$. In this case, it follows that on ∂U , $|f(z)| = M + \delta$ and so by Theorem 28.3.1 $|f(z)| \leq M + \delta$ for all $z \in U$ contradicting the definition of H . Next suppose ∂U contains a point of $\partial V, a$. Then in this case, a violates the condition on limsup. Either way you get a contradiction. Hence $H = \emptyset$ as claimed. Since $\delta > 0$ is arbitrary, this shows $|f(z)| \leq M$.

28.4 Extensions Of Maximum Modulus Theorem

28.4.1 Phragmén Lindelöf Theorem

This theorem is an extension of Theorem 28.3.5. It uses a growth condition near the extended boundary to conclude that f is bounded. I will present the version found in Conway [16]. It seems to be more of a method than an actual theorem. There are several versions of it.

Theorem 28.4.1 *Let Ω be a simply connected region in $\mathbb{C}$ and suppose f is analytic on Ω . Also suppose there exists a function, ϕ which is nonzero and uniformly bounded on Ω . Let M be a positive number. Now suppose $\partial_\infty \Omega = A \cup B$ such that for every $a \in A$, $\limsup_{z \rightarrow a} |f(z)| \leq M$ and for every $b \in B$, and $\eta > 0$, $\limsup_{z \rightarrow b} |f(z)| |\phi(z)|^\eta \leq M$. Then $|f(z)| \leq M$ for all $z \in \Omega$.*

Proof: By Theorem 28.2.1 there exists $\log(\phi(z))$ analytic on Ω . Now define $g(z) \equiv \exp(\eta \log(\phi(z)))$ so that $g(z) = \phi(z)^\eta$. Now also

$$|g(z)| = |\exp(\eta \log(\phi(z)))| = |\exp(\eta \ln |\phi(z)|)| = |\phi(z)|^\eta.$$

Let $m \geq |\phi(z)|$ for all $z \in \Omega$. Define $F(z) \equiv f(z)g(z)m^{-\eta}$. Thus F is analytic and for $b \in B$,

$$\limsup_{z \rightarrow b} |F(z)| = \limsup_{z \rightarrow b} |f(z)| |\phi(z)|^\eta m^{-\eta} \leq M m^{-\eta}$$

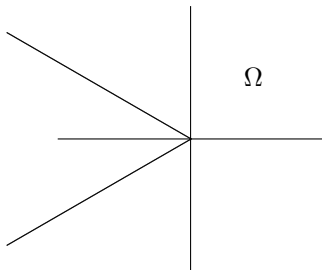
while for $a \in A$,

$$\limsup_{z \rightarrow a} |F(z)| \leq M.$$

Therefore, for $\alpha \in \partial_\infty \Omega$, $\limsup_{z \rightarrow \alpha} |F(z)| \leq \max(M, M\eta^{-\eta})$ and so by Theorem 28.3.5, $|f(z)| \leq \left(\frac{m^\eta}{|\phi(z)|^\eta}\right) \max(M, M\eta^{-\eta})$. Now let $\eta \rightarrow 0$ to obtain $|f(z)| \leq M$.

In applications, it is often the case that $B = \{\infty\}$.

Now here is an interesting case of this theorem. It involves a particular form for Ω , in this case $\Omega = \{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\}$ where $a \geq \frac{1}{2}$.



Then $\partial\Omega$ equals the two slanted lines. Also on Ω you can define a logarithm, $\log(z) = \ln|z| + i \arg(z)$ where $\arg(z)$ is the angle associated with z between $-\pi$

and π . Therefore, if c is a real number you can define z^c for such z in the usual way:

$$\begin{aligned} z^c &\equiv \exp(c \log(z)) = \exp(c[\ln|z| + i \arg(z)]) \\ &= |z|^c \exp(ic \arg(z)) = |z|^c (\cos(c \arg(z)) + i \sin(c \arg(z))). \end{aligned}$$

If $|c| < a$, then $|c \arg(z)| < \frac{\pi}{2}$ and so $\cos(c \arg(z)) > 0$. Therefore, for such c ,

$$\begin{aligned} |\exp(-(z^c))| &= |\exp(-|z|^c (\cos(c \arg(z)) + i \sin(c \arg(z))))| \\ &= |\exp(-|z|^c \cos(c \arg(z)))| \end{aligned}$$

which is bounded since $\cos(c \arg(z)) > 0$.

Corollary 28.4.2 Let $\Omega = \{z \in \mathbb{C} : |\arg(z)| < \frac{\pi}{2a}\}$ where $a \geq \frac{1}{2}$ and suppose f is analytic on Ω and satisfies $\limsup_{z \rightarrow a} |f(z)| \leq M$ on $\partial\Omega$ and suppose there are positive constants, P, b where $b < a$ and

$$|f(z)| \leq P \exp(|z|^b)$$

for all $|z|$ large enough. Then $|f(z)| \leq M$ for all $z \in \Omega$.

Proof: Let $b < c < a$ and let $\phi(z) \equiv \exp(-(z^c))$. Then as discussed above, $\phi(z) \neq 0$ on Ω and $|\phi(z)|$ is bounded on Ω . Now

$$|\phi(z)|^\eta = |\exp(-|z|^c \eta (\cos(c \arg(z))))|$$

$$\limsup_{z \rightarrow \infty} |f(z)| |\phi(z)|^\eta \leq \limsup_{z \rightarrow \infty} \frac{P \exp(|z|^b)}{|\exp(|z|^c \eta (\cos(c \arg(z))))|} = 0 \leq M$$

and so by Theorem 28.4.1 $|f(z)| \leq M$.

The following is another interesting case. This case is presented in Rudin [58]

Corollary 28.4.3 Let Ω be the open set consisting of $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ and suppose f is analytic on Ω , continuous on $\overline{\Omega}$, and bounded on Ω . Suppose also that $f(z) \leq 1$ on the two lines $\operatorname{Re} z = a$ and $\operatorname{Re} z = b$. Then $|f(z)| \leq 1$ for all $z \in \Omega$.

Proof: This time let $\phi(z) = \frac{1}{1+z-a}$. Thus $|\phi(z)| \leq 1$ because $\operatorname{Re}(z-a) > 0$ and $\phi(z) \neq 0$ for all $z \in \Omega$. Also, $\limsup_{z \rightarrow \infty} |\phi(z)|^\eta = 0$ for every $\eta > 0$. Therefore, if a is a point of the sides of Ω , $\limsup_{z \rightarrow a} |f(z)| \leq 1$ while $\limsup_{z \rightarrow \infty} |f(z)| |\phi(z)|^\eta = 0 \leq 1$ and so by Theorem 28.4.1, $|f(z)| \leq 1$ on Ω .

This corollary yields an interesting conclusion.

Corollary 28.4.4 Let Ω be the open set consisting of $\{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ and suppose f is analytic on Ω , continuous on $\overline{\Omega}$, and bounded on Ω . Define

$$M(x) \equiv \sup \{|f(z)| : \operatorname{Re} z = x\}$$

Then for $x \in (a, b)$.

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}.$$

Proof: Let $\varepsilon > 0$ and define

$$g(z) \equiv (M(a) + \varepsilon)^{\frac{b-z}{b-a}} (M(b) + \varepsilon)^{\frac{z-a}{b-a}}$$

where for $M > 0$ and $z \in \mathbb{C}$, $M^z \equiv \exp(z \ln(M))$. Thus $g \neq 0$ and so f/g is analytic on Ω and continuous on $\bar{\Omega}$. Also on the left side,

$$\left| \frac{f(a+iy)}{g(a+iy)} \right| = \left| \frac{f(a+iy)}{(M(a) + \varepsilon)^{\frac{b-a-iy}{b-a}}} \right| = \left| \frac{f(a+iy)}{(M(a) + \varepsilon)^{\frac{b-a}{b-a}}} \right| \leq 1$$

while on the right side a similar computation shows $\left| \frac{f}{g} \right| \leq 1$ also. Therefore, by Corollary 28.4.3 $|f/g| \leq 1$ on Ω . Therefore, letting $x + iy = z$,

$$|f(z)| \leq \left| (M(a) + \varepsilon)^{\frac{b-z}{b-a}} (M(b) + \varepsilon)^{\frac{z-a}{b-a}} \right| = \left| (M(a) + \varepsilon)^{\frac{b-x}{b-a}} (M(b) + \varepsilon)^{\frac{x-a}{b-a}} \right|$$

and so

$$M(x) \leq (M(a) + \varepsilon)^{\frac{b-x}{b-a}} (M(b) + \varepsilon)^{\frac{x-a}{b-a}}.$$

Since $\varepsilon > 0$ is arbitrary, it yields the conclusion of the corollary.

Another way of saying this is that $x \rightarrow \ln(M(x))$ is a convex function.

This corollary has an interesting application known as the Hadamard three circles theorem.

28.4.2 Hadamard Three Circles Theorem

Let $0 < R_1 < R_2$ and suppose f is analytic on $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$. Then letting $R_1 < a < b < R_2$, note that $g(z) \equiv \exp(z)$ maps the strip $\{z \in \mathbb{C} : \ln a < \operatorname{Re} z < b\}$ onto $\{z \in \mathbb{C} : a < |z| < b\}$ and that in fact, g maps the line $\ln r + iy$ onto the circle $re^{i\theta}$. Now let $M(x)$ be defined as above and m be defined by

$$m(r) \equiv \max_{\theta} |f(re^{i\theta})|.$$

Then for $a < r < b$, Corollary 28.4.4 implies

$$\begin{aligned} m(r) &= \sup_y |f(e^{\ln r + iy})| = M(\ln r) \leq M(\ln a)^{\frac{\ln b - \ln r}{\ln b - \ln a}} M(\ln b)^{\frac{\ln r - \ln a}{\ln b - \ln a}} \\ &= m(a)^{\ln(b/r)/\ln(b/a)} m(b)^{\ln(r/a)/\ln(b/a)} \end{aligned}$$

and so

$$m(r)^{\ln(b/a)} \leq m(a)^{\ln(b/r)} m(b)^{\ln(r/a)}.$$

Taking logarithms, this yields

$$\ln\left(\frac{b}{a}\right) \ln(m(r)) \leq \ln\left(\frac{b}{r}\right) \ln(m(a)) + \ln\left(\frac{r}{a}\right) \ln(m(b))$$

which says the same as $r \rightarrow \ln(m(r))$ is a convex function of $\ln r$.

The next example, also in Rudin [58] is very dramatic. An unbelievably weak assumption is made on the growth of the function and still you get a uniform bound in the conclusion.

Corollary 28.4.5 Let $\Omega = \{z \in \mathbb{C} : |\operatorname{Im}(z)| < \frac{\pi}{2}\}$. Suppose f is analytic on Ω , continuous on $\overline{\Omega}$, and there exist constants, $\alpha < 1$ and $A < \infty$ such that

$$|f(z)| \leq \exp(A \exp(\alpha|x|)) \text{ for } z = x + iy$$

and

$$\left| f\left(x \pm i\frac{\pi}{2}\right) \right| \leq 1$$

for all $x \in \mathbb{R}$. Then $|f(z)| \leq 1$ on Ω .

Proof: This time let $\phi(z) = [\exp(A \exp(\beta z)) \exp(A \exp(-\beta z))]^{-1}$ where $\alpha < \beta < 1$. Then $\phi(z) \neq 0$ on Ω and for $\eta > 0$

$$|\phi(z)|^\eta = \frac{1}{|\exp(\eta A \exp(\beta z)) \exp(\eta A \exp(-\beta z))|}$$

Now

$$\begin{aligned} & \exp(\eta A \exp(\beta z)) \exp(\eta A \exp(-\beta z)) \\ &= \exp(\eta A (\exp(\beta z) + \exp(-\beta z))) \\ &= \exp[\eta A (\cos(\beta y) (e^{\beta x} + e^{-\beta x}) + i \sin(\beta y) (e^{\beta x} - e^{-\beta x}))] \end{aligned}$$

and so

$$|\phi(z)|^\eta = \frac{1}{\exp[\eta A (\cos(\beta y) (e^{\beta x} + e^{-\beta x}))]}$$

Now $\cos \beta y > 0$ because $\beta < 1$ and $|y| < \frac{\pi}{2}$. Therefore,

$$\limsup_{z \rightarrow \infty} |f(z)| |\phi(z)|^\eta \leq 0 \leq 1$$

and so by Theorem 28.4.1, $|f(z)| \leq 1$.

28.4.3 Schwarz's Lemma

This interesting lemma comes from the maximum modulus theorem. It will be used later as part of the proof of the Riemann mapping theorem.

Lemma 28.4.6 Suppose $F : B(0, 1) \rightarrow B(0, 1)$, F is analytic, and $F(0) = 0$. Then for all $z \in B(0, 1)$,

$$|F(z)| \leq |z|, \quad (28.4.4)$$

and

$$|F'(0)| \leq 1. \quad (28.4.5)$$

If equality holds in 28.4.5 then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$F(z) = \lambda z. \quad (28.4.6)$$

Proof: First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,

$$\left| \frac{F(z)}{z} \right| \leq \max_{t \in [0, 2\pi]} \frac{|F(re^{it})|}{r} \leq \frac{1}{r}.$$

Then letting $r \rightarrow 1$,

$$\left| \frac{F(z)}{z} \right| \leq 1$$

this shows 28.4.4 and it also verifies 28.4.5 on taking the limit as $z \rightarrow 0$. If equality holds in 28.4.5, then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, λ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$.

Rudin [58] gives a memorable description of what this lemma says. It says that if an analytic function maps the unit ball to itself, keeping 0 fixed, then it must do one of two things, either be a rotation or move all points closer to 0. (This second part follows in case $|F'(0)| < 1$ because in this case, you must have $|F(z)| \neq |z|$ and so by 28.4.4, $|F(z)| < |z|$)

28.4.4 One To One Analytic Maps On The Unit Ball

The transformation in the next lemma is of fundamental importance.

Lemma 28.4.7 *Let $\alpha \in B(0, 1)$ and define*

$$\phi_\alpha(z) \equiv \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

Then $\phi_\alpha : B(0, 1) \rightarrow B(0, 1)$, $\phi_\alpha : \partial B(0, 1) \rightarrow \partial B(0, 1)$, and is one to one and onto. Also $\phi_{-\alpha} = \phi_\alpha^{-1}$. Also

$$\phi'_\alpha(0) = 1 - |\alpha|^2, \quad \phi'(\alpha) = \frac{1}{1 - |\alpha|^2}.$$

Proof: First of all, for $|z| < 1/|\alpha|$,

$$\phi_\alpha \circ \phi_{-\alpha}(z) \equiv \frac{\left(\frac{z + \alpha}{1 + \bar{\alpha}z} \right) - \alpha}{1 - \bar{\alpha} \left(\frac{z + \alpha}{1 + \bar{\alpha}z} \right)} = z$$

after a few computations. If I show that ϕ_α maps $B(0, 1)$ to $B(0, 1)$ for all $|\alpha| < 1$, this will have shown that ϕ_α is one to one and onto $B(0, 1)$.

Consider $|\phi_\alpha(e^{i\theta})|$. This yields

$$\left| \frac{e^{i\theta} - \alpha}{1 - \bar{\alpha}e^{i\theta}} \right| = \left| \frac{1 - \alpha e^{-i\theta}}{1 - \bar{\alpha}e^{i\theta}} \right| = 1$$

where the first equality is obtained by multiplying by $|e^{-i\theta}| = 1$. Therefore, ϕ_α maps $\partial B(0, 1)$ one to one and onto $\partial B(0, 1)$. Now notice that ϕ_α is analytic on $B(0, 1)$ because the only singularity, a pole is at $z = 1/\bar{\alpha}$. By the maximum modulus theorem, it follows

$$|\phi_\alpha(z)| < 1$$

whenever $|z| < 1$. The same is true of $\phi_{-\alpha}$.

It only remains to verify the assertions about the derivatives. Long division gives $\phi_\alpha(z) = (-\bar{\alpha})^{-1} + \left(\frac{-\alpha + (\bar{\alpha})^{-1}}{1 - \bar{\alpha}z}\right)$ and so

$$\begin{aligned}\phi'_\alpha(z) &= (-1)(1 - \bar{\alpha}z)^{-2} \left(-\alpha + (\bar{\alpha})^{-1}\right) (-\bar{\alpha}) \\ &= \bar{\alpha}(1 - \bar{\alpha}z)^{-2} \left(-\alpha + (\bar{\alpha})^{-1}\right) \\ &= (1 - \bar{\alpha}z)^{-2} \left(-|\alpha|^2 + 1\right)\end{aligned}$$

Hence the two formulas follow. This proves the lemma.

One reason these mappings are so important is the following theorem.

Theorem 28.4.8 *Suppose f is an analytic function defined on $B(0, 1)$ and f maps $B(0, 1)$ one to one and onto $B(0, 1)$. Then there exists θ such that*

$$f(z) = e^{i\theta} \phi_\alpha(z)$$

for some $\alpha \in B(0, 1)$.

Proof: Let $f(\alpha) = 0$. Then $h(z) \equiv f \circ \phi_{-\alpha}(z)$ maps $B(0, 1)$ one to one and onto $B(0, 1)$ and has the property that $h(0) = 0$. Therefore, by the Schwarz lemma,

$$|h(z)| \leq |z|.$$

but it is also the case that $h^{-1}(0) = 0$ and h^{-1} maps $B(0, 1)$ to $B(0, 1)$. Therefore, the same inequality holds for h^{-1} . Therefore,

$$|z| = |h^{-1}(h(z))| \leq |h(z)|$$

and so $|h(z)| = |z|$. By the Schwarz lemma again, $h(z) \equiv f(\phi_{-\alpha}(z)) = e^{i\theta}z$. Letting $z = \phi_\alpha$, you get $f(z) = e^{i\theta} \phi_\alpha(z)$.

28.5 Exercises

1. Consider the function, $g(z) = \frac{z-i}{z+i}$. Show this is analytic on the upper half plane, $P+$ and maps the upper half plane one to one and onto $B(0, 1)$. **Hint:** First show g maps the real axis to $\partial B(0, 1)$. This is really easy because you end up looking at a complex number divided by its conjugate. Thus $|g(z)| = 1$ for z on $\partial(P+)$. Now show that $\limsup_{z \rightarrow \infty} |g(z)| = 1$. Then apply a version of the maximum modulus theorem. You might note that $g(z) = 1 + \frac{-2i}{z+i}$. This will show $|g(z)| \leq 1$. Next pick $w \in B(0, 1)$ and solve $g(z) = w$. You just have to show there exists a unique solution and its imaginary part is positive.

2. Does there exist an entire function f which maps $\mathbb{C}$ onto the upper half plane?
3. Letting g be the function of Problem 1 show that $(g^{-1})'(0) = 2$. Also note that $g^{-1}(0) = i$. Now suppose f is an analytic function defined on the upper half plane which has the property that $|f(z)| \leq 1$ and $f(i) = \beta$ where $|\beta| < 1$. Find an upper bound to $|f'(i)|$. Also find all functions, f which satisfy the condition, $f(i) = \beta, |f(z)| \leq 1$, and achieve this maximum value. **Hint:** You could consider the function, $h(z) \equiv \phi_\beta \circ f \circ g^{-1}(z)$ and check the conditions for the Schwarz lemma for this function, h .
4. This and the next two problems follow a presentation of an interesting topic in Rudin [58]. Let ϕ_α be given in Lemma 28.4.7. Suppose f is an analytic function defined on $B(0, 1)$ which satisfies $|f(z)| \leq 1$. Suppose also there are $\alpha, \beta \in B(0, 1)$ and it is required $f(\alpha) = \beta$. If f is such a function, show that $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** To show this consider $g = \phi_\beta \circ f \circ \phi_{-\alpha}$. Show $g(0) = 0$ and $|g(z)| \leq 1$ on $B(0, 1)$. Now use Lemma 28.4.6.
5. In Problem 4 show there exists a function, f analytic on $B(0, 1)$ such that $f(\alpha) = \beta, |f(z)| \leq 1$, and $|f'(\alpha)| = \frac{1-|\beta|^2}{1-|\alpha|^2}$. **Hint:** You do this by choosing g in the above problem such that equality holds in Lemma 28.4.6. Thus you need $g(z) = \lambda z$ where $|\lambda| = 1$ and solve $g = \phi_\beta \circ f \circ \phi_{-\alpha}$ for f .
6. Suppose that $f : B(0, 1) \rightarrow B(0, 1)$ and that f is analytic, one to one, and onto with $f(\alpha) = 0$. Show there exists $\lambda, |\lambda| = 1$ such that $f(z) = \lambda \phi_\alpha(z)$. This gives a different way to look at Theorem 28.4.8. **Hint:** Let $g = f^{-1}$. Then $g'(0)f'(\alpha) = 1$. However, $f(\alpha) = 0$ and $g(0) = \alpha$. From Problem 4 with $\beta = 0$, you can conclude an inequality for $|f'(\alpha)|$ and another one for $|g'(0)|$. Then use the fact that the product of these two equals 1 which comes from the chain rule to conclude that equality must take place. Now use Problem 5 to obtain the form of f .
7. In Corollary 28.4.5 show that it is essential that $\alpha < 1$. That is, show there exists an example where the conclusion is not satisfied with a slightly weaker growth condition. **Hint:** Consider $\exp(\exp(z))$.
8. Suppose $\{f_n\}$ is a sequence of functions which are analytic on Ω , a bounded region such that each f_n is also continuous on $\bar{\Omega}$. Suppose that $\{f_n\}$ converges uniformly on $\partial\Omega$. Show that then $\{f_n\}$ converges uniformly on $\bar{\Omega}$ and that the function to which the sequence converges is analytic on Ω and continuous on $\bar{\Omega}$.
9. Suppose Ω is a bounded region and there exists a point $z_0 \in \Omega$ such that $|f(z_0)| = \min\{|f(z)| : z \in \bar{\Omega}\}$. Can you conclude f must equal a constant?
10. Suppose f is continuous on $\bar{B}(a, r)$ and analytic on $B(a, r)$ and that f is not constant. Suppose also $|f(z)| = C \neq 0$ for all $|z - a| = r$. Show that there exists $\alpha \in B(a, r)$ such that $f(\alpha) = 0$. **Hint:** If not, consider f/C and C/f . Both would be analytic on $B(a, r)$ and are equal to 1 on the boundary.

11. Suppose f is analytic on $B(0, 1)$ but for every $a \in \partial B(0, 1)$, $\lim_{z \rightarrow a} |f(z)| = \infty$. Show there exists a sequence, $\{z_n\} \subseteq B(0, 1)$ such that $\lim_{n \rightarrow \infty} |z_n| = 1$ and $f(z_n) = 0$.

28.6 Counting Zeros

The above proof of the open mapping theorem relies on the very important inverse function theorem from real analysis. There are other approaches to this important theorem which do not rely on the big theorems from real analysis and are more oriented toward the use of the Cauchy integral formula and specialized techniques from complex analysis. One of these approaches is given next which involves the notion of “counting zeros”. The next theorem is the one about counting zeros. It will also be used later in the proof of the Riemann mapping theorem.

Theorem 28.6.1 *Let Ω be an open set in $\mathbb{C}$ and let $\gamma : [a, b] \rightarrow \Omega$ be closed, continuous, bounded variation, and $n(\gamma, z) = 0$ for all $z \notin \Omega$. Suppose also that f is analytic on Ω having zeros $a_1, \dots, a_m$ where the zeros are repeated according to multiplicity, and suppose that none of these zeros are on γ^* . Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: Let $f(z) = \prod_{j=1}^m (z - a_j) g(z)$ where $g(z) \neq 0$ on Ω . Hence

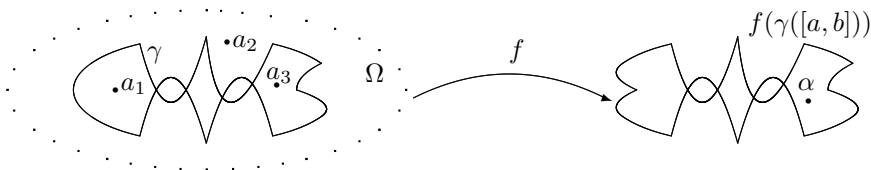
$$\frac{f'(z)}{f(z)} = \sum_{j=1}^m \frac{1}{z - a_j} + \frac{g'(z)}{g(z)}$$

and so

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^m n(\gamma, a_j) + \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} dz.$$

But the function, $z \rightarrow \frac{g'(z)}{g(z)}$ is analytic and so by Corollary 27.7.20, the last integral in the above expression equals 0. Therefore, this proves the theorem.

The following picture is descriptive of the situation described in the next theorem.



Theorem 28.6.2 *Let Ω be a region, let $\gamma : [a, b] \rightarrow \Omega$ be closed continuous, and bounded variation such that $n(\gamma, z) = 0$ for all $z \notin \Omega$. Also suppose $f : \Omega \rightarrow \mathbb{C}$ is*

analytic and that $\alpha \notin f(\gamma^*)$. Then $f \circ \gamma : [a, b] \rightarrow \mathbb{C}$ is continuous, closed, and bounded variation. Also suppose $\{a_1, \dots, a_m\} = f^{-1}(\alpha)$ where these points are counted according to their multiplicities as zeros of the function $f - \alpha$. Then

$$n(f \circ \gamma, \alpha) = \sum_{k=1}^m n(\gamma, a_k).$$

Proof: It is clear that $f \circ \gamma$ is continuous. It only remains to verify that it is of bounded variation. Suppose first that $\gamma^* \subseteq B \subseteq \overline{B} \subseteq \Omega$ where B is a ball. Then

$$|f(\gamma(t)) - f(\gamma(s))| =$$

$$\left| \int_0^1 f'(\gamma(s) + \lambda(\gamma(t) - \gamma(s))) (\gamma(t) - \gamma(s)) d\lambda \right|$$

$$\leq C |\gamma(t) - \gamma(s)|$$

where $C \geq \max \{|f'(z)| : z \in \overline{B}\}$. Hence, in this case,

$$V(f \circ \gamma, [a, b]) \leq CV(\gamma, [a, b]).$$

Now let ε denote the distance between γ^* and $\mathbb{C} \setminus \Omega$. Since γ^* is compact, $\varepsilon > 0$. By uniform continuity there exists $\delta = \frac{b-a}{p}$ for p a positive integer such that if $|s - t| < \delta$, then $|\gamma(s) - \gamma(t)| < \frac{\varepsilon}{2}$. Then

$$\gamma([t, t + \delta]) \subseteq \overline{B\left(\gamma(t), \frac{\varepsilon}{2}\right)} \subseteq \Omega.$$

Let $C \geq \max \left\{ |f'(z)| : z \in \bigcup_{j=1}^p \overline{B\left(\gamma(t_j), \frac{\varepsilon}{2}\right)} \right\}$ where $t_j \equiv \frac{j}{p}(b-a) + a$. Then from what was just shown,

$$\begin{aligned} V(f \circ \gamma, [a, b]) &\leq \sum_{j=0}^{p-1} V(f \circ \gamma, [t_j, t_{j+1}]) \\ &\leq C \sum_{j=0}^{p-1} V(\gamma, [t_j, t_{j+1}]) < \infty \end{aligned}$$

showing that $f \circ \gamma$ is bounded variation as claimed. Now from Theorem 27.7.15 there exists $\eta \in C^1([a, b])$ such that

$$\eta(a) = \gamma(a) = \gamma(b) = \eta(b), \quad \eta([a, b]) \subseteq \Omega,$$

and

$$n(\eta, a_k) = n(\gamma, a_k), \quad n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha) \quad (28.6.7)$$

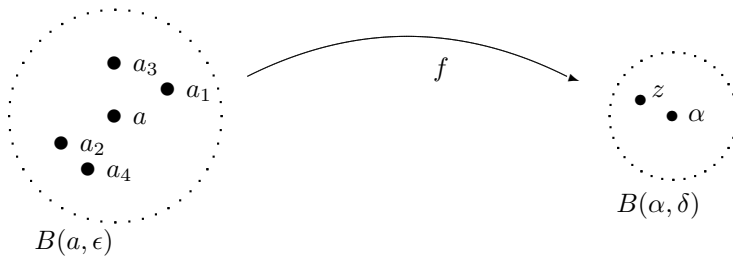
for $k = 1, \dots, m$. Then

$$n(f \circ \gamma, \alpha) = n(f \circ \eta, \alpha)$$

$$\begin{aligned}
&= \frac{1}{2\pi i} \int_{f \circ \eta} \frac{dw}{w - \alpha} \\
&= \frac{1}{2\pi i} \int_a^b \frac{f'(\eta(t))}{f(\eta(t)) - \alpha} \eta'(t) dt \\
&= \frac{1}{2\pi i} \int_{\eta} \frac{f'(z)}{f(z) - \alpha} dz \\
&= \sum_{k=1}^m n(\eta, a_k)
\end{aligned}$$

By Theorem 28.6.1. By 28.6.7, this equals $\sum_{k=1}^m n(\gamma, a_k)$ which proves the theorem.

The next theorem is incredible and is very interesting for its own sake. The following picture is descriptive of the situation of this theorem.



Theorem 28.6.3 Let $f : B(a, R) \rightarrow \mathbb{C}$ be analytic and let

$$f(z) - \alpha = (z - a)^m g(z), \quad \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a, R)$. ($f(z) - \alpha$ has a zero of order m at $z = a$.) Then there exist $\varepsilon, \delta > 0$ with the property that for each z satisfying $0 < |z - \alpha| < \delta$, there exist points,

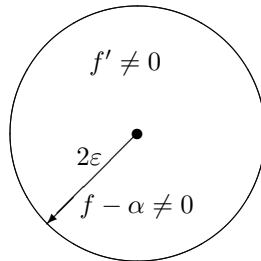
$$\{a_1, \dots, a_m\} \subseteq B(a, \varepsilon),$$

such that

$$f^{-1}(z) \cap B(a, \varepsilon) = \{a_1, \dots, a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot) - z$.

Proof: By Theorem 27.5.3 f is not constant on $B(a, R)$ because it has a zero of order m . Therefore, using this theorem again, there exists $\varepsilon > 0$ such that $\overline{B(a, 2\varepsilon)} \subseteq B(a, R)$ and there are no solutions to the equation $f(z) - \alpha = 0$ for $z \in \overline{B(a, 2\varepsilon)}$ except a . Also assume ε is small enough that for $0 < |z - a| \leq 2\varepsilon$, $f'(z) \neq 0$. This can be done since otherwise, a would be a limit point of a sequence of points, z_n , having $f'(z_n) = 0$ which would imply, by Theorem 27.5.3 that $f' = 0$ on $B(a, R)$, contradicting the assumption that $f - \alpha$ has a zero of order m and is therefore not constant. Thus the situation is described by the following picture.



Now pick $\gamma(t) = a + \varepsilon e^{it}$, $t \in [0, 2\pi]$. Then $\alpha \notin f(\gamma^*)$ so there exists $\delta > 0$ with

$$B(\alpha, \delta) \cap f(\gamma^*) = \emptyset. \quad (28.6.8)$$

Therefore, $B(\alpha, \delta)$ is contained on one component of $\mathbb{C} \setminus f(\gamma([0, 2\pi]))$. Therefore, $n(f \circ \gamma, \alpha) = n(f \circ \gamma, z)$ for all $z \in B(\alpha, \delta)$. Now consider f restricted to $B(a, 2\varepsilon)$. For $z \in B(\alpha, \delta)$, $f^{-1}(z)$ must consist of a finite set of points because $f'(w) \neq 0$ for all w in $\overline{B(a, 2\varepsilon)} \setminus \{a\}$ implying that the zeros of $f(\cdot) - z$ in $\overline{B(a, 2\varepsilon)}$ have no limit point. Since $\overline{B(a, 2\varepsilon)}$ is compact, this means there are only finitely many. By Theorem 28.6.2,

$$n(f \circ \gamma, z) = \sum_{k=1}^p n(\gamma, a_k) \quad (28.6.9)$$

where $\{a_1, \dots, a_p\} = f^{-1}(z)$. Each point, a_k of $f^{-1}(z)$ is either inside the circle traced out by γ , yielding $n(\gamma, a_k) = 1$, or it is outside this circle yielding $n(\gamma, a_k) = 0$ because of 28.6.8. It follows the sum in 28.6.9 reduces to the number of points of $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$. Thus, letting those points in $f^{-1}(z)$ which are contained in $B(a, \varepsilon)$ be denoted by $\{a_1, \dots, a_r\}$

$$n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r.$$

Also, by Theorem 28.6.1, $m = n(f \circ \gamma, \alpha)$ because a is a zero of $f - \alpha$ of order m . Therefore, for $z \in B(\alpha, \delta)$

$$m = n(f \circ \gamma, \alpha) = n(f \circ \gamma, z) = r$$

It is required to show $r = m$, the order of the zero of $f - \alpha$. Therefore, $r = m$. Each of these a_k is a zero of order 1 of the function $f(\cdot) - z$ because $f'(a_k) \neq 0$. This proves the theorem.

This is a very fascinating result partly because it implies that for values of f near a value, α , at which $f(\cdot) - \alpha$ has a zero of order m for $m > 1$, the inverse image of these values includes at least m points, not just one. Thus the topological properties of the inverse image changes radically. This theorem also shows that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$.

Theorem 28.6.4 (open mapping theorem) *Let Ω be a region and $f : \Omega \rightarrow \mathbb{C}$ be analytic. Then $f(\Omega)$ is either a point or a region. If f is one to one, then $f^{-1} : f(\Omega) \rightarrow \Omega$ is analytic.*

Proof: If f is not constant, then for every $\alpha \in f(\Omega)$, it follows from Theorem 27.5.3 that $f(\cdot) - \alpha$ has a zero of order $m < \infty$ and so from Theorem 28.6.3 for each $a \in \Omega$ there exist $\varepsilon, \delta > 0$ such that $f(B(a, \varepsilon)) \supseteq B(\alpha, \delta)$ which clearly implies that f maps open sets to open sets. Therefore, $f(\Omega)$ is open, connected because f is continuous. If f is one to one, Theorem 28.6.3 implies that for every $\alpha \in f(\Omega)$ the zero of $f(\cdot) - \alpha$ is of order 1. Otherwise, that theorem implies that for z near α , there are m points which f maps to z contradicting the assumption that f is one to one. Therefore, $f'(z) \neq 0$ and since f^{-1} is continuous, due to f being an open map, it follows

$$\begin{aligned} (f^{-1})'(f(z)) &= \lim_{f(z_1) \rightarrow f(z)} \frac{f^{-1}(f(z_1)) - f^{-1}(f(z))}{f(z_1) - f(z)} \\ &= \lim_{z_1 \rightarrow z} \frac{z_1 - z}{f(z_1) - f(z)} = \frac{1}{f'(z)}. \end{aligned}$$

This proves the theorem.

28.7 An Application To Linear Algebra

Gerschgorin's theorem gives a convenient way to estimate eigenvalues of a matrix from easy to obtain information. For A an $n \times n$ matrix, denote by $\sigma(A)$ the collection of all eigenvalues of A .

Theorem 28.7.1 *Let A be an $n \times n$ matrix. Consider the n Gerschgorin discs defined as*

$$D_i \equiv \left\{ \lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}.$$

Then every eigenvalue is contained in some Gerschgorin disc.

This theorem says to add up the absolute values of the entries of the i^{th} row which are off the main diagonal and form the disc centered at a_{ii} having this radius. The union of these discs contains $\sigma(A)$.

Proof: Suppose $A\mathbf{x} = \lambda\mathbf{x}$ where $\mathbf{x} \neq \mathbf{0}$. Then for $A = (a_{ij})$

$$\sum_{j \neq i} a_{ij}x_j = (\lambda - a_{ii})x_i.$$

Therefore, if we pick k such that $|x_k| \geq |x_j|$ for all x_j , it follows that $|x_k| \neq 0$ since $|\mathbf{x}| \neq 0$ and

$$|x_k| \sum_{j \neq k} |a_{kj}| \geq \sum_{j \neq k} |a_{kj}| |x_j| \geq |\lambda - a_{kk}| |x_k|.$$

Now dividing by $|x_k|$ we see that λ is contained in the k^{th} Gerschgorin disc.

More can be said using the theory about counting zeros. To begin with the distance between two $n \times n$ matrices, $A = (a_{ij})$ and $B = (b_{ij})$ as follows.

$$\|A - B\|^2 \equiv \sum_{ij} |a_{ij} - b_{ij}|^2.$$

Thus two matrices are close if and only if their corresponding entries are close.

Let A be an $n \times n$ matrix. Recall the eigenvalues of A are given by the zeros of the polynomial, $p_A(z) = \det(zI - A)$ where I is the $n \times n$ identity. Then small changes in A will produce small changes in $p_A(z)$ and $p'_A(z)$. Let γ_k denote a very small closed circle which winds around z_k , one of the eigenvalues of A , in the counter clockwise direction so that $n(\gamma_k, z_k) = 1$. This circle is to enclose only z_k and is to have no other eigenvalue on it. Then apply Theorem 28.6.1. According to this theorem

$$\frac{1}{2\pi i} \int_{\gamma} \frac{p'_A(z)}{p_A(z)} dz$$

is always an integer equal to the multiplicity of z_k as a root of $p_A(t)$. Therefore, small changes in A result in no change to the above contour integral because it must be an integer and small changes in A result in small changes in the integral. Therefore whenever every entry of the matrix B is close enough to the corresponding entry of the matrix A , the two matrices have the same number of zeros inside γ_k under the usual convention that zeros are to be counted according to multiplicity. By making the radius of the small circle equal to ε where ε is less than the minimum distance between any two distinct eigenvalues of A , this shows that if B is close enough to A , every eigenvalue of B is closer than ε to some eigenvalue of A . The next theorem is about continuous dependence of eigenvalues.

Theorem 28.7.2 *If λ is an eigenvalue of A , then if $\|B - A\|$ is small enough, some eigenvalue of B will be within ε of λ .*

Consider the situation that $A(t)$ is an $n \times n$ matrix and that $t \rightarrow A(t)$ is continuous for $t \in [0, 1]$.

Lemma 28.7.3 *Let $\lambda(t) \in \sigma(A(t))$ for $t < 1$ and let $\Sigma_t = \cup_{s \geq t} \sigma(A(s))$. Also let K_t be the connected component of $\lambda(t)$ in Σ_t . Then there exists $\eta > 0$ such that $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t + \eta]$.*

Proof: Denote by $D(\lambda(t), \delta)$ the disc centered at $\lambda(t)$ having radius $\delta > 0$, with other occurrences of this notation being defined similarly. Thus

$$D(\lambda(t), \delta) \equiv \{z \in \mathbb{C} : |\lambda(t) - z| \leq \delta\}.$$

Suppose $\delta > 0$ is small enough that $\lambda(t)$ is the only element of $\sigma(A(t))$ contained in $D(\lambda(t), \delta)$ and that $p_{A(t)}$ has no zeroes on the boundary of this disc. Then by continuity, and the above discussion and theorem, there exists $\eta > 0$, $t + \eta < 1$, such that for $s \in [t, t + \eta]$, $p_{A(s)}$ also has no zeroes on the boundary of this disc and that

$A(s)$ has the same number of eigenvalues, counted according to multiplicity, in the disc as $A(t)$. Thus $\sigma(A(s)) \cap D(\lambda(t), \delta) \neq \emptyset$ for all $s \in [t, t + \eta]$. Now let

$$H = \bigcup_{s \in [t, t + \eta]} \sigma(A(s)) \cap D(\lambda(t), \delta).$$

I will show H is connected. Suppose not. Then $H = P \cup Q$ where P, Q are separated and $\lambda(t) \in P$. Let

$$s_0 \equiv \inf \{s : \lambda(s) \in Q \text{ for some } \lambda(s) \in \sigma(A(s))\}.$$

There exists $\lambda(s_0) \in \sigma(A(s_0)) \cap D(\lambda(t), \delta)$. If $\lambda(s_0) \notin Q$, then from the above discussion there are

$$\lambda(s) \in \sigma(A(s)) \cap Q$$

for $s > s_0$ arbitrarily close to $\lambda(s_0)$. Therefore, $\lambda(s_0) \in Q$ which shows that $s_0 > t$ because $\lambda(t)$ is the only element of $\sigma(A(t))$ in $D(\lambda(t), \delta)$ and $\lambda(t) \in P$. Now let $s_n \uparrow s_0$. Then $\lambda(s_n) \in P$ for any

$$\lambda(s_n) \in \sigma(A(s_n)) \cap D(\lambda(t), \delta)$$

and from the above discussion, for some choice of $s_n \rightarrow s_0$, $\lambda(s_n) \rightarrow \lambda(s_0)$ which contradicts P and Q separated and nonempty. Since P is nonempty, this shows $Q = \emptyset$. Therefore, H is connected as claimed. But $K_t \supseteq H$ and so $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t + \eta]$. This proves the lemma.

The following is the necessary theorem.

Theorem 28.7.4 *Suppose $A(t)$ is an $n \times n$ matrix and that $t \rightarrow A(t)$ is continuous for $t \in [0, 1]$. Let $\lambda(0) \in \sigma(A(0))$ and define $\Sigma \equiv \bigcup_{t \in [0, 1]} \sigma(A(t))$. Let $K_{\lambda(0)} = K_0$ denote the connected component of $\lambda(0)$ in Σ . Then $K_0 \cap \sigma(A(t)) \neq \emptyset$ for all $t \in [0, 1]$.*

Proof: Let $S \equiv \{t \in [0, 1] : K_0 \cap \sigma(A(s)) \neq \emptyset \text{ for all } s \in [0, t]\}$. Then $0 \in S$. Let $t_0 = \sup(S)$. Say $\sigma(A(t_0)) = \lambda_1(t_0), \dots, \lambda_r(t_0)$. I claim at least one of these is a limit point of K_0 and consequently must be in K_0 which will show that S has a last point. Why is this claim true? Let $s_n \uparrow t_0$ so $s_n \in S$. Now let the discs, $D(\lambda_i(t_0), \delta)$, $i = 1, \dots, r$ be disjoint with $p_{A(t_0)}$ having no zeroes on γ_i the boundary of $D(\lambda_i(t_0), \delta)$. Then for n large enough it follows from Theorem 28.6.1 and the discussion following it that $\sigma(A(s_n))$ is contained in $\bigcup_{i=1}^r D(\lambda_i(t_0), \delta)$. Therefore, $K_0 \cap (\sigma(A(t_0)) + D(0, \delta)) \neq \emptyset$ for all δ small enough. This requires at least one of the $\lambda_i(t_0)$ to be in $\bar{K}_0$. Therefore, $t_0 \in S$ and S has a last point.

Now by Lemma 28.7.3, if $t_0 < 1$, then $K_0 \cup K_t$ would be a strictly larger connected set containing $\lambda(0)$. (The reason this would be strictly larger is that $K_0 \cap \sigma(A(s)) = \emptyset$ for some $s \in (t, t + \eta)$ while $K_t \cap \sigma(A(s)) \neq \emptyset$ for all $s \in [t, t + \eta]$.) Therefore, $t_0 = 1$ and this proves the theorem.

The following is an interesting corollary of the Gerschgorin theorem.

Corollary 28.7.5 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains an eigenvalue of A . Also, if there are n disjoint Gerschgorin discs, then each one contains an eigenvalue of A .*

Proof: Denote by $A(t)$ the matrix (a_{ij}^t) where if $i \neq j$, $a_{ij}^t = ta_{ij}$ and $a_{ii}^t = a_{ii}$. Thus to get $A(t)$ we multiply all non diagonal terms by t . Let $t \in [0, 1]$. Then $A(0) = \text{diag}(a_{11}, \dots, a_{nn})$ and $A(1) = A$. Furthermore, the map, $t \rightarrow A(t)$ is continuous. Denote by D_j^t the Gerschgorin disc obtained from the j^{th} row for the matrix, $A(t)$. Then it is clear that $D_j^t \subseteq D_j$ the j^{th} Gerschgorin disc for A . Then a_{ii} is the eigenvalue for $A(0)$ which is contained in the disc, consisting of the single point a_{ii} which is contained in D_i . Letting K be the connected component in Σ for Σ defined in Theorem 28.7.4 which is determined by a_{ii} , it follows by Gerschgorin's theorem that $K \cap \sigma(A(t)) \subseteq \bigcup_{j=1}^n D_j^t \subseteq \bigcup_{j=1}^n D_j = D_i \cup (\bigcup_{j \neq i} D_j)$ and also, since K is connected, there are no points of K in both D_i and $(\bigcup_{j \neq i} D_j)$. Since at least one point of K is in $D_i, (a_{ii})$ it follows all of K must be contained in D_i . Now by Theorem 28.7.4 this shows there are points of $K \cap \sigma(A)$ in D_i . The last assertion follows immediately.

Actually, this can be improved slightly. It involves the following lemma.

Lemma 28.7.6 *In the situation of Theorem 28.7.4 suppose $\lambda(0) = K_0 \cap \sigma(A(0))$ and that $\lambda(0)$ is a simple root of the characteristic equation of $A(0)$. Then for all $t \in [0, 1]$,*

$$\sigma(A(t)) \cap K_0 = \lambda(t)$$

where $\lambda(t)$ is a simple root of the characteristic equation of $A(t)$.

Proof: Let $S \equiv$

$$\{t \in [0, 1] : K_0 \cap \sigma(A(s)) = \lambda(s), \text{ a simple eigenvalue for all } s \in [0, t]\}.$$

Then $0 \in S$ so it is nonempty. Let $t_0 = \sup(S)$ and suppose $\lambda_1 \neq \lambda_2$ are two elements of $\sigma(A(t_0)) \cap K_0$. Then choosing $\eta > 0$ small enough, and letting D_i be disjoint discs containing λ_i respectively, similar arguments to those of Lemma 28.7.3 imply

$$H_i \equiv \bigcup_{s \in [t_0 - \eta, t_0]} \sigma(A(s)) \cap D_i$$

is a connected and nonempty set for $i = 1, 2$ which would require that $H_i \subseteq K_0$. But then there would be two different eigenvalues of $A(s)$ contained in K_0 , contrary to the definition of t_0 . Therefore, there is at most one eigenvalue, $\lambda(t_0) \in K_0 \cap \sigma(A(t_0))$. The possibility that it could be a repeated root of the characteristic equation must be ruled out. Suppose then that $\lambda(t_0)$ is a repeated root of the characteristic equation. As before, choose a small disc, D centered at $\lambda(t_0)$ and η small enough that

$$H \equiv \bigcup_{s \in [t_0 - \eta, t_0]} \sigma(A(s)) \cap D$$

is a nonempty connected set containing either multiple eigenvalues of $A(s)$ or else a single repeated root to the characteristic equation of $A(s)$. But since H is connected and contains $\lambda(t_0)$ it must be contained in K_0 which contradicts the condition for

$s \in S$ for all these $s \in [t_0 - \eta, t_0]$. Therefore, $t_0 \in S$ as hoped. If $t_0 < 1$, there exists a small disc centered at $\lambda(t_0)$ and $\eta > 0$ such that for all $s \in [t_0, t_0 + \eta]$, $A(s)$ has only simple eigenvalues in D and the only eigenvalues of $A(s)$ which could be in K_0 are in D . (This last assertion follows from noting that $\lambda(t_0)$ is the only eigenvalue of $A(t_0)$ in K_0 and so the others are at a positive distance from K_0 . For s close enough to t_0 , the eigenvalues of $A(s)$ are either close to these eigenvalues of $A(t_0)$ at a positive distance from K_0 or they are close to the eigenvalue, $\lambda(t_0)$ in which case it can be assumed they are in D .) But this shows that t_0 is not really an upper bound to S . Therefore, $t_0 = 1$ and the lemma is proved.

With this lemma, the conclusion of the above corollary can be improved.

Corollary 28.7.7 *Suppose one of the Gerschgorin discs, D_i is disjoint from the union of the others. Then D_i contains exactly one eigenvalue of A and this eigenvalue is a simple root to the characteristic polynomial of A .*

Proof: In the proof of Corollary 28.7.5, first note that a_{ii} is a simple root of $A(0)$ since otherwise the i^{th} Gerschgorin disc would not be disjoint from the others. Also, K , the connected component determined by a_{ii} must be contained in D_i because it is connected and by Gerschgorin's theorem above, $K \cap \sigma(A(t))$ must be contained in the union of the Gerschgorin discs. Since all the other eigenvalues of $A(0)$, the a_{jj} , are outside D_i , it follows that $K \cap \sigma(A(0)) = a_{ii}$. Therefore, by Lemma 28.7.6, $K \cap \sigma(A(1)) = K \cap \sigma(A)$ consists of a single simple eigenvalue. This proves the corollary.

Example 28.7.8 *Consider the matrix,*

$$\begin{pmatrix} 5 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The Gerschgorin discs are $D(5, 1)$, $D(1, 2)$, and $D(0, 1)$. Then $D(5, 1)$ is disjoint from the other discs. Therefore, there should be an eigenvalue in $D(5, 1)$. The actual eigenvalues are not easy to find. They are the roots of the characteristic equation, $t^3 - 6t^2 + 3t + 5 = 0$. The numerical values of these are -0.66966 , 1.4231 , and 5.24655 , verifying the predictions of Gerschgorin's theorem.

28.8 Exercises

1. Use Theorem 28.6.1 to give an alternate proof of the fundamental theorem of algebra. **Hint:** Take a contour of the form $\gamma_r = re^{it}$ where $t \in [0, 2\pi]$. Consider $\int_{\gamma_r} \frac{p'(z)}{p(z)} dz$ and consider the limit as $r \rightarrow \infty$.
2. Let M be an $n \times n$ matrix. Recall that the eigenvalues of M are given by the zeros of the polynomial, $p_M(z) = \det(M - zI)$ where I is the $n \times n$ identity. Formulate a theorem which describes how the eigenvalues depend on small

changes in M . **Hint:** You could define a norm on the space of $n \times n$ matrices as $\|M\| \equiv \operatorname{tr}(MM^*)^{1/2}$ where M^* is the conjugate transpose of M . Thus

$$\|M\| = \left(\sum_{j,k} |M_{jk}|^2 \right)^{1/2}.$$

Argue that small changes will produce small changes in $p_M(z)$. Then apply Theorem 28.6.1 using γ_k a very small circle surrounding z_k , the k^{th} eigenvalue.

3. Suppose that two analytic functions defined on a region are equal on some set, S which contains a limit point. (Recall p is a limit point of S if every open set which contains p , also contains infinitely many points of S .) Show the two functions coincide. We defined $e^z \equiv e^x (\cos y + i \sin y)$ earlier and we showed that e^z , defined this way was analytic on $\mathbb{C}$. Is there any other way to define e^z on all of $\mathbb{C}$ such that the function coincides with e^x on the real axis?
4. You know various identities for real valued functions. For example $\cosh^2 x - \sinh^2 x = 1$. If you define $\cosh z \equiv \frac{e^z + e^{-z}}{2}$ and $\sinh z \equiv \frac{e^z - e^{-z}}{2}$, does it follow that

$$\cosh^2 z - \sinh^2 z = 1$$

for all $z \in \mathbb{C}$? What about

$$\sin(z + w) = \sin z \cos w + \cos z \sin w?$$

Can you verify these sorts of identities just from your knowledge about what happens for real arguments?

5. Was it necessary that U be a region in Theorem 27.5.3? Would the same conclusion hold if U were only assumed to be an open set? Why? What about the open mapping theorem? Would it hold if U were not a region?
6. Let $f : U \rightarrow \mathbb{C}$ be analytic and one to one. Show that $f'(z) \neq 0$ for all $z \in U$. Does this hold for a function of a real variable?
7. We say a real valued function, u is subharmonic if $u_{xx} + u_{yy} \geq 0$. Show that if u is subharmonic on a bounded region, (open connected set) U , and continuous on $\bar{U}$ and $u \leq m$ on ∂U , then $u \leq m$ on U . **Hint:** If not, u achieves its maximum at $(x_0, y_0) \in U$. Let $u(x_0, y_0) > m + \delta$ where $\delta > 0$. Now consider $u_\varepsilon(x, y) = \varepsilon x^2 + u(x, y)$ where ε is small enough that $0 < \varepsilon x^2 < \delta$ for all $(x, y) \in U$. Show that u_ε also achieves its maximum at some point of U and that therefore, $u_{\varepsilon xx} + u_{\varepsilon yy} \leq 0$ at that point implying that $u_{xx} + u_{yy} \leq -\varepsilon$, a contradiction.
8. If u is harmonic on some region, U , show that u coincides locally with the real part of an analytic function and that therefore, u has infinitely many

derivatives on U . **Hint:** Consider the case where $0 \in U$. You can always reduce to this case by a suitable translation. Now let $B(0, r) \subseteq U$ and use the Schwarz formula to obtain an analytic function whose real part coincides with u on $\partial B(0, r)$. Then use Problem 7.

9. Show the solution to the Dirichlet problem of Problem 8 on Page 806 is unique. You need to formulate this precisely and then prove uniqueness.

Residues

Definition 29.0.1 *The residue of f at an isolated singularity α which is a pole, written $\text{res}(f, \alpha)$ is the coefficient of $(z - \alpha)^{-1}$ where*

$$f(z) = g(z) + \sum_{k=1}^m \frac{b_k}{(z - \alpha)^k}.$$

Thus $\text{res}(f, \alpha) = b_1$ in the above.

At this point, recall Corollary 27.7.20 which is stated here for convenience.

Corollary 29.0.2 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \rightarrow \Omega$, $k = 1, \dots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^m n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \rightarrow \mathbb{C}$ is analytic,

$$\sum_{k=1}^m \int_{\gamma_k} f(w) dw = 0.$$

The following theorem is called the residue theorem. Note the resemblance to Corollary 27.7.20.

Theorem 29.0.3 *Let Ω be an open set and let $\gamma_k : [a_k, b_k] \rightarrow \Omega$, $k = 1, \dots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^m n(\gamma_k, z) = 0$$

for all $z \notin \Omega$. Then if $f : \Omega \rightarrow \hat{\mathbb{C}}$ is meromorphic such that no γ_k^* contains any poles of f ,

$$\frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} f(w) dw = \sum_{\alpha \in A} \text{res}(f, \alpha) \sum_{k=1}^m n(\gamma_k, \alpha) \quad (29.0.1)$$

where here A denotes the set of poles of f in Ω . The sum on the right is a finite sum.

Proof: First note that there are at most finitely many α which are not in the unbounded component of $\mathbb{C} \setminus \cup_{k=1}^m \gamma_k([a_k, b_k])$. Thus there exists a finite set, $\{\alpha_1, \dots, \alpha_N\} \subseteq A$ such that these are the only possibilities for which $\sum_{k=1}^n n(\gamma_k, \alpha)$ might not equal zero. Therefore, 29.0.1 reduces to

$$\frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} f(w) dw = \sum_{j=1}^N \operatorname{res}(f, \alpha_j) \sum_{k=1}^n n(\gamma_k, \alpha_j)$$

and it is this last equation which is established. Near α_j ,

$$f(z) = g_j(z) + \sum_{r=1}^{m_j} \frac{b_r^j}{(z - \alpha_j)^r} \equiv g_j(z) + Q_j(z).$$

where g_j is analytic at and near α_j . Now define

$$G(z) \equiv f(z) - \sum_{j=1}^N Q_j(z).$$

It follows that $G(z)$ has a removable singularity at each α_j . Therefore, by Corollary 27.7.20,

$$0 = \sum_{k=1}^m \int_{\gamma_k} G(z) dz = \sum_{k=1}^m \int_{\gamma_k} f(z) dz - \sum_{j=1}^N \sum_{k=1}^m \int_{\gamma_k} Q_j(z) dz.$$

Now

$$\begin{aligned} \sum_{k=1}^m \int_{\gamma_k} Q_j(z) dz &= \sum_{k=1}^m \int_{\gamma_k} \left(\frac{b_1^j}{(z - \alpha_j)} + \sum_{r=2}^{m_j} \frac{b_r^j}{(z - \alpha_j)^r} \right) dz \\ &= \sum_{k=1}^m \int_{\gamma_k} \frac{b_1^j}{(z - \alpha_j)} dz \equiv \sum_{k=1}^m n(\gamma_k, \alpha_j) \operatorname{res}(f, \alpha_j) (2\pi i). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^m \int_{\gamma_k} f(z) dz &= \sum_{j=1}^N \sum_{k=1}^m \int_{\gamma_k} Q_j(z) dz \\ &= \sum_{j=1}^N \sum_{k=1}^m n(\gamma_k, \alpha_j) \operatorname{res}(f, \alpha_j) (2\pi i) \\ &= 2\pi i \sum_{j=1}^N \operatorname{res}(f, \alpha_j) \sum_{k=1}^m n(\gamma_k, \alpha_j) \\ &= (2\pi i) \sum_{\alpha \in A} \operatorname{res}(f, \alpha) \sum_{k=1}^m n(\gamma_k, \alpha) \end{aligned}$$

which proves the theorem.

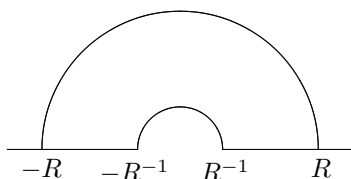
The following is an important example. This example can also be done by real variable methods and there are some who think that real variable methods are always to be preferred to complex variable methods. However, I will use the above theorem to work this example.

Example 29.0.4 Find $\lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin(x)}{x} dx$

Things are easier if you write it as

$$\lim_{R \rightarrow \infty} \frac{1}{i} \left(\int_{-R}^{-R^{-1}} \frac{e^{ix}}{x} dx + \int_{R^{-1}}^R \frac{e^{ix}}{x} dx \right).$$

This gives the same answer because $\cos(x)/x$ is odd. Consider the following contour in which the orientation involves counterclockwise motion exactly once around.



Denote by $\gamma_{R^{-1}}$ the little circle and γ_R the big one. Then on the inside of this contour there are no singularities of e^{iz}/z and it is contained in an open set with the property that the winding number with respect to this contour about any point not in the open set equals zero. By Theorem 27.7.22

$$\frac{1}{i} \left(\int_{-R}^{-R^{-1}} \frac{e^{ix}}{x} dx + \int_{\gamma_{R^{-1}}} \frac{e^{iz}}{z} dz + \int_{R^{-1}}^R \frac{e^{ix}}{x} dx + \int_{\gamma_R} \frac{e^{iz}}{z} dz \right) = 0 \quad (29.0.2)$$

Now

$$\left| \int_{\gamma_R} \frac{e^{iz}}{z} dz \right| = \left| \int_0^\pi e^{R(i \cos \theta - \sin \theta)} i d\theta \right| \leq \int_0^\pi e^{-R \sin \theta} d\theta$$

and this last integral converges to 0 by the dominated convergence theorem. Now consider the other circle. By the dominated convergence theorem again,

$$\int_{\gamma_{R^{-1}}} \frac{e^{iz}}{z} dz = \int_\pi^0 e^{R^{-1}(i \cos \theta - \sin \theta)} i d\theta \rightarrow -i\pi$$

as $R \rightarrow \infty$. Then passing to the limit in 29.0.2,

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{-R}^R \frac{\sin(x)}{x} dx \\ &= \lim_{R \rightarrow \infty} \frac{1}{i} \left(\int_{-R}^{-R^{-1}} \frac{e^{ix}}{x} dx + \int_{R^{-1}}^R \frac{e^{ix}}{x} dx \right) \\ &= \lim_{R \rightarrow \infty} \frac{1}{i} \left(- \int_{\gamma_{R^{-1}}} \frac{e^{iz}}{z} dz - \int_{\gamma_R} \frac{e^{iz}}{z} dz \right) = \frac{-1}{i} (-i\pi) = \pi. \end{aligned}$$

Example 29.0.5 Find $\lim_{R \rightarrow \infty} \int_{-R}^R e^{ixt} \frac{\sin x}{x} dx$. Note this is essentially finding the inverse Fourier transform of the function, $\sin(x)/x$.

This equals

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{-R}^R (\cos(xt) + i \sin(xt)) \frac{\sin(x)}{x} dx \\ &= \lim_{R \rightarrow \infty} \int_{-R}^R \cos(xt) \frac{\sin(x)}{x} dx \\ &= \lim_{R \rightarrow \infty} \int_{-R}^R \cos(xt) \frac{\sin(x)}{x} dx \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} \int_{-R}^R \frac{\sin(x(t+1)) + \sin(x(1-t))}{x} dx \end{aligned}$$

Let $t \neq 1, -1$. Then changing variables yields

$$\lim_{R \rightarrow \infty} \left(\frac{1}{2} \int_{-R(1+t)}^{R(1+t)} \frac{\sin(u)}{u} du + \frac{1}{2} \int_{-R(1-t)}^{R(1-t)} \frac{\sin(u)}{u} du \right).$$

In case $|t| < 1$ Example 29.0.4 implies this limit is π . However, if $t > 1$ the limit equals 0 and this is also the case if $t < -1$. Summarizing,

$$\lim_{R \rightarrow \infty} \int_{-R}^R e^{ixt} \frac{\sin x}{x} dx = \begin{cases} \pi & \text{if } |t| < 1 \\ 0 & \text{if } |t| > 1 \end{cases}.$$

29.1 Rouché's Theorem And The Argument Principle

29.1.1 Argument Principle

A simple closed curve is just one which is homeomorphic to the unit circle. The Jordan Curve theorem states that every simple closed curve in the plane divides the plane into exactly two connected components, one bounded and the other unbounded. This is a very hard theorem to prove. However, in most applications the

conclusion is obvious. Nevertheless, to avoid using this big topological result and to attain some extra generality, I will state the following theorem in terms of the winding number to avoid using it. This theorem is called the argument principle. First recall that f has a zero of order m at α if $f(z) = g(z)(z - \alpha)^m$ where g is an analytic function which is not equal to zero at α . This is equivalent to having $f(z) = \sum_{k=m}^{\infty} a_k (z - \alpha)^k$ for z near α where $a_m \neq 0$. Also recall that f has a pole of order m at α if for z near α , $f(z)$ is of the form

$$f(z) = h(z) + \sum_{k=1}^m \frac{b_k}{(z - \alpha)^k} \quad (29.1.3)$$

where $b_m \neq 0$ and h is a function analytic near α .

Theorem 29.1.1 (*argument principle*) *Let f be meromorphic in Ω . Also suppose γ^* is a closed bounded variation curve containing none of the poles or zeros of f with the property that for all $z \notin \Omega$, $n(\gamma, z) = 0$ and for all $z \in \Omega$, $n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \dots, p_m\}$ and $\{z_1, \dots, z_n\}$ be respectively the poles and zeros for which the winding number of γ about these points equals 1. Let z_k be a zero of order r_k and let p_k be a pole of order l_k . Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^n r_k - \sum_{k=1}^m l_k$$

Proof: This theorem follows from computing the residues of f'/f . It has residues at poles and zeros. I will do this now. First suppose f has a pole of order p at α . Then f has the form given in 29.1.3. Therefore,

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{h'(z) - \sum_{k=1}^p \frac{kb_k}{(z-\alpha)^{k+1}}}{h(z) + \sum_{k=1}^p \frac{b_k}{(z-\alpha)^k}} \\ &= \frac{h'(z)(z-\alpha)^p - \sum_{k=1}^{p-1} kb_k(z-\alpha)^{-k-1+p} - \frac{pb_p}{(z-\alpha)}}{h(z)(z-\alpha)^p + \sum_{k=1}^{p-1} b_k(z-\alpha)^{p-k} + b_p} \end{aligned}$$

This is of the form

$$= \frac{b_p}{s(z) + b_p} \frac{r(z) - \frac{pb_p}{(z-\alpha)}}{b_p} = \frac{b_p}{s(z) + b_p} \left(\frac{r(z)}{b_p} - \frac{p}{(z-\alpha)} \right)$$

where $s(\alpha) = r(\alpha) = 0$. From this, it is clear $\text{res}(f'/f, \alpha) = -p$, the order of the pole.

Next suppose f has a zero of order p at α . Then

$$\frac{f'(z)}{f(z)} = \frac{\sum_{k=p}^{\infty} a_k k (z - \alpha)^{k-1}}{\sum_{k=p}^{\infty} a_k (z - \alpha)^k} = \frac{\sum_{k=p}^{\infty} a_k k (z - \alpha)^{k-1-p}}{\sum_{k=p}^{\infty} a_k (z - \alpha)^{k-p}}$$

and from this it is clear $\text{res}(f'/f) = p$, the order of the zero. The conclusion of this theorem now follows from Theorem 29.0.3.

One can also generalize the theorem to the case where there are many closed curves involved. This is proved in the same way as the above.

Theorem 29.1.2 (*argument principle*) *Let f be meromorphic in Ω and let $\gamma_k : [a_k, b_k] \rightarrow \Omega$, $k = 1, \dots, m$, be closed, continuous and of bounded variation. Suppose also that*

$$\sum_{k=1}^m n(\gamma_k, z) = 0$$

and for all $z \notin \Omega$ and for $z \in \Omega$, $\sum_{k=1}^m n(\gamma_k, z)$ either equals 0 or 1. Now let $\{p_1, \dots, p_m\}$ and $\{z_1, \dots, z_n\}$ be respectively the poles and zeros for which the above sum of winding numbers equals 1. Let z_k be a zero of order r_k and let p_k be a pole of order l_k . Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^n r_k - \sum_{k=1}^m l_k$$

There is also a simple extension of this important principle which I found in [32].

Theorem 29.1.3 (*argument principle*) *Let f be meromorphic in Ω . Also suppose γ^* is a closed bounded variation curve with the property that for all $z \notin \Omega$, $n(\gamma, z) = 0$ and for all $z \in \Omega$, $n(\gamma, z)$ either equals 0 or 1. Now let $\{p_1, \dots, p_m\}$ and $\{z_1, \dots, z_n\}$ be respectively the poles and zeros for which the winding number of γ about these points equals 1 listed according to multiplicity. Thus if there is a pole of order m there will be this value repeated m times in the list for the poles. Also let $g(z)$ be an analytic function. Then*

$$\frac{1}{2\pi i} \int_{\gamma} g(z) \frac{f'(z)}{f(z)} dz = \sum_{k=1}^n g(z_k) - \sum_{k=1}^m g(p_k)$$

Proof: This theorem follows from computing the residues of $g(f'/f)$. It has residues at poles and zeros. I will do this now. First suppose f has a pole of order m at α . Then f has the form given in 29.1.3. Therefore,

$$\begin{aligned} & g(z) \frac{f'(z)}{f(z)} \\ &= \frac{g(z) \left(h'(z) - \sum_{k=1}^m \frac{k b_k}{(z-\alpha)^{k+1}} \right)}{h(z) + \sum_{k=1}^m \frac{b_k}{(z-\alpha)^k}} \\ &= g(z) \frac{h'(z)(z-\alpha)^m - \sum_{k=1}^{m-1} k b_k (z-\alpha)^{-k-1+m} - \frac{m b_m}{(z-\alpha)}}{h(z)(z-\alpha)^m + \sum_{k=1}^{m-1} b_k (z-\alpha)^{m-k} + b_m} \end{aligned}$$

From this, it is clear $\text{res}(g(f'/f), \alpha) = -mg(\alpha)$, where m is the order of the pole. Thus α would have been listed m times in the list of poles. Hence the residue at this point is equivalent to adding $-g(\alpha)$ m times.

Next suppose f has a zero of order m at α . Then

$$g(z) \frac{f'(z)}{f(z)} = g(z) \frac{\sum_{k=m}^{\infty} a_k k (z - \alpha)^{k-1}}{\sum_{k=m}^{\infty} a_k (z - \alpha)^k} = g(z) \frac{\sum_{k=m}^{\infty} a_k k (z - \alpha)^{k-1-m}}{\sum_{k=m}^{\infty} a_k (z - \alpha)^{k-m}}$$

and from this it is clear $\text{res}(g(f'/f)) = g(\alpha)m$, where m is the order of the zero. The conclusion of this theorem now follows from the residue theorem, Theorem 29.0.3.

The way people usually apply these theorems is to suppose γ^* is a simple closed bounded variation curve, often a circle. Thus it has an inside and an outside, the outside being the unbounded component of $\mathbb{C} \setminus \gamma^*$. The orientation of the curve is such that you go around it once in the counterclockwise direction. Then letting r_k and l_k be as described, the conclusion of the theorem follows. In applications, this is likely the way it will be.

29.1.2 Rouché's Theorem

With the argument principle, it is possible to prove Rouché's theorem. In the argument principle, denote by Z_f the quantity $\sum_{k=1}^m r_k$ and by P_f the quantity $\sum_{k=1}^n l_k$. Thus Z_f is the number of zeros of f counted according to the order of the zero with a similar definition holding for P_f . Thus the conclusion of the argument principle is.

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = Z_f - P_f$$

Rouché's theorem allows the comparison of $Z_h - P_h$ for $h = f, g$. It is a wonderful and amazing result.

Theorem 29.1.4 (*Rouché's theorem*) Let f, g be meromorphic in an open set Ω . Also suppose γ^* is a closed bounded variation curve with the property that for all $z \notin \Omega$, $n(\gamma, z) = 0$, no zeros or poles are on γ^* , and for all $z \in \Omega$, $n(\gamma, z)$ either equals 0 or 1. Let Z_f and P_f denote respectively the numbers of zeros and poles of f , which have the property that the winding number equals 1, counted according to order, with Z_g and P_g being defined similarly. Also suppose that for $z \in \gamma^*$

$$|f(z) + g(z)| < |f(z)| + |g(z)|. \quad (29.1.4)$$

Then

$$Z_f - P_f = Z_g - P_g.$$

Proof: From the hypotheses,

$$\left| 1 + \frac{f(z)}{g(z)} \right| < 1 + \left| \frac{f(z)}{g(z)} \right|$$

which shows that for all $z \in \gamma^*$,

$$\frac{f(z)}{g(z)} \in \mathbb{C} \setminus [0, \infty).$$

Letting l denote a branch of the logarithm defined on $\mathbb{C} \setminus [0, \infty)$, it follows that $l\left(\frac{f(z)}{g(z)}\right)$ is a primitive for the function,

$$\frac{(f/g)'}{(f/g)} = \frac{f'}{f} - \frac{g'}{g}.$$

Therefore, by the argument principle,

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_{\gamma} \frac{(f/g)'}{(f/g)} dz = \frac{1}{2\pi i} \int_{\gamma} \left(\frac{f'}{f} - \frac{g'}{g} \right) dz \\ &= Z_f - P_f - (Z_g - P_g). \end{aligned}$$

This proves the theorem.

Often another condition other than 29.1.4 is used.

Corollary 29.1.5 *In the situation of Theorem 29.1.4 change 29.1.4 to the condition,*

$$|f(z) - g(z)| < |f(z)|$$

for $z \in \gamma^*$. Then the conclusion is the same.

Proof: The new condition implies $\left|1 - \frac{g}{f}(z)\right| < \left|\frac{g(z)}{f(z)}\right|$ on γ^* . Therefore, $\frac{g(z)}{f(z)} \notin (-\infty, 0]$ and so you can do the same argument with a branch of the logarithm.

29.1.3 A Different Formulation

In [60] I found this modification for Rouché's theorem concerned with the counting of zeros of analytic functions. This is a very useful form of Rouché's theorem because it makes no mention of a contour.

Theorem 29.1.6 *Let Ω be a bounded open set and suppose f, g are continuous on $\bar{\Omega}$ and analytic on Ω . Also suppose $|f(z)| < |g(z)|$ on $\partial\Omega$. Then g and $f+g$ have the same number of zeros in Ω provided each zero is counted according to multiplicity.*

Proof: Let $K = \{z \in \bar{\Omega} : |f(z)| \geq |g(z)|\}$. Then letting $\lambda \in [0, 1]$, if $z \notin K$, then $|f(z)| < |g(z)|$ and so

$$0 < |g(z)| - |f(z)| \leq |g(z)| - \lambda|f(z)| \leq |g(z) + \lambda f(z)|$$

which shows that all zeros of $g + \lambda f$ are contained in K which must be a compact subset of Ω due to the assumption that $|f(z)| < |g(z)|$ on $\partial\Omega$. By Theorem 27.7.25 on Page 825 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k^* \subseteq \Omega \setminus K$, $\sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then as above, it follows from the residue theorem or more directly, Theorem 29.1.2,

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda f'(z) + g'(z)}{\lambda f(z) + g(z)} dz = \sum_{j=1}^p m_j$$

where m_j is the order of the j^{th} zero of $\lambda f + g$ in K , hence in Ω . However,

$$\lambda \rightarrow \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda f'(z) + g'(z)}{\lambda f(z) + g(z)} dz$$

is integer valued and continuous so it gives the same value when $\lambda = 0$ as when $\lambda = 1$. When $\lambda = 0$ this gives the number of zeros of g in Ω and when $\lambda = 1$ it is the number of zeros of $f + g$. This proves the theorem.

Here is another formulation of this theorem.

Corollary 29.1.7 *Let Ω be a bounded open set and suppose f, g are continuous on $\overline{\Omega}$ and analytic on Ω . Also suppose $|f(z) - g(z)| < |g(z)|$ on $\partial\Omega$. Then f and g have the same number of zeros in Ω provided each zero is counted according to multiplicity.*

Proof: You let $f - g$ play the role of f in Theorem 29.1.6. Thus $f - g + g = f$ and g have the same number of zeros. Alternatively, you can give a proof of this directly as follows.

Let $K = \{z \in \Omega : |f(z) - g(z)| \geq |g(z)|\}$. Then if $g(z) + \lambda(f(z) - g(z)) = 0$ it follows

$$\begin{aligned} 0 &= |g(z) + \lambda(f(z) - g(z))| \geq |g(z)| - \lambda |f(z) - g(z)| \\ &\geq |g(z)| - |f(z) - g(z)| \end{aligned}$$

and so $z \in K$. Thus all zeros of $g(z) + \lambda(f(z) - g(z))$ are contained in K . By Theorem 27.7.25 on Page 825 there exists a cycle, $\{\gamma_k\}_{k=1}^n$ such that $\cup_{k=1}^n \gamma_k^* \subseteq \Omega \setminus K$, $\sum_{k=1}^n n(\gamma_k, z) = 1$ for every $z \in K$ and $\sum_{k=1}^n n(\gamma_k, z) = 0$ for all $z \notin \Omega$. Then by Theorem 29.1.2,

$$\sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{\lambda(f'(z) - g'(z)) + g'(z)}{\lambda(f(z) - g(z)) + g(z)} dz = \sum_{j=1}^p m_j$$

where m_j is the order of the j^{th} zero of $\lambda(f - g) + g$ in K , hence in Ω . The left side is continuous as a function of λ and so the number of zeros of g corresponding to $\lambda = 0$ equals the number of zeros of f corresponding to $\lambda = 1$. This proves the corollary.

29.2 Singularities And The Laurent Series

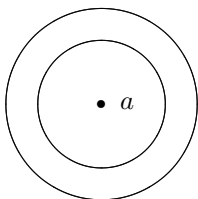
29.2.1 What Is An Annulus?

In general, when you consider singularities, isolated or not, the fundamental tool is the Laurent series. This series is important for many other reasons also. In particular, it is fundamental to the spectral theory of various operators in functional analysis and is one way to obtain relationships between algebraic and analytical

conditions essential in various convergence theorems. A Laurent series lives on an annulus. In all this f has values in X where X is a complex Banach space. If you like, let $X = \mathbb{C}$.

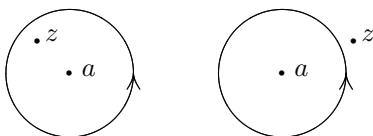
Definition 29.2.1 Define $\text{ann}(a, R_1, R_2) \equiv \{z : R_1 < |z - a| < R_2\}$.

Thus $\text{ann}(a, 0, R)$ would denote the punctured ball, $B(a, R) \setminus \{0\}$ and when $R_1 > 0$, the annulus looks like the following.



The annulus is the stuff between the two circles.

Here is an important lemma which is concerned with the situation described in the following picture.



Lemma 29.2.2 Let $\gamma_r(t) \equiv a + re^{it}$ for $t \in [0, 2\pi]$ and let $|z - a| < r$. Then $n(\gamma_r, z) = 1$. If $|z - a| > r$, then $n(\gamma_r, z) = 0$.

Proof: For the first claim, consider for $t \in [0, 1]$,

$$f(t) \equiv n(\gamma_r, a + t(z - a)).$$

Then from properties of the winding number derived earlier, $f(t) \in \mathbb{Z}$, f is continuous, and $f(0) = 1$. Therefore, $f(t) = 1$ for all $t \in [0, 1]$. This proves the first claim because $f(1) = n(\gamma_r, z)$.

For the second claim,

$$\begin{aligned} n(\gamma_r, z) &= \frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{w - z} dw \\ &= \frac{1}{2\pi i} \int_{\gamma_r} \frac{1}{w - a - (z - a)} dw \\ &= \frac{1}{2\pi i} \frac{-1}{z - a} \int_{\gamma_r} \frac{1}{1 - \left(\frac{w - a}{z - a}\right)} dw \\ &= \frac{-1}{2\pi i (z - a)} \int_{\gamma_r} \sum_{k=0}^{\infty} \left(\frac{w - a}{z - a}\right)^k dw. \end{aligned}$$

The series converges uniformly for $w \in \gamma_r$ because

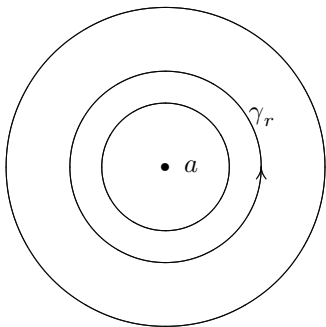
$$\left| \frac{w-a}{z-a} \right| = \frac{r}{r+c}$$

for some $c > 0$ due to the assumption that $|z-a| > r$. Therefore, the sum and the integral can be interchanged to give

$$n(\gamma_r, z) = \frac{-1}{2\pi i(z-a)} \sum_{k=0}^{\infty} \int_{\gamma_r} \left(\frac{w-a}{z-a} \right)^k dw = 0$$

because $w \rightarrow \left(\frac{w-a}{z-a} \right)^k$ has an antiderivative. This proves the lemma.

Now consider the following picture which pertains to the next lemma.



Lemma 29.2.3 *Let g be analytic on $\text{ann}(a, R_1, R_2)$. Then if $\gamma_r(t) \equiv a + re^{it}$ for $t \in [0, 2\pi]$ and $r \in (R_1, R_2)$, then $\int_{\gamma_r} g(z) dz$ is independent of r .*

Proof: Let $R_1 < r_1 < r_2 < R_2$ and denote by $-\gamma_r(t)$ the curve, $-\gamma_r(t) \equiv a + re^{i(2\pi-t)}$ for $t \in [0, 2\pi]$. Then if $z \in \overline{B}(a, R_1)$, Lemma 29.2.2 implies both $n(\gamma_{r_2}, z)$ and $n(\gamma_{r_1}, z) = 1$ and so

$$n(-\gamma_{r_1}, z) + n(\gamma_{r_2}, z) = -1 + 1 = 0.$$

Also if $z \notin \overline{B}(a, R_2)$, then Lemma 29.2.2 implies $n(\gamma_{r_j}, z) = 0$ for $j = 1, 2$. Therefore, whenever $z \notin \text{ann}(a, R_1, R_2)$, the sum of the winding numbers equals zero. Therefore, by Theorem 27.7.19 applied to the function, $f(w) = g(z)(w-z)$ and $z \in \text{ann}(a, R_1, R_2) \setminus \cup_{j=1}^2 \gamma_{r_j}([0, 2\pi])$,

$$\begin{aligned} f(z) (n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)) &= 0 (n(\gamma_{r_2}, z) + n(-\gamma_{r_1}, z)) = \\ &= \frac{1}{2\pi i} \int_{\gamma_{r_2}} \frac{g(w)(w-z)}{w-z} dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} \frac{g(w)(w-z)}{w-z} dw \\ &= \frac{1}{2\pi i} \int_{\gamma_{r_2}} g(w) dw - \frac{1}{2\pi i} \int_{\gamma_{r_1}} g(w) dw \end{aligned}$$

which proves the desired result.

29.2.2 The Laurent Series

The Laurent series is like a power series except it allows for negative exponents. First here is a definition of what is meant by the convergence of such a series.

Definition 29.2.4 $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ converges if both the series,

$$\sum_{n=0}^{\infty} a_n (z-a)^n \text{ and } \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$$

converge. When this is the case, the symbol, $\sum_{n=-\infty}^{\infty} a_n (z-a)^n$ is defined as

$$\sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}.$$

Lemma 29.2.5 Suppose

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

for all $|z-a| \in (R_1, R_2)$. Then both $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converge absolutely and uniformly on $\{z : r_1 \leq |z-a| \leq r_2\}$ for any $r_1 < r_2$ satisfying $R_1 < r_1 < r_2 < R_2$.

Proof: Let $R_1 < |w-a| = r_1 - \delta < r_1$. Then $\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}$ converges and so

$$\lim_{n \rightarrow \infty} |a_{-n}| |w-a|^{-n} = \lim_{n \rightarrow \infty} |a_{-n}| (r_1 - \delta)^{-n} = 0$$

which implies that for all n sufficiently large,

$$|a_{-n}| (r_1 - \delta)^{-n} < 1.$$

Therefore,

$$\sum_{n=1}^{\infty} |a_{-n}| |z-a|^{-n} = \sum_{n=1}^{\infty} |a_{-n}| (r_1 - \delta)^{-n} (r_1 - \delta)^n |z-a|^{-n}.$$

Now for $|z-a| \geq r_1$,

$$|z-a|^{-n} \leq \frac{1}{r_1^n}$$

and so for all sufficiently large n

$$|a_{-n}| |z-a|^{-n} \leq \frac{(r_1 - \delta)^n}{r_1^n}.$$

Therefore, by the Weierstrass M test, the series, $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ converges absolutely and uniformly on the set

$$\{z \in \mathbb{C} : |z-a| \geq r_1\}.$$

Similar reasoning shows the series, $\sum_{n=0}^{\infty} a_n (z - a)^n$ converges uniformly on the set

$$\{z \in \mathbb{C} : |z - a| \leq r_2\}.$$

This proves the Lemma.

Theorem 29.2.6 *Let f be analytic on $\text{ann}(a, R_1, R_2)$. Then there exist numbers, $a_n \in \mathbb{C}$ such that for all $z \in \text{ann}(a, R_1, R_2)$,*

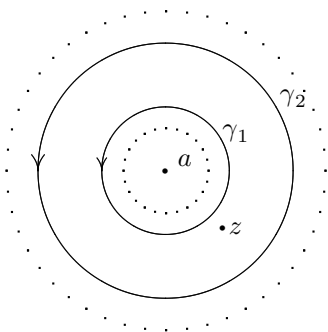
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n, \quad (29.2.5)$$

where the series converges absolutely and uniformly on $\overline{\text{ann}(a, r_1, r_2)}$ whenever $R_1 < r_1 < r_2 < R_2$. Also

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - a)^{n+1}} dw \quad (29.2.6)$$

where $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$ for any $r \in (R_1, R_2)$. Furthermore the series is unique in the sense that if 29.2.5 holds for $z \in \text{ann}(a, R_1, R_2)$, then a_n is given in 29.2.6.

Proof: Let $R_1 < r_1 < r_2 < R_2$ and define $\gamma_1(t) \equiv a + (r_1 - \varepsilon)e^{it}$ and $\gamma_2(t) \equiv a + (r_2 + \varepsilon)e^{it}$ for $t \in [0, 2\pi]$ and ε chosen small enough that $R_1 < r_1 - \varepsilon < r_2 + \varepsilon < R_2$.



Then using Lemma 29.2.2, if $z \notin \text{ann}(a, R_1, R_2)$ then

$$n(-\gamma_1, z) + n(\gamma_2, z) = 0$$

and if $z \in \text{ann}(a, r_1, r_2)$,

$$n(-\gamma_1, z) + n(\gamma_2, z) = 1.$$

Therefore, by Theorem 27.7.19, for $z \in \text{ann}(a, r_1, r_2)$

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi i} \left[\int_{-\gamma_1} \frac{f(w)}{w-z} dw + \int_{\gamma_2} \frac{f(w)}{w-z} dw \right] \\
 &= \frac{1}{2\pi i} \left[\int_{\gamma_1} \frac{f(w)}{(z-a) \left[1 - \frac{w-a}{z-a}\right]} dw + \int_{\gamma_2} \frac{f(w)}{(w-a) \left[1 - \frac{z-a}{w-a}\right]} dw \right] \\
 &= \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-a} \sum_{n=0}^{\infty} \left(\frac{z-a}{w-a} \right)^n dw + \\
 &\quad \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(z-a)} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a} \right)^n dw. \tag{29.2.7}
 \end{aligned}$$

From the formula 29.2.7, it follows that for $z \in \overline{\text{ann}(a, r_1, r_2)}$, the terms in the first sum are bounded by an expression of the form $C \left(\frac{r_2}{r_2+\varepsilon} \right)^n$ while those in the second are bounded by one of the form $C \left(\frac{r_1-\varepsilon}{r_1} \right)^n$ and so by the Weierstrass M test, the convergence is uniform and so the integrals and the sums in the above formula may be interchanged and after renaming the variable of summation, this yields

$$\begin{aligned}
 f(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n + \\
 &\quad \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n. \tag{29.2.8}
 \end{aligned}$$

Therefore, by Lemma 29.2.3, for any $r \in (R_1, R_2)$,

$$\begin{aligned}
 f(z) &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n + \\
 &\quad \sum_{n=-\infty}^{-1} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n. \tag{29.2.9}
 \end{aligned}$$

and so

$$f(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n.$$

where $r \in (R_1, R_2)$ is arbitrary. This proves the existence part of the theorem. It remains to characterize a_n .

If $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$ on $\text{ann}(a, R_1, R_2)$ let

$$f_n(z) \equiv \sum_{k=-n}^n a_k (z-a)^k. \tag{29.2.10}$$

This function is analytic in $\text{ann}(a, R_1, R_2)$ and so from the above argument,

$$f_n(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k. \quad (29.2.11)$$

Also if $k > n$ or if $k < -n$,

$$\left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) = 0.$$

and so

$$f_n(z) = \sum_{k=-n}^n \left(\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \right) (z-a)^k$$

which implies from 29.2.10 that for each $k \in [-n, n]$,

$$\frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw = a_k$$

However, from the uniform convergence of the series,

$$\sum_{n=0}^{\infty} a_n (w-a)^n$$

and

$$\sum_{n=1}^{\infty} a_{-n} (w-a)^{-n}$$

ensured by Lemma 29.2.5 which allows the interchange of sums and integrals, if $k \in [-n, n]$,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw \\ &= \frac{1}{2\pi i} \int_{\gamma_r} \frac{\sum_{m=0}^{\infty} a_m (w-a)^m + \sum_{m=1}^{\infty} a_{-m} (w-a)^{-m}}{(w-a)^{k+1}} dw \\ &= \sum_{m=0}^{\infty} a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{m-(k+1)} dw \\ & \quad + \sum_{m=1}^{\infty} a_{-m} \int_{\gamma_r} (w-a)^{-m-(k+1)} dw \\ &= \sum_{m=0}^n a_m \frac{1}{2\pi i} \int_{\gamma_r} (w-a)^{m-(k+1)} dw \\ & \quad + \sum_{m=1}^n a_{-m} \int_{\gamma_r} (w-a)^{-m-(k+1)} dw \\ &= \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_n(w)}{(w-a)^{k+1}} dw \end{aligned}$$

because if $l > n$ or $l < -n$,

$$\int_{\gamma_r} \frac{a_l (w-a)^l}{(w-a)^{k+1}} dw = 0$$

for all $k \in [-n, n]$. Therefore,

$$a_k = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{(w-a)^{k+1}} dw$$

and so this establishes uniqueness. This proves the theorem.

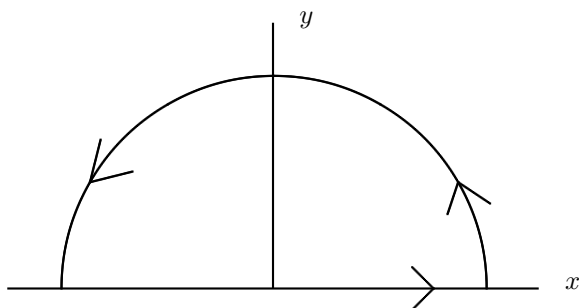
29.2.3 Contour Integrals And Evaluation Of Integrals

Here are some examples of hard integrals which can be evaluated by using residues. This will be done by integrating over various closed curves having bounded variation.

Example 29.2.7 *The first example we consider is the following integral.*

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$$

One could imagine evaluating this integral by the method of partial fractions and it should work out by that method. However, we will consider the evaluation of this integral by the method of residues instead. To do so, consider the following picture.



Let $\gamma_r(t) = re^{it}$, $t \in [0, \pi]$ and let $\sigma_r(t) = t : t \in [-r, r]$. Thus γ_r parameterizes the top curve and σ_r parameterizes the straight line from $-r$ to r along the x axis. Denoting by Γ_r the closed curve traced out by these two, we see from simple estimates that

$$\lim_{r \rightarrow \infty} \int_{\gamma_r} \frac{1}{1+z^4} dz = 0.$$

This follows from the following estimate.

$$\left| \int_{\gamma_r} \frac{1}{1+z^4} dz \right| \leq \frac{1}{r^4-1} \pi r.$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \lim_{r \rightarrow \infty} \int_{\Gamma_r} \frac{1}{1+z^4} dz.$$

We compute $\int_{\Gamma_r} \frac{1}{1+z^4} dz$ using the method of residues. The only residues of the integrand are located at points, z where $1+z^4=0$. These points are

$$\begin{aligned} z &= -\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}, z = \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}, \\ z &= \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}, z = -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2} \end{aligned}$$

and it is only the last two which are found in the inside of Γ_r . Therefore, we need to calculate the residues at these points. Clearly this function has a pole of order one at each of these points and so we may calculate the residue at α in this list by evaluating

$$\lim_{z \rightarrow \alpha} (z - \alpha) \frac{1}{1+z^4}$$

Thus

$$\begin{aligned} & \text{Res} \left(f, \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2} \right) \\ &= \lim_{z \rightarrow \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}} \left(z - \left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2} \right) \right) \frac{1}{1+z^4} \\ &= -\frac{1}{8}\sqrt{2} - \frac{1}{8}i\sqrt{2} \end{aligned}$$

Similarly we may find the other residue in the same way

$$\begin{aligned} & \text{Res} \left(f, -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2} \right) \\ &= \lim_{z \rightarrow -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}} \left(z - \left(-\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2} \right) \right) \frac{1}{1+z^4} \\ &= -\frac{1}{8}i\sqrt{2} + \frac{1}{8}\sqrt{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\Gamma_r} \frac{1}{1+z^4} dz &= 2\pi i \left(-\frac{1}{8}i\sqrt{2} + \frac{1}{8}\sqrt{2} + \left(-\frac{1}{8}\sqrt{2} - \frac{1}{8}i\sqrt{2} \right) \right) \\ &= \frac{1}{2}\pi\sqrt{2}. \end{aligned}$$

Thus, taking the limit we obtain $\frac{1}{2}\pi\sqrt{2} = \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx$.

Obviously many different variations of this are possible. The main idea being that the integral over the semicircle converges to zero as $r \rightarrow \infty$.

Sometimes we don't blow up the curves and take limits. Sometimes the problem of interest reduces directly to a complex integral over a closed curve. Here is an example of this.

Example 29.2.8 *The integral is*

$$\int_0^\pi \frac{\cos \theta}{2 + \cos \theta} d\theta$$

This integrand is even and so it equals

$$\frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos \theta}{2 + \cos \theta} d\theta.$$

For z on the unit circle, $z = e^{i\theta}$, $\bar{z} = \frac{1}{z}$ and therefore, $\cos \theta = \frac{1}{2} \left(z + \frac{1}{z} \right)$. Thus $dz = ie^{i\theta} d\theta$ and so $d\theta = \frac{dz}{iz}$. Note this is proceeding formally to get a complex integral which reduces to the one of interest. It follows that a complex integral which reduces to the one desired is

$$\frac{1}{2i} \int_{\gamma} \frac{\frac{1}{2} \left(z + \frac{1}{z} \right)}{2 + \frac{1}{2} \left(z + \frac{1}{z} \right)} \frac{dz}{z} = \frac{1}{2i} \int_{\gamma} \frac{z^2 + 1}{z(4z + z^2 + 1)} dz$$

where γ is the unit circle. Now the integrand has poles of order 1 at those points where $z(4z + z^2 + 1) = 0$. These points are

$$0, -2 + \sqrt{3}, -2 - \sqrt{3}.$$

Only the first two are inside the unit circle. It is also clear the function has simple poles at these points. Therefore,

$$\text{Res}(f, 0) = \lim_{z \rightarrow 0} z \left(\frac{z^2 + 1}{z(4z + z^2 + 1)} \right) = 1.$$

$$\text{Res}(f, -2 + \sqrt{3}) =$$

$$\lim_{z \rightarrow -2 + \sqrt{3}} \left(z - (-2 + \sqrt{3}) \right) \frac{z^2 + 1}{z(4z + z^2 + 1)} = -\frac{2}{3}\sqrt{3}.$$

It follows

$$\begin{aligned} \int_0^\pi \frac{\cos \theta}{2 + \cos \theta} d\theta &= \frac{1}{2i} \int_{\gamma} \frac{z^2 + 1}{z(4z + z^2 + 1)} dz \\ &= \frac{1}{2i} 2\pi i \left(1 - \frac{2}{3}\sqrt{3} \right) \\ &= \pi \left(1 - \frac{2}{3}\sqrt{3} \right). \end{aligned}$$

Other rational functions of the trig functions will work out by this method also.

Sometimes you have to be clever about which version of an analytic function that reduces to a real function you should use. The following is such an example.

Example 29.2.9 *The integral here is*

$$\int_0^\infty \frac{\ln x}{1+x^4} dx.$$

The same curve used in the integral involving $\frac{\sin x}{x}$ earlier will create problems with the log since the usual version of the log is not defined on the negative real axis. This does not need to be of concern however. Simply use another branch of the logarithm. Leave out the ray from 0 along the negative y axis and use Theorem 28.2.3 to define $L(z)$ on this set. Thus $L(z) = \ln|z| + i \arg_1(z)$ where $\arg_1(z)$ will be the angle, θ , between $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$ such that $z = |z|e^{i\theta}$. Now the only singularities contained in this curve are

$$\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}, -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$$

and the integrand, f has simple poles at these points. Thus using the same procedure as in the other examples,

$$\operatorname{Res}\left(f, \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi$$

and

$$\operatorname{Res}\left(f, -\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) =$$

$$\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi.$$

Consider the integral along the small semicircle of radius r . This reduces to

$$\int_\pi^0 \frac{\ln|r| + it}{1 + (re^{it})^4} (rie^{it}) dt$$

which clearly converges to zero as $r \rightarrow 0$ because $r \ln r \rightarrow 0$. Therefore, taking the limit as $r \rightarrow 0$,

$$\begin{aligned} & \int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz + \lim_{r \rightarrow 0+} \int_{-R}^{-r} \frac{\ln(-t) + i\pi}{1+t^4} dt + \\ & \lim_{r \rightarrow 0+} \int_r^R \frac{\ln t}{1+t^4} dt = 2\pi i \left(\frac{3}{32}\sqrt{2}\pi + \frac{3}{32}i\sqrt{2}\pi + \frac{1}{32}\sqrt{2}\pi - \frac{1}{32}i\sqrt{2}\pi \right). \end{aligned}$$

Observing that $\int_{\text{large semicircle}} \frac{L(z)}{1+z^4} dz \rightarrow 0$ as $R \rightarrow \infty$,

$$e(R) + 2 \lim_{r \rightarrow 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \int_{-\infty}^0 \frac{1}{1+t^4} dt = \left(-\frac{1}{8} + \frac{1}{4}i\right) \pi^2 \sqrt{2}$$

where $e(R) \rightarrow 0$ as $R \rightarrow \infty$. From an earlier example this becomes

$$e(R) + 2 \lim_{r \rightarrow 0+} \int_r^R \frac{\ln t}{1+t^4} dt + i\pi \left(\frac{\sqrt{2}}{4}\pi\right) = \left(-\frac{1}{8} + \frac{1}{4}i\right) \pi^2 \sqrt{2}.$$

Now letting $r \rightarrow 0+$ and $R \rightarrow \infty$,

$$\begin{aligned} 2 \int_0^\infty \frac{\ln t}{1+t^4} dt &= \left(-\frac{1}{8} + \frac{1}{4}i\right) \pi^2 \sqrt{2} - i\pi \left(\frac{\sqrt{2}}{4}\pi\right) \\ &= -\frac{1}{8} \sqrt{2} \pi^2, \end{aligned}$$

and so

$$\int_0^\infty \frac{\ln t}{1+t^4} dt = -\frac{1}{16} \sqrt{2} \pi^2,$$

which is probably not the first thing you would think of. You might try to imagine how this could be obtained using elementary techniques.

The next example illustrates the use of what is referred to as a branch cut. It includes many examples.

Example 29.2.10 *Mellin transformations are of the form*

$$\int_0^\infty f(x) x^\alpha \frac{dx}{x}.$$

Sometimes it is possible to evaluate such a transform in terms of the constant, α .

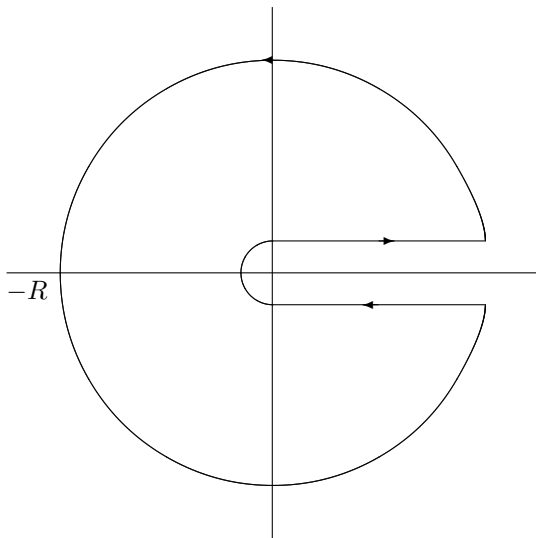
Assume f is an analytic function except at isolated singularities, none of which are on $(0, \infty)$. Also assume that f has the growth conditions,

$$|f(z)| \leq \frac{C}{|z|^b}, b > \alpha$$

for all large $|z|$ and assume that

$$|f(z)| \leq \frac{C'}{|z|^{b_1}}, b_1 < \alpha$$

for all $|z|$ sufficiently small. It turns out there exists an explicit formula for this Mellin transformation under these conditions. Consider the following contour.



In this contour the small semicircle in the center has radius ε which will converge to 0. Denote by γ_R the large circular path which starts at the upper edge of the slot and continues to the lower edge. Denote by γ_ε the small semicircular contour and denote by $\gamma_{\varepsilon R+}$ the straight part of the contour from 0 to R which provides the top edge of the slot. Finally denote by $\gamma_{\varepsilon R-}$ the straight part of the contour from R to 0 which provides the bottom edge of the slot. The interesting aspect of this problem is the definition of $f(z)z^{\alpha-1}$. Let

$$z^{\alpha-1} \equiv e^{(\ln|z| + i \arg(z))(\alpha-1)} = e^{(\alpha-1) \log(z)}$$

where $\arg(z)$ is the angle of z in $(0, 2\pi)$. Thus you use a branch of the logarithm which is defined on $\mathbb{C} \setminus (0, \infty)$. Then it is routine to verify from the assumed estimates that

$$\lim_{R \rightarrow \infty} \int_{\gamma_R} f(z) z^{\alpha-1} dz = 0$$

and

$$\lim_{\varepsilon \rightarrow 0+} \int_{\gamma_\varepsilon} f(z) z^{\alpha-1} dz = 0.$$

Also, it is routine to verify

$$\lim_{\varepsilon \rightarrow 0+} \int_{\gamma_{\varepsilon R+}} f(z) z^{\alpha-1} dz = \int_0^R f(x) x^{\alpha-1} dx$$

and

$$\lim_{\varepsilon \rightarrow 0+} \int_{\gamma_{\varepsilon R-}} f(z) z^{\alpha-1} dz = -e^{i2\pi(\alpha-1)} \int_0^R f(x) x^{\alpha-1} dx.$$

Therefore, letting Σ_R denote the sum of the residues of $f(z)z^{\alpha-1}$ which are contained in the disk of radius R except for the possible residue at 0,

$$e(R) + \left(1 - e^{i2\pi(\alpha-1)}\right) \int_0^R f(x) x^{\alpha-1} dx = 2\pi i \Sigma_R$$

where $e(R) \rightarrow 0$ as $R \rightarrow \infty$. Now letting $R \rightarrow \infty$,

$$\lim_{R \rightarrow \infty} \int_0^R f(x) x^{\alpha-1} dx = \frac{2\pi i}{1 - e^{i2\pi(\alpha-1)}} \Sigma = \frac{\pi e^{-\pi i \alpha}}{\sin(\pi \alpha)} \Sigma$$

where Σ denotes the sum of all the residues of $f(z)z^{\alpha-1}$ except for the residue at 0.

The next example is similar to the one on the Mellin transform. In fact it is a Mellin transform but is worked out independently of the above to emphasize a slightly more informal technique related to the contour.

Example 29.2.11 $\int_0^\infty \frac{x^{p-1}}{1+x} dx$, $p \in (0, 1)$.

Since the exponent of x in the numerator is larger than -1 . The integral does converge. However, the techniques of real analysis don't tell us what it converges to. The contour to be used is as follows: From $(\varepsilon, 0)$ to $(r, 0)$ along the x axis and then from $(r, 0)$ to $(r, 0)$ counter clockwise along the circle of radius r , then from $(r, 0)$ to $(\varepsilon, 0)$ along the x axis and from $(\varepsilon, 0)$ to $(\varepsilon, 0)$, clockwise along the circle of radius ε . You should draw a picture of this contour. The interesting thing about this is that z^{p-1} cannot be defined all the way around 0. Therefore, use a branch of z^{p-1} corresponding to the branch of the logarithm obtained by deleting the positive x axis. Thus

$$z^{p-1} = e^{(\ln|z| + iA(z))(p-1)}$$

where $z = |z|e^{iA(z)}$ and $A(z) \in (0, 2\pi)$. Along the integral which goes in the positive direction on the x axis, let $A(z) = 0$ while on the one which goes in the negative direction, take $A(z) = 2\pi$. This is the appropriate choice obtained by replacing the line from $(\varepsilon, 0)$ to $(r, 0)$ with two lines having a small gap joined by a circle of radius ε and then taking a limit as the gap closes. You should verify that the two integrals taken along the circles of radius ε and r converge to 0 as $\varepsilon \rightarrow 0$ and as $r \rightarrow \infty$. Therefore, taking the limit,

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx + \int_\infty^0 \frac{x^{p-1}}{1+x} \left(e^{2\pi i(p-1)}\right) dx = 2\pi i \operatorname{Res}(f, -1).$$

Calculating the residue of the integrand at -1 , and simplifying the above expression,

$$\left(1 - e^{2\pi i(p-1)}\right) \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i e^{(p-1)i\pi}.$$

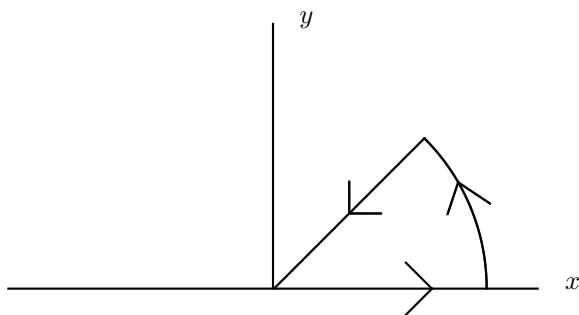
Upon simplification

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}.$$

Example 29.2.12 *The Fresnel integrals are*

$$\int_0^\infty \cos(x^2) dx, \int_0^\infty \sin(x^2) dx.$$

To evaluate these integrals consider $f(z) = e^{iz^2}$ on the curve which goes from the origin to the point r on the x axis and from this point to the point $r\left(\frac{1+i}{\sqrt{2}}\right)$ along a circle of radius r , and from there back to the origin as illustrated in the following picture.



Thus the curve to integrate over is shaped like a slice of pie. Denote by γ_r the curved part. Since f is analytic,

$$\begin{aligned} 0 &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{i\left(t\left(\frac{1+i}{\sqrt{2}}\right)\right)^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\ &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \int_0^r e^{-t^2} \left(\frac{1+i}{\sqrt{2}}\right) dt \\ &= \int_{\gamma_r} e^{iz^2} dz + \int_0^r e^{ix^2} dx - \frac{\sqrt{\pi}}{2} \left(\frac{1+i}{\sqrt{2}}\right) + e(r) \end{aligned}$$

where $e(r) \rightarrow 0$ as $r \rightarrow \infty$. Here we used the fact that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. Now consider the first of these integrals.

$$\begin{aligned} \left| \int_{\gamma_r} e^{iz^2} dz \right| &= \left| \int_0^{\frac{\pi}{4}} e^{i(re^{it})^2} rie^{it} dt \right| \\ &\leq r \int_0^{\frac{\pi}{4}} e^{-r^2 \sin 2t} dt \\ &= \frac{r}{2} \int_0^1 \frac{e^{-r^2 u}}{\sqrt{1-u^2}} du \\ &\leq \frac{r}{2} \int_0^{r^{-(3/2)}} \frac{1}{\sqrt{1-u^2}} du + \frac{r}{2} \left(\int_0^1 \frac{1}{\sqrt{1-u^2}} du \right) e^{-(r^{1/2})} \end{aligned}$$

which converges to zero as $r \rightarrow \infty$. Therefore, taking the limit as $r \rightarrow \infty$,

$$\frac{\sqrt{\pi}}{2} \left(\frac{1+i}{\sqrt{2}} \right) = \int_0^\infty e^{ix^2} dx$$

and so

$$\int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \int_0^\infty \cos x^2 dx.$$

The following example is one of the most interesting. By an auspicious choice of the contour it is possible to obtain a very interesting formula for $\cot \pi z$ known as the Mittag-Leffler expansion of $\cot \pi z$.

Example 29.2.13 Let γ_N be the contour which goes from $-N - \frac{1}{2} - Ni$ horizontally to $N + \frac{1}{2} - Ni$ and from there, vertically to $N + \frac{1}{2} + Ni$ and then horizontally to $-N - \frac{1}{2} + Ni$ and finally vertically to $-N - \frac{1}{2} - Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the following integral.

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z (\alpha^2 - z^2)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag-Leffler,

$$\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \frac{2}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}. \quad (29.2.12)$$

You should verify that $\cot \pi z$ is bounded on this contour and that therefore, $I_N \rightarrow 0$ as $N \rightarrow \infty$. Now you compute the residues of the integrand at $\pm \alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, you can evaluate I_N by using these. It is left as an exercise to calculate these residues and find that the residue at $\pm \alpha$ is

$$\frac{-\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore,

$$0 = \lim_{N \rightarrow \infty} I_N = \lim_{N \rightarrow \infty} 2\pi i \left[\sum_{n=-N}^N \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi \alpha}{\alpha} \right]$$

which establishes the following formula of Mittag-Leffler.

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

Writing this in a slightly nicer form, yields 29.2.12.

29.3 Exercises

1. Example 29.2.7 found the integral of a rational function of a certain sort. The technique used in this example typically works for rational functions of the form $\frac{f(x)}{g(x)}$ where $\deg(g(x)) \geq \deg f(x) + 2$ provided the rational function has no poles on the real axis. State and prove a theorem based on these observations.
2. Fill in the missing details of Example 29.2.13 about $I_N \rightarrow 0$. Note how important it was that the contour was chosen just right for this to happen. Also verify the claims about the residues.
3. Suppose f has a pole of order m at $z = a$. Define $g(z)$ by

$$g(z) = (z - a)^m f(z).$$

Show

$$\operatorname{Res}(f, a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Hint: Use the Laurent series.

4. Give a proof of Theorem 29.1.1. **Hint:** Let p be a pole. Show that near p , a pole of order m ,

$$\frac{f'(z)}{f(z)} = \frac{-m + \sum_{k=1}^{\infty} b_k (z-p)^k}{(z-p) + \sum_{k=2}^{\infty} c_k (z-p)^k}$$

Show that $\operatorname{Res}(f, p) = -m$. Carry out a similar procedure for the zeros.

5. Use Rouché's theorem to prove the fundamental theorem of algebra which says that if $p(z) = z^n + a_{n-1}z^{n-1} \cdots + a_1z + a_0$, then p has n zeros in $\mathbb{C}$. **Hint:** Let $q(z) = -z^n$ and let γ be a large circle, $\gamma(t) = re^{it}$ for r sufficiently large.
6. Consider the two polynomials $z^5 + 3z^2 - 1$ and $z^5 + 3z^2$. Show that on $|z| = 1$, the conditions for Rouché's theorem hold. Now use Rouché's theorem to verify that $z^5 + 3z^2 - 1$ must have two zeros in $|z| < 1$.
7. Consider the polynomial, $z^{11} + 7z^5 + 3z^2 - 17$. Use Rouché's theorem to find a bound on the zeros of this polynomial. In other words, find r such that if z is a zero of the polynomial, $|z| < r$. Try to make r fairly small if possible.
8. Verify that $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$. **Hint:** Use polar coordinates.
9. Use the contour described in Example 29.2.7 to compute the exact values of the following improper integrals.

(a) $\int_{-\infty}^{\infty} \frac{x}{(x^2 + 4x + 13)^2} dx$

- (b) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2} dx$
 (c) $\int_{-\infty}^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}, a, b > 0$

10. Evaluate the following improper integrals.

- (a) $\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2} dx$
 (b) $\int_0^\infty \frac{x \sin x}{(x^2+a^2)^2} dx$

11. Find the Cauchy principle value of the integral

$$\int_{-\infty}^\infty \frac{\sin x}{(x^2+1)(x-1)} dx$$

defined as

$$\lim_{\varepsilon \rightarrow 0+} \left(\int_{-\infty}^{1-\varepsilon} \frac{\sin x}{(x^2+1)(x-1)} dx + \int_{1+\varepsilon}^\infty \frac{\sin x}{(x^2+1)(x-1)} dx \right).$$

12. Find a formula for the integral $\int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}}$ where n is a nonnegative integer.

13. Find $\int_{-\infty}^\infty \frac{\sin^2 x}{x^2} dx$.

14. If $m < n$ for m and n integers, show

$$\int_0^\infty \frac{x^{2m}}{1+x^{2n}} dx = \frac{\pi}{2n} \frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)}.$$

15. Find $\int_{-\infty}^\infty \frac{1}{(1+x^4)^2} dx$.

16. Find $\int_0^\infty \frac{\ln(x)}{1+x^2} dx = 0$.

17. Suppose f has an isolated singularity at α . Show the singularity is essential if and only if the principal part of the Laurent series of f has infinitely many terms. That is, show $f(z) = \sum_{k=0}^\infty a_k (z-\alpha)^k + \sum_{k=1}^\infty \frac{b_k}{(z-\alpha)^k}$ where infinitely many of the b_k are nonzero.

18. Suppose Ω is a bounded open set and f_n is analytic on Ω and continuous on $\bar{\Omega}$. Suppose also that $f_n \rightarrow f$ uniformly on $\bar{\Omega}$ and that $f \neq 0$ on $\partial\Omega$. Show that for all n large enough, f_n and f have the same number of zeros on Ω provided the zeros are counted according to multiplicity.

Some Important Functional Analysis Applications

30.1 The Spectral Radius Of A Bounded Linear Transformation

As a very important application of the theory of Laurent series, I will give a short description of the spectral radius. This is a fundamental result which must be understood in order to prove convergence of various important numerical methods such as the Gauss Seidel or Jacobi methods.

Definition 30.1.1 *Let X be a complex Banach space and let $A \in \mathcal{L}(X, X)$. Then*

$$r(A) \equiv \left\{ \lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(X, X) \right\}$$

This is called the resolvent set. The spectrum of A , denoted by $\sigma(A)$ is defined as all the complex numbers which are not in the resolvent set. Thus

$$\sigma(A) \equiv \mathbb{C} \setminus r(A)$$

Lemma 30.1.2 *$\lambda \in r(A)$ if and only if $\lambda I - A$ is one to one and onto X . Also if $|\lambda| > \|A\|$, then $\lambda \in \sigma(A)$. If the Neumann series,*

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k$$

converges, then

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k = (\lambda I - A)^{-1}.$$

Proof: Note that to be in $r(A)$, $\lambda I - A$ must be one to one and map X onto X since otherwise, $(\lambda I - A)^{-1} \notin \mathcal{L}(X, X)$.

By the open mapping theorem, if these two algebraic conditions hold, then $(\lambda I - A)^{-1}$ is continuous and so this proves the first part of the lemma. Now suppose $|\lambda| > \|A\|$. Consider the Neumann series

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k.$$

By the root test, Theorem 27.1.3 on Page 792 this series converges to an element of $\mathcal{L}(X, X)$ denoted here by B . Now suppose the series converges. Letting $B_n \equiv \frac{1}{\lambda} \sum_{k=0}^n \left(\frac{A}{\lambda} \right)^k$,

$$\begin{aligned} (\lambda I - A) B_n &= B_n (\lambda I - A) = \sum_{k=0}^n \left(\frac{A}{\lambda} \right)^k - \sum_{k=0}^n \left(\frac{A}{\lambda} \right)^{k+1} \\ &= I - \left(\frac{A}{\lambda} \right)^{n+1} \rightarrow I \end{aligned}$$

as $n \rightarrow \infty$ because the convergence of the series requires the n^{th} term to converge to 0. Therefore,

$$(\lambda I - A) B = B (\lambda I - A) = I$$

which shows $\lambda I - A$ is both one to one and onto and the Neumann series converges to $(\lambda I - A)^{-1}$. This proves the lemma.

This lemma also shows that $\sigma(A)$ is bounded. In fact, $\sigma(A)$ is closed.

Lemma 30.1.3 *$r(A)$ is open. In fact, if $\lambda \in r(A)$ and $|\mu - \lambda| < \left\| (\lambda I - A)^{-1} \right\|^{-1}$, then $\mu \in r(A)$.*

Proof: First note

$$(\mu I - A) = \left(I - (\lambda - \mu)(\lambda I - A)^{-1} \right) (\lambda I - A) \quad (30.1.1)$$

$$= (\lambda I - A) \left(I - (\lambda - \mu)(\lambda I - A)^{-1} \right) \quad (30.1.2)$$

Also from the assumption about $|\lambda - \mu|$,

$$\left\| (\lambda - \mu)(\lambda I - A)^{-1} \right\| \leq |\lambda - \mu| \left\| (\lambda I - A)^{-1} \right\| < 1$$

and so by the root test,

$$\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda I - A)^{-1} \right)^k$$

converges to an element of $\mathcal{L}(X, X)$. As in Lemma 30.1.2,

$$\sum_{k=0}^{\infty} \left((\lambda - \mu)(\lambda I - A)^{-1} \right)^k = \left(I - (\lambda - \mu)(\lambda I - A)^{-1} \right)^{-1}.$$

Therefore, from 30.1.1,

$$(\mu I - A)^{-1} = (\lambda I - A)^{-1} \left(I - (\lambda - \mu)(\lambda I - A)^{-1} \right)^{-1}.$$

This proves the lemma.

Corollary 30.1.4 $\sigma(A)$ is a compact set.

Proof: Lemma 30.1.2 shows $\sigma(A)$ is bounded and Lemma 30.1.3 shows it is closed.

Definition 30.1.5 The spectral radius, denoted by $\rho(A)$ is defined by

$$\rho(A) \equiv \max \{ |\lambda| : \lambda \in \sigma(A) \}.$$

Since $\sigma(A)$ is compact, this maximum exists. Note from Lemma 30.1.2, $\rho(A) \leq \|A\|$.

There is a simple formula for the spectral radius.

Lemma 30.1.6 If $|\lambda| > \rho(A)$, then the Neumann series,

$$\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k$$

converges.

Proof: This follows directly from Theorem 29.2.6 on Page 867 and the observation above that $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k = (\lambda I - A)^{-1}$ for all $|\lambda| > \|A\|$. Thus the analytic function, $\lambda \rightarrow (\lambda I - A)^{-1}$ has a Laurent expansion on $|\lambda| > \rho(A)$ by Theorem 29.2.6 and it must coincide with $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k$ on $|\lambda| > \|A\|$ so the Laurent expansion of $\lambda \rightarrow (\lambda I - A)^{-1}$ must equal $\frac{1}{\lambda} \sum_{k=0}^{\infty} \left(\frac{A}{\lambda} \right)^k$ on $|\lambda| > \rho(A)$. This proves the lemma.

The theorem on the spectral radius follows. It is due to Gelfand.

Theorem 30.1.7 $\rho(A) = \lim_{n \rightarrow \infty} \|A^n\|^{1/n}$.

Proof: If

$$|\lambda| < \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}$$

then by the root test, the Neumann series does not converge and so by Lemma 30.1.6 $|\lambda| \leq \rho(A)$. Thus

$$\rho(A) \geq \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}.$$

Now let p be a positive integer. Then $\lambda \in \sigma(A)$ implies $\lambda^p \in \sigma(A^p)$ because

$$\begin{aligned}\lambda^p I - A^p &= (\lambda I - A)(\lambda^{p-1} I + \lambda^{p-2} A + \cdots + A^{p-1}) \\ &= (\lambda^{p-1} I + \lambda^{p-2} A + \cdots + A^{p-1})(\lambda I - A)\end{aligned}$$

It follows from Lemma 30.1.2 applied to A^p that for $\lambda \in \sigma(A)$, $|\lambda^p| \leq \|A^p\|$ and so $|\lambda| \leq \|A^p\|^{1/p}$. Therefore, $\rho(A) \leq \|A^p\|^{1/p}$ and since p is arbitrary,

$$\liminf_{p \rightarrow \infty} \|A^p\|^{1/p} \geq \rho(A) \geq \limsup_{n \rightarrow \infty} \|A^n\|^{1/n}.$$

This proves the theorem.

30.2 Analytic Semigroups

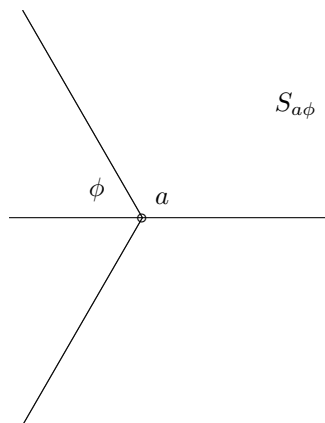
30.2.1 Sectorial Operators And Analytic Semigroups

With the theory of functions of a complex variable, it is time to consider the notion of analytic semigroups. These are better than continuous semigroups. I am mostly following the presentation in Henry [30]. In what follows H will be a Banach space unless specified to be a Hilbert space.

Definition 30.2.1 Let $\phi < \pi/2$ and for $a \in \mathbb{R}$, let $S_{a\phi}$ denote the sector in the complex plane

$$\{z \in \mathbb{C} \setminus \{a\} : |\arg(z - a)| \leq \pi - \phi\}$$

This sector is as shown below.



A closed, densely defined linear operator, A is called sectorial if for some sector as described above, it follows that for all $\lambda \in S_{a\phi}$,

$$(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$$

and for some M it satisfies

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{M}{|\lambda - a|}$$

To begin with it is interesting to have a perturbation theorem for sectorial operators. First note that for $\lambda \in S_{a\phi}$,

$$A(\lambda I - A)^{-1} = -I + \lambda(\lambda I - A)^{-1}$$

Proposition 30.2.2 *Suppose A is a sectorial operator as defined above so it is a densely defined closed operator on $D(A) \subseteq H$ which satisfies*

$$\left\| A(\lambda I - A)^{-1} \right\| \leq C \quad (30.2.3)$$

whenever $|\lambda|, \lambda \in S_{a\phi}$, is sufficiently large and suppose B is a densely defined closed operator such that $D(B) \supseteq D(A)$ and for all $x \in D(A)$,

$$\|Bx\| \leq \varepsilon \|Ax\| + K \|x\| \quad (30.2.4)$$

where $\varepsilon C < 1$. Then $A + B$ is also sectorial.

Proof: I need to consider $(\lambda I - (A + B))^{-1}$. This equals

$$\left(\left(I - B(\lambda I - A)^{-1} \right) (\lambda I - A) \right)^{-1}. \quad (30.2.5)$$

The issue is whether this makes any sense for all $\lambda \in S_{b\phi}$ for some $b \in \mathbb{R}$. Let $b > a$ be very large so that if $\lambda \in S_{b\phi}$, then 30.2.3 holds. Then from 30.2.4, it follows that for $\|x\| \leq 1$,

$$\begin{aligned} \left\| B(\lambda I - A)^{-1} x \right\| &\leq \varepsilon \left\| A(\lambda I - A)^{-1} x \right\| + K \left\| (\lambda I - A)^{-1} x \right\| \\ &\leq \varepsilon C + K/|\lambda - a| \end{aligned}$$

and so if b is made still larger, it follows this is less than $r < 1$ for all $\|x\| \leq 1$. Therefore, for such b ,

$$\left(I - B(\lambda I - A)^{-1} \right)^{-1}$$

exists and so for such b , the expression in 30.2.5 makes sense and equals

$$(\lambda I - A)^{-1} \left(I - B(\lambda I - A)^{-1} \right)^{-1}$$

and furthermore,

$$\left\| (\lambda I - A)^{-1} \left(I - B(\lambda I - A)^{-1} \right)^{-1} \right\| \leq \frac{M}{|\lambda - a|} \frac{1}{1 - r} \leq \frac{M'}{|\lambda - b|}$$

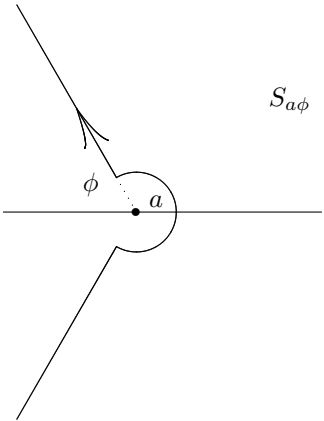
by adjusting the constants because

$$\frac{M}{|\lambda - a|} \frac{|\lambda - b|}{1 - r}$$

is bounded for $\lambda \in S_{b\phi}$. This proves the proposition.

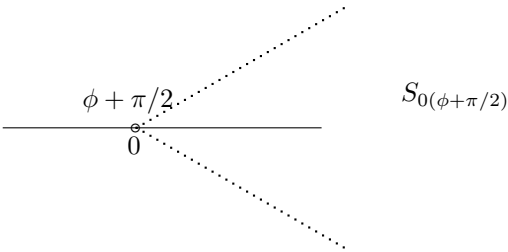
It is an interesting proposition because when you have compact embeddings, such inequalities tend to hold.

Definition 30.2.3 *Let $\varepsilon > 0$ and for a sectorial operator as defined above, let the contour $\gamma_{\varepsilon,\phi}$ be as shown next where the orientation is also as shown by the arrow.*



The little circle has radius ε in the above contour.

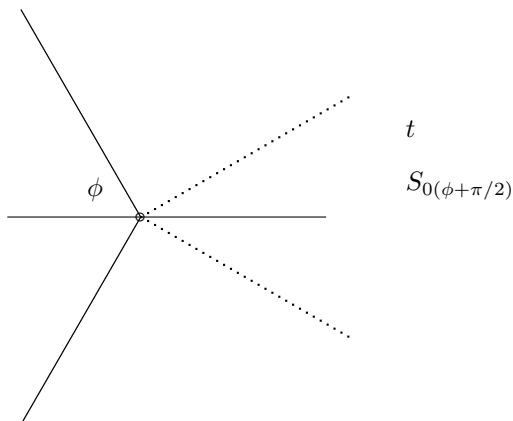
Definition 30.2.4 *For $t \in S^0_{0(\phi+\pi/2)}$ the open sector shown in the following picture,*



define

$$S(t) \equiv \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} (\lambda I - A)^{-1} d\lambda \quad (30.2.6)$$

where ε is chosen such that t is a positive distance from the set of points included in $\gamma_{\varepsilon, \phi}$. The situation is described by the following picture which shows $S_{0(\phi+\pi/2)}^0$ and $S_{0\phi}$. Note how the dotted line is at right angles to the solid line.



Also define $S(0) \equiv I$. It isn't necessary that ε be small, just that $\gamma_{\varepsilon, \phi}$ not contain t . This is because the integrand in 30.2.6 is analytic.

Then it is necessary to show the above definition is well defined.

Lemma 30.2.5 *The above definition is well defined for $t \in S_{0(\phi+\pi/2)}^0$. Also there is a constant, M_r such that*

$$\|S(t)\| \leq M_r$$

for every $t \in S_{0(\phi+\pi/2)}^0$ such that $|\arg t| \leq r < (\frac{\pi}{2} - \phi)$.

Proof: In the definition of $S(t)$ one can take $\varepsilon = 1/|t|$. Then on the little circle which is part of $\gamma_{\varepsilon, \phi}$ the contour integral equals

$$\frac{1}{2\pi} \int_{\phi-\pi}^{\pi-\phi} e^{e^{i(\theta+\arg(t))}} \left(\frac{1}{|t|} e^{i\theta} - A \right)^{-1} \frac{1}{|t|} e^{i\theta} d\theta$$

and by assumption the norm of the integrand is no larger than

$$\frac{M}{1/|t|} \frac{1}{|t|}$$

and so the norm of this integral is dominated by

$$\frac{M}{2\pi} \int_{\phi-\pi}^{\pi-\phi} d\theta = \frac{M}{2\pi} (2\pi - 2\phi) \leq M$$

which is independent of t .

Now consider the part of the contour used to define $S(t)$ which is the top line segment. This equals

$$\frac{1}{2\pi i} \int_{1/|t|}^{\infty} e^{ywt} (ywI - A)^{-1} w dy$$

where w is a fixed complex number of unit length which gives a direction for motion along this segment, $\arg(w) = \pi - \phi$. Then the norm of this is dominated by

$$\frac{1}{2\pi} \int_{1/|t|}^{\infty} |e^{ywt}| \frac{M}{y} dy = \frac{1}{2\pi} \int_{1/|t|}^{\infty} \exp(y|t| \cos(\arg(w) + \arg(t))) \frac{M}{y} dy$$

By assumption $|\arg(t)| \leq r < (\frac{\pi}{2} - \phi)$ and so

$$\begin{aligned} \arg(w) + \arg(t) &\geq (\pi - \phi) - (-r) \\ &= \frac{\pi}{2} + \left(\left(\frac{\pi}{2} - \phi \right) - r \right) \\ &\equiv \frac{\pi}{2} + \delta(r), \quad \delta(r) > 0. \end{aligned}$$

It follows the integral dominated by

$$\begin{aligned} &\frac{1}{2\pi} \int_{1/|t|}^{\infty} \exp(-c(r)|t|y) \frac{M}{y} dy \\ &= \frac{1}{2\pi} \int_1^{\infty} \exp(-c(r)x) \frac{M|t|}{x} \frac{1}{|t|} dx \\ &= \frac{1}{2\pi} \int_1^{\infty} \exp(-c(r)x) \frac{M}{x} dx \end{aligned}$$

where $c(r) < 0$ independent of $|\arg(t)| \leq r$. A similar estimate holds for the integral on the bottom segment. Thus for $|\arg(t)| \leq r$, $\|S(t)\|$ is bounded. This proves the Lemma.

Also note that if the contour is shifted to the right slightly, the integral over the shifted contour, $\gamma'_{\varepsilon, \phi}$ coincides with the integral over $\gamma_{\varepsilon, \phi}$ thanks to the Cauchy integral formula and an approximation argument involving truncating the infinite contours and joining them at either end. Also note that in particular, $\|S(t)\|$ is bounded for all positive real t . The following is the main result.

Theorem 30.2.6 *Let A be a sectorial operator as defined in Definition 30.2.1 for the sector $S_{0\phi}$.*

1. *Then $S(t)$ given above in 30.2.6 is analytic for $t \in S_{0(\phi+\pi/2)}^0$.*
2. *For any $x \in H$ and $t > 0$, then for n a positive integer,*

$$S^{(n)}(t)x = A^n S(t)x$$

3. S is a semigroup on the open sector, $S_{0(\phi+\pi/2)}^0$. That is, for all $t, s \in S_{0(\phi+\pi/2)}^0$,

$$S(t+s) = S(t)S(s)$$

4. $t \rightarrow S(t)x$ is continuous at $t=0$ for all $x \in H$.

5. For some constants M, N such that if t is positive and real,

$$\|S(t)\| \leq M$$

$$\|AS(t)\| \leq \frac{N}{t}$$

Proof: Consider the first claim. This follows right away from the formula.

$$S(t) \equiv \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} (\lambda I - A)^{-1} d\lambda$$

The estimates for uniform convergence do not change for small changes in t and so the formula can be differentiated with respect to the complex variable t using the dominated convergence theorem to obtain

$$\begin{aligned} S'(t) &\equiv \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} \lambda e^{\lambda t} (\lambda I - A)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} \left(I + A(\lambda I - A)^{-1} \right) d\lambda \\ &= \frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} A(\lambda I - A)^{-1} d\lambda \end{aligned}$$

because of the Cauchy integral theorem and an approximation result. Now approximating the infinite contour with a finite one and then the integral with Riemann sums, one can use the fact A is closed to take A out of the integral and write

$$S'(t) = A \left(\frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} (\lambda I - A)^{-1} d\lambda \right) = AS(t)$$

To get the higher derivatives, note $S(t)$ has infinitely many derivatives due to t being a complex variable. Therefore,

$$S''(t) = \lim_{h \rightarrow 0} \frac{S'(t+h) - S'(t)}{h} = \lim_{h \rightarrow 0} A \frac{S(t+h) - S(t)}{h}$$

and

$$\frac{S(t+h) - S(t)}{h} \rightarrow AS(t)$$

and so since A is closed, $AS(t) \in D(A)$ and

$$A^2S(t)$$

Continuing this way yields the claims 1.) and 2.). Note this also implies $S(t)x \in D(A)$ for each $t \in S_{0(\phi+\pi/2)}^0$.

Next consider the semigroup property. Let $s, t \in S_{0(\phi+\pi/2)}^0$ and let ε be sufficiently small that $\gamma_{\varepsilon, \phi}$ is at a positive distance from both s and t . As described above let $\gamma'_{\varepsilon, \phi}$ denote the contour shifted slightly to the right, still at a positive distance from t . Then

$$S(t)S(s) = \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\varepsilon, \phi}} \int_{\gamma'_{\varepsilon, \phi}} e^{\lambda t} (\lambda I - A)^{-1} e^{\mu s} (\mu I - A)^{-1} d\mu d\lambda$$

At this point note that

$$(\lambda I - A)^{-1} (\mu I - A)^{-1} = (\mu - \lambda)^{-1} \left((\lambda I - A)^{-1} - (\mu I - A)^{-1} \right).$$

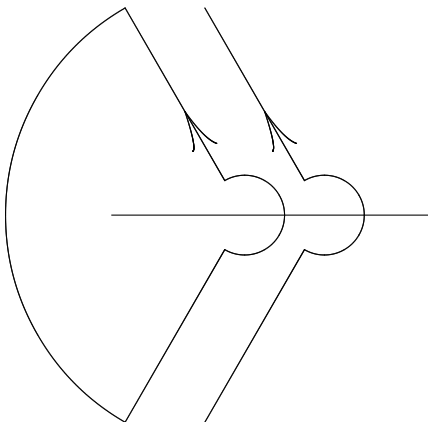
Then substituting this in the integrals above, it equals

$$\begin{aligned} & \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\varepsilon, \phi}} \int_{\gamma'_{\varepsilon, \phi}} e^{\mu s} e^{\lambda t} \left((\mu - \lambda)^{-1} \left((\lambda I - A)^{-1} - (\mu I - A)^{-1} \right) \right) d\mu d\lambda \\ &= - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} \int_{\gamma'_{\varepsilon, \phi}} e^{\mu s} (\mu - \lambda)^{-1} (\mu I - A)^{-1} d\mu d\lambda \\ & \quad + \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\varepsilon, \phi}} \int_{\gamma'_{\varepsilon, \phi}} e^{\mu s} e^{\lambda t} (\mu - \lambda)^{-1} (\lambda I - A)^{-1} d\mu d\lambda \end{aligned}$$

The order of integration can be interchanged because of the absolute convergence and Fubini's theorem. Then this reduces to

$$\begin{aligned} &= - \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma'_{\varepsilon, \phi}} (\mu I - A)^{-1} e^{\mu s} \int_{\gamma_{\varepsilon, \phi}} e^{\lambda t} (\mu - \lambda)^{-1} d\lambda d\mu \\ & \quad + \left(\frac{1}{2\pi i}\right)^2 \int_{\gamma_{\varepsilon, \phi}} (\lambda I - A)^{-1} e^{\lambda t} \int_{\gamma'_{\varepsilon, \phi}} e^{\mu s} (\mu - \lambda)^{-1} d\mu d\lambda \end{aligned}$$

Now the following diagram might help in drawing some interesting conclusions.



The first iterated integral equals 0. This can be seen from the above picture. The inner integral taken over $\gamma_{\varepsilon, \phi}$ is essentially equal to the integral over the closed contour in the above picture provided the radius of the part of the circle in the above closed contour is large enough. This closed contour integral equals 0 by the Cauchy integral theorem. The second iterated integral equals

$$\frac{1}{2\pi i} \int_{\gamma_{\varepsilon, \phi}} (\lambda I - A)^{-1} e^{\lambda t} e^{\lambda s} d\lambda$$

This can be seen from considering a similar closed contour, using the Cauchy integral formula. This verifies the semigroup identity.

Next consider 4.), the continuity at $t = 0$. This requires showing

$$\lim_{t \rightarrow 0+} S(t)x = x$$

where the limit is taken through positive real values of t . It suffices to let $x \in D(A)$ because by Lemma 30.2.5, $\|S(t)\|$ is bounded for these values of t . Also in doing the computation, let $\gamma_{\varepsilon, \phi}$ equal $\gamma_{t^{-1}, \phi}$ and it will be assumed $t < 1$. Then one must estimate

$$\|S(t)x - x\| \equiv \left\| \frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} e^{\lambda t} (\lambda I - A)^{-1} x d\lambda - Ix \right\| \quad (30.2.7)$$

By the Cauchy integral formula and approximating the integral with a contour integral over a finite closed contour,

$$\frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} \frac{e^{\lambda t}}{\lambda} d\lambda = 1$$

and so 30.2.7 equals

$$\begin{aligned} & \left\| \frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} e^{\lambda t} \left((\lambda I - A)^{-1} - I\lambda^{-1} \right) x d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} e^{\lambda t} \lambda^{-1} (\lambda I - A)^{-1} A x d\lambda \right\| \end{aligned}$$

Changing the variable letting $\lambda t = \mu$, the above equals

$$\left\| \frac{1}{2\pi i} \int_{\gamma_{1, \phi}} e^{\mu t} (\mu)^{-1} ((\mu/t) I - A)^{-1} A x \frac{1}{t} d\mu \right\|$$

which is dominated by

$$\frac{1}{2\pi} \int_{\gamma_{1, \phi}} e^{|\mu| \cos \psi} \frac{t^2 M}{|\mu|^2} \|Ax\| \frac{1}{t} d|\mu| = tC \|Ax\|.$$

Therefore, $\lim_{t \rightarrow 0+} \|S(t)x - x\| = 0$ whenever $x \in D(A)$. Since $S(t)$ is bounded for positive t , if $y \in H$ is arbitrary, choose $x \in D(A)$ close enough to y such that

$$\begin{aligned} \|S(t)y - y\| &\leq \|S(t)(y - x)\| + \|S(t)x - x\| + \|x - y\| \\ &\leq \varepsilon/2 + \|S(t)x - x\| \end{aligned}$$

and now if t is close enough to 0 it follows the right side is less than ε . This proves 4.).

Finally consider 5.). The first part follows from Lemma 30.2.5. It remains to show the second part. Let $x \in H$. First suppose $t \neq 1$.

$$\begin{aligned} \|AS(t)x\| &= \left\| \frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} e^{\lambda t} A (\lambda I - A)^{-1} x d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} e^{\lambda t} \left(-I + \lambda (\lambda I - A)^{-1} \right) x d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma_{t^{-1}, \phi}} e^{\lambda t} \lambda (\lambda I - A)^{-1} x d\lambda \right\| \\ &= \left\| \frac{1}{2\pi i} \int_{\gamma_{1, \phi}} e^{\mu} \frac{\mu}{t} \left(\frac{\mu}{t} I - A \right)^{-1} x \frac{1}{t} d\mu \right\| \end{aligned}$$

and this is dominated by

$$\leq \frac{1}{2\pi} \int_{\gamma_{1, \phi}} e^{|\mu| \cos \psi} M d|\mu| \frac{1}{t} = \frac{N}{t}.$$

This proves the theorem.

What if A is sectorial in the more general sense with $S_{a\phi}$ taking the place of $S_{0\phi}$ and the resolvent estimate being

$$\left\| (\lambda I - A)^{-1} \right\| \leq \frac{M}{|\lambda - a|}, \quad \lambda \in S_{a\phi}?$$

Then $S_{a\phi} = a + S_{0\phi}$ and so for $\lambda \in S_{0\phi}$, $(\lambda + a) \in S_{a\phi}$ and so

$$(\lambda I - (A - aI))^{-1} = ((\lambda + a)I - A)^{-1} \in \mathcal{L}(H, H)$$

and

$$\begin{aligned} \left\| (\lambda I - (A - aI))^{-1} \right\| &= \left\| ((\lambda + a)I - A)^{-1} \right\| \\ &\leq \frac{M}{|(\lambda + a) - a|} = \frac{M}{|\lambda|} \end{aligned}$$

Therefore, letting $A_a \equiv A - aI$, it follows the result of Theorem 30.2.6 holds for this operator. There exists a semigroup, $S_a(t)$ satisfying

1. $S_a(t)$ is analytic for $t \in S_{0(\phi+\pi/2)}^0$.
2. For any $x \in H$ and $t > 0$, then for n a positive integer,

$$S_a^{(n)}(t)x = A_a^n S_a(t)x$$

3. S_a is a semigroup on the open sector, $S_{0(\phi+\pi/2)}^0$. That is, for all $t, s \in S_{0(\phi+\pi/2)}^0$,

$$S_a(t+s) = S_a(t)S_a(s)$$

4. $t \rightarrow S_a(t)x$ is continuous at $t = 0$ for all $x \in H$.
5. For some constants M, N such that if t is positive and real,

$$\|S_a(t)\| \leq M$$

$$\|A_a S_a(t)\| \leq \frac{N}{t}$$

Define

$$S(t) \equiv e^{at} S_a(t).$$

This satisfies the semigroup identity, is continuous at 0, and satisfies the inequality

$$\|S(t)\| \leq M e^{at}$$

for all t real and nonnegative. What is its derivative?

$$\begin{aligned} S'(t) &= a e^{at} S_a(t) + (A - aI) e^{at} S_a(t) \\ &= A e^{at} S_a(t) = AS(t) \end{aligned}$$

and by continuing this way using A is closed as before, it follows

$$S^{(n)}(t) = A^n S(t).$$

Also for $t > 0$,

$$\begin{aligned} \|AS(t)\| &= \|e^{at}AS_a(t)\| \\ &= \|e^{at}(A - aI)S_a(t) + e^{at}aS_a(t)\| \\ &= \|e^{at}A_aS_a(t) + e^{at}aS_a(t)\| \\ &\leq e^{at}\frac{N}{t} + ae^{at}M = e^{at}\left(\frac{N}{t} + aM\right). \end{aligned}$$

It is clear $S(t)x$ is continuous at 0 since this is true of $S_a(t)$.

This proves the following corollary which is a useful generalization of the above major theorem.

Corollary 30.2.7 *Let A be sectorial satisfying*

$$(\lambda I - A)^{-1} \in \mathcal{L}(H, H) \text{ for } \lambda \in S_{a\phi}$$

for $\phi < \pi/2$ and

$$\left\|(\lambda I - A)^{-1}\right\| \leq \frac{M}{|\lambda - a|}$$

for all $\lambda \in S_{a\lambda}$. Then there exists a semigroup, $S(t)$ defined on $S_{0(\phi+\pi/2)}^0$ which satisfies

1. $S(t)$ is analytic for $t \in S_{0(\phi+\pi/2)}^0$.
2. For any $x \in H$ and $t > 0$, then for n a positive integer,

$$S^{(n)}(t)x = A^n S(t)x$$

3. $S(t)$ is a semigroup on the open sector, $S_{0(\phi+\pi/2)}^0$. That is, for all $t, s \in S_{0(\phi+\pi/2)}^0$,

$$S(t+s) = S(t)S(s)$$

4. $t \rightarrow S(t)x$ is continuous at $t = 0$ for all $x \in H$.
5. For some constants M, N such that if t is positive and real,

$$\|S(t)\| \leq Me^{at}$$

$$\|AS(t)\| \leq e^{at}\left(\frac{N}{t} + aM\right)$$

30.2.2 The Numerical Range

In Hilbert space, there is a useful easy to check criterion which implies an operator is sectorial.

Definition 30.2.8 Let A be a closed densely defined operator $A : D(A) \rightarrow H$ for H a Hilbert space. The numerical range is the following set.

$$\{(Au, u) : u \in D(A)\}$$

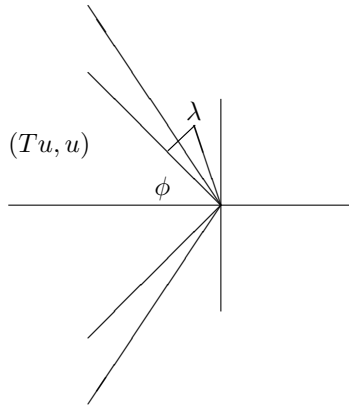
Also recall the resolvent set, $r(A)$ consists of those $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1} \in \mathcal{L}(H, H)$. Thus, to be in this set $\lambda I - A$ is one to one and onto with continuous inverse.

Proposition 30.2.9 Suppose the numerical range of A , a closed densely defined operator $A : D(A) \rightarrow H$ for H a Hilbert space is contained in the set

$$\{z \in \mathbb{C} : |\arg(z)| \geq \pi - \phi\}$$

where $0 < \phi < \pi/2$ and suppose $A^{-1} \in \mathcal{L}(H, H)$, $(0 \in r(A))$. Then A is sectorial with the sector $S_{0, \phi'}$ where $\pi/2 > \phi' > \phi$.

Proof: Here is a picture of the situation along with details used to motivate the proof.



In the picture the angle which is a little larger than ϕ is ϕ' . Let λ be as shown with $|\arg \lambda| \leq \pi - \phi'$. Then from the picture and trigonometry, if $u \in D(A)$,

$$|\lambda| \sin(\phi' - \phi) < \left| \lambda - \left(A \frac{u}{|u|}, \frac{u}{|u|} \right) \right|$$

and so

$$|u| |\lambda| \sin(\phi' - \phi) < \left| \left(\lambda u - Au, \frac{u}{|u|} \right) \right| \leq \|(\lambda I - A)u\|$$

Hence for all λ such that $|\arg \lambda| \leq \pi - \phi'$ and $u \in D(A)$,

$$\begin{aligned} |u| &< \left(\frac{1}{\sin(\phi' - \phi)} \right) \frac{1}{|\lambda|} |(\lambda I - A)u| \\ &\equiv \frac{M}{|\lambda|} |(\lambda I - A)u| \end{aligned}$$

Thus $(\lambda I - A)$ is one to one on $S_{0,\phi'}$ and if $\lambda \in r(A)$, then

$$\left\| (\lambda I - A)^{-1} \right\| < \frac{M}{|\lambda|}.$$

By assumption $0 \in r(A)$. Now if $|\mu|$ is small,

$$(\mu I - A)^{-1}$$

must exist because it equals

$$((\mu A^{-1} - I)A)^{-1}$$

and for $|\mu| < \|A^{-1}\|$, $(\mu A^{-1} - I)^{-1} \in \mathcal{L}(H, H)$ since the infinite series

$$\sum_{k=0}^{\infty} (-1)^k (\mu A^{-1})^k$$

converges and must equal to $(\mu A^{-1} - I)^{-1}$. Therefore, there exists $\mu \in S_{0,\phi'}$ such that $\mu \neq 0$ and $\mu \in r(A)$. Also if $\mu \neq 0$ and $\mu \in S_{0,\phi'}$, then if $|\lambda - \mu| < \frac{|\mu|}{M}$, $(\lambda I - A)^{-1}$ must exist because

$$(\lambda I - A)^{-1} = \left[\left((\lambda - \mu)(\mu I - A)^{-1} - I \right) (\mu I - A) \right]^{-1}$$

where $\left((\lambda - \mu)(\mu I - A)^{-1} - I \right)^{-1}$ exists because

$$\begin{aligned} \left\| (\lambda - \mu)(\mu I - A)^{-1} \right\| &= |\lambda - \mu| \left\| (\mu I - A)^{-1} \right\| \\ &< \frac{|\mu|}{M} \cdot \frac{M}{|\mu|} = 1. \end{aligned}$$

It follows that if $S \equiv \{\lambda \in S_{0,\phi'} : \lambda \in r(A)\}$, then S is open in $S_{0,\phi}$. However, S is also closed because if $\lambda = \lim_{n \rightarrow \infty} \lambda_n$ where $\lambda_n \in S$, then if $\lambda = 0$, it is given $\lambda \in S$. If $\lambda \neq 0$, then for large enough n ,

$$|\lambda - \lambda_n| < \frac{|\lambda_n|}{M}$$

and so $\lambda \in S$. Since $S_{0,\phi'}$ is connected, it follows $S = S_{0,\phi'}$. This proves the proposition.

Corollary 30.2.10 *If for some $a \in \mathbb{R}$, the numerical values of $-aI + A$ are in the set $\{\lambda : |\lambda| \geq \pi - \phi\}$ where $0 < \phi < \pi/2$, and $a \in r(A)$ then A is sectorial.*

Proof: By assumption, $0 \in r(-aI + A)$ and also from Proposition 30.2.9, for $\mu \in S_{0,\phi'}$ where $\pi/2 > \phi' > \phi$,

$$\begin{aligned} ((-aI + A) - \mu I)^{-1} &\in \mathcal{L}(H, H), \\ \left\| ((-aI + A) - \mu I)^{-1} \right\| &\leq \frac{M}{|\mu|} \end{aligned}$$

Therefore, for $\mu \in S_{0,\phi'}$, $\mu + a \in r(A)$. Therefore, if $\lambda \in S_{a,\phi'}$, $\lambda - a \in S_{0,\phi'}$

$$\left\| (A - \lambda I)^{-1} \right\| = \left\| (A - aI - (\lambda - a)I)^{-1} \right\| \leq \frac{M}{|\lambda - a|}$$

This proves the corollary.

30.2.3 An Interesting Example

In this section related to this example, for V a Banach space, V' will denote the space of continuous conjugate linear functions defined on V . Usually the symbol has meant the space of continuous linear functions but here they will be conjugate linear. That is $f \in V'$ means

$$f(ax + by) = \bar{a}f(x) + \bar{b}f(y)$$

and f is continuous.

Let Ω be a bounded open set in $\mathbb{R}^n$ and define

$$V_0 \equiv \{u \in C^\infty(\bar{\Omega}) : u = 0 \text{ on } \Gamma\}$$

where Γ is some measurable subset of the boundary of Ω and $C^\infty(\bar{\Omega})$ denotes the restrictions of functions in $C_c^\infty(\mathbb{R}^n)$ to Ω . By Corollary 12.5.11 V_0 is dense in $L^2(\Omega)$. Now define the following for $u, v \in V_0$.

$$A_0 u(v) \equiv -a \int_{\Omega} u \bar{v} dx - \int_{\Omega} a(x) \nabla u \cdot \bar{\nabla} v dx$$

where $a > 0$ and $a(x) \geq 0$ is a $C^1(\bar{\Omega})$ function. Also define the following inner product on V_0 .

$$(u, v)_1 \equiv \int_{\Omega} (au\bar{v} + a(x) \nabla u \cdot \bar{\nabla} v) dx$$

Let $\|\cdot\|_1$ denote the corresponding norm.

Of course V_0 is not a Banach space because it fails to be complete. $u \in V$ will mean that $u \in L^2(\Omega)$ and there exists a sequence $\{u_n\} \subseteq V_0$ such that

$$\lim_{m,n \rightarrow \infty} \|u_n - u_m\|_1 = 0$$

and

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{L^2(\Omega)} = 0.$$

For $u \in V$, define ∇u to be that element of $L^2(\Omega; \mathbb{C}^n, a(x) dm_n)$, the space of vector valued L^2 functions taken with respect to the measure $a(x) dm_n$ which satisfies

$$\|\nabla u - \nabla u_n\|_{L^2(\Omega; \mathbb{C}^n, a(x) dm_n)} \rightarrow 0.$$

Denote this space by W for simplicity of notation.

Observation 30.2.11 *V is a Hilbert space with inner product given by*

$$(u, v)_1 \equiv \int_{\Omega} (au\bar{v} + a(x) \nabla u \cdot \overline{\nabla v}) dx$$

Everything is obvious except completeness. Suppose then that $\{u_n\}$ is a Cauchy sequence in V . Then there exists a unique $u \in L^2(\Omega)$ such that $\|u_n - u\|_{L^2(\Omega)} \rightarrow 0$. Now let

$$\|w_n - u_n\|_{L^2(\Omega)} + \|\nabla w_n - \nabla u_n\|_W < 1/2^n$$

It follows $\{\nabla w_n\}$ is also a Cauchy sequence in W while $\{w_n\}$ is a Cauchy sequence in $L^2(\Omega)$ converging to u . Thus the thing to which ∇w_n converges in W is the definition of ∇u and $u \in V$. Thus

$$\begin{aligned} \|u_n - u\|_1 &\leq \|u_n - w_n\|_1 + \|w_n - u\|_1 \\ &< \frac{1}{2^n} + \|w_n - u\|_1 \end{aligned}$$

and the last term converges to 0. Hence V is complete as claimed.

Then it is clear V is a Hilbert space. The next observation is a simple one involving the Riesz map.

Definition 30.2.12 *Let V be a Hilbert space and let V' be the space of continuous conjugate linear functions defined on V . Then define $R : V \rightarrow V'$ by*

$$Rx(y) \equiv (x, y).$$

This is called the Riesz map.

Lemma 30.2.13 *The Riesz map is one to one and onto and linear.*

Proof: It is obvious it is one to one and linear. The only challenge is to show it is onto. Let $z^* \in V'$. If $z^*(V) = \{0\}$, then letting $z = 0$, it follows $Rz = z^*$. If $z^*(V) \neq \{0\}$, then

$$\ker(z^*) \equiv \{x \in V : z^*(x) = 0\}$$

is a closed subspace. It is closed because z^* is continuous and it is just $z^{*-1}(0)$. Since $\ker(z^*)$ is not everything in V there exists

$$w \in \ker(z^*)^\perp \equiv \{x : (x, y) = 0 \text{ for all } y \in \ker(z^*)\}$$

and $w \neq 0$. Then

$$z^* \left(\overline{z^*(x)w - z^*(w)x} \right) = z^*(x)z^*(w) - z^*(w)z^*(x) = 0$$

and so $\overline{z^*(x)w - z^*(w)x} \in \ker(z^*)$. Therefore, for any $x \in V$,

$$\begin{aligned} 0 &= \left(w, \overline{z^*(x)w - z^*(w)x} \right) \\ &= z^*(x)(w, w) - z^*(w)(w, x) \end{aligned}$$

and so

$$z^*(x) = \left(\frac{z^*(w)}{\|w\|^2}, x \right)$$

so let $z = w/\|w\|^2$. Then $Rz = z^*$ and so R is onto. This proves the lemma.

Now for the V described above,

$$Ru(v) = \int_{\Omega} (au\bar{v} + a(x) \nabla u \cdot \overline{\nabla v}) dx$$

Also, as noted above V is dense in $H \equiv L^2(\Omega)$ and so if H is identified with H' , it follows

$$V \subseteq H = H' \subseteq V'.$$

Let $A : D(A) \rightarrow H$ be given by

$$D(A) \equiv \{u \in V : Ru \in H\}$$

and

$$A \equiv -R$$

on $D(A)$. Then the numerical range for A is contained in $(-\infty, -a]$ and so A is sectorial by Proposition 30.2.9 provided A is closed and densely defined.

Why is $D(A)$ dense? It is because it contains $C_c^\infty(\Omega)$ which is dense in $L^2(\Omega)$. This follows from integration by parts which shows that for $u, v \in C_c^\infty(\Omega)$,

$$\begin{aligned} & - \int_{\Omega} au\bar{v} dx - \int_{\Omega} a(x) \nabla u \cdot \overline{\nabla v} dx \\ &= - \int_{\Omega} au\bar{v} dx + \int_{\Omega} \nabla \cdot (a(x) \nabla u) \bar{v} dx \end{aligned}$$

and since $C_c^\infty(\Omega)$ is dense in H ,

$$Au = -au + \nabla \cdot (a(x) \nabla u) \in L^2(\Omega) = H.$$

Why is A closed? If $u_n \in D(A)$ and $u_n \rightarrow u$ in H while $Au_n \rightarrow \xi$ in H , then it follows from the definition that $Ru_n \rightarrow -\xi$ and $\{u_n\}$ converges to u in V so for any $v \in V$,

$$Ru(v) = \lim_{n \rightarrow \infty} Ru_n(v) = \lim_{n \rightarrow \infty} (Ru_n, v)_H = (-\xi, v)_H$$

which shows $Ru = -\xi \in H$ and so $u \in D(A)$ and $Au = \xi$. Thus A is closed. This completes the example.

Obviously you could follow identical reasoning to include many other examples of more complexity. What does it mean for $u \in D(A)$? It means that in a weak sense

$$-au + \nabla \cdot (a(x) \nabla u) \in H.$$

Since A is sectorial for $S_{-a, \phi}$ for any $0 < \phi < \pi/2$, this has shown the existence of a weak solution to the partial differential equation along with appropriate boundary conditions,

$$-au + \nabla \cdot (a(x) \nabla u) = f, \quad u \in V.$$

What are these appropriate boundary conditions? $u = 0$ on Γ is one. the other would be a variational boundary condition which comes from integration by parts. Letting $v \in V$, formally do the following using the divergence theorem.

$$\begin{aligned} (f, v)_H &= \int_{\Omega} (-au + \nabla \cdot (a(x) \nabla u)) v dx \\ &= \int_{\Omega} -auv dx + \int_{\partial\Omega} (a(x) \nabla u) \cdot n ds - \int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) dx \\ &= (f, v)_H + \int_{\partial\Omega \setminus \Gamma} (a(x) \nabla u) \cdot n v ds \end{aligned}$$

and so the other boundary condition is

$$a(x) \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \setminus \Gamma.$$

To what extent this weak solution is really a classical solution depends on more technical considerations.

30.2.4 Fractional Powers Of Sectorial Operators

It will always be assumed in this section that A is sectorial for the sector $S_{-a, \phi}$ where $a > 0$. To begin with, here is a useful lemma which will be used in the presentation of these fractional powers.

Lemma 30.2.14 *The following holds for $\alpha \in (0, 1)$ and $\sigma < t$.*

$$\int_{\sigma}^t (t-s)^{\alpha-1} (s-\sigma)^{-\alpha} ds = \frac{\pi}{\sin(\pi\alpha)}$$

In particular,

$$\int_0^1 (1-s)^{\alpha-1} s^{-\alpha} ds = \frac{\pi}{\sin(\pi\alpha)}.$$

Also for $\alpha, \beta > 0$

$$\Gamma(\alpha) \Gamma(\beta) = \left(\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \right) \Gamma(\alpha + \beta).$$

Proof: First change variables to get rid of the σ . Let $y = (t - \sigma)^{-1}(s - \sigma)$. Then the integral becomes

$$\begin{aligned} & \int_0^1 (t - [(t - \sigma)y + \sigma])^{\alpha-1} (t - \sigma)^{-\alpha} y^{-\alpha} (t - \sigma) dy \\ &= \int_0^1 ((t - \sigma)(1 - y))^{\alpha-1} (t - \sigma)^{-\alpha} y^{-\alpha} (t - \sigma) dy \\ &= \int_0^1 (1 - y)^{\alpha-1} y^{-\alpha} dy \end{aligned}$$

Next let $y = x^2$. The integral is

$$2 \int_0^1 (1 - x^2)^{\alpha-1} x^{1-2\alpha} dx$$

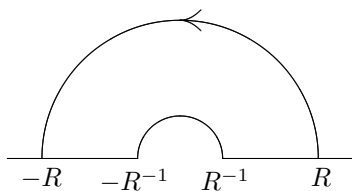
Next let $x = \sin \theta$

$$2 \int_0^{\frac{1}{2}\pi} (\cos(\theta))^{2\alpha-1} \sin^{(1-2\alpha)}(\theta) d\theta = 2 \int_0^{\frac{1}{2}\pi} \left(\frac{\cos(\theta)}{\sin(\theta)} \right)^{2\alpha-1} d\theta$$

Now change the variable again. Let $u = \cot(\theta)$. Then this yields

$$2 \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} du$$

This is fairly easy to evaluate using contour integrals. Consider the following contour called Γ_R for large R . As $R \rightarrow \infty$, the integral over the little circle converges to 0 and so does the integral over the big circle. There is one singularity at i .



Thus

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{e^{(\ln|z| + i \arg(z))(1-2\alpha)}}{1 + z^2} dz = \\ &= (1 + \cos(1 - 2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} du \\ & \quad + i \sin((1 - 2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1 + u^2} du \end{aligned}$$

$$= \pi \left(\cos \left(\frac{\pi}{2} (1 - 2\alpha) \right) + i \sin \left(\frac{\pi}{2} (1 - 2\alpha) \right) \right)$$

Then equating the imaginary parts yields

$$\sin((1 - 2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1+u^2} du = \pi \sin \left(\frac{\pi}{2} (1 - 2\alpha) \right)$$

and so using the trig identities for the sum of two angles,

$$\begin{aligned} \int_0^\infty \frac{u^{2\alpha-1}}{1+u^2} du &= \frac{\pi \left(\sin \left(\frac{\pi}{2} (1 - 2\alpha) \right) \right)}{2 \sin \left(\frac{\pi}{2} (1 - 2\alpha) \right) \cos \left(\frac{\pi}{2} (1 - 2\alpha) \right)} \\ &= \frac{\pi}{2 \cos \left(\frac{\pi}{2} (1 - 2\alpha) \right)} = \frac{\pi}{2 \sin(\pi\alpha)} \end{aligned}$$

It remains to verify the last identity.

$$\begin{aligned} \Gamma(\alpha) \Gamma(\beta) &\equiv \int_0^\infty \int_0^\infty t^{\alpha-1} e^{-t} s^{\beta-1} e^{-s} ds dt \\ &= \int_0^\infty \int_t^\infty t^{\alpha-1} e^{-u} (u-t)^{\beta-1} du dt \\ &= \int_0^\infty e^{-u} \int_0^u t^{\alpha-1} (u-t)^{\beta-1} dt du \\ &= \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \int_0^\infty e^{-u} u^{\alpha+\beta-1} du \\ &= \left(\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \right) \Gamma(\alpha + \beta) \end{aligned}$$

This proves the lemma.

If it is not stated otherwise, in all that follows $\alpha > 0$.

Definition 30.2.15 Let A be a sectorial operator corresponding to the sector $S_{-a\phi}$ where $-a < 0$. Then define for $\alpha > 0$,

$$(-A)^{-\alpha} \equiv \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt$$

where $S(t)$ is the analytic semigroup generated by A as in Corollary 30.2.7. Note that from the estimate, $\|S(t)\| \leq Me^{-at}$ of this corollary, the integral is well defined and is in $\mathcal{L}(H, H)$.

Theorem 30.2.16 For $(-A)^{-\alpha}$ as defined in Definition 30.2.15

$$(-A)^{-\alpha} (-A)^{-\beta} = (-A)^{-(\alpha+\beta)} \quad (30.2.8)$$

Also

$$(-A)^{-1} (-A) = I, \quad (-A) (-A)^{-1} = I \quad (30.2.9)$$

and $(-A)^{-\alpha}$ is one to one if $\alpha \geq 0$, defining $A^0 \equiv I$.

If $\alpha < \beta$, then

$$(-A)^{-\beta}(H) \subseteq (-A)^{-\alpha}(H). \quad (30.2.10)$$

If $\alpha \in (0, 1)$, then

$$(-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda \quad (30.2.11)$$

Proof: Consider 30.2.8.

$$(-A)^{-\alpha}(-A)^{-\beta} \equiv \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty t^{\alpha-1} s^{\beta-1} S(t+s) ds dt$$

Changing variables and using Fubini's theorem which is justified because of the absolute convergence of the iterated integrals, which follows from Corollary 30.2.7, this becomes

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_t^\infty t^{\alpha-1} (u-t)^{\beta-1} S(u) du dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^u t^{\alpha-1} (u-t)^{\beta-1} S(u) dt du \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty S(u) \int_0^1 (ux)^{\alpha-1} (u-ux)^{\beta-1} u dx du \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \right) \int_0^\infty S(u) u^{\alpha+\beta-1} du \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left(\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} dx \right) \Gamma(\alpha+\beta) (-A)^{-(\alpha+\beta)} \\ &= (-A)^{-(\alpha+\beta)} \end{aligned}$$

This proves the first part of the theorem.

Consider 30.2.9. Since A is a closed operator, and approximating the integral with an appropriate sequence of Riemann sums, $(-A)$ can be taken inside the integral and so

$$\begin{aligned} (-A) \frac{1}{\Gamma(1)} \int_0^\infty t^{1-1} S(t) dt &= \int_0^\infty (-A) S(t) dt \\ &= \int_0^\infty -\frac{d}{dt} (S(t)) dt = S(0) = I. \end{aligned}$$

Next let $x \in D(-A)$. Then

$$\begin{aligned} \frac{1}{\Gamma(1)} \int_0^\infty t^{1-1} S(t) dt (-A)x &= - \int_0^\infty S(t) A x dt \\ &= - \int_0^\infty A S(t) x dt = \int_0^\infty -\frac{d}{dt} (S(t)) dt = Ix \end{aligned}$$

This shows that the integral in which $\alpha = 1$ deserves to be called A^{-1} so the definition is not bad notation. Also, by assumption, A^{-1} is one to one. Thus

$$(-A)^{-1}(-A)^{-1}x = 0$$

implies

$$(-A)^{-1}x = 0$$

hence $x = 0$ so that $(-A)^{-2}$ is also one to one. Similarly, $(-A)^{-m}$ is one to one for all positive integers m .

From what was just shown, if $(-A)^{-\alpha}x = 0$ for $\alpha \in (0, 1)$, then

$$(-A)^{-1}x = (-A)^{-(1-\alpha)}(-A)^{-\alpha}x = 0$$

and so $x = 0$. This shows $(-A)^{-\alpha}$ is one to one for all $\alpha \in [0, 1]$ if is defined as $(-A)^0 \equiv I$.

What about $\alpha > 1$? For such α , it is of the form $m + \beta$ where $\beta \in [0, 1)$ and m is a positive integer. Therefore, if

$$(-A)^{-(m+\beta)}x = 0$$

then

$$(-A)^{-\beta}((-A)^{-m})x = 0$$

and so from what was just shown,

$$((-A)^{-m})x = 0$$

and now this implies $x = 0$ so that $(-A)^{-\alpha}$ is one to one for all $\alpha \geq 0$.

Consider 30.2.10. It was shown above that

$$(-A)^{-\alpha}(-A)^{-\beta} = (-A)^{-(\alpha+\beta)}$$

Let $x = (-A)^{-(\alpha+\beta)}y$. Then

$$x = (-A)^{-\alpha}(-A)^{-\beta}y \subseteq (-A)^{-\alpha}(-A)^{-\beta}(H) \subseteq (-A)^{-\alpha}(H).$$

This proves 30.2.10. If $\alpha < \beta$, $(-A)^{-\beta}(H) \subseteq (-A)^{-\alpha}(H)$.

Now consider the problem of writing $(-A)^{-\alpha}$ for $\alpha \in (0, 1)$ in terms of A , not mentioning $S(t)$. By Proposition 15.12.5,

$$(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} S(t) x dt$$

Then

$$\begin{aligned} \int_0^\infty \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda &= \int_0^\infty \lambda^{-\alpha} \int_0^\infty e^{-\lambda t} S(t) dt d\lambda \\ &= \int_0^\infty S(t) \int_0^\infty \lambda^{-\alpha} e^{-\lambda t} d\lambda dt \\ &= \int_0^\infty S(t) \int_0^\infty \lambda^{\beta-1} e^{-\lambda t} d\lambda dt \end{aligned}$$

where $\beta \equiv 1 - \alpha$. Then using Lemma 30.2.14, this equals

$$\begin{aligned} \int_0^\infty S(t) \int_0^\infty \mu^{\beta-1} t^{1-\beta} e^{-\mu} t^{-1} d\mu dt &= \int_0^\infty t^{-\beta} S(t) \int_0^\infty \mu^{\beta-1} e^{-\mu} d\mu dt \\ &= \Gamma(1-\alpha) \int_0^\infty t^{\alpha-1} S(t) dt = \Gamma(\alpha) \Gamma(1-\alpha) (-A)^{-\alpha} \\ &= \left(\int_0^1 x^{\alpha-1} (1-x)^{-\alpha} dx \right) (-A)^{-\alpha} = \frac{\pi}{\sin(\pi\alpha)} (-A)^{-\alpha} \end{aligned}$$

and so this gives the formula

$$(-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda.$$

This proves 30.2.11.

Definition 30.2.17 For $\alpha \geq 0$, define $(-A)^\alpha$ on $D((-A)^\alpha) \equiv (-A)^{-\alpha}(H)$ by

$$(-A)^\alpha \equiv \left((-A)^{-\alpha} \right)^{-1}$$

Note that if $\alpha, \beta > 0$, then if $x \in D((-A)^{\alpha+\beta})$,

$$\begin{aligned} (-A)^{\alpha+\beta} x &= \left((-A)^{-(\alpha+\beta)} \right)^{-1} x = \\ &= \left((-A)^{-\alpha} (-A)^{-\beta} \right)^{-1} x = (-A)^\beta (-A)^\alpha x. \end{aligned} \quad (30.2.12)$$

Next let $\beta > \alpha > 0$ and let $x \in D((-A)^\beta)$. Then from what was just shown,

$$(-A)^\alpha (-A)^{\beta-\alpha} x = (-A)^\beta x$$

and so

$$(-A)^{\beta-\alpha} x = (-A)^{-\alpha} (-A)^\beta x$$

If $x \in D((-A)^\beta)$, does it follow that $(-A)^{-\alpha} x \in D((-A)^\beta)$? Note $x = (-A)^{-\beta} y$ and so

$$(-A)^{-\alpha} x = (-A)^{-\alpha} (-A)^{-\beta} y = (-A)^{-(\alpha+\beta)} y \in D((-A)^{\alpha+\beta}).$$

Therefore, from 30.2.12,

$$(-A)^{\beta-\alpha} x = (-A)^{\beta-\alpha} (-A)^\alpha \left((-A)^{-\alpha} x \right) = (-A)^\beta (-A)^{-\alpha} x.$$

Theorem 30.2.18 *The definition of $(-A)^\alpha$ is well defined and $(-A)^\alpha$ is densely defined and closed. Also for any $\alpha > 0$,*

$$\|(-A)^\alpha S(t)\| \leq \frac{C_\alpha}{\delta} \frac{1}{t^\alpha} e^{-\delta t} \quad (30.2.13)$$

where $-\delta > -a$. Furthermore, C_α is bounded as $\alpha \rightarrow 0+$ and is bounded on compact intervals of $(0, \infty)$. Also for $\alpha \in (0, 1)$ and $x \in D((-A)^\alpha)$,

$$\|(S(t) - I)x\| \leq \frac{C_{1-\alpha}}{\alpha \delta} t^\alpha \|(-A)^\alpha x\| \quad (30.2.14)$$

There exists a constant C independent of $\alpha \in [0, 1]$ such that for $x \in D(A)$ and $\varepsilon > 0$,

$$\|(-A)^\alpha x\| \leq \varepsilon \|(-A)x\| + C\varepsilon^{-\alpha/(1-\alpha)} \|x\| \quad (30.2.15)$$

There exists a constant C' independent of $\alpha \in [0, 1]$ such that for $x \in D(A)$,

$$\|(-A)^\alpha x\| \leq C' \|(-A)x\|^\alpha \|x\|^{1-\alpha} \quad (30.2.16)$$

The formula 30.2.16 is called an interpolation inequality.

Proof: It is obvious $(-A)^\alpha$ is densely defined because its domain is at least as large as $D(A)$ which was assumed to be dense. It is a closed operator because if $x_n \in D((-A)^\alpha)$ and

$$x_n \rightarrow x, \quad (-A)^\alpha x_n \rightarrow y,$$

then

$$(-A)^{-\alpha} x_n \rightarrow (-A)^{-\alpha} x, \quad x_n = (-A)^{-\alpha} (-A)^\alpha x_n \rightarrow (-A)^{-\alpha} y$$

and so

$$(-A)^{-\alpha} y = x$$

showing $x \in D((-A)^\alpha)$ and $y = (-A)^{-\alpha} x$. Thus $(-A)^\alpha$ is closed and densely defined.

Let $-\delta > -a$ where the sector for A was $S_{-a, \phi}$, $a > 0$. Then recall from Corollary 30.2.7 there is a constant, N such that

$$\|(-A) S(t)\| \leq \frac{N}{t} e^{-\delta t}$$

What about $\|(-A)^\alpha S(t)\|$? First note that for $\alpha \in [0, 1]$ this at least makes sense because $S(t)$ maps into $D(A)$. For any $\alpha > 0$,

$$S(t)(-A)^{-\alpha} = (-A)^{-\alpha} S(t)$$

follows from the definition of $(-A)^{-\alpha}$. Therefore,

$$(-A)^\alpha S(t)(-A)^{-\alpha} = S(t). \quad (30.2.17)$$

Note this implies that on $D((-A)^\alpha)$,

$$(-A)^\alpha S(t) = S(t)(-A)^\alpha.$$

Also

$$(-A)^{-1} S(t) = S(t)(-A)^{-1} = S(t)(-A)^{-\alpha}(-A)^{-(1-\alpha)}$$

and so

$$S(t) = (-A) S(t)(-A)^{-\alpha}(-A)^{-(1-\alpha)}$$

From 30.2.17 it follows

$$\begin{aligned} (-A)^\alpha S(t) &= (-A)(-A)^\alpha S(t)(-A)^{-\alpha}(-A)^{-(1-\alpha)} \\ &= (-A) S(t)(-A)^{-(1-\alpha)} \end{aligned} \quad (30.2.18)$$

Then with this formula,

$$\begin{aligned} \|(-A)^\alpha S(t)\| &= \left\| (-A) S(t)(-A)^{-(1-\alpha)} \right\| \\ &= \left\| \frac{1}{\Gamma(1-\alpha)} \int_0^\infty s^{1-\alpha} (-A) S(t+s) ds \right\| \\ &\leq \frac{N}{\Gamma(1-\alpha)} \int_0^\infty \frac{s^{1-\alpha}}{(t+s)} e^{-\delta(s+t)} ds \\ &= \frac{N}{\Gamma(1-\alpha)} \int_t^\infty \frac{(u-t)^{1-\alpha}}{u} e^{-\delta u} ds \\ &\leq \frac{N}{\Gamma(1-\alpha)} \int_t^\infty \left(1 - \frac{t}{u}\right)^{1-\alpha} \frac{1}{u^\alpha} e^{-\delta u} ds \\ &\leq \frac{N}{\Gamma(1-\alpha)} \frac{1}{t^\alpha} \int_t^\infty e^{-\delta u} ds = \frac{N}{\Gamma(1-\alpha)} \frac{1}{\delta t^\alpha} e^{-\delta t} \\ &\equiv \frac{C_\alpha}{\delta} \frac{1}{t^\alpha} e^{-\delta t}. \end{aligned}$$

this establishes the formula when $\alpha \in [0, 1)$. Next suppose $\alpha = m$, a positive integer.

$$\begin{aligned} \|A^m S(t)\| &= \left\| A^m S\left(\frac{t}{m}\right)^m \right\| \\ &= \left\| \left(AS\left(\frac{t}{m}\right)\right)^m \right\| \leq \frac{N}{t^m} m^m. \end{aligned}$$

This is why the above inequality holds.

If $\alpha, \beta > 0$,

$$\begin{aligned} \|A^{\alpha+\beta} S(t)\| &= \left\| A^{\alpha+\beta} S\left(\frac{t}{2}\right) S\left(\frac{t}{2}\right) \right\| \\ &= \left\| A^{\alpha} S\left(\frac{t}{2}\right) A^{\beta} S\left(\frac{t}{2}\right) \right\| \\ &\leq \frac{C_{\alpha}}{t^{\alpha}} \frac{C_{\beta}}{t^{\beta}} e^{-2\delta t} = \frac{C}{t^{\alpha+\beta}} e^{-\delta t} \end{aligned}$$

Suppose now that $\alpha > 0$. Then

$$\alpha = m + \beta$$

where $\beta \in [0, 1)$. Then from what was just shown,

$$\|A^{m+\beta} S(t)\| \leq \frac{C}{t^{m+\beta}} e^{-\delta t}.$$

Next consider 30.2.14. First note that whenever $\alpha > 0$,

$$(-A)^{-\alpha} S(s) = S(s) (-A)^{-\alpha}$$

and so on $D((-A)^{\alpha})$,

$$S(s) = (-A)^{\alpha} S(s) (-A)^{-\alpha}, \quad S(s) (-A)^{\alpha} = (-A)^{\alpha} S(s)$$

Now for $x \in D((-A)^{\alpha})$,

$$\begin{aligned} \|(S(t) - I)x\| &= \left\| -\int_0^t (-A) S(s) x ds \right\| \\ &= \left\| -\int_0^t (-A)^{1-\alpha} (-A)^{\alpha} S(s) x ds \right\| \\ &= \left\| -\int_0^t (-A)^{1-\alpha} S(s) (-A)^{\alpha} x ds \right\| \\ &\leq \int_0^t \left\| (-A)^{1-\alpha} S(s) \right\| ds \|(-A)^{\alpha} x\| \\ &\leq \int_0^t \frac{C_{1-\alpha}}{\delta} \frac{1}{s^{1-\alpha}} e^{-\delta s} ds \|(-A)^{\alpha} x\| \\ &\leq \frac{C_{1-\alpha}}{\delta} \frac{1}{\alpha} t^{\alpha} \|(-A)^{\alpha} x\| \end{aligned}$$

and this shows 30.2.14.

Next consider 30.2.15. Let $x \in H$ and $\beta \in (0, 1)$. Then

$$\|(-A)^{-\beta} x\| = \frac{1}{\Gamma(\beta)} \left\| \int_0^{\infty} t^{\beta-1} S(t) x dt \right\|$$

$$\begin{aligned}
&= \frac{1}{\Gamma(\beta)} \left\| \int_0^\eta t^{\beta-1} S(t) x dt + \int_\eta^\infty t^{\beta-1} S(t) x dt \right\| \\
&\leq \frac{1}{\Gamma(\beta)} \int_0^\eta t^{\beta-1} \|S(t) x\| dt + \frac{1}{\Gamma(\beta)} \left\| \int_\eta^\infty t^{\beta-1} S(t) x dt \right\| \\
&\leq \frac{C}{\Gamma(\beta)} \frac{\eta^\beta}{\beta} \|x\| + \frac{1}{\Gamma(\beta)} \left\| \int_\eta^\infty t^{\beta-1} S(t) x dt \right\| \\
&\leq \frac{C}{\Gamma(\beta)} \frac{\eta^\beta}{\beta} \|x\| + \\
&\quad \frac{1}{\Gamma(\beta)} \left\| \eta^{\beta-1} S(\eta) A^{-1} x + (1-\beta) \int_\eta^\infty t^{\beta-2} S(t) A^{-1} x dt \right\| \\
&\leq \frac{1}{\Gamma(\beta)} \left(\frac{C\eta^\beta}{\beta} \|x\| + \eta^{\beta-1} \|A^{-1} x\| + (1-\beta) \|A^{-1} x\| \int_\eta^\infty t^{\beta-2} dt \right) \\
&= \frac{1}{\Gamma(\beta)} \left(\frac{C\eta^\beta}{\beta} \|x\| + 2\eta^{\beta-1} \|A^{-1} x\| \right).
\end{aligned}$$

Now let $\delta = C\eta^\beta$ so $\eta = C^{-1/\beta} \delta^{1/\beta}$ and $\eta^{\beta-1} = C^{\frac{1-\beta}{\beta}} \delta^{(\beta-1)/\beta}$. Thus for all $x \in H$,

$$\left\| (-A)^{-\beta} x \right\| \leq \frac{1}{\Gamma(\beta)} \left(\frac{\delta}{\beta} \|x\| + 2C^{\frac{1-\beta}{\beta}} \delta^{(\beta-1)/\beta} \|A^{-1} x\| \right).$$

Let $\varepsilon = \frac{\delta}{\beta\Gamma(\beta)} = \frac{\delta}{\Gamma(1+\beta)}$. Then the above is of the form

$$\begin{aligned}
\left\| (-A)^{-\beta} x \right\| &\leq \varepsilon \|x\| + 2 \frac{C^{\frac{1-\beta}{\beta}}}{\Gamma(\beta)} (\varepsilon \Gamma(1+\beta))^{(\beta-1)/\beta} \|A^{-1} x\| \\
&\leq \varepsilon \|x\| + 2C^{\frac{1-\beta}{\beta}} (\varepsilon \Gamma(1+\beta))^{(\beta-1)/\beta} \|A^{-1} x\|
\end{aligned}$$

because Γ is decreasing on $(0, 1)$. I need to verify that for $\beta \in (0, 1)$,

$$\Gamma(1+\beta)^{(\beta-1)/\beta}$$

is bounded. It is continuous on $(0, 1]$ and so if I can show $\lim_{\beta \rightarrow 0+} \Gamma(1+\beta)^{(\beta-1)/\beta}$ exists, then it will follow the function is bounded. It suffices to show

$$\lim_{\beta \rightarrow 0+} \frac{\beta-1}{\beta} \ln \Gamma(1+\beta) = - \lim_{\beta \rightarrow 0+} \frac{\ln \Gamma(1+\beta)}{\beta}$$

exists. Consider this. By L'Hospital's rule and dominated convergence theorem, this is

$$\begin{aligned}
\lim_{\beta \rightarrow 0+} \frac{\int_0^\infty \ln(t) t^\beta e^{-t} dt}{\Gamma(1+\beta)} &= \lim_{\beta \rightarrow 0+} \int_0^\infty \ln(t) t^\beta e^{-t} dt \\
&= \lim_{\beta \rightarrow 0+} \int_0^\infty \ln(t) e^{-t} dt.
\end{aligned}$$

Thus the function is bounded independent of $\beta \in (0, 1)$. This shows there is a constant C which is independent of $\beta \in (0, 1)$ such that for any $x \in H$,

$$\left\| (-A)^{-\beta} x \right\| \leq \varepsilon \|x\| + C\varepsilon^{(\beta-1)/\beta} \|A^{-1}x\|. \quad (30.2.19)$$

Now let $y \in D(A) = D((-A))$ and let $x = (-A)y$. Then the above becomes

$$\left\| (-A)^{-\beta} (-A)y \right\| \leq \varepsilon \|(-A)y\| + C\varepsilon^{(\beta-1)/\beta} \|y\|$$

I claim that

$$(-A)^{-\beta} (-A)y = (-A)^{1-\beta} y.$$

The reason for this is as follows.

$$(-A)^\beta (-A)^{1-\beta} y = (-A)y$$

and so the desired result follows from multiplying on the left by $(-A)^{-\beta}$. Hence

$$\left\| (-A)^{1-\beta} y \right\| \leq \varepsilon \|(-A)y\| + C\varepsilon^{(\beta-1)/\beta} \|y\|$$

Now let $1 - \beta = \alpha$ and obtain

$$\|(-A)^\alpha y\| \leq \varepsilon \|(-A)y\| + C\varepsilon^{-\alpha/(1-\alpha)} \|y\|$$

This proves 30.2.15.

Finally choose ε to minimize the right side of the above expression. Thus let

$$\varepsilon = \left(\frac{\alpha \|y\| C}{\|(-A)y\| (1-\alpha)} \right)^{1-\alpha}$$

Then the above expression becomes

$$\begin{aligned} \|(-A)^\alpha y\| &\leq \|(-A)y\| \left(\frac{\alpha \|y\| C}{\|(-A)y\| (1-\alpha)} \right)^{1-\alpha} \\ &\quad + C \left(\left(\frac{\alpha \|y\| C}{\|(-A)y\| (1-\alpha)} \right)^{1-\alpha} \right)^{-\alpha/(1-\alpha)} \|y\| \\ &= \|(-A)y\|^\alpha \|y\|^{1-\alpha} \left(\frac{\alpha C}{(1-\alpha)} \right)^{1-\alpha} \\ &\quad + \|(-A)y\|^\alpha \|y\|^{1-\alpha} \left(\frac{\alpha C}{(1-\alpha)} \right)^{-\alpha} \\ &= \left(\left(\frac{\alpha C}{(1-\alpha)} \right)^{1-\alpha} + \left(\frac{\alpha C}{(1-\alpha)} \right)^{-\alpha} \right) \|(-A)y\|^\alpha \|y\|^{1-\alpha} \\ &\leq C' \|(-A)y\|^\alpha \|y\|^{1-\alpha} \end{aligned}$$

where C' does not depend on $\alpha \in (0, 1)$. To see such a constant exists, note

$$\lim_{\alpha \rightarrow 1} \left(\frac{\alpha C}{(1 - \alpha)} \right)^{1 - \alpha} = 1$$

and

$$\lim_{\alpha \rightarrow 1} \left(\frac{\alpha C}{(1 - \alpha)} \right)^{-\alpha} = 0$$

while

$$\lim_{\alpha \rightarrow 0} \left(\frac{\alpha C}{(1 - \alpha)} \right)^{1 - \alpha} = 0, \quad \lim_{\alpha \rightarrow 0} \left(\frac{\alpha C}{(1 - \alpha)} \right)^{-\alpha} = 1$$

Of course C' depends on C but as shown above, this did not depend on $\alpha \in (0, 1)$. This proves 30.2.16.

The following corollary follows from the proof of the above theorem.

Corollary 30.2.19 *Let $\alpha \in (0, 1)$. Then for all $\varepsilon > 0$, there exists a constant $C(\alpha, \varepsilon)$ such that*

$$\left\| (-A)^{-\alpha} x \right\| \leq \varepsilon \|x\| + C(\varepsilon, \alpha) \left\| (-A)^{-1} x \right\|$$

Also if A^{-1} is compact, then so is $(-A)^{-\alpha}$ for all $\alpha \in (0, 1)$.

Proof: The first part is done in the above theorem. Let S be a bounded set and let $\eta > 0$. Then let $\varepsilon > 0$ be small enough that for all $x \in S$, $\varepsilon \|x\| < \eta/4$. Let $\left\{ (-A)^{-1} x_n \right\}$ be a $\eta/(2 + 2C(\varepsilon, \alpha))$ net for $(-A)^{-1}(S)$. Then if $(-A)^{-\alpha} x \in (-A)^{-\alpha} S$, there exists x_n such that

$$\left\| (-A)^{-1} x_n - (-A)^{-1} x \right\| < \frac{\eta}{2 + 2C(\varepsilon, \alpha)}.$$

Then

$$\begin{aligned} \left\| (-A)^{-\alpha} x_n - (-A)^{-\alpha} x \right\| &\leq \varepsilon \|x_n - x\| + C(\varepsilon, \alpha) \left\| (-A)^{-1} x_n - (-A)^{-1} x \right\| \\ &< \frac{\eta}{2} + \frac{\eta}{2} = \eta \end{aligned}$$

showing $(-A)^{-\alpha}(S)$ has a η net. Thus $(-A)^{-\alpha}$ is compact. This proves the corollary.

The next proposition gives a general interpolation inequality.

Proposition 30.2.20 *Let $0 < \alpha < \beta$ and let*

$$\gamma = \theta\beta + (1 - \theta)\alpha, \quad \theta \in (0, 1).$$

Then there exists a constant, C such that for all $x \in D\left((-A)^\beta\right)$,

$$\|(-A)^\gamma x\| \leq C \left\| (-A)^\beta x \right\|^\theta \|(-A)^\alpha x\|^{1 - \theta}.$$

Proof: This is an exercise in using 30.2.13. Letting $x \in D((-A)^\beta)$,

$$(-A)^\gamma x = (-A)^\theta (-A)^{-\theta} (-A)^\gamma x$$

Therefore, letting C denote a generic constant, it follows since $(-A)^\theta$ is closed,

$$\begin{aligned} \Gamma(\theta) \|(-A)^\gamma x\| &= \left\| \int_0^\infty t^{\theta-1} (-A)^\theta S(t) (-A)^\gamma x dt \right\| \\ &\leq \int_0^\eta t^{\theta-1} \left\| (-A)^\theta (-A)^{\gamma-\beta} S(t) (-A)^\beta x \right\| dt \\ &\quad + \int_\eta^\infty t^{\theta-1} \left\| (-A)^\theta (-A)^{\gamma-\alpha} S(t) (-A)^\alpha x \right\| dt \\ &\leq C \int_0^\eta t^{\theta-1} t^{-\theta} t^{\beta-\gamma} dt \left\| (-A)^\beta x \right\| + C \int_\eta^\infty t^{\theta-1} t^{-\theta} t^{\alpha-\gamma} dt \|(-A)^\alpha x\| \\ &= C \left(\frac{\eta^{\beta-\gamma}}{\beta-\gamma} \left\| (-A)^\beta x \right\| + \frac{\eta^{-(\gamma-\alpha)}}{\gamma-\alpha} \|(-A)^\alpha x\| \right) \end{aligned}$$

and now writing in what γ is in terms of θ yields

$$\Gamma(\theta) \|(-A)^\gamma x\| \leq C \left(\frac{1}{\beta-\alpha} \right) \left(\frac{(\eta^{\beta-\alpha})^{1-\theta}}{(1-\theta)} \left\| (-A)^\beta x \right\| + \frac{(\eta^{\beta-\alpha})^{-\theta}}{\theta} \|(-A)^\alpha x\| \right)$$

Letting $\lambda = \eta^{\beta-\alpha}$, it follows

$$\Gamma(\theta) \|(-A)^\gamma x\| \leq C \left(\frac{1}{\beta-\alpha} \right) \left(\frac{\lambda^{1-\theta}}{(1-\theta)} \left\| (-A)^\beta x \right\| + \frac{\lambda^{-\theta}}{\theta} \|(-A)^\alpha x\| \right)$$

then let

$$\lambda = \frac{\|(-A)^\alpha x\|}{\left\| (-A)^\beta x \right\|}$$

which is obtained from minimizing the expression on the right in the above. then placing this in the inequality yields

$$\begin{aligned} &\Gamma(\theta) \|(-A)^\gamma x\| \\ &\leq C \left(\frac{1}{\beta-\alpha} \right) \left(\frac{\left(\frac{\|(-A)^\alpha x\|}{\left\| (-A)^\beta x \right\|} \right)^{1-\theta}}{(1-\theta)} \left\| (-A)^\beta x \right\| \right. \\ &\quad \left. + \frac{\left(\frac{\|(-A)^\alpha x\|}{\left\| (-A)^\beta x \right\|} \right)^{-\theta}}{\theta} \|(-A)^\alpha x\| \right) \end{aligned}$$

$$= C \left(\frac{1}{\beta - \alpha} \right) \left(\frac{1}{(1 - \theta)} + \frac{1}{\theta} \right) \|(-A)^\alpha x\|^{1-\theta} \left\| (-A)^\beta x \right\|^\theta$$

and this proves the proposition.

Note that the constant is not bounded as $\theta \rightarrow 1$.

Here is another interesting result about compactness.

Proposition 30.2.21 *Let A be sectorial for $S_{-a, \phi}$ where $-a < 0$. Then the following are equivalent.*

1. $(-A)^{-\alpha}$ is compact for all $\alpha > 0$.
2. $S(t)$ is compact for each $t > 0$.

Proof: First suppose $(-A)^{-\alpha}$ is compact for all $\alpha > 0$. Then

$$\begin{aligned} \Gamma(\alpha)(-A)^{-\alpha} &= \int_0^t s^{\alpha-1} S(s) ds + \int_t^\infty s^{\alpha-1} S(s) ds \\ &= \frac{t^\alpha}{\alpha} S(t) - \int_0^t \frac{s^\alpha}{\alpha} AS(s) ds + s^{\alpha-1} S(s) A^{-1} \Big|_t^\infty \\ &\quad - (\alpha - 1) \int_t^\infty s^{\alpha-2} S(s) A^{-1} ds \end{aligned}$$

Now

$$\left\| \frac{s^\alpha}{\alpha} AS(s) \right\| \leq C \frac{s^{\alpha-1}}{\alpha}$$

and so the second integral satisfies

$$\left\| \int_0^t \frac{s^\alpha}{\alpha} AS(s) ds \right\| \leq C \frac{t^\alpha}{\alpha^2}$$

$$\begin{aligned} \Gamma(\alpha)(-A)^{-\alpha} &= O\left(\frac{t^\alpha}{\alpha^2}\right) + \frac{t^\alpha}{\alpha} S(t) \\ &\quad - t^{\alpha-1} A^{-1} S(t) - (\alpha - 1) \int_t^\infty s^{\alpha-2} S(s) ds A^{-1} \end{aligned}$$

It follows that for $t > 0$, and $\varepsilon > 0$ given,

$$\begin{aligned} S(t) &= \left(\frac{t^\alpha}{\alpha} - t^{\alpha-1} \right)^{-1} \left(\Gamma(\alpha)(-A)^{-\alpha} \right. \\ &\quad \left. + (\alpha - 1) \int_t^\infty s^{\alpha-2} S(s) ds A^{-1} + O\left(\frac{t^\alpha}{\alpha^2}\right) \right) \\ &= \left(\frac{t^\alpha}{\alpha} - t^{\alpha-1} \right)^{-1} \left(\Gamma(\alpha)(-A)^{-\alpha} \right. \\ &\quad \left. + (\alpha - 1) \int_t^\infty s^{\alpha-2} S(s) ds A^{-1} \right) + O\left(\frac{1}{\alpha}\right) \\ &= N_\alpha + O\left(\frac{1}{\alpha}\right). \end{aligned}$$

where N_α is a compact operator. Now let B be a bounded set in H , $\|x\| \leq M$ for all $x \in B$ and let $\eta > 0$ be given. Then choose α large enough that $\|O(\frac{1}{\alpha})\| < \frac{\eta}{4+4M}$. Then there exists a $\eta/2$ net, $\{N_\alpha x_n\}_{n=1}^N$ for $N_\alpha(B)$. Then consider $\{S(t)x_n\}_{n=1}^N$. For $x \in B$, there exists x_n such that $\|N_\alpha x_n - N_\alpha x\| < \eta/2$. Then

$$\begin{aligned} \|S(t)x - S(t)x_n\| &\leq \|S(t)x - N_\alpha x\| \\ &\quad + \|N_\alpha x - N_\alpha x_n\| + \|N_\alpha x_n - S(t)x_n\| \\ &\leq \frac{\eta}{4+4M}M + \frac{\eta}{2} + \frac{\eta}{4+4M}M < \eta \end{aligned}$$

Thus $S(t)(B)$ has an η net for every $\eta > 0$ and so $S(t)$ is compact.

Next suppose $S(t)$ is compact for all $t > 0$. Then

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t) dt$$

and the integral is a limit in norm of Riemann sums of the form

$$\sum_{k=1}^m t_k^{\alpha-1} S(t_k) \Delta t_k$$

and each of these operators is compact. Since $(-A)^{-\alpha}$ is the limit in norm of compact operators, it must also be compact. This proves the proposition.

Here are some observations which are listed in the book by Henry [30]. Like the above proposition, these are exercises in this book.

Observation 30.2.22 *For each $x \in H$, $t \rightarrow tAS(t)$ is continuous and $\lim_{t \rightarrow 0+} tAS(t)x = 0$.*

The reason for this is that if $x \in D(A)$, then

$$tAS(t)x = |tS(t)Ax| \rightarrow 0$$

as $t \rightarrow 0$. Now suppose $y \in H$ is arbitrary. Then letting $x \in D(A)$,

$$\begin{aligned} |tAS(t)y| &\leq |tAS(t)(y-x)| + |tAS(t)x| \\ &\leq \varepsilon + |tAS(t)x| \end{aligned}$$

provided x is close enough to y . The last term converges to 0 and so

$$\limsup_{t \rightarrow 0+} |tAS(t)y| \leq \varepsilon$$

where $\varepsilon > 0$ is arbitrary. Thus

$$\lim_{t \rightarrow 0+} |tAS(t)y| = 0.$$

Why is $t \rightarrow tAS(t)x$ continuous on $[0, T]$? This is true if $x \in D(A)$ because $t \rightarrow tS(t)Ax$ is continuous. If $y \in H$ is arbitrary, let x_n converge to y in H where $x_n \in D(A)$. Then

$$|tAS(t)y - tAS(t)x_n| \leq C|y - x_n|$$

and so the convergence is uniform. Thus $t \rightarrow tAS(t)y$ is continuous because it is the uniform limit of a sequence of continuous functions.

Observation 30.2.23 *If $x \in H$ and A is sectorial for $S_{-a, \phi}$, $-a < 0$, then for any $\alpha \in [0, 1]$,*

$$\lim_{t \rightarrow 0+} t^\alpha \|(-A)^\alpha S(t)x\| = 0.$$

This follows as above because you can verify this is true for $x \in D(A)$ and then use the fact shown above that

$$t^\alpha \|(-A)^\alpha S(t)\| \leq C$$

to extend it to x arbitrary.

30.2.5 A Scale Of Banach Spaces

Next I will present an important and interesting theorem which can be used to prove equivalence of certain norms.

Theorem 30.2.24 *Let A, B be sectorial for $S_{-a, \phi}$ where $-a < 0$ and suppose $D(A) = D(B)$. Also suppose*

$$(A - B)(-A)^{-\alpha}, (A - B)(-B)^{-\alpha}$$

are both bounded on $D(A)$ for some $\alpha \in (0, 1)$. Then for all $\beta \in [0, 1]$,

$$(-A)^\beta (-B)^{-\beta}, (-B)^\beta (-A)^{-\beta}$$

are both bounded on $D(A) = D(B)$. Also $D((-A)^\beta) = D((-B)^\beta)$.

Proof: First of all it is a good idea to verify $(A - B)(-A)^{-\alpha}, (A - B)(-B)^{-\alpha}$ make sense on $D(A)$. If $x \in D(A)$, then why is $(-A)^{-\alpha}x \in D(A)$? Here is why. Since $x \in D(A)$,

$$x = (-A)^{-1}y$$

for some $y \in H$. Then

$$(-A)^{-\alpha}x = (-A)^{-\alpha}(-A)^{-1}y = (-A)^{-1}(-A)^{-\alpha}y \in D(A).$$

The case of $(A - B)(-B)^{-\alpha}$ is similar.

Next for $\beta \in (0, 1)$ and $\lambda > 0$, use 30.2.16 to write

$$\begin{aligned}
 & \left\| (-A)^\beta (\lambda I - A)^{-1} x \right\| \\
 & \leq C \left\| (-A) (\lambda I - A)^{-1} x \right\|^\beta \left\| (\lambda I - A)^{-1} x \right\|^{1-\beta} \\
 & \leq C \left\| (-A) (\lambda I - A)^{-1} \right\|^\beta \left\| (\lambda I - A)^{-1} \right\|^{1-\beta} \|x\| \\
 & \leq C \left\| I - \lambda (\lambda I - A)^{-1} \right\|^\beta \frac{M}{(\lambda + \delta)^{1-\beta}} \|x\| \\
 & \leq C \left(1 + \frac{\lambda}{(\lambda + \delta)} \right)^\beta \frac{M}{(\lambda + \delta)^{1-\beta}} \|x\| \equiv \frac{C}{(\lambda + \delta)^{1-\beta}} \|x\| \quad (30.2.20)
 \end{aligned}$$

where $-a < -\delta < 0$ where C denotes a generic constant. Similarly, for all $\beta \in (0, 1)$,

$$\left\| (-B)^\beta (\lambda I - B)^{-1} x \right\| \leq \frac{C}{(\lambda + \delta)^{1-\beta}} \|x\| \quad (30.2.21)$$

Now from Theorem 30.2.16 and letting $\beta \in (0, 1)$,

$$\begin{aligned}
 (-B)^{-\beta} - (-A)^{-\beta} &= \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \lambda^{-\beta} \left((\lambda I - B)^{-1} - (\lambda I - A)^{-1} \right) d\lambda \\
 &= \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \lambda^{-\beta} (\lambda I - B)^{-1} (A - B) (\lambda I - A)^{-1} d\lambda. \quad (30.2.22)
 \end{aligned}$$

Therefore, letting $x \in D(A)$ and letting C denote a generic constant which can be changed from line to line and using 30.2.20 and 30.2.21,

$$\begin{aligned}
 & \left\| x - (-B)^\beta (-A)^{-\beta} x \right\| \\
 & \leq C \int_0^\infty \frac{1}{\lambda^\beta} \left\| (-B)^\beta (\lambda I - B)^{-1} (A - B) (\lambda I - A)^{-1} x \right\| d\lambda
 \end{aligned}$$

The reason $(-B)^\beta$ goes inside the integral is that it is a closed operator. Then the above

$$\begin{aligned}
 & \leq C \int_0^\infty \frac{1}{\lambda^\beta (\lambda + \delta)^{1-\beta}} \left\| (A - B) (-A)^{-\alpha} (-A)^\alpha (\lambda I - A)^{-1} x \right\| d\lambda \\
 & \leq C \int_0^\infty \frac{1}{\lambda^\beta (\lambda + \delta)^{1-\beta}} \left\| (-A)^\alpha (\lambda I - A)^{-1} x \right\| d\lambda \\
 & \leq C \int_0^\infty \frac{1}{\lambda^\beta (\lambda + \delta)^{1-\beta}} \frac{1}{(\lambda + \delta)^{1-\alpha}} d\lambda \|x\| = C \|x\|.
 \end{aligned}$$

It follows $(-B)^\beta (-A)^{-\beta}$ is bounded on $D(A)$.

Next reverse A and B in 30.2.22. This yields

$$(-A)^{-\beta} - (-B)^{-\beta} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \lambda^{-\beta} (\lambda I - A)^{-1} (B - A) (\lambda I - B)^{-1} d\lambda.$$

Letting $x \in D(A)$,

$$\begin{aligned} & \left\| x - (-A)^\beta (-B)^{-\beta} x \right\| \\ & \leq C \int_0^\infty \lambda^{-\beta} \left\| (-A)^\beta (\lambda I - A)^{-1} (B - A) (\lambda I - B)^{-1} x \right\| d\lambda \\ & \leq C \int_0^\infty \frac{1}{\lambda^\beta (\lambda + \delta)^{1-\beta}} \left\| (B - A) (-B)^{-\alpha} (-B)^\alpha (\lambda I - B)^{-1} x \right\| d\lambda \quad (30.2.23) \\ & \leq C \int_0^\infty \frac{1}{\lambda^\beta (\lambda + \delta)^{1-\beta} (\lambda + \delta)^{1-\alpha}} d\lambda \|x\| = C \|x\| \quad (30.2.24) \end{aligned}$$

This shows $(-A)^\beta (-B)^{-\beta}$ is bounded on $D(A) = D(B)$. Note the assertion these are bounded refers to the norm on H .

It remains to verify $D((-A)^\beta) = D((-B)^\beta)$. Since $D(A)$ is dense in H there exists a unique $L(A, B) \in \mathcal{L}(H, H)$ such that $L(A, B) = (-A)^\beta (-B)^{-\beta}$ on $D(A)$. Let $L(B, A)$ be defined similarly as a continuous linear map which equals $(-B)^\beta (-A)^{-\beta}$ on $D(A)$. Then

$$\begin{aligned} (-A)^{-\beta} L(A, B) &= (-B)^{-\beta} \\ (-B)^{-\beta} L(B, A) &= (-A)^{-\beta} \end{aligned}$$

The first of these equations shows $D((-B)^\beta) \subseteq D((-A)^\beta)$ and the second turns the inclusion around. Thus they are equal as claimed.

Next consider the case where $\beta = 1$. In this case

$$(A - B) B^{-\alpha}$$

is bounded on $D(A)$ and so

$$(A - B) B^{-\alpha} B^{-1+\alpha}$$

is also bounded on $D(A)$. But this equals

$$(A - B) B^{-1}.$$

Thus AB^{-1} is bounded on $D(A)$. Similarly you can show

$$(B - A) A^{-1}$$

is bounded which implies BA^{-1} is bounded on $D(A)$. This proves the theorem.

Definition 30.2.25 Let A be sectorial for the sector $S_{a,\phi}$. Let $b > a$ so that $A - bI$ is sectorial for $S_{-\delta,\phi}$ where $\delta = b - a$. Then for each $\alpha \in [0, 1]$, define a norm on $D((bI - A)^\alpha) \equiv H_\alpha$ by

$$\|x\|_\alpha \equiv \|(bI - A)^\alpha x\|$$

The $\{H_\alpha\}_{\alpha \in [0,1]}$ is called a scale of Banach spaces.

Proposition 30.2.26 The H_α above are Banach spaces and they decrease in α . Furthermore, if $b_i > a$ for $i = 1, 2$ then the two norms associated with the b_i are equivalent.

Proof: That the H_α are decreasing was shown above in Theorem 30.2.16. They are Banach spaces because $(bI - A)^\alpha$ is a closed mapping which is also one to one.

It only remains to verify the claim about the equivalence of the norms. Let $b_2 > b_1 > a$. Then if $\alpha \in (0, 1)$,

$$\begin{aligned} & ((b_1I - A) - (b_2I - A)) (b_2I - A)^{-\alpha} \\ &= (b_1 - b_2) (b_2I - A)^{-\alpha} \in \mathcal{L}(H, H) \end{aligned}$$

and so by Theorem 30.2.24, for each $\beta \in [0, 1]$,

$$D((b_1I - A)^\beta) = D((b_2I - A)^\beta)$$

so the spaces, H_β are the same for either choice of $b > a$. Also from this theorem,

$$(b_1I - A)^\beta (b_2I - A)^{-\beta}, (b_2I - A)^\beta (b_1I - A)^{-\beta}$$

are both bounded on $D(A)$. Therefore, for $x \in H_\beta$

$$\begin{aligned} \left\| (b_1I - A)^\beta x \right\| &= \left\| (b_1I - A)^\beta (b_2I - A)^{-\beta} (b_2I - A)^\beta x \right\| \\ &\leq C \left\| (b_2I - A)^\beta x \right\| \end{aligned}$$

Similarly using the boundedness of $(b_2I - A)^\beta (b_1I - A)^{-\beta}$, it follows

$$\left\| (b_2I - A)^\beta x \right\| \leq C' \left\| (b_1I - A)^\beta x \right\|$$

Thus showing the two norms are equivalent. This proves the proposition.

Complex Mappings

31.1 Conformal Maps

If $\gamma(t) = x(t) + iy(t)$ is a C^1 curve having values in U , an open set of $\mathbb{C}$, and if $f : U \rightarrow \mathbb{C}$ is analytic, consider $f \circ \gamma$, another C^1 curve having values in $\mathbb{C}$. Also, $\gamma'(t)$ and $(f \circ \gamma)'(t)$ are complex numbers so these can be considered as vectors in $\mathbb{R}^2$ as follows. The complex number, $x + iy$ corresponds to the vector, (x, y) . Suppose that γ and η are two such C^1 curves having values in U and that $\gamma(t_0) = \eta(s_0) = z$ and suppose that $f : U \rightarrow \mathbb{C}$ is analytic. What can be said about the angle between $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$? It turns out this angle is the same as the angle between $\gamma'(t_0)$ and $\eta'(s_0)$ assuming that $f'(z) \neq 0$. To see this, note $(x, y) \cdot (a, b) = \frac{1}{2}(z\bar{w} + \bar{z}w)$ where $z = x + iy$ and $w = a + ib$. Therefore, letting θ be the cosine between the two vectors, $(f \circ \gamma)'(t_0)$ and $(f \circ \eta)'(s_0)$, it follows from calculus that

$$\begin{aligned} & \cos \theta \\ = & \frac{(f \circ \gamma)'(t_0) \cdot (f \circ \eta)'(s_0)}{|(f \circ \eta)'(s_0)| |(f \circ \gamma)'(t_0)|} \\ = & \frac{\frac{1}{2} \frac{f'(\gamma(t_0)) \gamma'(t_0) \overline{f'(\eta(s_0)) \eta'(s_0)} + \overline{f'(\gamma(t_0)) \gamma'(t_0)} f'(\eta(s_0)) \eta'(s_0)}{|f'(\gamma(t_0))| |f'(\eta(s_0))|} \\ = & \frac{\frac{1}{2} \frac{f'(z) \overline{f'(z)} \gamma'(t_0) \overline{\eta'(s_0)} + \overline{f'(z)} f'(z) \overline{\gamma'(t_0)} \eta'(s_0)}{|f'(z)| |f'(z)|} \\ = & \frac{\frac{1}{2} \frac{\gamma'(t_0) \overline{\eta'(s_0)} + \eta'(s_0) \overline{\gamma'(t_0)}}{1}}{1} \end{aligned}$$

which equals the angle between the vectors, $\gamma'(t_0)$ and $\eta'(s_0)$. Thus analytic mappings preserve angles at points where the derivative is nonzero. Such mappings are called isogonal.

Actually, they also preserve orientations. If $z = x + iy$ and $w = a + ib$ are two complex numbers, then $(x, y, 0)$ and $(a, b, 0)$ are two vectors in $\mathbb{R}^3$. Recall that the cross product, $(x, y, 0) \times (a, b, 0)$, yields a vector normal to the two given vectors such that the triple, $(x, y, 0)$, $(a, b, 0)$, and $(x, y, 0) \times (a, b, 0)$ satisfies the right hand

rule and has magnitude equal to the product of the sine of the included angle times the product of the two norms of the vectors. In this case, the cross product will produce a vector which is a multiple of $\mathbf{k}$, the unit vector in the direction of the z axis. In fact, you can verify by computing both sides that, letting $z = x + iy$ and $w = a + ib$,

$$(x, y, 0) \times (a, b, 0) = \operatorname{Re}(z i \bar{w}) \mathbf{k}.$$

Therefore, in the above situation,

$$\begin{aligned} & (f \circ \gamma)'(t_0) \times (f \circ \eta)'(s_0) \\ &= \operatorname{Re} \left(f'(\gamma(t_0)) \gamma'(t_0) \overline{i f'(\eta(s_0)) \eta'(s_0)} \right) \mathbf{k} \\ &= |f'(z)|^2 \operatorname{Re} \left(\gamma'(t_0) \overline{i \eta'(s_0)} \right) \mathbf{k} \end{aligned}$$

which shows that the orientation of $\gamma'(t_0), \eta'(s_0)$ is the same as the orientation of $(f \circ \gamma)'(t_0), (f \circ \eta)'(s_0)$. Mappings which preserve both orientation and angles are called conformal mappings and this has shown that analytic functions are conformal mappings if the derivative does not vanish.

31.2 Fractional Linear Transformations

31.2.1 Circles And Lines

These mappings map lines and circles to either lines or circles.

Definition 31.2.1 *A fractional linear transformation is a function of the form*

$$f(z) = \frac{az + b}{cz + d} \quad (31.2.1)$$

where $ad - bc \neq 0$.

Note that if $c = 0$, this reduces to a linear transformation $(a/d)z + (b/d)$. Special cases of these are defined as follows.

$$\text{dilations: } z \rightarrow \delta z, \delta \neq 0, \text{ inversions: } z \rightarrow \frac{1}{z},$$

$$\text{translations: } z \rightarrow z + \rho.$$

The next lemma is the key to understanding fractional linear transformations.

Lemma 31.2.2 *The fractional linear transformation, 31.2.1 can be written as a finite composition of dilations, inversions, and translations.*

Proof: Let

$$S_1(z) = z + \frac{d}{c}, S_2(z) = \frac{1}{z}, S_3(z) = \frac{(bc - ad)}{c^2} z$$

and

$$S_4(z) = z + \frac{a}{c}$$

in the case where $c \neq 0$. Then $f(z)$ given in 31.2.1 is of the form

$$f(z) = S_4 \circ S_3 \circ S_2 \circ S_1.$$

Here is why.

$$S_2(S_1(z)) = S_2\left(z + \frac{d}{c}\right) \equiv \frac{1}{z + \frac{d}{c}} = \frac{c}{zc + d}.$$

Now consider

$$S_3\left(\frac{c}{zc + d}\right) \equiv \frac{(bc - ad)}{c^2} \left(\frac{c}{zc + d}\right) = \frac{bc - ad}{c(zc + d)}.$$

Finally, consider

$$S_4\left(\frac{bc - ad}{c(zc + d)}\right) \equiv \frac{bc - ad}{c(zc + d)} + \frac{a}{c} = \frac{b + az}{zc + d}.$$

In case that $c = 0$, $f(z) = \frac{a}{d}z + \frac{b}{d}$ which is a translation composed with a dilation. Because of the assumption that $ad - bc \neq 0$, it follows that since $c = 0$, both a and $d \neq 0$. This proves the lemma.

This lemma implies the following corollary.

Corollary 31.2.3 *Fractional linear transformations map circles and lines to circles or lines.*

Proof: It is obvious that dilations and translations map circles to circles and lines to lines. What of inversions? If inversions have this property, the above lemma implies a general fractional linear transformation has this property as well.

Note that all circles and lines may be put in the form

$$\alpha(x^2 + y^2) - 2ax - 2by = r^2 - (a^2 + b^2)$$

where $\alpha = 1$ gives a circle centered at (a, b) with radius r and $\alpha = 0$ gives a line. In terms of complex variables you may therefore consider all possible circles and lines in the form

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0, \quad (31.2.2)$$

To see this let $\beta = \beta_1 + i\beta_2$ where $\beta_1 \equiv -a$ and $\beta_2 \equiv b$. Note that even if α is not 0 or 1 the expression still corresponds to either a circle or a line because you can divide by α if $\alpha \neq 0$. Now I verify that replacing z with $\frac{1}{\bar{z}}$ results in an expression of the form in 31.2.2. Thus, let $w = \frac{1}{\bar{z}}$ where z satisfies 31.2.2. Then

$$(\alpha + \beta\bar{w} + \bar{\beta}w + \gamma w\bar{w}) = \frac{1}{z\bar{z}} (\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma) = 0$$

and so w also satisfies a relation like 31.2.2. One simply switches α with γ and β with $\bar{\beta}$. Note the situation is slightly different than with dilations and translations. In the case of an inversion, a circle becomes either a line or a circle and similarly, a line becomes either a circle or a line. This proves the corollary.

The next example is quite important.

Example 31.2.4 Consider the fractional linear transformation, $w = \frac{z-i}{z+i}$.

First consider what this mapping does to the points of the form $z = x + i0$. Substituting into the expression for w ,

$$w = \frac{x-i}{x+i} = \frac{x^2-1-2xi}{x^2+1},$$

a point on the unit circle. Thus this transformation maps the real axis to the unit circle.

The upper half plane is composed of points of the form $x + iy$ where $y > 0$. Substituting in to the transformation,

$$w = \frac{x+i(y-1)}{x+i(y+1)},$$

which is seen to be a point on the interior of the unit disk because $|y-1| < |y+1|$ which implies $|x+i(y+1)| > |x+i(y-1)|$. Therefore, this transformation maps the upper half plane to the interior of the unit disk.

One might wonder whether the mapping is one to one and onto. The mapping is clearly one to one because it has an inverse, $z = -i\frac{w+1}{w-1}$ for all w in the interior of the unit disk. Also, a short computation verifies that z so defined is in the upper half plane. Therefore, this transformation maps $\{z \in \mathbb{C} \text{ such that } \operatorname{Im} z > 0\}$ one to one and onto the unit disk $\{z \in \mathbb{C} \text{ such that } |z| < 1\}$.

A fancy way to do part of this is to use Theorem 28.3.5. $\limsup_{z \rightarrow a} \left| \frac{z-i}{z+i} \right| \leq 1$ whenever a is the real axis or ∞ . Therefore, $\left| \frac{z-i}{z+i} \right| \leq 1$. This is a little shorter.

31.2.2 Three Points To Three Points

There is a simple procedure for determining fractional linear transformations which map a given set of three points to another set of three points. The problem is as follows: There are three distinct points in the extended complex plane, z_1, z_2 , and z_3 and it is desired to find a fractional linear transformation such that $z_i \rightarrow w_i$ for $i = 1, 2, 3$ where here w_1, w_2 , and w_3 are three distinct points in the extended complex plane. Then the procedure says that to find the desired fractional linear transformation solve the following equation for w .

$$\frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1} = \frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1}$$

The result will be a fractional linear transformation with the desired properties. If any of the points equals ∞ , then the quotient containing this point should be adjusted.

Why should this procedure work? Here is a heuristic argument to indicate why you would expect this to happen rather than a rigorous proof. The reader may want to tighten the argument to give a proof. First suppose $z = z_1$. Then the right side equals zero and so the left side also must equal zero. However, this requires $w = w_1$. Next suppose $z = z_2$. Then the right side equals 1. To get a 1 on the left, you need $w = w_2$. Finally suppose $z = z_3$. Then the right side involves division by 0. To get the same bad behavior, on the left, you need $w = w_3$.

Example 31.2.5 Let $\text{Im } \xi > 0$ and consider the fractional linear transformation which takes ξ to 0, $\bar{\xi}$ to ∞ and 0 to $\xi/\bar{\xi}$.

The equation for w is

$$\frac{w - 0}{w - (\xi/\bar{\xi})} = \frac{z - \xi}{z - 0} \cdot \frac{\bar{\xi} - 0}{\bar{\xi} - \xi}$$

After some computations,

$$w = \frac{z - \xi}{z - \bar{\xi}}.$$

Note that this has the property that $\frac{x - \xi}{x - \bar{\xi}}$ is always a point on the unit circle because it is a complex number divided by its conjugate. Therefore, this fractional linear transformation maps the real line to the unit circle. It also takes the point, ξ to 0 and so it must map the upper half plane to the unit disk. You can verify the mapping is onto as well.

Example 31.2.6 Let $z_1 = 0, z_2 = 1$, and $z_3 = 2$ and let $w_1 = 0, w_2 = i$, and $w_3 = 2i$.

Then the equation to solve is

$$\frac{w}{w - 2i} \cdot \frac{-i}{i} = \frac{z}{z - 2} \cdot \frac{-1}{1}$$

Solving this yields $w = iz$ which clearly works.

31.3 Riemann Mapping Theorem

From the open mapping theorem analytic functions map regions to other regions or else to single points. The Riemann mapping theorem states that for every simply connected region, Ω which is not equal to all of $\mathbb{C}$ there exists an analytic function, f such that $f(\Omega) = B(0, 1)$ and in addition to this, f is one to one. The proof involves several ideas which have been developed up to now. The proof is based on the following important theorem, a case of Montel's theorem. Before, beginning,

note that the Riemann mapping theorem is a classic example of a major existence theorem. In mathematics there are two sorts of questions, those related to whether something exists and those involving methods for finding it. The real questions are often related to questions of existence. There is a long and involved history for proofs of this theorem. The first proofs were based on the Dirichlet principle and turned out to be incorrect, thanks to Weierstrass who pointed out the errors. For more on the history of this theorem, see Hille [32].

The following theorem is really wonderful. It is about the existence of a subsequence having certain salubrious properties. It is this wonderful result which will give the existence of the mapping desired. The other parts of the argument are technical details to set things up and use this theorem.

31.3.1 Montel's Theorem

Theorem 31.3.1 *Let Ω be an open set in $\mathbb{C}$ and let $\mathcal{F}$ denote a set of analytic functions mapping Ω to $B(0, M) \subseteq \mathbb{C}$. Then there exists a sequence of functions from $\mathcal{F}$, $\{f_n\}_{n=1}^\infty$ and an analytic function, f such that $f_n^{(k)}$ converges uniformly to $f^{(k)}$ on every compact subset of Ω .*

Proof: First note there exists a sequence of compact sets, K_n such that $K_n \subseteq \text{int } K_{n+1} \subseteq \Omega$ for all n where here $\text{int } K$ denotes the interior of the set K , the union of all open sets contained in K and $\cup_{n=1}^\infty K_n = \Omega$. In fact, you can verify that $\overline{B(0, n)} \cap \{z \in \Omega : \text{dist}(z, \Omega^C) \leq \frac{1}{n}\}$ works for K_n . Then there exist positive numbers, δ_n such that if $z \in K_n$, then $\overline{B(z, \delta_n)} \subseteq \text{int } K_{n+1}$. Now denote by $\mathcal{F}_n$ the set of restrictions of functions of $\mathcal{F}$ to K_n . Then let $z \in K_n$ and let $\gamma(t) \equiv z + \delta_n e^{it}$, $t \in [0, 2\pi]$. It follows that for $z_1 \in B(z, \delta_n)$, and $f \in \mathcal{F}$,

$$\begin{aligned} |f(z) - f(z_1)| &= \left| \frac{1}{2\pi i} \int_\gamma f(w) \left(\frac{1}{w-z} - \frac{1}{w-z_1} \right) dw \right| \\ &\leq \frac{1}{2\pi} \left| \int_\gamma f(w) \frac{z-z_1}{(w-z)(w-z_1)} dw \right| \end{aligned}$$

Letting $|z_1 - z| < \frac{\delta_n}{2}$,

$$\begin{aligned} |f(z) - f(z_1)| &\leq \frac{M}{2\pi} 2\pi \delta_n \frac{|z-z_1|}{\delta_n^2/2} \\ &\leq 2M \frac{|z-z_1|}{\delta_n}. \end{aligned}$$

It follows that $\mathcal{F}_n$ is equicontinuous and uniformly bounded so by the Arzela Ascoli theorem there exists a sequence, $\{f_{nk}\}_{k=1}^\infty \subseteq \mathcal{F}$ which converges uniformly on K_n . Let $\{f_{1k}\}_{k=1}^\infty$ converge uniformly on K_1 . Then use the Arzela Ascoli theorem applied to this sequence to get a subsequence, denoted by $\{f_{2k}\}_{k=1}^\infty$ which also converges uniformly on K_2 . Continue in this way to obtain $\{f_{nk}\}_{k=1}^\infty$ which converges uniformly on $K_1, \dots, K_n$. Now the sequence $\{f_{nn}\}_{n=m}^\infty$ is a subsequence of $\{f_{mk}\}_{k=1}^\infty$

and so it converges uniformly on K_m for all m . Denoting f_{nn} by f_n for short, this is the sequence of functions promised by the theorem. It is clear $\{f_n\}_{n=1}^\infty$ converges uniformly on every compact subset of Ω because every such set is contained in K_m for all m large enough. Let $f(z)$ be the point to which $f_n(z)$ converges. Then f is a continuous function defined on Ω . Is f analytic? Yes it is by Lemma 27.3.12. Alternatively, you could let $T \subseteq \Omega$ be a triangle. Then

$$\int_{\partial T} f(z) dz = \lim_{n \rightarrow \infty} \int_{\partial T} f_n(z) dz = 0.$$

Therefore, by Morera's theorem, f is analytic.

As for the uniform convergence of the derivatives of f , recall Theorem 27.7.25 about the existence of a cycle. Let K be a compact subset of $\text{int}(K_n)$ and let $\{\gamma_k\}_{k=1}^m$ be closed oriented curves contained in

$$\text{int}(K_n) \setminus K$$

such that $\sum_{k=1}^m n(\gamma_k, z) = 1$ for every $z \in K$. Also let η denote the distance between $\cup_j \gamma_j^*$ and K . Then for $z \in K$,

$$\begin{aligned} \left| f^{(k)}(z) - f_n^{(k)}(z) \right| &= \left| \frac{k!}{2\pi i} \sum_{j=1}^m \int_{\gamma_j} \frac{f(w) - f_n(w)}{(w-z)^{k+1}} dw \right| \\ &\leq \frac{k!}{2\pi} \|f_k - f\|_{K_n} \sum_{j=1}^m (\text{length of } \gamma_k) \frac{1}{\eta^{k+1}}. \end{aligned}$$

where here $\|f_k - f\|_{K_n} \equiv \sup \{|f_k(z) - f(z)| : z \in K_n\}$. Thus you get uniform convergence of the derivatives.

Since the family, $\mathcal{F}$ satisfies the conclusion of Theorem 31.3.1 it is known as a normal family of functions. More generally,

Definition 31.3.2 Let $\mathcal{F}$ denote a collection of functions which are analytic on Ω , a region. Then $\mathcal{F}$ is normal if every sequence contained in $\mathcal{F}$ has a subsequence which converges uniformly on compact subsets of Ω .

The following result is about a certain class of fractional linear transformations. Recall Lemma 28.4.7 which is listed here for convenience.

Lemma 31.3.3 For $\alpha \in B(0, 1)$, let

$$\phi_\alpha(z) \equiv \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

Then ϕ_α maps $B(0, 1)$ one to one and onto $B(0, 1)$, $\phi_\alpha^{-1} = \phi_{-\alpha}$, and

$$\phi'_\alpha(\alpha) = \frac{1}{1 - |\alpha|^2}.$$

The next lemma, known as Schwarz's lemma is interesting for its own sake but will also be an important part of the proof of the Riemann mapping theorem. It was stated and proved earlier but for convenience it is given again here.

Lemma 31.3.4 *Suppose $F : B(0, 1) \rightarrow B(0, 1)$, F is analytic, and $F(0) = 0$. Then for all $z \in B(0, 1)$,*

$$|F(z)| \leq |z|, \quad (31.3.3)$$

and

$$|F'(0)| \leq 1. \quad (31.3.4)$$

If equality holds in 31.3.4 then there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and

$$F(z) = \lambda z. \quad (31.3.5)$$

Proof: First note that by assumption, $F(z)/z$ has a removable singularity at 0 if its value at 0 is defined to be $F'(0)$. By the maximum modulus theorem, if $|z| < r < 1$,

$$\left| \frac{F(z)}{z} \right| \leq \max_{t \in [0, 2\pi]} \frac{|F(re^{it})|}{r} \leq \frac{1}{r}.$$

Then letting $r \rightarrow 1$,

$$\left| \frac{F(z)}{z} \right| \leq 1$$

this shows 31.3.3 and it also verifies 31.3.4 on taking the limit as $z \rightarrow 0$. If equality holds in 31.3.4, then $|F(z)/z|$ achieves a maximum at an interior point so $F(z)/z$ equals a constant, λ by the maximum modulus theorem. Since $F(z) = \lambda z$, it follows $F'(0) = \lambda$ and so $|\lambda| = 1$. This proves the lemma.

Definition 31.3.5 *A region, Ω has the square root property if whenever $f, \frac{1}{f} : \Omega \rightarrow \mathbb{C}$ are both analytic¹, it follows there exists $\phi : \Omega \rightarrow \mathbb{C}$ such that ϕ is analytic and $f(z) = \phi^2(z)$.*

The next theorem will turn out to be equivalent to the Riemann mapping theorem.

31.3.2 Regions With Square Root Property

Theorem 31.3.6 *Let $\Omega \neq \mathbb{C}$ for Ω a region and suppose Ω has the square root property. Then for $z_0 \in \Omega$ there exists $h : \Omega \rightarrow B(0, 1)$ such that h is one to one, onto, analytic, and $h(z_0) = 0$.*

Proof: Define $\mathcal{F}$ to be the set of functions, f such that $f : \Omega \rightarrow B(0, 1)$ is one to one and analytic. The first task is to show $\mathcal{F}$ is nonempty. Then, using Montel's theorem it will be shown there is a function in $\mathcal{F}$, h , such that $|h'(z_0)| \geq |\psi'(z_0)|$

¹This implies f has no zero on Ω .

for all $\psi \in \mathcal{F}$. When this has been done it will be shown that h is actually onto. This will prove the theorem.

Claim 1: $\mathcal{F}$ is nonempty.

Proof of Claim 1: Since $\Omega \neq \mathbb{C}$ it follows there exists $\xi \notin \Omega$. Then it follows $z - \xi$ and $\frac{1}{z - \xi}$ are both analytic on Ω . Since Ω has the square root property, there exists an analytic function, $\phi : \Omega \rightarrow \mathbb{C}$ such that $\phi^2(z) = z - \xi$ for all $z \in \Omega$, $\phi(z) = \sqrt{z - \xi}$. Since $z - \xi$ is not constant, neither is ϕ and it follows from the open mapping theorem that $\phi(\Omega)$ is a region. Note also that ϕ is one to one because if $\phi(z_1) = \phi(z_2)$, then you can square both sides and conclude $z_1 - \xi = z_2 - \xi$ implying $z_1 = z_2$.

Now pick $a \in \phi(\Omega)$. Thus $\sqrt{z_a - \xi} = a$. I claim there exists a positive lower bound to $|\sqrt{z - \xi} + a|$ for $z \in \Omega$. If not, there exists a sequence, $\{z_n\} \subseteq \Omega$ such that

$$\sqrt{z_n - \xi} + a = \sqrt{z_n - \xi} + \sqrt{z_a - \xi} \equiv \varepsilon_n \rightarrow 0.$$

Then

$$\sqrt{z_n - \xi} = (\varepsilon_n - \sqrt{z_a - \xi}) \quad (31.3.6)$$

and squaring both sides,

$$z_n - \xi = \varepsilon_n^2 + z_a - \xi - 2\varepsilon_n \sqrt{z_a - \xi}.$$

Consequently, $(z_n - z_a) = \varepsilon_n^2 - 2\varepsilon_n \sqrt{z_a - \xi}$ which converges to 0. Taking the limit in 31.3.6, it follows $2\sqrt{z_a - \xi} = 0$ and so $\xi = z_a$, a contradiction to $\xi \notin \Omega$. Choose $r > 0$ such that for all $z \in \Omega$, $|\sqrt{z - \xi} + a| > r > 0$. Then consider

$$\psi(z) \equiv \frac{r}{\sqrt{z - \xi} + a}. \quad (31.3.7)$$

This is one to one, analytic, and maps Ω into $B(0, 1)$ ($|\sqrt{z - \xi} + a| > r$). Thus $\mathcal{F}$ is not empty and this proves the claim.

Claim 2: Let $z_0 \in \Omega$. There exists a finite positive real number, η , defined by

$$\eta \equiv \sup \{ |\psi'(z_0)| : \psi \in \mathcal{F} \} \quad (31.3.8)$$

and an analytic function, $h \in \mathcal{F}$ such that $|h'(z_0)| = \eta$. Furthermore, $h(z_0) = 0$.

Proof of Claim 2: First you show $\eta < \infty$. Let $\gamma(t) = z_0 + re^{it}$ for $t \in [0, 2\pi]$ and r is small enough that $B(z_0, r) \subseteq \Omega$. Then for $\psi \in \mathcal{F}$, the Cauchy integral formula for the derivative implies

$$\psi'(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{\psi(w)}{(w - z_0)^2} dw$$

and so $|\psi'(z_0)| \leq (1/2\pi) 2\pi r (1/r^2) = 1/r$. Therefore, $\eta < \infty$ as desired. For ψ defined above in 31.3.7

$$\psi'(z_0) = \frac{-r\phi'(z_0)}{(\phi(z_0) + a)^2} = \frac{-r(1/2)(\sqrt{z_0 - \xi})^{-1}}{(\phi(z_0) + a)^2} \neq 0.$$

Therefore, $\eta > 0$. It remains to verify the existence of the function, h .

By Theorem 31.3.1, there exists a sequence, $\{\psi_n\}$, of functions in $\mathcal{F}$ and an analytic function, h , such that

$$|\psi'_n(z_0)| \rightarrow \eta \quad (31.3.9)$$

and

$$\psi_n \rightarrow h, \psi'_n \rightarrow h', \quad (31.3.10)$$

uniformly on all compact subsets of Ω . It follows

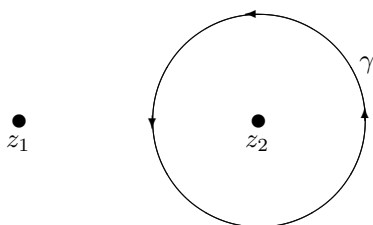
$$|h'(z_0)| = \lim_{n \rightarrow \infty} |\psi'_n(z_0)| = \eta \quad (31.3.11)$$

and for all $z \in \Omega$,

$$|h(z)| = \lim_{n \rightarrow \infty} |\psi_n(z)| \leq 1. \quad (31.3.12)$$

By 31.3.11, h is not a constant. Therefore, in fact, $|h(z)| < 1$ for all $z \in \Omega$ in 31.3.12 by the open mapping theorem.

Next it must be shown that h is one to one in order to conclude $h \in \mathcal{F}$. Pick $z_1 \in \Omega$ and suppose z_2 is another point of Ω . Since the zeros of $h - h(z_1)$ have no limit point, there exists a circular contour bounding a circle which contains z_2 but not z_1 such that γ^* contains no zeros of $h - h(z_1)$.



Using the theorem on counting zeros, Theorem 28.6.1, and the fact that ψ_n is one to one,

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma} \frac{\psi'_n(w)}{\psi_n(w) - \psi_n(z_1)} dw \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{h'(w)}{h(w) - h(z_1)} dw, \end{aligned}$$

which shows that $h - h(z_1)$ has no zeros in $B(z_2, r)$. In particular z_2 is not a zero of $h - h(z_1)$. This shows that h is one to one since $z_2 \neq z_1$ was arbitrary. Therefore, $h \in \mathcal{F}$. It only remains to verify that $h(z_0) = 0$.

If $h(z_0) \neq 0$, consider $\phi_{h(z_0)} \circ h$ where ϕ_{α} is the fractional linear transformation defined in Lemma 31.3.3. By this lemma it follows $\phi_{h(z_0)} \circ h \in \mathcal{F}$. Now using the

chain rule,

$$\begin{aligned}
 \left| \left(\phi_{h(z_0)} \circ h \right)' (z_0) \right| &= \left| \phi'_{h(z_0)} (h(z_0)) \right| |h'(z_0)| \\
 &= \left| \frac{1}{1 - |h(z_0)|^2} \right| |h'(z_0)| \\
 &= \left| \frac{1}{1 - |h(z_0)|^2} \right| \eta > \eta
 \end{aligned}$$

Contradicting the definition of η . This proves Claim 2.

Claim 3: The function, h just obtained maps Ω onto $B(0, 1)$.

Proof of Claim 3: To show h is onto, use the fractional linear transformation of Lemma 31.3.3. Suppose h is not onto. Then there exists $\alpha \in B(0, 1) \setminus h(\Omega)$. Then $0 \neq \phi_\alpha \circ h(z)$ for all $z \in \Omega$ because

$$\phi_\alpha \circ h(z) = \frac{h(z) - \alpha}{1 - \bar{\alpha}h(z)}$$

and it is assumed $\alpha \notin h(\Omega)$. Therefore, since Ω has the square root property, you can consider an analytic function $z \rightarrow \sqrt{\phi_\alpha \circ h(z)}$. This function is one to one because both ϕ_α and h are. Also, the values of this function are in $B(0, 1)$ by Lemma 31.3.3 so it is in $\mathcal{F}$.

Now let

$$\psi \equiv \phi_{\sqrt{\phi_\alpha \circ h(z_0)}} \circ \sqrt{\phi_\alpha \circ h}. \quad (31.3.13)$$

Thus

$$\psi(z_0) = \phi_{\sqrt{\phi_\alpha \circ h(z_0)}} \circ \sqrt{\phi_\alpha \circ h(z_0)} = 0$$

and ψ is a one to one mapping of Ω into $B(0, 1)$ so ψ is also in $\mathcal{F}$. Therefore,

$$|\psi'(z_0)| \leq \eta, \quad \left| \left(\sqrt{\phi_\alpha \circ h} \right)' (z_0) \right| \leq \eta. \quad (31.3.14)$$

Define $s(w) \equiv w^2$. Then using Lemma 31.3.3, in particular, the description of $\phi_\alpha^{-1} = \phi_{-\alpha}$, you can solve 31.3.13 for h to obtain

$$\begin{aligned}
 h(z) &= \phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}} \circ \psi \\
 &= \left(\overbrace{\phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}}^{\equiv F} \circ \psi \right) (z) \\
 &= (F \circ \psi)(z)
 \end{aligned} \quad (31.3.15)$$

Now

$$F(0) = \phi_{-\alpha} \circ s \circ \phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(0) = \phi_\alpha^{-1}(\phi_\alpha \circ h(z_0)) = h(z_0) = 0$$

and F maps $B(0, 1)$ into $B(0, 1)$. Also, F is not one to one because it maps $B(0, 1)$ to $B(0, 1)$ and has s in its definition. Thus there exists $z_1 \in B(0, 1)$ such that $\phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(z_1) = -\frac{1}{2}$ and another point $z_2 \in B(0, 1)$ such that $\phi_{-\sqrt{\phi_\alpha \circ h(z_0)}}(z_2) = \frac{1}{2}$. However, thanks to s , $F(z_1) = F(z_2)$.

Since $F(0) = h(z_0) = 0$, you can apply the Schwarz lemma to F . Since F is not one to one, it can't be true that $F(z) = \lambda z$ for $|\lambda| = 1$ and so by the Schwarz lemma it must be the case that $|F'(0)| < 1$. But this implies from 31.3.15 and 31.3.14 that

$$\begin{aligned}\eta &= |h'(z_0)| = |F'(\psi(z_0))| |\psi'(z_0)| \\ &= |F'(0)| |\psi'(z_0)| < |\psi'(z_0)| \leq \eta,\end{aligned}$$

a contradiction. This proves the theorem.

The following lemma yields the usual form of the Riemann mapping theorem.

Lemma 31.3.7 *Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$. Then Ω has the square root property.*

Proof: Let f and $\frac{1}{f}$ both be analytic on Ω . Then $\frac{f'}{f}$ is analytic on Ω so by Corollary 27.7.23, there exists $\tilde{F}$, analytic on Ω such that $\tilde{F}' = \frac{f'}{f}$ on Ω . Then $(fe^{-\tilde{F}})' = 0$ and so $f(z) = Ce^{\tilde{F}} = e^{a+ib}e^{\tilde{F}}$. Now let $F = \tilde{F} + a + ib$. Then F is still a primitive of f'/f and $f(z) = e^{F(z)}$. Now let $\phi(z) \equiv e^{\frac{1}{2}F(z)}$. Then ϕ is the desired square root and so Ω has the square root property.

Corollary 31.3.8 (*Riemann mapping theorem*) *Let Ω be a simply connected region with $\Omega \neq \mathbb{C}$ and let $z_0 \in \Omega$. Then there exists a function, $f : \Omega \rightarrow B(0, 1)$ such that f is one to one, analytic, and onto with $f(z_0) = 0$. Furthermore, f^{-1} is also analytic.*

Proof: From Theorem 31.3.6 and Lemma 31.3.7 there exists a function, $f : \Omega \rightarrow B(0, 1)$ which is one to one, onto, and analytic such that $f(z_0) = 0$. The assertion that f^{-1} is analytic follows from the open mapping theorem.

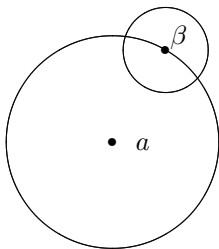
31.4 Analytic Continuation

31.4.1 Regular And Singular Points

Given a function which is analytic on some set, can you extend it to an analytic function defined on a larger set? Sometimes you can do this. It was done in the proof of the Cauchy integral formula. There are also reflection theorems like those discussed in the exercises starting with Problem 10 on Page 828. Here I will give a systematic way of extending an analytic function to a larger set. I will emphasize simply connected regions. The subject of analytic continuation is much larger than the introduction given here. A good source for much more on this is found in Alfors

[2]. The approach given here is suggested by Rudin [58] and avoids many of the standard technicalities.

Definition 31.4.1 Let f be analytic on $B(a, r)$ and let $\beta \in \partial B(a, r)$. Then β is called a *regular point* of f if there exists some $\delta > 0$ and a function, g analytic on $B(\beta, \delta)$ such that $g = f$ on $B(\beta, \delta) \cap B(a, r)$. Those points of $\partial B(a, r)$ which are not regular are called *singular*.



Theorem 31.4.2 Suppose f is analytic on $B(a, r)$ and the power series

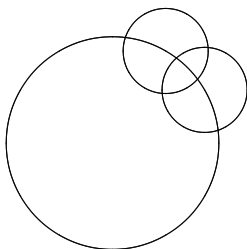
$$f(z) = \sum_{k=0}^{\infty} a_k (z - a)^k$$

has radius of convergence r . Then there exists a singular point on $\partial B(a, r)$.

Proof: If not, then for every $z \in \partial B(a, r)$ there exists $\delta_z > 0$ and g_z analytic on $B(z, \delta_z)$ such that $g_z = f$ on $B(z, \delta_z) \cap B(a, r)$. Since $\partial B(a, r)$ is compact, there exist $z_1, \dots, z_n$, points in $\partial B(a, r)$ such that $\{B(z_k, \delta_{z_k})\}_{k=1}^n$ covers $\partial B(a, r)$. Now define

$$g(z) \equiv \begin{cases} f(z) & \text{if } z \in B(a, r) \\ g_{z_k}(z) & \text{if } z \in B(z_k, \delta_{z_k}) \end{cases}$$

Is this well defined? If $z \in B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j})$, is $g_{z_i}(z) = g_{z_j}(z)$? Consider the following picture representing this situation.



You see that if $z \in B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j})$ then $I \equiv B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j}) \cap B(a, r)$ is a nonempty open set. Both g_{z_i} and g_{z_j} equal f on I . Therefore, they must be equal on $B(z_i, \delta_{z_i}) \cap B(z_j, \delta_{z_j})$ because I has a limit point. Therefore, g is well defined and analytic on an open set containing $\overline{B(a, r)}$. Since g agrees

with f on $B(a, r)$, the power series for g is the same as the power series for f and converges on a ball which is larger than $B(a, r)$ contrary to the assumption that the radius of convergence of the above power series equals r . This proves the theorem.

31.4.2 Continuation Along A Curve

Next I will describe what is meant by continuation along a curve. The following definition is standard and is found in Rudin [58].

Definition 31.4.3 A function element is an ordered pair, (f, D) where D is an open ball and f is analytic on D . (f_0, D_0) and (f_1, D_1) are direct continuations of each other if $D_1 \cap D_0 \neq \emptyset$ and $f_0 = f_1$ on $D_1 \cap D_0$. In this case I will write $(f_0, D_0) \sim (f_1, D_1)$. A chain is a finite sequence, of disks, $\{D_0, \dots, D_n\}$ such that $D_{i-1} \cap D_i \neq \emptyset$. If (f_0, D_0) is a given function element and there exist function elements, (f_i, D_i) such that $\{D_0, \dots, D_n\}$ is a chain and $(f_{j-1}, D_{j-1}) \sim (f_j, D_j)$ then (f_n, D_n) is called the analytic continuation of (f_0, D_0) along the chain $\{D_0, \dots, D_n\}$. Now suppose γ is an oriented curve with parameter interval $[a, b]$ and there exists a chain, $\{D_0, \dots, D_n\}$ such that $\gamma^* \subseteq \bigcup_{k=1}^n D_k$, $\gamma(a)$ is the center of D_0 , $\gamma(b)$ is the center of D_n , and there is an increasing list of numbers in $[a, b]$, $a = s_0 < s_1 < \dots < s_n = b$ such that $\gamma([s_i, s_{i+1}]) \subseteq D_i$ and (f_n, D_n) is an analytic continuation of (f_0, D_0) along the chain. Then (f_n, D_n) is called an analytic continuation of (f_0, D_0) along the curve γ . (γ will always be a continuous curve. Nothing more is needed.)

In the above situation it does not follow that if $D_n \cap D_0 \neq \emptyset$, that $f_n = f_0$! However, there are some cases where this will happen. This is the monodromy theorem which follows. This is as far as I will go on the subject of analytic continuation. For more on this subject including a development of the concept of Riemann surfaces, see Alfors [2].

Lemma 31.4.4 Suppose $(f, B(0, r))$ for $r < 1$ is a function element and $(f, B(0, r))$ can be analytically continued along every curve in $B(0, 1)$ that starts at 0. Then there exists an analytic function, g defined on $B(0, 1)$ such that $g = f$ on $B(0, r)$.

Proof: Let

$$R = \sup\{r_1 \geq r \text{ such that there exists } g_{r_1} \text{ analytic on } B(0, r_1) \text{ which agrees with } f \text{ on } B(0, r)\}.$$

Define $g_R(z) \equiv g_{r_1}(z)$ where $|z| < r_1$. This is well defined because if you use r_1 and r_2 , both g_{r_1} and g_{r_2} agree with f on $B(0, r)$, a set with a limit point and so the two functions agree at every point in both $B(0, r_1)$ and $B(0, r_2)$. Thus g_R is analytic on $B(0, R)$. If $R < 1$, then by the assumption there are no singular points on $B(0, R)$ and so Theorem 31.4.2 implies the radius of convergence of the power series for g_R is larger than R contradicting the choice of R . Therefore, $R = 1$ and this proves the lemma. Let $g = g_R$.

The following theorem is the main result in this subject, the monodromy theorem.

Theorem 31.4.5 *Let Ω be a simply connected proper subset of $\mathbb{C}$ and suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through a . Then there exists G analytic on Ω such that G agrees with f on $B(a, r)$.*

Proof: By the Riemann mapping theorem, there exists $h : \Omega \rightarrow B(0, 1)$ which is analytic, one to one and onto such that $f(a) = 0$. Since h is an open map, there exists $\delta > 0$ such that

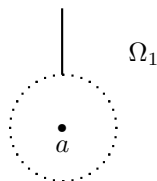
$$B(0, \delta) \subseteq h(B(a, r)).$$

It follows $f \circ h^{-1}$ can be analytically continued along every curve through 0. By Lemma 31.4.4 there exists g analytic on $B(0, 1)$ which agrees with $f \circ h^{-1}$ on $B(0, \delta)$. Define $G(z) \equiv g(h(z))$. For $z = h^{-1}(w)$, it follows $G(h^{-1}(w)) = g(w)$. If $w \in B(0, \delta)$, then $G(h^{-1}(w)) = f \circ h^{-1}(w)$ and so $G = f$ on $h^{-1}(B(0, \delta))$, an open set contained in $B(a, r)$. Therefore, $G = f$ on $B(a, r)$ because $h^{-1}(B(0, \delta))$ has a limit point. This proves the theorem.

Actually, you sometimes want to consider the case where $\Omega = \mathbb{C}$. This requires a small modification to obtain from the above theorem.

Corollary 31.4.6 *Suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \mathbb{C}$. Suppose also that this function element can be analytically continued along every curve through a . Then there exists G analytic on $\mathbb{C}$ such that G agrees with f on $B(a, r)$.*

Proof: Let $\Omega_1 \equiv \{z \in \mathbb{C} : a + it : t > a\}$ and $\Omega_2 \equiv \{z \in \mathbb{C} : a - it : t > a\}$. Here is a picture of Ω_1 .



A picture of Ω_2 is similar except the line extends down from the boundary of $B(a, r)$.

Thus $B(a, r) \subseteq \Omega_i$ and Ω_i is simply connected and proper. By Theorem 31.4.5 there exist analytic functions, G_i analytic on Ω_i such that $G_i = f$ on $B(a, r)$. Thus $G_1 = G_2$ on $B(a, r)$, a set with a limit point. Therefore, $G_1 = G_2$ on $\Omega_1 \cap \Omega_2$. Now let $G(z) = G_i(z)$ where $z \in \Omega_i$. This is well defined and analytic on $\mathbb{C}$. This proves the corollary.

31.5 The Picard Theorems

The Picard theorem says that if f is an entire function and there are two complex numbers not contained in $f(\mathbb{C})$, then f is constant. This is certainly one of the most amazing things which could be imagined. However, this is only the little

Picard theorem. The big Picard theorem is even more incredible. This one asserts that to be non constant the entire function must take every value of $\mathbb{C}$ but two infinitely many times! I will begin with the little Picard theorem. The method of proof I will use is the one found in Saks and Zygmund [60], Conway [16] and Hille [32]. This is not the way Picard did it in 1879. That approach is very different and is presented at the end of the material on elliptic functions. This approach is much more recent dating it appears from around 1924.

Lemma 31.5.1 *Let f be analytic on a region containing $\overline{B(0, r)}$ and suppose*

$$|f'(0)| = b > 0, f(0) = 0,$$

and $|f(z)| \leq M$ for all $z \in \overline{B(0, r)}$. Then $f(B(0, r)) \supseteq B\left(0, \frac{r^2 b^2}{6M}\right)$.

Proof: By assumption,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |z| \leq r. \quad (31.5.16)$$

Then by the Cauchy integral formula for the derivative,

$$a_k = \frac{1}{2\pi i} \int_{\partial B(0, r)} \frac{f(w)}{w^{k+1}} dw$$

where the integral is in the counter clockwise direction. Therefore,

$$|a_k| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(re^{i\theta})|}{r^k} d\theta \leq \frac{M}{r^k}.$$

In particular, $br \leq M$. Therefore, from 31.5.16

$$\begin{aligned} |f(z)| &\geq b|z| - \sum_{k=2}^{\infty} \frac{M}{r^k} |z|^k = b|z| - \frac{M \left(\frac{|z|}{r}\right)^2}{1 - \frac{|z|}{r}} \\ &= b|z| - \frac{M|z|^2}{r^2 - r|z|} \end{aligned}$$

Suppose $|z| = \frac{r^2 b}{4M} < r$. Then this is no larger than

$$\frac{1}{4} b^2 r^2 \frac{3M - br}{M(4M - br)} \geq \frac{1}{4} b^2 r^2 \frac{3M - M}{M(4M - M)} = \frac{r^2 b^2}{6M}.$$

Let $|w| < \frac{r^2 b}{4M}$. Then for $|z| = \frac{r^2 b}{4M}$ and the above,

$$|w| = |(f(z) - w) - f(z)| < \frac{r^2 b}{4M} \leq |f(z)|$$

and so by Rouché's theorem, $z \rightarrow f(z) - w$ and $z \rightarrow f(z)$ have the same number of zeros in $B\left(0, \frac{r^2 b}{4M}\right)$. But f has at least one zero in this ball and so this shows there exists at least one $z \in B\left(0, \frac{r^2 b}{4M}\right)$ such that $f(z) - w = 0$. This proves the lemma.

31.5.1 Two Competing Lemmas

Lemma 31.5.1 is a really nice lemma but there is something even better, Bloch's lemma. This lemma does not depend on the bound of f . Like the above two lemmas it is interesting for its own sake and in addition is the key to a fairly short proof of Picard's theorem. It features the number $\frac{1}{24}$. The best constant is not currently known.

Lemma 31.5.2 *Let f be analytic on an open set containing $\overline{B(0, R)}$ and suppose $|f'(0)| > 0$. Then there exists $a \in B(0, R)$ such that*

$$f(B(0, R)) \supseteq B\left(f(a), \frac{|f'(0)|R}{24}\right).$$

Proof: Let $K(\rho) \equiv \max\{|f'(z)| : |z| = \rho\}$. For simplicity, let $C_\rho \equiv \{z : |z| = \rho\}$.

Claim: K is continuous from the left.

Proof of claim: Let $z_\rho \in C_\rho$ such that $|f'(z_\rho)| = K(\rho)$. Then by the maximum modulus theorem, if $\lambda \in (0, 1)$,

$$|f'(\lambda z_\rho)| \leq K(\lambda \rho) \leq K(\rho) = |f'(z_\rho)|.$$

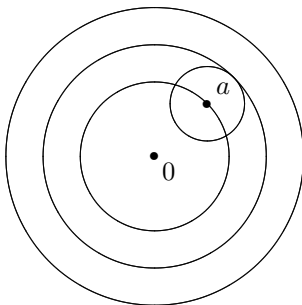
Letting $\lambda \rightarrow 1$ yields the claim.

Let ρ_0 be the largest such that $(R - \rho_0)K(\rho_0) = R|f'(0)|$. (Note $(R - 0)K(0) = R|f'(0)|$.) Thus $\rho_0 < R$ because $(R - R)K(R) = 0$. Let $|a| = \rho_0$ such that $|f'(a)| = K(\rho_0)$. Thus

$$|f'(a)|(R - \rho_0) = |f'(0)|R \quad (31.5.17)$$

Now let $r = \frac{R - \rho_0}{2}$. From 31.5.17,

$$|f'(a)|r = \frac{1}{2}|f'(0)|R, \quad B(a, r) \subseteq B(0, \rho_0 + r) \subseteq B(0, R). \quad (31.5.18)$$



Therefore, if $z \in B(a, r)$, it follows from the maximum modulus theorem and the definition of ρ_0 that

$$\begin{aligned} |f'(z)| &\leq K(\rho_0 + r) < \frac{R|f'(0)|}{R - \rho_0 - r} = \frac{2R|f'(0)|}{R - \rho_0} \\ &= \frac{2R|f'(0)|}{2r} = \frac{R|f'(0)|}{r} \end{aligned} \quad (31.5.19)$$

Let $g(z) = f(a + z) - f(a)$ where $z \in B(0, r)$. Then $|g'(0)| = |f'(a)| > 0$ and for $z \in B(0, r)$,

$$|g(z)| \leq \left| \int_{\gamma(a, z)} g'(w) dw \right| \leq |z - a| \frac{R|f'(0)|}{r} = R|f'(0)|.$$

By Lemma 31.5.1 and 31.5.18,

$$\begin{aligned} g(B(0, r)) &\supseteq B\left(0, \frac{r^2 |f'(a)|^2}{6R|f'(0)|}\right) \\ &= B\left(0, \frac{r^2 \left(\frac{1}{2r} |f'(0)| R\right)^2}{6R|f'(0)|}\right) = B\left(0, \frac{|f'(0)| R}{24}\right) \end{aligned}$$

Now $g(B(0, r)) = f(B(a, r)) - f(a)$ and so this implies

$$f(B(0, R)) \supseteq f(B(a, r)) \supseteq B\left(f(a), \frac{|f'(0)| R}{24}\right).$$

This proves the lemma.

Here is a slightly more general version which allows the center of the open set to be arbitrary.

Lemma 31.5.3 *Let f be analytic on an open set containing $\overline{B(z_0, R)}$ and suppose $|f'(z_0)| > 0$. Then there exists $a \in B(z_0, R)$ such that*

$$f(B(z_0, R)) \supseteq B\left(f(a), \frac{|f'(z_0)| R}{24}\right).$$

Proof: You look at $g(z) \equiv f(z_0 + z) - f(z_0)$ for $z \in B(0, R)$. Then $g'(0) = f'(z_0)$ and so by Lemma 31.5.2 there exists $a_1 \in B(0, R)$ such that

$$g(B(0, R)) \supseteq B\left(g(a_1), \frac{|f'(z_0)| R}{24}\right).$$

Now $g(B(0, R)) = f(B(z_0, R)) - f(z_0)$ and $g(a_1) = f(a) - f(z_0)$ for some $a \in B(z_0, R)$ and so

$$\begin{aligned} f(B(z_0, R)) - f(z_0) &\supseteq B\left(g(a_1), \frac{|f'(z_0)| R}{24}\right) \\ &= B\left(f(a) - f(z_0), \frac{|f'(z_0)| R}{24}\right) \end{aligned}$$

which implies

$$f(B(z_0, R)) \supseteq B\left(f(a), \frac{|f'(z_0)|R}{24}\right)$$

as claimed. This proves the lemma.

No attempt was made to find the best number to multiply by $R|f'(z_0)|$. A discussion of this is given in Conway [16]. See also [32]. Much larger numbers than $1/24$ are available and there is a conjecture due to Alfors about the best value. The conjecture is that $1/24$ can be replaced with

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{(1+\sqrt{3})^{1/2}\Gamma\left(\frac{1}{4}\right)} \approx .47186$$

You can see there is quite a gap between the constant for which this lemma is proved above and what is thought to be the best constant.

Bloch's lemma above gives the existence of a ball of a certain size inside the image of a ball. By contrast the next lemma leads to conditions under which the values of a function do not contain a ball of certain radius. It concerns analytic functions which do not achieve the values 0 and 1.

Lemma 31.5.4 *Let $\mathcal{F}$ denote the set of functions, f defined on Ω , a simply connected region which do not achieve the values 0 and 1. Then for each such function, it is possible to define a function analytic on Ω , $H(z)$ by the formula*

$$H(z) \equiv \log \left[\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right].$$

There exists a constant C independent of $f \in \mathcal{F}$ such that $H(\Omega)$ does not contain any ball of radius C .

Proof: Let $f \in \mathcal{F}$. Then since f does not take the value 0, there exists g_1 a primitive of f'/f . Thus

$$\frac{d}{dz}(e^{-g_1}f) = 0$$

so there exists a, b such that $f(z)e^{-g_1(z)} = e^{a+bi}$. Letting $g(z) = g_1(z) + a + ib$, it follows $e^{g(z)} = f(z)$. Let $\log(f(z)) = g(z)$. Then for $n \in \mathbb{Z}$, the integers,

$$\frac{\log(f(z))}{2\pi i}, \frac{\log(f(z))}{2\pi i} - 1 \neq n$$

because if equality held, then $f(z) = 1$ which does not happen. It follows $\frac{\log(f(z))}{2\pi i}$ and $\frac{\log(f(z))}{2\pi i} - 1$ are never equal to zero. Therefore, using the same reasoning, you can define a logarithm of these two quantities and therefore, a square root. Hence there exists a function analytic on Ω ,

$$\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1}. \quad (31.5.20)$$

For n a positive integer, this function cannot equal $\sqrt{n} \pm \sqrt{n-1}$ because if it did, then

$$\left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right) = \sqrt{n} \pm \sqrt{n-1} \quad (31.5.21)$$

and you could take reciprocals of both sides to obtain

$$\left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right) = \sqrt{n} \mp \sqrt{n-1}. \quad (31.5.22)$$

Then adding 31.5.21 and 31.5.22

$$2\sqrt{\frac{\log(f(z))}{2\pi i}} = 2\sqrt{n}$$

which contradicts the above observation that $\frac{\log(f(z))}{2\pi i}$ is not equal to an integer.

Also, the function of 31.5.20 is never equal to zero. Therefore, you can define the logarithm of this function also. It follows

$$H(z) \equiv \log \left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right) \neq \ln(\sqrt{n} \pm \sqrt{n-1}) + 2m\pi i$$

where m is an arbitrary integer and n is a positive integer. Now

$$\lim_{n \rightarrow \infty} \ln(\sqrt{n} + \sqrt{n-1}) = \infty$$

and $\lim_{n \rightarrow \infty} \ln(\sqrt{n} - \sqrt{n-1}) = -\infty$ and so $\mathbb{C}$ is covered by rectangles having vertices at points $\ln(\sqrt{n} \pm \sqrt{n-1}) + 2m\pi i$ as described above. Each of these rectangles has height equal to 2π and a short computation shows their widths are bounded. Therefore, there exists C independent of $f \in \mathcal{F}$ such that C is larger than the diameter of all these rectangles. Hence $H(\Omega)$ cannot contain any ball of radius larger than C .

31.5.2 The Little Picard Theorem

Now here is the little Picard theorem. It is easy to prove from the above.

Theorem 31.5.5 *If h is an entire function which omits two values then h is a constant.*

Proof: Suppose the two values omitted are a and b and that h is not constant. Let $f(z) = (h(z) - a)/(b - a)$. Then f omits the two values 0 and 1. Let H be defined in Lemma 31.5.4. Then $H(z)$ is clearly not of the form $az + b$ because then it would have values equal to the vertices $\ln(\sqrt{n} \pm \sqrt{n-1}) + 2m\pi i$ or else be constant neither of which happen if h is not constant. Therefore, by Liouville's theorem, H' must be unbounded. Pick ξ such that $|H'(\xi)| > 24C$ where C is such that $H(\mathbb{C})$

contains no balls of radius larger than C . But by Lemma 31.5.3 $H(B(\xi, 1))$ must contain a ball of radius $\frac{|H'(\xi)|}{24} > \frac{24C}{24} = C$, a contradiction. This proves Picard's theorem.

The following is another formulation of this theorem.

Corollary 31.5.6 *If f is a meromorphic function defined on $\mathbb{C}$ which omits three distinct values, a, b, c , then f is a constant.*

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c} \frac{b-c}{b-a}$. Then $\phi(c) = \infty$, $\phi(a) = 0$, and $\phi(b) = 1$. Now consider the function, $h = \phi \circ f$. Then h misses the three points $\infty, 0$, and 1 . Since h is meromorphic and does not have ∞ in its values, it must actually be analytic. Thus h is an entire function which misses the two values 0 and 1 . Therefore, h is constant by Theorem 31.5.5.

31.5.3 Schottky's Theorem

Lemma 31.5.7 *Let f be analytic on an open set containing $\overline{B(0, R)}$ and suppose that f does not take on either of the two values 0 or 1 . Also suppose $|f(0)| \leq \beta$. Then letting $\theta \in (0, 1)$, it follows*

$$|f(z)| \leq M(\beta, \theta)$$

for all $z \in B(0, \theta R)$, where $M(\beta, \theta)$ is a function of only the two variables β, θ . (In particular, there is no dependence on R .)

Proof: Consider the function, $H(z)$ used in Lemma 31.5.4 given by

$$H(z) \equiv \log \left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right). \quad (31.5.23)$$

You notice there are two explicit uses of logarithms. Consider first the logarithm inside the radicals. Choose this logarithm such that

$$\log(f(0)) = \ln|f(0)| + i \arg(f(0)), \quad \arg(f(0)) \in (-\pi, \pi]. \quad (31.5.24)$$

You can do this because

$$e^{\log(f(0))} = f(0) = e^{\ln|f(0)|} e^{i\alpha} = e^{\ln|f(0)| + i\alpha}$$

and by replacing α with $\alpha + 2m\pi$ for a suitable integer, m it follows the above equation still holds. Therefore, you can assume 31.5.24. Similar reasoning applies to the logarithm on the outside of the parenthesis. It can be assumed $H(0)$ equals

$$\ln \left| \sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1} \right| + i \arg \left(\sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1} \right) \quad (31.5.25)$$

where the imaginary part is no larger than π in absolute value.

Now if $\xi \in B(0, R)$ is a point where $H'(\xi) \neq 0$, then by Lemma 31.5.2

$$H(B(\xi, R - |\xi|)) \supseteq B\left(H(a), \frac{|H'(\xi)|(R - |\xi|)}{24}\right)$$

where a is some point in $B(\xi, R - |\xi|)$. But by Lemma 31.5.4 $H(B(\xi, R - |\xi|))$ contains no balls of radius C where C depended only on the maximum diameters of those rectangles having vertices $\ln(\sqrt{n} \pm \sqrt{n-1}) + 2m\pi i$ for n a positive integer and m an integer. Therefore,

$$\frac{|H'(\xi)|(R - |\xi|)}{24} < C$$

and consequently

$$|H'(\xi)| < \frac{24C}{R - |\xi|}.$$

Even if $H'(\xi) = 0$, this inequality still holds. Therefore, if $z \in B(0, R)$ and $\gamma(0, z)$ is the straight segment from 0 to z ,

$$\begin{aligned} |H(z) - H(0)| &= \left| \int_{\gamma(0, z)} H'(w) dw \right| = \left| \int_0^1 H'(tz) z dt \right| \\ &\leq \int_0^1 |H'(tz)| |z| dt \leq \int_0^1 \frac{24C}{R - t|z|} |z| dt \\ &= 24C \ln \left(\frac{R}{R - |z|} \right). \end{aligned}$$

Therefore, for $z \in \partial B(0, \theta R)$,

$$|H(z)| \leq |H(0)| + 24C \ln \left(\frac{1}{1 - \theta} \right). \quad (31.5.26)$$

By the maximum modulus theorem, the above inequality holds for all $|z| < \theta R$ also.

Next I will use 31.5.23 to get an inequality for $|f(z)|$ in terms of $|H(z)|$. From 31.5.23,

$$H(z) = \log \left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right)$$

and so

$$\begin{aligned} 2H(z) &= \log \left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right)^2 \\ -2H(z) &= \log \left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right)^{-2} \\ &= \log \left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right)^2 \end{aligned}$$

Therefore,

$$\begin{aligned} & \left(\sqrt{\frac{\log(f(z))}{2\pi i}} + \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right)^2 \\ & + \left(\sqrt{\frac{\log(f(z))}{2\pi i}} - \sqrt{\frac{\log(f(z))}{2\pi i} - 1} \right)^2 \\ & = \exp(2H(z)) + \exp(-2H(z)) \end{aligned}$$

and

$$\left(\frac{\log(f(z))}{\pi i} - 1 \right) = \frac{1}{2} (\exp(2H(z)) + \exp(-2H(z))).$$

Thus

$$\log(f(z)) = \pi i + \frac{\pi i}{2} (\exp(2H(z)) + \exp(-2H(z)))$$

which shows

$$\begin{aligned} |f(z)| &= \left| \exp \left[\frac{\pi i}{2} (\exp(2H(z)) + \exp(-2H(z))) \right] \right| \\ &\leq \exp \left| \frac{\pi i}{2} (\exp(2H(z)) + \exp(-2H(z))) \right| \\ &\leq \exp \left| \frac{\pi}{2} (|\exp(2H(z))| + |\exp(-2H(z))|) \right| \\ &\leq \exp \left| \frac{\pi}{2} (\exp(2|H(z)|) + \exp(|-2H(z)|)) \right| \\ &= \exp(\pi \exp 2|H(z)|). \end{aligned}$$

Now from 31.5.26 this is dominated by

$$\begin{aligned} & \exp \left(\pi \exp 2 \left(|H(0)| + 24C \ln \left(\frac{1}{1-\theta} \right) \right) \right) \\ & = \exp \left(\pi \exp(2|H(0)|) \exp \left(48C \ln \left(\frac{1}{1-\theta} \right) \right) \right) \end{aligned} \quad (31.5.27)$$

Consider $\exp(2|H(0)|)$. I want to obtain an inequality for this which involves β . This is where I will use the convention about the logarithms discussed above. From 31.5.25,

$$2|H(0)| = 2 \left| \log \left(\sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1} \right) \right|$$

$$\begin{aligned}
&\leq 2 \left(\left(\ln \left| \sqrt{\frac{\log(f(0))}{2\pi i}} - \sqrt{\frac{\log(f(0))}{2\pi i} - 1} \right| \right)^2 + \pi^2 \right)^{1/2} \\
&\leq 2 \left(\left| \ln \left(\left| \sqrt{\frac{\log(f(0))}{2\pi i}} \right| + \left| \sqrt{\frac{\log(f(0))}{2\pi i} - 1} \right| \right) \right|^2 + \pi^2 \right)^{1/2} \\
&\leq 2 \left| \ln \left(\left| \sqrt{\frac{\log(f(0))}{2\pi i}} \right| + \left| \sqrt{\frac{\log(f(0))}{2\pi i} - 1} \right| \right) \right| + 2\pi \\
&\leq \ln \left(2 \left(\left| \frac{\log(f(0))}{2\pi i} \right| + \left| \frac{\log(f(0))}{2\pi i} - 1 \right| \right) \right) + 2\pi \\
&= \ln \left(\left(\left| \frac{\log(f(0))}{\pi i} \right| + \left| \frac{\log(f(0))}{\pi i} - 2 \right| \right) \right) + 2\pi \tag{31.5.28}
\end{aligned}$$

Consider $\left| \frac{\log(f(0))}{\pi i} \right|$

$$\frac{\log(f(0))}{\pi i} = -\frac{\ln|f(0)|}{\pi}i + \frac{\arg(f(0))}{\pi}$$

and so

$$\begin{aligned}
\left| \frac{\log(f(0))}{\pi i} \right| &= \left(\left| \frac{\ln|f(0)|}{\pi} \right|^2 + \left(\frac{\arg(f(0))}{\pi} \right)^2 \right)^{1/2} \\
&\leq \left(\left| \frac{\ln\beta}{\pi} \right|^2 + \left(\frac{\pi}{\pi} \right)^2 \right)^{1/2} \\
&= \left(\left| \frac{\ln\beta}{\pi} \right|^2 + 1 \right)^{1/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left| \frac{\log(f(0))}{\pi i} - 2 \right| &\leq \left(\left| \frac{\ln\beta}{\pi} \right|^2 + (2+1)^2 \right)^{1/2} \\
&= \left(\left| \frac{\ln\beta}{\pi} \right|^2 + 9 \right)^{1/2}
\end{aligned}$$

It follows from 31.5.28 that

$$2|H(0)| \leq \ln \left(2 \left(\left| \frac{\ln\beta}{\pi} \right|^2 + 9 \right)^{1/2} \right) + 2\pi.$$

Hence from 31.5.27

$$|f(z)| \leq$$

$$\exp \left(\pi \exp \left(\ln \left(2 \left(\left| \frac{\ln \beta}{\pi} \right|^2 + 9 \right)^{1/2} \right) + 2\pi \right) \exp \left(48C \ln \left(\frac{1}{1-\theta} \right) \right) \right)$$

and so, letting $M(\beta, \theta)$ be given by the above expression on the right, the lemma is proved.

The following theorem will be referred to as Schottky's theorem. It looks just like the above lemma except it is only assumed that f is analytic on $B(0, R)$ rather than on an open set containing $\overline{B(0, R)}$. Also, the case of an arbitrary center is included along with arbitrary points which are not attained as values of the function.

Theorem 31.5.8 *Let f be analytic on $B(z_0, R)$ and suppose that f does not take on either of the two distinct values a or b . Also suppose $|f(z_0)| \leq \beta$. Then letting $\theta \in (0, 1)$, it follows*

$$|f(z)| \leq M(a, b, \beta, \theta)$$

for all $z \in B(z_0, \theta R)$, where $M(a, b, \beta, \theta)$ is a function of only the variables β, θ, a, b . (In particular, there is no dependence on R .)

Proof: First you can reduce to the case where the two values are 0 and 1 by considering

$$h(z) \equiv \frac{f(z) - a}{b - a}.$$

If there exists an estimate of the desired sort for h , then there exists such an estimate for f . Of course here the function, M would depend on a and b . Therefore, there is no loss of generality in assuming the points which are missed are 0 and 1.

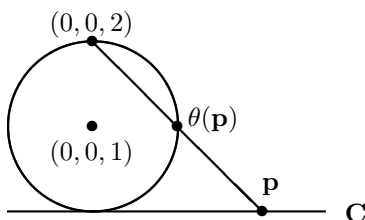
Apply Lemma 31.5.7 to $B(0, R_1)$ for the function, $g(z) \equiv f(z_0 + z)$ and $R_1 < R$. Then if $\beta \geq |f(z_0)| = |g(0)|$, it follows $|g(z)| = |f(z_0 + z)| \leq M(\beta, \theta)$ for every $z \in B(0, \theta R_1)$. Now let $\theta \in (0, 1)$ and choose $R_1 < R$ large enough that $\theta R = \theta_1 R_1$ where $\theta_1 \in (0, 1)$. Then if $|z - z_0| < \theta R$, it follows

$$|f(z)| \leq M(\beta, \theta_1).$$

Now let $R_1 \rightarrow R$ so $\theta_1 \rightarrow \theta$.

31.5.4 A Brief Review

First recall the definition of the metric on $\widehat{\mathbb{C}}$. For convenience it is listed here again. Consider the unit sphere, S^2 given by $(z - 1)^2 + y^2 + x^2 = 1$. Define a map from the complex plane to the surface of this sphere as follows. Extend a line from the point, p in the complex plane to the point $(0, 0, 2)$ on the top of this sphere and let $\theta(p)$ denote the point of this sphere which the line intersects. Define $\theta(\infty) \equiv (0, 0, 2)$.



Then θ^{-1} is sometimes called stereographic projection. The mapping θ is clearly continuous because it takes converging sequences, to converging sequences. Furthermore, it is clear that θ^{-1} is also continuous. In terms of the extended complex plane, $\widehat{\mathbb{C}}$, a sequence, z_n converges to ∞ if and only if θz_n converges to $(0, 0, 2)$ and a sequence, z_n converges to $z \in \mathbb{C}$ if and only if $\theta(z_n) \rightarrow \theta(z)$.

In fact this makes it easy to define a metric on $\widehat{\mathbb{C}}$.

Definition 31.5.9 Let $z, w \in \widehat{\mathbb{C}}$. Then let $d(x, y) \equiv |\theta(z) - \theta(w)|$ where this last distance is the usual distance measured in $\mathbb{R}^3$.

Theorem 31.5.10 $(\widehat{\mathbb{C}}, d)$ is a compact, hence complete metric space.

Proof: Suppose $\{z_n\}$ is a sequence in $\widehat{\mathbb{C}}$. This means $\{\theta(z_n)\}$ is a sequence in S^2 which is compact. Therefore, there exists a subsequence, $\{\theta z_{n_k}\}$ and a point, $z \in S^2$ such that $\theta z_{n_k} \rightarrow \theta z$ in S^2 which implies immediately that $d(z_{n_k}, z) \rightarrow 0$. A compact metric space must be complete.

Also recall the interesting fact that meromorphic functions are continuous with values in $\widehat{\mathbb{C}}$ which is reviewed here for convenience. It came from the theory of classification of isolated singularities.

Theorem 31.5.11 Let Ω be an open subset of $\mathbb{C}$ and let $f : \Omega \rightarrow \widehat{\mathbb{C}}$ be meromorphic. Then f is continuous with respect to the metric, d on $\widehat{\mathbb{C}}$.

Proof: Let $z_n \rightarrow z$ where $z \in \Omega$. Then if z is a pole, it follows from Theorem 27.7.11 that

$$d(f(z_n), \infty) \equiv d(f(z_n), f(z)) \rightarrow 0.$$

If z is not a pole, then $f(z_n) \rightarrow f(z)$ in $\mathbb{C}$ which implies $|\theta(f(z_n)) - \theta(f(z))| = d(f(z_n), f(z)) \rightarrow 0$. Recall that θ is continuous on $\mathbb{C}$.

The fundamental result behind all the theory about to be presented is the Ascoli Arzela theorem also listed here for convenience.

Definition 31.5.12 Let (X, d) be a complete metric space. Then it is said to be locally compact if $\overline{B(x, r)}$ is compact for each $r > 0$.

Thus if you have a locally compact metric space, then if $\{a_n\}$ is a bounded sequence, it must have a convergent subsequence.

Let K be a compact subset of $\mathbb{R}^n$ and consider the continuous functions which have values in a locally compact metric space, (X, d) where d denotes the metric on X . Denote this space as $C(K, X)$.

Definition 31.5.13 For $f, g \in C(K, X)$, where K is a compact subset of $\mathbb{R}^n$ and X is a locally compact complete metric space define

$$\rho_K(f, g) \equiv \sup \{d(f(\mathbf{x}), g(\mathbf{x})) : \mathbf{x} \in K\}.$$

The Ascoli Arzela theorem, Theorem 6.4.4 is a major result which tells which subsets of $C(K, X)$ are sequentially compact.

Definition 31.5.14 Let $A \subseteq C(K, X)$ for K a compact subset of $\mathbb{R}^n$. Then A is said to be uniformly equicontinuous if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $\mathbf{x}, \mathbf{y} \in K$ with $|\mathbf{x} - \mathbf{y}| < \delta$ and $f \in A$,

$$d(f(\mathbf{x}), f(\mathbf{y})) < \varepsilon.$$

The set, A is said to be uniformly bounded if for some $M < \infty$, and $a \in X$,

$$f(\mathbf{x}) \in B(a, M)$$

for all $f \in A$ and $\mathbf{x} \in K$.

The Ascoli Arzela theorem follows.

Theorem 31.5.15 Suppose K is a nonempty compact subset of $\mathbb{R}^n$ and $A \subseteq C(K, X)$, is uniformly bounded and uniformly equicontinuous where X is a locally compact complete metric space. Then if $\{f_k\} \subseteq A$, there exists a function, $f \in C(K, X)$ and a subsequence, f_{k_l} such that

$$\lim_{l \rightarrow \infty} \rho_K(f_{k_l}, f) = 0.$$

In the cases of interest here, $X = \hat{\mathbb{C}}$ with the metric defined above.

31.5.5 Montel's Theorem

The following lemma is another version of Montel's theorem. It is this which will make possible a proof of the big Picard theorem.

Lemma 31.5.16 Let Ω be a region and let $\mathcal{F}$ be a set of functions analytic on Ω none of which achieve the two distinct values, a and b . If $\{f_n\} \subseteq \mathcal{F}$ then one of the following hold: Either there exists a function, f analytic on Ω and a subsequence, $\{f_{n_k}\}$ such that for any compact subset, K of Ω ,

$$\lim_{k \rightarrow \infty} \|f_{n_k} - f\|_{K, \infty} = 0. \quad (31.5.29)$$

or there exists a subsequence $\{f_{n_k}\}$ such that for all compact subsets K ,

$$\lim_{k \rightarrow \infty} \rho_K(f_{n_k}, \infty) = 0. \quad (31.5.30)$$

Proof: Let $B(z_0, 2R) \subseteq \Omega$. There are two cases to consider. The first case is that there exists a subsequence, n_k such that $\{f_{n_k}(z_0)\}$ is bounded. The second case is that $\lim_{n \rightarrow \infty} |f_{n_k}(z_0)| = \infty$.

Consider the first case. By Theorem 31.5.8 $\{f_{n_k}(z)\}$ is uniformly bounded on $\overline{B(z_0, R)}$ because by this theorem, and letting $\theta = 1/2$ applied to $B(z_0, 2R)$, it follows $|f_{n_k}(z)| \leq M(a, b, \frac{1}{2}, \beta)$ where β is an upper bound to the numbers, $|f_{n_k}(z_0)|$. The Cauchy integral formula implies the existence of a uniform bound on the $\{f'_{n_k}\}$ which implies the functions are equicontinuous and uniformly bounded. Therefore, by the Ascoli Arzela theorem there exists a further subsequence which converges uniformly on $\overline{B(z_0, R)}$ to a function, f analytic on $B(z_0, R)$. Thus denoting this subsequence by $\{f_{n_k}\}$ to save on notation,

$$\lim_{k \rightarrow \infty} \|f_{n_k} - f\|_{\overline{B(z_0, R)}, \infty} = 0. \quad (31.5.31)$$

Consider the second case. In this case, it follows $\{1/f_n(z_0)\}$ is bounded on $\overline{B(z_0, R)}$ and so by the same argument just given $\{1/f_n(z)\}$ is uniformly bounded on $\overline{B(z_0, R)}$. Therefore, a subsequence converges uniformly on $\overline{B(z_0, R)}$. But $\{1/f_n(z)\}$ converges to 0 and so this requires that $\{1/f_n(z)\}$ must converge uniformly to 0. Therefore,

$$\lim_{k \rightarrow \infty} \rho_{\overline{B(z_0, R)}}(f_{n_k}, \infty) = 0. \quad (31.5.32)$$

Now let $\{D_k\}$ denote a countable set of closed balls, $D_k = \overline{B(z_k, R_k)}$ such that $B(z_k, 2R_k) \subseteq \Omega$ and $\bigcup_{k=1}^{\infty} \text{int}(D_k) = \Omega$. Using a Cantor diagonal process, there exists a subsequence, $\{f_{n_k}\}$ of $\{f_n\}$ such that for each D_j , one of the above two alternatives holds. That is, either

$$\lim_{k \rightarrow \infty} \|f_{n_k} - g_j\|_{D_j, \infty} = 0 \quad (31.5.33)$$

or,

$$\lim_{k \rightarrow \infty} \rho_{D_j}(f_{n_k}, \infty). \quad (31.5.34)$$

Let $A = \{\bigcup \text{int}(D_j) : 31.5.33 \text{ holds}\}$, $B = \{\bigcup \text{int}(D_j) : 31.5.34 \text{ holds}\}$. Note that the balls whose union is A cannot intersect any of the balls whose union is B . Therefore, one of A or B must be empty since otherwise, Ω would not be connected.

If K is any compact subset of Ω , it follows K must be a subset of some finite collection of the D_j . Therefore, one of the alternatives in the lemma must hold. That the limit function, f must be analytic follows easily in the same way as the proof in Theorem 31.3.1 on Page 924. You could also use Morera's theorem. This proves the lemma.

31.5.6 The Great Big Picard Theorem

The next theorem is the main result which the above lemmas lead to. It is the Big Picard theorem, also called the Great Picard theorem. Recall $B'(a, r)$ is the deleted ball consisting of all the points of the ball except the center.

Theorem 31.5.17 *Suppose f has an isolated essential singularity at 0. Then for every $R > 0$, and $\beta \in \mathbb{C}$, $f^{-1}(\beta) \cap B'(0, R)$ is an infinite set except for one possible exceptional β .*

Proof: Suppose this is not true. Then there exists $R_1 > 0$ and two points, α and β such that $f^{-1}(\beta) \cap B'(0, R_1)$ and $f^{-1}(\alpha) \cap B'(0, R_1)$ are both finite sets. Then shrinking R_1 and calling the result R , there exists $B(0, R)$ such that

$$f^{-1}(\beta) \cap B'(0, R) = \emptyset, \quad f^{-1}(\alpha) \cap B'(0, R) = \emptyset.$$

Now let A_0 denote the annulus $\{z \in \mathbb{C} : \frac{R}{2^2} < |z| < \frac{3R}{2^2}\}$ and let A_n denote the annulus $\{z \in \mathbb{C} : \frac{R}{2^{2+n}} < |z| < \frac{3R}{2^{2+n}}\}$. The reason for the 3 is to insure that $A_n \cap A_{n+1} \neq \emptyset$. This follows from the observation that $3R/2^{2+1+n} > R/2^{2+n}$. Now define a set of functions on A_0 as follows:

$$f_n(z) \equiv f\left(\frac{z}{2^n}\right).$$

By the choice of R , this set of functions missed the two points α and β . Therefore, by Lemma 31.5.16 there exists a subsequence such that one of the two options presented there holds.

First suppose $\lim_{k \rightarrow \infty} \|f_{n_k} - f\|_{K, \infty} = 0$ for all K a compact subset of A_0 and f is analytic on A_0 . In particular, this happens for γ_0 the circular contour having radius $R/2$. Thus f_{n_k} must be bounded on this contour. But this says the same thing as $f(z/2^{n_k})$ is bounded for $|z| = R/2$, this holding for each $k = 1, 2, \dots$. Thus there exists a constant, M such that on each of a shrinking sequence of concentric circles whose radii converge to 0, $|f(z)| \leq M$. By the maximum modulus theorem, $|f(z)| \leq M$ at every point between successive circles in this sequence. Therefore, $|f(z)| \leq M$ in $B'(0, R)$ contradicting the Weierstrass Casorati theorem.

The other option which might hold from Lemma 31.5.16 is that $\lim_{k \rightarrow \infty} \rho_K(f_{n_k}, \infty) = 0$ for all K compact subset of A_0 . Since f has an essential singularity at 0 the zeros of f in $B(0, R)$ are isolated. Therefore, for all k large enough, f_{n_k} has no zeros for $|z| < 3R/2^2$. This is because the values of f_{n_k} are the values of f on A_{n_k} , a small annulus which avoids all the zeros of f whenever k is large enough. Only consider k this large. Then use the above argument on the analytic functions $1/f_{n_k}$. By the assumption that $\lim_{k \rightarrow \infty} \rho_K(f_{n_k}, \infty) = 0$, it follows $\lim_{k \rightarrow \infty} \|1/f_{n_k} - 0\|_{K, \infty} = 0$ and so as above, there exists a shrinking sequence of concentric circles whose radii converge to 0 and a constant, M such that for z on any of these circles, $|1/f(z)| \leq M$. This implies that on some deleted ball, $B'(0, r)$ where $r \leq R$, $|f(z)| \geq 1/M$ which again violates the Weierstrass Casorati theorem. This proves the theorem.

As a simple corollary, here is what this remarkable theorem says about entire functions.

Corollary 31.5.18 *Suppose f is entire and nonconstant and not a polynomial. Then f assumes every complex value infinitely many times with the possible exception of one.*

Proof: Since f is entire, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Define for $z \neq 0$,

$$g(z) \equiv f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n.$$

Thus 0 is an isolated essential singular point of g . By the big Picard theorem, Theorem 31.5.17 it follows g takes every complex number but possibly one an infinite number of times. This proves the corollary.

Note the difference between this and the little Picard theorem which says that an entire function which is not constant must achieve every value but two.

31.6 Exercises

1. Prove that in Theorem 31.3.1 it suffices to assume $\mathcal{F}$ is uniformly bounded on each compact subset of Ω .
2. Find conditions on a, b, c, d such that the fractional linear transformation, $\frac{az+b}{cz+d}$ maps the upper half plane onto the upper half plane.
3. Let D be a simply connected region which is a proper subset of $\mathbb{C}$. Does there exist an entire function, f which maps $\mathbb{C}$ onto D ? Why or why not?
4. Verify the conclusion of Theorem 31.3.1 involving the higher order derivatives.
5. What if $\Omega = \mathbb{C}$? Does there exist an analytic function, f mapping Ω one to one and onto $B(0, 1)$? Explain why or why not. Was $\Omega \neq \mathbb{C}$ used in the proof of the Riemann mapping theorem?
6. Verify that $|\phi_{\alpha}(z)| = 1$ if $|z| = 1$. Apply the maximum modulus theorem to conclude that $|\phi_{\alpha}(z)| \leq 1$ for all $|z| < 1$.
7. Suppose that $|f(z)| \leq 1$ for $|z| = 1$ and $f(\alpha) = 0$ for $|\alpha| < 1$. Show that $|f(z)| \leq |\phi_{\alpha}(z)|$ for all $z \in B(0, 1)$. **Hint:** Consider $\frac{f(z)(1-\bar{\alpha}z)}{z-\alpha}$ which has a removable singularity at α . Show the modulus of this function is bounded by 1 on $|z| = 1$. Then apply the maximum modulus theorem.
8. Let U and V be open subsets of $\mathbb{C}$ and suppose $u : U \rightarrow \mathbb{R}$ is harmonic while h is an analytic map which takes V one to one onto U . Show that $u \circ h$ is harmonic on V .
9. Show that for a harmonic function, u defined on $B(0, R)$, there exists an analytic function, $h = u + iv$ where

$$v(x, y) \equiv \int_0^y u_x(x, t) dt - \int_0^x u_y(t, 0) dt.$$

10. Suppose Ω is a simply connected region and u is a real valued function defined on Ω such that u is harmonic. Show there exists an analytic function, f such that $u = \operatorname{Re} f$. Show this is not true if Ω is not a simply connected region. **Hint:** You might use the Riemann mapping theorem and Problems 8 and 9. For the second part it might be good to try something like $u(x, y) = \ln(x^2 + y^2)$ on the annulus $1 < |z| < 2$.
11. Show that $w = \frac{1+z}{1-z}$ maps $\{z \in \mathbb{C} : \operatorname{Im} z > 0 \text{ and } |z| < 1\}$ to the first quadrant, $\{z = x + iy : x, y > 0\}$.
12. Let $f(z) = \frac{az+b}{cz+d}$ and let $g(z) = \frac{a_1z+b_1}{c_1z+d_1}$. Show that $f \circ g(z)$ equals the quotient of two expressions, the numerator being the top entry in the vector

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix}$$

and the denominator being the bottom entry. Show that if you define

$$\phi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \equiv \frac{az+b}{cz+d},$$

then $\phi(AB) = \phi(A) \circ \phi(B)$. Find an easy way to find the inverse of $f(z) = \frac{az+b}{cz+d}$ and give a condition on the a, b, c, d which insures this function has an inverse.

13. The modular group² is the set of fractional linear transformations, $\frac{az+b}{cz+d}$ such that a, b, c, d are integers and $ad - bc = 1$. Using Problem 12 or brute force show this modular group is really a group with the group operation being composition. Also show the inverse of $\frac{az+b}{cz+d}$ is $\frac{dz-b}{-cz+a}$.
14. Let Ω be a region and suppose f is analytic on Ω and that the functions f_n are also analytic on Ω and converge to f uniformly on compact subsets of Ω . Suppose f is one to one. Can it be concluded that for an arbitrary compact set, $K \subseteq \Omega$ that f_n is one to one for all n large enough?
15. The Vitali theorem says that if Ω is a region and $\{f_n\}$ is a uniformly bounded sequence of functions which converges pointwise on a set, $S \subseteq \Omega$ which has a limit point in Ω , then in fact, $\{f_n\}$ must converge uniformly on compact subsets of Ω to an analytic function. Prove this theorem. **Hint:** If the sequence fails to converge, show you can get two different subsequences converging uniformly on compact sets to different functions. Then argue these two functions coincide on S .
16. Does there exist a function analytic on $B(0, 1)$ which maps $B(0, 1)$ onto $B'(0, 1)$, the open unit ball in which 0 has been deleted?

²This is the terminology used in Rudin's book Real and Complex Analysis.

Approximation By Rational Functions

32.1 Runge's Theorem

Consider the function, $\frac{1}{z} = f(z)$ for z defined on $\Omega \equiv B(0, 1) \setminus \{0\} = B'(0, 1)$. Clearly f is analytic on Ω . Suppose you could approximate f uniformly by polynomials on $\text{ann}(0, \frac{1}{2}, \frac{3}{4})$, a compact subset of Ω . Then, there would exist a suitable polynomial $p(z)$, such that $\left| \frac{1}{2\pi i} \int_{\gamma} f(z) - p(z) dz \right| < \frac{1}{10}$ where here γ is a circle of radius $\frac{2}{3}$. However, this is impossible because $\frac{1}{2\pi i} \int_{\gamma} f(z) dz = 1$ while $\frac{1}{2\pi i} \int_{\gamma} p(z) dz = 0$. This shows you can't expect to be able to uniformly approximate analytic functions on compact sets using polynomials. This is just horrible! In real variables, you can approximate any **continuous function** on a compact set with a polynomial. However, that is just the way it is. It turns out that the ability to approximate an analytic function on Ω with polynomials is dependent on Ω being simply connected.

All these theorems work for f having values in a complex Banach space. However, I will present them in the context of functions which have values in $\mathbb{C}$. The changes necessary to obtain the extra generality are very minor.

Definition 32.1.1 *Approximation will be taken with respect to the following norm.*

$$\|f - g\|_{K, \infty} \equiv \sup \{ \|f(z) - g(z)\| : z \in K \}$$

32.1.1 Approximation With Rational Functions

It turns out you can approximate analytic functions by rational functions, quotients of polynomials. The resulting theorem is one of the most profound theorems in complex analysis. The basic idea is simple. The Riemann sums for the Cauchy integral formula are rational functions. The idea used to implement this observation is that if you have a compact subset, K of an open set, Ω there exists a cycle composed of closed oriented curves $\{\gamma_j\}_{j=1}^n$ which are contained in $\Omega \setminus K$ such that

for every $z \in K$, $\sum_{k=1}^n n(\gamma_k, z) = 1$. One more ingredient is needed and this is a theorem which lets you keep the approximation but move the poles.

To begin with, consider the part about the cycle of closed oriented curves. Recall Theorem 27.7.25 which is stated for convenience.

Theorem 32.1.2 *Let K be a compact subset of an open set, Ω . Then there exist continuous, closed, bounded variation oriented curves $\{\gamma_j\}_{j=1}^m$ for which $\gamma_j^* \cap K = \emptyset$ for each j , $\gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^m n(p, \gamma_k) = 1.$$

and

$$\sum_{k=1}^m n(z, \gamma_k) = 0$$

for all $z \notin \Omega$.

This theorem implies the following.

Theorem 32.1.3 *Let $K \subseteq \Omega$ where K is compact and Ω is open. Then there exist oriented closed curves, γ_k such that $\gamma_k^* \cap K = \emptyset$ but $\gamma_k^* \subseteq \Omega$, such that for all $z \in K$,*

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^p \int_{\gamma_k} \frac{f(w)}{w-z} dw. \quad (32.1.1)$$

Proof: This follows from Theorem 27.7.25 and the Cauchy integral formula. As shown in the proof, you can assume the γ_k are linear mappings but this is not important.

Next I will show how the Cauchy integral formula leads to approximation by rational functions, quotients of polynomials.

Lemma 32.1.4 *Let K be a compact subset of an open set, Ω and let f be analytic on Ω . Then there exists a rational function, Q whose poles are not in K such that*

$$\|Q - f\|_{K, \infty} < \varepsilon.$$

Proof: By Theorem 32.1.3 there are oriented curves, γ_k described there such that for all $z \in K$,

$$f(z) = \frac{1}{2\pi i} \sum_{k=1}^p \int_{\gamma_k} \frac{f(w)}{w-z} dw. \quad (32.1.2)$$

Defining $g(w, z) \equiv \frac{f(w)}{w-z}$ for $(w, z) \in \cup_{k=1}^p \gamma_k^* \times K$, it follows since the distance between K and $\cup_k \gamma_k^*$ is positive that g is uniformly continuous and so there exists a $\delta > 0$ such that if $\|\mathcal{P}\| < \delta$, then for all $z \in K$,

$$\left| f(z) - \frac{1}{2\pi i} \sum_{k=1}^p \sum_{j=1}^n \frac{f(\gamma_k(\tau_j))(\gamma_k(t_i) - \gamma_k(t_{i-1}))}{\gamma_k(\tau_j) - z} \right| < \frac{\varepsilon}{2}.$$

The complicated expression is obtained by replacing each integral in 32.1.2 with a Riemann sum. Simplifying the appearance of this, it follows there exists a rational function of the form

$$R(z) = \sum_{k=1}^M \frac{A_k}{w_k - z}$$

where the w_k are elements of components of $\mathbb{C} \setminus K$ and A_k are complex numbers or in the case where f has values in X , these would be elements of X such that

$$\|R - f\|_{K, \infty} < \frac{\varepsilon}{2}.$$

This proves the lemma.

32.1.2 Moving The Poles And Keeping The Approximation

Lemma 32.1.4 is a nice lemma but needs refining. In this lemma, the Riemann sum handed you the poles. It is much better if you can pick the poles. The following theorem from advanced calculus, called Merten's theorem, will be used

32.1.3 Merten's Theorem.

Theorem 32.1.5 *Suppose $\sum_{i=r}^{\infty} a_i$ and $\sum_{j=r}^{\infty} b_j$ both converge absolutely¹. Then*

$$\left(\sum_{i=r}^{\infty} a_i \right) \left(\sum_{j=r}^{\infty} b_j \right) = \sum_{n=r}^{\infty} c_n$$

where

$$c_n = \sum_{k=r}^n a_k b_{n-k+r}.$$

Proof: Let $p_{nk} = 1$ if $r \leq k \leq n$ and $p_{nk} = 0$ if $k > n$. Then

$$c_n = \sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r}.$$

¹Actually, it is only necessary to assume one of the series converges and the other converges absolutely. This is known as Merten's theorem and may be read in the 1974 book by Apostol listed in the bibliography.

Also,

$$\begin{aligned}
 \sum_{k=r}^{\infty} \sum_{n=r}^{\infty} p_{nk} |a_k| |b_{n-k+r}| &= \sum_{k=r}^{\infty} |a_k| \sum_{n=r}^{\infty} p_{nk} |b_{n-k+r}| \\
 &= \sum_{k=r}^{\infty} |a_k| \sum_{n=k}^{\infty} |b_{n-k+r}| \\
 &= \sum_{k=r}^{\infty} |a_k| \sum_{n=k}^{\infty} |b_{n-(k-r)}| \\
 &= \sum_{k=r}^{\infty} |a_k| \sum_{m=r}^{\infty} |b_m| < \infty.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \sum_{n=r}^{\infty} c_n &= \sum_{n=r}^{\infty} \sum_{k=r}^n a_k b_{n-k+r} = \sum_{n=r}^{\infty} \sum_{k=r}^{\infty} p_{nk} a_k b_{n-k+r} \\
 &= \sum_{k=r}^{\infty} a_k \sum_{n=r}^{\infty} p_{nk} b_{n-k+r} = \sum_{k=r}^{\infty} a_k \sum_{n=k}^{\infty} b_{n-k+r} \\
 &= \sum_{k=r}^{\infty} a_k \sum_{m=r}^{\infty} b_m
 \end{aligned}$$

and this proves the theorem.

It follows that $\sum_{n=r}^{\infty} c_n$ converges absolutely. Also, you can see by induction that you can multiply any number of absolutely convergent series together and obtain a series which is absolutely convergent. Next, here are some similar results related to Merten's theorem.

Lemma 32.1.6 *Let $\sum_{n=0}^{\infty} a_n(z)$ and $\sum_{n=0}^{\infty} b_n(z)$ be two convergent series for $z \in K$ which satisfy the conditions of the Weierstrass M test. Thus there exist positive constants, A_n and B_n such that $|a_n(z)| \leq A_n, |b_n(z)| \leq B_n$ for all $z \in K$ and $\sum_{n=0}^{\infty} A_n < \infty, \sum_{n=0}^{\infty} B_n < \infty$. Then defining the Cauchy product,*

$$c_n(z) \equiv \sum_{k=0}^n a_{n-k}(z) b_k(z),$$

it follows $\sum_{n=0}^{\infty} c_n(z)$ also converges absolutely and uniformly on K because $c_n(z)$ satisfies the conditions of the Weierstrass M test. Therefore,

$$\sum_{n=0}^{\infty} c_n(z) = \left(\sum_{k=0}^{\infty} a_k(z) \right) \left(\sum_{n=0}^{\infty} b_n(z) \right). \quad (32.1.3)$$

Proof:

$$|c_n(z)| \leq \sum_{k=0}^n |a_{n-k}(z)| |b_k(z)| \leq \sum_{k=0}^n A_{n-k} B_k.$$

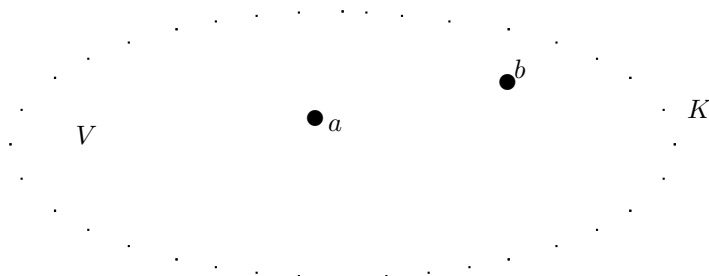
Also,

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=0}^n A_{n-k} B_k &= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} A_{n-k} B_k \\ &= \sum_{k=0}^{\infty} B_k \sum_{n=0}^{\infty} A_n < \infty. \end{aligned}$$

The claim of 32.1.3 follows from Merten's theorem. This proves the lemma.

Corollary 32.1.7 *Let P be a polynomial and let $\sum_{n=0}^{\infty} a_n(z)$ converge uniformly and absolutely on K such that the a_n satisfy the conditions of the Weierstrass M test. Then there exists a series for $P(\sum_{n=0}^{\infty} a_n(z))$, $\sum_{n=0}^{\infty} c_n(z)$, which also converges absolutely and uniformly for $z \in K$ because $c_n(z)$ also satisfies the conditions of the Weierstrass M test.*

The following picture is descriptive of the following lemma. This lemma says that if you have a rational function with one pole off a compact set, then you can approximate on the compact set with another rational function which has a different pole.



Lemma 32.1.8 *Let R be a rational function which has a pole only at $a \in V$, a component of $\mathbb{C} \setminus K$ where K is a compact set. Suppose $b \in V$. Then for $\varepsilon > 0$ given, there exists a rational function, Q , having a pole only at b such that*

$$\|R - Q\|_{K, \infty} < \varepsilon. \quad (32.1.4)$$

If it happens that V is unbounded, then there exists a polynomial, P such that

$$\|R - P\|_{K, \infty} < \varepsilon. \quad (32.1.5)$$

Proof: Say that $b \in V$ satisfies $\mathcal{P}$ if for all $\varepsilon > 0$ there exists a rational function, Q_b , having a pole only at b such that

$$\|R - Q_b\|_{K, \infty} < \varepsilon$$

Now define a set,

$$S \equiv \{b \in V : b \text{ satisfies } \mathcal{P}\}.$$

Observe that $S \neq \emptyset$ because $a \in S$.

I claim S is open. Suppose $b_1 \in S$. Then there exists a $\delta > 0$ such that

$$\left| \frac{b_1 - b}{z - b} \right| < \frac{1}{2} \quad (32.1.6)$$

for all $z \in K$ whenever $b \in B(b_1, \delta)$. In fact, it suffices to take $|b - b_1| < \text{dist}(b_1, K)/4$ because then

$$\begin{aligned} \left| \frac{b_1 - b}{z - b} \right| &< \left| \frac{\text{dist}(b_1, K)/4}{z - b} \right| \leq \frac{\text{dist}(b_1, K)/4}{|z - b_1| - |b_1 - b|} \\ &\leq \frac{\text{dist}(b_1, K)/4}{\text{dist}(b_1, K) - \text{dist}(b_1, K)/4} \leq \frac{1}{3} < \frac{1}{2}. \end{aligned}$$

Since b_1 satisfies $\mathcal{P}$, there exists a rational function Q_{b_1} with the desired properties. It is shown next that you can approximate Q_{b_1} with Q_b thus yielding an approximation to R by the use of the triangle inequality,

$$\|R - Q_{b_1}\|_{K, \infty} + \|Q_{b_1} - Q_b\|_{K, \infty} \geq \|R - Q_b\|_{K, \infty}.$$

Since Q_{b_1} has poles only at b_1 , it follows it is a sum of functions of the form $\frac{\alpha_n}{(z - b_1)^n}$. Therefore, it suffices to consider the terms of Q_{b_1} or that Q_{b_1} is of the special form

$$Q_{b_1}(z) = \frac{1}{(z - b_1)^n}.$$

However,

$$\frac{1}{(z - b_1)^n} = \frac{1}{(z - b)^n \left(1 - \frac{b_1 - b}{z - b}\right)^n}$$

Now from the choice of b_1 , the series

$$\sum_{k=0}^{\infty} \left(\frac{b_1 - b}{z - b} \right)^k = \frac{1}{\left(1 - \frac{b_1 - b}{z - b}\right)}$$

converges absolutely independent of the choice of $z \in K$ because

$$\left| \left(\frac{b_1 - b}{z - b} \right)^k \right| < \frac{1}{2^k}.$$

By Corollary 32.1.7 the same is true of the series for $\frac{1}{\left(1 - \frac{b_1 - b}{z - b}\right)^n}$. Thus a suitable partial sum can be made uniformly on K as close as desired to $\frac{1}{(z - b_1)^n}$. This shows that b satisfies $\mathcal{P}$ whenever b is close enough to b_1 verifying that S is open.

Next it is shown S is closed in V . Let $b_n \in S$ and suppose $b_n \rightarrow b \in V$. Then since $b_n \in S$, there exists a rational function, Q_{b_n} such that

$$\|Q_{b_n} - R\|_{K, \infty} < \frac{\varepsilon}{2}.$$

Then for all n large enough,

$$\frac{1}{2} \operatorname{dist}(b, K) \geq |b_n - b|$$

and so for all n large enough,

$$\left| \frac{b - b_n}{z - b_n} \right| < \frac{1}{2},$$

for all $z \in K$. Pick such a b_n . As before, it suffices to assume Q_{b_n} is of the form $\frac{1}{(z - b_n)^n}$. Then

$$Q_{b_n}(z) = \frac{1}{(z - b_n)^n} = \frac{1}{(z - b)^n \left(1 - \frac{b_n - b}{z - b}\right)^n}$$

and because of the estimate, there exists M such that for all $z \in K$

$$\left| \frac{1}{\left(1 - \frac{b_n - b}{z - b}\right)^n} - \sum_{k=0}^M a_k \left(\frac{b_n - b}{z - b}\right)^k \right| < \frac{\varepsilon (\operatorname{dist}(b, K))^n}{2}. \quad (32.1.7)$$

Therefore, for all $z \in K$

$$\begin{aligned} \left| Q_{b_n}(z) - \frac{1}{(z - b)^n} \sum_{k=0}^M a_k \left(\frac{b_n - b}{z - b}\right)^k \right| &= \\ \left| \frac{1}{(z - b)^n \left(1 - \frac{b_n - b}{z - b}\right)^n} - \frac{1}{(z - b)^n} \sum_{k=0}^M a_k \left(\frac{b_n - b}{z - b}\right)^k \right| &\leq \\ \frac{\varepsilon (\operatorname{dist}(b, K))^n}{2} \frac{1}{\operatorname{dist}(b, K)^n} &= \frac{\varepsilon}{2} \end{aligned}$$

and so, letting $Q_b(z) = \frac{1}{(z - b)^n} \sum_{k=0}^M a_k \left(\frac{b_n - b}{z - b}\right)^k$,

$$\begin{aligned} \|R - Q_b\|_{K, \infty} &\leq \|R - Q_{b_n}\|_{K, \infty} + \|Q_{b_n} - Q_b\|_{K, \infty} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

showing that $b \in S$. Since S is both open and closed in V it follows that, since $S \neq \emptyset$, $S = V$. Otherwise V would fail to be connected.

It remains to consider the case where V is unbounded. Pick $b \in V$ large enough that

$$\left| \frac{z}{b} \right| < \frac{1}{2} \quad (32.1.8)$$

for all $z \in K$. From what was just shown, there exists a rational function, Q_b having a pole only at b such that $\|Q_b - R\|_{K, \infty} < \frac{\varepsilon}{2}$. It suffices to assume that Q_b is of the

form

$$\begin{aligned} Q_b(z) &= \frac{p(z)}{(z-b)^n} = p(z) (-1)^n \frac{1}{b^n} \frac{1}{\left(1 - \frac{z}{b}\right)^n} \\ &= p(z) (-1)^n \frac{1}{b^n} \left(\sum_{k=0}^{\infty} \left(\frac{z}{b}\right)^k \right)^n \end{aligned}$$

Then by an application of Corollary 32.1.7 there exists a partial sum of the power series for Q_b which is uniformly close to Q_b on K . Therefore, you can approximate Q_b and therefore also R uniformly on K by a polynomial consisting of a partial sum of the above infinite sum. This proves the theorem.

If f is a polynomial, then f has a pole at ∞ . This will be discussed more later.

32.1.4 Runge's Theorem

Now what follows is the first form of Runge's theorem.

Theorem 32.1.9 *Let K be a compact subset of an open set, Ω and let $\{b_j\}$ be a set which consists of one point from each component of $\widehat{\mathbb{C}} \setminus K$. Let f be analytic on Ω . Then for each $\varepsilon > 0$, there exists a rational function, Q whose poles are all contained in the set, $\{b_j\}$ such that*

$$\|Q - f\|_{K,\infty} < \varepsilon. \quad (32.1.9)$$

If $\widehat{\mathbb{C}} \setminus K$ has only one component, then Q may be taken to be a polynomial.

Proof: By Lemma 32.1.4 there exists a rational function of the form

$$R(z) = \sum_{k=1}^M \frac{A_k}{w_k - z}$$

where the w_k are elements of components of $\mathbb{C} \setminus K$ and A_k are complex numbers such that

$$\|R - f\|_{K,\infty} < \frac{\varepsilon}{2}.$$

Consider the rational function, $R_k(z) \equiv \frac{A_k}{w_k - z}$ where $w_k \in V_j$, one of the components of $\mathbb{C} \setminus K$, the given point of V_j being b_j . By Lemma 32.1.8, there exists a function, Q_k which is either a rational function having its only pole at b_j or a polynomial, depending on whether V_j is bounded such that

$$\|R_k - Q_k\|_{K,\infty} < \frac{\varepsilon}{2M}.$$

Letting $Q(z) \equiv \sum_{k=1}^M Q_k(z)$,

$$\|R - Q\|_{K,\infty} < \frac{\varepsilon}{2}.$$

It follows

$$\|f - Q\|_{K, \infty} \leq \|f - R\|_{K, \infty} + \|R - Q\|_{K, \infty} < \varepsilon.$$

In the case of only one component of $\mathbb{C} \setminus K$, this component is the unbounded component and so you can take Q to be a polynomial. This proves the theorem.

The next version of Runge's theorem concerns the case where the given points are contained in $\widehat{\mathbb{C}} \setminus \Omega$ for Ω an open set rather than a compact set. Note that here there could be uncountably many components of $\widehat{\mathbb{C}} \setminus \Omega$ because the components are no longer open sets. An easy example of this phenomenon in one dimension is where $\Omega = [0, 1] \setminus P$ for P the Cantor set. Then you can show that $\mathbb{R} \setminus \Omega$ has uncountably many components. Nevertheless, Runge's theorem will follow from Theorem 32.1.9 with the aid of the following interesting lemma.

Lemma 32.1.10 *Let Ω be an open set in $\mathbb{C}$. Then there exists a sequence of compact sets, $\{K_n\}$ such that*

$$\Omega = \bigcup_{k=1}^{\infty} K_n, \dots, K_n \subseteq \text{int } K_{n+1} \dots, \quad (32.1.10)$$

and for any $K \subseteq \Omega$,

$$K \subseteq K_n, \quad (32.1.11)$$

for all n sufficiently large, and every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$.

Proof: Let

$$V_n \equiv \{z : |z| > n\} \cup \bigcup_{z \notin \Omega} B\left(z, \frac{1}{n}\right).$$

Thus $\{z : |z| > n\}$ contains the point, ∞ . Now let

$$K_n \equiv \widehat{\mathbb{C}} \setminus V_n = \mathbb{C} \setminus V_n \subseteq \Omega.$$

You should verify that 32.1.10 and 32.1.11 hold. It remains to show that every component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$. Let D be a component of $\widehat{\mathbb{C}} \setminus K_n \equiv V_n$.

If $\infty \notin D$, then D contains no point of $\{z : |z| > n\}$ because this set is connected and D is a component. (If it did contain a point of this set, it would have to contain the whole set.) Therefore, $D \subseteq \bigcup_{z \notin \Omega} B\left(z, \frac{1}{n}\right)$ and so D contains some point

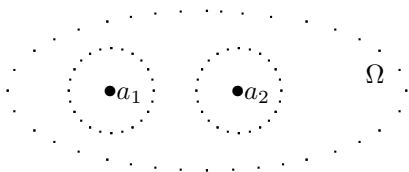
of $B\left(z, \frac{1}{n}\right)$ for some $z \notin \Omega$. Therefore, since this ball is connected, it follows D must contain the whole ball and consequently D contains some point of Ω^C . (The point z at the center of the ball will do.) Since D contains $z \notin \Omega$, it must contain the component, H_z , determined by this point. The reason for this is that

$$H_z \subseteq \widehat{\mathbb{C}} \setminus \Omega \subseteq \widehat{\mathbb{C}} \setminus K_n$$

and H_z is connected. Therefore, H_z can only have points in one component of $\widehat{\mathbb{C}} \setminus K_n$. Since it has a point in D , it must therefore, be totally contained in D . This verifies the desired condition in the case where $\infty \notin D$.

Now suppose that $\infty \in D$. $\infty \notin \Omega$ because Ω is given to be a set in $\mathbb{C}$. Letting H_∞ denote the component of $\widehat{\mathbb{C}} \setminus \Omega$ determined by ∞ , it follows both D and H_∞ contain ∞ . Therefore, the connected set, H_∞ cannot have any points in another component of $\widehat{\mathbb{C}} \setminus K_n$ and it is a set which is contained in $\widehat{\mathbb{C}} \setminus K_n$ so it must be contained in D . This proves the lemma.

The following picture is a very simple example of the sort of thing considered by Runge's theorem. The picture is of a region which has a couple of holes.



However, there could be many more holes than two. In fact, there could be infinitely many. Nor does it follow that the components of the complement of Ω need to have any interior points. Therefore, the picture is certainly not representative.

Theorem 32.1.11 (*Runge*) *Let Ω be an open set, and let A be a set which has one point in each component of $\widehat{\mathbb{C}} \setminus \Omega$ and let f be analytic on Ω . Then there exists a sequence of rational functions, $\{R_n\}$ having poles only in A such that R_n converges uniformly to f on compact subsets of Ω .*

Proof: Let K_n be the compact sets of Lemma 32.1.10 where each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a component of $\widehat{\mathbb{C}} \setminus \Omega$. It follows each component of $\widehat{\mathbb{C}} \setminus K_n$ contains a point of A . Therefore, by Theorem 32.1.9 there exists R_n a rational function with poles only in A such that

$$\|R_n - f\|_{K_n, \infty} < \frac{1}{n}.$$

It follows, since a given compact set, K is a subset of K_n for all n large enough, that $R_n \rightarrow f$ uniformly on K . This proves the theorem.

Corollary 32.1.12 *Let Ω be simply connected and f analytic on Ω . Then there exists a sequence of polynomials, $\{p_n\}$ such that $p_n \rightarrow f$ uniformly on compact sets of Ω .*

Proof: By definition of what is meant by simply connected, $\widehat{\mathbb{C}} \setminus \Omega$ is connected and so there are no bounded components of $\widehat{\mathbb{C}} \setminus \Omega$. Therefore, in the proof of Theorem 32.1.11 when you use Theorem 32.1.9, you can always have R_n be a polynomial by Lemma 32.1.8.

32.2 The Mittag-Leffler Theorem

32.2.1 A Proof From Runge's Theorem

This theorem is fairly easy to prove once you have Theorem 32.1.9. Given a set of complex numbers, does there exist a meromorphic function having its poles equal

to this set of numbers? The Mittag-Leffler theorem provides a very satisfactory answer to this question. Actually, it says somewhat more. You can specify, not just the location of the pole but also the kind of singularity the meromorphic function is to have at that pole.

Theorem 32.2.1 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in an open subset of $\mathbb{C}$, Ω . Suppose also that $P \subseteq \Omega \subseteq \mathbb{C}$. For each z_k , denote by $S_k(z)$ a function of the form*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{a_j^k}{(z - z_k)^j}.$$

Then there exists a meromorphic function, Q defined on Ω such that the poles of Q are the points, $\{z_k\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of Q at z_k equals $S_k(z)$. In other words, for z near z_k , $Q(z) = g_k(z) + S_k(z)$ for some function, g_k analytic near z_k .

Proof: Let $\{K_n\}$ denote the sequence of compact sets described in Lemma 32.1.10. Thus $\cup_{n=1}^{\infty} K_n = \Omega$, $K_n \subseteq \text{int}(K_{n+1}) \subseteq K_{n+1} \cdots$, and the components of $\widehat{\mathbb{C}} \setminus K_n$ contain the components of $\widehat{\mathbb{C}} \setminus \Omega$. Renumbering if necessary, you can assume each $K_n \neq \emptyset$. Also let $K_0 = \emptyset$. Let $P_m \equiv P \cap (K_m \setminus K_{m-1})$ and consider the rational function, R_m defined by

$$R_m(z) \equiv \sum_{z_k \in K_m \setminus K_{m-1}} S_k(z).$$

Since each K_m is compact, it follows P_m is finite and so the above really is a rational function. Now for $m > 1$, this rational function is analytic on some open set containing K_{m-1} . There exists a set of points, A one point in each component of $\widehat{\mathbb{C}} \setminus \Omega$. Consider $\widehat{\mathbb{C}} \setminus K_{m-1}$. Each of its components contains a component of $\widehat{\mathbb{C}} \setminus \Omega$ and so for each of these components of $\widehat{\mathbb{C}} \setminus K_{m-1}$, there exists a point of A which is contained in it. Denote the resulting set of points by A' . By Theorem 32.1.9 there exists a rational function, Q_m whose poles are all contained in the set, $A' \subseteq \Omega^C$ such that

$$\|R_m - Q_m\|_{K_{m-1}, \infty} < \frac{1}{2^m}.$$

The meromorphic function is

$$Q(z) \equiv R_1(z) + \sum_{k=2}^{\infty} (R_k(z) - Q_k(z)).$$

It remains to verify this function works. First consider K_1 . Then on K_1 , the above sum converges uniformly. Furthermore, the terms of the sum are analytic in some open set containing K_1 . Therefore, the infinite sum is analytic on this open set and so for $z \in K_1$ The function, f is the sum of a rational function, R_1 , having poles at

P_1 with the specified singular terms and an analytic function. Therefore, Q works on K_1 . Now consider K_m for $m > 1$. Then

$$Q(z) = R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z)) + \sum_{k=m+2}^{\infty} (R_k(z) - Q_k(z)).$$

As before, the infinite sum converges uniformly on K_{m+1} and hence on some open set, O containing K_m . Therefore, this infinite sum equals a function which is analytic on O . Also,

$$R_1(z) + \sum_{k=2}^{m+1} (R_k(z) - Q_k(z))$$

is a rational function having poles at $\cup_{k=1}^m P_k$ with the specified singularities because the poles of each Q_k are not in Ω . It follows this function is meromorphic because it is analytic except for the points in P . It also has the property of retaining the specified singular behavior.

32.2.2 A Direct Proof Without Runge's Theorem

There is a direct proof of this important theorem which is not dependent on Runge's theorem in the case where $\Omega = \mathbb{C}$. I think it is arguably easier to understand and the Mittag-Leffler theorem is very important so I will give this proof here.

Theorem 32.2.2 *Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in $\mathbb{C}$ which satisfies $\lim_{n \rightarrow \infty} |z_n| = \infty$. For each z_k , denote by $S_k(z)$ a polynomial in $\frac{1}{z-z_k}$ which is of the form*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{a_j^k}{(z-z_k)^j}.$$

Then there exists a meromorphic function, Q defined on $\mathbb{C}$ such that the poles of Q are the points, $\{z_k\}_{k=1}^{\infty}$ and the singular part of the Laurent expansion of Q at z_k equals $S_k(z)$. In other words, for z near z_k ,

$$Q(z) = g_k(z) + S_k(z)$$

for some function, g_k analytic in some open set containing z_k .

Proof: First consider the case where none of the $z_k = 0$. Letting

$$K_k \equiv \{z : |z| \leq |z_k|/2\},$$

there exists a power series for $\frac{1}{z-z_k}$ which converges uniformly and absolutely on this set. Here is why:

$$\frac{1}{z-z_k} = \left(\frac{-1}{1 - \frac{z}{z_k}} \right) \frac{1}{z_k} = \frac{-1}{z_k} \sum_{l=0}^{\infty} \left(\frac{z}{z_k} \right)^l$$

and the Weierstrass M test can be applied because

$$\left| \frac{z}{z_k} \right| < \frac{1}{2}$$

on this set. Therefore, by Corollary 32.1.7, $S_k(z)$, being a polynomial in $\frac{1}{z-z_k}$, has a power series which converges uniformly to $S_k(z)$ on K_k . Therefore, there exists a polynomial, $P_k(z)$ such that

$$\|P_k - S_k\|_{\overline{B(0, |z_k|/2)}, \infty} < \frac{1}{2^k}.$$

Let

$$Q(z) \equiv \sum_{k=1}^{\infty} (S_k(z) - P_k(z)). \quad (32.2.12)$$

Consider $z \in K_m$ and let N be large enough that if $k > N$, then $|z_k| > 2|z|$

$$Q(z) = \sum_{k=1}^N (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).$$

On K_m , the second sum converges uniformly to a function analytic on $\text{int}(K_m)$ (interior of K_m) while the first is a rational function having poles at $z_1, \dots, z_N$. Since any compact set is contained in K_m for large enough m , this shows $Q(z)$ is meromorphic as claimed and has poles with the given singularities.

Now consider the case where the poles are at $\{z_k\}_{k=0}^{\infty}$ with $z_0 = 0$. Everything is similar in this case. Let

$$Q(z) \equiv S_0(z) + \sum_{k=1}^{\infty} (S_k(z) - P_k(z)).$$

The series converges uniformly on every compact set because of the assumption that $\lim_{n \rightarrow \infty} |z_n| = \infty$ which implies that any compact set is contained in K_k for k large enough. Choose N such that $z \in \text{int}(K_N)$ and $z_n \notin K_N$ for all $n \geq N+1$. Then

$$Q(z) = S_0(z) + \sum_{k=1}^N (S_k(z) - P_k(z)) + \sum_{k=N+1}^{\infty} (S_k(z) - P_k(z)).$$

The last sum is analytic on $\text{int}(K_N)$ because each function in the sum is analytic due to the fact that none of its poles are in K_N . Also, $S_0(z) + \sum_{k=1}^N (S_k(z) - P_k(z))$ is a finite sum of rational functions so it is a rational function and P_k is a polynomial so z_m is a pole of this function with the correct singularity whenever $z_m \in \text{int}(K_N)$.

32.2.3 Functions Meromorphic On $\widehat{\mathbb{C}}$

Sometimes it is useful to think of isolated singular points at ∞ .

Definition 32.2.3 Suppose f is analytic on $\{z \in \mathbb{C} : |z| > r\}$. Then f is said to have a removable singularity at ∞ if the function, $g(z) \equiv f(\frac{1}{z})$ has a removable singularity at 0. f is said to have a pole at ∞ if the function, $g(z) = f(\frac{1}{z})$ has a pole at 0. Then f is said to be meromorphic on $\widehat{\mathbb{C}}$ if all its singularities are isolated and either poles or removable.

So what is f like for these cases? First suppose f has a removable singularity at ∞ . Then $zg(z)$ converges to 0 as $z \rightarrow 0$. It follows $g(z)$ must be analytic near 0 and so can be given as a power series. Thus $f(z)$ is of the form $f(z) = g(\frac{1}{z}) = \sum_{n=0}^{\infty} a_n (\frac{1}{z})^n$. Next suppose f has a pole at ∞ . This means $g(z)$ has a pole at 0 so $g(z)$ is of the form $g(z) = \sum_{k=1}^m \frac{b_k}{z^k} + h(z)$ where $h(z)$ is analytic near 0. Thus in the case of a pole at ∞ , $f(z)$ is of the form $f(z) = g(\frac{1}{z}) = \sum_{k=1}^m b_k z^k + \sum_{n=0}^{\infty} a_n (\frac{1}{z})^n$.

It turns out that the functions which are meromorphic on $\widehat{\mathbb{C}}$ are all rational functions. To see this suppose f is meromorphic on $\widehat{\mathbb{C}}$ and note that there exists $r > 0$ such that $f(z)$ is analytic for $|z| > r$. This is required if ∞ is to be isolated. Therefore, there are only finitely many poles of f for $|z| \leq r, \{a_1, \dots, a_m\}$, because by assumption, these poles are isolated and this is a compact set. Let the singular part of f at a_k be denoted by $S_k(z)$. Then $f(z) - \sum_{k=1}^m S_k(z)$ is analytic on all of $\mathbb{C}$. Therefore, it is bounded on $|z| \leq r$. In one case, f has a removable singularity at ∞ . In this case, f is bounded as $z \rightarrow \infty$ and $\sum_k S_k$ also converges to 0 as $z \rightarrow \infty$. Therefore, by Liouville's theorem, $f(z) - \sum_{k=1}^m S_k(z)$ equals a constant and so $f - \sum_k S_k$ is a constant. Thus f is a rational function. In the other case that f has a pole at ∞ , $f(z) - \sum_{k=1}^m S_k(z) - \sum_{k=1}^m b_k z^k = \sum_{n=0}^{\infty} a_n (\frac{1}{z})^n - \sum_{k=1}^m S_k(z)$. Now $f(z) - \sum_{k=1}^m S_k(z) - \sum_{k=1}^m b_k z^k$ is analytic on $\mathbb{C}$ and so is bounded on $|z| \leq r$. But now $\sum_{n=0}^{\infty} a_n (\frac{1}{z})^n - \sum_{k=1}^m S_k(z)$ converges to 0 as $z \rightarrow \infty$ and so by Liouville's theorem, $f(z) - \sum_{k=1}^m S_k(z) - \sum_{k=1}^m b_k z^k$ must equal a constant and again, $f(z)$ equals a rational function.

32.2.4 A Great And Glorious Theorem About Simply Connected Regions

Here is given a laundry list of properties which are equivalent to an open set being simply connected. Recall Definition 27.7.21 on Page 823 which said that an open set, Ω is simply connected means $\widehat{\mathbb{C}} \setminus \Omega$ is connected. Recall also that this is not the same thing at all as saying $\mathbb{C} \setminus \Omega$ is connected. Consider the outside of a disk for example. I will continue to use this definition for simply connected because it is the most convenient one for complex analysis. However, there are many other equivalent conditions. First here is an interesting lemma which is interesting for its own sake. Recall $n(p, \gamma)$ means the winding number of γ about p . Now recall Theorem 27.7.25 implies the following lemma in which B^C is playing the role of Ω in Theorem 27.7.25.

Lemma 32.2.4 *Let K be a compact subset of B^C , the complement of a closed set. Then there exist continuous, closed, bounded variation oriented curves $\{\Gamma_j\}_{j=1}^m$ for which $\Gamma_j^* \cap K = \emptyset$ for each j , $\Gamma_j^* \subseteq \Omega$, and for all $p \in K$,*

$$\sum_{k=1}^m n(\Gamma_k, p) = 1.$$

while for all $z \in B$

$$\sum_{k=1}^m n(\Gamma_k, z) = 0.$$

Definition 32.2.5 *Let γ be a closed curve in an open set, Ω , $\gamma : [a, b] \rightarrow \Omega$. Then γ is said to be homotopic to a point, p in Ω if there exists a continuous function, $H : [0, 1] \times [a, b] \rightarrow \Omega$ such that $H(0, t) = p$, $H(\alpha, a) = H(\alpha, b)$, and $H(1, t) = \gamma(t)$. This function, H is called a homotopy.*

Lemma 32.2.6 *Suppose γ is a closed continuous bounded variation curve in an open set, Ω which is homotopic to a point. Then if $a \notin \Omega$, it follows $n(a, \gamma) = 0$.*

Proof: Let H be the homotopy described above. The problem with this is that it is not known that $H(\alpha, \cdot)$ is of bounded variation. There is no reason it should be. Therefore, it might not make sense to take the integral which defines the winding number. There are various ways around this. Extend H as follows. $H(\alpha, t) = H(\alpha, a)$ for $t < a$, $H(\alpha, t) = H(\alpha, b)$ for $t > b$. Let $\varepsilon > 0$.

$$H_\varepsilon(\alpha, t) \equiv \frac{1}{2\varepsilon} \int_{-2\varepsilon+t+\frac{2\varepsilon}{(b-a)}(t-a)}^{t+\frac{2\varepsilon}{(b-a)}(t-a)} H(\alpha, s) ds, \quad H_\varepsilon(0, t) = p.$$

Thus $H_\varepsilon(\alpha, \cdot)$ is a closed curve which has bounded variation and when $\alpha = 1$, this converges to γ uniformly on $[a, b]$. Therefore, for ε small enough, $n(a, H_\varepsilon(1, \cdot)) = n(a, \gamma)$ because they are both integers and as $\varepsilon \rightarrow 0$, $n(a, H_\varepsilon(1, \cdot)) \rightarrow n(a, \gamma)$. Also, $H_\varepsilon(\alpha, t) \rightarrow H(\alpha, t)$ uniformly on $[0, 1] \times [a, b]$ because of uniform continuity of H . Therefore, for small enough ε , you can also assume $H_\varepsilon(\alpha, t) \in \Omega$ for all α, t . Now $\alpha \rightarrow n(a, H_\varepsilon(\alpha, \cdot))$ is continuous. Hence it must be constant because the winding number is integer valued. But

$$\lim_{\alpha \rightarrow 0} \frac{1}{2\pi i} \int_{H_\varepsilon(\alpha, \cdot)} \frac{1}{z - a} dz = 0$$

because the length of $H_\varepsilon(\alpha, \cdot)$ converges to 0 and the integrand is bounded because $a \notin \Omega$. Therefore, the constant can only equal 0. This proves the lemma.

Now it is time for the great and glorious theorem on simply connected regions. The following equivalence of properties is taken from Rudin [58]. There is a slightly different list in Conway [16] and a shorter list in Ash [6].

Theorem 32.2.7 *The following are equivalent for an open set, Ω .*

1. Ω is homeomorphic to the unit disk, $B(0, 1)$.
2. Every closed curve contained in Ω is homotopic to a point in Ω .
3. If $z \notin \Omega$, and if γ is a closed bounded variation continuous curve in Ω , then $n(\gamma, z) = 0$.
4. Ω is simply connected, ($\widehat{\mathbb{C}} \setminus \Omega$ is connected and Ω is connected.)
5. Every function analytic on Ω can be uniformly approximated by polynomials on compact subsets.
6. For every f analytic on Ω and every closed continuous bounded variation curve, γ ,

$$\int_{\gamma} f(z) dz = 0.$$
7. Every function analytic on Ω has a primitive on Ω .
8. If $f, 1/f$ are both analytic on Ω , then there exists an analytic, g on Ω such that $f = \exp(g)$.
9. If $f, 1/f$ are both analytic on Ω , then there exists ϕ analytic on Ω such that $f = \phi^2$.

Proof: $1 \Rightarrow 2$. Assume 1 and let γ be a closed curve in Ω . Let h be the homeomorphism, $h : B(0, 1) \rightarrow \Omega$. Let $H(\alpha, t) = h(\alpha(h^{-1}\gamma(t)))$. This works.

$2 \Rightarrow 3$ This is Lemma 32.2.6.

$3 \Rightarrow 4$. Suppose 3 but 4 fails to hold. Then if $\widehat{\mathbb{C}} \setminus \Omega$ is not connected, there exist disjoint nonempty sets, A and B such that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. It follows each of these sets must be closed because neither can have a limit point in Ω nor in the other. Also, one and only one of them contains ∞ . Let this set be B . Thus A is a closed set which must also be bounded. Otherwise, there would exist a sequence of points in A , $\{a_n\}$ such that $\lim_{n \rightarrow \infty} a_n = \infty$ which would contradict the requirement that no limit points of A can be in B . Therefore, A is a compact set contained in the open set, $B^C \equiv \{z \in \mathbb{C} : z \notin B\}$. Pick $p \in A$. By Lemma 32.2.4 there exist continuous bounded variation closed curves $\{\Gamma_k\}_{k=1}^m$ which are contained in B^C , do not intersect A and such that

$$1 = \sum_{k=1}^m n(p, \Gamma_k)$$

However, if these curves do not intersect A and they also do not intersect B then they must be all contained in Ω . Since $p \notin \Omega$, it follows by 3 that for each k , $n(p, \Gamma_k) = 0$, a contradiction.

$4 \Rightarrow 5$ This is Corollary 32.1.12 on Page 960.

5 $\Rightarrow$ 6 Every polynomial has a primitive and so the integral over any closed bounded variation curve of a polynomial equals 0. Let f be analytic on Ω . Then let $\{f_n\}$ be a sequence of polynomials converging uniformly to f on γ^* . Then

$$0 = \lim_{n \rightarrow \infty} \int_{\gamma} f_n(z) dz = \int_{\gamma} f(z) dz.$$

6 $\Rightarrow$ 7 Pick $z_0 \in \Omega$. Letting $\gamma(z_0, z)$ be a bounded variation continuous curve joining z_0 to z in Ω , you define a primitive for f as follows.

$$F(z) = \int_{\gamma(z_0, z)} f(w) dw.$$

This is well defined by 6 and is easily seen to be a primitive. You just write the difference quotient and take a limit using 6.

$$\begin{aligned} \lim_{w \rightarrow 0} \frac{F(z+w) - F(z)}{w} &= \lim_{w \rightarrow 0} \frac{1}{w} \left(\int_{\gamma(z_0, z+w)} f(u) du - \int_{\gamma(z_0, z)} f(u) du \right) \\ &= \lim_{w \rightarrow 0} \frac{1}{w} \int_{\gamma(z, z+w)} f(u) du \\ &= \lim_{w \rightarrow 0} \frac{1}{w} \int_0^1 f(z+tw) w dt = f(z). \end{aligned}$$

7 $\Rightarrow$ 8 Suppose then that $f, 1/f$ are both analytic. Then f'/f is analytic and so it has a primitive by 7. Let this primitive be g_1 . Then

$$\begin{aligned} (e^{-g_1} f)' &= e^{-g_1} (-g_1') f + e^{-g_1} f' \\ &= -e^{-g_1} \left(\frac{f'}{f} \right) f + e^{-g_1} f' = 0. \end{aligned}$$

Therefore, since Ω is connected, it follows $e^{-g_1} f$ must equal a constant. (Why?) Let the constant be e^{a+ib} . Then $f(z) = e^{g_1(z)} e^{a+ib}$. Therefore, you let $g(z) = g_1(z) + a + ib$.

8 $\Rightarrow$ 9 Suppose then that $f, 1/f$ are both analytic on Ω . Then by 8 $f(z) = e^{g(z)}$. Let $\phi(z) \equiv e^{g(z)/2}$.

9 $\Rightarrow$ 1 There are two cases. First suppose $\Omega = \mathbb{C}$. This satisfies condition 9 because if $f, 1/f$ are both analytic, then the same argument involved in 8 $\Rightarrow$ 9 gives the existence of a square root. A homeomorphism is $h(z) \equiv \frac{z}{\sqrt{1+|z|^2}}$. It obviously maps onto $B(0, 1)$ and is continuous. To see it is 1-1 consider the case of z_1 and z_2 having different arguments. Then $h(z_1) \neq h(z_2)$. If $z_2 = tz_1$ for a positive $t \neq 1$, then it is also clear $h(z_1) \neq h(z_2)$. To show h^{-1} is continuous, note that if you have an open set in $\mathbb{C}$ and a point in this open set, you can get a small open set containing this point by allowing the modulus and the argument to lie in some open interval. Reasoning this way, you can verify h maps open sets to open sets. In the case where $\Omega \neq \mathbb{C}$, there exists a one to one analytic map which maps Ω onto $B(0, 1)$ by the Riemann mapping theorem. This proves the theorem.

32.3 Exercises

1. Let $a \in \mathbb{C}$. Show there exists a sequence of polynomials, $\{p_n\}$ such that $p_n(a) = 1$ but $p_n(z) \rightarrow 0$ for all $z \neq a$.
2. Let l be a line in $\mathbb{C}$. Show there exists a sequence of polynomials $\{p_n\}$ such that $p_n(z) \rightarrow 1$ on one side of this line and $p_n(z) \rightarrow -1$ on the other side of the line. **Hint:** The complement of this line is simply connected.
3. Suppose Ω is a simply connected region, f is analytic on Ω , $f \neq 0$ on Ω , and $n \in \mathbb{N}$. Show that there exists an analytic function, g such that $g(z)^n = f(z)$ for all $z \in \Omega$. That is, you can take the n^{th} root of $f(z)$. If Ω is a region which contains 0, is it possible to find $g(z)$ such that g is analytic on Ω and $g(z)^2 = z$?
4. Suppose Ω is a region (connected open set) and f is an analytic function defined on Ω such that $f(z) \neq 0$ for any $z \in \Omega$. Suppose also that for every positive integer, n there exists an analytic function, g_n defined on Ω such that $g_n^n(z) = f(z)$. Show that then it is possible to define an analytic function, L on $f(\Omega)$ such that $e^{L(f(z))} = f(z)$ for all $z \in \Omega$.
5. You know that $\phi(z) \equiv \frac{z-i}{z+i}$ maps the upper half plane onto the unit ball. Its inverse, $\psi(z) = i\frac{1+z}{1-z}$ maps the unit ball onto the upper half plane. Also for z in the upper half plane, you can define a square root as follows. If $z = |z|e^{i\theta}$ where $\theta \in (0, \pi)$, let $z^{1/2} \equiv |z|^{1/2}e^{i\theta/2}$ so the square root maps the upper half plane to the first quadrant. Now consider

$$z \rightarrow \exp \left(-i \log \left[i \left(\frac{1+z}{1-z} \right) \right]^{1/2} \right). \quad (32.3.13)$$

Show this is an analytic function which maps the unit ball onto an annulus. Is it possible to find a one to one analytic map which does this?

Infinite Products

The Mittag-Leffler theorem gives existence of a meromorphic function which has specified singular part at various poles. It would be interesting to do something similar to zeros of an analytic function. That is, given the order of the zero at various points, does there exist an analytic function which has these points as zeros with the specified orders? You know that if you have the zeros of the polynomial, you can factor it. Can you do something similar with analytic functions which are just limits of polynomials? These questions involve the concept of an infinite product.

Definition 33.0.1 $\prod_{n=1}^{\infty} (1 + u_n) \equiv \lim_{n \rightarrow \infty} \prod_{k=1}^n (1 + u_k)$ whenever this limit exists. If $u_n = u_n(z)$ for $z \in H$, we say the infinite product converges uniformly on H if the partial products, $\prod_{k=1}^n (1 + u_k(z))$ converge uniformly on H .

The main theorem is the following.

Theorem 33.0.2 Let $H \subseteq \mathbb{C}$ and suppose that $\sum_{n=1}^{\infty} |u_n(z)|$ converges uniformly on H where $u_n(z)$ bounded on H . Then

$$P(z) \equiv \prod_{n=1}^{\infty} (1 + u_n(z))$$

converges uniformly on H . If $(n_1, n_2, \dots)$ is any permutation of $(1, 2, \dots)$, then for all $z \in H$,

$$P(z) = \prod_{k=1}^{\infty} (1 + u_{n_k}(z))$$

and P has a zero at z_0 if and only if $u_n(z_0) = -1$ for some n .

Proof: First a simple estimate:

$$\begin{aligned}
 & \prod_{k=m}^n (1 + |u_k(z)|) \\
 = & \exp \left(\ln \left(\prod_{k=m}^n (1 + |u_k(z)|) \right) \right) = \exp \left(\sum_{k=m}^n \ln (1 + |u_k(z)|) \right) \\
 \leq & \exp \left(\sum_{k=m}^{\infty} |u_k(z)| \right) < e
 \end{aligned}$$

for all $z \in H$ provided m is large enough. Since $\sum_{k=1}^{\infty} |u_k(z)|$ converges uniformly on H , $|u_k(z)| < \frac{1}{2}$ for all $z \in H$ provided k is large enough. Thus you can take $\log(1 + u_k(z))$. Pick N_0 such that for $n > m \geq N_0$,

$$|u_m(z)| < \frac{1}{2}, \quad \prod_{k=m}^n (1 + |u_k(z)|) < e. \quad (33.0.1)$$

Now having picked N_0 , the assumption the u_n are bounded on H implies there exists a constant, C , independent of $z \in H$ such that for all $z \in H$,

$$\prod_{k=1}^{N_0} (1 + |u_k(z)|) < C. \quad (33.0.2)$$

Let $N_0 < M < N$. Then

$$\begin{aligned}
 & \left| \prod_{k=1}^N (1 + u_k(z)) - \prod_{k=1}^M (1 + u_k(z)) \right| \\
 \leq & \left| \prod_{k=1}^{N_0} (1 + |u_k(z)|) \right| \left| \prod_{k=N_0+1}^N (1 + u_k(z)) - \prod_{k=N_0+1}^M (1 + u_k(z)) \right| \\
 \leq & C \left| \prod_{k=N_0+1}^N (1 + u_k(z)) - \prod_{k=N_0+1}^M (1 + u_k(z)) \right| \\
 \leq & C \left(\prod_{k=N_0+1}^M (1 + |u_k(z)|) \right) \left| \prod_{k=M+1}^N (1 + u_k(z)) - 1 \right| \\
 \leq & Ce \left| \prod_{k=M+1}^N (1 + |u_k(z)|) - 1 \right|.
 \end{aligned}$$

Since $1 \leq \prod_{k=M+1}^N (1 + |u_k(z)|) \leq e$, it follows the term on the far right is dominated by

$$\begin{aligned} & Ce^2 \left| \ln \left(\prod_{k=M+1}^N (1 + |u_k(z)|) \right) - \ln 1 \right| \\ & \leq Ce^2 \sum_{k=M+1}^N \ln(1 + |u_k(z)|) \\ & \leq Ce^2 \sum_{k=M+1}^N |u_k(z)| < \varepsilon \end{aligned}$$

uniformly in $z \in H$ provided M is large enough. This follows from the simple observation that if $1 < x < e$, then $x-1 \leq e(\ln x - \ln 1)$. Therefore, $\{\prod_{k=1}^m (1 + u_k(z))\}_{m=1}^\infty$ is uniformly Cauchy on H and therefore, converges uniformly on H . Let $P(z)$ denote the function it converges to.

What about the permutations? Let $\{n_1, n_2, \dots\}$ be a permutation of the indices. Let $\varepsilon > 0$ be given and let N_0 be such that if $n > N_0$,

$$\left| \prod_{k=1}^n (1 + u_k(z)) - P(z) \right| < \varepsilon$$

for all $z \in H$. Let $\{1, 2, \dots, n\} \subseteq \{n_1, n_2, \dots, n_{p(n)}\}$ where $p(n)$ is an increasing sequence. Then from 33.0.1 and 33.0.2,

$$\begin{aligned} & \left| P(z) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| \\ & \leq \left| P(z) - \prod_{k=1}^n (1 + u_k(z)) \right| + \left| \prod_{k=1}^n (1 + u_k(z)) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| \\ & \leq \varepsilon + \left| \prod_{k=1}^n (1 + u_k(z)) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| \\ & \leq \varepsilon + \left| \prod_{k=1}^n (1 + |u_k(z)|) \right| \left| 1 - \prod_{n_k > n} (1 + u_{n_k}(z)) \right| \\ & \leq \varepsilon + \left| \prod_{k=1}^{N_0} (1 + |u_k(z)|) \right| \left| \prod_{k=N_0+1}^n (1 + |u_k(z)|) \right| \left| 1 - \prod_{n_k > n} (1 + u_{n_k}(z)) \right| \\ & \leq \varepsilon + Ce \left| \prod_{n_k > n} (1 + |u_{n_k}(z)|) - 1 \right| \leq \varepsilon + Ce \left| \prod_{k=n+1}^{M(p(n))} (1 + |u_{n_k}(z)|) - 1 \right| \end{aligned}$$

where $M(p(n))$ is the largest index in the permuted list, $\{n_1, n_2, \dots, n_{p(n)}\}$. then from 33.0.1, this last term is dominated by

$$\begin{aligned} & \varepsilon + Ce^2 \left| \ln \left(\prod_{k=n+1}^{M(p(n))} (1 + |u_{n_k}(z)|) \right) - \ln 1 \right| \\ & \leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} \ln(1 + |u_{n_k}|) \leq \varepsilon + Ce^2 \sum_{k=n+1}^{\infty} |u_{n_k}| < 2\varepsilon \end{aligned}$$

for all n large enough uniformly in $z \in H$. Therefore, $\left| P(z) - \prod_{k=1}^{p(n)} (1 + u_{n_k}(z)) \right| < 2\varepsilon$ whenever n is large enough. This proves the part about the permutation.

It remains to verify the assertion about the points, z_0 , where $P(z_0) = 0$. Obviously, if $u_n(z_0) = -1$, then $P(z_0) = 0$. Suppose then that $P(z_0) = 0$ and $M > N_0$. Then

$$\begin{aligned} & \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| = \\ & \left| \prod_{k=1}^M (1 + u_k(z_0)) - \prod_{k=1}^{\infty} (1 + u_k(z_0)) \right| \\ & \leq \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| \left| 1 - \prod_{k=M+1}^{\infty} (1 + u_k(z_0)) \right| \\ & \leq \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| \left| \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - 1 \right| \\ & \leq e \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| \left| \ln \prod_{k=M+1}^{\infty} (1 + |u_k(z_0)|) - \ln 1 \right| \\ & \leq e \left(\sum_{k=M+1}^{\infty} \ln(1 + |u_k(z)|) \right) \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| \\ & \leq e \sum_{k=M+1}^{\infty} |u_k(z)| \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| \\ & \leq \frac{1}{2} \left| \prod_{k=1}^M (1 + u_k(z_0)) \right| \end{aligned}$$

whenever M is large enough. Therefore, for such M ,

$$\prod_{k=1}^M (1 + u_k(z_0)) = 0$$

and so $u_k(z_0) = -1$ for some $k \leq M$. This proves the theorem.

33.1 Analytic Function With Prescribed Zeros

Suppose you are given complex numbers, $\{z_n\}$ and you want to find an analytic function, f such that these numbers are the zeros of f . How can you do it? The problem is easy if there are only finitely many of these zeros, $\{z_1, z_2, \dots, z_m\}$. You just write $(z - z_1)(z - z_2) \cdots (z - z_m)$. Now if none of the $z_k = 0$ you could also write it as $\prod_{k=1}^m \left(1 - \frac{z}{z_k}\right)$ and this might have a better chance of success in the case of infinitely many prescribed zeros. However, you would need to verify something like $\sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right| < \infty$ which might not be so. The way around this is to adjust the product, making it $\prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{g_k(z)}$ where $g_k(z)$ is some analytic function. Recall also that for $|x| < 1$, $\ln\left((1-x)^{-1}\right) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. If you had x/x_n small and real, then $1 = (1 - x/x_n) \exp\left(\ln\left((1 - x/x_n)^{-1}\right)\right)$ and $\prod_{k=1}^{\infty} 1$ of course converges but loses all the information about zeros. However, this is why it is not too unreasonable to consider factors of the form

$$\left(1 - \frac{z}{z_k}\right) e^{\sum_{k=1}^{p_k} \left(\frac{z}{z_k}\right)^k \frac{1}{k}}$$

where p_k is suitably chosen.

First here are some estimates.

Lemma 33.1.1 For $z \in \mathbb{C}$,

$$|e^z - 1| \leq |z| e^{|z|}, \quad (33.1.3)$$

and if $|z| \leq 1/2$,

$$\left|\sum_{k=m}^{\infty} \frac{z^k}{k}\right| \leq \frac{1}{m} \frac{|z|^m}{1 - |z|} \leq \frac{2}{m} |z|^m \leq \frac{1}{m} \frac{1}{2^{m-1}}. \quad (33.1.4)$$

Proof: Consider 33.1.3.

$$|e^z - 1| = \left|\sum_{k=1}^{\infty} \frac{z^k}{k!}\right| \leq \sum_{k=1}^{\infty} \frac{|z|^k}{k!} = e^{|z|} - 1 \leq |z| e^{|z|}$$

the last inequality holding by the mean value theorem. Now consider 33.1.4.

$$\begin{aligned} \left|\sum_{k=m}^{\infty} \frac{z^k}{k}\right| &\leq \sum_{k=m}^{\infty} \frac{|z|^k}{k} \leq \frac{1}{m} \sum_{k=m}^{\infty} |z|^k \\ &= \frac{1}{m} \frac{|z|^m}{1 - |z|} \leq \frac{2}{m} |z|^m \leq \frac{1}{m} \frac{1}{2^{m-1}}. \end{aligned}$$

This proves the lemma.

The functions, E_p in the next definition are called the elementary factors.

Definition 33.1.2 Let $E_0(z) \equiv 1 - z$ and for $p \geq 1$,

$$E_p(z) \equiv (1 - z) \exp \left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right)$$

In terms of this new symbol, here is another estimate. A sharper inequality is available in Rudin [58] but it is more difficult to obtain.

Corollary 33.1.3 For E_p defined above and $|z| \leq 1/2$,

$$|E_p(z) - 1| \leq 3|z|^{p+1}.$$

Proof: From elementary calculus, $\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$ for all $|x| < 1$. Therefore, for $|z| < 1$,

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}, \quad \log((1 - z)^{-1}) = \sum_{n=1}^{\infty} \frac{z^n}{n},$$

because the function $\log(1 - z)$ and the analytic function, $-\sum_{n=1}^{\infty} \frac{z^n}{n}$ both are equal to $\ln(1 - x)$ on the real line segment $(-1, 1)$, a set which has a limit point. Therefore, using Lemma 33.1.1,

$$\begin{aligned} & |E_p(z) - 1| \\ &= \left| (1 - z) \exp \left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right) - 1 \right| \\ &= \left| (1 - z) \exp \left(\log((1 - z)^{-1}) - \sum_{n=p+1}^{\infty} \frac{z^n}{n} \right) - 1 \right| \\ &= \left| \exp \left(-\sum_{n=p+1}^{\infty} \frac{z^n}{n} \right) - 1 \right| \\ &\leq \left| -\sum_{n=p+1}^{\infty} \frac{z^n}{n} \right| e^{\left| -\sum_{n=p+1}^{\infty} \frac{z^n}{n} \right|} \\ &\leq \frac{1}{p+1} \cdot 2 \cdot e^{1/(p+1)} |z|^{p+1} \leq 3|z|^{p+1} \end{aligned}$$

This proves the corollary.

With this estimate, it is easy to prove the Weierstrass product formula.

Theorem 33.1.4 Let $\{z_n\}$ be a sequence of nonzero complex numbers which have no limit point in $\mathbb{C}$ and let $\{p_n\}$ be a sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty} \left(\frac{R}{|z_n|} \right)^{p_n+1} < \infty \quad (33.1.5)$$

for all $R \in \mathbb{R}$. Then

$$P(z) \equiv \prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{z_n} \right)$$

is analytic on $\mathbb{C}$ and has a zero at each point, z_n and at no others. If w occurs m times in $\{z_n\}$, then P has a zero of order m at w .

Proof: Since $\{z_n\}$ has no limit point, it follows $\lim_{n \rightarrow \infty} |z_n| = \infty$. Therefore, if $p_n = n - 1$ the condition, 33.1.5 holds for this choice of p_n . Now by Theorem 33.0.2, the infinite product in this theorem will converge uniformly on $|z| \leq R$ if the same is true of the sum,

$$\sum_{n=1}^{\infty} \left| E_{p_n} \left(\frac{z}{z_n} \right) - 1 \right|. \quad (33.1.6)$$

But by Corollary 33.1.3 the n^{th} term of this sum satisfies

$$\left| E_{p_n} \left(\frac{z}{z_n} \right) - 1 \right| \leq 3 \left| \frac{z}{z_n} \right|^{p_n+1}.$$

Since $|z_n| \rightarrow \infty$, there exists N such that for $n > N$, $|z_n| > 2R$. Therefore, for $|z| < R$ and letting $0 < a = \min \{|z_n| : n \leq N\}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \left| E_{p_n} \left(\frac{z}{z_n} \right) - 1 \right| &\leq 3 \sum_{n=1}^N \left| \frac{R}{a} \right|^{p_n+1} \\ &+ 3 \sum_{n=N}^{\infty} \left(\frac{R}{2R} \right)^{p_n+1} < \infty. \end{aligned}$$

By the Weierstrass M test, the series in 33.1.6 converges uniformly for $|z| < R$ and so the same is true of the infinite product. It follows from Lemma 27.3.12 on Page 802 that $P(z)$ is analytic on $|z| < R$ because it is a uniform limit of analytic functions.

Also by Theorem 33.0.2 the zeros of the analytic $P(z)$ are exactly the points, $\{z_n\}$, listed according to multiplicity. That is, if z_n is a zero of order m , then if it is listed m times in the formula for $P(z)$, then it is a zero of order m for P . This proves the theorem.

The following corollary is an easy consequence and includes the case where there is a zero at 0.

Corollary 33.1.5 *Let $\{z_n\}$ be a sequence of nonzero complex numbers which have no limit point and let $\{p_n\}$ be a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|} \right)^{1+p_n} < \infty \quad (33.1.7)$$

for all $r \in \mathbb{R}$. Then

$$P(z) \equiv z^m \prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{z_n} \right)$$

is analytic Ω and has a zero at each point, z_n and at no others along with a zero of order m at 0. If w occurs m times in $\{z_n\}$, then P has a zero of order m at w .

The above theory can be generalized to include the case of an arbitrary open set. First, here is a lemma.

Lemma 33.1.6 *Let Ω be an open set. Also let $\{z_n\}$ be a sequence of points in Ω which is bounded and which has no point repeated more than finitely many times such that $\{z_n\}$ has no limit point in Ω . Then there exist $\{w_n\} \subseteq \partial\Omega$ such that $\lim_{n \rightarrow \infty} |z_n - w_n| = 0$.*

Proof: Since $\partial\Omega$ is closed, there exists $w_n \in \partial\Omega$ such that $\text{dist}(z_n, \partial\Omega) = |z_n - w_n|$. Now if there is a subsequence, $\{z_{n_k}\}$ such that $|z_{n_k} - w_{n_k}| \geq \varepsilon$ for all k , then $\{z_{n_k}\}$ must possess a limit point because it is a bounded infinite set of points. However, this limit point can only be in Ω because $\{z_{n_k}\}$ is bounded away from $\partial\Omega$. This is a contradiction. Therefore, $\lim_{n \rightarrow \infty} |z_n - w_n| = 0$. This proves the lemma.

Corollary 33.1.7 *Let $\{z_n\}$ be a sequence of complex numbers contained in Ω , an open subset of $\mathbb{C}$ which has no limit point in Ω . Suppose each z_n is repeated no more than finitely many times. Then there exists a function f which is analytic on Ω whose zeros are exactly $\{z_n\}$. If $w \in \{z_n\}$ and w is listed m times, then w is a zero of order m of f .*

Proof: There is nothing to prove if $\{z_n\}$ is finite. You just let $f(z) = \prod_{j=1}^m (z - z_j)$ where $\{z_n\} = \{z_1, \dots, z_m\}$.

Pick $w \in \Omega \setminus \{z_n\}_{n=1}^{\infty}$ and let $h(z) \equiv \frac{1}{z-w}$. Since w is not a limit point of $\{z_n\}$, there exists $r > 0$ such that $B(w, r)$ contains no points of $\{z_n\}$. Let $\Omega_1 \equiv \Omega \setminus \{w\}$. Now h is not constant and so $h(\Omega_1)$ is an open set by the open mapping theorem. In fact, h maps each component of Ω to a region. $|z_n - w| > r$ for all z_n and so $|h(z_n)| < r^{-1}$. Thus the sequence, $\{h(z_n)\}$ is a bounded sequence in the open set $h(\Omega_1)$. It has no limit point in $h(\Omega_1)$ because this is true of $\{z_n\}$ and Ω_1 . By Lemma 33.1.6 there exist $w_n \in \partial(h(\Omega_1))$ such that $\lim_{n \rightarrow \infty} |w_n - h(z_n)| = 0$. Consider for $z \in \Omega_1$

$$f(z) \equiv \prod_{n=1}^{\infty} E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right). \quad (33.1.8)$$

Letting K be a compact subset of Ω_1 , $h(K)$ is a compact subset of $h(\Omega_1)$ and so if $z \in K$, then $|h(z) - w_n|$ is bounded below by a positive constant. Therefore, there exists N large enough that for all $z \in K$ and $n \geq N$,

$$\left| \frac{h(z_n) - w_n}{h(z) - w_n} \right| < \frac{1}{2}$$

and so by Corollary 33.1.3, for all $z \in K$ and $n \geq N$,

$$\left| E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right) - 1 \right| \leq 3 \left(\frac{1}{2} \right)^n. \quad (33.1.9)$$

Therefore,

$$\sum_{n=1}^{\infty} \left| E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right) - 1 \right|$$

converges uniformly for $z \in K$. This implies $\prod_{n=1}^{\infty} E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right)$ also converges uniformly for $z \in K$ by Theorem 33.0.2. Since K is arbitrary, this shows f defined in 33.1.8 is analytic on Ω_1 .

Also if z_n is listed m times so it is a zero of multiplicity m and w_n is the point from $\partial(h(\Omega_1))$ closest to $h(z_n)$, then there are m factors in 33.1.8 which are of the form

$$\begin{aligned} E_n \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right) &= \left(1 - \frac{h(z_n) - w_n}{h(z) - w_n} \right) e^{g_n(z)} \\ &= \left(\frac{h(z) - h(z_n)}{h(z) - w_n} \right) e^{g_n(z)} \\ &= \frac{z_n - z}{(z - w)(z_n - w)} \left(\frac{1}{h(z) - w_n} \right) e^{g_n(z)} \\ &= (z - z_n) G_n(z) \end{aligned} \quad (33.1.10)$$

where G_n is an analytic function which is not zero at and near z_n . Therefore, f has a zero of order m at z_n . This proves the theorem except for the point, w which has been left out of Ω_1 . It is necessary to show f is analytic at this point also and right now, f is not even defined at w .

The $\{w_n\}$ are bounded because $\{h(z_n)\}$ is bounded and $\lim_{n \rightarrow \infty} |w_n - h(z_n)| = 0$ which implies $|w_n - h(z_n)| \leq C$ for some constant, C . Therefore, there exists $\delta > 0$ such that if $z \in B'(w, \delta)$, then for all n ,

$$\left| \frac{h(z_n) - w}{\left(\frac{1}{z-w} \right) - w_n} \right| = \left| \frac{h(z_n) - w_n}{h(z) - w_n} \right| < \frac{1}{2}.$$

Thus 33.1.9 holds for all $z \in B'(w, \delta)$ and n so by Theorem 33.0.2, the infinite product in 33.1.8 converges uniformly on $B'(w, \delta)$. This implies f is bounded in $B'(w, \delta)$ and so w is a removable singularity and f can be extended to w such that the result is analytic. It only remains to verify $f(w) \neq 0$. After all, this would not do because it would be another zero other than those in the given list. By 33.1.10, a partial product is of the form

$$\prod_{n=1}^N \left(\frac{h(z) - h(z_n)}{h(z) - w_n} \right) e^{g_n(z)} \quad (33.1.11)$$

where

$$g_n(z) \equiv \left(\frac{h(z_n) - w_n}{h(z) - w_n} + \frac{1}{2} \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right)^2 + \cdots + \frac{1}{n} \left(\frac{h(z_n) - w_n}{h(z) - w_n} \right)^n \right)$$

Each of the quotients in the definition of $g_n(z)$ converges to 0 as $z \rightarrow w$ and so the partial product of 33.1.11 converges to 1 as $z \rightarrow w$ because $\left(\frac{h(z) - h(z_n)}{h(z) - w_n} \right) \rightarrow 1$ as $z \rightarrow w$.

If $f(w) = 0$, then if z is close enough to w , it follows $|f(z)| < \frac{1}{2}$. Also, by the uniform convergence on $B'(w, \delta)$, it follows that for some N , the partial product up to N must also be less than $1/2$ in absolute value for all z close enough to w and as noted above, this does not occur because such partial products converge to 1 as $z \rightarrow w$. Hence $f(w) \neq 0$. This proves the corollary.

Recall the definition of a meromorphic function on Page 816. It was a function which is analytic everywhere except at a countable set of isolated points at which the function has a pole. It is clear that the quotient of two analytic functions yields a meromorphic function but is this the only way it can happen?

Theorem 33.1.8 *Suppose Q is a meromorphic function on an open set, Ω . Then there exist analytic functions on Ω , $f(z)$ and $g(z)$ such that $Q(z) = f(z)/g(z)$ for all z not in the set of poles of Q .*

Proof: Let Q have a pole of order $m(z)$ at z . Then by Corollary 33.1.7 there exists an analytic function, g which has a zero of order $m(z)$ at every $z \in \Omega$. It follows gQ has a removable singularity at the poles of Q . Therefore, there is an analytic function, f such that $f(z) = g(z)Q(z)$. This proves the theorem.

Corollary 33.1.9 *Suppose Ω is a region and Q is a meromorphic function defined on Ω such that the set, $\{z \in \Omega : Q(z) = c\}$ has a limit point in Ω . Then $Q(z) = c$ for all $z \in \Omega$.*

Proof: From Theorem 33.1.8 there are analytic functions, f, g such that $Q = \frac{f}{g}$. Therefore, the zero set of the function, $f(z) - cg(z)$ has a limit point in Ω and so $f(z) - cg(z) = 0$ for all $z \in \Omega$. This proves the corollary.

33.2 Factoring A Given Analytic Function

The next theorem is the Weierstrass factorization theorem which can be used to factor a given analytic function f . If f has a zero of order m when $z = 0$, then you could factor out a z^m and from there consider the factorization of what remains when you have factored out the z^m . Therefore, the following is the main thing of interest.

Theorem 33.2.1 *Let f be analytic on $\mathbb{C}$, $f(0) \neq 0$, and let the zeros of f , be $\{z_k\}$, listed according to order. (Thus if z is a zero of order m , it will be listed m times in the list, $\{z_k\}$.) Choosing nonnegative integers, p_n such that for all $r > 0$,*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|} \right)^{p_n+1} < \infty,$$

There exists an entire function, g such that

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{z_n} \right). \quad (33.2.12)$$

Note that $e^{g(z)} \neq 0$ for any z and this is the interesting thing about this function.

Proof: $\{z_n\}$ cannot have a limit point because if there were a limit point of this sequence, it would follow from Theorem 27.5.3 that $f(z) = 0$ for all z , contradicting the hypothesis that $f(0) \neq 0$. Hence $\lim_{n \rightarrow \infty} |z_n| = \infty$ and so

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|} \right)^{1+n-1} = \sum_{n=1}^{\infty} \left(\frac{r}{|z_n|} \right)^n < \infty$$

by the root test. Therefore, by Theorem 33.1.4

$$P(z) = \prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{z_n} \right)$$

a function analytic on $\mathbb{C}$ by picking $p_n = n - 1$ or perhaps some other choice. ($p_n = n - 1$ works but there might be another choice that would work.) Then f/P has only removable singularities in $\mathbb{C}$ and no zeros thanks to Theorem 33.1.4. Thus, letting $h(z) = f(z)/P(z)$, Corollary 27.7.23 implies that h'/h has a primitive, $\tilde{g}$. Then

$$(he^{-\tilde{g}})' = 0$$

and so

$$h(z) = e^{a+ib} e^{\tilde{g}(z)}$$

for some constants, a, b . Therefore, letting $g(z) = \tilde{g}(z) + a + ib$, $h(z) = e^{g(z)}$ and thus 33.2.12 holds. This proves the theorem.

Corollary 33.2.2 *Let f be analytic on $\mathbb{C}$, f has a zero of order m at 0, and let the other zeros of f be $\{z_k\}$, listed according to order. (Thus if z is a zero of order l , it will be listed l times in the list, $\{z_k\}$.) Also let*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|z_n|} \right)^{1+p_n} < \infty \quad (33.2.13)$$

for any choice of $r > 0$. Then there exists an entire function, g such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n} \left(\frac{z}{z_n} \right). \quad (33.2.14)$$

Proof: Since f has a zero of order m at 0, it follows from Theorem 27.5.3 that $\{z_k\}$ cannot have a limit point in $\mathbb{C}$ and so you can apply Theorem 33.2.1 to the function, $f(z)/z^m$ which has a removable singularity at 0. This proves the corollary.

33.2.1 Factoring Some Special Analytic Functions

Factoring a polynomial is in general a hard task. It is true it is easy to prove the factors exist but finding them is another matter. Corollary 33.2.2 gives the existence of factors of a certain form but it does not tell how to find them. This should not be surprising. You can't expect things to get easier when you go from polynomials to analytic functions. Nevertheless, it is possible to factor some popular analytic functions. These factorizations are based on the following Mittag-Leffler expansions. By an auspicious choice of the contour and the method of residues it is possible to obtain a very interesting formula for $\cot \pi z$.

Example 33.2.3 Let γ_N be the contour which goes from $-N - \frac{1}{2} - Ni$ horizontally to $N + \frac{1}{2} - Ni$ and from there, vertically to $N + \frac{1}{2} + Ni$ and then horizontally to $-N - \frac{1}{2} + Ni$ and finally vertically to $-N - \frac{1}{2} - Ni$. Thus the contour is a large rectangle and the direction of integration is in the counter clockwise direction. Consider the integral

$$I_N \equiv \int_{\gamma_N} \frac{\pi \cos \pi z}{\sin \pi z (\alpha^2 - z^2)} dz$$

where $\alpha \in \mathbb{R}$ is not an integer. This will be used to verify the formula of Mittag-Leffler,

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi \alpha. \quad (33.2.15)$$

First you show that $\cot \pi z$ is bounded on this contour. This is easy using the formula for $\cot(z) = \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$. Therefore, $I_N \rightarrow 0$ as $N \rightarrow \infty$ because the integrand is of order $1/N^2$ while the diameter of γ_N is of order N . Next you compute the residues of the integrand at $\pm \alpha$ and at n where $|n| < N + \frac{1}{2}$ for n an integer. These are the only singularities of the integrand in this contour and therefore, using the residue theorem, you can evaluate I_N by using these. You can calculate these residues and find that the residue at $\pm \alpha$ is

$$\frac{-\pi \cos \pi \alpha}{2\alpha \sin \pi \alpha}$$

while the residue at n is

$$\frac{1}{\alpha^2 - n^2}.$$

Therefore

$$0 = \lim_{N \rightarrow \infty} I_N = \lim_{N \rightarrow \infty} 2\pi i \left[\sum_{n=-N}^N \frac{1}{\alpha^2 - n^2} - \frac{\pi \cot \pi \alpha}{\alpha} \right]$$

which establishes the following formula of Mittag Leffler.

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N \frac{1}{\alpha^2 - n^2} = \frac{\pi \cot \pi \alpha}{\alpha}.$$

Writing this in a slightly nicer form, you obtain 33.2.15.

This is a very interesting formula. This will be used to factor $\sin(\pi z)$. The zeros of this function are at the integers. Therefore, considering 33.2.13 you can pick $p_n = 1$ in the Weierstrass factorization formula. Therefore, by Corollary 33.2.2 there exists an analytic function $g(z)$ such that

$$\sin(\pi z) = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n} \quad (33.2.16)$$

where the z_n are the nonzero integers. Remember you can permute the factors in these products. Therefore, this can be written more conveniently as

$$\sin(\pi z) = z e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

and it is necessary to find $g(z)$. Differentiating both sides of 33.2.16

$$\begin{aligned} \pi \cos(\pi z) &= e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) + z g'(z) e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right) \\ &\quad + z e^{g(z)} \sum_{n=1}^{\infty} -\left(\frac{2z}{n^2}\right) \prod_{k \neq n} \left(1 - \left(\frac{z}{k}\right)^2\right) \end{aligned}$$

Now divide both sides by $\sin(\pi z)$ to obtain

$$\begin{aligned} \pi \cot(\pi z) &= \frac{1}{z} + g'(z) - \sum_{n=1}^{\infty} \frac{2z/n^2}{(1 - z^2/n^2)} \\ &= \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \end{aligned}$$

By 33.2.15, this yields $g'(z) = 0$ for z not an integer and so $g(z) = c$, a constant. So far this yields

$$\sin(\pi z) = z e^c \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$$

and it only remains to find c . Divide both sides by πz and take a limit as $z \rightarrow 0$. Using the power series of $\sin(\pi z)$, this yields

$$1 = \frac{e^c}{\pi}$$

and so $c = \ln \pi$. Therefore,

$$\sin(\pi z) = z\pi \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right). \quad (33.2.17)$$

Example 33.2.4 Find an interesting formula for $\tan(\pi z)$.

This is easy to obtain from the formula for $\cot(\pi z)$.

$$\cot\left(\pi\left(z + \frac{1}{2}\right)\right) = -\tan \pi z$$

for z real and therefore, this formula holds for z complex also. Therefore, for $z + \frac{1}{2}$ not an integer

$$\pi \cot\left(\pi\left(z + \frac{1}{2}\right)\right) = \frac{2}{2z+1} + \sum_{n=1}^{\infty} \frac{2z+1}{\left(\frac{2z+1}{2}\right)^2 - n^2}$$

33.3 The Existence Of An Analytic Function With Given Values

The Weierstrass product formula, Theorem 33.1.4, along with the Mittag-Leffler theorem, Theorem 32.2.1 can be used to obtain an analytic function which has given values on a countable set of points, having no limit point. This is clearly an amazing result and indicates how potent these theorems are. In fact, you can show that it isn't just the values of the function which may be specified at the points in this countable set of points but the derivatives up to any finite order.

Theorem 33.3.1 Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in $\mathbb{C}$, which has no limit point. For each z_k , consider

$$\sum_{j=0}^{m_k} a_j^k (z - z_k)^j. \quad (33.3.18)$$

Then there exists an analytic function defined on $\mathbb{C}$ such that the Taylor series of f at z_k has the first m_k terms given by 33.3.18.¹

Proof: By the Weierstrass product theorem, Theorem 33.1.4, there exists an analytic function, f defined on all of Ω such that f has a zero of order $m_k + 1$ at z_k . Consider this z_k . Thus for z near z_k ,

$$f(z) = \sum_{j=m_k+1}^{\infty} c_j (z - z_k)^j$$

¹This says you can specify the first m_k derivatives of the function at the point z_k .

where $c_{m_k+1} \neq 0$. You choose $b_1, b_2, \dots, b_{m_k+1}$ such that

$$f(z) \left(\sum_{l=1}^{m_k+1} \frac{b_l}{(z-z_k)^k} \right) = \sum_{j=0}^{m_k} a_j^k (z-z_k)^j + \sum_{k=m_k+1}^{\infty} c_j^k (z-z_k)^j.$$

Thus you need

$$\sum_{l=1}^{m_k+1} \sum_{j=m_k+1}^{\infty} c_j b_l (z-z_k)^{j-l} = \sum_{r=0}^{m_k} a_r^k (z-z_k)^r + \text{Higher order terms.}$$

It follows you need to solve the following system of equations for $b_1, \dots, b_{m_k+1}$.

$$\begin{aligned} c_{m_k+1} b_{m_k+1} &= a_0^k \\ c_{m_k+2} b_{m_k+1} + c_{m_k+1} b_{m_k} &= a_1^k \\ c_{m_k+3} b_{m_k+1} + c_{m_k+2} b_{m_k} + c_{m_k+1} b_{m_k-1} &= a_2^k \\ &\vdots \\ c_{m_k+m_k+1} b_{m_k+1} + c_{m_k+m_k} b_{m_k} + \dots + c_{m_k+1} b_1 &= a_{m_k}^k \end{aligned}$$

Since $c_{m_k+1} \neq 0$, it follows there exists a unique solution to the above system. You first solve for b_{m_k+1} in the top. Then, having found it, you go to the next and use $c_{m_k+1} \neq 0$ again to find b_{m_k} and continue in this manner. Let $S_k(z)$ be determined in this manner for each z_k . By the Mittag-Leffler theorem, there exists a Meromorphic function, g such that g has exactly the singularities, $S_k(z)$. Therefore, $f(z)g(z)$ has removable singularities at each z_k and for z near z_k , the first m_k terms of fg are as prescribed. This proves the theorem.

Corollary 33.3.2 Let $P \equiv \{z_k\}_{k=1}^{\infty}$ be a set of points in Ω , an open set such that P has no limit points in Ω . For each z_k , consider

$$\sum_{j=0}^{m_k} a_j^k (z-z_k)^j. \quad (33.3.19)$$

Then there exists an analytic function defined on Ω such that the Taylor series of f at z_k has the first m_k terms given by 33.3.19.

Proof: The proof is identical to the above except you use the versions of the Mittag-Leffler theorem and Weierstrass product which pertain to open sets.

Definition 33.3.3 Denote by $H(\Omega)$ the analytic functions defined on Ω , an open subset of $\mathbb{C}$. Then $H(\Omega)$ is a commutative ring² with the usual operations of addition and multiplication. A set, $I \subseteq H(\Omega)$ is called a finitely generated ideal of the ring if I is of the form

$$\left\{ \sum_{k=1}^n g_k f_k : f_k \in H(\Omega) \text{ for } k = 1, 2, \dots, n \right\}$$

²It is not a field because you can't divide two analytic functions and get another one.

where $g_1, \dots, g_n$ are given functions in $H(\Omega)$. This ideal is also denoted as $[g_1, \dots, g_n]$ and is called the ideal generated by the functions, $\{g_1, \dots, g_n\}$. Since there are finitely many of these functions it is called a finitely generated ideal. A principal ideal is one which is generated by a single function. An example of such a thing is $[1] = H(\Omega)$.

Then there is the following interesting theorem.

Theorem 33.3.4 *Every finitely generated ideal in $H(\Omega)$ for Ω a connected open set (region) is a principal ideal.*

Proof: Let $I = [g_1, \dots, g_n]$ be a finitely generated ideal as described above. Then if any of the functions has no zeros, this ideal would consist of $H(\Omega)$ because then $g_i^{-1} \in H(\Omega)$ and so $1 \in I$. It follows all the functions have zeros. If any of the functions has a zero of infinite order, then the function equals zero on Ω because Ω is connected and can be deleted from the list. Similarly, if the zeros of any of these functions have a limit point in Ω , then the function equals zero and can be deleted from the list. Thus, without loss of generality, all zeros are of finite order and there are no limit points of the zeros in Ω . Let $m(g_i, z)$ denote the order of the zero of g_i at z . If g_i has no zero at z , then $m(g_i, z) = 0$.

I claim that if no point of Ω is a zero of all the g_i , then the conclusion of the theorem is true and in fact $[g_1, \dots, g_n] = [1] = H(\Omega)$. The claim is obvious if $n = 1$ because this assumption that no point is a zero of all the functions implies $g \neq 0$ and so g^{-1} is analytic. Hence $1 \in [g_1]$. Suppose it is true for $n - 1$ and consider $[g_1, \dots, g_n]$ where no point of Ω is a zero of all the g_i . Even though this may be true of $\{g_1, \dots, g_n\}$, it may not be true of $\{g_1, \dots, g_{n-1}\}$. By Corollary 33.1.7 there exists ϕ , a function analytic on Ω such that $m(\phi, z) = \min\{m(g_i, z), i = 1, 2, \dots, n - 1\}$. Thus the functions $\{g_1/\phi, \dots, g_{n-1}/\phi\}$ are all analytic. Could they all equal zero at some point, z ? If so, pick i where $m(\phi, z) = m(g_i, z)$. Thus g_i/ϕ is not equal to zero at z after all and so these functions are analytic there is no point of Ω which is a zero of all of them. By induction, $[g_1/\phi, \dots, g_{n-1}/\phi] = H(\Omega)$. (Also there are no new zeros obtained in this way.)

Now this means there exist functions $f_i \in H(\Omega)$ such that

$$\sum_{i=1}^n f_i \left(\frac{g_i}{\phi} \right) = 1$$

and so $\phi = \sum_{i=1}^n f_i g_i$. Therefore, $[\phi] \subseteq [g_1, \dots, g_{n-1}]$. On the other hand, if $\sum_{k=1}^{n-1} h_k g_k \in [g_1, \dots, g_{n-1}]$ you could define $h \equiv \sum_{k=1}^{n-1} h_k (g_k/\phi)$, an analytic function with the property that $h\phi = \sum_{k=1}^{n-1} h_k g_k$ which shows $[\phi] = [g_1, \dots, g_{n-1}]$. Therefore,

$$[g_1, \dots, g_n] = [\phi, g_n]$$

Now ϕ has no zeros in common with g_n because the zeros of ϕ are contained in the set of zeros for $g_1, \dots, g_{n-1}$. Now consider a zero, α of ϕ . It is not a zero of g_n and

so near α , these functions have the form

$$\phi(z) = \sum_{k=m}^{\infty} a_k (z - \alpha)^k, \quad g_n(z) = \sum_{k=0}^{\infty} b_k (z - \alpha)^k, \quad b_0 \neq 0.$$

I want to determine coefficients for an analytic function, h such that

$$m(1 - hg_n, \alpha) \geq m(\phi, \alpha). \quad (33.3.20)$$

Let

$$h(z) = \sum_{k=0}^{\infty} c_k (z - \alpha)^k$$

and the c_k must be determined. Using Merten's theorem, the power series for $1 - hg_n$ is of the form

$$1 - b_0 c_0 - \sum_{j=1}^{\infty} \left(\sum_{r=0}^j b_{j-r} c_r \right) (z - \alpha)^j.$$

First determine c_0 such that $1 - c_0 b_0 = 0$. This is no problem because $b_0 \neq 0$. Next you need to get the coefficients of $(z - \alpha)$ to equal zero. This requires

$$b_1 c_0 + b_0 c_1 = 0.$$

Again, there is no problem because $b_0 \neq 0$. In fact, $c_1 = (-b_1 c_0 / b_0)$. Next consider the second order terms if $m \geq 2$.

$$b_2 c_0 + b_1 c_1 + b_0 c_2 = 0$$

Again there is no problem in solving, this time for c_2 because $b_0 \neq 0$. Continuing this way, you see that in every step, the c_k which needs to be solved for is multiplied by $b_0 \neq 0$. Therefore, by Corollary 33.1.7 there exists an analytic function, h satisfying 33.3.20. Therefore, $(1 - hg_n)/\phi$ has a removable singularity at every zero of ϕ and so may be considered an analytic function. Therefore,

$$1 = \frac{1 - hg_n}{\phi} \phi + hg_n \in [\phi, g_n] = [g_1 \cdots g_n]$$

which shows $[g_1 \cdots g_n] = H(\Omega) = [1]$. It follows the claim is established.

Now suppose $\{g_1 \cdots g_n\}$ are just elements of $H(\Omega)$. As explained above, it can be assumed they all have zeros of finite order and the zeros have no limit point in Ω since if these occur, you can delete the function from the list. By Corollary 33.1.7 there exists $\phi \in H(\Omega)$ such that $m(\phi, z) \leq \min \{m(g_i, z) : i = 1, \dots, n\}$. Then g_k/ϕ has a removable singularity at each zero of g_k and so can be regarded as an analytic function. Also, as before, there is no point which is a zero of each g_k/ϕ and so by the first part of this argument, $[g_1/\phi \cdots g_n/\phi] = H(\Omega)$. As in the first part of the argument, this implies $[g_1 \cdots g_n] = [\phi]$ which proves the theorem. $[g_1 \cdots g_n]$ is a principal ideal as claimed.

The following corollary follows from the above theorem. You don't need to assume Ω is connected.

Corollary 33.3.5 *Every finitely generated ideal in $H(\Omega)$ for Ω an open set is a principal ideal.*

Proof: Let $[g_1, \dots, g_n]$ be a finitely generated ideal in $H(\Omega)$. Let $\{U_k\}$ be the components of Ω . Then applying the above to each component, there exists $h_k \in H(U_k)$ such that restricting each g_i to U_k , $[g_1, \dots, g_n] = [h_k]$. Then let $h(z) = h_k(z)$ for $z \in U_k$. This is an analytic function which works.

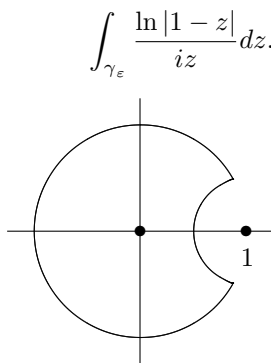
33.4 Jensen's Formula

This interesting formula relates the zeros of an analytic function to an integral. The proof given here follows Alfors, [2]. First, here is a technical lemma.

Lemma 33.4.1

$$\int_{-\pi}^{\pi} \ln |1 - e^{i\theta}| d\theta = 0.$$

Proof: First note that the only problem with the integrand occurs when $\theta = 0$. However, this is an integrable singularity so the integral will end up making sense. Letting $z = e^{i\theta}$, you could get the above integral as a limit as $\varepsilon \rightarrow 0$ of the following contour integral where γ_ε is the contour shown in the following picture with the radius of the big circle equal to 1 and the radius of the little circle equal to ε .



On the indicated contour, $1 - z$ lies in the half plane $\operatorname{Re} z > 0$ and so $\log(1 - z) = \ln |1 - z| + i \arg(1 - z)$. The above integral equals

$$\int_{\gamma_\varepsilon} \frac{\log(1 - z)}{iz} dz - \int_{\gamma_\varepsilon} \frac{\arg(1 - z)}{z} dz$$

The first of these integrals equals zero because the integrand has a removable singularity at 0. The second equals

$$\begin{aligned} & i \int_{-\pi}^{-\eta_\varepsilon} \arg(1 - e^{i\theta}) d\theta + i \int_{\eta_\varepsilon}^{\pi} \arg(1 - e^{i\theta}) d\theta \\ & + \varepsilon i \int_{-\frac{\pi}{2} - \lambda_\varepsilon}^{-\pi} \theta d\theta + \varepsilon i \int_{\pi}^{\frac{\pi}{2} - \lambda_\varepsilon} \theta d\theta \end{aligned}$$

where $\eta_\varepsilon, \lambda_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. The last two terms converge to 0 as $\varepsilon \rightarrow 0$ while the first two add to zero. To see this, change the variable in the first integral and then recall that when you multiply complex numbers you add the arguments. Thus you end up integrating $\arg$ (real valued function) which equals zero.

In this material on Jensen's equation, ε will denote a small positive number. Its value is not important as long as it is positive. Therefore, it may change from place to place. Now suppose f is analytic on $B(0, r + \varepsilon)$, and f has no zeros on $\overline{B(0, r)}$. Then you can define a branch of the logarithm which makes sense for complex numbers near $f(z)$. Thus $z \rightarrow \log(f(z))$ is analytic on $B(0, r + \varepsilon)$. Therefore, its real part, $u(x, y) \equiv \ln|f(x + iy)|$ must be harmonic. Consider the following lemma.

Lemma 33.4.2 *Let u be harmonic on $B(0, r + \varepsilon)$. Then*

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta.$$

Proof: For a harmonic function, u defined on $B(0, r + \varepsilon)$, there exists an analytic function, $h = u + iv$ where

$$v(x, y) \equiv \int_0^y u_x(x, t) dt - \int_0^x u_y(t, 0) dt.$$

By the Cauchy integral theorem,

$$h(0) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{h(z)}{z} dz = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(re^{i\theta}) d\theta.$$

Therefore, considering the real part of h ,

$$u(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta}) d\theta.$$

This proves the lemma.

Now this shows the following corollary.

Corollary 33.4.3 *Suppose f is analytic on $B(0, r + \varepsilon)$ and has no zeros on $\overline{B(0, r)}$. Then*

$$\ln|f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|f(re^{i\theta})| d\theta \quad (33.4.21)$$

What if f has some zeros on $|z| = r$ but none on $B(0, r)$? It turns out 33.4.21 is still valid. Suppose the zeros are at $\{re^{i\theta_k}\}_{k=1}^m$, listed according to multiplicity. Then let

$$g(z) = \frac{f(z)}{\prod_{k=1}^m (z - re^{i\theta_k})}.$$

It follows g is analytic on $B(0, r + \varepsilon)$ but has no zeros in $\overline{B(0, r)}$. Then 33.4.21 holds for g in place of f . Thus

$$\begin{aligned} & \ln |f(0)| - \sum_{k=1}^m \ln |r| \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^m \ln |re^{i\theta} - re^{i\theta_k}| d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^m \ln |e^{i\theta} - e^{i\theta_k}| d\theta - \sum_{k=1}^m \ln |r| \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^m \ln |e^{i\theta} - 1| d\theta - \sum_{k=1}^m \ln |r| \end{aligned}$$

Therefore, 33.4.21 will continue to hold exactly when $\frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=1}^m \ln |e^{i\theta} - 1| d\theta = 0$. But this is the content of Lemma 33.4.1. This proves the following lemma.

Lemma 33.4.4 *Suppose f is analytic on $B(0, r + \varepsilon)$ and has no zeros on $B(0, r)$. Then*

$$\ln |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| d\theta \quad (33.4.22)$$

With this preparation, it is now not too hard to prove Jensen's formula. Suppose there are n zeros of f in $B(0, r)$, $\{a_k\}_{k=1}^n$, listed according to multiplicity, none equal to zero. Let

$$F(z) \equiv f(z) \prod_{i=1}^n \frac{r^2 - \overline{a_i}z}{r(z - a_i)}.$$

Then F is analytic on $B(0, r + \varepsilon)$ and has no zeros in $B(0, r)$. The reason for this is that $f(z) / \prod_{i=1}^n r(z - a_i)$ has no zeros there and $r^2 - \overline{a_i}z$ cannot equal zero if $|z| < r$ because if this expression equals zero, then

$$|z| = \frac{r^2}{|a_i|} > r.$$

The other interesting thing about $F(z)$ is that when $z = re^{i\theta}$,

$$\begin{aligned} F(re^{i\theta}) &= f(re^{i\theta}) \prod_{i=1}^n \frac{r^2 - \overline{a_i}re^{i\theta}}{r(re^{i\theta} - a_i)} \\ &= f(re^{i\theta}) \prod_{i=1}^n \frac{r - \overline{a_i}e^{i\theta}}{(re^{i\theta} - a_i)} = f(re^{i\theta}) e^{i\theta} \prod_{i=1}^n \frac{re^{-i\theta} - \overline{a_i}}{re^{i\theta} - a_i} \end{aligned}$$

so $|F(re^{i\theta})| = |f(re^{i\theta})|$.

Theorem 33.4.5 *Let f be analytic on $B(0, r + \varepsilon)$ and suppose $f(0) \neq 0$. If the zeros of f in $B(0, r)$ are $\{a_k\}_{k=1}^n$, listed according to multiplicity, then*

$$\ln |f(0)| = - \sum_{i=1}^n \ln \left(\frac{r}{|a_i|} \right) + \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta.$$

Proof: From the above discussion and Lemma 33.4.4,

$$\ln |F(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |f(re^{i\theta})| d\theta$$

But $F(0) = f(0) \prod_{i=1}^n \frac{r}{a_i}$ and so $\ln |F(0)| = \ln |f(0)| + \sum_{i=1}^n \ln \left| \frac{r}{a_i} \right|$. Therefore,

$$\ln |f(0)| = - \sum_{i=1}^n \ln \left| \frac{r}{a_i} \right| + \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta$$

as claimed.

Written in terms of exponentials this is

$$|f(0)| \prod_{k=1}^n \left| \frac{r}{a_k} \right| = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta \right).$$

33.5 Blaschke Products

The Blaschke³ product is a way to produce a function which is bounded and analytic on $B(0, 1)$ which also has given zeros in $B(0, 1)$. The interesting thing here is that there may be infinitely many of these zeros. Thus, unlike the above case of Jensen's inequality, the function is not analytic on $\overline{B(0, 1)}$. Recall for purposes of comparison, Liouville's theorem which says bounded entire functions are constant. The Blaschke product gives examples of bounded functions on $B(0, 1)$ which are definitely not constant.

Theorem 33.5.1 *Let $\{\alpha_n\}$ be a sequence of nonzero points in $B(0, 1)$ with the property that*

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty.$$

Then for $k \geq 0$, an integer

$$B(z) \equiv z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n}$$

is a bounded function which is analytic on $B(0, 1)$ which has zeros only at 0 if $k > 0$ and at the α_n .

³Wilhelm Blaschke, 1915

Proof: From Theorem 33.0.2 the above product will converge uniformly on $B(0, r)$ for $r < 1$ to an analytic function if

$$\sum_{k=1}^{\infty} \left| \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n} - 1 \right|$$

converges uniformly on $B(0, r)$. But for $|z| < r$,

$$\begin{aligned} & \left| \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n} - 1 \right| \\ &= \left| \frac{\alpha_n - z}{1 - \overline{\alpha_n} z} \frac{|\alpha_n|}{\alpha_n} - \frac{\alpha_n (1 - \overline{\alpha_n} z)}{\alpha_n (1 - \overline{\alpha_n} z)} \right| \\ &= \left| \frac{|\alpha_n| \alpha_n - |\alpha_n| z - \alpha_n + |\alpha_n|^2 z}{(1 - \overline{\alpha_n} z) \alpha_n} \right| \\ &= \left| \frac{|\alpha_n| \alpha_n - \alpha_n - |\alpha_n| z + |\alpha_n|^2 z}{(1 - \overline{\alpha_n} z) \alpha_n} \right| \\ &= ||\alpha_n| - 1| \left| \frac{\alpha_n + z |\alpha_n|}{(1 - \overline{\alpha_n} z) \alpha_n} \right| \\ &= ||\alpha_n| - 1| \left| \frac{1 + z (|\alpha_n| / \alpha_n)}{(1 - \overline{\alpha_n} z)} \right| \\ &\leq ||\alpha_n| - 1| \left| \frac{1 + |z|}{1 - |z|} \right| \leq ||\alpha_n| - 1| \left| \frac{1 + r}{1 - r} \right| \end{aligned}$$

and so the assumption on the sum gives uniform convergence of the product on $B(0, r)$ to an analytic function. Since $r < 1$ is arbitrary, this shows $B(z)$ is analytic on $B(0, 1)$ and has the specified zeros because the only place the factors equal zero are at the α_n or 0.

Now consider the factors in the product. The claim is that they are all no larger in absolute value than 1. This is very easy to see from the maximum modulus theorem. Let $|\alpha| < 1$ and $\phi(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$. Then ϕ is analytic near $B(0, 1)$ because its only pole is $1/\overline{\alpha}$. Consider $z = e^{i\theta}$. Then

$$|\phi(e^{i\theta})| = \left| \frac{\alpha - e^{i\theta}}{1 - \overline{\alpha} e^{i\theta}} \right| = \left| \frac{1 - \alpha e^{-i\theta}}{1 - \overline{\alpha} e^{i\theta}} \right| = 1.$$

Thus the modulus of $\phi(z)$ equals 1 on $\partial B(0, 1)$. Therefore, by the maximum modulus theorem, $|\phi(z)| < 1$ if $|z| < 1$. This proves the claim that the terms in the product are no larger than 1 and shows the function determined by the Blaschke product is bounded. This proves the theorem.

Note in the conditions for this theorem the one for the sum, $\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty$. The Blaschke product gives an analytic function, whose absolute value is bounded by 1 and which has the α_n as zeros. What if you had a bounded function, analytic on $B(0, 1)$ which had zeros at $\{\alpha_k\}$? Could you conclude the condition on the sum?

The answer is yes. In fact, you can get by with less than the assumption that f is bounded but this will not be presented here. See Rudin [58]. This theorem is an exciting use of Jensen's equation.

Theorem 33.5.2 *Suppose f is an analytic function on $B(0,1)$, $f(0) \neq 0$, and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the zeros of f are $\{\alpha_k\}_{k=1}^\infty$, listed according to multiplicity. Then $\sum_{k=1}^\infty (1 - |\alpha_k|) < \infty$.*

Proof: If there are only finitely many zeros, there is nothing to prove so assume there are infinitely many. Also let the zeros be listed such that $|\alpha_n| \leq |\alpha_{n+1}| \cdots$. Let $n(r)$ denote the number of zeros in $B(0,r)$. By Jensen's formula,

$$\ln |f(0)| + \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta \leq \ln(M).$$

Therefore, by the mean value theorem,

$$\sum_{i=1}^{n(r)} \frac{1}{r} (r - |\alpha_i|) \leq \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i| \leq \ln(M) - \ln |f(0)|$$

As $r \rightarrow 1-$, $n(r) \rightarrow \infty$, and so an application of Fatous lemma yields

$$\sum_{i=1}^\infty (1 - |\alpha_i|) \leq \liminf_{r \rightarrow 1-} \sum_{i=1}^{n(r)} \frac{1}{r} (r - |\alpha_i|) \leq \ln(M) - \ln |f(0)|.$$

This proves the theorem.

You don't need the assumption that $f(0) \neq 0$.

Corollary 33.5.3 *Suppose f is an analytic function on $B(0,1)$ and $|f(z)| \leq M$ for all $z \in B(0,1)$. Suppose also that the nonzero zeros⁴ of f are $\{\alpha_k\}_{k=1}^\infty$, listed according to multiplicity. Then $\sum_{k=1}^\infty (1 - |\alpha_k|) < \infty$.*

Proof: Suppose f has a zero of order m at 0. Then consider the analytic function, $g(z) \equiv f(z)/z^m$ which has the same zeros except for 0. The argument goes the same way except here you use g instead of f and only consider $r > r_0 > 0$.

⁴This is a fun thing to say: nonzero zeros.

Thus from Jensen's equation,

$$\begin{aligned}
 & \ln |g(0)| + \sum_{i=1}^{n(r)} \ln r - \ln |\alpha_i| \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \ln |g(re^{i\theta})| d\theta \\
 &= \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_0^{2\pi} m \ln(r) \\
 &\leq M + \frac{1}{2\pi} \int_0^{2\pi} m \ln(r^{-1}) \\
 &\leq M + m \ln\left(\frac{1}{r_0}\right).
 \end{aligned}$$

Now the rest of the argument is the same.

An interesting restatement yields the following amazing result.

Corollary 33.5.4 *Suppose f is analytic and bounded on $B(0, 1)$ having zeros $\{\alpha_n\}$. Then if $\sum_{k=1}^{\infty} (1 - |\alpha_n|) = \infty$, it follows f is identically equal to zero.*

33.5.1 The Müntz-Szasz Theorem Again

Corollary 33.5.4 makes possible an easy proof of a remarkable theorem named above which yields a wonderful generalization of the Weierstrass approximation theorem. In what follows $b > 0$. The Weierstrass approximation theorem states that linear combinations of $1, t, t^2, t^3, \dots$ (polynomials) are dense in $C([0, b])$. Let $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ be an increasing list of positive real numbers. This theorem tells when linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ are dense in $C([0, b])$. The proof which follows is like the one given in Rudin [58]. There is a much longer one in Cheney [17] which discusses more aspects of the subject. See also Page 438 where the version given in Cheney is presented. This other approach is much more elementary and does not depend in any way on the theory of functions of a complex variable. There are those of us who automatically prefer real variable techniques. Nevertheless, this proof by Rudin is a very nice and insightful application of the preceding material. Cheney refers to the theorem as the second Müntz theorem. I guess Szasz must also have been involved.

Theorem 33.5.5 *Let $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ be an increasing list of positive real numbers and let $a > 0$. If*

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty, \quad (33.5.23)$$

then linear combinations of $1, t^{\lambda_1}, t^{\lambda_2}, \dots$ are dense in $C([0, b])$.

Proof: Let X denote the closure of linear combinations of $\{1, t^{\lambda_1}, t^{\lambda_2}, \dots\}$ in $C([0, b])$. If $X \neq C([0, b])$, then letting $f \in C([0, b]) \setminus X$, define $\Lambda \in C([0, b])'$ as follows. First let $\Lambda_0 : X + \mathbb{C}f$ be given by $\Lambda_0(g + \alpha f) = \alpha \|f\|_\infty$. Then

$$\begin{aligned} \sup_{\|g + \alpha f\| \leq 1} |\Lambda_0(g + \alpha f)| &= \sup_{\|g + \alpha f\| \leq 1} |\alpha| \|f\|_\infty \\ &= \sup_{\|g/\alpha + f\| \leq \frac{1}{|\alpha|}} |\alpha| \|f\|_\infty \\ &= \sup_{\|g + f\| \leq \frac{1}{|\alpha|}} |\alpha| \|f\|_\infty \end{aligned}$$

Now $\text{dist}(f, X) > 0$ because X is closed. Therefore, there exists a lower bound, $\eta > 0$ to $\|g + f\|$ for $g \in X$. Therefore, the above is no larger than

$$\sup_{|\alpha| \leq \frac{1}{\eta}} |\alpha| \|f\|_\infty = \left(\frac{1}{\eta}\right) \|f\|_\infty$$

which shows that $\|\Lambda_0\| \leq \left(\frac{1}{\eta}\right) \|f\|_\infty$. By the Hahn Banach theorem Λ_0 can be extended to $\Lambda \in C([0, b])'$ which has the property that $\Lambda(X) = 0$ but $\Lambda(f) = \|f\| \neq 0$. By the Weierstrass approximation theorem, Theorem 7.1.6 or one of its cases, there exists a polynomial, p such that $\Lambda(p) \neq 0$. Therefore, if it can be shown that whenever $\Lambda(X) = 0$, it is the case that $\Lambda(p) = 0$ for all polynomials, it must be the case that X is dense in $C([0, b])$.

By the Riesz representation theorem the elements of $C([0, b])'$ are complex measures. Suppose then that for μ a complex measure it follows that for all t^{λ_k} ,

$$\int_{[0, b]} t^{\lambda_k} d\mu = 0.$$

I want to show that then

$$\int_{[0, b]} t^k d\mu = 0$$

for all positive integers. It suffices to modify μ is necessary to have $\mu(\{0\}) = 0$ since this will not change any of the above integrals. Let $\mu_1(E) = \mu(E \cap (0, b])$ and use μ_1 . I will continue using the symbol, μ .

For $\text{Re}(z) > 0$, define

$$F(z) \equiv \int_{[0, b]} t^z d\mu = \int_{(0, b]} t^z d\mu$$

The function $t^z = \exp(z \ln(t))$ is analytic. I claim that $F(z)$ is also analytic for $\text{Re } z > 0$. Apply Morera's theorem. Let T be a triangle in $\text{Re } z > 0$. Then

$$\int_{\partial T} F(z) dz = \int_{\partial T} \int_{(0, b]} e^{(z \ln(t))} \xi d|\mu| dz$$

Now $\int_{\partial T}$ can be split into three integrals over intervals of $\mathbb{R}$ and so this integral is essentially a Lebesgue integral taken with respect to Lebesgue measure. Furthermore, $e^{(z \ln(t))}$ is a continuous function of the two variables and ξ is a function of only the one variable, t . Thus the integrand is product measurable. The iterated integral is also absolutely integrable because $|e^{(z \ln(t))}| \leq e^{x \ln t} \leq e^{x \ln b}$ where $x + iy = z$ and x is given to be positive. Thus the integrand is actually bounded. Therefore, you can apply Fubini's theorem and write

$$\begin{aligned} \int_{\partial T} F(z) dz &= \int_{\partial T} \int_{(0,b]} e^{(z \ln(t))} \xi d|\mu| dz \\ &= \int_{(0,b]} \xi \int_{\partial T} e^{(z \ln(t))} dz d|\mu| = 0. \end{aligned}$$

By Morera's theorem, F is analytic on $\operatorname{Re} z > 0$ which is given to have zeros at the λ_k .

Now let $\phi(z) = \frac{1+z}{1-z}$. Then ϕ maps $B(0,1)$ one to one onto $\operatorname{Re} z > 0$. To see this let $0 < r < 1$.

$$\phi(re^{i\theta}) = \frac{1+re^{i\theta}}{1-re^{i\theta}} = \frac{1-r^2+i2r\sin\theta}{1+r^2-2r\cos\theta}$$

and so $\operatorname{Re} \phi(re^{i\theta}) > 0$. Now the inverse of ϕ is $\phi^{-1}(z) = \frac{z-1}{z+1}$. For $\operatorname{Re} z > 0$,

$$|\phi^{-1}(z)|^2 = \frac{z-1}{z+1} \cdot \frac{\bar{z}-1}{\bar{z}+1} = \frac{|z|^2 - 2\operatorname{Re} z + 1}{|z|^2 + 2\operatorname{Re} z + 1} < 1.$$

Consider $F \circ \phi$, an analytic function defined on $B(0,1)$. This function is given to have zeros at z_n where $\phi(z_n) = \frac{1+z_n}{1-z_n} = \lambda_n$. This reduces to $z_n = \frac{-1+\lambda_n}{1+\lambda_n}$. Now

$$1 - |z_n| \geq \frac{c}{1 + \lambda_n}$$

for a positive constant, c . It is given that $\sum \frac{1}{\lambda_n} = \infty$, so it follows $\sum (1 - |z_n|) = \infty$ also. Therefore, by Corollary 33.5.4, $F \circ \phi = 0$. It follows $F = 0$ also. In particular, $F(k)$ for k a positive integer equals zero. This has shown that if $\Lambda \in C([0,b])'$ and Λ sends 1 and all the t^{λ_n} to 0, then Λ sends 1 and all t^k for k a positive integer to zero. As explained above, X is dense in $C((0,b])$.

The converse of this theorem is also true and is proved in Rudin [58].

33.6 Exercises

1. Suppose f is an entire function with $f(0) = 1$. Let

$$M(r) = \max \{|f(z)| : |z| = r\}.$$

Use Jensen's equation to establish the following inequality.

$$M(2r) \geq 2^{n(r)}$$

where $n(r)$ is the number of zeros of f in $\overline{B(0,r)}$.

2. The version of the Blaschke product presented above is that found in most complex variable texts. However, there is another one in [47]. Instead of $\frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n}$ you use

$$\frac{\alpha_n - z}{\frac{1}{\bar{\alpha}_n} - z}$$

Prove a version of Theorem 33.5.1 using this modification.

3. The Weierstrass approximation theorem holds for polynomials of n variables on any compact subset of $\mathbb{R}^n$. Give a multidimensional version of the Müntz-Szasz theorem which will generalize the Weierstrass approximation theorem for n dimensions. You might just pick a compact subset of $\mathbb{R}^n$ in which all components are positive. You have to do something like this because otherwise, t^λ might not be defined.
4. Show $\cos(\pi z) = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2}\right)$.
5. Recall $\sin(\pi z) = z\pi \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{n}\right)^2\right)$. Use this to derive Wallis product, $\frac{\pi}{2} = \prod_{k=1}^{\infty} \frac{4k^2}{(2k-1)(2k+1)}$.
6. The order of an entire function, f is defined as

$$\inf \left\{ a \geq 0 : |f(z)| \leq e^{|z|^a} \text{ for all large enough } |z| \right\}$$

If no such a exists, the function is said to be of infinite order. Show the order of an entire function is also equal to $\limsup_{r \rightarrow \infty} \frac{\ln(\ln(M(r)))}{\ln(r)}$ where $M(r) \equiv \max\{|f(z)| : |z| = r\}$.

7. Suppose Ω is a simply connected region and let f be meromorphic on Ω . Suppose also that the set, $S \equiv \{z \in \Omega : f(z) = c\}$ has a limit point in Ω . Can you conclude $f(z) = c$ for all $z \in \Omega$?
8. This and the next collection of problems are dealing with the gamma function. Show that

$$\left| \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}} - 1 \right| \leq \frac{C(z)}{n^2}$$

and therefore,

$$\sum_{n=1}^{\infty} \left| \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}} - 1 \right| < \infty$$

with the convergence uniform on compact sets.

9. $\uparrow$ Show $\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}}$ converges to an analytic function on $\mathbb{C}$ which has zeros only at the negative integers and that therefore,

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

is a meromorphic function (Analytic except for poles) having simple poles at the negative integers.

10. ↑ Show there exists γ such that if

$$\Gamma(z) \equiv \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}},$$

then $\Gamma(1) = 1$. Thus Γ is a meromorphic function having simple poles at the negative integers. **Hint:** $\prod_{n=1}^{\infty} (1+n) e^{-1/n} = c = e^{\gamma}$.

11. ↑ Now show that

$$\gamma = \lim_{n \rightarrow \infty} \left[\sum_{k=1}^n \frac{1}{k} - \ln n \right]$$

12. ↑ Justify the following argument leading to Gauss's formula

$$\begin{aligned} \Gamma(z) &= \lim_{n \rightarrow \infty} \left(\prod_{k=1}^n \left(\frac{k}{k+z} \right) e^{\frac{z}{k}} \right) \frac{e^{-\gamma z}}{z} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n!}{(1+z)(2+z) \cdots (n+z)} e^{z(\sum_{k=1}^n \frac{1}{k})} \right) \frac{e^{-\gamma z}}{z} \\ &= \lim_{n \rightarrow \infty} \frac{n!}{(1+z)(2+z) \cdots (n+z)} e^{z(\sum_{k=1}^n \frac{1}{k})} e^{-z[\sum_{k=1}^n \frac{1}{k} - \ln n]} \\ &= \lim_{n \rightarrow \infty} \frac{n! n^z}{(1+z)(2+z) \cdots (n+z)}. \end{aligned}$$

13. ↑ Verify from the Gauss formula above that $\Gamma(z+1) = \Gamma(z)z$ and that for n a nonnegative integer, $\Gamma(n+1) = n!$.
14. ↑ The usual definition of the gamma function for positive x is

$$\Gamma_1(x) \equiv \int_0^{\infty} e^{-t} t^{x-1} dt.$$

Show $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for $t \in [0, n]$. Then show

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt = \frac{n! n^x}{x(x+1) \cdots (x+n)}.$$

Use the first part to conclude that

$$\Gamma_1(x) = \lim_{n \rightarrow \infty} \frac{n! n^x}{x(x+1) \cdots (x+n)} = \Gamma(x).$$

Hint: To show $\left(1 - \frac{t}{n}\right)^n \leq e^{-t}$ for $t \in [0, n]$, verify this is equivalent to showing $(1-u)^n \leq e^{-nu}$ for $u \in [0, 1]$.

15. ↑ Show $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$, whenever $\operatorname{Re} z > 0$. **Hint:** You have already shown that this is true for positive real numbers. Verify this formula for $\operatorname{Re} z > 0$ yields an analytic function.
16. ↑ Show $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then find $\Gamma\left(\frac{5}{2}\right)$.
17. Show that $\int_{-\infty}^\infty e^{-\frac{s^2}{2}} ds = \sqrt{2\pi}$. **Hint:** Denote this integral by I and observe that $I^2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2} dx dy$. Then change variables to polar coordinates, $x = r \cos(\theta)$, $y = r \sin \theta$.
18. ↑ Now that you know what the gamma function is, consider in the formula for $\Gamma(\alpha + 1)$ the following change of variables. $t = \alpha + \alpha^{1/2}s$. Then in terms of the new variable, s , the formula for $\Gamma(\alpha + 1)$ is

$$\begin{aligned} e^{-\alpha} \alpha^{\alpha+1/2} \int_{-\sqrt{\alpha}}^\infty e^{-\sqrt{\alpha}s} \left(1 + \frac{s}{\sqrt{\alpha}}\right)^\alpha ds \\ = e^{-\alpha} \alpha^{\alpha+1/2} \int_{-\sqrt{\alpha}}^\infty e^{\alpha \left[\ln\left(1 + \frac{s}{\sqrt{\alpha}}\right) - \frac{s}{\sqrt{\alpha}}\right]} ds \end{aligned}$$

Show the integrand converges to $e^{-\frac{s^2}{2}}$. Show that then

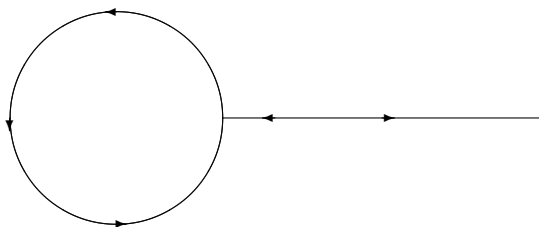
$$\lim_{\alpha \rightarrow \infty} \frac{\Gamma(\alpha + 1)}{e^{-\alpha} \alpha^{\alpha+1/2}} = \int_{-\infty}^\infty e^{-\frac{s^2}{2}} ds = \sqrt{2\pi}.$$

Hint: You will need to obtain a dominating function for the integral so that you can use the dominated convergence theorem. You might try considering $s \in (-\sqrt{\alpha}, \sqrt{\alpha})$ first and consider something like $e^{1-(s^2/4)}$ on this interval. Then look for another function for $s > \sqrt{\alpha}$. This formula is known as Stirling's formula.

19. This and the next several problems develop the zeta function and give a relation between the zeta and the gamma function. Define for $0 < r < 2\pi$

$$\begin{aligned} I_r(z) \equiv & \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^{re^{i\theta}} - 1} i r e^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt \\ & + \int_\infty^r \frac{e^{(z-1)\ln t}}{e^t - 1} dt \end{aligned} \quad (33.6.24)$$

Show that I_r is an entire function. The reason $0 < r < 2\pi$ is that this prevents $e^{re^{i\theta}} - 1$ from equaling zero. The above is just a precise description of the contour integral, $\int_\gamma \frac{w^{z-1}}{e^w - 1} dw$ where γ is the contour shown below.

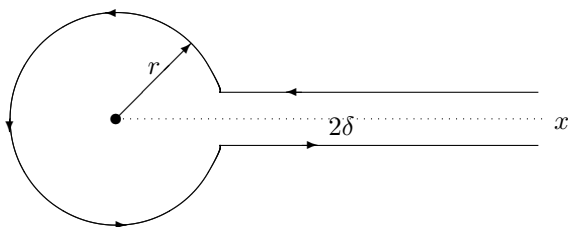


in which on the integrals along the real line, the argument is different in going from r to ∞ than it is in going from ∞ to r . Now I have not defined such contour integrals over contours which have infinite length and so have chosen to simply write out explicitly what is involved. You have to work with these integrals given above anyway but the contour integral just mentioned is the motivation for them. **Hint:** You may want to use convergence theorems from real analysis if it makes this more convenient but you might not have to.

20. $\uparrow$ In the context of Problem 19 define for small $\delta > 0$

$$I_{r\delta}(z) \equiv \int_{\gamma_{r,\delta}} \frac{w^{z-1}}{e^w - 1} dw$$

where $\gamma_{r\delta}$ is shown below.

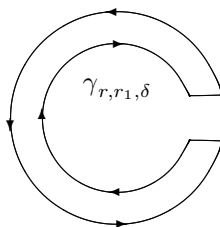


Show that $\lim_{\delta \rightarrow 0} I_{r\delta}(z) = I_r(z)$. **Hint:** Use the dominated convergence theorem if it makes this go easier. This is not a hard problem if you use these theorems but you can probably do it without them with more work.

21. $\uparrow$ In the context of Problem 20 show that for $r_1 < r$, $I_{r\delta}(z) - I_{r_1\delta}(z)$ is a contour integral,

$$\int_{\gamma_{r,r_1,\delta}} \frac{w^{z-1}}{e^w - 1} dw$$

where the oriented contour is shown below.



In this contour integral, w^{z-1} denotes $e^{(z-1)\log(w)}$ where $\log(w) = \ln|w| + i\arg(w)$ for $\arg(w) \in (0, 2\pi)$. Explain why this integral equals zero. From Problem 20 it follows that $I_r = I_{r_1}$. Therefore, you can define an entire function, $I(z) \equiv I_r(z)$ for all r positive but sufficiently small. **Hint:** Remember the Cauchy integral formula for analytic functions defined on simply connected regions. You could argue there is a simply connected region containing $\gamma_{r, r_1, \delta}$.

22. $\uparrow$ In case $\operatorname{Re} z > 1$, you can get an interesting formula for $I(z)$ by taking the limit as $r \rightarrow 0$. Recall that

$$I_r(z) \equiv \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^{re^{i\theta}} - 1} ire^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt \quad (33.6.25)$$

$$+ \int_\infty^r \frac{e^{(z-1)\ln t}}{e^t - 1} dt$$

and now it is desired to take a limit in the case where $\operatorname{Re} z > 1$. Show the first integral above converges to 0 as $r \rightarrow 0$. Next argue the sum of the two last integrals converges to

$$\left(e^{(z-1)2\pi i} - 1\right) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt.$$

Thus

$$I(z) = \left(e^{z2\pi i} - 1\right) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt \quad (33.6.26)$$

when $\operatorname{Re} z > 1$.

23. $\uparrow$ So what does all this have to do with the zeta function and the gamma function? The zeta function is defined for $\operatorname{Re} z > 1$ by

$$\sum_{n=1}^{\infty} \frac{1}{n^z} \equiv \zeta(z).$$

By Problem 15, whenever $\operatorname{Re} z > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Change the variable and conclude

$$\Gamma(z) \frac{1}{n^z} = \int_0^\infty e^{-ns} s^{z-1} ds.$$

Therefore, for $\operatorname{Re} z > 1$,

$$\zeta(z) \Gamma(z) = \sum_{n=1}^\infty \int_0^\infty e^{-ns} s^{z-1} ds.$$

Now show that you can interchange the order of the sum and the integral. This is possibly most easily done by using Fubini's theorem. Show that $\sum_{n=1}^\infty \int_0^\infty |e^{-ns} s^{z-1}| ds < \infty$ and then use Fubini's theorem. I think you could do it other ways though. It is possible to do it without any reference to Lebesgue integration. Thus

$$\begin{aligned} \zeta(z) \Gamma(z) &= \int_0^\infty s^{z-1} \sum_{n=1}^\infty e^{-ns} ds \\ &= \int_0^\infty \frac{s^{z-1} e^{-s}}{1 - e^{-s}} ds = \int_0^\infty \frac{s^{z-1}}{e^s - 1} ds \end{aligned}$$

By 33.6.26,

$$\begin{aligned} I(z) &= (e^{z2\pi i} - 1) \int_0^\infty \frac{e^{(z-1)\ln(t)}}{e^t - 1} dt \\ &= (e^{z2\pi i} - 1) \zeta(z) \Gamma(z) \\ &= (e^{2\pi iz} - 1) \zeta(z) \Gamma(z) \end{aligned}$$

whenever $\operatorname{Re} z > 1$.

24. $\uparrow$ Now show there exists an entire function, $h(z)$ such that

$$\zeta(z) = \frac{1}{z-1} + h(z)$$

for $\operatorname{Re} z > 1$. Conclude $\zeta(z)$ extends to a meromorphic function defined on all of $\mathbb{C}$ which has a simple pole at $z = 1$, namely, the right side of the above formula. **Hint:** Use Problem 10 to observe that $\Gamma(z)$ is never equal to zero but has simple poles at every nonnegative integer. Then for $\operatorname{Re} z > 1$,

$$\zeta(z) \equiv \frac{I(z)}{(e^{2\pi iz} - 1) \Gamma(z)}.$$

By 33.6.26 ζ has no poles for $\operatorname{Re} z > 1$. The right side of the above equation is defined for all z . There are no poles except possibly when z is a nonnegative integer. However, these points are not poles either because of Problem 10 which states that Γ has simple poles at these points thus cancelling the simple

zeros of $(e^{2\pi iz} - 1)$. The only remaining possibility for a pole for ζ is at $z = 1$. Show it has a simple pole at this point. You can use the formula for $I(z)$

$$I(z) \equiv \int_0^{2\pi} \frac{e^{(z-1)(\ln r + i\theta)}}{e^{re^{i\theta}} - 1} ire^{i\theta} d\theta + \int_r^\infty \frac{e^{(z-1)(\ln t + 2\pi i)}}{e^t - 1} dt \quad (33.6.27) \\ + \int_\infty^r \frac{e^{(z-1)\ln t}}{e^t - 1} dt$$

Thus $I(1)$ is given by

$$I(1) \equiv \int_0^{2\pi} \frac{1}{e^{re^{i\theta}} - 1} ire^{i\theta} d\theta + \int_r^\infty \frac{1}{e^t - 1} dt + \int_\infty^r \frac{1}{e^t - 1} dt$$

$= \int_{\gamma_r} \frac{dw}{e^w - 1}$ where γ_r is the circle of radius r . This contour integral equals $2\pi i$ by the residue theorem. Therefore,

$$\frac{I(z)}{(e^{2\pi iz} - 1)\Gamma(z)} = \frac{1}{z-1} + h(z)$$

where $h(z)$ is an entire function. People worry a lot about where the zeros of ζ are located. In particular, the zeros for $\operatorname{Re} z \in (0, 1)$ are of special interest. The Riemann hypothesis says they are all on the line $\operatorname{Re} z = 1/2$. This is a good problem for you to do next.

25. There is an important relation between prime numbers and the zeta function due to Euler. Let $\{p_n\}_{n=1}^\infty$ be the prime numbers. Then for $\operatorname{Re} z > 1$,

$$\prod_{n=1}^\infty \frac{1}{1 - p_n^{-z}} = \zeta(z).$$

To see this, consider a partial product.

$$\prod_{n=1}^N \frac{1}{1 - p_n^{-z}} = \prod_{n=1}^N \sum_{j_n=1}^\infty \left(\frac{1}{p_n^z} \right)^{j_n}.$$

Let S_N denote all positive integers which use only $p_1, \dots, p_N$ in their prime factorization. Then the above equals $\sum_{n \in S_N} \frac{1}{n^z}$. Letting $N \rightarrow \infty$ and using the fact that $\operatorname{Re} z > 1$ so that the order in which you sum is not important (See Theorem 34.0.1 or recall advanced calculus.) you obtain the desired equation. Show $\sum_{n=1}^\infty \frac{1}{p_n} = \infty$.

Elliptic Functions

This chapter is to give a short introduction to elliptic functions. There is much more available. There are books written on elliptic functions. What I am presenting here follows Alfors [2] although the material is found in many books on complex analysis. Hille, [32] has a much more extensive treatment than what I will attempt here. There are also many references and historical notes available in the book by Hille. Another good source for more having much the same emphasis as what is presented here is in the book by Saks and Zygmund [60]. This is a very interesting subject because it has considerable overlap with algebra.

Before beginning, recall that an absolutely convergent series can be summed in any order and you always get the same answer. The easy way to see this is to think of the series as a Lebesgue integral with respect to counting measure and apply convergence theorems as needed. The following theorem provides the necessary results.

Theorem 34.0.1 *Suppose $\sum_{n=1}^{\infty} |a_n| < \infty$ and let $\theta, \phi : \mathbb{N} \rightarrow \mathbb{N}$ be one to one and onto mappings. Then $\sum_{n=1}^{\infty} a_{\phi(n)}$ and $\sum_{n=1}^{\infty} a_{\theta(n)}$ both converge and the two sums are equal.*

Proof: By the monotone convergence theorem,

$$\sum_{n=1}^{\infty} |a_n| = \lim_{n \rightarrow \infty} \sum_{k=1}^n |a_{\phi(k)}| = \lim_{n \rightarrow \infty} \sum_{k=1}^n |a_{\theta(k)}|$$

but these last two equal $\sum_{k=1}^{\infty} |a_{\phi(k)}|$ and $\sum_{k=1}^{\infty} |a_{\theta(k)}|$ respectively. Therefore, $\sum_{k=1}^{\infty} a_{\theta(k)}$ and $\sum_{k=1}^{\infty} a_{\phi(k)}$ exist ($n \rightarrow a_{\theta(n)}$ is in L^1 with respect to counting measure.) It remains to show the two are equal. There exists M such that if $n > M$ then

$$\begin{aligned} \sum_{k=n+1}^{\infty} |a_{\theta(k)}| &< \varepsilon, & \sum_{k=n+1}^{\infty} |a_{\phi(k)}| &< \varepsilon \\ \left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^n a_{\phi(k)} \right| &< \varepsilon, & \left| \sum_{k=1}^{\infty} a_{\theta(k)} - \sum_{k=1}^n a_{\theta(k)} \right| &< \varepsilon \end{aligned}$$

Pick such an n denoted by n_1 . Then pick $n_2 > n_1 > M$ such that

$$\{\theta(1), \dots, \theta(n_1)\} \subseteq \{\phi(1), \dots, \phi(n_2)\}.$$

Then

$$\sum_{k=1}^{n_2} a_{\phi(k)} = \sum_{k=1}^{n_1} a_{\theta(k)} + \sum_{\phi(k) \notin \{\theta(1), \dots, \theta(n_1)\}} a_{\phi(k)}.$$

Therefore,

$$\left| \sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)} \right| = \left| \sum_{\phi(k) \notin \{\theta(1), \dots, \theta(n_1)\}, k \leq n_2} a_{\phi(k)} \right|$$

Now all of these $\phi(k)$ in the last sum are contained in $\{\theta(n_1 + 1), \dots\}$ and so the last sum above is dominated by

$$\leq \sum_{k=n_1+1}^{\infty} |a_{\theta(k)}| < \varepsilon.$$

Therefore,

$$\begin{aligned} \left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{\infty} a_{\theta(k)} \right| &\leq \left| \sum_{k=1}^{\infty} a_{\phi(k)} - \sum_{k=1}^{n_2} a_{\phi(k)} \right| \\ &\quad + \left| \sum_{k=1}^{n_2} a_{\phi(k)} - \sum_{k=1}^{n_1} a_{\theta(k)} \right| \\ &\quad + \left| \sum_{k=1}^{n_1} a_{\theta(k)} - \sum_{k=1}^{\infty} a_{\theta(k)} \right| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \end{aligned}$$

and since ε is arbitrary, it follows $\sum_{k=1}^{\infty} a_{\phi(k)} = \sum_{k=1}^{\infty} a_{\theta(k)}$ as claimed. This proves the theorem.

34.1 Periodic Functions

Definition 34.1.1 A function defined on $\mathbb{C}$ is said to be periodic if there exists w such that $f(z + w) = f(z)$ for all $z \in \mathbb{C}$. Denote by M the set of all periods. Thus if $w_1, w_2 \in M$ and $a, b \in \mathbb{Z}$, then $aw_1 + bw_2 \in M$. For this reason M is called the module of periods.¹ In all which follows it is assumed f is meromorphic.

Theorem 34.1.2 Let f be a meromorphic function and let M be the module of periods. Then if M has a limit point, then f equals a constant. If this does not happen then either there exists $w_1 \in M$ such that $\mathbb{Z}w_1 = M$ or there exist $w_1, w_2 \in M$ such that $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$ and w_1/w_2 is not real. Also if $\tau = w_2/w_1$,

$$|\tau| \geq 1, \quad \frac{-1}{2} \leq \operatorname{Re} \tau \leq \frac{1}{2}.$$

¹A module is like a vector space except instead of a field of scalars, you have a ring of scalars.

Proof: Suppose f is meromorphic and M has a limit point, w_0 . By Theorem 33.1.8 on Page 978 there exist analytic functions, p, q such that $f(z) = \frac{p(z)}{q(z)}$. Now pick z_0 such that z_0 is not a pole of f . Then letting $w_n \rightarrow w_0$ where $\{w_n\} \subseteq M$, $f(z_0 + w_n) = f(z_0)$. Therefore, $p(z_0 + w_n) = f(z_0)q(z_0 + w_n)$ and so the analytic function, $p(z) - f(z_0)q(z)$ has a zero set which has a limit point. Therefore, this function is identically equal to zero because of Theorem 27.5.3 on Page 807. Thus f equals a constant as claimed.

This has shown that if f is not constant, then M is discrete. Therefore, there exists $w_1 \in M$ such that $|w_1| = \min\{|w| : w \in M\}$. Suppose first that every element of M is a real multiple of w_1 . Thus, if $w \in M$, it follows there exists a real number, x such that $w = xw_1$. Then there exist positive integers, $k, k+1$ such that $k \leq x < k+1$. If $x > k$, then $w - kw_1 = (x - k)w_1$ is a period having smaller absolute value than $|w_1|$ which would be a contradiction. Hence, $x = k$ and so $M = \mathbb{Z}w_1$.

Now suppose there exists $w_2 \in M$ which is not a real multiple of w_1 . You can let w_2 be the element of M having this property which has smallest absolute value. Now let $w \in M$. Since w_1 and w_2 point in different directions, it follows $w = xw_1 + yw_2$ for some real numbers, x, y . Let $|m - x| \leq \frac{1}{2}$ and $|n - y| \leq \frac{1}{2}$ where m, n are integers. Therefore,

$$w = mw_1 + nw_2 + (x - m)w_1 + (y - n)w_2$$

and so

$$w - mw_1 - nw_2 = (x - m)w_1 + (y - n)w_2 \quad (34.1.1)$$

Now since $w_2/w_1 \notin \mathbb{R}$,

$$\begin{aligned} |(x - m)w_1 + (y - n)w_2| &< |(x - m)w_1| + |(y - n)w_2| \\ &= \frac{1}{2}|w_1| + \frac{1}{2}|w_2|. \end{aligned}$$

Therefore, from 34.1.1,

$$\begin{aligned} |w - mw_1 - nw_2| &= |(x - m)w_1 + (y - n)w_2| \\ &< \frac{1}{2}|w_1| + \frac{1}{2}|w_2| \leq |w_2| \end{aligned}$$

and so the period, $w - mw_1 - nw_2$ cannot be a non real multiple of w_1 because w_2 is the one which has smallest absolute value and this period has smaller absolute value than w_2 . Therefore, the ratio $w - mw_1 - nw_2/w_1$ must be a real number, x . Thus

$$w - mw_1 - nw_2 = xw_1$$

Since w_1 has minimal absolute value of all periods, it follows $|x| \geq 1$. Let $k \leq x < k+1$ for some integer, k . If $x > k$, then

$$w - mw_1 - nw_2 - kw_1 = (x - k)w_1$$

which would contradict the choice of w_1 as being the period having minimal absolute value because the expression on the left in the above is a period and it equals

something which has absolute value less than $|w_1|$. Therefore, $x = k$ and w is an integer linear combination of w_1 and w_2 . It only remains to verify the claim about τ .

From the construction, $|w_1| \leq |w_2|$ and $|w_2| \leq |w_1 - w_2|, |w_2| \leq |w_1 + w_2|$. Therefore,

$$|\tau| \geq 1, |\tau| \leq |1 - \tau|, |\tau| \leq |1 + \tau|.$$

The last two of these inequalities imply $-1/2 \leq \operatorname{Re} \tau \leq 1/2$.

This proves the theorem.

Definition 34.1.3 For f a meromorphic function which has the last of the above alternatives holding in which $M = \{aw_1 + bw_2 : a, b \in \mathbb{Z}\}$, the function, f is called elliptic. This is also called doubly periodic.

Theorem 34.1.4 Suppose f is an elliptic function which has no poles. Then f is constant.

Proof: Since f has no poles it is analytic. Now consider the parallelograms determined by the vertices, $mw_1 + nw_2$ for $m, n \in \mathbb{Z}$. By periodicity of f it must be bounded because its values are identical on each of these parallelograms. Therefore, it equals a constant by Liouville's theorem.

Definition 34.1.5 Define P_a to be the parallelogram determined by the points

$$\begin{aligned} &a + mw_1 + nw_2, a + (m + 1)w_1 + nw_2, a + mw_1 + (n + 1)w_2, \\ &a + (m + 1)w_1 + (n + 1)w_2 \end{aligned}$$

Such P_a will be referred to as a period parallelogram. The sum of the orders of the poles in a period parallelogram which contains no poles or zeros of f on its boundary is called the order of the function. This is well defined because of the periodic property of f .

Theorem 34.1.6 The sum of the residues of any elliptic function, f equals zero on every P_a if a is chosen so that there are no poles on ∂P_a .

Proof: Choose a such that there are no poles of f on the boundary of P_a . By periodicity,

$$\int_{\partial P_a} f(z) dz = 0$$

because the integrals over opposite sides of the parallelogram cancel out because the values of f are the same on these sides and the orientations are opposite. It follows from the residue theorem that the sum of the residues in P_a equals 0.

Theorem 34.1.7 Let P_a be a period parallelogram for a nonconstant elliptic function, f which has order equal to m . Then f assumes every value in $f(P_a)$ exactly m times.

Proof: Let $c \in f(P_a)$ and consider $P_{a'}$ such that $f^{-1}(c) \cap P_{a'} = f^{-1}(c) \cap P_a$ and $P_{a'}$ contains the same poles and zeros of $f - c$ as P_a but $P_{a'}$ has no zeros of $f(z) - c$ or poles of f on its boundary. Thus $f'(z) / (f(z) - c)$ is also an elliptic function and so Theorem 34.1.6 applies. Consider

$$\frac{1}{2\pi i} \int_{\partial P_{a'}} \frac{f'(z)}{f(z) - c} dz.$$

By the argument principle, this equals $N_z - N_p$ where N_z equals the number of zeros of $f(z) - c$ and N_p equals the number of the poles of $f(z)$. From Theorem 34.1.6 this must equal zero because it is the sum of the residues of $f' / (f - c)$ and so $N_z = N_p$. Now N_p equals the number of poles in P_a counted according to multiplicity.

There is an even better theorem than this one.

Theorem 34.1.8 *Let f be a non constant elliptic function and suppose it has poles $p_1, \dots, p_m$ and zeros, $z_1, \dots, z_m$ in P_α , listed according to multiplicity where ∂P_α contains no poles or zeros of f . Then $\sum_{k=1}^m z_k - \sum_{k=1}^m p_k \in M$, the module of periods.*

Proof: You can assume ∂P_a contains no poles or zeros of f because if it did, then you could consider a slightly shifted period parallelogram, $P_{a'}$ which contains no new zeros and poles but which has all the old ones but no poles or zeros on its boundary. By Theorem 29.1.3 on Page 860

$$\frac{1}{2\pi i} \int_{\partial P_a} z \frac{f'(z)}{f(z)} dz = \sum_{k=1}^m z_k - \sum_{k=1}^m p_k. \quad (34.1.2)$$

Denoting by $\gamma(z, w)$ the straight oriented line segment from z to w ,

$$\begin{aligned} & \int_{\partial P_a} z \frac{f'(z)}{f(z)} dz \\ &= \int_{\gamma(a, a+w_1)} z \frac{f'(z)}{f(z)} dz + \int_{\gamma(a+w_1+w_2, a+w_2)} z \frac{f'(z)}{f(z)} dz \\ & \quad + \int_{\gamma(a+w_1, a+w_2+w_1)} z \frac{f'(z)}{f(z)} dz + \int_{\gamma(a+w_2, a)} z \frac{f'(z)}{f(z)} dz \\ &= \int_{\gamma(a, a+w_1)} (z - (z + w_2)) \frac{f'(z)}{f(z)} dz \\ & \quad + \int_{\gamma(a, a+w_2)} (z - (z + w_1)) \frac{f'(z)}{f(z)} dz \end{aligned}$$

Now near these line segments $\frac{f'(z)}{f(z)}$ is analytic and so there exists a primitive, $g_{w_i}(z)$ on $\gamma(a, a + w_i)$ by Corollary 27.7.5 on Page 813 which satisfies $e^{g_{w_i}(z)} = f(z)$. Therefore,

$$= -w_2 (g_{w_1}(a + w_1) - g_{w_1}(a)) - w_1 (g_{w_2}(a + w_2) - g_{w_2}(a)).$$

Now by periodicity of f it follows $f(a + w_1) = f(a) = f(a + w_2)$. Hence

$$g_{w_i}(a + w_1) - g_{w_i}(a) = 2m\pi i$$

for some integer, m because

$$e^{g_{w_i}(a+w_i)} - e^{g_{w_i}(a)} = f(a + w_i) - f(a) = 0.$$

Therefore, from 34.1.2, there exist integers, k, l such that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\partial P_\alpha} z \frac{f'(z)}{f(z)} dz \\ &= \frac{1}{2\pi i} [-w_2 (g_{w_1}(a + w_1) - g_{w_1}(a)) - w_1 (g_{w_2}(a + w_2) - g_{w_2}(a))] \\ &= \frac{1}{2\pi i} [-w_2 (2k\pi i) - w_1 (2l\pi i)] \\ &= -w_2 k - w_1 l \in M. \end{aligned}$$

From 34.1.2 it follows

$$\sum_{k=1}^m z_k - \sum_{k=1}^m p_k \in M.$$

This proves the theorem.

Hille says this relation is due to Liouville. There is also a simple corollary which follows from the above theorem applied to the elliptic function, $f(z) - c$.

Corollary 34.1.9 *Let f be a non constant elliptic function and suppose the function, $f(z) - c$ has poles $p_1, \dots, p_m$ and zeros, $z_1, \dots, z_m$ on P_α , listed according to multiplicity where ∂P_α contains no poles or zeros of $f(z) - c$. Then $\sum_{k=1}^m z_k - \sum_{k=1}^m p_k \in M$, the module of periods.*

34.1.1 The Unimodular Transformations

Definition 34.1.10 *Suppose f is a nonconstant elliptic function and the module of periods is of the form $\{aw_1 + bw_2\}$ where a, b are integers and w_1/w_2 is not real. Then by analogy with linear algebra, $\{w_1, w_2\}$ is referred to as a basis. The unimodular transformations will refer to matrices of the form*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where all entries are integers and

$$ad - bc = \pm 1.$$

These linear transformations are also called the modular group.

The following is an interesting lemma which ties matrices with the fractional linear transformations.

Lemma 34.1.11 *Define*

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \equiv \frac{az+b}{cz+d}.$$

Then

$$\phi(AB) = \phi(A) \circ \phi(B), \quad (34.1.3)$$

$\phi(A)(z) = z$ if and only if

$$A = cI$$

where I is the identity matrix and $c \neq 0$. Also if $f(z) = \frac{az+b}{cz+d}$, then $f^{-1}(z)$ exists if and only if $ad - cb \neq 0$. Furthermore it is easy to find f^{-1} .

Proof: The equation in 34.1.3 is just a simple computation. Now suppose $\phi(A)(z) = z$. Then letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, this requires

$$az + b = z(cz + d)$$

and so $az + b = cz^2 + dz$. Since this is to hold for all z it follows $c = 0 = b$ and $a = d$. The other direction is obvious.

Consider the claim about the existence of an inverse. Let $ad - cb \neq 0$ for $f(z) = \frac{az+b}{cz+d}$. Then

$$f(z) = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

It follows $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ exists and equals $\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Therefore,

$$\begin{aligned} z &= \phi(I)(z) = \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left(\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)\right)(z) \\ &= \phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \circ \phi\left(\left(\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right)\right)(z) \\ &= f \circ f^{-1}(z) \end{aligned}$$

which shows f^{-1} exists and it is easy to find.

Next suppose f^{-1} exists. I need to verify the condition $ad - cb \neq 0$. If f^{-1} exists, then from the process used to find it, you see that it must be a fractional linear transformation. Letting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so $\phi(A) = f$, it follows there exists a matrix B such that

$$\phi(BA)(z) = \phi(B) \circ \phi(A)(z) = z.$$

However, it was shown that this implies BA is a nonzero multiple of I which requires that A^{-1} must exist. Hence the condition must hold.

Theorem 34.1.12 *If f is a nonconstant elliptic function with a basis $\{w_1, w_2\}$ for the module of periods, then $\{w'_1, w'_2\}$ is another basis, if and only if there exists a unimodular transformation, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$ such that*

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (34.1.4)$$

Proof: Since $\{w_1, w_2\}$ is a basis, there exist integers, a, b, c, d such that 34.1.4 holds. It remains to show the transformation determined by the matrix is unimodular. Taking conjugates,

$$\begin{pmatrix} \overline{w'_1} \\ \overline{w'_2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w'_1 & \overline{w'_1} \\ w'_2 & \overline{w'_2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix}$$

Now since $\{w'_1, w'_2\}$ is also given to be a basis, there exists another matrix having all integer entries, $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ such that

$$\begin{pmatrix} \overline{w_1} \\ \overline{w_2} \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} \overline{w'_1} \\ \overline{w'_2} \end{pmatrix}$$

and

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} w'_1 & \overline{w'_1} \\ w'_2 & \overline{w'_2} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} w'_1 & \overline{w'_1} \\ w'_2 & \overline{w'_2} \end{pmatrix}.$$

However, since w'_1/w'_2 is not real, it is routine to verify that

$$\det \begin{pmatrix} w'_1 & \overline{w'_1} \\ w'_2 & \overline{w'_2} \end{pmatrix} \neq 0.$$

Therefore,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

and so $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} e & f \\ g & h \end{pmatrix} = 1$. But the two matrices have all integer entries and so both determinants must equal either 1 or -1 .

Next suppose

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad (34.1.5)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unimodular. I need to verify that $\{w'_1, w'_2\}$ is a basis. If $w \in M$, there exist integers, m, n such that

$$w = mw_1 + nw_2 = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

From 34.1.5

$$\pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and so

$$w = \pm \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix}$$

which is an integer linear combination of $\{w'_1, w'_2\}$. It only remains to verify that w'_1/w'_2 is not real.

Claim: Let w_1 and w_2 be nonzero complex numbers. Then w_2/w_1 is not real if and only if

$$w_1\overline{w_2} - \overline{w_1}w_2 = \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0$$

Proof of the claim: Let $\lambda = w_2/w_1$. Then

$$w_1\overline{w_2} - \overline{w_1}w_2 = \overline{\lambda}w_1\overline{w_1} - \overline{w_1}\lambda w_1 = (\overline{\lambda} - \lambda)|w_1|^2$$

Thus the ratio is not real if and only if $(\overline{\lambda} - \lambda) \neq 0$ if and only if $w_1\overline{w_2} - \overline{w_1}w_2 \neq 0$.

Now to verify w'_2/w'_1 is not real,

$$\begin{aligned} & \det \begin{pmatrix} w'_1 & \overline{w'_1} \\ w'_2 & \overline{w'_2} \end{pmatrix} \\ &= \det \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \right) \\ &= \pm \det \begin{pmatrix} w_1 & \overline{w_1} \\ w_2 & \overline{w_2} \end{pmatrix} \neq 0 \end{aligned}$$

This proves the theorem.

34.1.2 The Search For An Elliptic Function

By Theorem 34.1.4 and 34.1.6 if you want to find a nonconstant elliptic function it must fail to be analytic and also have either no terms in its Laurent expansion which are of the form $b_1(z-a)^{-1}$ or else these terms must cancel out. It is simplest to look for a function which simply does not have them. Weierstrass looked for a function of the form

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \quad (34.1.6)$$

where w consists of all numbers of the form $aw_1 + bw_2$ for a, b integers. Sometimes people write this as $\wp(z, w_1, w_2)$ to emphasize its dependence on the periods, w_1 and w_2 but I won't do so. It is understood there exist these periods, which are given. This is a reasonable thing to try. Suppose you formally differentiate the right side. Never mind whether this is justified for now. This yields

$$\wp'(z) = \frac{-2}{z^3} - \sum_{w \neq 0} \frac{-2}{(z-w)^3} = \sum_w \frac{-2}{(z-w)^3}$$

which is clearly periodic having both periods w_1 and w_2 . Therefore, $\wp(z + w_1) - \wp(z)$ and $\wp(z + w_2) - \wp(z)$ are both constants, c_1 and c_2 respectively. The reason for this is that since $\wp'$ is periodic with periods w_1 and w_2 , it follows $\wp'(z + w_i) - \wp'(z) = 0$ as long as z is not a period. From 34.1.6 you can see right away that

$$\wp(z) = \wp(-z)$$

Indeed

$$\begin{aligned} \wp(-z) &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z-w)^2} - \frac{1}{w^2} \right) \\ &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z+w)^2} - \frac{1}{w^2} \right) = \wp(z). \end{aligned}$$

and so

$$\begin{aligned} c_1 &= \wp\left(-\frac{w_1}{2} + w_1\right) - \wp\left(-\frac{w_1}{2}\right) \\ &= \wp\left(\frac{w_1}{2}\right) - \wp\left(-\frac{w_1}{2}\right) = 0 \end{aligned}$$

which shows the constant for $\wp(z + w_1) - \wp(z)$ must equal zero. Similarly the constant for $\wp(z + w_2) - \wp(z)$ also equals zero. Thus $\wp$ is periodic having the two periods w_1, w_2 .

Of course to justify this, you need to consider whether the series of 34.1.6 converges. Consider the terms of the series.

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = |z| \left| \frac{2w-z}{(z-w)^2 w^2} \right|$$

If $|w| > 2|z|$, this can be estimated more. For such w ,

$$\begin{aligned} &\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| \\ &= |z| \left| \frac{2w-z}{(z-w)^2 w^2} \right| \leq |z| \frac{(5/2)|w|}{|w|^2 (|w| - |z|)^2} \\ &\leq |z| \frac{(5/2)|w|}{|w|^2 ((1/2)|w|)^2} = |z| \frac{10}{|w|^3}. \end{aligned}$$

It follows the series in 34.1.6 converges uniformly and absolutely on every compact set, K provided $\sum_{w \neq 0} \frac{1}{|w|^3}$ converges. This question is considered next.

Claim: There exists a positive number, k such that for all pairs of integers, m, n , not both equal to zero,

$$\frac{|mw_1 + nw_2|}{|m| + |n|} \geq k > 0.$$

Proof of claim: If not, there exists m_k and n_k such that

$$\lim_{k \rightarrow \infty} \frac{m_k}{|m_k| + |n_k|} w_1 + \frac{n_k}{|m_k| + |n_k|} w_2 = 0$$

However, $\left(\frac{m_k}{|m_k| + |n_k|}, \frac{n_k}{|m_k| + |n_k|} \right)$ is a bounded sequence in $\mathbb{R}^2$ and so, taking a subsequence, still denoted by k , you can have

$$\left(\frac{m_k}{|m_k| + |n_k|}, \frac{n_k}{|m_k| + |n_k|} \right) \rightarrow (x, y) \in \mathbb{R}^2$$

and so there are real numbers, x, y such that $xw_1 + yw_2 = 0$ contrary to the assumption that w_2/w_1 is not equal to a real number. This proves the claim.

Now from the claim,

$$\begin{aligned} & \sum_{w \neq 0} \frac{1}{|w|^3} \\ &= \sum_{(m,n) \neq (0,0)} \frac{1}{|mw_1 + nw_2|^3} \leq \sum_{(m,n) \neq (0,0)} \frac{1}{k^3 (|m| + |n|)^3} \\ &= \frac{1}{k^3} \sum_{j=1}^{\infty} \sum_{|m|+|n|=j} \frac{1}{(|m| + |n|)^3} = \frac{1}{k^3} \sum_{j=1}^{\infty} \frac{4j}{j^3} < \infty. \end{aligned}$$

Now consider the series in 34.1.6. Letting $z \in B(0, R)$,

$$\begin{aligned} \wp(z) &\equiv \frac{1}{z^2} + \sum_{w \neq 0, |w| \leq R} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \\ &\quad + \sum_{w \neq 0, |w| > R} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \end{aligned}$$

and the last series converges uniformly on $B(0, R)$ to an analytic function. Thus $\wp$ is a meromorphic function and also the argument given above involving differentiation of the series termwise is valid. Thus $\wp$ is an elliptic function as claimed. This is called the Weierstrass $\wp$ function. This has proved the following theorem.

Theorem 34.1.13 *The function $\wp$ defined above is an example of an elliptic function. On any compact set, $\wp$ equals a rational function added to a series which is uniformly and absolutely convergent on the compact set.*

34.1.3 The Differential Equation Satisfied By $\wp$

For z not a pole,

$$\wp'(z) = \frac{-2}{z^3} - \sum_{w \neq 0} \frac{2}{(z-w)^3}$$

Also since there are no poles of order 1 you can obtain a primitive for $\wp, -\zeta$.² To do so, recall

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

where for $|z| < R$ this is the sum of a rational function with a uniformly convergent series. Therefore, you can take the integral along any path, $\gamma(0, z)$ from 0 to z which misses the poles of $\wp$. By the uniform convergence of the above integral, you can interchange the sum with the integral and obtain

$$\zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w} \quad (34.1.7)$$

This function is odd. Here is why.

$$\zeta(-z) = \frac{1}{-z} + \sum_{w \neq 0} \frac{1}{-z-w} - \frac{z}{w^2} + \frac{1}{w}$$

while

$$\begin{aligned} -\zeta(z) &= \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{z-w} - \frac{z}{w^2} - \frac{1}{w} \\ &= \frac{1}{-z} + \sum_{w \neq 0} \frac{-1}{z+w} - \frac{z}{w^2} + \frac{1}{w}. \end{aligned}$$

Now consider 34.1.7. It will be used to find the Laurent expansion about the origin for ζ which will then be differentiated to obtain the Laurent expansion for $\wp$ at the origin. Since $w \neq 0$ and the interest is for z near 0 so $|z| < |w|$,

$$\begin{aligned} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w} &= \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w} \frac{1}{1 - \frac{z}{w}} \\ &= \frac{z}{w^2} + \frac{1}{w} - \frac{1}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w} \right)^k \\ &= -\frac{1}{w} \sum_{k=2}^{\infty} \left(\frac{z}{w} \right)^k \end{aligned}$$

²I don't know why it is traditional to refer to this antiderivative as $-\zeta$ rather than ζ but I am following the convention. I think it is to minimize the number of minus signs in the next expression.

From 34.1.7

$$\begin{aligned}\zeta(z) &= \frac{1}{z} + \sum_{w \neq 0} \left(- \sum_{k=2}^{\infty} \frac{z^k}{w^{k+1}} \right) \\ &= \frac{1}{z} - \sum_{k=2}^{\infty} \sum_{w \neq 0} \frac{z^k}{w^{k+1}} = \frac{1}{z} - \sum_{k=2}^{\infty} \sum_{w \neq 0} \frac{z^{2k-1}}{w^{2k}}\end{aligned}$$

because the sum over odd powers must be zero because for each $w \neq 0$, there exists $-w \neq 0$ such that the two terms $\frac{z^{2k}}{w^{2k+1}}$ and $\frac{z^{2k}}{(-w)^{2k+1}}$ cancel each other. Hence

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} G_k z^{2k-1}$$

where $G_k = \sum_{w \neq 0} \frac{1}{w^{2k}}$. Now with this,

$$\begin{aligned}-\zeta'(z) &= \wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} G_k (2k-1) z^{2k-2} \\ &= \frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \dots\end{aligned}$$

Therefore,

$$\begin{aligned}\wp'(z) &= \frac{-2}{z^3} + 6G_2 z + 20G_3 z^3 + \dots \\ \wp'(z)^2 &= \frac{4}{z^6} - \frac{24G_2}{z^2} - 80G_3 + \dots \\ 4\wp(z)^3 &= 4 \left(\frac{1}{z^2} + 3G_2 z^2 + 5G_3 z^4 + \dots \right)^3 \\ &= \frac{4}{z^6} + \frac{36}{z^2} G_2 + 60G_3 + \dots\end{aligned}$$

and finally

$$60G_2 \wp(z) = \frac{60G_2}{z^2} + 0 + \dots$$

where in the above, the positive powers of z are not listed explicitly. Therefore,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2 \wp(z) + 140G_3 = \sum_{n=1}^{\infty} a_n z^n$$

In deriving the equation it was assumed $|z| < |w|$ for all $w = aw_1 + bw_2$ where a, b are integers not both zero. The left side of the above equation is periodic with respect to w_1 and w_2 where w_2/w_1 is not a real number. The only possible poles of the left side are at 0, w_1 , w_2 , and $w_1 + w_2$, the vertices of the parallelogram determined by w_1 and w_2 . This follows from the original formula for $\wp(z)$. However, the above

equation shows the left side has no pole at 0. Since the left side is periodic with periods w_1 and w_2 , it follows it has no pole at the other vertices of this parallelogram either. Therefore, the left side is periodic and has no poles. Consequently, it equals a constant by Theorem 34.1.4. But the right side of the above equation shows this constant is 0 because this side equals zero when $z = 0$. Therefore, $\wp$ satisfies the differential equation,

$$\wp'(z)^2 - 4\wp(z)^3 + 60G_2\wp(z) + 140G_3 = 0.$$

It is traditional to define $60G_2 \equiv g_2$ and $140G_3 \equiv g_3$. Then in terms of these new quantities the differential equation is

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Suppose e_1, e_2 and e_3 are zeros of the polynomial $4w^3 - g_2w - g_3 = 0$. Then the above equation can be written in the form

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3). \quad (34.1.8)$$

34.1.4 A Modular Function

The next task is to find the e_i in 34.1.8. First recall that $\wp$ is an even function. That is $\wp(-z) = \wp(z)$. This follows from 34.1.6 which is listed here for convenience.

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right) \quad (34.1.9)$$

Thus

$$\begin{aligned} \wp(-z) &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z-w)^2} - \frac{1}{w^2} \right) \\ &= \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(-z+w)^2} - \frac{1}{w^2} \right) = \wp(z). \end{aligned}$$

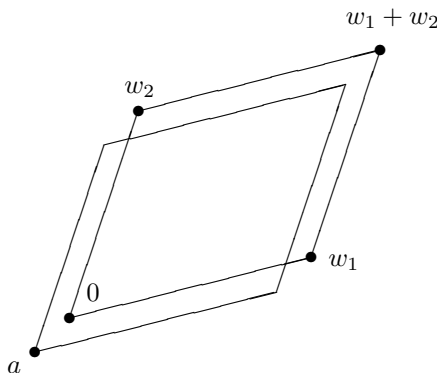
Therefore, $\wp(w_1 - z) = \wp(z - w_1) = \wp(z)$ and so $-\wp'(w_1 - z) = \wp'(z)$. Letting $z = w_1/2$, it follows $\wp'(w_1/2) = 0$. Similarly, $\wp'(w_2/2) = 0$ and $\wp'((w_1 + w_2)/2) = 0$. Therefore, from 34.1.8

$$0 = 4(\wp(w_1/2) - e_1)(\wp(w_1/2) - e_2)(\wp(w_1/2) - e_3).$$

It follows one of the e_i must equal $\wp(w_1/2)$. Similarly, one of the e_i must equal $\wp(w_2/2)$ and one must equal $\wp((w_1 + w_2)/2)$.

Lemma 34.1.14 *The numbers, $\wp(w_1/2)$, $\wp(w_2/2)$, and $\wp((w_1 + w_2)/2)$ are distinct.*

Proof: Choose P_a , a period parallelogram which contains the pole 0, and the points $w_1/2$, $w_2/2$, and $(w_1 + w_2)/2$ but no other pole of $\wp(z)$. Also ∂P_a^* does not contain any zeros of the elliptic function, $z \rightarrow \wp(z) - \wp(w_1/2)$. This can be done by shifting P_0 slightly because the poles are only at the points $aw_1 + bw_2$ for a, b integers and the zeros of $\wp(z) - \wp(w_1/2)$ are discrete.



If $\wp(w_2/2) = \wp(w_1/2)$, then $\wp(z) - \wp(w_1/2)$ has two zeros, $w_2/2$ and $w_1/2$ and since the pole at 0 is of order 2, this is the order of $\wp(z) - \wp(w_1/2)$ on P_a hence by Theorem 34.1.7 on Page 1006 these are the only zeros of this function on P_a . It follows by Corollary 34.1.9 on Page 1008 which says the sum of the zeros minus the sum of the poles is in M , $\frac{w_1}{2} + \frac{w_2}{2} \in M$. Thus there exist integers, a, b such that

$$\frac{w_1 + w_2}{2} = aw_1 + bw_2$$

which implies $(2a - 1)w_1 + (2b - 1)w_2 = 0$ contradicting w_2/w_1 not being real. Similar reasoning applies to the other pairs of points in $\{w_1/2, w_2/2, (w_1 + w_2)/2\}$. For example, consider $(w_1 + w_2)/2$ and choose P_a such that its boundary contains no zeros of the elliptic function, $z \rightarrow \wp(z) - \wp((w_1 + w_2)/2)$ and P_a contains no poles of $\wp$ on its interior other than 0. Then if $\wp(w_2/2) = \wp((w_1 + w_2)/2)$, it follows from Theorem 34.1.7 on Page 1006 $w_2/2$ and $(w_1 + w_2)/2$ are the only two zeros of $\wp(z) - \wp((w_1 + w_2)/2)$ on P_a and by Corollary 34.1.9 on Page 1008

$$\frac{w_1 + w_1 + w_2}{2} = aw_1 + bw_2 \in M$$

for some integers a, b which leads to the same contradiction as before about w_1/w_2 not being real. The other cases are similar. This proves the lemma.

Lemma 34.1.14 proves the e_i are distinct. Number the e_i such that

$$e_1 = \wp(w_1/2), e_2 = \wp(w_2/2)$$

and

$$e_3 = \wp((w_1 + w_2)/2).$$

To summarize, it has been shown that for complex numbers, w_1 and w_2 with w_2/w_1 not real, an elliptic function, $\wp$ has been defined. Denote this function as

$\wp(z) = \wp(z, w_1, w_2)$. This in turn determined numbers, e_i as described above. Thus these numbers depend on w_1 and w_2 and as described above,

$$\begin{aligned} e_1(w_1, w_2) &= \wp\left(\frac{w_1}{2}, w_1, w_2\right), \quad e_2(w_1, w_2) = \wp\left(\frac{w_2}{2}, w_1, w_2\right) \\ e_3(w_1, w_2) &= \wp\left(\frac{w_1 + w_2}{2}, w_1, w_2\right). \end{aligned}$$

Therefore, using the formula for $\wp$, 34.1.9,

$$\wp(z) \equiv \frac{1}{z^2} + \sum_{w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

you see that if the two periods w_1 and w_2 are replaced with tw_1 and tw_2 respectively, then

$$e_i(tw_1, tw_2) = t^{-2} e_i(w_1, w_2).$$

Let τ denote the complex number which equals the ratio, w_2/w_1 which was assumed in all this to not be real. Then

$$e_i(w_1, w_2) = w_1^{-2} e_i(1, \tau)$$

Now define the function, $\lambda(\tau)$

$$\lambda(\tau) \equiv \frac{e_3(1, \tau) - e_2(1, \tau)}{e_1(1, \tau) - e_2(1, \tau)} \left(= \frac{e_3(w_1, w_2) - e_2(w_1, w_2)}{e_1(w_1, w_2) - e_2(w_1, w_2)} \right). \quad (34.1.10)$$

This function is meromorphic for $\text{Im } \tau > 0$ or for $\text{Im } \tau < 0$. However, since the denominator is never equal to zero the function must actually be analytic on both the upper half plane and the lower half plane. It never is equal to 0 because $e_3 \neq e_2$ and it never equals 1 because $e_3 \neq e_1$. This is stated as an observation.

Observation 34.1.15 *The function, $\lambda(\tau)$ is analytic for τ in the upper half plane and never assumes the values 0 and 1.*

This is a very interesting function. Consider what happens when

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and the matrix is unimodular. By Theorem 34.1.12 on Page 1010 $\{w'_1, w'_2\}$ is just another basis for the same module of periods. Therefore, $\wp(z, w_1, w_2) = \wp(z, w'_1, w'_2)$ because both are defined as sums over the same values of w , just in different order which does not matter because of the absolute convergence of the sums on compact subsets of $\mathbb{C}$. Since $\wp$ is unchanged, it follows $\wp'(z)$ is also unchanged and so the numbers, e_i are also the same. However, they might be permuted in which case the function $\lambda(\tau)$ defined above would change. What would it take for $\lambda(\tau)$ to not change? In other words, for which unimodular transformations will λ be left

unchanged? This happens if and only if no permuting takes place for the e_i . This occurs if $\wp\left(\frac{w_1}{2}\right) = \wp\left(\frac{w'_1}{2}\right)$ and $\wp\left(\frac{w_2}{2}\right) = \wp\left(\frac{w'_2}{2}\right)$. If

$$\frac{w'_1}{2} - \frac{w_1}{2} \in M, \quad \frac{w'_2}{2} - \frac{w_2}{2} \in M$$

then $\wp\left(\frac{w_1}{2}\right) = \wp\left(\frac{w'_1}{2}\right)$ and so e_1 will be unchanged and similarly for e_2 and e_3 . This occurs exactly when

$$\frac{1}{2}((a-1)w_1 + bw_2) \in M, \quad \frac{1}{2}(cw_1 + (d-1)w_2) \in M.$$

This happens if a and d are odd and if b and c are even. Of course the stylish way to say this is

$$a \equiv 1 \pmod{2}, \quad d \equiv 1 \pmod{2}, \quad b \equiv 0 \pmod{2}, \quad c \equiv 0 \pmod{2}. \quad (34.1.11)$$

This has shown that for unimodular transformations satisfying 34.1.11 λ is unchanged. Letting τ be defined as above,

$$\tau' = \frac{w'_2}{w'_1} \equiv \frac{cw_1 + dw_2}{aw_1 + bw_2} = \frac{c + d\tau}{a + b\tau}.$$

Thus for unimodular transformations, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying 34.1.11, or more succinctly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \quad (34.1.12)$$

it follows that

$$\lambda\left(\frac{c + d\tau}{a + b\tau}\right) = \lambda(\tau). \quad (34.1.13)$$

Furthermore, this is the only way this can happen.

Lemma 34.1.16 $\lambda(\tau) = \lambda(\tau')$ if and only if

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where 34.1.12 holds.

Proof: It only remains to verify that if $\wp(w'_1/2) = \wp(w_1/2)$ then it is necessary that

$$\frac{w'_1}{2} - \frac{w_1}{2} \in M$$

with a similar requirement for w_2 and w'_2 . If $\frac{w'_1}{2} - \frac{w_1}{2} \notin M$, then there exist integers, m, n such that

$$\frac{-w'_1}{2} + mw_1 + nw_2$$

is in the interior of P_0 , the period parallelogram whose vertices are $0, w_1, w_1 + w_2$, and w_2 . Therefore, it is possible to choose small a such that P_a contains the pole, $0, \frac{w_1}{2}$, and $\frac{-w'_1}{2} + mw_1 + nw_2$ but no other poles of $\wp$ and in addition, ∂P_a^* contains no zeros of $z \rightarrow \wp(z) - \wp(\frac{w_1}{2})$. Then the order of this elliptic function is 2. By assumption, and the fact that $\wp$ is even,

$$\wp\left(\frac{-w'_1}{2} + mw_1 + nw_2\right) = \wp\left(\frac{-w'_1}{2}\right) = \wp\left(\frac{w'_1}{2}\right) = \wp\left(\frac{w_1}{2}\right).$$

It follows both $\frac{-w'_1}{2} + mw_1 + nw_2$ and $\frac{w_1}{2}$ are zeros of $\wp(z) - \wp(\frac{w_1}{2})$ and so by Theorem 34.1.7 on Page 1006 these are the only two zeros of this function in P_a . Therefore, from Corollary 34.1.9 on Page 1008

$$\frac{w_1}{2} - \frac{w'_1}{2} + mw_1 + nw_2 \in M$$

which shows $\frac{w_1}{2} - \frac{w'_1}{2} \in M$. This completes the proof of the lemma.

Note the condition in the lemma is equivalent to the condition 34.1.13 because you can relabel the coefficients. The message of either version is that the coefficient of τ in the numerator and denominator is odd while the constant in the numerator and denominator is even.

Next, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}$ and therefore,

$$\lambda\left(\frac{2+\tau}{1}\right) = \lambda(\tau + 2) = \lambda(\tau). \quad (34.1.14)$$

Thus λ is periodic of period 2.

Thus λ leaves invariant a certain subgroup of the unimodular group. According to the next definition, λ is an example of something called a modular function.

Definition 34.1.17 *When an analytic or meromorphic function is invariant under a group of linear transformations, it is called an automorphic function. A function which is automorphic with respect to a subgroup of the modular group is called a modular function or an elliptic modular function.*

Now consider what happens for some other unimodular matrices which are not congruent to the identity mod 2. This will yield other functional equations for λ in addition to the fact that λ is periodic of period 2. As before, these functional equations come about because $\wp$ is unchanged when you change the basis for M , the module of periods. In particular, consider the unimodular matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (34.1.15)$$

Consider the first of these. Thus

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_1 + w_2 \end{pmatrix}$$

Hence $\tau' = w'_2/w'_1 = (w_1 + w_2)/w_1 = 1 + \tau$. Then from the definition of λ ,

$$\begin{aligned}
 \lambda(\tau') &= \lambda(1 + \tau) \\
 &= \frac{\wp\left(\frac{w'_1 + w'_2}{2}\right) - \wp\left(\frac{w'_2}{2}\right)}{\wp\left(\frac{w'_1}{2}\right) - \wp\left(\frac{w'_2}{2}\right)} \\
 &= \frac{\wp\left(\frac{w_1 + w_2 + w_1}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)} \\
 &= \frac{\wp\left(\frac{w_2}{2} + w_1\right) - \wp\left(\frac{w_1 + w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)} \\
 &= \frac{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)} \\
 &= -\frac{\wp\left(\frac{w_1 + w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)} \\
 &= -\frac{\wp\left(\frac{w_1 + w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right) + \wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)} \\
 &= -\frac{\left(\frac{\wp\left(\frac{w_1 + w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)}{1 + \left(\frac{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1 + w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)} \\
 &= \frac{\left(\frac{\wp\left(\frac{w_1 + w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right)}{\left(\frac{\wp\left(\frac{w_1 + w_2}{2}\right) - \wp\left(\frac{w_2}{2}\right)}{\wp\left(\frac{w_1}{2}\right) - \wp\left(\frac{w_2}{2}\right)}\right) - 1} \\
 &= \frac{\lambda(\tau)}{\lambda(\tau) - 1}. \tag{34.1.16}
 \end{aligned}$$

Summarizing the important feature of the above,

$$\lambda(1 + \tau) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}. \tag{34.1.17}$$

Next consider the other unimodular matrix in 34.1.15. In this case $w'_1 = w_2$ and $w'_2 = w_1$. Therefore, $\tau' = w'_2/w'_1 = w_1/w_2 = 1/\tau$. Then

$$\begin{aligned}
 \lambda(\tau') &= \lambda(1/\tau) \\
 &= \frac{\wp\left(\frac{w'_1+w'_2}{2}\right) - \wp\left(\frac{w'_2}{2}\right)}{\wp\left(\frac{w'_1}{2}\right) - \wp\left(\frac{w'_2}{2}\right)} \\
 &= \frac{\wp\left(\frac{w_1+w_2}{2}\right) - \wp\left(\frac{w_1}{2}\right)}{\wp\left(\frac{w_2}{2}\right) - \wp\left(\frac{w_1}{2}\right)} \\
 &= \frac{e_3 - e_1}{e_2 - e_1} = -\frac{e_3 - e_2 + e_2 - e_1}{e_1 - e_2} \\
 &= -(\lambda(\tau) - 1) = -\lambda(\tau) + 1.
 \end{aligned} \tag{34.1.18}$$

You could try other unimodular matrices and attempt to find other functional equations if you like but this much will suffice here.

34.1.5 A Formula For λ

Recall the formula of Mittag-Leffler for $\cot(\pi\alpha)$ given in 33.2.15. For convenience, here it is.

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \frac{2\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha.$$

As explained in the derivation of this formula it can also be written as

$$\sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2 - n^2} = \pi \cot \pi\alpha.$$

Differentiating both sides yields

$$\begin{aligned}
 \pi^2 \csc^2(\pi\alpha) &= \sum_{n=-\infty}^{\infty} \frac{\alpha^2 + n^2}{(\alpha^2 - n^2)^2} \\
 &= \sum_{n=-\infty}^{\infty} \frac{(\alpha + n)^2 - 2\alpha n}{(\alpha + n)^2 (\alpha - n)^2} \\
 &= \sum_{n=-\infty}^{\infty} \frac{(\alpha + n)^2}{(\alpha + n)^2 (\alpha - n)^2} - \overbrace{\sum_{n=-\infty}^{\infty} \frac{2\alpha n}{(\alpha^2 - n^2)^2}}^{=0} \\
 &= \sum_{n=-\infty}^{\infty} \frac{1}{(\alpha - n)^2}.
 \end{aligned} \tag{34.1.19}$$

Now this formula can be used to obtain a formula for $\lambda(\tau)$. As pointed out above, λ depends only on the ratio w_2/w_1 and so it suffices to take $w_1 = 1$ and

$w_2 = \tau$. Thus

$$\lambda(\tau) = \frac{\wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right)}{\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right)}. \quad (34.1.20)$$

From the original formula for $\wp$,

$$\begin{aligned} & \wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\ &= \frac{1}{\left(\frac{1+\tau}{2}\right)^2} - \frac{1}{\left(\frac{\tau}{2}\right)^2} + \sum_{(k,m) \neq (0,0)} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + \left(-m - \frac{1}{2}\right)\tau\right)^2} - \frac{1}{\left(k + \left(-m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(\frac{1}{2} + \left(m + \frac{1}{2}\right)\tau - k\right)^2} - \frac{1}{\left(\left(m + \frac{1}{2}\right)\tau - k\right)^2}. \end{aligned} \quad (34.1.21)$$

Similarly,

$$\begin{aligned} & \wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\ &= \frac{1}{\left(\frac{1}{2}\right)^2} - \frac{1}{\left(\frac{\tau}{2}\right)^2} + \sum_{(k,m) \neq (0,0)} \frac{1}{\left(k - \frac{1}{2} + m\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} + m\tau\right)^2} - \frac{1}{\left(k + \left(m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(k - \frac{1}{2} - m\tau\right)^2} - \frac{1}{\left(k + \left(-m - \frac{1}{2}\right)\tau\right)^2} \\ &= \sum_{(k,m) \in \mathbb{Z}^2} \frac{1}{\left(\frac{1}{2} + m\tau - k\right)^2} - \frac{1}{\left(\left(m + \frac{1}{2}\right)\tau - k\right)^2}. \end{aligned} \quad (34.1.22)$$

Now use 34.1.19 to sum these over k . This yields,

$$\begin{aligned} & \wp\left(\frac{1+\tau}{2}\right) - \wp\left(\frac{\tau}{2}\right) \\ &= \sum_m \frac{\pi^2}{\sin^2\left(\pi\left(\frac{1}{2} + \left(m + \frac{1}{2}\right)\tau\right)\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} \\ &= \sum_m \frac{\pi^2}{\cos^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} \end{aligned}$$

and

$$\begin{aligned}\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau}{2}\right) &= \sum_m \frac{\pi^2}{\sin^2\left(\pi\left(\frac{1}{2} + m\tau\right)\right)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} \\ &= \sum_m \frac{\pi^2}{\cos^2(\pi m\tau)} - \frac{\pi^2}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)}.\end{aligned}$$

The following interesting formula for λ results.

$$\lambda(\tau) = \frac{\sum_m \frac{1}{\cos^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)} - \frac{1}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)}}{\sum_m \frac{1}{\cos^2(\pi m\tau)} - \frac{1}{\sin^2\left(\pi\left(m + \frac{1}{2}\right)\tau\right)}}. \quad (34.1.23)$$

From this it is obvious $\lambda(-\tau) = \lambda(\tau)$. Therefore, from 34.1.18,

$$-\lambda(\tau) + 1 = \lambda\left(\frac{1}{\tau}\right) = \lambda\left(\frac{-1}{\tau}\right) \quad (34.1.24)$$

(It is good to recall that λ has been defined for $\tau \notin \mathbb{R}$.)

34.1.6 Mapping Properties Of λ

The two functional equations, 34.1.24 and 34.1.17 along with some other properties presented above are of fundamental importance. For convenience, they are summarized here in the following lemma.

Lemma 34.1.18 *The following functional equations hold for λ .*

$$\lambda(1 + \tau) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}, 1 = \lambda(\tau) + \lambda\left(\frac{-1}{\tau}\right) \quad (34.1.25)$$

$$\lambda(\tau + 2) = \lambda(\tau), \quad (34.1.26)$$

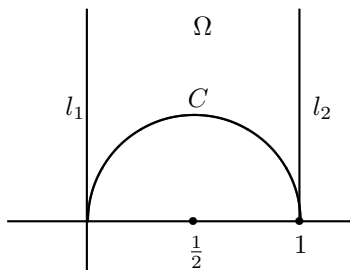
$\lambda(z) = \lambda(w)$ if and only if there exists a unimodular matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}$$

such that

$$w = \frac{az + b}{cz + d} \quad (34.1.27)$$

Consider the following picture.



In this picture, l_1 is the y axis and l_2 is the line, $x = 1$ while C is the top half of the circle centered at $(\frac{1}{2}, 0)$ which has radius $1/2$. Note the above formula implies λ has real values on l_1 which are between 0 and 1. This is because 34.1.23 implies

$$\begin{aligned}\lambda(ib) &= \frac{\sum_m \frac{1}{\cos^2(\pi(m+\frac{1}{2})ib)} - \frac{1}{\sin^2(\pi(m+\frac{1}{2})ib)}}{\sum_m \frac{1}{\cos^2(\pi mib)} - \frac{1}{\sin^2(\pi(m+\frac{1}{2})ib)}} \\ &= \frac{\sum_m \frac{1}{\cosh^2(\pi(m+\frac{1}{2})b)} + \frac{1}{\sinh^2(\pi(m+\frac{1}{2})b)}}{\sum_m \frac{1}{\cosh^2(\pi mb)} + \frac{1}{\sinh^2(\pi(m+\frac{1}{2})b)}} \in (0, 1). \quad (34.1.28)\end{aligned}$$

This follows from the observation that

$$\cos(ix) = \cosh(x), \quad \sin(ix) = i \sinh(x).$$

Thus it is clear from 34.1.28 that $\lim_{b \rightarrow 0+} \lambda(ib) = 1$.

Next I need to consider the behavior of $\lambda(\tau)$ as $\text{Im}(\tau) \rightarrow \infty$. From 34.1.23 listed here for convenience,

$$\lambda(\tau) = \frac{\sum_m \frac{1}{\cos^2(\pi(m+\frac{1}{2})\tau)} - \frac{1}{\sin^2(\pi(m+\frac{1}{2})\tau)}}{\sum_m \frac{1}{\cos^2(\pi m\tau)} - \frac{1}{\sin^2(\pi(m+\frac{1}{2})\tau)}}, \quad (34.1.29)$$

it follows

$$\begin{aligned}\lambda(\tau) &= \frac{\frac{1}{\cos^2(\pi(-\frac{1}{2})\tau)} - \frac{1}{\sin^2(\pi(-\frac{1}{2})\tau)} + \frac{1}{\cos^2(\pi\frac{1}{2}\tau)} - \frac{1}{\sin^2(\pi\frac{1}{2}\tau)} + A(\tau)}{1 + B(\tau)} \\ &= \frac{\frac{2}{\cos^2(\pi(\frac{1}{2})\tau)} - \frac{2}{\sin^2(\pi(\frac{1}{2})\tau)} + A(\tau)}{1 + B(\tau)} \quad (34.1.30)\end{aligned}$$

Where $A(\tau), B(\tau) \rightarrow 0$ as $\text{Im}(\tau) \rightarrow \infty$. I took out the $m = 0$ term involving $1/\cos^2(\pi m\tau)$ in the denominator and the $m = -1$ and $m = 0$ terms in the numerator of 34.1.29. In fact, $e^{-i\pi(a+ib)}A(a+ib), e^{-i\pi(a+ib)}B(a+ib)$ converge to zero uniformly in a as $b \rightarrow \infty$.

Lemma 34.1.19 *For A, B defined in 34.1.30, $e^{-i\pi(a+ib)}C(a+ib) \rightarrow 0$ uniformly in a for $C = A, B$.*

Proof: From 34.1.23,

$$e^{-i\pi\tau}A(\tau) = \sum_{\substack{m \neq 0 \\ m \neq -1}} \frac{e^{-i\pi\tau}}{\cos^2(\pi(m+\frac{1}{2})\tau)} - \frac{e^{-i\pi\tau}}{\sin^2(\pi(m+\frac{1}{2})\tau)}$$

Now let $\tau = a + ib$. Then letting $\alpha_m = \pi(m + \frac{1}{2})$,

$$\begin{aligned}\cos(\alpha_m a + i\alpha_m b) &= \cos(\alpha_m a) \cosh(\alpha_m b) - i \sinh(\alpha_m b) \sin(\alpha_m a) \\ \sin(\alpha_m a + i\alpha_m b) &= \sin(\alpha_m a) \cosh(\alpha_m b) + i \cos(\alpha_m a) \sinh(\alpha_m b)\end{aligned}$$

Therefore,

$$\begin{aligned} |\cos^2(\alpha_m a + i\alpha_m b)| &= \cos^2(\alpha_m a) \cosh^2(\alpha_m b) + \sinh^2(\alpha_m b) \sin^2(\alpha_m a) \\ &\geq \sinh^2(\alpha_m b). \end{aligned}$$

Similarly,

$$\begin{aligned} |\sin^2(\alpha_m a + i\alpha_m b)| &= \sin^2(\alpha_m a) \cosh^2(\alpha_m b) + \cos^2(\alpha_m a) \sinh^2(\alpha_m b) \\ &\geq \sinh^2(\alpha_m b). \end{aligned}$$

It follows that for $\tau = a + ib$ and b large

$$\begin{aligned} &|e^{-i\pi\tau} A(\tau)| \\ &\leq \sum_{\substack{m \neq 0 \\ m \neq -1}} \frac{2e^{\pi b}}{\sinh^2(\pi(m + \frac{1}{2})b)} \\ &\leq \sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2(\pi(m + \frac{1}{2})b)} + \sum_{m=-\infty}^{-2} \frac{2e^{\pi b}}{\sinh^2(\pi(m + \frac{1}{2})b)} \\ &= 2 \sum_{m=1}^{\infty} \frac{2e^{\pi b}}{\sinh^2(\pi(m + \frac{1}{2})b)} = 4 \sum_{m=1}^{\infty} \frac{e^{\pi b}}{\sinh^2(\pi(m + \frac{1}{2})b)} \end{aligned}$$

Now a short computation shows

$$\frac{\frac{e^{\pi b}}{\sinh^2(\pi(m+1+\frac{1}{2})b)}}{\frac{e^{\pi b}}{\sinh^2(\pi(m+\frac{1}{2})b)}} = \frac{\sinh^2(\pi(m + \frac{1}{2})b)}{\sinh^2(\pi(m + \frac{3}{2})b)} \leq \frac{1}{e^{3\pi b}}.$$

Therefore, for $\tau = a + ib$,

$$\begin{aligned} |e^{-i\pi\tau} A(\tau)| &\leq 4 \frac{e^{\pi b}}{\sinh(\frac{3\pi b}{2})} \sum_{m=1}^{\infty} \left(\frac{1}{e^{3\pi b}} \right)^m \\ &\leq 4 \frac{e^{\pi b}}{\sinh(\frac{3\pi b}{2})} \frac{1/e^{3\pi b}}{1 - (1/e^{3\pi b})} \end{aligned}$$

which converges to zero as $b \rightarrow \infty$. Similar reasoning will establish the claim about $B(\tau)$. This proves the lemma.

Lemma 34.1.20 $\lim_{b \rightarrow \infty} \lambda(a + ib) e^{-i\pi(a+ib)} = 16$ uniformly in $a \in \mathbb{R}$.

Proof: From 34.1.30 and Lemma 34.1.19, this lemma will be proved if it is shown

$$\lim_{b \rightarrow \infty} \left(\frac{2}{\cos^2(\pi(\frac{1}{2})(a + ib))} - \frac{2}{\sin^2(\pi(\frac{1}{2})(a + ib))} \right) e^{-i\pi(a+ib)} = 16$$

uniformly in $a \in \mathbb{R}$. Let $\tau = a + ib$ to simplify the notation. Then the above expression equals

$$\begin{aligned}
 & \left(\frac{8}{(e^{i\frac{\pi}{2}\tau} + e^{-i\frac{\pi}{2}\tau})^2} + \frac{8}{(e^{i\frac{\pi}{2}\tau} - e^{-i\frac{\pi}{2}\tau})^2} \right) e^{-i\pi\tau} \\
 &= \left(\frac{8e^{i\pi\tau}}{(e^{i\pi\tau} + 1)^2} + \frac{8e^{i\pi\tau}}{(e^{i\pi\tau} - 1)^2} \right) e^{-i\pi\tau} \\
 &= \frac{8}{(e^{i\pi\tau} + 1)^2} + \frac{8}{(e^{i\pi\tau} - 1)^2} \\
 &= 16 \frac{1 + e^{2\pi i\tau}}{(1 - e^{2\pi i\tau})^2}.
 \end{aligned}$$

Now

$$\begin{aligned}
 \left| \frac{1 + e^{2\pi i\tau}}{(1 - e^{2\pi i\tau})^2} - 1 \right| &= \left| \frac{1 + e^{2\pi i\tau}}{(1 - e^{2\pi i\tau})^2} - \frac{(1 - e^{2\pi i\tau})^2}{(1 - e^{2\pi i\tau})^2} \right| \\
 &\leq \frac{|3e^{2\pi i\tau} - e^{4\pi i\tau}|}{(1 - e^{-2\pi b})^2} \leq \frac{3e^{-2\pi b} + e^{-4\pi b}}{(1 - e^{-2\pi b})^2}
 \end{aligned}$$

and this estimate proves the lemma.

Corollary 34.1.21 $\lim_{b \rightarrow \infty} \lambda(a + ib) = 0$ uniformly in $a \in \mathbb{R}$. Also $\lambda(ib)$ for $b > 0$ is real and is between 0 and 1, λ is real on the line, l_2 and on the curve, C and $\lim_{b \rightarrow 0+} \lambda(1 + ib) = -\infty$.

Proof: From Lemma 34.1.20,

$$\left| \lambda(a + ib) e^{-i\pi(a+ib)} - 16 \right| < 1$$

for all a provided b is large enough. Therefore, for such b ,

$$|\lambda(a + ib)| \leq 17e^{-\pi b}.$$

34.1.28 proves the assertion about $\lambda(-bi)$ real.

By the first part, $\lim_{b \rightarrow \infty} |\lambda(ib)| = 0$. Now from 34.1.24

$$\lim_{b \rightarrow 0+} \lambda(ib) = \lim_{b \rightarrow 0+} \left(1 - \lambda\left(\frac{-1}{ib}\right) \right) = \lim_{b \rightarrow 0+} \left(1 - \lambda\left(\frac{i}{b}\right) \right) = 1. \quad (34.1.31)$$

by Corollary 34.1.21.

Next consider the behavior of λ on line l_2 in the above picture. From 34.1.17 and 34.1.28,

$$\lambda(1 + ib) = \frac{\lambda(ib)}{\lambda(ib) - 1} < 0$$

and so as $b \rightarrow 0+$ in the above, $\lambda(1 + ib) \rightarrow -\infty$.

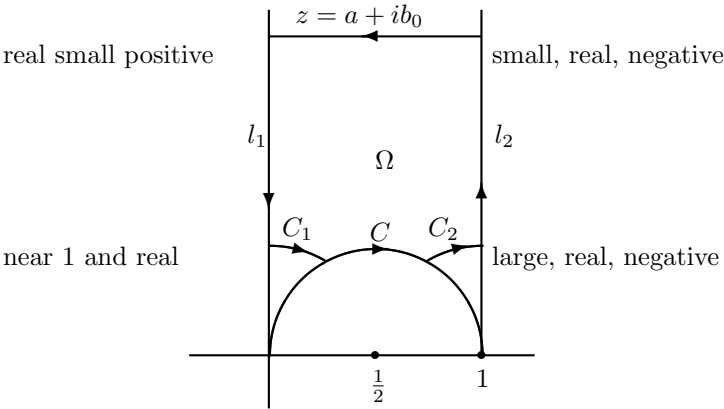
It is left as an exercise to show that the map $\tau \rightarrow 1 - \frac{1}{\tau}$ maps l_2 onto the curve, C . Therefore, by 34.1.25, for $\tau \in l_2$,

$$\lambda\left(1 - \frac{1}{\tau}\right) = \frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right) - 1} \tag{34.1.32}$$

$$= \frac{1 - \lambda(\tau)}{(1 - \lambda(\tau)) - 1} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \in \mathbb{R} \tag{34.1.33}$$

It follows λ is real on the boundary of Ω in the above picture. This proves the corollary.

Now, following Alfors [2], cut off Ω by considering the horizontal line segment, $z = a + ib_0$ where b_0 is very large and positive and $a \in [0, 1]$. Also cut Ω off by the images of this horizontal line, under the transformations $z = \frac{1}{\tau}$ and $z = 1 - \frac{1}{\tau}$. These are arcs of circles because the two transformations are fractional linear transformations. It is left as an exercise for you to verify these arcs are situated as shown in the following picture. The important thing to notice is that for b_0 large the points of these circles are close to the origin and $(1, 0)$ respectively. The following picture is a summary of what has been obtained so far on the mapping by λ .



In the picture, the descriptions are of λ acting on points of the indicated boundary of Ω . Consider the oriented contour which results from $\lambda(z)$ as z moves first up l_2 as indicated, then along the line $z = a + ib$ and then down l_1 and then along C_1 to C and along C till C_2 and then along C_2 to l_2 . As indicated in the picture, this involves going from a large negative real number to a small negative real number and then over a smooth curve which stays small to a real positive number and from there to a real number near 1. $\lambda(z)$ stays fairly near 1 on C_1 provided b_0 is large so that the circle, C_1 has very small radius. Then along C , $\lambda(z)$ is real until it hits C_2 . What about the behavior of λ on C_2 ? For $z \in C_2$, it follows from the definition of C_2 that $z = 1 - \frac{1}{\tau}$ where τ is on the line, $a + ib_0$. Therefore, by Lemma 34.1.20,

34.1.17, and 34.1.24

$$\begin{aligned}\lambda(z) &= \lambda\left(1 - \frac{1}{\tau}\right) = \frac{\lambda\left(\frac{-1}{\tau}\right)}{\lambda\left(\frac{-1}{\tau}\right) - 1} = \frac{\lambda\left(\frac{1}{\tau}\right)}{\lambda\left(\frac{1}{\tau}\right) - 1} \\ &= \frac{1 - \lambda(\tau)}{(1 - \lambda(\tau)) - 1} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} = 1 - \frac{1}{\lambda(\tau)}\end{aligned}$$

which is approximately equal to

$$1 - \frac{1}{16e^{i\pi(a+ib_0)}} = 1 - \frac{e^{\pi b_0} e^{-ia\pi}}{16}.$$

These points are essentially on a large half circle in the upper half plane which has radius approximately $\frac{e^{\pi b_0}}{16}$.

Now let $w \in \mathbb{C}$ with $\text{Im}(w) \neq 0$. Then for b_0 large enough, the motion over the boundary of the truncated region indicated in the above picture results in λ tracing out a large simple closed curve oriented in the counter clockwise direction which includes w on its interior if $\text{Im}(w) > 0$ but which excludes w if $\text{Im}(w) < 0$.

Theorem 34.1.22 *Let Ω be the domain described above. Then λ maps Ω one to one and onto the upper half plane of $\mathbb{C}$, $\{z \in \mathbb{C} \text{ such that } \text{Im}(z) > 0\}$. Also, the line $\lambda(l_1) = (0, 1)$, $\lambda(l_2) = (-\infty, 0)$, and $\lambda(C) = (1, \infty)$.*

Proof: Let $\text{Im}(w) > 0$ and denote by γ the oriented contour described above and illustrated in the above picture. Then the winding number of $\lambda \circ \gamma$ about w equals 1. Thus

$$\frac{1}{2\pi i} \int_{\lambda \circ \gamma} \frac{1}{z - w} dz = 1.$$

But, splitting the contour integrals into l_2 , the top line, l_1 , C_1 , C , and C_2 and changing variables on each of these, yields

$$1 = \frac{1}{2\pi i} \int_{\gamma} \frac{\lambda'(z)}{\lambda(z) - w} dz$$

and by the theorem on counting zeros, Theorem 28.6.1 on Page 844, the function, $z \rightarrow \lambda(z) - w$ has exactly one zero inside the truncated Ω . However, this shows this function has exactly one zero inside Ω because b_0 was arbitrary as long as it is sufficiently large. Since w was an arbitrary element of the upper half plane, this verifies the first assertion of the theorem. The remaining claims follow from the above description of λ , in particular the estimate for λ on C_2 . This proves the theorem.

Note also that the argument in the above proof shows that if $\text{Im}(w) < 0$, then w is not in $\lambda(\Omega)$. However, if you consider the reflection of Ω about the y axis, then it will follow that λ maps this set one to one onto the lower half plane. The argument will make significant use of Theorem 28.6.3 on Page 846 which is stated here for convenience.

Theorem 34.1.23 Let $f : B(a, R) \rightarrow \mathbb{C}$ be analytic and let

$$f(z) - \alpha = (z - a)^m g(z), \quad \infty > m \geq 1$$

where $g(z) \neq 0$ in $B(a, R)$. ($f(z) - \alpha$ has a zero of order m at $z = a$.) Then there exist $\varepsilon, \delta > 0$ with the property that for each z satisfying $0 < |z - \alpha| < \delta$, there exist points,

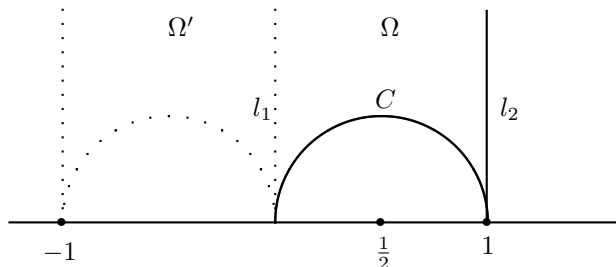
$$\{a_1, \dots, a_m\} \subseteq B(a, \varepsilon),$$

such that

$$f^{-1}(z) \cap B(a, \varepsilon) = \{a_1, \dots, a_m\}$$

and each a_k is a zero of order 1 for the function $f(\cdot) - z$.

Corollary 34.1.24 Let Ω be the region above. Consider the set of points, $Q = \overline{\Omega} \cup \Omega' \setminus \{0, 1\}$ described by the following picture.



Then $\lambda(Q) = \mathbb{C} \setminus \{0, 1\}$. Also $\lambda'(z) \neq 0$ for every z in $\cup_{k=-\infty}^{\infty} (Q + 2k) \equiv H$.

Proof: By Theorem 34.1.22, this will be proved if it can be shown that $\lambda(\Omega') = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$. Consider λ_1 defined on Ω' by

$$\lambda_1(x + iy) \equiv \overline{\lambda(-x + iy)}.$$

Claim: λ_1 is analytic.

Proof of the claim: You just verify the Cauchy Riemann equations. Letting $\lambda(x + iy) = u(x, y) + iv(x, y)$,

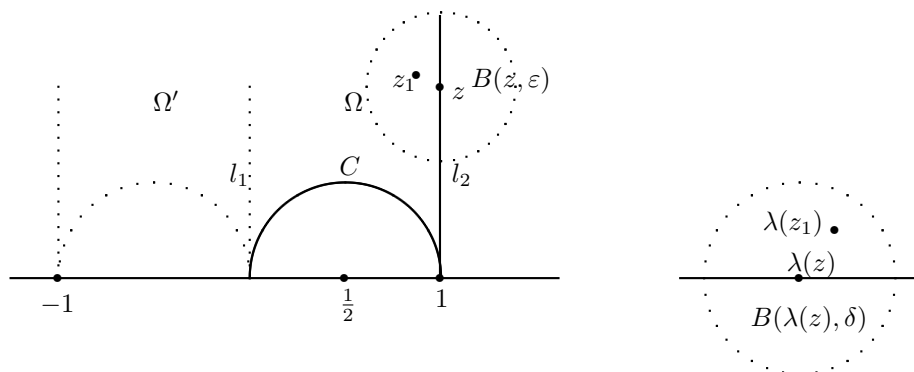
$$\begin{aligned} \lambda_1(x + iy) &= u(-x, y) - iv(-x, y) \\ &\equiv u_1(x, y) + iv(x, y). \end{aligned}$$

Then $u_{1x}(x, y) = -u_x(-x, y)$ and $v_{1y}(x, y) = -v_y(-x, y) = -u_x(-x, y)$ since λ is analytic. Thus $u_{1x} = v_{1y}$. Next, $u_{1y}(x, y) = u_y(-x, y)$ and $v_{1x}(x, y) = v_x(-x, y) = -u_y(-x, y)$ and so $u_{1y} = -v_x$.

Now recall that on l_1 , λ takes real values. Therefore, $\lambda_1 = \lambda$ on l_1 , a set with a limit point. It follows $\lambda = \lambda_1$ on $\Omega' \cup \Omega$. By Theorem 34.1.22 λ maps Ω one to one onto the upper half plane. Therefore, from the definition of $\lambda_1 = \lambda$, it follows λ maps Ω' one to one onto the lower half plane as claimed. This has shown that λ

is one to one on $\Omega \cup \Omega'$. This also verifies from Theorem 28.6.3 on Page 846 that $\lambda' \neq 0$ on $\Omega \cup \Omega'$.

Now consider the lines l_2 and C . If $\lambda'(z) = 0$ for $z \in l_2$, a contradiction can be obtained. Pick such a point. If $\lambda'(z) = 0$, then z is a zero of order $m \geq 2$ of the function, $\lambda - \lambda(z)$. Then by Theorem 28.6.3 there exist $\delta, \varepsilon > 0$ such that if $w \in B(\lambda(z), \delta)$, then $\lambda^{-1}(w) \cap B(z, \varepsilon)$ contains at least m points.



In particular, for $z_1 \in \Omega \cap B(z, \varepsilon)$ sufficiently close to z , $\lambda(z_1) \in B(\lambda(z), \delta)$ and so the function $\lambda - \lambda(z_1)$ has at least two distinct zeros. These zeros must be in $B(z, \varepsilon) \cap \Omega$ because $\lambda(z_1)$ has positive imaginary part and the points on l_2 are mapped by λ to a real number while the points of $B(z, \varepsilon) \setminus \overline{\Omega}$ are mapped by λ to the lower half plane thanks to the relation, $\lambda(z+2) = \lambda(z)$. This contradicts λ one to one on Ω . Therefore, $\lambda' \neq 0$ on l_2 . Consider C . Points on C are of the form $1 - \frac{1}{\tau}$ where $\tau \in l_2$. Therefore, using 34.1.33,

$$\lambda \left(1 - \frac{1}{\tau} \right) = \frac{\lambda(\tau) - 1}{\lambda(\tau)}.$$

Taking the derivative of both sides,

$$\lambda' \left(1 - \frac{1}{\tau} \right) \left(\frac{1}{\tau^2} \right) = \frac{\lambda'(\tau)}{\lambda(\tau)^2} \neq 0.$$

Since λ is periodic of period 2 it follows $\lambda'(z) \neq 0$ for all $z \in \cup_{k=-\infty}^{\infty} (Q + 2k)$.

Lemma 34.1.25 *If $\text{Im}(\tau) > 0$ then there exists a unimodular $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that*

$$\frac{c + d\tau}{a + b\tau}$$

is contained in the interior of Q . In fact, $\left| \frac{c+d\tau}{a+b\tau} \right| \geq 1$ and

$$-1/2 \leq \operatorname{Re} \left(\frac{c+d\tau}{a+b\tau} \right) \leq 1/2.$$

Proof: Letting a basis for the module of periods of $\wp$ be $\{1, \tau\}$, it follows from Theorem 34.1.2 on Page 1004 that there exists a basis for the same module of periods, $\{w'_1, w'_2\}$ with the property that for $\tau' = w'_2/w'_1$

$$|\tau'| \geq 1, \quad \frac{-1}{2} \leq \operatorname{Re} \tau' \leq \frac{1}{2}.$$

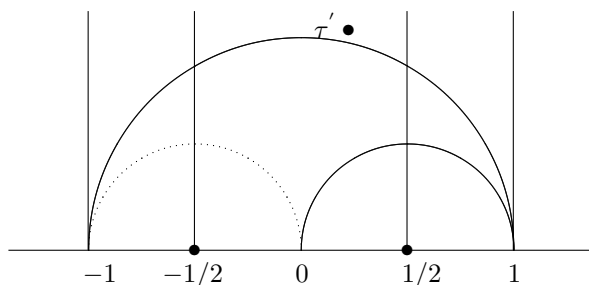
Since this is a basis for the same module of periods, there exists a unimodular matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}.$$

Hence,

$$\tau' = \frac{w'_2}{w'_1} = \frac{c+d\tau}{a+b\tau}.$$

Thus τ' is in the interior of H . In fact, it is on the interior of $\Omega' \cup \Omega \equiv Q$.



34.1.7 A Short Review And Summary

With this lemma, it is easy to extend Corollary 34.1.24. First, a simple observation and review is a good idea. Recall that when you change the basis for the module of periods, the Weierstrass $\wp$ function does not change and so the set of e_i used in defining λ also do not change. Letting the new basis be $\{w'_1, w'_2\}$, it was shown that

$$\begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

for some unimodular transformation, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Letting $\tau = w_2/w_1$ and $\tau' = w'_2/w'_1$

$$\tau' = \frac{c + d\tau}{a + b\tau} \equiv \phi(\tau)$$

Now as discussed earlier

$$\begin{aligned} \lambda(\tau') &= \lambda(\phi(\tau)) \equiv \frac{\wp\left(\frac{w'_1 + w'_2}{2}\right) - \wp\left(\frac{w'_2}{2}\right)}{\wp\left(\frac{w'_1}{2}\right) - \wp\left(\frac{w'_2}{2}\right)} \\ &= \frac{\wp\left(\frac{1+\tau'}{2}\right) - \wp\left(\frac{\tau'}{2}\right)}{\wp\left(\frac{1}{2}\right) - \wp\left(\frac{\tau'}{2}\right)} \end{aligned}$$

These numbers in the above fraction must be the same as $\wp\left(\frac{1+\tau}{2}\right)$, $\wp\left(\frac{\tau}{2}\right)$, and $\wp\left(\frac{1}{2}\right)$ but they might occur differently. This is because $\wp$ does not change and these numbers are the zeros of a polynomial having coefficients involving only numbers and $\wp(z)$. It could happen for example that $\wp\left(\frac{1+\tau'}{2}\right) = \wp\left(\frac{\tau}{2}\right)$ in which case this would change the value of λ . In effect, you can keep track of all possibilities by simply permuting the e_i in the formula for $\lambda(\tau)$ given by $\frac{e_3 - e_2}{e_1 - e_2}$. Thus consider the following permutation table.

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{array}.$$

Corresponding to this list of 6 permutations, all possible formulas for $\lambda(\phi(\tau))$ can be obtained as follows. Letting $\tau' = \phi(\tau)$ where ϕ is a unimodular matrix corresponding to a change of basis,

$$\lambda(\tau') = \frac{e_3 - e_2}{e_1 - e_2} = \lambda(\tau) \quad (34.1.34)$$

$$\lambda(\tau') = \frac{e_1 - e_3}{e_2 - e_3} = \frac{e_3 - e_2 + e_2 - e_1}{e_3 - e_2} = 1 - \frac{1}{\lambda(\tau)} = \frac{\lambda(\tau) - 1}{\lambda(\tau)} \quad (34.1.35)$$

$$\begin{aligned} \lambda(\tau') &= \frac{e_2 - e_1}{e_3 - e_1} = - \left[\frac{e_3 - e_2 - (e_1 - e_2)}{e_1 - e_2} \right]^{-1} \\ &= -[\lambda(\tau) - 1]^{-1} = \frac{1}{1 - \lambda(\tau)} \end{aligned} \quad (34.1.36)$$

$$\begin{aligned} \lambda(\tau') &= \frac{e_3 - e_1}{e_2 - e_1} = - \left[\frac{e_3 - e_2 - (e_1 - e_2)}{e_1 - e_2} \right] \\ &= -[\lambda(\tau) - 1] = 1 - \lambda(\tau) \end{aligned} \quad (34.1.37)$$

$$\lambda(\tau') = \frac{e_2 - e_3}{e_1 - e_3} = \frac{e_3 - e_2}{e_3 - e_2 - (e_1 - e_2)} = \frac{1}{1 - \frac{1}{\lambda(\tau)}} = \frac{\lambda(\tau)}{\lambda(\tau) - 1} \quad (34.1.38)$$

$$\lambda(\tau') = \frac{e_1 - e_3}{e_3 - e_2} = \frac{1}{\lambda(\tau)} \quad (34.1.39)$$

Corollary 34.1.26 $\lambda'(\tau) \neq 0$ for all τ in the upper half plane, denoted by P_+ .

Proof: Let $\tau \in P_+$. By Lemma 34.1.25 there exists ϕ a unimodular transformation and τ' in the interior of Q such that $\tau' = \phi(\tau)$. Now from the definition of λ in terms of the e_i , there is at worst a permutation of the e_i and so it might be the case that $\lambda(\phi(\tau)) \neq \lambda(\tau)$ but it is the case that $\lambda(\phi(\tau)) = \xi(\lambda(\tau))$ where $\xi'(z) \neq 0$. Here ξ is one of the functions determined by 34.1.34 - 34.1.39. (Since $\lambda(\tau) \notin \{0, 1\}$, $\xi'(\lambda(\tau)) \neq 0$. This follows from the above possibilities for ξ listed above in 34.1.34 - 34.1.39.) All the possibilities are $\xi(z) =$

$$z, \frac{z-1}{z}, \frac{1}{1-z}, 1-z, \frac{z}{z-1}, \frac{1}{z}$$

and these are the same as the possibilities for ξ^{-1} . Therefore, $\lambda'(\phi(\tau))\phi'(\tau) = \xi'(\lambda(\tau))\lambda'(\tau)$ and so $\lambda'(\tau) \neq 0$ as claimed.

Now I will present a lemma which is of major significance. It depends on the remarkable mapping properties of the modular function and the monodromy theorem from analytic continuation. A review of the monodromy theorem will be listed here for convenience. First recall the definition of the concept of function elements and analytic continuation.

Definition 34.1.27 A function element is an ordered pair, (f, D) where D is an open ball and f is analytic on D . (f_0, D_0) and (f_1, D_1) are direct continuations of each other if $D_1 \cap D_0 \neq \emptyset$ and $f_0 = f_1$ on $D_1 \cap D_0$. In this case I will write $(f_0, D_0) \sim (f_1, D_1)$. A chain is a finite sequence, of disks, $\{D_0, \dots, D_n\}$ such that $D_{i-1} \cap D_i \neq \emptyset$. If (f_0, D_0) is a given function element and there exist function elements, (f_i, D_i) such that $\{D_0, \dots, D_n\}$ is a chain and $(f_{j-1}, D_{j-1}) \sim (f_j, D_j)$ then (f_n, D_n) is called the analytic continuation of (f_0, D_0) along the chain $\{D_0, \dots, D_n\}$. Now suppose γ is an oriented curve with parameter interval $[a, b]$ and there exists a chain, $\{D_0, \dots, D_n\}$ such that $\gamma^* \subseteq \cup_{k=1}^n D_k$, $\gamma(a)$ is the center of D_0 , $\gamma(b)$ is the center of D_n , and there is an increasing list of numbers in $[a, b]$, $a = s_0 < s_1 < \dots < s_n = b$ such that $\gamma([s_i, s_{i+1}]) \subseteq D_i$ and (f_n, D_n) is an analytic continuation of (f_0, D_0) along the chain. Then (f_n, D_n) is called an analytic continuation of (f_0, D_0) along the curve γ . (γ will always be a continuous curve. Nothing more is needed.)

Then the main theorem is the monodromy theorem listed next, Theorem 31.4.5 and its corollary on Page 933.

Theorem 34.1.28 Let Ω be a simply connected subset of $\mathbb{C}$ and suppose $(f, B(a, r))$ is a function element with $B(a, r) \subseteq \Omega$. Suppose also that this function element can be analytically continued along every curve through a . Then there exists G analytic on Ω such that G agrees with f on $B(a, r)$.

Here is the lemma.

Lemma 34.1.29 *Let λ be the modular function defined on P_+ the upper half plane. Let V be a simply connected region in $\mathbb{C}$ and let $f : V \rightarrow \mathbb{C} \setminus \{0, 1\}$ be analytic and nonconstant. Then there exists an analytic function, $g : V \rightarrow P_+$ such that $\lambda \circ g = f$.*

Proof: Let $a \in V$ and choose r_0 small enough that $f(B(a, r_0))$ contains neither 0 nor 1. You need only let $B(a, r_0) \subseteq V$. Now there exists a unique point in Q, τ_0 such that $\lambda(\tau_0) = f(a)$. By Corollary 34.1.24, $\lambda'(\tau_0) \neq 0$ and so by the open mapping theorem, Theorem 28.6.3 on Page 846, There exists $B(\tau_0, R_0) \subseteq P_+$ such that λ is one to one on $B(\tau_0, R_0)$ and has a continuous inverse. Then picking r_0 still smaller, it can be assumed $f(B(a, r_0)) \subseteq \lambda(B(\tau_0, R_0))$. Thus there exists a local inverse for λ , λ_0^{-1} defined on $f(B(a, r_0))$ having values in $B(\tau_0, R_0) \cap \lambda^{-1}(f(B(a, r_0)))$. Then defining $g_0 \equiv \lambda_0^{-1} \circ f$, $(g_0, B(a, r_0))$ is a function element. I need to show this can be continued along every curve starting at a in such a way that each function in each function element has values in P_+ .

Let $\gamma : [\alpha, \beta] \rightarrow V$ be a continuous curve starting at $a, (\gamma(\alpha) = a)$ and suppose that if $t < T$ there exists a nonnegative integer m and a function element $(g_m, B(\gamma(t), r_m))$ which is an analytic continuation of $(g_0, B(a, r_0))$ along γ where $g_m(\gamma(t)) \in P_+$ and each function in every function element for $j \leq m$ has values in P_+ . Thus for some small $T > 0$ this has been achieved.

Then consider $f(\gamma(T)) \in \mathbb{C} \setminus \{0, 1\}$. As in the first part of the argument, there exists a unique $\tau_T \in Q$ such that $\lambda(\tau_T) = f(\gamma(T))$ and for r small enough there is an analytic local inverse, λ_T^{-1} between $f(B(\gamma(T), r))$ and $\lambda^{-1}(f(B(\gamma(T), r))) \cap B(\tau_T, R_T) \subseteq P_+$ for some $R_T > 0$. By the assumption that the analytic continuation can be carried out for $t < T$, there exists $\{t_0, \dots, t_m = t\}$ and function elements $(g_j, B(\gamma(t_j), r_j)), j = 0, \dots, m$ as just described with $g_j(\gamma(t_j)) \in P_+, \lambda \circ g_j = f$ on $B(\gamma(t_j), r_j)$ such that for $t \in [t_m, T], \gamma(t) \in B(\gamma(T), r)$. Let

$$I = B(\gamma(t_m), r_m) \cap B(\gamma(T), r).$$

Then since λ_T^{-1} is a local inverse, it follows for all $z \in I$

$$\lambda(g_m(z)) = f(z) = \lambda(\lambda_T^{-1} \circ f(z))$$

Pick $z_0 \in I$. Then by Lemma 34.1.18 on Page 1024 there exists a unimodular mapping of the form

$$\phi(z) = \frac{az + b}{cz + d}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2}$$

such that

$$g_m(z_0) = \phi(\lambda_T^{-1} \circ f(z_0)).$$

Since both $g_m(z_0)$ and $\phi(\lambda_T^{-1} \circ f(z_0))$ are in the upper half plane, it follows $ad - cb = 1$ and ϕ maps the upper half plane to the upper half plane. Note the pole of

ϕ is real and all the sets being considered are contained in the upper half plane so ϕ is analytic where it needs to be.

Claim: For all $z \in I$,

$$g_m(z) = \phi \circ \lambda_T^{-1} \circ f(z). \quad (34.1.40)$$

Proof: For $z = z_0$ the equation holds. Let

$$A = \{z \in I : g_m(z) = \phi(\lambda_T^{-1} \circ f(z))\}.$$

Thus $z_0 \in I$. If $z \in I$ and if w is close enough to z , then $w \in I$ also and so both sides of 34.1.40 with w in place of z are in $\lambda_m^{-1}(f(I))$. But by construction, λ is one to one on this set and since λ is invariant with respect to ϕ ,

$$\lambda(g_m(w)) = \lambda(\lambda_T^{-1} \circ f(w)) = \lambda(\phi \circ \lambda_T^{-1} \circ f(w))$$

and consequently, $w \in A$. This shows A is open. But A is also closed in I because the functions are continuous. Therefore, $A = I$ and so 34.1.40 is obtained.

Letting $f(z) \in f(B(\gamma(T)), r)$,

$$\lambda(\phi(\lambda_T^{-1}(f(z)))) = \lambda(\lambda_T^{-1}(f(z))) = f(z)$$

and so $\phi \circ \lambda_T^{-1}$ is a local inverse for λ on $f(B(\gamma(T)), r)$. Let the new function element be $\left(\overbrace{\phi \circ \lambda_T^{-1} \circ f}^{g_{m+1}}, B(\gamma(T), r) \right)$. This has shown the initial function element can be continued along every curve through a .

By the monodromy theorem, there exists g analytic on V such that g has values in P_+ and $g = g_0$ on $B(a, r_0)$. By the construction, it also follows $\lambda \circ g = f$. This last claim is easy to see because $\lambda \circ g = f$ on $B(a, r_0)$, a set with a limit point so the equation holds for all $z \in V$. This proves the lemma.

34.2 The Picard Theorem Again

Having done all this work on the modular function which is important for its own sake, there is an easy proof of the Picard theorem. In fact, this is the way Picard did it in 1879. I will state it slightly differently since it is no trouble to do so, [32].

Theorem 34.2.1 *Let f be meromorphic on $\mathbb{C}$ and suppose f misses three distinct points, a, b, c . Then f is a constant function.*

Proof: Let $\phi(z) \equiv \frac{z-a}{z-c} \frac{b-c}{b-a}$. Then $\phi(c) = \infty$, $\phi(a) = 0$, and $\phi(b) = 1$. Now consider the function, $h = \phi \circ f$. Then h misses the three points $\infty, 0$, and 1 . Since h is meromorphic and does not have ∞ in its values, it must actually be analytic. Thus h is an entire function which misses the two values 0 and 1 . If h is not constant, then by Lemma 34.1.29 there exists a function, g analytic on $\mathbb{C}$ which has values

in the upper half plane, P_+ such that $\lambda \circ g = h$. However, g must be a constant because there exists ψ an analytic map on the upper half plane which maps the upper half plane to $B(0, 1)$. You can use the Riemann mapping theorem or more simply, $\psi(z) = \frac{z-i}{z+i}$. Thus $\psi \circ g$ equals a constant by Liouville's theorem. Hence g is a constant and so h must also be a constant because $\lambda(g(z)) = h(z)$. This proves f is a constant also. This proves the theorem.

34.3 Exercises

1. Show the set of modular transformations is a group. Also show those modular transformations which are congruent mod 2 to the identity as described above is a subgroup.
2. Suppose f is an elliptic function with period module M . If $\{w_1, w_2\}$ and $\{w'_1, w'_2\}$ are two bases, show that the resulting period parallelograms resulting from the two bases have the same area.
3. Given a module of periods with basis $\{w_1, w_2\}$ and letting a typical element of this module be denoted by w as described above, consider the product

$$\sigma(z) \equiv z \prod_{w \neq 0} \left(1 - \frac{z}{w}\right) e^{(z/w) + \frac{1}{2}(z/w)^2}.$$

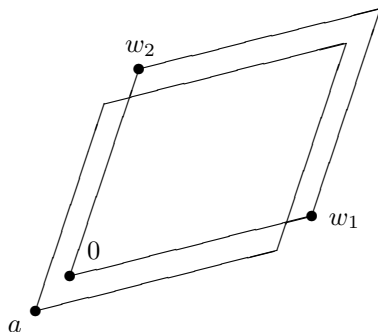
Show this product converges uniformly on compact sets, is an entire function, and satisfies

$$\sigma'(z) / \sigma(z) = \zeta(z)$$

where $\zeta(z)$ was defined above as a primitive of $\wp(z)$ and is given by

$$\zeta(z) = \frac{1}{z} + \sum_{w \neq 0} \frac{1}{z-w} + \frac{z}{w^2} + \frac{1}{w}.$$

4. Show $\zeta(z + w_i) = \zeta(z) + \eta_i$ where η_i is a constant.
5. Let P_a be the parallelogram shown in the following picture.



Show that $\frac{1}{2\pi i} \int_{\partial P_a} \zeta(z) dz = 1$ where the contour is taken once around the parallelogram in the counter clockwise direction. Next evaluate this contour integral directly to obtain Legendre's relation,

$$\eta_1 w_2 - \eta_2 w_1 = 2\pi i.$$

6. For σ defined in Problem 3, 4 explain the following steps. For $j = 1, 2$

$$\frac{\sigma'(z + w_j)}{\sigma(z + w_j)} = \zeta(z + w_j) = \zeta(z) + \eta_j = \frac{\sigma'(z)}{\sigma(z)} + \eta_j$$

Therefore, there exists a constant, C_j such that

$$\sigma(z + w_j) = C_j \sigma(z) e^{\eta_j z}.$$

Next show σ is an odd function, $(\sigma(-z) = -\sigma(z))$ and then let $z = -w_j/2$ to find $C_j = -e^{\frac{\eta_j w_j}{2}}$ and so

$$\sigma(z + w_j) = -\sigma(z) e^{\eta_j(z + \frac{w_j}{2})}. \quad (34.3.41)$$

7. Show any **even** elliptic function, f with periods w_1 and w_2 for which 0 is neither a pole nor a zero can be expressed in the form

$$f(z) = C \prod_{k=1}^n \frac{\wp(z) - \wp(a_k)}{\wp(z) - \wp(b_k)}$$

where C is some constant. Here $\wp$ is the Weierstrass function which comes from the two periods, w_1 and w_2 . **Hint:** You might consider the above function in terms of the poles and zeros on a period parallelogram and recall that an entire function which is elliptic is a constant.

8. Suppose f is any elliptic function with $\{w_1, w_2\}$ a basis for the module of periods. Using Theorem 34.1.8 and 34.3.41 show that there exists constants $a_1, \dots, a_n$ and $b_1, \dots, b_n$ such that for some constant C ,

$$f(z) = C \prod_{k=1}^n \frac{\sigma(z - a_k)}{\sigma(z - b_k)}.$$

Hint: You might try something like this: By Theorem 34.1.8, it follows that if $\{\alpha_k\}$ are the zeros and $\{b_k\}$ the poles in an appropriate period parallelogram, $\sum \alpha_k - \sum b_k$ equals a period. Replace α_k with a_k such that $\sum a_k - \sum b_k = 0$. Then use 34.3.41 to show that the given formula for f is bi periodic. Anyway, you try to arrange things such that the given formula has the same poles as f . Remember an entire elliptic function equals a constant.

9. Show that the map $\tau \rightarrow 1 - \frac{1}{\tau}$ maps l_2 onto the curve, C in the above picture on the mapping properties of λ .
10. Modify the proof of Theorem 34.1.22 to show that $\lambda(\Omega) \cap \{z \in \mathbb{C} : \text{Im}(z) < 0\} = \emptyset$.

Part IV

Sobolev Spaces

Weak Derivatives

35.1 Weak * Convergence

A very important sort of convergence in applications of functional analysis is the concept of weak or weak * convergence. It is important because it allows you to assert the existence of a convergent subsequence of a given bounded sequence. The only problem is the convergence is very weak so it does not tell you as much as you would like. Nevertheless, it is a very useful concept. The big theorems in the subject are the Eberlein Smulian theorem and the Banach Alaoglu theorem about the weak or weak * compactness of the closed unit balls in either a Banach space or its dual space. These theorems are proved in Yosida [67]. Here I will present a special case which turns out to be by far the most important in applications and it is not hard to get from the Riesz representation theorem for L^p . First I define weak and weak * convergence.

Definition 35.1.1 *Let X' be the dual of a Banach space X and let $\{x_n^*\}$ be a sequence of elements of X' . Then x_n^* converges weak * to x^* if and only if for all $x \in X$,*

$$\lim_{n \rightarrow \infty} x_n^*(x) = x^*(x).$$

A sequence in X , $\{x_n\}$ converges weakly to $x \in X$ if and only if for all $x^* \in X'$

$$\lim_{n \rightarrow \infty} x^*(x_n) = x^*(x).$$

The main result is contained in the following lemma.

Lemma 35.1.2 *Let X' be the dual of a Banach space, X and suppose X is separable. Then if $\{x_n^*\}$ is a bounded sequence in X' , there exists a weak * convergent subsequence.*

Proof: Let D be a dense countable set in X . Then the sequence, $\{x_n^*(x)\}$ is bounded for all x and in particular for all $x \in D$. Use the Cantor diagonal process to obtain a subsequence, still denoted by n such that $x_n^*(d)$ converges for each $d \in D$.

Now let $x \in X$ be completely arbitrary. In fact $\{x_n^*(x)\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be given and pick $d \in D$ such that for all n

$$|x_n^*(x) - x_n^*(d)| < \frac{\varepsilon}{3}.$$

This is possible because D is dense. By the first part of the proof, there exists N_ε such that for all $m, n > N_\varepsilon$,

$$|x_n^*(d) - x_m^*(d)| < \frac{\varepsilon}{3}.$$

Then for such m, n ,

$$\begin{aligned} |x_n^*(x) - x_m^*(x)| &\leq |x_n^*(x) - x_n^*(d)| + |x_n^*(d) - x_m^*(d)| \\ &\quad + |x_m^*(d) - x_m^*(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows $\{x_n^*(x)\}$ is a Cauchy sequence for all $x \in X$.

Now define $f(x) \equiv \lim_{n \rightarrow \infty} x_n^*(x)$. Since each x_n^* is linear, it follows f is also linear. In addition to this,

$$|f(x)| = \lim_{n \rightarrow \infty} |x_n^*(x)| \leq K \|x\|$$

where K is some constant which is larger than all the norms of the x_n^* . Such a constant exists because the sequence, $\{x_n^*\}$ was bounded. This proves the lemma.

The lemma implies the following important theorem.

Theorem 35.1.3 *Let Ω be a measurable subset of $\mathbb{R}^n$ and let $\{f_k\}$ be a bounded sequence in $L^p(\Omega)$ where $1 < p \leq \infty$. Then there exists a weak * convergent subsequence.*

Proof: Since $L^{p'}(\Omega)$ is separable, this follows from the Riesz representation theorem.

Note that from the Riesz representation theorem, it follows that if $p < \infty$, then the sequence converges weakly.

35.2 Test Functions And Weak Derivatives

In elementary courses in mathematics, functions are often thought of as things which have a formula associated with them and it is the formula which receives the most attention. For example, in beginning calculus courses the derivative of a function is defined as the limit of a difference quotient. You start with one function which tends to be identified with a formula and, by taking a limit, you get another formula for the derivative. A jump in abstraction occurs as soon as you encounter the derivative of a function of n variables where the derivative is defined as a certain linear transformation which is determined not by a formula but by what it does to vectors. When this is understood, it reduces to the usual idea in one dimension. The

idea of weak partial derivatives goes further in the direction of defining something in terms of what it does rather than by a formula, and extra generality is obtained when it is used. In particular, it is possible to differentiate almost anything if the notion of what is meant by the derivative is sufficiently weak. This has the advantage of allowing the consideration of the weak partial derivative of a function without having to agonize over the important question of existence but it has the disadvantage of not being able to say much about the derivative. Nevertheless, it is the idea of weak partial derivatives which makes it possible to use functional analytic techniques in the study of partial differential equations and it is shown in this chapter that the concept of weak derivative is useful for unifying the discussion of some very important theorems. Certain things which should be true are.

Let $\Omega \subseteq \mathbb{R}^n$. A distribution on Ω is defined to be a linear functional on $C_c^\infty(\Omega)$, called the space of test functions. The space of all such linear functionals will be denoted by $\mathcal{D}'(\Omega)$. Actually, more is sometimes done here. One imposes a topology on $C_c^\infty(\Omega)$ making it into a topological vector space, and when this has been done, $\mathcal{D}'(\Omega)$ is defined as the dual space of this topological vector space. To see this, consult the book by Yosida [67] or the book by Rudin [59].

Example: The space $L_{loc}^1(\Omega)$ may be considered as a subset of $\mathcal{D}'(\Omega)$ as follows.

$$f(\phi) \equiv \int_{\Omega} f(\mathbf{x}) \phi(\mathbf{x}) dx$$

for all $\phi \in C_c^\infty(\Omega)$. Recall that $f \in L_{loc}^1(\Omega)$ if $f\chi_K \in L^1(\Omega)$ whenever K is compact.

Example: $\delta_x \in \mathcal{D}'(\Omega)$ where $\delta_x(\phi) \equiv \phi(\mathbf{x})$.

It will be observed from the above two examples and a little thought that $\mathcal{D}'(\Omega)$ is truly enormous. The derivative of a distribution will be defined in such a way that it agrees with the usual notion of a derivative on those distributions which are also continuously differentiable functions. With this in mind, let f be the restriction to Ω of a smooth function defined on $\mathbb{R}^n$. Then $D_{x_i}f$ makes sense and for $\phi \in C_c^\infty(\Omega)$

$$D_{x_i}f(\phi) \equiv \int_{\Omega} D_{x_i}f(\mathbf{x}) \phi(\mathbf{x}) dx = - \int_{\Omega} f D_{x_i}\phi dx = -f(D_{x_i}\phi).$$

This motivates the following definition.

Definition 35.2.1 For $T \in \mathcal{D}'(\Omega)$

$$D_{x_i}T(\phi) \equiv -T(D_{x_i}\phi).$$

Of course one can continue taking derivatives indefinitely. Thus,

$$D_{x_i x_j}T \equiv D_{x_i}(D_{x_j}T)$$

and it is clear that all mixed partial derivatives are equal because this holds for the functions in $C_c^\infty(\Omega)$. In this weak sense, the derivative of almost anything exists, even functions that may be discontinuous everywhere. However the notion of “derivative” is very weak, hence the name, “weak derivatives”.

Example: Let $\Omega = \mathbb{R}$ and let

$$H(x) \equiv \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then

$$DH(\phi) = - \int H(x) \phi'(x) dx = \phi(0) = \delta_0(\phi).$$

Note that in this example, DH is not a function.

What happens when Df is a function?

Theorem 35.2.2 *Let $\Omega = (a, b)$ and suppose that f and Df are both in $L^1(a, b)$. Then f is equal to a continuous function a.e., still denoted by f and*

$$f(x) = f(a) + \int_a^x Df(t) dt.$$

In proving Theorem 35.2.2 the following lemma is useful.

Lemma 35.2.3 *Let $T \in \mathcal{D}^*(a, b)$ and suppose $DT = 0$. Then there exists a constant C such that*

$$T(\phi) = \int_a^b C\phi dx.$$

Proof: $T(D\phi) = 0$ for all $\phi \in C_c^\infty(a, b)$ from the definition of $DT = 0$. Let

$$\phi_0 \in C_c^\infty(a, b), \quad \int_a^b \phi_0(x) dx = 1,$$

and let

$$\psi_\phi(x) = \int_a^x [\phi(t) - \left(\int_a^b \phi(y) dy \right) \phi_0(t)] dt$$

for $\phi \in C_c^\infty(a, b)$. Thus $\psi_\phi \in C_c^\infty(a, b)$ and

$$D\psi_\phi = \phi - \left(\int_a^b \phi(y) dy \right) \phi_0.$$

Therefore,

$$\phi = D\psi_\phi + \left(\int_a^b \phi(y) dy \right) \phi_0$$

and so

$$T(\phi) = T(D\psi_\phi) + \left(\int_a^b \phi(y) dy \right) T(\phi_0) = \int_a^b T(\phi_0) \phi(y) dy.$$

Let $C = T\phi_0$. This proves the lemma.

Proof of Theorem 35.2.2 Since f and Df are both in $L^1(a, b)$,

$$Df(\phi) - \int_a^b Df(x) \phi(x) dx = 0.$$

Consider

$$f(\cdot) - \int_a^{(\cdot)} Df(t) dt$$

and let $\phi \in C_c^\infty(a, b)$.

$$\begin{aligned} & D \left(f(\cdot) - \int_a^{(\cdot)} Df(t) dt \right) (\phi) \\ & \equiv - \int_a^b f(x) \phi'(x) dx + \int_a^b \left(\int_a^x Df(t) dt \right) \phi'(x) dx \\ & = Df(\phi) + \int_a^b \int_t^b Df(t) \phi'(x) dx dt \\ & = Df(\phi) - \int_a^b Df(t) \phi(t) dt = 0. \end{aligned}$$

By Lemma 35.2.3, there exists a constant, C , such that

$$\left(f(\cdot) - \int_a^{(\cdot)} Df(t) dt \right) (\phi) = \int_a^b C \phi(x) dx$$

for all $\phi \in C_c^\infty(a, b)$. Thus

$$\int_a^b \left\{ \left(f(x) - \int_a^x Df(t) dt \right) - C \right\} \phi(x) dx = 0$$

for all $\phi \in C_c^\infty(a, b)$. It follows from Lemma 35.3.3 in the next section that

$$f(x) - \int_a^x Df(t) dt - C = 0 \text{ a.e. } x.$$

Thus let $f(a) = C$ and write

$$f(x) = f(a) + \int_a^x Df(t) dt.$$

This proves Theorem 35.2.2.

Theorem 35.2.2 says that

$$f(x) = f(a) + \int_a^x Df(t) dt$$

whenever it makes sense to write $\int_a^x Df(t) dt$, if Df is interpreted as a weak derivative. Somehow, this is the way it ought to be. It follows from the fundamental theorem of calculus that $f'(x)$ exists for a.e. x where the derivative is taken in the sense of a limit of difference quotients and $f'(x) = Df(x)$. This raises an interesting question. Suppose f is continuous on $[a, b]$ and $f'(x)$ exists in the classical sense for a.e. x . Does it follow that

$$f(x) = f(a) + \int_a^x f'(t) dt?$$

The answer is no. To see an example, consider Problem 4 on Page 542 which gives an example of a function which is continuous on $[0, 1]$, has a zero derivative for a.e. x but climbs from 0 to 1 on $[0, 1]$. Thus this function is not recovered from integrating its classical derivative.

In summary, if the notion of weak derivative is used, one can at least give meaning to the derivative of almost anything, the mixed partial derivatives are always equal, and, in one dimension, one can recover the function from integrating its derivative. None of these claims are true for the classical derivative. Thus weak derivatives are convenient and rule out pathologies.

35.3 Weak Derivatives In L_{loc}^p

Definition 35.3.1 Let U be an open set in $\mathbb{R}^n$. $f \in L_{loc}^p(U)$ if $f\chi_K \in L^p$ whenever K is a compact subset of U .

Definition 35.3.2 For $\alpha = (k_1, \dots, k_n)$ where the k_i are nonnegative integers, define

$$|\alpha| \equiv \sum_{i=1}^n |k_{x_i}|, \quad D^\alpha f(\mathbf{x}) \equiv \frac{\partial^{|\alpha|} f(\mathbf{x})}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}.$$

Also define ϕ_k to be a mollifier if $\text{spt}(\phi_k) \subseteq B(\mathbf{0}, \frac{1}{k})$, $\phi_k \geq 0$, $\int \phi_k dx = 1$, and $\phi_k \in C_c^\infty(B(\mathbf{0}, \frac{1}{k}))$. In the case a Greek letter like δ or ε is used as a subscript, it will mean $\text{spt}(\phi_\delta) \subseteq B(\mathbf{0}, \delta)$, $\phi_\delta \geq 0$, $\int \phi_\delta dx = 1$, and $\phi_\delta \in C_c^\infty(B(\mathbf{0}, \delta))$. You can always get a mollifier by letting $\phi \geq 0$, $\phi \in C_c^\infty(B(\mathbf{0}, 1))$, $\int \phi dx = 1$, and then defining $\phi_k(\mathbf{x}) \equiv k^n \phi(k\mathbf{x})$ or in the case of a Greek subscript, $\phi_\delta(\mathbf{x}) = \frac{1}{\delta^n} \phi(\frac{\mathbf{x}}{\delta})$.

Consider the case where u and $D^\alpha u$ for $|\alpha| = 1$ are each in $L_{loc}^p(\mathbb{R}^n)$. The next lemma is the one alluded to in the proof of Theorem 35.2.2.

Lemma 35.3.3 Suppose $f \in L_{loc}^1(U)$ and suppose

$$\int f \phi dx = 0$$

for all $\phi \in C_c^\infty(U)$. Then $f(\mathbf{x}) = 0$ a.e. $\mathbf{x}$.

Proof: Without loss of generality f is real valued. Let

$$E \equiv \{\mathbf{x} : f(\mathbf{x}) > \varepsilon\}$$

and let

$$E_m \equiv E \cap B(0, m).$$

Is $m(E_m) = 0$? If not, there exists an open set, V , and a compact set K satisfying

$$K \subseteq E_m \subseteq V \subseteq B(0, m), \quad m(V \setminus K) < 4^{-1}m(E_m),$$

$$\int_{V \setminus K} |f| dx < \varepsilon 4^{-1}m(E_m).$$

Let H and W be open sets satisfying

$$K \subseteq H \subseteq \overline{H} \subseteq W \subseteq \overline{W} \subseteq V$$

and let

$$\overline{H} \prec g \prec W$$

where the symbol, $\prec$, in the above implies $\text{spt}(g) \subseteq W$, g has all values in $[0, 1]$, and $g(\mathbf{x}) = 1$ on $\overline{H}$. Then let ϕ_δ be a mollifier and let $h \equiv g * \phi_\delta$ for δ small enough that

$$K \prec h \prec V.$$

Thus

$$\begin{aligned} 0 &= \int f h dx = \int_K f dx + \int_{V \setminus K} f h dx \\ &\geq \varepsilon m(K) - \varepsilon 4^{-1}m(E_m) \\ &\geq \varepsilon (m(E_m) - 4^{-1}m(E_m)) - \varepsilon 4^{-1}m(E_m) \\ &\geq 2^{-1}\varepsilon m(E_m). \end{aligned}$$

Therefore, $m(E_m) = 0$, a contradiction. Thus

$$m(E) \leq \sum_{m=1}^{\infty} m(E_m) = 0$$

and so, since $\varepsilon > 0$ is arbitrary,

$$m(\{\mathbf{x} : f(\mathbf{x}) > 0\}) = 0.$$

Similarly $m(\{\mathbf{x} : f(\mathbf{x}) < 0\}) = 0$. This proves the lemma.

This lemma allows the following definition.

Definition 35.3.4 Let U be an open subset of $\mathbb{R}^n$ and let $u \in L^1_{loc}(U)$. Then $D^\alpha u \in L^1_{loc}(U)$ if there exists a function $g \in L^1_{loc}(U)$, necessarily unique by Lemma 35.3.3, such that for all $\phi \in C_c^\infty(U)$,

$$\int_U g \phi dx = D^\alpha u(\phi) \equiv \int_U (-1)^{|\alpha|} u(D^\alpha \phi) dx.$$

Then $D^\alpha u$ is defined to equal g when this occurs.

Lemma 35.3.5 *Let $u \in L^1_{loc}(\mathbb{R}^n)$ and suppose $u_{,i} \in L^1_{loc}(\mathbb{R}^n)$, where the subscript on the u following the comma denotes the i^{th} weak partial derivative. Then if ϕ_ε is a mollifier and $u_\varepsilon \equiv u * \phi_\varepsilon$, it follows $u_{\varepsilon,i} \equiv u_{,i} * \phi_\varepsilon$.*

Proof: If $\psi \in C_c^\infty(\mathbb{R}^n)$, then

$$\begin{aligned} \int u(\mathbf{x} - \mathbf{y}) \psi_{,i}(\mathbf{x}) dx &= \int u(\mathbf{z}) \psi_{,i}(\mathbf{z} + \mathbf{y}) dz \\ &= - \int u_{,i}(\mathbf{z}) \psi(\mathbf{z} + \mathbf{y}) dz \\ &= - \int u_{,i}(\mathbf{x} - \mathbf{y}) \psi(\mathbf{x}) dx. \end{aligned}$$

Therefore,

$$\begin{aligned} u_{\varepsilon,i}(\psi) &= - \int u_\varepsilon \psi_{,i} = - \int \int u(\mathbf{x} - \mathbf{y}) \phi_\varepsilon(\mathbf{y}) \psi_{,i}(\mathbf{x}) d\mathbf{y} d\mathbf{x} \\ &= - \int \int u(\mathbf{x} - \mathbf{y}) \psi_{,i}(\mathbf{x}) \phi_\varepsilon(\mathbf{y}) d\mathbf{x} d\mathbf{y} \\ &= \int \int u_{,i}(\mathbf{x} - \mathbf{y}) \psi(\mathbf{x}) \phi_\varepsilon(\mathbf{y}) d\mathbf{x} d\mathbf{y} \\ &= \int u_{,i} * \phi_\varepsilon(\mathbf{x}) \psi(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

The technical questions about product measurability in the use of Fubini's theorem may be resolved by picking a Borel measurable representative for u . This proves the lemma.

What about the product rule? Does it have some form in the context of weak derivatives?

Lemma 35.3.6 *Let U be an open set, $\psi \in C^\infty(U)$ and suppose $u, u_{,i} \in L^p_{loc}(U)$. Then $(u\psi)_{,i}$ and $u\psi$ are in $L^p_{loc}(U)$ and*

$$(u\psi)_{,i} = u_{,i}\psi + u\psi_{,i}.$$

Proof: Let $\phi \in C_c^\infty(U)$ then

$$\begin{aligned} (u\psi)_{,i}(\phi) &\equiv - \int_U u\psi\phi_{,i} dx \\ &= - \int_U u[(\psi\phi)_{,i} - \phi\psi_{,i}] dx \\ &= \int_U (u_{,i}\psi\phi + u\psi_{,i}\phi) dx \\ &= \int_U (u_{,i}\psi + u\psi_{,i}) \phi dx \end{aligned}$$

This proves the lemma.

Recall the notation for the gradient of a function.

$$\nabla u(\mathbf{x}) \equiv (u_{,1}(\mathbf{x}) \cdots u_{,n}(\mathbf{x}))^T$$

thus

$$Du(\mathbf{x}) \mathbf{v} = \nabla u(\mathbf{x}) \cdot \mathbf{v}.$$

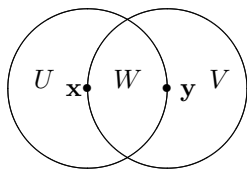
35.4 Morrey's Inequality

The following inequality will be called Morrey's inequality. It relates an expression which is given pointwise to an integral of the p^{th} power of the derivative.

Lemma 35.4.1 *Let $u \in C^1(\mathbb{R}^n)$ and $p > n$. Then there exists a constant, C , depending only on n such that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$\begin{aligned} & |u(\mathbf{x}) - u(\mathbf{y})| \\ & \leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z})|^p dz \right)^{1/p} \left(|\mathbf{x} - \mathbf{y}|^{(1-n/p)} \right). \end{aligned} \quad (35.4.1)$$

Proof: In the argument C will be a generic constant which depends on n . Consider the following picture.



This is a picture of two balls of radius r in $\mathbb{R}^n$, U and V having centers at $\mathbf{x}$ and $\mathbf{y}$ respectively, which intersect in the set, W . The center of U is on the boundary of V and the center of V is on the boundary of U as shown in the picture. There exists a constant, C , independent of r depending only on n such that

$$\frac{m(W)}{m(U)} = \frac{m(W)}{m(V)} = C.$$

You could compute this constant if you desired but it is not important here.

Define the average of a function over a set, $E \subseteq \mathbb{R}^n$ as follows.

$$\oint_E f dx \equiv \frac{1}{m(E)} \int_E f dx.$$

Then

$$\begin{aligned}
 |u(\mathbf{x}) - u(\mathbf{y})| &= \oint_W |u(\mathbf{x}) - u(\mathbf{y})| dz \\
 &\leq \oint_W |u(\mathbf{x}) - u(\mathbf{z})| dz + \oint_W |u(\mathbf{z}) - u(\mathbf{y})| dz \\
 &= \frac{C}{m(U)} \left[\int_W |u(\mathbf{x}) - u(\mathbf{z})| dz + \int_W |u(\mathbf{z}) - u(\mathbf{y})| dz \right] \\
 &\leq C \left[\oint_U |u(\mathbf{x}) - u(\mathbf{z})| dz + \oint_V |u(\mathbf{y}) - u(\mathbf{z})| dz \right]
 \end{aligned}$$

Now consider these two terms. Using spherical coordinates and letting U_0 denote the ball of the same radius as U but with center at $\mathbf{0}$,

$$\begin{aligned}
 &\oint_U |u(\mathbf{x}) - u(\mathbf{z})| dz \\
 &= \frac{1}{m(U_0)} \int_{U_0} |u(\mathbf{x}) - u(\mathbf{z} + \mathbf{x})| dz \\
 &= \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} |u(\mathbf{x}) - u(\rho \mathbf{w} + \mathbf{x})| d\sigma(w) d\rho \\
 &\leq \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} \int_0^\rho |\nabla u(\mathbf{x} + t\mathbf{w}) \cdot \mathbf{w}| dt d\sigma d\rho \\
 &\leq \frac{1}{m(U_0)} \int_0^r \rho^{n-1} \int_{S^{n-1}} \int_0^\rho |\nabla u(\mathbf{x} + t\mathbf{w})| dt d\sigma d\rho \\
 &\leq C \frac{1}{r} \int_0^r \int_{S^{n-1}} \int_0^r |\nabla u(\mathbf{x} + t\mathbf{w})| dt d\sigma d\rho \\
 &= C \frac{1}{r} \int_0^r \int_{S^{n-1}} \int_0^r \frac{|\nabla u(\mathbf{x} + t\mathbf{w})|}{t^{n-1}} t^{n-1} dt d\sigma d\rho \\
 &= C \int_{S^{n-1}} \int_0^r \frac{|\nabla u(\mathbf{x} + t\mathbf{w})|}{t^{n-1}} t^{n-1} dt d\sigma \\
 &= C \int_{U_0} \frac{|\nabla u(\mathbf{x} + \mathbf{z})|}{|\mathbf{z}|^{n-1}} dz \\
 &\leq C \left(\int_{U_0} |\nabla u(\mathbf{x} + \mathbf{z})|^p dz \right)^{1/p} \left(\int_U |\mathbf{z}|^{p' - np'} \right)^{1/p'} \\
 &= C \left(\int_U |\nabla u(\mathbf{z})|^p dz \right)^{1/p} \left(\int_{S^{n-1}} \int_0^r \rho^{p' - np'} \rho^{n-1} d\rho d\sigma \right)^{(p-1)/p} \\
 &= C \left(\int_U |\nabla u(\mathbf{z})|^p dz \right)^{1/p} \left(\int_{S^{n-1}} \int_0^r \frac{1}{\rho^{\frac{n-1}{p-1}}} d\rho d\sigma \right)^{(p-1)/p}
 \end{aligned}$$

$$\begin{aligned}
&= C \left(\frac{p-1}{p-n} \right)^{(p-1)/p} \left(\int_U |\nabla u(\mathbf{z})|^p dz \right)^{1/p} r^{1-\frac{n}{p}} \\
&= C \left(\frac{p-1}{p-n} \right)^{(p-1)/p} \left(\int_U |\nabla u(\mathbf{z})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{1-\frac{n}{p}}
\end{aligned}$$

Similarly,

$$\int_V |u(\mathbf{y}) - u(\mathbf{z})| dz \leq C \left(\frac{p-1}{p-n} \right)^{(p-1)/p} \left(\int_V |\nabla u(\mathbf{z})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{1-\frac{n}{p}}$$

Therefore,

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C \left(\frac{p-1}{p-n} \right)^{(p-1)/p} \left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{1-\frac{n}{p}}$$

because $B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|) \supseteq V \cup U$. This proves the lemma.

The following corollary is also interesting

Corollary 35.4.2 Suppose $u \in C^1(\mathbb{R}^n)$. Then

$$\begin{aligned}
&|u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})| \\
&\leq C \left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|. \quad (35.4.2)
\end{aligned}$$

Proof: This follows easily from letting $g(\mathbf{y}) \equiv u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})$. Then $g \in C^1(\mathbb{R}^n)$, $g(\mathbf{x}) = 0$, and $\nabla g(\mathbf{z}) = \nabla u(\mathbf{z}) - \nabla u(\mathbf{x})$. From Lemma 35.4.1,

$$\begin{aligned}
&|u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})| \\
&= |g(\mathbf{y})| = |g(\mathbf{y}) - g(\mathbf{x})| \\
&\leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{1-\frac{n}{p}} \\
&= C \left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x}-\mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|.
\end{aligned}$$

This proves the corollary.

It may be interesting at this point to recall the definition of differentiability on Page 121. If you knew the above inequality held for ∇u having components in $L^1_{loc}(\mathbb{R}^n)$, then at Lebesgue points of ∇u , the above would imply $Du(\mathbf{x})$ exists. This is exactly the approach taken below.

35.5 Rademacher's Theorem

The inequality of Corollary 35.4.2 can be extended to the case where u and $u_{,i}$ are in $L^p_{loc}(\mathbb{R}^n)$ for $p > n$. This leads to an elegant proof of the differentiability a.e. of a Lipschitz continuous function as well as a more general theorem.

Theorem 35.5.1 *Suppose u and all its weak partial derivatives, $u_{,i}$ are in $L^p_{loc}(\mathbb{R}^n)$. Then there exists a set of measure zero, E such that if $\mathbf{x}, \mathbf{y} \notin E$ then inequalities 35.4.2 and 35.4.1 are both valid. Furthermore, u equals a continuous function a.e.*

Proof: Let $u \in L^p_{loc}(\mathbb{R}^n)$ and $\psi_k \in C^\infty_c(\mathbb{R}^n)$, $\psi_k \geq 0$, and $\psi_k(\mathbf{z}) = 1$ for all $\mathbf{z} \in B(\mathbf{0}, k)$. Then it is routine to verify that

$$u\psi_k, (u\psi_k)_{,i} \in L^p(\mathbb{R}^n).$$

Here is why:

$$\begin{aligned} (u\psi_k)_{,i}(\phi) &\equiv - \int_{\mathbb{R}^n} u\psi_k\phi_{,i} dx \\ &= - \int_{\mathbb{R}^n} u\psi_k\phi_{,i} dx - \int_{\mathbb{R}^n} u\psi_{k,i}\phi dx + \int_{\mathbb{R}^n} u\psi_{k,i}\phi dx \\ &= - \int_{\mathbb{R}^n} u(\psi_k\phi)_{,i} dx + \int_{\mathbb{R}^n} u\psi_{k,i}\phi dx \\ &= \int_{\mathbb{R}^n} (u_{,i}\psi_k + u\psi_{k,i})\phi dx \end{aligned}$$

which shows

$$(u\psi_k)_{,i} = u_{,i}\psi_k + u\psi_{k,i}$$

as expected.

Let ϕ_ε be a mollifier and consider

$$(u\psi_k)_\varepsilon \equiv u\psi_k * \phi_\varepsilon.$$

By Lemma 35.3.5 on Page 1048,

$$(u\psi_k)_{\varepsilon,i} = (u\psi_k)_{,i} * \phi_\varepsilon.$$

Therefore

$$(u\psi_k)_{\varepsilon,i} \rightarrow (u\psi_k)_{,i} \text{ in } L^p(\mathbb{R}^n) \quad (35.5.3)$$

and

$$(u\psi_k)_\varepsilon \rightarrow u\psi_k \text{ in } L^p(\mathbb{R}^n) \quad (35.5.4)$$

as $\varepsilon \rightarrow 0$. By 35.5.4, there exists a subsequence $\varepsilon \rightarrow 0$ such that for $|\mathbf{z}| < k$ and for each $i = 1, 2, \dots, n$

$$(u\psi_k)_{\varepsilon,i}(\mathbf{z}) \rightarrow (u\psi_k)_{,i}(\mathbf{z}) = u_{,i}(\mathbf{z}) \text{ a.e.}$$

$$(u\psi_k)_\varepsilon(\mathbf{z}) \rightarrow u\psi_k(\mathbf{z}) = u(\mathbf{z}) \text{ a.e.} \quad (35.5.5)$$

Denoting the exceptional set by E_k , let

$$\mathbf{x}, \mathbf{y} \notin \bigcup_{k=1}^{\infty} E_k \equiv E$$

and let k be so large that

$$B(\mathbf{0}, k) \supseteq B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|).$$

Then by 35.4.1 and for $\mathbf{x}, \mathbf{y} \notin E$,

$$\begin{aligned} & |(u\psi_k)_\varepsilon(\mathbf{x}) - (u\psi_k)_\varepsilon(\mathbf{y})| \\ & \leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{y} - \mathbf{x}|)} |\nabla(u\psi_k)_\varepsilon|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{(1-n/p)} \end{aligned}$$

where C depends only on n . Similarly, by 35.4.2,

$$\begin{aligned} & |(u\psi_k)_\varepsilon(\mathbf{x}) - (u\psi_k)_\varepsilon(\mathbf{y}) - \nabla(u\psi_k)_\varepsilon(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})| \leq \\ & C \left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|)} |\nabla(u\psi_k)_\varepsilon(\mathbf{z}) - \nabla(u\psi_k)_\varepsilon(\mathbf{x})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|. \end{aligned}$$

Now by 35.5.5 and 35.5.3 passing to the limit as $\varepsilon \rightarrow 0$ yields

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{y} - \mathbf{x}|)} |\nabla u|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{(1-n/p)} \quad (35.5.6)$$

and

$$\begin{aligned} & |u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})| \\ & \leq C \left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|. \end{aligned} \quad (35.5.7)$$

Redefining u on the set of measure zero, E yields 35.5.6 for all $\mathbf{x}, \mathbf{y}$. This proves the theorem.

Corollary 35.5.2 *Let $u, u_i \in L_{loc}^p(\mathbb{R}^n)$ for $i = 1, \dots, n$ and $p > n$. Then the representative of u described in Theorem 35.5.1 is differentiable a.e.*

Proof: From Theorem 35.5.1

$$\begin{aligned} & |u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})| \\ & \leq C \left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|. \end{aligned} \quad (35.5.8)$$

and at every Lebesgue point, $\mathbf{x}$ of ∇u

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \left(\frac{1}{m(B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|))} \int_{B(\mathbf{x}, 2|\mathbf{x} - \mathbf{y}|)} |\nabla u(\mathbf{z}) - \nabla u(\mathbf{x})|^p dz \right)^{1/p} = 0$$

and so at each of these points,

$$\lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{|u(\mathbf{y}) - u(\mathbf{x}) - \nabla u(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x})|}{|\mathbf{x} - \mathbf{y}|} = 0$$

which says that u is differentiable at $\mathbf{x}$ and $Du(\mathbf{x})(\mathbf{v}) = \nabla u(\mathbf{x}) \cdot (\mathbf{v})$. See Page 121. This proves the corollary.

Definition 35.5.3 Now suppose u is Lipschitz on $\mathbb{R}^n$,

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|$$

for some constant K . Define $\text{Lip}(u)$ as the smallest value of K that works in this inequality.

The following corollary is known as Rademacher's theorem. It states that every Lipschitz function is differentiable a.e.

Corollary 35.5.4 If u is Lipschitz continuous then u is differentiable a.e. and $\|u_{,i}\|_{\infty} \leq \text{Lip}(u)$.

Proof: This is done by showing that Lipschitz continuous functions have weak derivatives in $L^{\infty}(\mathbb{R}^n)$ and then using the previous results. Let

$$D_{\mathbf{e}_i}^h u(\mathbf{x}) \equiv h^{-1} [u(\mathbf{x} + h\mathbf{e}_i) - u(\mathbf{x})].$$

Then $D_{\mathbf{e}_i}^h u$ is bounded in $L^{\infty}(\mathbb{R}^n)$ and

$$\|D_{\mathbf{e}_i}^h u\|_{\infty} \leq \text{Lip}(u).$$

It follows that $D_{\mathbf{e}_i}^h u$ is contained in a ball in $L^{\infty}(\mathbb{R}^n)$, the dual space of $L^1(\mathbb{R}^n)$. By Theorem 35.1.3 on Page 1042, there is a subsequence $h \rightarrow 0$ such that

$$D_{\mathbf{e}_i}^h u \rightharpoonup w, \|w\|_{\infty} \leq \text{Lip}(u)$$

where the convergence takes place in the weak $*$ topology of $L^{\infty}(\mathbb{R}^n)$. Let $\phi \in C_c^{\infty}(\mathbb{R}^n)$. Then

$$\begin{aligned} \int w \phi dx &= \lim_{h \rightarrow 0} \int D_{\mathbf{e}_i}^h u \phi dx \\ &= \lim_{h \rightarrow 0} \int u(\mathbf{x}) \frac{(\phi(\mathbf{x} - h\mathbf{e}_i) - \phi(\mathbf{x}))}{h} dx \\ &= - \int u(\mathbf{x}) \phi_{,i}(\mathbf{x}) dx. \end{aligned}$$

Thus $w = u_{,i}$ and $u_{,i} \in L^{\infty}(\mathbb{R}^n)$ for each i . Hence $u, u_{,i} \in L_{loc}^p(\mathbb{R}^n)$ for all $p > n$ and so u is differentiable a.e. by Corollary 35.5.2. This proves the corollary.

35.6 Change Of Variables Formula Lipschitz Maps

With Rademacher's theorem, one can give a general change of variables formula involving Lipschitz maps.

Definition 35.6.1 Let E be a Lebesgue measurable set. $\mathbf{x} \in E$ is a point of density if

$$\lim_{r \rightarrow 0} \frac{m(E \cap B(\mathbf{x}, r))}{m(B(\mathbf{x}, r))} = 1.$$

You see that if $\mathbf{x}$ were an interior point of E , then this limit will equal 1. However, it is sometimes the case that the limit equals 1 even when $\mathbf{x}$ is not an interior point. In fact, these points of density make sense even for sets that have empty interior.

Lemma 35.6.2 Let E be a Lebesgue measurable set. Then there exists a set of measure zero, N , such that if $\mathbf{x} \in E \setminus N$, then $\mathbf{x}$ is a point of density of E .

Proof: Consider the function, $f(\mathbf{x}) = \chi_E(\mathbf{x})$. This function is in $L^1_{loc}(\mathbb{R}^n)$. Let N^C denote the Lebesgue points of f . Then for $\mathbf{x} \in E \setminus N$,

$$\begin{aligned} 1 &= \chi_E(\mathbf{x}) = \lim_{r \rightarrow 0} \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} \chi_E(\mathbf{y}) dm_n \\ &= \lim_{r \rightarrow 0} \frac{m_n(B(\mathbf{x}, r) \cap E)}{m_n(B(\mathbf{x}, r))}. \end{aligned}$$

In this section, Ω will be a Lebesgue measurable set in $\mathbb{R}^n$ and $\mathbf{h} : \Omega \rightarrow \mathbb{R}^n$ will be Lipschitz. Recall the following definition and theorems. See Page 10.4.2 for the proofs and more discussion.

Definition 35.6.3 Let $\mathcal{F}$ be a collection of balls that cover a set, E , which have the property that if $\mathbf{x} \in E$ and $\varepsilon > 0$, then there exists $B \in \mathcal{F}$, diameter of $B < \varepsilon$ and $\mathbf{x} \in B$. Such a collection covers E in the sense of Vitali.

Theorem 35.6.4 Let $E \subseteq \mathbb{R}^n$ and suppose $\overline{m}_n(E) < \infty$ where $\overline{m}_n$ is the outer measure determined by m_n , n dimensional Lebesgue measure, and let $\mathcal{F}$, be a collection of closed balls of bounded radii such that $\mathcal{F}$ covers E in the sense of Vitali. Then there exists a countable collection of disjoint balls from $\mathcal{F}$, $\{B_j\}_{j=1}^\infty$, such that $\overline{m}_n(E \setminus \cup_{j=1}^\infty B_j) = 0$.

Now this theorem implies a simple lemma which is what will be used.

Lemma 35.6.5 Let V be an open set in $\mathbb{R}^r$, $m_r(V) < \infty$. Then there exists a sequence of disjoint open balls $\{B_i\}$ having radii less than δ and a set of measure 0, T , such that

$$V = (\cup_{i=1}^\infty B_i) \cup T.$$

As in the proof of the change of variables theorem given earlier, the first step is to show that $\mathbf{h}$ maps Lebesgue measurable sets to Lebesgue measurable sets. In showing this the key result is the next lemma which states that $\mathbf{h}$ maps sets of measure zero to sets of measure zero.

Lemma 35.6.6 *If $m_n(T) = 0$ then $m_n(\mathbf{h}(T)) = 0$.*

Proof: Let V be an open set containing T whose measure is less than ε . Now using the Vitali covering theorem, there exists a sequence of disjoint balls $\{B_i\}$, $B_i = B(\mathbf{x}_i, r_i)$ which are contained in V such that the sequence of enlarged balls, $\{\widehat{B}_i\}$, having the same center but 5 times the radius, covers T . Then

$$\begin{aligned} m_n(\mathbf{h}(T)) &\leq m_n\left(\mathbf{h}\left(\bigcup_{i=1}^{\infty} \widehat{B}_i\right)\right) \\ &\leq \sum_{i=1}^{\infty} m_n\left(\mathbf{h}\left(\widehat{B}_i\right)\right) \\ &\leq \sum_{i=1}^{\infty} \alpha(n) (\text{Lip}(\mathbf{h}))^n 5^n r_i^n = 5^n (\text{Lip}(\mathbf{h}))^n \sum_{i=1}^{\infty} m_n(B_i) \\ &\leq (\text{Lip}(\mathbf{h}))^n 5^n m_n(V) \leq \varepsilon (\text{Lip}(\mathbf{h}))^n 5^n. \end{aligned}$$

Since ε is arbitrary, this proves the lemma.

With the conclusion of this lemma, the next lemma is fairly easy to obtain.

Lemma 35.6.7 *If A is Lebesgue measurable, then $\mathbf{h}(A)$ is m_n measurable. Furthermore,*

$$m_n(\mathbf{h}(A)) \leq (\text{Lip}(\mathbf{h}))^n m_n(A). \quad (35.6.9)$$

Proof: Let $A_k = A \cap B(\mathbf{0}, k)$, $k \in \mathbb{N}$. Let $V \supseteq A_k$ and let $m_n(V) < \infty$. By Lemma 35.6.5, there is a sequence of disjoint balls $\{B_i\}$ and a set of measure 0, T , such that

$$V = \bigcup_{i=1}^{\infty} B_i \cup T, \quad B_i = B(x_i, r_i).$$

By Lemma 35.6.6,

$$\begin{aligned} \overline{m_n}(\mathbf{h}(A_k)) &\leq \overline{m_n}(\mathbf{h}(V)) \\ &\leq \overline{m_n}(\mathbf{h}(\bigcup_{i=1}^{\infty} B_i)) + \overline{m_n}(\mathbf{h}(T)) = \overline{m_n}(\mathbf{h}(\bigcup_{i=1}^{\infty} B_i)) \\ &\leq \sum_{i=1}^{\infty} \overline{m_n}(\mathbf{h}(B_i)) \leq \sum_{i=1}^{\infty} \overline{m_n}(B(\mathbf{h}(x_i), \text{Lip}(\mathbf{h}) r_i)) \\ &\leq \sum_{i=1}^{\infty} \alpha(n) (\text{Lip}(\mathbf{h}) r_i)^n = \text{Lip}(\mathbf{h})^n \sum_{i=1}^{\infty} m_n(B_i) = \text{Lip}(\mathbf{h})^n m_n(V). \end{aligned}$$

Therefore,

$$\overline{m_n}(\mathbf{h}(A_k)) \leq \text{Lip}(\mathbf{h})^n m_n(V).$$

Since V is an arbitrary open set containing A_k , it follows from regularity of Lebesgue measure that

$$\overline{m_n}(\mathbf{h}(A_k)) \leq \text{Lip}(\mathbf{h})^n m_n(A_k). \quad (35.6.10)$$

Now let $k \rightarrow \infty$ to obtain 35.6.9. This proves the formula. It remains to show $\mathbf{h}(A)$ is measurable.

By inner regularity of Lebesgue measure, there exists a set, F , which is the countable union of compact sets and a set T with $m_n(T) = 0$ such that

$$F \cup T = A_k.$$

Then $\mathbf{h}(F) \subseteq \mathbf{h}(A_k) \subseteq \mathbf{h}(F) \cup \mathbf{h}(T)$. By continuity of $\mathbf{h}$, $\mathbf{h}(F)$ is a countable union of compact sets and so it is Borel. By 35.6.10 with T in place of A_k ,

$$\overline{m_n}(\mathbf{h}(T)) = 0$$

and so $\mathbf{h}(T)$ is m_n measurable. Therefore, $\mathbf{h}(A_k)$ is m_n measurable because m_n is a complete measure and this exhibits $\mathbf{h}(A_k)$ between two m_n measurable sets whose difference has measure 0. Now

$$\mathbf{h}(A) = \bigcup_{k=1}^{\infty} \mathbf{h}(A_k)$$

so $\mathbf{h}(A)$ is also m_n measurable and this proves the lemma.

The following lemma, depending on the Brouwer fixed point theorem and found in Rudin [58], will be important for the following arguments. A slightly more precise version was presented earlier on Page 583 but this version given below will suffice in this context. The idea is that if a continuous function mapping a ball in $\mathbb{R}^k$ to $\mathbb{R}^k$ doesn't move any point very much, then the image of the ball must contain a slightly smaller ball.

Lemma 35.6.8 *Let $B = B(\mathbf{0}, r)$, a ball in $\mathbb{R}^k$ and let $\mathbf{F} : \overline{B} \rightarrow \mathbb{R}^k$ be continuous and suppose for some $\varepsilon < 1$,*

$$|\mathbf{F}(\mathbf{v}) - \mathbf{v}| < \varepsilon r$$

for all $\mathbf{v} \in \overline{B}$. Then

$$\mathbf{F}(\overline{B}) \supseteq \overline{B(\mathbf{0}, r(1 - \varepsilon))}.$$

Proof: Suppose $\mathbf{a} \in \overline{B(\mathbf{0}, r(1 - \varepsilon))} \setminus \mathbf{F}(\overline{B})$ and let

$$\mathbf{G}(\mathbf{v}) \equiv \frac{r(\mathbf{a} - \mathbf{F}(\mathbf{v}))}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|}.$$

Then by the Brouwer fixed point theorem, $\mathbf{G}(\mathbf{v}) = \mathbf{v}$ for some $\mathbf{v} \in \overline{B}$. Using the formula for $\mathbf{G}$, it follows $|\mathbf{v}| = r$. Taking the inner product with $\mathbf{v}$,

$$\begin{aligned} (\mathbf{G}(\mathbf{v}), \mathbf{v}) &= |\mathbf{v}|^2 = r^2 = \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} (\mathbf{a} - \mathbf{v} + \mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v}) \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a} - \mathbf{v}, \mathbf{v}) + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\ &= \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [(\mathbf{a}, \mathbf{v}) - |\mathbf{v}|^2 + (\mathbf{v} - \mathbf{F}(\mathbf{v}), \mathbf{v})] \\ &\leq \frac{r}{|\mathbf{a} - \mathbf{F}(\mathbf{v})|} [r^2(1 - \varepsilon) - r^2 + r^2\varepsilon] = 0, \end{aligned}$$

a contradiction. Therefore, $\overline{B(\mathbf{0}, r(1 - \varepsilon))} \setminus \mathbf{F}(\overline{B}) = \emptyset$ and this proves the lemma.

Now let Ω be a Lebesgue measurable set and suppose $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Lipschitz continuous and one to one on Ω . Let

$$N \equiv \{\mathbf{x} \in \Omega : D\mathbf{h}(\mathbf{x}) \text{ does not exist}\} \quad (35.6.11)$$

$$S \equiv \left\{ \mathbf{x} \in \Omega \setminus N : D\mathbf{h}(\mathbf{x})^{-1} \text{ does not exist} \right\} \quad (35.6.12)$$

Lemma 35.6.9 *Let $\mathbf{x} \in \Omega \setminus (S \cup N)$. Then if $\varepsilon \in (0, 1)$ the following hold for all r small enough.*

$$m_n \left(\mathbf{h} \left(\overline{B(\mathbf{x}, r)} \right) \right) \geq m_n (D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon))), \quad (35.6.13)$$

$$\mathbf{h}(B(\mathbf{x}, r)) \subseteq \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1 + \varepsilon)), \quad (35.6.14)$$

$$m_n \left(\mathbf{h} \left(\overline{B(\mathbf{x}, r)} \right) \right) \leq m_n (D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1 + \varepsilon))) \quad (35.6.15)$$

If $\mathbf{x} \in \Omega \setminus (S \cup N)$ is also a point of density of Ω , then

$$\lim_{r \rightarrow 0} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} = 1. \quad (35.6.16)$$

If $\mathbf{x} \in \Omega \setminus N$, then

$$|\det D\mathbf{h}(\mathbf{x})| = \lim_{r \rightarrow 0} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \text{ a.e.} \quad (35.6.17)$$

Proof: Since $D\mathbf{h}(\mathbf{x})^{-1}$ exists,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(|\mathbf{v}|) \quad (35.6.18)$$

$$= \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x}) \left(\overbrace{\mathbf{v} + D\mathbf{h}(\mathbf{x})^{-1} o(|\mathbf{v}|)}^{=o(|\mathbf{v}|)} \right) \quad (35.6.19)$$

Consequently, when r is small enough, 35.6.14 holds. Therefore, 35.6.15 holds. From 35.6.19, and the assumption that $D\mathbf{h}(\mathbf{x})^{-1}$ exists,

$$D\mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x} + \mathbf{v}) - D\mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}) - \mathbf{v} = o(|\mathbf{v}|). \quad (35.6.20)$$

Letting

$$\mathbf{F}(\mathbf{v}) = D\mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x} + \mathbf{v}) - D\mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}),$$

apply Lemma 35.6.8 in 35.6.20 to conclude that for r small enough, whenever $|\mathbf{v}| < r$,

$$D\mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x} + \mathbf{v}) - D\mathbf{h}(\mathbf{x})^{-1} \mathbf{h}(\mathbf{x}) \supseteq B(\mathbf{0}, (1 - \varepsilon)r).$$

Therefore,

$$\mathbf{h}(\overline{B(\mathbf{x}, r)}) \supseteq \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, (1 - \varepsilon)r)$$

which implies

$$m_n(\mathbf{h}(\overline{B(\mathbf{x}, r)})) \geq m_n(D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon)))$$

which shows 35.6.13.

Now suppose that $\mathbf{x}$ is a point of density of Ω as well as being a point where $D\mathbf{h}(\mathbf{x})^{-1}$ and $D\mathbf{h}(\mathbf{x})$ exist. Then whenever r is small enough,

$$1 - \varepsilon < \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} \leq 1$$

and so

$$\begin{aligned} 1 - \varepsilon &< \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega^C))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} + \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} \\ &\leq \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega^C))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} + 1. \end{aligned}$$

which implies

$$m_n(B(\mathbf{x}, r) \setminus \Omega) < \varepsilon \alpha(n) r^n. \quad (35.6.21)$$

Then for such r ,

$$\begin{aligned} 1 &\geq \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} \\ &\geq \frac{m_n(\mathbf{h}(B(\mathbf{x}, r))) - m_n(\mathbf{h}(B(\mathbf{x}, r) \setminus \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))}. \end{aligned}$$

From Lemma 35.6.7, 35.6.21, and 35.6.13, this is no larger than

$$1 - \frac{\text{Lip}(\mathbf{h})^n \varepsilon \alpha(n) r^n}{m_n(D\mathbf{h}(\mathbf{x}) B(\mathbf{0}, r(1 - \varepsilon)))}.$$

By the theorem on the change of variables for a linear map, this expression equals

$$1 - \frac{\text{Lip}(\mathbf{h})^n \varepsilon \alpha(n) r^n}{|\det(D\mathbf{h}(\mathbf{x}))| r^n \alpha(n) (1 - \varepsilon)^n} \equiv 1 - g(\varepsilon)$$

where $\lim_{\varepsilon \rightarrow 0} g(\varepsilon) = 0$. Then for all r small enough,

$$1 \geq \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(\mathbf{h}(B(\mathbf{x}, r)))} \geq 1 - g(\varepsilon)$$

which shows 35.6.16 since ε is arbitrary. It remains to verify 35.6.17.

In case $\mathbf{x} \in S$, for small $|\mathbf{v}|$,

$$\mathbf{h}(\mathbf{x} + \mathbf{v}) = \mathbf{h}(\mathbf{x}) + D\mathbf{h}(\mathbf{x})\mathbf{v} + o(|\mathbf{v}|)$$

where $|o(|\mathbf{v}|)| < \varepsilon |\mathbf{v}|$. Therefore, for small enough r ,

$$\mathbf{h}(B(\mathbf{x}, r)) - \mathbf{h}(\mathbf{x}) \subseteq K + B(\mathbf{0}, r\varepsilon)$$

where K is a compact subset of an $n-1$ dimensional subspace contained in $D\mathbf{h}(\mathbf{x})(\mathbb{R}^n)$ which has diameter no more than $2\|D\mathbf{h}(\mathbf{x})\|r$. By Lemma 10.6.5 on Page 304,

$$\begin{aligned} m_n(\mathbf{h}(B(\mathbf{x}, r))) &= m_n(\mathbf{h}(B(\mathbf{x}, r)) - \mathbf{h}(\mathbf{x})) \\ &\leq 2^n \varepsilon r (2\|D\mathbf{h}(\mathbf{x})\|r + r\varepsilon)^{n-1} \end{aligned}$$

and so, in this case, letting r be small enough,

$$\frac{m_n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \leq \frac{2^n \varepsilon r (2\|D\mathbf{h}(\mathbf{x})\|r + r\varepsilon)^{n-1}}{\alpha(n)r^n} \leq C\varepsilon.$$

Since ε is arbitrary, the limit as $r \rightarrow 0$ of this quotient equals 0.

If $\mathbf{x} \notin S$, use 35.6.13 - 35.6.15 along with the change of variables formula for linear maps. This proves the Lemma.

Since $\mathbf{h}$ is one to one, there exists a measure, μ , defined by

$$\mu(E) \equiv m_n(\mathbf{h}(E))$$

on the Lebesgue measurable subsets of Ω . By Lemma 35.6.7 $\mu \ll m_n$ and so by the Radon Nikodym theorem, there exists a nonnegative function, $J(\mathbf{x})$ in $L^1_{loc}(\mathbb{R}^n)$ such that whenever E is Lebesgue measurable,

$$\mu(E) = m_n(\mathbf{h}(E \cap \Omega)) = \int_{E \cap \Omega} J(\mathbf{x}) dm_n. \quad (35.6.22)$$

Extend J to equal zero off Ω .

Lemma 35.6.10 *The function, $J(\mathbf{x})$ equals $|\det D\mathbf{h}(\mathbf{x})|$ a.e.*

Proof: Define

$$Q \equiv \{\mathbf{x} \in \Omega : \mathbf{x} \text{ is not a point of density of } \Omega\} \cup N \cup$$

$$\{\mathbf{x} \in \Omega : \mathbf{x} \text{ is not a Lebesgue point of } J\}.$$

Then Q is a set of measure zero and if $\mathbf{x} \notin Q$, then by 35.6.17, and 35.6.16,

$$\begin{aligned}
 & |\det D\mathbf{h}(\mathbf{x})| \\
 &= \lim_{r \rightarrow 0} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} \\
 &= \lim_{r \rightarrow 0} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))} \frac{m_n(\mathbf{h}(B(\mathbf{x}, r) \cap \Omega))}{m_n(B(\mathbf{x}, r))} \\
 &= \lim_{r \rightarrow 0} \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r) \cap \Omega} J(\mathbf{y}) dm_n \\
 &= \lim_{r \rightarrow 0} \frac{1}{m_n(B(\mathbf{x}, r))} \int_{B(\mathbf{x}, r)} J(\mathbf{y}) dm_n = J(\mathbf{x}).
 \end{aligned}$$

the last equality because J was extended to be zero off Ω . This proves the lemma.

Here is the change of variables formula for Lipschitz mappings. It is a special case of the area formula.

Theorem 35.6.11 *Let Ω be a Lebesgue measurable set, let $f \geq 0$ be Lebesgue measurable. Then for $\mathbf{h}$ a Lipschitz mapping defined on $\mathbb{R}^n$ which is one to one on Ω ,*

$$\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) dm_n = \int_{\Omega} f(\mathbf{h}(\mathbf{x})) |\det D\mathbf{h}(\mathbf{x})| dm_n. \quad (35.6.23)$$

Proof: Let F be a Borel set. It follows that $\mathbf{h}^{-1}(F)$ is a Lebesgue measurable set. Therefore, by 35.6.22,

$$\begin{aligned}
 & m_n(\mathbf{h}(\mathbf{h}^{-1}(F) \cap \Omega)) \\
 &= \int_{\mathbf{h}(\Omega)} \mathcal{X}_F(\mathbf{y}) dm_n = \int_{\Omega} \mathcal{X}_{\mathbf{h}^{-1}(F)}(\mathbf{x}) J(\mathbf{x}) dm_n \\
 &= \int_{\Omega} \mathcal{X}_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n.
 \end{aligned} \quad (35.6.24)$$

What if F is only Lebesgue measurable? Note there are no measurability problems with the above expression because $\mathbf{x} \rightarrow \mathcal{X}_F(\mathbf{h}(\mathbf{x}))$ is Borel measurable due to the assumption that $\mathbf{h}$ is continuous while J is given to be Lebesgue measurable. However, if F is Lebesgue measurable, not necessarily Borel measurable, then it is no longer clear that $\mathbf{x} \rightarrow \mathcal{X}_F(\mathbf{h}(\mathbf{x}))$ is measurable. In fact this is not always even true. However, $\mathbf{x} \rightarrow \mathcal{X}_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is measurable and 35.6.24 holds.

Let F be Lebesgue measurable. Then by inner regularity, $F = H \cup N$ where N has measure zero, H is the countable union of compact sets so it is a Borel set, and $H \cap N = \emptyset$. Therefore, letting N' denote a Borel set of measure zero which contains N ,

$$\begin{aligned}
 b(\mathbf{x}) &\equiv \mathcal{X}_H(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \leq \mathcal{X}_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \\
 &= \mathcal{X}_H(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) + \mathcal{X}_N(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \\
 &\leq \mathcal{X}_H(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) + \mathcal{X}_{N'}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) \equiv u(\mathbf{x})
 \end{aligned}$$

Now since N' is Borel,

$$\begin{aligned} \int_{\Omega} (u(\mathbf{x}) - b(\mathbf{x})) dm_n &= \int_{\Omega} \chi_{N'}(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \\ &= m_n(\mathbf{h}(\mathbf{h}^{-1}(N') \cap \Omega)) = m_n(N' \cap \mathbf{h}(\Omega)) = 0 \end{aligned}$$

and this shows $\chi_H(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) = \chi_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ except on a set of measure zero. By completeness of Lebesgue measure, it follows $\mathbf{x} \rightarrow \chi_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x})$ is Lebesgue measurable and also since $\mathbf{h}$ maps sets of measure zero to sets of measure zero,

$$\begin{aligned} \int_{\Omega} \chi_F(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n &= \int_{\Omega} \chi_H(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \\ &= \int_{\mathbf{h}(\Omega)} \chi_H(\mathbf{y}) dm_n \\ &= \int_{\mathbf{h}(\Omega)} \chi_F(\mathbf{y}) dm_n. \end{aligned}$$

It follows that if s is any nonnegative Lebesgue measurable simple function,

$$\int_{\Omega} s(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n = \int_{\mathbf{h}(\Omega)} s(\mathbf{y}) dm_n \quad (35.6.25)$$

and now, if $f \geq 0$ is Lebesgue measurable, let s_k be an increasing sequence of Lebesgue measurable simple functions converging pointwise to f . Then since 35.6.25 holds for s_k , the monotone convergence theorem applies and yields 35.6.23. This proves the theorem.

It turns out that a Lipschitz function defined on some subset of $\mathbb{R}^n$ always has a Lipschitz extension to all of $\mathbb{R}^n$. The next theorem gives a proof of this. For more on this sort of theorem see Federer [26]. He gives a better but harder theorem than what follows.

Theorem 35.6.12 *If $\mathbf{h} : \Omega \rightarrow \mathbb{R}^m$ is Lipschitz, then there exists $\bar{\mathbf{h}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ which extends $\mathbf{h}$ and is also Lipschitz.*

Proof: It suffices to assume $m = 1$ because if this is shown, it may be applied to the components of $\mathbf{h}$ to get the desired result. Suppose

$$|h(\mathbf{x}) - h(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|. \quad (35.6.26)$$

Define

$$\bar{h}(\mathbf{x}) \equiv \inf\{h(\mathbf{w}) + K |\mathbf{x} - \mathbf{w}| : \mathbf{w} \in \Omega\}. \quad (35.6.27)$$

If $\mathbf{x} \in \Omega$, then for all $\mathbf{w} \in \Omega$,

$$h(\mathbf{w}) + K |\mathbf{x} - \mathbf{w}| \geq h(\mathbf{x})$$

by 35.6.26. This shows $h(\mathbf{x}) \leq \bar{h}(\mathbf{x})$. But also you could take $\mathbf{w} = \mathbf{x}$ in 35.6.27 which yields $\bar{h}(\mathbf{x}) \leq h(\mathbf{x})$. Therefore $\bar{h}(\mathbf{x}) = h(\mathbf{x})$ if $\mathbf{x} \in \Omega$.

Now suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and consider $|\bar{h}(\mathbf{x}) - \bar{h}(\mathbf{y})|$. Without loss of generality assume $\bar{h}(\mathbf{x}) \geq \bar{h}(\mathbf{y})$. (If not, repeat the following argument with $\mathbf{x}$ and $\mathbf{y}$ interchanged.) Pick $\mathbf{w} \in \Omega$ such that

$$h(\mathbf{w}) + K|\mathbf{y} - \mathbf{w}| - \varepsilon < \bar{h}(\mathbf{y}).$$

Then

$$\begin{aligned} |\bar{h}(\mathbf{x}) - \bar{h}(\mathbf{y})| &= \bar{h}(\mathbf{x}) - \bar{h}(\mathbf{y}) \leq h(\mathbf{w}) + K|\mathbf{x} - \mathbf{w}| - \\ &\quad [h(\mathbf{w}) + K|\mathbf{y} - \mathbf{w}| - \varepsilon] \leq K|\mathbf{x} - \mathbf{y}| + \varepsilon. \end{aligned}$$

Since ε is arbitrary,

$$|\bar{h}(\mathbf{x}) - \bar{h}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$$

and this proves the theorem.

This yields a simple corollary to Theorem 35.6.11.

Corollary 35.6.13 *Let $\mathbf{h} : \Omega \rightarrow \mathbb{R}^n$ be Lipschitz continuous and one to one where Ω is a Lebesgue measurable set. Then if $f \geq 0$ is Lebesgue measurable,*

$$\int_{\mathbf{h}(\Omega)} f(\mathbf{y}) \, dm_n = \int_{\Omega} f(\mathbf{h}(\mathbf{x})) |\det D\bar{\mathbf{h}}(\mathbf{x})| \, dm_n. \quad (35.6.28)$$

where $\bar{\mathbf{h}}$ denotes a Lipschitz extension of $\mathbf{h}$.

The Area And Coarea Formulas

36.1 The Area Formula Again

Recall the area formula presented earlier. For convenience, here it is.

Theorem 36.1.1 *Let $g : \mathbf{h}(A) \rightarrow [0, \infty]$ be $\mathcal{H}^n$ measurable where $\mathbf{h}$ is a continuous function and A is a Lebesgue measurable set which satisfies 19.5.12 - 19.5.14. That is, U is an open set in $\mathbb{R}^n$ on which $\mathbf{h}$ is defined and $A \subseteq U$ is a Lebesgue measurable set, $m \geq n$, and*

$$\mathbf{h} : U \rightarrow \mathbb{R}^m \text{ is continuous,} \quad (36.1.1)$$

$$D\mathbf{h}(\mathbf{x}) \text{ exists for all } \mathbf{x} \in A, \quad (36.1.2)$$

Also assume that for every $\mathbf{x} \in A$, there exists $r_{\mathbf{x}}$ and $L_{\mathbf{x}}$ such that for all $\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, r_{\mathbf{x}})$,

$$|\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq L_{\mathbf{x}} |\mathbf{x} - \mathbf{y}| \quad (36.1.3)$$

Then

$$\mathbf{x} \rightarrow (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

Obviously, one can obtain improved versions of this important theorem by using Rademacher's theorem and condition 36.1.3. As mentioned earlier, a function which satisfies 36.1.3 is called locally Lipschitz at $\mathbf{x}$. Here is a simple lemma which is in the spirit of similar lemmas presented in the chapter on Hausdorff measures.

Lemma 36.1.2 *Let U be an open set in $\mathbb{R}^n$ and let $\mathbf{h} : U \rightarrow \mathbb{R}^m$ where $m \geq n$. Let $A \subseteq U$ and let $\mathbf{h}$ be locally Lipschitz at every point of A . Then if $N \subseteq A$ has Lebesgue measure zero, it follows that $\mathcal{H}^n(\mathbf{h}(N)) = 0$.*

Proof: Let N_k be defined as

$$N_k \equiv \{\mathbf{x} \in N : \text{for some } R_{\mathbf{x}} > 0, |\mathbf{h}(\mathbf{z}) - \mathbf{h}(\mathbf{y})| \leq k|\mathbf{z} - \mathbf{y}| \text{ for all } \mathbf{y}, \mathbf{z} \in B(\mathbf{x}, R_{\mathbf{x}})\}$$

Thus $N_k \uparrow N$. Let $\varepsilon > 0$ be given and let $U \supseteq V_k \supseteq N$ be open and $m_n(V_k) < \frac{\varepsilon}{5^n k^n}$. Now fix $\delta > 0$. For $\mathbf{x} \in N_k$ let $B(\mathbf{x}, 5r_{\mathbf{x}}) \subseteq V_k$ such that $r_{\mathbf{x}} < \min(\frac{\delta}{5k}, R_{\mathbf{x}})$. By the Vitali covering theorem, there exists a disjoint sequence of these balls, $\{B_i\}_{i=1}^{\infty}$ such that $\{\widehat{B_i}\}_{i=1}^{\infty}$, the corresponding sequence of balls having the same centers but five times the radius covers N_k . Then $\text{diam}(\widehat{B_i}) < 2\delta/k$. Hence $\{\mathbf{h}(\widehat{B_i})\}_{i=1}^{\infty}$ covers $\mathbf{h}(N_k)$ and $\text{diam}(\mathbf{h}(\widehat{B_i})) < 2\delta$. It follows

$$\begin{aligned} \mathcal{H}_{2\delta}^n(\mathbf{h}(N_k)) &\leq \sum_{i=1}^{\infty} \alpha(n) r(\mathbf{h}(\widehat{B_i}))^n \\ &\leq \sum_{i=1}^{\infty} \alpha(n) k^n 5^n r(B_i)^n \\ &= 5^n k^n \sum_{i=1}^{\infty} m_n(B_i) \leq 5^n k^n m_n(V_k) < \varepsilon \end{aligned}$$

Since δ was arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(N_k)) \leq \varepsilon$. Since k was arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(N)) = \lim_{k \rightarrow \infty} \mathcal{H}^n(\mathbf{h}(N_k)) \leq \varepsilon$. Since ε is arbitrary, this shows $\mathcal{H}^n(\mathbf{h}(N)) = 0$. This proves the lemma.

Now with this lemma, here is one of many possible generalizations of the area formula.

Theorem 36.1.3 *Let U be an open set in $\mathbb{R}^n$ and $\mathbf{h} : U \rightarrow \mathbb{R}^m$. Let $\mathbf{h}$ be locally Lipschitz and one to one on A , a Lebesgue measurable subset of U and let $g : \mathbf{h}(A) \rightarrow \mathbb{R}$ be a nonnegative $\mathcal{H}^n$ measurable function. Then*

$$\mathbf{x} \rightarrow (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable and

$$\int_{\mathbf{h}(A)} g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

Proof: For $\mathbf{x} \in A$, there exists a ball, $B_{\mathbf{x}}$ on which $\mathbf{h}$ is Lipschitz. By Rademacher's theorem, $\mathbf{h}$ is differentiable a.e. on $B_{\mathbf{x}}$. There is a countable cover of A consisting of such balls on which $\mathbf{h}$ is Lipschitz. Therefore, $\mathbf{h}$ is differentiable on $A_0 \subseteq A$ where $m_n(A \setminus A_0) = 0$. Then by the earlier area formula,

$$\int_{\mathbf{h}(A_0)} g(\mathbf{y}) d\mathcal{H}^n = \int_{A_0} g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n$$

By Lemma 36.1.2

$$\begin{aligned} \int_{\mathbf{h}(A)} g(\mathbf{y}) d\mathcal{H}^n &= \int_{\mathbf{h}(A_0)} g(\mathbf{y}) d\mathcal{H}^n \\ &= \int_{A_0} g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \end{aligned}$$

This proves the theorem.

Note how a special case of this occurs when $\mathbf{h}$ is one to one and C^1 . Of course this yields the earlier change of variables formula as a still more special case.

In addition to this, recall the divergence theorem, Theorem 19.7.14 on Page 613. This theorem was stated for bounded open sets which have a Lipschitz boundary. This definition of Lipschitz boundary involved an assumption that certain Lipschitz mappings had a derivative a.e. Rademacher's theorem makes this assumption redundant. Therefore, the statement of Theorem 19.7.14 remains valid with the following definition of a Lipschitz boundary.

Definition 36.1.4 *A bounded open set, $U \subseteq \mathbb{R}^n$ is said to have a Lipschitz boundary and to lie on one side of its boundary if the following conditions hold. There exist open boxes, $Q_1, \dots, Q_N$,*

$$Q_i = \prod_{j=1}^n (a_j^i, b_j^i)$$

such that $\partial U \equiv \overline{U} \setminus U$ is contained in their union. Also, for each Q_i , there exists k and a Lipschitz function, g_i such that $U \cap Q_i$ is of the form

$$\left\{ \mathbf{x} : (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i) \text{ and } a_k^i < x_k < g_i(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \right\} \quad (36.1.4)$$

or else of the form

$$\left\{ \mathbf{x} : (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \prod_{j=1}^{k-1} (a_j^i, b_j^i) \times \prod_{j=k+1}^n (a_j^i, b_j^i) \text{ and } g_i(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) < x_k < b_k^i \right\} \quad (36.1.5)$$

Also, there exists an open set, Q_0 such that $Q_0 \subseteq \overline{Q_0} \subseteq U$ and $\overline{U} \subseteq Q_0 \cup Q_1 \cup \dots \cup Q_N$.

36.2 Mappings That Are Not One To One

Next I will consider the case where $\mathbf{h}$ is not necessarily one to one. Recall the major theorem presented earlier on which the proof of the area formula depended, Theorem 19.5.10 on Page 587. Here it is.

Theorem 36.2.1 *Let $\mathbf{h} : U \rightarrow \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U . Also suppose that for every $\mathbf{x} \in A$, $D\mathbf{h}(\mathbf{x})$ exists. Then for $\mathbf{x} \in A$,*

$$J(\mathbf{x}) = \lim_{r \rightarrow 0} \frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))}, \quad (36.2.6)$$

where $J(\mathbf{x}) \equiv \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

The next lemma is a version of Sard's lemma.

Lemma 36.2.2 *Let $\mathbf{h} : U \rightarrow \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U . Let*

$$N \equiv \{\mathbf{x} \in A : D\mathbf{h}(\mathbf{x}) \text{ does not exist}\} \quad (36.2.7)$$

and let

$$S \equiv \{\mathbf{x} \in A_0 \equiv A \setminus N : J(\mathbf{x}) = 0\} \quad (36.2.8)$$

Then $\mathcal{H}^n(\mathbf{h}(S \cup N)) = 0$.

Proof: By Rademacher's theorem, N has measure 0. Therefore, $\mathcal{H}^n(\mathbf{h}(N)) = 0$ by Lemma 36.1.2.

It remains to show $\mathcal{H}^n(\mathbf{h}(S)) = 0$. Let $S_k = B(\mathbf{0}, k) \cap S$ for k a positive integer large enough that $S_k \neq \emptyset$. By Theorem 19.5.10 on Page 587 stated above, if $\mathbf{x} \in S_k$, there exists $r_{\mathbf{x}}$ such that $5r_{\mathbf{x}} < \min(R_{\mathbf{x}}, 1)$ and if $r \leq 5r_{\mathbf{x}}$,

$$\frac{\mathcal{H}^n(\mathbf{h}(B(\mathbf{x}, r)))}{m_n(B(\mathbf{x}, r))} < \frac{\varepsilon}{5^n k^n}, \quad B(\mathbf{x}, r) \subseteq B(\mathbf{0}, k) \cap U \quad (36.2.9)$$

Then by the Vitali covering theorem, there exists a sequence of disjoint balls of this sort, $\{B_i\}_{i=1}^\infty$ such that the balls having 5 times the radius but the same center, $\{\widehat{B}_i\}_{i=1}^\infty$ cover S_k . Then $\{\mathbf{h}(\widehat{B}_i)\}_{i=1}^\infty$ covers $\mathbf{h}(S_k)$. Then from 36.2.9

$$\begin{aligned} \mathcal{H}^n(\mathbf{h}(S_k)) &\leq \sum_{i=1}^\infty \mathcal{H}^n(\mathbf{h}(\widehat{B}_i)) \leq \sum_{i=1}^\infty 5^n \mathcal{H}^n(\mathbf{h}(B_i)) \\ &\leq \sum_{i=1}^\infty 5^n \frac{\varepsilon}{5^n k^n} m_n(B_i) \leq \frac{\varepsilon}{k^n} m_n(B(\mathbf{0}, k)) = \varepsilon \alpha(n) \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows $\mathcal{H}^n(\mathbf{h}(S_k)) = 0$ and now letting $k \rightarrow \infty$, it follows $\mathcal{H}^n(\mathbf{h}(S)) = 0$. This proves the lemma.

The following very technical lemma provides the necessary theory to generalize to functions which are not one to one.

Lemma 36.2.3 *Let $\mathbf{h} : U \rightarrow \mathbb{R}^m$ where U is an open set in $\mathbb{R}^n$ for $n \leq m$ and suppose $\mathbf{h}$ is locally Lipschitz at every point of a Lebesgue measurable subset, A of U . Let*

$$N \equiv \{\mathbf{x} \in A : D\mathbf{h}(\mathbf{x}) \text{ does not exist}\}$$

and let

$$S \equiv \{\mathbf{x} \in A_0 \equiv A \setminus N : J(\mathbf{x}) = 0\}$$

Let $B = A \setminus (S \cup N)$. Then there exist measurable disjoint sets, $\{E_i\}_{i=1}^{\infty}$ such that $A = \cup_{i=1}^{\infty} E_i$ and $\mathbf{h}$ is one to one on E_i . Furthermore, $\mathbf{h}^{-1}$ is Lipschitz on $\mathbf{h}(E_i)$.

Proof: Let $\mathcal{C}$ be a dense countable subset of B and let $\mathcal{F}$ be a countable dense subset of the invertible elements of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. For i a positive integer and $T \in \mathcal{F}, \mathbf{c} \in \mathcal{C}$

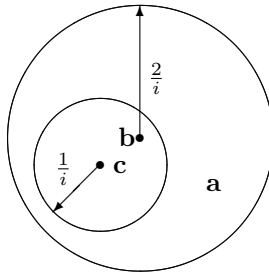
$$E(\mathbf{c}, T, i) \equiv \left\{ \mathbf{b} \in B \left(\mathbf{c}, \frac{1}{i} \right) \cap B : (a), (b) \text{ both hold} \right\}$$

where $(a), (b)$ are given by

$$\frac{2}{3} |T\mathbf{v}| \leq |U(\mathbf{b})\mathbf{v}| \text{ for all } \mathbf{v} \quad (a)$$

$$|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b}) - D\mathbf{h}(\mathbf{b})(\mathbf{a} - \mathbf{b})| \leq \frac{1}{2} |T(\mathbf{a} - \mathbf{b})|. \quad (b)$$

for all $\mathbf{a} \in B(\mathbf{b}, \frac{2}{i})$.



First I will show these sets, $E(\mathbf{c}, T, i)$ cover B and that they are measurable sets. To begin with consider the measurability question. Inequality (a) is the same as saying

$$\frac{2}{3} |T\mathbf{v}| \leq |D\mathbf{h}(\mathbf{b})\mathbf{v}| \text{ for all } \mathbf{v}$$

which is the same as saying

$$\frac{2}{3} |\mathbf{v}| \leq |D\mathbf{h}(\mathbf{b})T^{-1}\mathbf{v}| \text{ for all } \mathbf{v}.$$

Let $\{\mathbf{v}_i\}$ denote a dense countable subset of $\mathbb{R}^n$. Letting

$$S_i \equiv \left\{ \mathbf{b} : \frac{2}{3} |\mathbf{v}_i| \leq |D\mathbf{h}(\mathbf{b}) T^{-1} \mathbf{v}_i| \right\}$$

it follows easily that S_i is measurable because the component functions of the matrix of $D\mathbf{h}(\mathbf{b})$ are limits of difference quotients of continuous functions so they are Borel measurable. (Note that if B were Borel, then S_i would also be Borel.) Now by continuity,

$$\cup_{i=1}^{\infty} S_i = \left\{ \mathbf{b} : \frac{2}{3} |\mathbf{v}| \leq |D\mathbf{h}(\mathbf{b}) T^{-1} \mathbf{v}| \text{ for all } \mathbf{v} \right\}$$

and so this set is measurable also. Inequality (b) also determines a measurable set by similar reasoning. It is the same as saying that for all $|\mathbf{v}| < 2/i$,

$$|\mathbf{h}(\mathbf{b} + \mathbf{v}) - \mathbf{h}(\mathbf{b}) - D\mathbf{h}(\mathbf{b})(\mathbf{v})| \leq \frac{1}{2} |T(\mathbf{v})|$$

Use $\{\mathbf{v}_i\}$ a countable dense subset of $B(\mathbf{0}, 2/i)$ in a similar fashion to (a).

Next I need to show these sets cover B . Let $\mathbf{x} \in B$. Then pick $\mathbf{c}_i \in B(\mathbf{x}, \frac{1}{i})$ and $T_i \in B(U(\mathbf{x}), \frac{1}{i})$. I need to show that $\mathbf{x} \in E(\mathbf{c}_i, T_i, i)$ for i large enough. For i large enough, $\left\| T_i U(\mathbf{x})^{-1} \right\| < \frac{3}{2}$. Therefore, for such i

$$\left| T_i U(\mathbf{x})^{-1}(\mathbf{v}) \right| < \frac{3}{2} |\mathbf{v}|$$

for all $\mathbf{v}$ and so

$$|T_i \mathbf{w}| < \frac{3}{2} |U(\mathbf{x}) \mathbf{w}|$$

for all $\mathbf{w}$. Next consider (b). An equivalent norm is $\mathbf{v} \rightarrow |U(\mathbf{x}) \mathbf{v}|$ and so, for i large enough,

$$|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})(\mathbf{a} - \mathbf{x})| \leq \frac{1}{8} |U(\mathbf{x})(\mathbf{a} - \mathbf{x})| \quad (36.2.10)$$

whenever $|\mathbf{a} - \mathbf{x}| < 2/i$. Now also, for i large enough, $\|U(\mathbf{x}) T_i^{-1}\| < 4$ and so for all $\mathbf{w}$,

$$|U(\mathbf{x}) T_i^{-1} \mathbf{w}| < 4 |\mathbf{w}|$$

which implies

$$|U(\mathbf{x}) \mathbf{v}| < 4 |T_i \mathbf{v}|.$$

Applying this in 36.2.10 yields

$$|\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{x}) - D\mathbf{h}(\mathbf{x})(\mathbf{a} - \mathbf{x})| \leq \frac{1}{2} |T_i(\mathbf{a} - \mathbf{x})|$$

with implies $\mathbf{x} \in E(\mathbf{c}_i, T_i, i)$.

Next I need to show $\mathbf{h}$ is one to one on $E(\mathbf{c}, T, i)$. Suppose $\mathbf{b}_1, \mathbf{b}_2 \in E(\mathbf{c}, T, i)$. From (b) and (a),

$$\begin{aligned} |T(\mathbf{b}_2 - \mathbf{b}_1)| &\leq \frac{3}{2} |U(\mathbf{b}_1)(\mathbf{b}_2 - \mathbf{b}_1)| = \frac{3}{2} |D\mathbf{h}(\mathbf{b}_1)(\mathbf{b}_2 - \mathbf{b}_1)| \\ &\leq \frac{3}{2} \frac{1}{2} |T(\mathbf{b}_2 - \mathbf{b}_1)| \end{aligned}$$

which is a contradiction unless $\mathbf{b}_2 = \mathbf{b}_1$.

There are clearly countably many $E(\mathbf{c}, T, i)$. Denote them as $\{F_i\}_{i=1}^\infty$. Then let $E_1 = F_1$ and if $E_1, \dots, E_m$ have been chosen, let

$$E_{m+1} = F_{m+1} \setminus \cup_{i=1}^m E_i.$$

Thus the E_i are disjoint measurable sets whose union is B and $\mathbf{h}$ is one to one on each E_i .

Now consider one of the E_i . This is a subset of some $E(\mathbf{c}, T, i)$. Let $\mathbf{a}, \mathbf{b} \in E_i$. Then using (a) and (b),

$$\begin{aligned} |T(\mathbf{a} - \mathbf{b})| &\leq \frac{3}{2} |U(\mathbf{b})(\mathbf{a} - \mathbf{b})| \\ &= \frac{3}{2} |D\mathbf{h}(\mathbf{b})(\mathbf{a} - \mathbf{b})| \\ &\leq \frac{3}{2} |\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})| + \frac{3}{4} |T(\mathbf{a} - \mathbf{b})|. \end{aligned}$$

Hence

$$\frac{1}{4} |T(\mathbf{a} - \mathbf{b})| \leq \frac{3}{2} |\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})|$$

Since $\mathbf{v} \rightarrow |T\mathbf{v}|$ is an equivalent norm, there exists some $r > 0$ such that $|T\mathbf{v}| \geq r|\mathbf{v}|$ for all $\mathbf{v}$. Therefore,

$$|\mathbf{a} - \mathbf{b}| \leq \frac{6}{r} |\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})|.$$

In other words,

$$|\mathbf{h}^{-1}(\mathbf{h}(\mathbf{a})) - \mathbf{h}^{-1}(\mathbf{h}(\mathbf{b}))| = |\mathbf{a} - \mathbf{b}| \leq \frac{6}{r} |\mathbf{h}(\mathbf{a}) - \mathbf{h}(\mathbf{b})|.$$

which completes the proof.

With these lemmas, here is the main theorem which is a generalization of Theorem 36.1.3. First remember that from Lemma 19.5.3 on Page 583 a locally Lipschitz function maps Lebesgue measurable sets to Hausdorff measurable sets.

Theorem 36.2.4 *Let U be an open set in $\mathbb{R}^n$ and $\mathbf{h} : U \rightarrow \mathbb{R}^m$. Let $\mathbf{h}$ be locally Lipschitz on A , a Lebesgue measurable subset of U and let $g : \mathbf{h}(A) \rightarrow \mathbb{R}$ be a nonnegative $\mathcal{H}^n$ measurable function. Also let*

$$\#(\mathbf{y}) \equiv \text{Number of elements of } \mathbf{h}^{-1}(\mathbf{y})$$

Then $\#$ is $\mathcal{H}^n$ measurable,

$$\mathbf{x} \rightarrow (g \circ \mathbf{h})(\mathbf{x}) J(\mathbf{x})$$

is Lebesgue measurable, and

$$\int_{\mathbf{h}(A)} \#(\mathbf{y}) g(\mathbf{y}) d\mathcal{H}^n = \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n$$

where $J(\mathbf{x}) = \det(U(\mathbf{x})) = \det(D\mathbf{h}(\mathbf{x})^* D\mathbf{h}(\mathbf{x}))^{1/2}$.

Proof: Let $B = A \setminus (S \cup N)$ where S is the set of points where $J(\mathbf{x}) = 0$ and N is the set of points, $\mathbf{x}$ of A where $D\mathbf{h}(\mathbf{x})$ does not exist. Also from Lemma 36.2.3 there exists $\{E_i\}_{i=1}^\infty$, a sequence of disjoint measurable sets whose union equals B such that $\mathbf{h}$ is one to one on each E_i . Then from Theorem 36.1.3

$$\begin{aligned} & \int_A g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \\ &= \int_B g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n = \sum_{i=1}^\infty \int_{E_i} g(\mathbf{h}(\mathbf{x})) J(\mathbf{x}) dm_n \\ &= \sum_{i=1}^\infty \int_{\mathbf{h}(E_i)} g(\mathbf{y}) d\mathcal{H}^n = \int_{\mathbf{h}(B)} \left(\sum_{i=1}^\infty \chi_{\mathbf{h}(E_i)}(\mathbf{y}) \right) g(\mathbf{y}) d\mathcal{H}^n. \quad (36.2.11) \end{aligned}$$

Now $\#(\mathbf{y}) = (\sum_{i=1}^\infty \chi_{\mathbf{h}(E_i)}(\mathbf{y}))$ on $\mathbf{h}(B)$ and $\#$ differs from this $\mathcal{H}^n$ measurable function only on $\mathbf{h}(S \cup N)$, which by Lemma 36.2.2 is a set of $\mathcal{H}^n$ measure zero. Therefore, $\#$ is $\mathcal{H}^n$ measurable and the last term of 36.2.11 equals

$$\int_{\mathbf{h}(A)} \left(\sum_{i=1}^\infty \chi_{\mathbf{h}(E_i)}(\mathbf{y}) \right) g(\mathbf{y}) d\mathcal{H}^n = \int_{\mathbf{h}(A)} \#(\mathbf{y}) g(\mathbf{y}) d\mathcal{H}^n.$$

This proves the theorem.

36.3 The Coarea Formula

The coarea formula involves a function, $\mathbf{h}$ which maps a subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ where $m \leq n$ instead of $m \geq n$ as in the area formula. The symbol, $\text{Lip}(\mathbf{h})$ will denote the Lipschitz constant for $\mathbf{h}$.

It is possible to obtain the coarea formula as a computation involving the area formula and some simple linear algebra and this is the approach taken here. To begin with, here is the necessary linear algebra.

Theorem 36.3.1 *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix for $m \leq n$. Then*

$$p_{BA}(t) = t^{n-m} p_{AB}(t),$$

so the eigenvalues of BA and AB are the same including multiplicities except that BA has $n - m$ extra zero eigenvalues.

Proof: Use block multiplication to write

$$\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}$$

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix} = \begin{pmatrix} AB & ABA \\ B & BA \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$$

It follows that $\begin{pmatrix} 0 & 0 \\ B & BA \end{pmatrix}$ and $\begin{pmatrix} AB & 0 \\ B & 0 \end{pmatrix}$ have the same characteristic polynomials because the two matrices are similar. Thus

$$\det \begin{pmatrix} tI - AB & 0 \\ -B & tI \end{pmatrix} = \det \begin{pmatrix} tI & 0 \\ -B & tI - BA \end{pmatrix}$$

and so noting that BA is an $n \times n$ matrix and AB is an $m \times m$ matrix,

$$t^m \det(tI - BA) = t^n \det(tI - AB)$$

and so $\det(tI - BA) = p_{BA}(t) = t^{n-m} \det(tI - AB) = t^{n-m} p_{AB}(t)$. This proves the theorem.

The following corollary is what will be used to prove the coarea formula.

Corollary 36.3.2 *Let A be an $m \times n$ matrix. Then*

$$\det(I + AA^*) = \det(I + A^*A).$$

Proof: Assume $m \leq n$. From Theorem 36.3.1 AA^* and A^*A have the eigenvalues, $\lambda_1, \dots, \lambda_m$, necessarily nonnegative, with the same multiplicities and some zero eigenvalues which have differing multiplicities. The eigenvalues, $\lambda_1, \dots, \lambda_m$ are the zeros of $p_{AA^*}(t)$. Thus there is an orthogonal transformation, P such that

$$A^*A = P \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_m & & 0 \\ & & & 0 & \\ & 0 & & & \ddots \\ & & & & & 0 \end{pmatrix} P^*.$$

Therefore,

$$I + A^*A = P \begin{pmatrix} \lambda_1 + 1 & & & & \\ & \ddots & & & \\ & & \lambda_m + 1 & & 0 \\ & & & 1 & \\ & 0 & & & \ddots \\ & & & & & 1 \end{pmatrix} P^*$$

and so

$$\det(I + A^*A) = \det \begin{pmatrix} \lambda_1 + 1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m + 1 \end{pmatrix} = \det(I + AA^*).$$

This proves the corollary.

The other main ingredient is the following version of the chain rule.

Theorem 36.3.3 *Let $\mathbf{h}$ and $\mathbf{g}$ be locally Lipschitz mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $\mathbf{h}(\mathbf{g}(\mathbf{x})) = \mathbf{x}$ on A , a Lebesgue measurable set. Then for a.e. $\mathbf{x} \in A$, $D\mathbf{g}(\mathbf{h}(\mathbf{x}))$, $D\mathbf{h}(\mathbf{x})$, and $D(\mathbf{h} \circ \mathbf{g})(\mathbf{x})$ all exist and*

$$I = D(\mathbf{g} \circ \mathbf{h})(\mathbf{x}) = D\mathbf{g}(\mathbf{h}(\mathbf{x})) D\mathbf{h}(\mathbf{x}).$$

The proof of this theorem is based on the following lemma.

Lemma 36.3.4 *If $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz, then if $\mathbf{h}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in A$, then $\det(D\mathbf{h}(\mathbf{x})) = 0$ a.e.*

Proof: By the case of the Area formula which involves mappings which are not one to one, $0 = \int_{\{\mathbf{0}\}} \#(\mathbf{y}) d\mathbf{y} = \int_A |\det(D\mathbf{h}(\mathbf{x}))| d\mathbf{x}$ and so $\det(D\mathbf{h}(\mathbf{x})) = 0$ a.e.

Proof of the theorem: On A , $\mathbf{g}(\mathbf{h}(\mathbf{x})) - \mathbf{x} = \mathbf{0}$ and so by the lemma, there exists a set of measure zero, N_1 such that if $\mathbf{x} \notin N_1$, $D(\mathbf{g} \circ \mathbf{h})(\mathbf{x}) - I = \mathbf{0}$. Let M be the set of measure zero of points in $\mathbf{h}(\mathbb{R}^n)$ where $\mathbf{g}$ fails to be differentiable and let $N_2 \equiv \mathbf{g}(M) \cap A$, also a set of measure zero because locally Lipschitz maps take sets of measure zero to sets of measure zero. Finally let N_3 be the set of points where $\mathbf{h}$ fails to be differentiable. Then if $\mathbf{x} \notin N_1 \cup N_2 \cup N_3$, the chain rule implies $I = D(\mathbf{g} \circ \mathbf{h})(\mathbf{x}) = D\mathbf{g}(\mathbf{h}(\mathbf{x})) D\mathbf{h}(\mathbf{x})$. This proves the theorem.

Lemma 36.3.5 *Let $\mathbf{h} : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be Lipschitz continuous and $\delta > 0$. Then if $A \subseteq \mathbb{R}^p$ is either open or compact,*

$$\mathbf{y} \rightarrow \mathcal{H}_\delta^s(A \cap \mathbf{h}^{-1}(\mathbf{y}))$$

is Borel measurable.

Proof: Suppose first that A is compact and suppose for $\delta > 0$,

$$\mathcal{H}_\delta^s(A \cap \mathbf{h}^{-1}(\mathbf{y})) < t$$

Then there exist sets S_i , satisfying

$$\text{diam}(S_i) < \delta, A \cap \mathbf{h}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^\infty S_i,$$

and

$$\sum_{i=1}^\infty \alpha(s)(r(S_i))^s < t.$$

I claim these sets can be taken to be open sets. Choose $\lambda > 1$ but close enough to 1 that

$$\sum_{i=1}^{\infty} \alpha(s) (\lambda r(S_i))^s < t$$

Replace S_i with $S_i + B(0, \eta_i)$ where η_i is small enough that

$$\text{diam}(S_i) + 2\eta_i < \lambda \text{diam}(S_i).$$

Then

$$\text{diam}(S_i + B(0, \eta_i)) \leq \lambda \text{diam}(S_i)$$

and so $r(S_i + B(0, \eta_i)) \leq \lambda r(S_i)$. Thus

$$\sum_{i=1}^{\infty} \alpha(s) r(S_i + B(0, \eta_i))^s < t.$$

Hence you could replace S_i with $S_i + B(0, \eta_i)$ and so one can assume the sets S_i are open.

Claim: If $\mathbf{z}$ is close enough to $\mathbf{y}$, then $A \cap \mathbf{h}^{-1}(\mathbf{z}) \subseteq \cup_{i=1}^{\infty} S_i$.

Proof: If not, then there exists a sequence $\{\mathbf{z}_k\}$ such that

$$\mathbf{z}_k \rightarrow \mathbf{y},$$

and

$$\mathbf{x}_k \in (A \cap \mathbf{h}^{-1}(\mathbf{z}_k)) \setminus \cup_{i=1}^{\infty} S_i.$$

By compactness of A , there exists a subsequence still denoted by k such that

$$\mathbf{z}_k \rightarrow \mathbf{y}, \quad \mathbf{x}_k \rightarrow \mathbf{x} \in A \setminus \cup_{i=1}^{\infty} S_i.$$

Hence

$$\mathbf{h}(\mathbf{x}) = \lim_{k \rightarrow \infty} \mathbf{h}(\mathbf{x}_k) = \lim_{k \rightarrow \infty} \mathbf{z}_k = \mathbf{y}.$$

But $\mathbf{x} \notin \cup_{i=1}^{\infty} S_i$ contrary to the assumption that $A \cap \mathbf{h}^{-1}(\mathbf{y}) \subseteq \cup_{i=1}^{\infty} S_i$.

It follows from this claim that whenever $\mathbf{z}$ is close enough to $\mathbf{y}$,

$$\mathcal{H}_{\delta}^s(A \cap \mathbf{h}^{-1}(\mathbf{z})) < t.$$

This shows

$$\{\mathbf{z} \in \mathbb{R}^p : \mathcal{H}_{\delta}^s(A \cap \mathbf{h}^{-1}(\mathbf{z})) < t\}$$

is an open set and so $\mathbf{y} \rightarrow \mathcal{H}_{\delta}^s(A \cap \mathbf{h}^{-1}(\mathbf{y}))$ is Borel measurable whenever A is compact. Now let V be an open set and let

$$A_k \uparrow V, \quad A_k \text{ compact.}$$

Then

$$\mathcal{H}_{\delta}^s(V \cap \mathbf{h}^{-1}(\mathbf{y})) = \lim_{k \rightarrow \infty} \mathcal{H}_{\delta}^s(A_k \cap \mathbf{h}^{-1}(\mathbf{y}))$$

so $\mathbf{y} \rightarrow \mathcal{H}_{\delta}^s(V \cap \mathbf{h}^{-1}(\mathbf{y}))$ is Borel measurable for all V open. This proves the lemma.

Lemma 36.3.6 *Let $\mathbf{h} : \mathbb{R}^p \rightarrow \mathbb{R}^m$ be Lipschitz continuous. Suppose A is either open or compact in $\mathbb{R}^p$. Then $\mathbf{y} \rightarrow \mathcal{H}^s(A \cap \mathbf{h}^{-1}(\mathbf{y}))$ is also Borel measurable and*

$$\int_{\mathbb{R}^m} \mathcal{H}^s(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy \leq 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(s) \alpha(m)}{\alpha(s+m)} \mathcal{H}^{s+m}(A)$$

In particular, if $s = n - m$ and $p = n$

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy \leq 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(n-m) \alpha(m)}{\alpha(n)} m_n(A)$$

Proof: From Lemma 36.3.5 $\mathbf{y} \rightarrow \mathcal{H}_\delta^s(A \cap \mathbf{h}^{-1}(\mathbf{y}))$ is Borel measurable for each $\delta > 0$. Without loss of generality, $\mathcal{H}^{s+m}(A) < \infty$. Now let B_i be closed sets with $\text{diam}(B_i) < \delta$, $A \subseteq \bigcup_{i=1}^\infty B_i$, and

$$\mathcal{H}_\delta^{s+m}(A) + \varepsilon > \sum_{i=1}^\infty \alpha(s+m) r(B_i)^{s+m}.$$

Note each B_i is compact so $\mathbf{y} \rightarrow \mathcal{H}_\delta^s(B_i \cap \mathbf{h}^{-1}(\mathbf{y}))$ is Borel measurable. Thus

$$\begin{aligned} & \int_{\mathbb{R}^m} \mathcal{H}_\delta^s(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy \\ & \leq \int_{\mathbb{R}^m} \sum_i \mathcal{H}_\delta^s(B_i \cap \mathbf{h}^{-1}(\mathbf{y})) dy \\ & = \sum_i \int_{\mathbb{R}^m} \mathcal{H}_\delta^s(B_i \cap \mathbf{h}^{-1}(\mathbf{y})) dy \\ & \leq \sum_i \int_{\mathbf{h}(B_i)} \mathcal{H}_\delta^s(B_i) dy \\ & = \sum_i m_m(\mathbf{h}(B_i)) \mathcal{H}_\delta^s(B_i) \\ & \leq \sum_i (\text{Lip}(\mathbf{h}))^m 2^m \alpha(m) r(B_i)^m \alpha(s) r(B_i)^s \\ & = (\text{Lip}(\mathbf{h}))^m \frac{\alpha(m) \alpha(s)}{\alpha(m+s)} 2^m \sum_i \alpha(s+m) r(B_i)^{m+s} \\ & \leq (\text{Lip}(\mathbf{h}))^m \frac{\alpha(m) \alpha(s)}{\alpha(m+s)} 2^m (\mathcal{H}_\delta^{s+m}(A) + \varepsilon) \end{aligned}$$

Since ε is arbitrary,

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^s(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy \leq (\text{Lip}(\mathbf{h}))^m \frac{\alpha(m) \alpha(s)}{\alpha(m+s)} 2^m \mathcal{H}_\delta^{s+m}(A)$$

Taking a limit as $\delta \rightarrow 0$ this proves the lemma.

Next I will show that whenever A is Lebesgue measurable,

$$\mathbf{y} \rightarrow \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y}))$$

is m_m measurable and the above estimate holds.

Lemma 36.3.7 *Let A be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be Lipschitz. Then*

$$\mathbf{y} \rightarrow \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y}))$$

is Lebesgue measurable. Furthermore, for all A Lebesgue measurable,

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) \, dy \leq 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(A)$$

Proof: Let A be a bounded Lebesgue measurable set in $\mathbb{R}^n$. Then by inner and outer regularity of Lebesgue measure there exists an increasing sequence of compact sets, $\{K_k\}$ contained in A and a decreasing sequence of open sets, $\{V_k\}$ containing A such that $m_n(V_k \setminus K_k) < 2^{-k}$. Thus $m_n(V_1) \leq m_n(A) + 1$. By Lemma 36.3.6

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}(V_1 \cap \mathbf{h}^{-1}(\mathbf{y})) \, dy < 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} (m_n(A) + 1).$$

Then

$$\mathcal{H}_\delta^{n-m}(K_k \cap \mathbf{h}^{-1}(\mathbf{y})) \leq \mathcal{H}_\delta^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) \leq \mathcal{H}_\delta^{n-m}(V_k \cap \mathbf{h}^{-1}(\mathbf{y})) \quad (36.3.12)$$

By Lemma 36.3.6

$$\begin{aligned} &= \int_{\mathbb{R}^m} (\mathcal{H}_\delta^{n-m}(V_k \cap \mathbf{h}^{-1}(\mathbf{y})) - \mathcal{H}_\delta^{n-m}(K_k \cap \mathbf{h}^{-1}(\mathbf{y}))) \, dy \\ &= \int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}((V_k - K_k) \cap \mathbf{h}^{-1}(\mathbf{y})) \, dy \\ &\leq 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(V_k \setminus K_k) \\ &< 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} 2^{-k} \end{aligned}$$

Let the Borel measurable functions, g and f be defined by

$$g(\mathbf{y}) \equiv \lim_{k \rightarrow \infty} \mathcal{H}_\delta^{n-m}(V_k \cap \mathbf{h}^{-1}(\mathbf{y})), \quad f(\mathbf{y}) \equiv \lim_{k \rightarrow \infty} \mathcal{H}_\delta^{n-m}(K_k \cap \mathbf{h}^{-1}(\mathbf{y}))$$

It follows from the dominated convergence theorem and 36.3.12 that

$$f(\mathbf{y}) \leq \mathcal{H}_\delta^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) \leq g(\mathbf{y})$$

and

$$\int_{\mathbb{R}^m} (g(\mathbf{y}) - f(\mathbf{y})) \, dy = 0.$$

By completeness of m_m , this establishes $\mathbf{y} \rightarrow \mathcal{H}_\delta^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y}))$ is Lebesgue measurable. Then by Lemma 36.3.6 again,

$$\int_{\mathbb{R}^m} \mathcal{H}_\delta^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) \, dy \leq 2^m (\text{Lip}(\mathbf{h}))^m \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} m_n(A).$$

Letting $\delta \rightarrow 0$ and using the monotone convergence theorem yields the desired inequality for $\mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y}))$.

The case where A is not bounded can be handled by considering $A_r = A \cap B(\mathbf{0}, r)$ and letting $r \rightarrow \infty$. This proves the lemma.

By fussing with the isodiametric inequality one can remove the factor of 2^m in the above inequalities obtaining much more attractive formulas. This is done in [24]. See also [46] which follows [24] and [26]. This last reference probably has the most complete treatment of these topics.

With these lemmas, it is now possible to give a proof of the coarea formula.

Define $\Lambda(n, m)$ as all possible ordered lists of m numbers taken from $\{1, 2, \dots, n\}$.

Lemma 36.3.8 *Let A be a measurable set in $\mathbb{R}^n$ and let $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz map where $m \leq n$ which is differentiable at every point of A and for which*

$$J\mathbf{h}(\mathbf{x}) \equiv \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2} \neq 0.$$

Then the following formula holds along with all measurability assertions needed for it to make sense.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) d\mathbf{y} = \int_A J\mathbf{h}(\mathbf{x}) d\mathbf{x} \quad (36.3.13)$$

Proof: For $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{i} \in \Lambda(n, m)$, with $\mathbf{i} = (i_1, \dots, i_m)$, define $\mathbf{x}_{\mathbf{i}} \equiv (x_{i_1}, \dots, x_{i_m})$, and $\pi_{\mathbf{i}}\mathbf{x} \equiv \mathbf{x}_{\mathbf{i}}$. Also for $\mathbf{i} \in \Lambda(n, m)$, let $\mathbf{i}_c \in \Lambda(n, n-m)$ consist of the remaining indices taken in order. For $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ where $m \leq n$, define $J\mathbf{h}(\mathbf{x}) \equiv \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2}$. For each $\mathbf{i} \in \Lambda(n, m)$, define

$$\mathbf{h}^{\mathbf{i}}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{h}(\mathbf{x}) \\ \mathbf{x}_{\mathbf{i}_c} \end{pmatrix}.$$

By Lemma 36.2.3, there exist disjoint measurable sets $\{F_j^{\mathbf{i}}\}_{j=1}^{\infty}$ such that $\mathbf{h}^{\mathbf{i}}$ is one to one on $F_j^{\mathbf{i}}$, $(\mathbf{h}^{\mathbf{i}})^{-1}$ is Lipschitz on $\mathbf{h}^{\mathbf{i}}(F_j^{\mathbf{i}})$, and

$$\cup_{j=1}^{\infty} F_j^{\mathbf{i}} = \{\mathbf{x} \in A : \det(D\mathbf{h}^{\mathbf{i}}(\mathbf{x})) \neq 0\}.$$

For $\mathbf{x} \in A$, $\det(D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}(\mathbf{x})) \neq 0$ for some $\mathbf{i} \in \Lambda(n, m)$. But $\det(D_{\mathbf{x}_{\mathbf{i}}}\mathbf{h}(\mathbf{x})) = \det(D\mathbf{h}^{\mathbf{i}}(\mathbf{x}))$ and so $\mathbf{x} \in F_j^{\mathbf{i}}$ for some $\mathbf{i}$ and j . Hence

$$\cup_{\mathbf{i}, j} F_j^{\mathbf{i}} = A.$$

Now let $\{E_j^{\mathbf{i}}\}$ be measurable sets such that $E_j^{\mathbf{i}} \subseteq F_k^{\mathbf{i}}$ for some k , the sets are disjoint, and their union coincides with $\cup_{\mathbf{i}, j} F_j^{\mathbf{i}}$. Then

$$\int_A J\mathbf{h}(\mathbf{x}) d\mathbf{x} = \sum_{\mathbf{i} \in \Lambda(n, m)} \sum_{j=1}^{\infty} \int_{E_j^{\mathbf{i}} \cap A} \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2} d\mathbf{x}. \quad (36.3.14)$$

Let $\mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lipschitz extension of $(\mathbf{h}^i)^{-1}$ so $\mathbf{g} \circ \mathbf{h}^i(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in E_j^i$. First, using Theorem 36.3.3, and the fact that Lipschitz mappings take sets of measure zero to sets of measure zero, replace E_j^i with a measurable set, $\tilde{E}_j^i \subseteq E_j^i$ such that $E_j^i \setminus \tilde{E}_j^i$ has measure zero and

$$D\mathbf{h}^i(\mathbf{g}(\mathbf{y})) D\mathbf{g}(\mathbf{y}) = I$$

on $\mathbf{h}^i(\tilde{E}_j^i)$. Changing the variables using the area formula, the expression in 36.3.14 equals

$$\int_A J\mathbf{h}(\mathbf{x}) dx = \sum_{\mathbf{i} \in \Lambda(n, m)} \sum_{j=1}^{\infty} \int_{\mathbf{h}^i(\tilde{E}_j^i \cap A)} \det(D\mathbf{h}(\mathbf{g}(\mathbf{y})) D\mathbf{h}(\mathbf{g}(\mathbf{y}))^*)^{1/2} |\det D\mathbf{h}^i(\mathbf{g}(\mathbf{y}))|^{-1} dy. \quad (36.3.15)$$

Note the integrands are all Borel measurable functions because they are continuous functions of the entries of matrices which entries come from taking limits of difference quotients of continuous functions. Thus,

$$\begin{aligned} & \int_{\tilde{E}_j^i \cap A} \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2} dx = \\ & \int_{\mathbb{R}^n} \mathcal{X}_{\mathbf{h}^i(\tilde{E}_j^i \cap A)}(\mathbf{y}) \det(D\mathbf{h}(\mathbf{g}(\mathbf{y})) D\mathbf{h}(\mathbf{g}(\mathbf{y}))^*)^{1/2} |\det D\mathbf{h}^i(\mathbf{g}(\mathbf{y}))|^{-1} dy \\ & = \int_{\mathbb{R}^m} \int_{\pi_{\mathbf{i}_c}(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \tilde{E}_j^i \cap A)} \det(D\mathbf{h}(\mathbf{g}(\mathbf{y})) D\mathbf{h}(\mathbf{g}(\mathbf{y}))^*)^{1/2} |\det D\mathbf{h}_{\mathbf{x}_i}(\mathbf{g}(\mathbf{y}))|^{-1} dy_2 dy_1 \end{aligned} \quad (36.3.16)$$

where $\mathbf{y}_1 = \mathbf{h}(\mathbf{x})$ and $\mathbf{y}_2 = \mathbf{x}_{\mathbf{i}_c}$. Thus

$$\mathbf{y}_2 = \pi_{\mathbf{i}_c} \mathbf{g}(\mathbf{y}) = \pi_{\mathbf{i}_c} \mathbf{g}(\mathbf{h}^i(\mathbf{x})) = \mathbf{x}_{\mathbf{i}_c}. \quad (36.3.17)$$

Now consider the inner integral in 36.3.16 in which $\mathbf{y}_1$ is fixed. The integrand equals

$$\det \left[\begin{pmatrix} D_{\mathbf{x}_i} \mathbf{h}(\mathbf{g}(\mathbf{y})) & D_{\mathbf{x}_{\mathbf{i}_c}} \mathbf{h}(\mathbf{g}(\mathbf{y})) \end{pmatrix} \begin{pmatrix} D_{\mathbf{x}_i} \mathbf{h}(\mathbf{g}(\mathbf{y}))^* \\ D_{\mathbf{x}_{\mathbf{i}_c}} \mathbf{h}(\mathbf{g}(\mathbf{y}))^* \end{pmatrix} \right]^{1/2} |\det D\mathbf{h}_{\mathbf{x}_i}(\mathbf{g}(\mathbf{y}))|^{-1}. \quad (36.3.18)$$

I want to massage the above expression slightly. Since $\mathbf{y}_1$ is fixed, and $\mathbf{y}_1 = \mathbf{h}(\pi_{\mathbf{i}} \mathbf{g}(\mathbf{y}), \pi_{\mathbf{i}_c} \mathbf{g}(\mathbf{y})) = \mathbf{h}(\mathbf{g}(\mathbf{y}))$, it follows from 36.3.17 that

$$\begin{aligned} \mathbf{0} &= D_{\mathbf{x}_i} \mathbf{h}(\mathbf{g}(\mathbf{y})) D_{\mathbf{y}_2} \pi_{\mathbf{i}} \mathbf{g}(\mathbf{y}) + D_{\mathbf{x}_{\mathbf{i}_c}} \mathbf{h}(\mathbf{g}(\mathbf{y})) D_{\mathbf{y}_2} \pi_{\mathbf{i}_c} \mathbf{g}(\mathbf{y}) \\ &= D_{\mathbf{x}_i} \mathbf{h}(\mathbf{g}(\mathbf{y})) D_{\mathbf{y}_2} \pi_{\mathbf{i}} \mathbf{g}(\mathbf{y}) + D_{\mathbf{x}_{\mathbf{i}_c}} \mathbf{h}(\mathbf{g}(\mathbf{y})). \end{aligned}$$

Letting $A \equiv D_{\mathbf{x}_i} \mathbf{h}(\mathbf{g}(\mathbf{y}))$ and $B \equiv D_{\mathbf{y}_2} \pi_i \mathbf{g}(\mathbf{y})$ and using the above formula, 36.3.18 is of the form

$$\begin{aligned} & \det \left[\begin{pmatrix} A & -AB \end{pmatrix} \begin{pmatrix} A^* \\ -B^* A^* \end{pmatrix} \right]^{1/2} |\det A|^{-1} \\ &= \det [AA^* + ABB^*A^*]^{1/2} |\det A|^{-1} \\ &= \det [A(I + BB^*)A^*]^{1/2} |\det A|^{-1} \\ &= (\det(A) \det(A^*))^{1/2} \det(I + BB^*)^{1/2} |\det A|^{-1} \\ &= \det(I + BB^*)^{1/2}, \end{aligned}$$

which, by Corollary 36.3.2, equals $\det(I + B^*B)^{1/2}$. (Note the size of the identity changes in these two expressions, the first being an $m \times m$ matrix and the second being a $n - m \times n - m$ matrix.)

By 36.3.17 $\pi_{i_c} \mathbf{g}(\mathbf{y}) = \mathbf{y}_2$ and so,

$$\begin{aligned} \det(I + B^*B)^{1/2} &= \det \left[\begin{pmatrix} B^* & I \end{pmatrix} \begin{pmatrix} B \\ I \end{pmatrix} \right]^{1/2} \\ &= \det \left[\begin{pmatrix} D_{\mathbf{y}_2} \pi_i \mathbf{g}(\mathbf{y})^* & D_{\mathbf{y}_2} \pi_{i_c} \mathbf{g}(\mathbf{y})^* \end{pmatrix} \begin{pmatrix} D_{\mathbf{y}_2} \pi_i \mathbf{g}(\mathbf{y}) \\ D_{\mathbf{y}_2} \pi_{i_c} \mathbf{g}(\mathbf{y}) \end{pmatrix} \right]^{1/2} \\ &= \det(D_{\mathbf{y}_2} \mathbf{g}(\mathbf{y})^* D_{\mathbf{y}_2} \mathbf{g}(\mathbf{y}))^{1/2}. \end{aligned}$$

Therefore, 36.3.16 reduces to

$$\begin{aligned} & \int_{\tilde{E}_j^i \cap A} \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2} dx = \\ & \int_{\mathbb{R}^m} \int_{\pi_{i_c}(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \tilde{E}_j^i \cap A)} \det(D_{\mathbf{y}_2} \mathbf{g}(\mathbf{y})^* D_{\mathbf{y}_2} \mathbf{g}(\mathbf{y}))^{1/2} dy_2 dy_1. \end{aligned} \quad (36.3.19)$$

By the area formula applied to the inside integral, this integral equals

$$\mathcal{H}^{n-m}(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \tilde{E}_j^i \cap A)$$

and so

$$\begin{aligned} & \int_{\tilde{E}_j^i \cap A} \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2} dx \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\mathbf{h}^{-1}(\mathbf{y}_1) \cap \tilde{E}_j^i \cap A) dy_1. \end{aligned}$$

Using Lemma 36.3.7, along with the inner regularity of Lebesgue measure, $\tilde{E}_j^i$ can be replaced with E_j^i . Therefore, summing the terms over all i and j ,

$$\int_A \det(D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2} dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(\mathbf{h}^{-1}(\mathbf{y}) \cap A) dy.$$

This proves the lemma.

Now the following is the Coarea formula.

Corollary 36.3.9 *Let A be a measurable set in $\mathbb{R}^n$ and let $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a Lipschitz map where $m \leq n$. Then the following formula holds along with all measurability assertions needed for it to make sense.*

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m} (A \cap \mathbf{h}^{-1}(\mathbf{y})) \, d\mathbf{y} = \int_A J\mathbf{h}(\mathbf{x}) \, dx \quad (36.3.20)$$

where $J\mathbf{h}(\mathbf{x}) \equiv \det (D\mathbf{h}(\mathbf{x}) D\mathbf{h}(\mathbf{x})^*)^{1/2}$.

Proof: By Lemma 36.3.7 again, this formula is true for all measurable $A \subseteq \mathbb{R}^n \setminus S$. It remains to verify the formula for all measurable sets, A , whether or not they intersect S .

Consider the case where

$$A \subseteq S \equiv \{\mathbf{x} : J(D\mathbf{h}(\mathbf{x})) = 0\}.$$

Let A be compact so that by Lemma 36.3.6, $\mathbf{y} \rightarrow \mathcal{H}^{n-m} (A \cap \mathbf{h}^{-1}(\mathbf{y}))$ is Borel. For $\varepsilon > 0$, define $\mathbf{k}, \mathbf{p} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$\mathbf{k}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{h}(\mathbf{x}) + \varepsilon \mathbf{y}, \quad \mathbf{p}(\mathbf{x}, \mathbf{y}) \equiv \mathbf{y}.$$

Then

$$D\mathbf{k}(\mathbf{x}, \mathbf{y}) = (D\mathbf{h}(\mathbf{x}), \varepsilon I) = (UR, \varepsilon I)$$

where the dependence of U and R on $\mathbf{x}$ has been suppressed. Thus

$$\begin{aligned} J\mathbf{k}^2 &= \det (UR, \varepsilon I) \begin{pmatrix} R^* U \\ \varepsilon I \end{pmatrix} = \det (U^2 + \varepsilon^2 I) \\ &= \det (Q^* DQ Q^* DQ + \varepsilon^2 I) = \det (D^2 + \varepsilon^2 I) \\ &= \prod_{i=1}^m (\lambda_i^2 + \varepsilon^2) \in [\varepsilon^{2m}, C^2 \varepsilon^2] \end{aligned} \quad (36.3.21)$$

since one of the λ_i equals 0. All the eigenvalues must be bounded independent of $\mathbf{x}$, since $\|D\mathbf{h}(\mathbf{x})\|$ is bounded independent of $\mathbf{x}$ due to the assumption that $\mathbf{h}$ is Lipschitz. Since $J\mathbf{k} \neq 0$, the first part of the argument implies

$$\begin{aligned} \varepsilon C m_{n+m} \left(A \times \overline{B(\mathbf{0}, 1)} \right) &\geq \int_{A \times B(\mathbf{0}, 1)} |J\mathbf{k}| \, dm_{n+m} \\ &= \int_{\mathbb{R}^m} \mathcal{H}^n \left(\mathbf{k}^{-1}(\mathbf{y}) \cap A \times \overline{B(\mathbf{0}, 1)} \right) \, d\mathbf{y} \end{aligned}$$

Which by Lemma 36.3.6,

$$\geq C_{nm} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(\mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)} \right) \, dw d\mathbf{y} \quad (36.3.22)$$

where $C_{nm} = \frac{\alpha(n)}{\alpha(n-m)\alpha(m)}$.

Claim:

$$\begin{aligned} & \mathcal{H}^{n-m} \left(\mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)} \right) \\ & \geq \mathcal{X}_{\overline{B(\mathbf{0}, 1)}}(\mathbf{w}) \mathcal{H}^{n-m} \left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon \mathbf{w}) \cap A \right). \end{aligned}$$

Proof of the claim: If $\mathbf{w} \notin \overline{B(\mathbf{0}, 1)}$, there is nothing to prove so assume $\mathbf{w} \in \overline{B(\mathbf{0}, 1)}$. For such $\mathbf{w}$,

$$(\mathbf{x}, \mathbf{w}_1) \in \mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)}$$

if and only if $\mathbf{h}(\mathbf{x}) + \varepsilon \mathbf{w}_1 = \mathbf{y}$, $\mathbf{w}_1 = \mathbf{w}$, and $\mathbf{x} \in A$, if and only if

$$(\mathbf{x}, \mathbf{w}_1) \in \mathbf{h}^{-1}(\mathbf{y} - \varepsilon \mathbf{w}) \cap A \times \{\mathbf{w}\}.$$

Therefore for $\mathbf{w} \in \overline{B(\mathbf{0}, 1)}$,

$$\begin{aligned} & \mathcal{H}^{n-m} \left(\mathbf{k}^{-1}(\mathbf{y}) \cap \mathbf{p}^{-1}(\mathbf{w}) \cap A \times \overline{B(\mathbf{0}, 1)} \right) \\ & \geq \mathcal{H}^{n-m} \left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon \mathbf{w}) \cap A \times \{\mathbf{w}\} \right) = \mathcal{H}^{n-m} \left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon \mathbf{w}) \cap A \right). \end{aligned}$$

(Actually equality holds in the claim.) From the claim, 36.3.22 is at least as large as

$$C_{nm} \int_{\mathbb{R}^m} \int_{\overline{B(\mathbf{0}, 1)}} \mathcal{H}^{n-m} \left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon \mathbf{w}) \cap A \right) d\mathbf{w} d\mathbf{y} \quad (36.3.23)$$

$$\begin{aligned} & = C_{nm} \int_{\overline{B(\mathbf{0}, 1)}} \int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(\mathbf{h}^{-1}(\mathbf{y} - \varepsilon \mathbf{w}) \cap A \right) d\mathbf{y} d\mathbf{w} \\ & = \frac{\alpha(n)}{\alpha(n-m)} \int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(\mathbf{h}^{-1}(\mathbf{y}) \cap A \right) d\mathbf{y}. \end{aligned} \quad (36.3.24)$$

The use of Fubini's theorem is justified because the integrand is Borel measurable.

Now by 36.3.24, it follows since $\varepsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m} \left(A \cap \mathbf{h}^{-1}(\mathbf{y}) \right) d\mathbf{y} = 0 = \int_A J\mathbf{h}(\mathbf{x}) d\mathbf{x}.$$

Since this holds for arbitrary compact sets in S , it follows from Lemma 36.3.7 and inner regularity of Lebesgue measure that the equation holds for all measurable subsets of S . This completes the proof of the coarea formula. There is a simple corollary to this theorem in the case of locally Lipschitz maps.

Corollary 36.3.10 *Let $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ where $m \leq n$ and $\mathbf{h}$ is locally Lipschitz. Then the Coarea formula, 36.3.13, holds for $\mathbf{h}$.*

Proof: The assumption that $\mathbf{h}$ is locally Lipschitz implies that for each $r > 0$ it follows $\mathbf{h}$ is Lipschitz on $B(\mathbf{0}, r)$. To see this, cover the compact set, $\overline{B(\mathbf{0}, r)}$ with finitely many balls on which $\mathbf{h}$ is Lipschitz.

Let $A \subseteq B(\mathbf{0}, r)$ and let $\mathbf{h}_r$ be Lipschitz with

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}_r(\mathbf{x})$$

for $\mathbf{x} \in B(\mathbf{0}, r+1)$. Then

$$\begin{aligned} \int_A J(D\mathbf{h}(\mathbf{x})) dx &= \int_A J(D\mathbf{h}_r(\mathbf{x})) dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap \mathbf{h}_r^{-1}(\mathbf{y})) dy \\ &= \int_{\mathbf{h}_r(A)} \mathcal{H}^{n-m}(A \cap \mathbf{h}_r^{-1}(\mathbf{y})) dy = \int_{\mathbf{h}(A)} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy \end{aligned}$$

Now for arbitrary measurable A the above shows for $k = 1, 2, \dots$

$$\int_{A \cap B(\mathbf{0}, k)} J(D\mathbf{h}(\mathbf{x})) dx = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap B(\mathbf{0}, k) \cap \mathbf{h}^{-1}(\mathbf{y})) dy.$$

Use the monotone convergence theorem to obtain 36.3.13.

From the definition of Hausdorff measure it follows $\mathcal{H}^0(E)$ equals the number of elements in E . Thus, if $n = m$, the Coarea formula implies

$$\int_A J(D\mathbf{h}(\mathbf{x})) dx = \int_{\mathbf{h}(A)} \mathcal{H}^0(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy = \int_{\mathbf{h}(A)} \#(y) dy$$

This gives a version of Sard's theorem by letting $S = A$.

36.4 A Nonlinear Fubini's Theorem

Coarea formula holds for $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m, n \geq m$ if whenever A is a Lebesgue measurable subset of $\mathbb{R}^n$, the following formula is valid.

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy = \int_A J\mathbf{h}(\mathbf{x}) dx \quad (36.4.25)$$

Note this is the same as

$$\int_A J(D\mathbf{h}(\mathbf{x})) dx = \int_{\mathbf{h}(A)} \mathcal{H}^{n-m}(A \cap \mathbf{h}^{-1}(\mathbf{y})) dy$$

because if $\mathbf{y} \notin \mathbf{h}(A)$, then $\mathbf{h}^{-1}(\mathbf{y}) = \emptyset$. Now let

$$s(\mathbf{x}) = \sum_{i=1}^p c_i \mathcal{X}_{E_i}(\mathbf{x})$$

where E_i is measurable and $c_i \geq 0$. Then

$$\begin{aligned}
 \int_{\mathbb{R}^n} s(\mathbf{x}) J((D\mathbf{h}(\mathbf{x}))) dx &= \sum_{i=1}^p c_i \int_{E_i} J(D\mathbf{h}(\mathbf{x})) dx \\
 &= \sum_{i=1}^p c_i \int_{\mathbf{h}(E_i)} \mathcal{H}^{n-m}(E_i \cap \mathbf{h}^{-1}(\mathbf{y})) dy \\
 &= \int_{\mathbf{h}(\mathbb{R}^n)} \sum_{i=1}^p c_i \mathcal{H}^{n-m}(E_i \cap \mathbf{h}^{-1}(\mathbf{y})) dy \\
 &= \int_{\mathbf{h}(\mathbb{R}^n)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} s d\mathcal{H}^{n-m} \right] dy \\
 &= \int_{\mathbf{h}(\mathbb{R}^n)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} s d\mathcal{H}^{n-m} \right] dy. \tag{36.4.26}
 \end{aligned}$$

Theorem 36.4.1 *Let $g \geq 0$ be Lebesgue measurable and let*

$$\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad n \geq m$$

satisfy the Coarea formula. For example, it could be locally Lipschitz. Then

$$\int_{\mathbb{R}^n} g(\mathbf{x}) J((D\mathbf{h}(\mathbf{x}))) dx = \int_{\mathbf{h}(\mathbb{R}^n)} \left[\int_{\mathbf{h}^{-1}(\mathbf{y})} g d\mathcal{H}^{n-m} \right] dy.$$

Proof: Let $s_i \uparrow g$ where s_i is a simple function satisfying 36.4.26. Then let $i \rightarrow \infty$ and use the monotone convergence theorem to replace s_i with g . This proves the change of variables formula.

Note that this formula is a nonlinear version of Fubini's theorem. The “ $n - m$ dimensional surface”, $\mathbf{h}^{-1}(\mathbf{y})$, plays the role of $\mathbb{R}^{n-m}$ and $\mathcal{H}^{n-m}$ is like $n - m$ dimensional Lebesgue measure. The term, $J((D\mathbf{h}(\mathbf{x})))$, corrects for the error occurring because of the lack of flatness of $\mathbf{h}^{-1}(\mathbf{y})$.

Integration On Manifolds

You can do integration on various manifolds by using the Hausdorff measure of an appropriate dimension. However, it is possible to discuss this through the use of the Riesz representation theorem and some of the machinery for accomplishing this is interesting for its own sake so I will present this alternate point of view.

37.1 Partitions Of Unity

This material has already been mostly discussed starting on Page 602. However, that was a long time ago and it seems like it might be good to go over it again and so, for the sake of convenience, here it is again.

Definition 37.1.1 *Let $\mathfrak{C}$ be a set whose elements are subsets of $\mathbb{R}^n$.¹ Then $\mathfrak{C}$ is said to be locally finite if for every $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, $U_{\mathbf{x}}$ containing $\mathbf{x}$ such that $U_{\mathbf{x}}$ has nonempty intersection with only finitely many sets of $\mathfrak{C}$.*

Lemma 37.1.2 *Let $\mathfrak{C}$ be a set whose elements are open subsets of $\mathbb{R}^n$ and suppose $\cup \mathfrak{C} \supseteq H$, a closed set. Then there exists a countable list of open sets, $\{U_i\}_{i=1}^{\infty}$ such that each U_i is bounded, each U_i is a subset of some set of $\mathfrak{C}$, and $\cup_{i=1}^{\infty} U_i \supseteq H$.*

Proof: Let $W_k \equiv B(\mathbf{0}, k)$, $W_0 = W_{-1} = \emptyset$. For each $\mathbf{x} \in H \cap \overline{W_k}$ there exists an open set, $U_{\mathbf{x}}$ such that $U_{\mathbf{x}}$ is a subset of some set of $\mathfrak{C}$ and $U_{\mathbf{x}} \subseteq W_{k+1} \setminus \overline{W_{k-1}}$. Then since $H \cap \overline{W_k}$ is compact, there exist finitely many of these sets, $\{U_i^k\}_{i=1}^{m(k)}$ whose union contains $H \cap \overline{W_k}$. If $H \cap \overline{W_k} = \emptyset$, let $m(k) = 0$ and there are no such sets obtained. The desired countable list of open sets is $\cup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$. Each open set in this list is bounded. Furthermore, if $\mathbf{x} \in \mathbb{R}^n$, then $\mathbf{x} \in W_k$ where k is the first positive integer with $\mathbf{x} \in W_k$. Then $W_k \setminus \overline{W_{k-1}}$ is an open set containing $\mathbf{x}$ and this open set can have nonempty intersection only with with a set of $\{U_i^k\}_{i=1}^{m(k)} \cup \{U_i^{k-1}\}_{i=1}^{m(k-1)}$, a finite list of sets. Therefore, $\cup_{k=1}^{\infty} \{U_i^k\}_{i=1}^{m(k)}$ is locally finite.

The set, $\{U_i\}_{i=1}^{\infty}$ is said to be a locally finite cover of H . The following lemma gives some important reasons why a locally finite list of sets is so significant. First

¹The definition applies with no change to a general topological space in place of $\mathbb{R}^n$.

of all consider the rational numbers, $\{r_i\}_{i=1}^\infty$ each rational number is a closed set.

$$\mathbb{Q} = \{r_i\}_{i=1}^\infty = \bigcup_{i=1}^\infty \overline{\{r_i\}} \neq \overline{\bigcup_{i=1}^\infty \{r_i\}} = \mathbb{R}$$

The set of rational numbers is definitely not locally finite.

Lemma 37.1.3 *Let $\mathfrak{C}$ be locally finite. Then*

$$\overline{\bigcup \mathfrak{C}} = \bigcup \{\overline{H} : H \in \mathfrak{C}\}.$$

Next suppose the elements of $\mathfrak{C}$ are open sets and that for each $U \in \mathfrak{C}$, there exists a differentiable function, ψ_U having $\text{spt}(\psi_U) \subseteq U$. Then you can define the following finite sum for each $\mathbf{x} \in \mathbb{R}^n$

$$f(\mathbf{x}) \equiv \sum \{\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

Furthermore, f is also a differentiable function² and

$$Df(\mathbf{x}) = \sum \{D\psi_U(\mathbf{x}) : \mathbf{x} \in U \in \mathfrak{C}\}.$$

Proof: Let $\mathbf{p}$ be a limit point of $\bigcup \mathfrak{C}$ and let W be an open set which intersects only finitely many sets of $\mathfrak{C}$. Then $\overline{\mathbf{p}}$ must be a limit point of one of these sets. It follows $\mathbf{p} \in \bigcup \{\overline{H} : H \in \mathfrak{C}\}$ and so $\overline{\bigcup \mathfrak{C}} \subseteq \bigcup \{\overline{H} : H \in \mathfrak{C}\}$. The inclusion in the other direction is obvious.

Now consider the second assertion. Letting $\mathbf{x} \in \mathbb{R}^n$, there exists an open set, W intersecting only finitely many open sets of $\mathfrak{C}$, $U_1, U_2, \dots, U_m$. Then for all $\mathbf{y} \in W$,

$$f(\mathbf{y}) = \sum_{i=1}^m \psi_{U_i}(\mathbf{y})$$

and so the desired result is obvious. It merely says that a finite sum of differentiable functions is differentiable. Recall the following definition.

Definition 37.1.4 *Let K be a closed subset of an open set, U . $K \prec f \prec U$ if f is continuous, has values in $[0, 1]$, equals 1 on K , and has compact support contained in U .*

Lemma 37.1.5 *Let U be a bounded open set and let K be a closed subset of U . Then there exist an open set, W , such that $W \subseteq \overline{W} \subseteq U$ and a function, $f \in C_c^\infty(U)$ such that $K \prec f \prec U$.*

Proof: The set, K is compact so is at a positive distance from U^C . Let

$$W \equiv \{\mathbf{x} : \text{dist}(\mathbf{x}, K) < 3^{-1} \text{dist}(K, U^C)\}.$$

²If each ψ_U were only continuous, one could conclude f is continuous. Here the main interest is differentiable.

Also let

$$W_1 \equiv \{\mathbf{x} : \text{dist}(\mathbf{x}, K) < 2^{-1} \text{dist}(K, U^C)\}$$

Then it is clear

$$K \subseteq W \subseteq \overline{W} \subseteq W_1 \subseteq \overline{W_1} \subseteq U$$

Now consider the function,

$$h(\mathbf{x}) \equiv \frac{\text{dist}(\mathbf{x}, W_1^C)}{\text{dist}(\mathbf{x}, W_1^C) + \text{dist}(\mathbf{x}, \overline{W})}$$

Since $\overline{W}$ is compact it is at a positive distance from W_1^C and so h is a well defined continuous function which has compact support contained in $\overline{W_1}$, equals 1 on W , and has values in $[0, 1]$. Now let ϕ_k be a mollifier. Letting

$$k^{-1} < \min(\text{dist}(K, W^C), 2^{-1} \text{dist}(\overline{W_1}, U^C)),$$

it follows that for such k , the function, $h * \phi_k \in C_c^\infty(U)$, has values in $[0, 1]$, and equals 1 on K . Let $f = h * \phi_k$.

The above lemma is used repeatedly in the following.

Lemma 37.1.6 *Let K be a closed set and let $\{V_i\}_{i=1}^\infty$ be a locally finite list of bounded open sets whose union contains K . Then there exist functions, $\psi_i \in C_c^\infty(V_i)$ such that for all $\mathbf{x} \in K$,*

$$1 = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$$

and the function $f(\mathbf{x})$ given by

$$f(\mathbf{x}) = \sum_{i=1}^{\infty} \psi_i(\mathbf{x})$$

is in $C^\infty(\mathbb{R}^n)$.

Proof: Let $K_1 = K \setminus \bigcup_{i=2}^\infty V_i$. Thus K_1 is compact because $K_1 \subseteq V_1$. Let

$$K_1 \subseteq W_1 \subseteq \overline{W_1} \subseteq V_1$$

Thus $W_1, V_2, \dots, V_n$ covers K and $\overline{W_1} \subseteq V_1$. Suppose $W_1, \dots, W_r$ have been defined such that $\overline{W_i} \subseteq V_i$ for each i , and $W_1, \dots, W_r, V_{r+1}, \dots, V_n$ covers K . Then let

$$K_{r+1} \equiv K \setminus ((\bigcup_{i=r+2}^\infty V_i) \cup (\bigcup_{j=1}^r W_j)).$$

It follows K_{r+1} is compact because $K_{r+1} \subseteq V_{r+1}$. Let W_{r+1} satisfy

$$K_{r+1} \subseteq W_{r+1} \subseteq \overline{W_{r+1}} \subseteq V_{r+1}$$

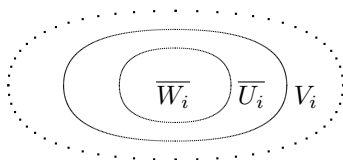
Continuing this way defines a sequence of open sets, $\{W_i\}_{i=1}^\infty$ with the property

$$\overline{W_i} \subseteq V_i, \quad K \subseteq \bigcup_{i=1}^\infty W_i.$$

Note $\{W_i\}_{i=1}^\infty$ is locally finite because the original list, $\{V_i\}_{i=1}^\infty$ was locally finite. Now let U_i be open sets which satisfy

$$\overline{W_i} \subseteq U_i \subseteq \overline{U_i} \subseteq V_i.$$

Similarly, $\{U_i\}_{i=1}^\infty$ is locally finite.



Since the set, $\{W_i\}_{i=1}^\infty$ is locally finite, it follows $\overline{\bigcup_{i=1}^\infty W_i} = \bigcup_{i=1}^\infty \overline{W_i}$ and so it is possible to define ϕ_i and γ , infinitely differentiable functions having compact support such that

$$\overline{U_i} \prec \phi_i \prec V_i, \quad \bigcup_{i=1}^\infty \overline{W_i} \prec \gamma \prec \bigcup_{i=1}^\infty U_i.$$

Now define

$$\psi_i(\mathbf{x}) = \begin{cases} \gamma(\mathbf{x})\phi_i(\mathbf{x}) / \sum_{j=1}^\infty \phi_j(\mathbf{x}) & \text{if } \sum_{j=1}^\infty \phi_j(\mathbf{x}) \neq 0, \\ 0 & \text{if } \sum_{j=1}^\infty \phi_j(\mathbf{x}) = 0. \end{cases}$$

If $\mathbf{x}$ is such that $\sum_{j=1}^\infty \phi_j(\mathbf{x}) = 0$, then $\mathbf{x} \notin \bigcup_{i=1}^\infty \overline{U_i}$ because ϕ_i equals one on $\overline{U_i}$. Consequently $\gamma(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$ thanks to the fact that $\bigcup_{i=1}^\infty \overline{U_i}$ is closed and so $\psi_i(\mathbf{y}) = 0$ for all $\mathbf{y}$ near $\mathbf{x}$. Hence ψ_i is infinitely differentiable at such $\mathbf{x}$. If $\sum_{j=1}^\infty \phi_j(\mathbf{x}) \neq 0$, this situation persists near $\mathbf{x}$ because each ϕ_j is continuous and so ψ_i is infinitely differentiable at such points also thanks to Lemma 37.1.3. Therefore ψ_i is infinitely differentiable. If $\mathbf{x} \in K$, then $\gamma(\mathbf{x}) = 1$ and so $\sum_{j=1}^\infty \psi_j(\mathbf{x}) = 1$. Clearly $0 \leq \psi_i(\mathbf{x}) \leq 1$ and $\text{spt}(\psi_j) \subseteq V_j$. This proves the theorem.

The method of proof of this lemma easily implies the following useful corollary.

Corollary 37.1.7 *If H is a compact subset of V_i for some V_i there exists a partition of unity such that $\psi_i(x) = 1$ for all $x \in H$ in addition to the conclusion of Lemma 37.1.6.*

Proof: Keep V_i the same but replace V_j with $\widetilde{V_j} \equiv V_j \setminus H$. Now in the proof above, applied to this modified collection of open sets, if $j \neq i$, $\phi_j(x) = 0$ whenever $x \in H$. Therefore, $\psi_i(x) = 1$ on H .

Theorem 37.1.8 *Let H be any closed set and let $\mathfrak{C}$ be any open cover of H . Then there exist functions $\{\psi_i\}_{i=1}^\infty$ such that $\text{spt}(\psi_i)$ is contained in some set of $\mathfrak{C}$ and ψ_i is infinitely differentiable having values in $[0, 1]$ such that on H , $\sum_{i=1}^\infty \psi_i(\mathbf{x}) = 1$. Furthermore, the function, $f(\mathbf{x}) \equiv \sum_{i=1}^\infty \psi_i(\mathbf{x})$ is infinitely differentiable on $\mathbb{R}^n$. Also, $\text{spt}(\psi_i) \subseteq U_i$ where U_i is a bounded open set with the property that $\{U_i\}_{i=1}^\infty$ is locally finite and each U_i is contained in some set of $\mathfrak{C}$.*

Proof: By Lemma 37.1.2 there exists an open cover of H composed of bounded open sets, U_i such that each U_i is a subset of some set of $\mathfrak{C}$ and the collection, $\{U_i\}_{i=1}^\infty$ is locally finite. Then the result follows from Lemma 37.1.6 and Lemma 37.1.3.

Corollary 37.1.9 *Let H be any closed set and let $\{V_i\}_{i=1}^m$ be a finite open cover of H . Then there exist functions $\{\phi_i\}_{i=1}^m$ such that $\text{spt}(\phi_i) \subseteq V_i$ and ϕ_i is infinitely differentiable having values in $[0, 1]$ such that on H , $\sum_{i=1}^m \phi_i(\mathbf{x}) = 1$.*

Proof: By Theorem 37.1.8 there exists a set of functions, $\{\psi_i\}_{i=1}^\infty$ having the properties listed in this theorem relative to the open covering, $\{V_i\}_{i=1}^m$. Let $\phi_1(\mathbf{x})$ equal the sum of all $\psi_j(\mathbf{x})$ such that $\text{spt}(\psi_j) \subseteq V_1$. Next let $\phi_2(\mathbf{x})$ equal the sum of all $\psi_j(\mathbf{x})$ which have not already been included and for which $\text{spt}(\psi_j) \subseteq V_2$. Continue in this manner. Since the open sets, $\{U_i\}_{i=1}^\infty$ mentioned in Theorem 37.1.8 are locally finite, it follows from Lemma 37.1.3 that each ϕ_i is infinitely differentiable having support in V_i . This proves the corollary.

37.2 Integration On Manifolds

Manifolds are things which locally appear to be $\mathbb{R}^n$ for some n . The extent to which they have such a local appearance varies according to various analytical characteristics which the manifold possesses.

Definition 37.2.1 *Let $U \subseteq \mathbb{R}^n$ be an open set and let $\mathbf{h} : U \rightarrow \mathbb{R}^m$. Then for $r \in [0, 1)$, $\mathbf{h} \in C^{k,r}(U)$ for k a nonnegative integer means that $D^\alpha \mathbf{h}$ exists for all $|\alpha| \leq k$ and each $D^\alpha \mathbf{h}$ is Holder continuous with exponent r . That is*

$$|D^\alpha \mathbf{h}(\mathbf{x}) - D^\alpha \mathbf{h}(\mathbf{y})| \leq K |\mathbf{x} - \mathbf{y}|^r.$$

Also $\mathbf{h} \in C^{k,r}(\bar{U})$ if it is the restriction of a function of $C^{k,r}(\mathbb{R}^n)$ to U .

Definition 37.2.2 *Let Γ be a closed subset of $\mathbb{R}^p$ where $p \geq n$. Suppose $\Gamma = \bigcup_{i=1}^\infty \Gamma_i$ where $\Gamma_i = \Gamma \cap W_i$ for W_i a bounded open set. Suppose also $\{W_i\}_{i=1}^\infty$ is locally finite. This means every bounded open set intersects only finitely many. Also suppose there are open bounded sets, U_i having Lipschitz boundaries and functions $\mathbf{h}_i : U_i \rightarrow \Gamma_i$ which are one to one, onto, and in $C^{m,1}(U_i)$. Suppose also there exist functions, $\mathbf{g}_i : W_i \rightarrow U_i$ such that $\mathbf{g}_i$ is $C^{m,1}(W_i)$, and $\mathbf{g}_i \circ \mathbf{h}_i = \text{id}$ on U_i while $\mathbf{h}_i \circ \mathbf{g}_i = \text{id}$ on Γ_i . The collection of sets, Γ_j and mappings, $\mathbf{g}_j, \{(\Gamma_j, \mathbf{g}_j)\}$ is called an atlas and an individual entry in the atlas is called a chart. Thus $(\Gamma_j, \mathbf{g}_j)$ is a chart. Then Γ as just described is called a $C^{m,1}$ manifold. The number, m is just a nonnegative integer. When $m = 0$ this would be called a Lipschitz manifold, the least smooth of the manifolds discussed here.*

For example, take $p = n + 1$ and let

$$\mathbf{h}_i(\mathbf{u}) = (u_1, \dots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \dots, u_n)^T$$

for $\mathbf{u} = (u_1, \dots, u_i, u_{i+1}, \dots, u_n)^T \in U_i$ for $\phi_i \in C^{m,1}(U_i)$ and $\mathbf{g}_i : U_i \times \mathbb{R} \rightarrow U_i$ given by

$$\mathbf{g}_i(u_1, \dots, u_i, y, u_{i+1}, \dots, u_n) \equiv \mathbf{u}$$

for $i = 1, 2, \dots, p$. Then for $\mathbf{u} \in U_i$, the definition gives

$$\mathbf{g}_i \circ \mathbf{h}_i(\mathbf{u}) = \mathbf{g}_i(u_1, \dots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \dots, u_n) = \mathbf{u}$$

and for $\Gamma_i \equiv \mathbf{h}_i(U_i)$ and $(u_1, \dots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \dots, u_n)^T \in \Gamma_i$,

$$\begin{aligned} \mathbf{h}_i \circ \mathbf{g}_i(u_1, \dots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \dots, u_n) \\ = \mathbf{h}_i(\mathbf{u}) = (u_1, \dots, u_i, \phi_i(\mathbf{u}), u_{i+1}, \dots, u_n)^T. \end{aligned}$$

This example can be used to describe the boundary of a bounded open set and since $\phi_i \in C^{m,1}(U_i)$, such an open set is said to have a $C^{m,1}$ boundary. Note also that in this example, U_i could be taken to be $\mathbb{R}^n$ or if U_i is given, both $\mathbf{h}_i$ and $\mathbf{g}_i$ can be taken as restrictions of functions defined on all of $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively.

The symbol, I will refer to an increasing list of n indices taken from $\{1, \dots, p\}$. Denote by $\Lambda(p, n)$ the set of all such increasing lists of n indices.

Let

$$J_i(\mathbf{u}) \equiv \left[\sum_{I \in \Lambda(p, n)} \left(\frac{\partial(x^{i_1} \dots x^{i_n})}{\partial(u^1 \dots u^n)} \right)^2 \right]^{1/2}$$

where here the sum is taken over all possible increasing lists of n indices, I , from $\{1, \dots, p\}$ and $\mathbf{x} = \mathbf{h}_i \mathbf{u}$. Thus there are $\binom{p}{n}$ terms in the sum. In this formula, $\frac{\partial(x^{i_1} \dots x^{i_n})}{\partial(u^1 \dots u^n)}$ is defined to be the determinant of the following matrix.

$$\begin{pmatrix} \frac{\partial x^{i_1}}{\partial u_1} & \dots & \frac{\partial x^{i_1}}{\partial u_n} \\ \vdots & & \vdots \\ \frac{\partial x^{i_n}}{\partial u_1} & \dots & \frac{\partial x^{i_n}}{\partial u_n} \end{pmatrix}.$$

Note that if $p = n$ there is only one term in the sum, the absolute value of the determinant of $D\mathbf{x}(\mathbf{u})$. Define a positive linear functional, Λ on $C_c(\Gamma)$ as follows: First let $\{\psi_i\}$ be a C^∞ partition of unity subordinate to the open sets, $\{W_i\}$. Thus $\psi_i \in C_c^\infty(W_i)$ and $\sum_i \psi_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Gamma$. Then

$$\Lambda f \equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} f \psi_i(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) du. \quad (37.2.1)$$

Is this well defined?

Lemma 37.2.3 *The functional defined in 37.2.1 does not depend on the choice of atlas or the partition of unity.*

Proof: In 37.2.1, let $\{\psi_i\}$ be a C^∞ partition of unity which is associated with the atlas $(\Gamma_i, \mathbf{g}_i)$ and let $\{\eta_i\}$ be a C^∞ partition of unity associated in the same manner with the atlas $(\Gamma'_i, \mathbf{g}'_i)$. In the following argument, the local finiteness of the Γ_i implies that all sums are finite. Using the change of variables formula with $\mathbf{u} = (\mathbf{g}_i \circ \mathbf{h}'_j) \mathbf{v}$

$$\sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} \psi_i f(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) d\mathbf{u} = \quad (37.2.2)$$

$$\begin{aligned} & \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} \eta_j \psi_i f(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) d\mathbf{u} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}'_j(\Gamma_i \cap \Gamma'_j)} \\ & \eta_j(\mathbf{h}'_j(\mathbf{v})) \psi_i(\mathbf{h}'_j(\mathbf{v})) f(\mathbf{h}'_j(\mathbf{v})) J_i(\mathbf{u}) \left| \frac{\partial(u^1 \cdots u^n)}{\partial(v^1 \cdots v^n)} \right| d\mathbf{v} \\ & = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}'_j(\Gamma_i \cap \Gamma'_j)} \eta_j(\mathbf{h}'_j(\mathbf{v})) \psi_i(\mathbf{h}'_j(\mathbf{v})) f(\mathbf{h}'_j(\mathbf{v})) J_j(\mathbf{v}) d\mathbf{v}. \end{aligned} \quad (37.2.3)$$

Thus

the definition of Λf using $(\Gamma_i, \mathbf{g}_i) \equiv$

$$\begin{aligned} & \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} \psi_i f(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) d\mathbf{u} = \\ & \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{\mathbf{g}'_j(\Gamma_i \cap \Gamma'_j)} \eta_j(\mathbf{h}'_j(\mathbf{v})) \psi_i(\mathbf{h}'_j(\mathbf{v})) f(\mathbf{h}'_j(\mathbf{v})) J_j(\mathbf{v}) d\mathbf{v} \\ & = \sum_{j=1}^{\infty} \int_{\mathbf{g}'_j(\Gamma'_j)} \eta_j(\mathbf{h}'_j(\mathbf{v})) f(\mathbf{h}'_j(\mathbf{v})) J_j(\mathbf{v}) d\mathbf{v} \end{aligned}$$

the definition of Λf using $(V_i, \mathbf{g}'_i)$.

This proves the lemma.

This lemma and the Riesz representation theorem for positive linear functionals implies the part of the following theorem which says the functional is well defined.

Theorem 37.2.4 *Let Γ be a $C^{m,1}$ manifold. Then there exists a unique Radon measure, μ , defined on Γ such that whenever f is a continuous function having compact support which is defined on Γ and $(\Gamma_i, \mathbf{g}_i)$ denotes an atlas and $\{\psi_i\}$ a partition of unity subordinate to this atlas,*

$$\Lambda f = \int_{\Gamma} f d\mu = \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} \psi_i f(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) d\mathbf{u}. \quad (37.2.4)$$

Also, a subset, A , of Γ is μ measurable if and only if for all r , $\mathbf{g}_r(\Gamma_r \cap A)$ is ν_r measurable where ν_r is the measure defined by

$$\nu_r(\mathbf{g}_r(\Gamma_r \cap A)) \equiv \int_{\mathbf{g}_r(\Gamma_r \cap A)} J_r(\mathbf{u}) d\mathbf{u}$$

Proof: To begin, here is a claim.

Claim : A set, $S \subseteq \Gamma_i$, has μ measure zero if and only if $\mathbf{g}_i S$ has measure zero in $\mathbf{g}_i \Gamma_i$ with respect to the measure, ν_i .

Proof of the claim: Let $\varepsilon > 0$ be given. By outer regularity, there exists a set, $V \subseteq \Gamma_i$, open³ in Γ such that $\mu(V) < \varepsilon$ and $S \subseteq V \subseteq \Gamma_i$. Then $\mathbf{g}_i V$ is open in $\mathbb{R}^n$ and contains $\mathbf{g}_i S$. Letting $h \prec \mathbf{g}_i V$ and $h_1(\mathbf{x}) \equiv h(\mathbf{g}_i(\mathbf{x}))$ for $\mathbf{x} \in \Gamma_i$ it follows $h_1 \prec V$. By Corollary 37.1.7 on Page 1088 there exists a partition of unity such that $\text{spt}(h_1) \subseteq \{\mathbf{x} \in \mathbb{R}^p : \psi_i(\mathbf{x}) = 1\}$. Thus $\psi_j h_1(\mathbf{h}_j(u)) = 0$ unless $j = i$ when this reduces to $h_1(\mathbf{h}_i(u))$. It follows

$$\begin{aligned} \varepsilon &\geq \mu(V) \geq \int_V h_1 d\mu = \int_{\Gamma} h_1 d\mu \\ &= \sum_{j=1}^{\infty} \int_{\mathbf{g}_j \Gamma_j} \psi_j h_1(\mathbf{h}_j(\mathbf{u})) J_j(\mathbf{u}) du \\ &= \int_{\mathbf{g}_i \Gamma_i} h_1(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) du = \int_{\mathbf{g}_i \Gamma_i} h(\mathbf{u}) J_i(\mathbf{u}) du \\ &= \int_{\mathbf{g}_i V} h(\mathbf{u}) J_i(\mathbf{u}) du \end{aligned}$$

Now this holds for all $h \prec \mathbf{g}_i V$ and so

$$\int_{\mathbf{g}_i V} J_i(\mathbf{u}) du \leq \varepsilon.$$

Since ε is arbitrary, this shows $\mathbf{g}_i V$ has measure no more than ε with respect to the measure, ν_i . Since ε is arbitrary, $\mathbf{g}_i S$ has measure zero.

Consider the converse. Suppose $\mathbf{g}_i S$ has ν_i measure zero. Then there exists an open set, $O \subseteq \mathbf{g}_i \Gamma_i$ such that $O \supseteq \mathbf{g}_i S$ and

$$\int_O J_i(\mathbf{u}) du < \varepsilon.$$

Thus $\mathbf{h}_i(O)$ is open in Γ and contains S . Let $h \prec \mathbf{h}_i(O)$ be such that

$$\int_{\Gamma} h d\mu + \varepsilon > \mu(\mathbf{h}_i(O)) \geq \mu(S) \quad (37.2.5)$$

As in the first part, Corollary 37.1.7 on Page 1088 implies there exists a partition of unity such that $h(\mathbf{x}) = 0$ off the set,

$$\{\mathbf{x} \in \mathbb{R}^p : \psi_i(\mathbf{x}) = 1\}$$

³This means V is the intersection of an open set with Γ . Equivalently, it means that V is an open set in the traditional way regarding Γ as a metric space with the metric it inherits from $\mathbb{R}^m$.

and so as in this part of the argument,

$$\begin{aligned}
 \int_{\Gamma} h d\mu &\equiv \sum_{j=1}^{\infty} \int_{\mathbf{g}_j U_j} \psi_j h(\mathbf{h}_j(\mathbf{u})) J_j(\mathbf{u}) du \\
 &= \int_{\mathbf{g}_i \Gamma_i} h(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) du \\
 &= \int_{O \cap \mathbf{g}_i \Gamma_i} h(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) du \\
 &\leq \int_O J_i(\mathbf{u}) du < \varepsilon
 \end{aligned} \tag{37.2.6}$$

and so from 37.2.5 and 37.2.6 $\mu(S) \leq 2\varepsilon$. Since ε is arbitrary, this proves the claim.

For the last part of the theorem, it suffices to let $A \subseteq \Gamma_r$ because otherwise, the above argument would apply to $A \cap \Gamma_r$. Thus let $A \subseteq \Gamma_r$ be μ measurable. By the regularity of the measure, there exists an F_σ set, F and a G_δ set, G such that $\Gamma_r \supseteq G \supseteq A \supseteq F$ and $\mu(G \setminus F) = 0$. (Recall a G_δ set is a countable intersection of open sets and an F_σ set is a countable union of closed sets.) Then since $\bar{\Gamma}_r$ is compact, it follows each of the closed sets whose union equals F is a compact set. Thus if $F = \cup_{k=1}^{\infty} F_k$, $\mathbf{g}_r(F_k)$ is also a compact set and so $\mathbf{g}_r(F) = \cup_{k=1}^{\infty} \mathbf{g}_r(F_k)$ is a Borel set. Similarly, $\mathbf{g}_r(G)$ is also a Borel set. Now by the claim,

$$\int_{\mathbf{g}_r(G \setminus F)} J_r(\mathbf{u}) du = 0.$$

Since $\mathbf{g}_r$ is one to one,

$$\mathbf{g}_r G \setminus \mathbf{g}_r F = \mathbf{g}_r(G \setminus F)$$

and so

$$\mathbf{g}_r(F) \subseteq \mathbf{g}_r(A) \subseteq \mathbf{g}_r(G)$$

where $\mathbf{g}_r(G) \setminus \mathbf{g}_r(F)$ has measure zero. By completeness of the measure, ν_r , $\mathbf{g}_r(A)$ is measurable. It follows that if $A \subseteq \Gamma$ is μ measurable, then $\mathbf{g}_r(\Gamma_r \cap A)$ is ν_r measurable for all r . The converse is entirely similar. This proves the theorem.

Corollary 37.2.5 *Let $f \in L^1(\Gamma; \mu)$ and suppose $f(\mathbf{x}) = 0$ for all $\mathbf{x} \notin \Gamma_r$ where $(\Gamma_r, \mathbf{g}_r)$ is a chart. Then*

$$\int_{\Gamma} f d\mu = \int_{\Gamma_r} f d\mu = \int_{\mathbf{g}_r \Gamma_r} f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du. \tag{37.2.7}$$

Furthermore, if $\{(\Gamma_i, \mathbf{g}_i)\}$ is an atlas and $\{\psi_i\}$ is a partition of unity as described earlier, then for any $f \in L^1(\Gamma, \mu)$,

$$\int_{\Gamma} f d\mu = \sum_{r=1}^{\infty} \int_{\mathbf{g}_r \Gamma_r} \psi_r f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du. \tag{37.2.8}$$

Proof: Let $f \in L^1(\Gamma, \mu)$ with $f = 0$ off Γ_r . Without loss of generality assume $f \geq 0$ because if the formulas can be established for this case, the same formulas are obtained for an arbitrary complex valued function by splitting it up into positive and negative parts of the real and imaginary parts in the usual way. Also, let $K \subseteq \Gamma_r$ a compact set. Since μ is a Radon measure there exists a sequence of continuous functions, $\{f_k\}$, $f_k \in C_c(\Gamma_r)$, which converges to f in $L^1(\Gamma, \mu)$ and for μ a.e. $\mathbf{x}$. Take the partition of unity, $\{\psi_i\}$ to be such that

$$K \subseteq \{\mathbf{x} : \psi_r(\mathbf{x}) = 1\}.$$

Therefore, the sequence $\{f_k(\mathbf{h}_r(\cdot))\}$ is a Cauchy sequence in the sense that

$$\lim_{k, l \rightarrow \infty} \int_{\mathbf{g}_r(K)} |f_k(\mathbf{h}_r(\mathbf{u})) - f_l(\mathbf{h}_r(\mathbf{u}))| J_r(\mathbf{u}) du = 0$$

It follows there exists g such that

$$\int_{\mathbf{g}_r(K)} |f_k(\mathbf{h}_r(\mathbf{u})) - g(\mathbf{u})| J_r(\mathbf{u}) du \rightarrow 0,$$

and

$$g \in L^1(\mathbf{g}_r K; \nu_r).$$

By the pointwise convergence and the claim used in the proof of Theorem 37.2.4,

$$g(\mathbf{u}) = f(\mathbf{h}_r(\mathbf{u}))$$

for μ a.e. $\mathbf{h}_r(\mathbf{u}) \in K$. Therefore,

$$\begin{aligned} \int_K f d\mu &= \lim_{k \rightarrow \infty} \int_K f_k d\mu = \lim_{k \rightarrow \infty} \int_{\mathbf{g}_r(K)} f_k(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du \\ &= \int_{\mathbf{g}_r(K)} g(\mathbf{u}) J_r(\mathbf{u}) du = \int_{\mathbf{g}_r(K)} f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du. \end{aligned} \quad (37.2.9)$$

Now let $\cdots K_j \subseteq K_{j+1} \cdots$ and $\cup_{j=1}^\infty K_j = \Gamma_r$ where K_j is compact for all j . Replace K in 37.2.9 with K_j and take a limit as $j \rightarrow \infty$. By the monotone convergence theorem,

$$\int_{\Gamma_r} f d\mu = \int_{\mathbf{g}_r(\Gamma_r)} f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du.$$

This establishes 37.2.7.

To establish 37.2.8, let $f \in L^1(\Gamma, \mu)$ and let $\{(\Gamma_i, \mathbf{g}_i)\}$ be an atlas and $\{\psi_i\}$ be a partition of unity. Then $f\psi_i \in L^1(\Gamma, \mu)$ and is zero off Γ_i . Therefore, from what was just shown,

$$\begin{aligned} \int_\Gamma f d\mu &= \sum_{i=1}^\infty \int_{\Gamma_i} f\psi_i d\mu \\ &= \sum_{r=1}^\infty \int_{\mathbf{g}_r(\Gamma_r)} \psi_r f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du \end{aligned}$$

37.3 Comparison With $\mathcal{H}^n$

The above gives a measure on a manifold, Γ . I will now show that the measure obtained is nothing more than $\mathcal{H}^n$, the n dimensional Hausdorff measure. Recall $\Lambda(p, n)$ was the set of all increasing lists of n indices taken from $\{1, 2, \dots, p\}$

Recall

$$J_i(\mathbf{u}) \equiv \left[\sum_{I \in \Lambda(p, n)} \left(\frac{\partial (x^{i_1} \dots x^{i_n})}{\partial (u^1 \dots u^n)} \right)^2 \right]^{1/2}$$

where here the sum is taken over all possible increasing lists of n indices, I , from $\{1, \dots, p\}$ and $\mathbf{x} = \mathbf{h}_i \mathbf{u}$ and the functional was given as

$$\Lambda f \equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} f \psi_i(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) du \quad (37.3.10)$$

where the $\{\psi_i\}_{i=1}^{\infty}$ was a partition of unity subordinate to the open sets, $\{W_i\}_{i=1}^{\infty}$ as described above. I will show

$$J_i(\mathbf{u}) = \det(D\mathbf{h}(\mathbf{u})^* D\mathbf{h}(\mathbf{u}))^{1/2}$$

and then use the area formula. The key result is really a special case of the Binet Cauchy theorem and this special case is presented in the next lemma.

Lemma 37.3.1 *Let $A = (a_{ij})$ be a real $p \times n$ matrix in which $p \geq n$. For $I \in \Lambda(p, n)$ denote by A_I the $n \times n$ matrix obtained by deleting from A all rows except for those corresponding to an element of I . Then*

$$\sum_{I \in \Lambda(p, n)} \det(A_I)^2 = \det(A^* A)$$

Proof: For $(j_1, \dots, j_n) \in \Lambda(p, n)$, define $\theta(j_k) \equiv k$. Then let for $\{k_1, \dots, k_n\} = \{j_1, \dots, j_n\}$ define

$$\text{sgn}(k_1, \dots, k_n) \equiv \text{sgn}(\theta(k_1), \dots, \theta(k_n)).$$

Then from the definition of the determinant and matrix multiplication,

$$\begin{aligned} \det(A^* A) &= \sum_{i_1, \dots, i_n} \text{sgn}(i_1, \dots, i_n) \sum_{k_1=1}^p a_{k_1 i_1} a_{k_1 1} \sum_{k_2=1}^p a_{k_2 i_2} a_{k_2 2} \\ &\quad \dots \sum_{k_n=1}^p a_{k_n i_n} a_{k_n n} \\ &= \sum_{J \in \Lambda(p, n)} \sum_{\{k_1, \dots, k_n\} = J} \sum_{i_1, \dots, i_n} \text{sgn}(i_1, \dots, i_n) a_{k_1 i_1} a_{k_1 1} a_{k_2 i_2} a_{k_2 2} \dots a_{k_n i_n} a_{k_n n} \end{aligned}$$

$$\begin{aligned}
&= \sum_{J \in \Lambda(p, n)} \sum_{\{k_1, \dots, k_n\} = J} \sum_{i_1, \dots, i_n} \operatorname{sgn}(i_1, \dots, i_n) a_{k_1 i_1} a_{k_2 i_2} \cdots a_{k_n i_n} \cdot a_{k_1 1} a_{k_2 2} \cdots a_{k_n n} \\
&= \sum_{J \in \Lambda(p, n)} \sum_{\{k_1, \dots, k_n\} = J} \operatorname{sgn}(k_1, \dots, k_n) \det(A_J) a_{k_1 1} a_{k_2 2} \cdots a_{k_n n} \\
&= \sum_{J \in \Lambda(p, n)} \det(A_J) \det(A_J)
\end{aligned}$$

and this proves the lemma.

It follows from this lemma that

$$J_i(\mathbf{u}) = \det(D\mathbf{h}(\mathbf{u})^* D\mathbf{h}(\mathbf{u}))^{1/2}.$$

From 37.3.10 and the area formula, the functional equals

$$\begin{aligned}
\Lambda f &\equiv \sum_{i=1}^{\infty} \int_{\mathbf{g}_i \Gamma_i} f \psi_i(\mathbf{h}_i(\mathbf{u})) J_i(\mathbf{u}) du \\
&= \sum_{i=1}^{\infty} \int_{\Gamma_i} f \psi_i(\mathbf{y}) d\mathcal{H}^n = \int_{\Gamma} f(\mathbf{y}) d\mathcal{H}^n.
\end{aligned}$$

Now $\mathcal{H}^n$ is a Borel measure defined on Γ which is finite on all compact subsets of Γ . This finiteness follows from the above formula. If K is a compact subset of Γ , then there exists an open set, W whose closure is compact and a continuous function with compact support, f such that $K \prec f \prec W$. Then $\mathcal{H}^n(K) \leq \int_{\Gamma} f(\mathbf{y}) d\mathcal{H}^n < \infty$ because of the above formula.

Lemma 37.3.2 $\mu = \mathcal{H}^n$ on every μ measurable set.

Proof: The Riesz representation theorem shows that

$$\int_{\Gamma} f d\mu = \int_{\Gamma} f d\mathcal{H}^n$$

for every continuous function having compact support. Therefore, since every open set is the countable union of compact sets, it follows $\mu = \mathcal{H}^n$ on all open sets. Since compact sets can be obtained as the countable intersection of open sets, these two measures are also equal on all compact sets. It follows they are also equal on all countable unions of compact sets. Suppose now that E is a μ measurable set of finite measure. Then there exist sets, F, G such that G is the countable intersection of open sets each of which has finite measure and F is the countable union of compact sets such that $\mu(G \setminus F) = 0$ and $F \subseteq E \subseteq G$. Thus $\mathcal{H}^n(G \setminus F) = 0$,

$$\mathcal{H}^n(G) = \mu(G) = \mu(F) = \mathcal{H}^n(F)$$

By completeness of $\mathcal{H}^n$ it follows E is $\mathcal{H}^n$ measurable and $\mathcal{H}^n(E) = \mu(E)$. If E is not of finite measure, consider $E_r \equiv E \cap B(\mathbf{0}, r)$. This is contained in the compact set $\Gamma \cap \bar{B}(\mathbf{0}, r)$ and so $\mu(E_r)$ is finite. Thus from what was just shown, $\mathcal{H}^n(E_r) = \mu(E_r)$ and so, taking $r \rightarrow \infty$ $\mathcal{H}^n(E) = \mu(E)$.

This shows you can simply use $\mathcal{H}^n$ for the measure on Γ .

Basic Theory Of Sobolev Spaces

Definition 38.0.3 Let U be an open set of $\mathbb{R}^n$. Define $X^{m,p}(U)$ as the set of all functions in $L^p(U)$ whose weak partial derivatives up to order m are also in $L^p(U)$ where $1 \leq p$. The norm¹ in this space is given by

$$\|u\|_{m,p} \equiv \left(\int_U \sum_{|\alpha| \leq m} |D^\alpha u|^p dx \right)^{1/p}.$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $|\alpha| \equiv \sum \alpha_i$. Here $D^0 u \equiv u$. $C^\infty(\overline{U})$ is defined to be the set of functions which are restrictions to U of a function in $C_c^\infty(\mathbb{R}^n)$. Thus $C^\infty(\overline{U}) \subseteq W^{m,p}(U)$. The Sobolev space, $W^{m,p}(U)$ is defined to be the closure of $C^\infty(\overline{U})$ in $X^{m,p}(U)$ with respect to the above norm. Denote this norm by $\|u\|_{W^{m,p}(U)}$, $\|u\|_{X^{m,p}(U)}$, or $\|u\|_{m,p,U}$ when it is important to identify the open set, U .

Also the following notation will be used pretty consistently.

Definition 38.0.4 Let u be a function defined on U . Define

$$\tilde{u}(\mathbf{x}) \equiv \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ 0 & \text{if } \mathbf{x} \notin U \end{cases}.$$

Theorem 38.0.5 Both $X^{m,p}(U)$ and $W^{m,p}(U)$ are separable reflexive Banach spaces provided $p > 1$.

Proof: Define $\Lambda : X^{m,p}(U) \rightarrow L^p(U)^w$ where w equals the number of multi indices, α , such that $|\alpha| \leq m$ as follows. Letting $\{\alpha_i\}_{i=1}^w$ be the set of all multi indices with $\alpha_1 = 0$,

$$\Lambda(u) \equiv (D^{\alpha_1} u, D^{\alpha_2} u, \dots, D^{\alpha_w} u) = (u, D^{\alpha_2} u, \dots, D^{\alpha_w} u).$$

¹You could also let the norm be given by $\|u\|_{m,p} \equiv \sum_{|\alpha| \leq m} \|D^\alpha u\|_p$ or $\|u\|_{m,p} \equiv \max \{ \|D^\alpha u\|_p : |\alpha| \leq m \}$ because all norms are equivalent on $\mathbb{R}^p$ where p is the number of multi indices no larger than m . This is used whenever convenient.

Then Λ is one to one because one of the multi indices is $\mathbf{0}$. Also

$$\Lambda(X^{m,p}(U))$$

is a closed subspace of $L^p(U)^w$. To see this, suppose

$$(u_k, D^{\alpha_2} u_k, \dots, D^{\alpha_w} u_k) \rightarrow (f_1, f_2, \dots, f_w)$$

in $L^p(U)^w$. Then $u_k \rightarrow f_1$ in $L^p(U)$ and $D^{\alpha_j} u_k \rightarrow f_j$ in $L^p(U)$. Therefore, letting $\phi \in C_c^\infty(U)$ and letting $k \rightarrow \infty$,

$$\begin{array}{rcl} \int_U (D^{\alpha_j} u_k) \phi dx & = & (-1)^{|\alpha|} \int_U u_k D^{\alpha_j} \phi dx \\ \downarrow & & \downarrow \\ \int_U f_j \phi dx & & (-1)^{|\alpha|} \int_U f_1 D^{\alpha_j} \phi dx \quad \equiv D^{\alpha_j}(f_1)(\phi) \end{array}$$

It follows $D^{\alpha_j}(f_1) = f_j$ and so $\Lambda(X^{m,p}(U))$ is closed as claimed. This is clearly also a subspace of $L^p(U)^w$ and so it follows that $\Lambda(X^{m,p}(U))$ is a reflexive Banach space. This is because $L^p(U)^w$, being the product of reflexive Banach spaces, is reflexive and any closed subspace of a reflexive Banach space is reflexive. Now Λ is an isometry of $X^{m,p}(U)$ and $\Lambda(X^{m,p}(U))$ which shows that $X^{m,p}(U)$ is a reflexive Banach space. Finally, $W^{m,p}(U)$ is a closed subspace of the reflexive Banach space, $X^{m,p}(U)$ and so it is also reflexive. To see $X^{m,p}(U)$ is separable, note that $L^p(U)^w$ is separable because it is the finite product of the separable hence completely separable metric space, $L^p(U)$ and $\Lambda(X^{m,p}(U))$ is a subset of $L^p(U)^w$. Therefore, $\Lambda(X^{m,p}(U))$ is separable and since Λ is an isometry, it follows $X^{m,p}(U)$ is separable also. Now $W^{m,p}(U)$ must also be separable because it is a subset of $X^{m,p}(U)$.

The following theorem is obvious but is worth noting because it says that if a function has a weak derivative in $L^p(U)$ on a large open set, U then the restriction of this weak derivative is also the weak derivative for any smaller open set.

Theorem 38.0.6 *Suppose U is an open set and $U_0 \subseteq U$ is another open set. Suppose also $D^\alpha u \in L^p(U)$. Then for all $\psi \in C_c^\infty(U_0)$,*

$$\int_{U_0} (D^\alpha u) \psi dx = (-1)^{|\alpha|} \int_{U_0} u (D^\alpha \psi).$$

The following theorem is a fundamental approximation result for functions in $X^{m,p}(U)$.

Theorem 38.0.7 *Let U be an open set and let U_0 be an open subset of U with the property that $\text{dist}(\overline{U_0}, U^C) > 0$. Then if $u \in X^{m,p}(U)$ and $\tilde{u}$ denotes the zero extension of u off U ,*

$$\lim_{l \rightarrow \infty} \|\tilde{u} * \phi_l - u\|_{X^{m,p}(U_0)} = 0.$$

Proof: Always assume l is large enough that $1/l < \text{dist}(\overline{U_0}, U^C)$. Thus for $\mathbf{x} \in U_0$,

$$\tilde{u} * \phi_l(\mathbf{x}) = \int_{B(\mathbf{0}, \frac{1}{l})} u(\mathbf{x} - \mathbf{y}) \phi_l(\mathbf{y}) d\mathbf{y}. \quad (38.0.1)$$

The theorem is proved if it can be shown that $D^\alpha(\tilde{u} * \phi_l) \rightarrow D^\alpha u$ in $L^p(U_0)$. Let $\psi \in C_c^\infty(U_0)$

$$\begin{aligned} D^\alpha(\tilde{u} * \phi_l)(\psi) &\equiv (-1)^{|\alpha|} \int_{U_0} (\tilde{u} * \phi_l)(D^\alpha \psi) dx \\ &= (-1)^{|\alpha|} \int_{U_0} \int \tilde{u}(\mathbf{y}) \phi_l(\mathbf{x} - \mathbf{y}) (D^\alpha \psi)(\mathbf{x}) dy d\mathbf{x} \\ &= (-1)^{|\alpha|} \int_U u(\mathbf{y}) \int_{U_0} \phi_l(\mathbf{x} - \mathbf{y}) (D^\alpha \psi)(\mathbf{x}) dx dy. \end{aligned}$$

Also,

$$\begin{aligned} (\widetilde{D^\alpha u} * \phi_l)(\psi) &\equiv \int_{U_0} \left(\int \widetilde{D^\alpha u}(\mathbf{y}) \phi_l(\mathbf{x} - \mathbf{y}) dy \right) \psi(\mathbf{x}) dx \\ &= \int_{U_0} \left(\int_U D^\alpha u(\mathbf{y}) \phi_l(\mathbf{x} - \mathbf{y}) dy \right) \psi(\mathbf{x}) dx \\ &= \int_{U_0} \left(\int_U u(\mathbf{y}) (D^\alpha \phi_l)(\mathbf{x} - \mathbf{y}) dy \right) \psi(\mathbf{x}) dx \\ &= \int_U u(\mathbf{y}) \int_{U_0} (D^\alpha \phi_l)(\mathbf{x} - \mathbf{y}) \psi(\mathbf{x}) dx dy \\ &= (-1)^{|\alpha|} \int_U u(\mathbf{y}) \int_{U_0} \phi_l(\mathbf{x} - \mathbf{y}) (D^\alpha \psi)(\mathbf{x}) dx dy. \end{aligned}$$

It follows that $D^\alpha(\tilde{u} * \phi_l) = (\widetilde{D^\alpha u} * \phi_l)$ as weak derivatives defined on $C_c^\infty(U_0)$. Therefore,

$$\begin{aligned} \|D^\alpha(\tilde{u} * \phi_l) - D^\alpha u\|_{L^p(U_0)} &= \left\| \widetilde{D^\alpha u} * \phi_l - D^\alpha u \right\|_{L^p(U_0)} \\ &\leq \left\| \widetilde{D^\alpha u} * \phi_l - \widetilde{D^\alpha u} \right\|_{L^p(\mathbb{R}^n)} \rightarrow 0. \end{aligned}$$

This proves the theorem.

As part of the proof of the theorem, the following corollary was established.

Corollary 38.0.8 *Let U_0 and U be as in the above theorem. Then for all l large enough and ϕ_l a mollifier,*

$$D^\alpha(\tilde{u} * \phi_l) = (\widetilde{D^\alpha u} * \phi_l) \quad (38.0.2)$$

as distributions on $C_c^\infty(U_0)$.

Definition 38.0.9 Let U be an open set. $C^\infty(U)$ denotes the set of functions which are defined and infinitely differentiable on U .

Note that $f(x) = \frac{1}{x}$ is a function in $C^\infty(0,1)$. However, it is not equal to the restriction to $(0,1)$ of some function which is in $C_c^\infty(\mathbb{R})$. This illustrates the distinction between $C^\infty(U)$ and $C^\infty(\overline{U})$. The set, $C^\infty(\overline{U})$ is a subset of $C^\infty(U)$. The following theorem is known as the Meyer Serrin theorem.

Theorem 38.0.10 (Meyer Serrin) Let U be an open subset of $\mathbb{R}^n$. Then if $\delta > 0$ and $u \in X^{m,p}(U)$, there exists $J \in C^\infty(U)$ such that $\|J - u\|_{m,p,U} < \delta$.

Proof: Let $\cdots U_k \subseteq \overline{U_k} \subseteq U_{k+1} \cdots$ be a sequence of open subsets of U whose union equals U such that $\overline{U_k}$ is compact for all k . Also let $U_{-3} = U_{-2} = U_{-1} = U_0 = \emptyset$. Now define $V_k \equiv U_{k+1} \setminus \overline{U_{k-1}}$. Thus $\{V_k\}_{k=1}^\infty$ is an open cover of U . Note the open cover is locally finite and therefore, there exists a partition of unity subordinate to this open cover, $\{\eta_k\}_{k=1}^\infty$ such that each $\text{spt}(\eta_k) \in C_c(V_k)$. Let ψ_m denote the sum of all the η_k which are non zero at some point of V_m . Thus

$$\text{spt}(\psi_m) \subseteq U_{m+2} \setminus \overline{U_{m-2}}, \psi_m \in C_c^\infty(U), \sum_{m=1}^\infty \psi_m(\mathbf{x}) = 1 \quad (38.0.3)$$

for all $\mathbf{x} \in U$, and $\psi_m u \in W^{m,p}(U_{m+2})$.

Now let ϕ_l be a mollifier and consider

$$J \equiv \sum_{m=0}^\infty u \psi_m * \phi_{l_m} \quad (38.0.4)$$

where l_m is chosen large enough that the following two conditions hold:

$$\text{spt}(u \psi_m * \phi_{l_m}) \subseteq U_{m+3} \setminus \overline{U_{m-3}}, \quad (38.0.5)$$

$$\|(u \psi_m) * \phi_{l_m} - u \psi_m\|_{m,p,U_{m+3}} = \|(u \psi_m) * \phi_{l_m} - u \psi_m\|_{m,p,U} < \frac{\delta}{2^{m+5}}, \quad (38.0.6)$$

where 38.0.6 is obtained from Theorem 38.0.7. Because of 38.0.3 only finitely many terms of the series in 38.0.4 are nonzero and therefore, $J \in C^\infty(U)$. Now let $N > 10$, some large value.

$$\begin{aligned} \|J - u\|_{m,p,U_{N-3}} &= \left\| \sum_{k=0}^N (u \psi_k * \phi_{l_k} - u \psi_k) \right\|_{m,p,U_{N-3}} \\ &\leq \sum_{k=0}^N \|u \psi_k * \phi_{l_k} - u \psi_k\|_{m,p,U_{N-3}} \\ &\leq \sum_{k=0}^N \frac{\delta}{2^{k+5}} < \delta. \end{aligned}$$

Now apply the monotone convergence theorem to conclude that $\|J - u\|_{m,p,U} \leq \delta$. This proves the theorem.

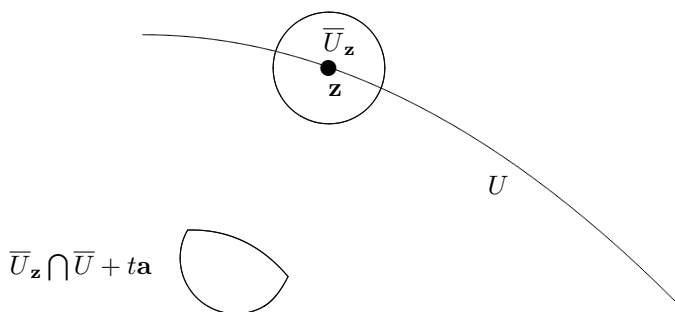
Note that $J = 0$ on ∂U . Later on, you will see that this is pathological.

In the study of partial differential equations it is the space $W^{m,p}(U)$ which is of the most use, not the space $X^{m,p}(U)$. This is because of the density of $C^\infty(\bar{U})$. Nevertheless, for reasonable open sets, U , the two spaces coincide.

Definition 38.0.11 An open set, $U \subseteq \mathbb{R}^n$ is said to satisfy the segment condition if for all $\mathbf{z} \in \bar{U}$, there exists an open set $U_{\mathbf{z}}$ containing $\mathbf{z}$ and a vector $\mathbf{a}$ such that

$$\bar{U} \cap \bar{U}_{\mathbf{z}} + t\mathbf{a} \subseteq U$$

for all $t \in (0, 1)$.



You can imagine open sets which do not satisfy the segment condition. For example, a pair of circles which are tangent at their boundaries. The condition in the above definition breaks down at their point of tangency.

Here is a simple lemma which will be used in the proof of the following theorem.

Lemma 38.0.12 If $u \in W^{m,p}(U)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$, then $u\psi \in W^{m,p}(U)$.

Proof: Let $|\alpha| \leq m$ and let $\phi \in C_c^\infty(U)$. Then

$$\begin{aligned} (D_{x_i}(u\psi))(\phi) &\equiv - \int_U u\psi\phi_{,x_i} dx \\ &= - \int_U u \left((\psi\phi)_{,x_i} - \phi\psi_{,x_i} \right) dx \\ &= (D_{x_i}u)(\psi\phi) + \int_U u\psi_{,x_i}\phi dx \\ &= \int_U (\psi D_{x_i}u + u\psi_{,x_i})\phi dx \end{aligned}$$

Therefore, $D_{x_i}(u\psi) = \psi D_{x_i}u + u\psi_{,x_i} \in L^p(U)$. In other words, the product rule holds. Now considering the terms in the last expression, you can do the same

argument with each of these as long as they all have derivatives in $L^p(U)$. Therefore, continuing this process the lemma is proved.

Theorem 38.0.13 *Let U be an open set and suppose there exists a locally finite covering² of $\overline{U}$ which is of the form $\{U_i\}_{i=1}^\infty$ such that each U_i is a bounded open set which satisfies the conditions of Definition 38.0.11. Thus there exist vectors, $\mathbf{a}_i$ such that for all $t \in (0, 1)$,*

$$\overline{U_i} \cap U + t\mathbf{a}_i \subseteq U.$$

Then $C^\infty(\overline{U})$ is dense in $X^{m,p}(U)$ and so $W^{m,p}(U) = X^{m,p}(U)$.

Proof: Let $\{\psi_i\}_{i=1}^\infty$ be a partition of unity subordinate to the given open cover with $\psi_i \in C_c^\infty(U_i)$ and let $u \in X^{m,p}(U)$. Thus

$$u = \sum_{k=1}^\infty \psi_k u.$$

Consider U_k for some k . Let $\mathbf{a}_k$ be the special vector associated with U_k such that

$$t\mathbf{a}_k + \overline{U} \cap \overline{U_k} \subseteq U \quad (38.0.7)$$

for all $t \in (0, 1)$ and consider only t small enough that

$$\text{spt}(\psi_k) - t\mathbf{a}_k \subseteq U_k \quad (38.0.8)$$

Pick $l(t) > 1/t$ which is also large enough that

$$t\mathbf{a}_k + \overline{U} \cap \overline{U_k} + B\left(\mathbf{0}, \frac{1}{l(t)}\right) \subseteq U, \quad \overline{\text{spt}(\psi_k) + B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) - t\mathbf{a}_k} \subseteq U_k. \quad (38.0.9)$$

This can be done because $t\mathbf{a}_k + \overline{U} \cap \overline{U_k}$ is a compact subset of U and so has positive distance to U^C and $\text{spt}(\psi_k) - t\mathbf{a}_k$ is a compact subset of U_k having positive distance to U_k^C . Let t_k be such a value for t and for ϕ_l a mollifier, define

$$v_{t_k}(\mathbf{x}) \equiv \int_{\mathbb{R}^n} \tilde{u}(\mathbf{x} + t_k\mathbf{a}_k - \mathbf{y}) \psi_k(\mathbf{x} + t_k\mathbf{a}_k - \mathbf{y}) \phi_{l(t_k)}(\mathbf{y}) d\mathbf{y} \quad (38.0.10)$$

where as usual, $\tilde{u}$ is the zero extension of u off U . For $v_{t_k}(\mathbf{x}) \neq 0$, it is necessary that $\mathbf{x} + t_k\mathbf{a}_k - \mathbf{y} \in \text{spt}(\psi_k)$ for some $\mathbf{y} \in B\left(\mathbf{0}, \frac{1}{l(t_k)}\right)$. Therefore, using 38.0.9, for $v_{t_k}(\mathbf{x}) \neq 0$, it is necessary that

$$\mathbf{x} \in \mathbf{y} - t_k\mathbf{a}_k + U \cap \text{spt}(\psi_k) \subseteq B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) + \text{spt}(\psi_k) - t_k\mathbf{a}_k$$

²This is never a problem in $\mathbb{R}^n$. In fact, every open covering has a locally finite subcovering in $\mathbb{R}^n$ or more generally in any metric space due to Stone's theorem. These are issues best left to you in case you are interested. I am usually interested in bounded sets, U , and for these, there is a finite covering.

$$\subseteq \overline{B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) + \text{spt}(\psi_k) - t_k \mathbf{a}_k} \subseteq U_k$$

showing that v_{t_k} has compact support in U_k . Now change variables in 38.0.10 to obtain

$$v_{t_k}(\mathbf{x}) \equiv \int_{\mathbb{R}^n} \tilde{u}(\mathbf{y}) \psi_k(\mathbf{y}) \phi_{l(t_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) dy. \quad (38.0.11)$$

For $\mathbf{x} \in U \cap U_k$, the above equals zero unless

$$\mathbf{y} - t_k \mathbf{a}_k - \mathbf{x} \in B\left(\mathbf{0}, \frac{1}{l(t_k)}\right)$$

which implies by 38.0.9 that

$$\mathbf{y} \in t_k \mathbf{a}_k + U \cap U_k + B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) \subseteq U$$

Therefore, for such $\mathbf{x} \in U \cap U_k$, 38.0.11 reduces to

$$\begin{aligned} v_{t_k}(\mathbf{x}) &= \int_{\mathbb{R}^n} u(\mathbf{y}) \psi_k(\mathbf{y}) \phi_{l(t_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) dy \\ &= \int_U u(\mathbf{y}) \psi_k(\mathbf{y}) \phi_{l(t_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) dy. \end{aligned}$$

It follows that for $|\alpha| \leq m$, and $\mathbf{x} \in U \cap U_k$

$$\begin{aligned} D^\alpha v_{t_k}(\mathbf{x}) &= \int_U u(\mathbf{y}) \psi_k(\mathbf{y}) D^\alpha \phi_{l(t_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) dy \\ &= \int_U D^\alpha (u \psi_k)(\mathbf{y}) \phi_{l(t_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) dy \\ &= \int_{\mathbb{R}^n} \widetilde{D^\alpha (u \psi_k)}(\mathbf{y}) \phi_{l(t_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) dy \\ &= \int_{\mathbb{R}^n} \widetilde{D^\alpha (u \psi_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) \phi_{l(t_k)}(\mathbf{y}) dy. \end{aligned} \quad (38.0.12)$$

Actually, this formula holds for all $\mathbf{x} \in U$. If $\mathbf{x} \in U$ but $\mathbf{x} \notin U_k$, then the left side of the above formula equals zero because, as noted above, $\text{spt}(v_{t_k}) \subseteq U_k$. The integrand of the right side equals zero unless

$$\mathbf{x} \in B\left(\mathbf{0}, \frac{1}{l(t_k)}\right) + \text{spt}(\psi_k) - t_k \mathbf{a}_k \subseteq U_k$$

by 38.0.9 and here $\mathbf{x} \notin U_k$.

Next an estimate is obtained for $\|D^\alpha v_{t_k} - D^\alpha (u \psi_k)\|_{L^p(U)}$. By 38.0.12,

$$\|D^\alpha v_{t_k} - D^\alpha (u \psi_k)\|_{L^p(U)} \leq$$

$$\begin{aligned}
& \left(\int_U \left(\int_{\mathbb{R}^n} \left| D^\alpha \widetilde{(u\psi_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) - D^\alpha \widetilde{(u\psi_k)}(\mathbf{x}) \right| \phi_{l(t_k)}(\mathbf{y}) dy \right)^p dx \right)^{1/p} \\
& \leq \int_{\mathbb{R}^n} \phi_{l(t_k)}(\mathbf{y}) \left(\int_U \left| D^\alpha \widetilde{(u\psi_k)}(\mathbf{x} + t_k \mathbf{a}_k - \mathbf{y}) - D^\alpha \widetilde{(u\psi_k)}(\mathbf{x}) \right|^p dx \right)^{1/p} dy \\
& \leq \frac{\varepsilon}{2^k}
\end{aligned}$$

whenever t_k is taken small enough. Pick t_k this small and let $w_k \equiv v_{t_k}$. Thus

$$\|D^\alpha w_k - D^\alpha(u\psi_k)\|_{L^p(U)} \leq \frac{\varepsilon}{2^k}$$

and $w_k \in C_c^\infty(\mathbb{R}^n)$. Now let

$$J(\mathbf{x}) \equiv \sum_{k=1}^{\infty} w_k.$$

Since the U_k are locally finite and $\text{spt}(w_k) \subseteq U_k$ for each k , it follows

$$D^\alpha J = \sum_{k=0}^{\infty} D^\alpha w_k$$

and the sum is always finite. Similarly,

$$D^\alpha \sum_{k=1}^{\infty} (\psi_k u) = \sum_{k=1}^{\infty} D^\alpha (\psi_k u)$$

and the sum is always finite. Therefore,

$$\begin{aligned}
\|D^\alpha J - D^\alpha u\|_{L^p(U)} &= \left\| \sum_{k=1}^{\infty} D^\alpha w_k - D^\alpha (\psi_k u) \right\|_{L^p(U)} \\
&\leq \sum_{k=1}^{\infty} \|D^\alpha w_k - D^\alpha (\psi_k u)\|_{L^p(U)} \leq \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.
\end{aligned}$$

By choosing t_k small enough, such an inequality can be obtained for

$$\|D^\beta J - D^\beta u\|_{L^p(U)}$$

for each multi index, β such that $|\beta| \leq m$. Therefore, there exists

$$J \in C_c^\infty(\mathbb{R}^n)$$

such that

$$\|J - u\|_{W^{m,p}(U)} \leq \varepsilon K$$

where K equals the number of multi indices no larger than m . Since ε is arbitrary, this proves the theorem.

Corollary 38.0.14 *Let U be an open set which has the segment property. Then $W^{m,p}(U) = X^{m,p}(U)$.*

Proof: Start with an open covering of $\bar{U}$ whose sets satisfy the segment condition and obtain a locally finite refinement consisting of bounded sets which are of the sort in the above theorem.

Now consider a situation where $\mathbf{h} : U \rightarrow V$ where U and V are two open sets in $\mathbb{R}^n$ and $D^\alpha \mathbf{h}$ exists and is continuous and bounded if $|\alpha| < m - 1$ and $D^\alpha \mathbf{h}$ is Lipschitz if $|\alpha| = m - 1$.

Definition 38.0.15 *Whenever $\mathbf{h} : U \rightarrow V$, define $\mathbf{h}^*$ mapping the functions which are defined on V to the functions which are defined on U as follows.*

$$\mathbf{h}^* f(\mathbf{x}) \equiv f(\mathbf{h}(\mathbf{x})).$$

$\mathbf{h} : U \rightarrow V$ is bilipschitz if $\mathbf{h}$ is one to one, onto and Lipschitz and $\mathbf{h}^{-1}$ is also one to one, onto and Lipschitz.

Theorem 38.0.16 *Let $\mathbf{h} : U \rightarrow V$ be one to one and onto where U and V are two open sets. Also suppose that $D^\alpha \mathbf{h}$ and $D^\alpha (\mathbf{h}^{-1})$ exist and are Lipschitz continuous if $|\alpha| \leq m - 1$ for m a positive integer. Then*

$$\mathbf{h}^* : W^{m,p}(V) \rightarrow W^{m,p}(U)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $(\mathbf{h}^{-1})^*$.

Proof: It is clear that $\mathbf{h}^*$ is linear. It is required to show it is one to one and continuous. First suppose $\mathbf{h}^* f = 0$. Then

$$0 = \int_V |f(\mathbf{h}(\mathbf{x}))|^p dx$$

and so $f(\mathbf{h}(\mathbf{x})) = 0$ for a.e. $\mathbf{x} \in U$. Since $\mathbf{h}$ is Lipschitz, it takes sets of measure zero to sets of measure zero. Therefore, $f(\mathbf{y}) = 0$ a.e. This shows $\mathbf{h}^*$ is one to one.

By the Meyer Serrin theorem, Theorem 38.0.10, it suffices to verify that $\mathbf{h}^*$ is continuous on functions in $C^\infty(V)$. Let f be such a function. Then using the chain rule and product rule, $(\mathbf{h}^* f)_{,i}(\mathbf{x}) = f_{,k}(\mathbf{h}(\mathbf{x})) h_{k,i}(\mathbf{x})$,

$$\begin{aligned} (\mathbf{h}^* f)_{,ij}(\mathbf{x}) &= (f_{,k}(\mathbf{h}(\mathbf{x})) h_{k,i}(\mathbf{x}))_{,j} \\ &= f_{,kl}(\mathbf{h}(\mathbf{x})) h_{l,j}(\mathbf{x}) h_{k,i}(\mathbf{x}) + f_{,k}(\mathbf{h}(\mathbf{x})) h_{k,ij}(\mathbf{x}) \end{aligned}$$

etc. In general, for $|\alpha| \leq m - 1$, successive applications of the product rule and chain rule yield that $D^\alpha (\mathbf{h}^* f)(\mathbf{x})$ has the form

$$D^\alpha (\mathbf{h}^* f)(\mathbf{x}) = \sum_{|\beta| \leq |\alpha|} \mathbf{h}^* (D^\beta f)(\mathbf{x}) g_\beta(\mathbf{x})$$

where g_β is a bounded Lipschitz function with Lipschitz constant dependent on $\mathbf{h}$ and its derivatives. It only remains to take one more derivative of the functions, $D^\alpha f$ for $|\alpha| = m - 1$. This can be done again but this time you have to use Rademacher's theorem which assures you that the derivative of a Lipschitz function exists a.e. in order to take the partial derivative of the $g_\beta(\mathbf{x})$. When this is done, the above formula remains valid for all $|\alpha| \leq m$. Therefore, using the change of variables formula for multiple integrals, Corollary 35.6.13 on Page 1063,

$$\begin{aligned} \int_U |D^\alpha (\mathbf{h}^* f)(\mathbf{x})|^p dx &\leq C_{m,p,\mathbf{h}} \sum_{|\beta| \leq m} \int_U |\mathbf{h}^* (D^\beta f)(\mathbf{x})|^p dx \\ &= C_{m,p,\mathbf{h}} \sum_{|\beta| \leq m} \int_U |(D^\beta f)(\mathbf{h}(\mathbf{x}))|^p dx \\ &= C_{m,p,\mathbf{h}} \sum_{|\beta| \leq m} \int_V |(D^\beta f)(\mathbf{y})|^p |\det D\mathbf{h}^{-1}(\mathbf{y})| dy \\ &\leq C_{m,p,\mathbf{h},\mathbf{h}^{-1}} \|f\|_{m,p,V} \end{aligned}$$

This shows $\mathbf{h}^*$ is continuous on $C^\infty(V) \cap W^{m,p}(U)$ and since this set is dense, this proves $\mathbf{h}^*$ is continuous. The same argument applies to $(\mathbf{h}^{-1})^*$ and now the definitions of $\mathbf{h}^*$ and $(\mathbf{h}^{-1})^*$ show these are inverses.

38.1 Embedding Theorems For $W^{m,p}(\mathbb{R}^n)$

Recall Theorem 35.5.1 which is listed here for convenience.

Theorem 38.1.1 Suppose $u, u_i \in L^p_{loc}(\mathbb{R}^n)$ for $i = 1, \dots, n$ and $p > n$. Then u has a representative, still denoted by u , such that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq C \left(\int_{B(\mathbf{x}, 2|\mathbf{y}-\mathbf{x}|)} |\nabla u|^p dz \right)^{1/p} |\mathbf{x} - \mathbf{y}|^{(1-n/p)}. \quad (38.1.13)$$

This amazing result shows that every $u \in W^{m,p}(\mathbb{R}^n)$ has a representative which is continuous provided $p > n$.

Using the above inequality, one can give an important embedding theorem.

Definition 38.1.2 Let X, Y be two Banach spaces and let $f : X \rightarrow Y$ be a function. Then f is a compact map if whenever S is a bounded set in X , it follows that $f(S)$ is precompact in Y .

Theorem 38.1.3 Let U be a bounded open set and for u a function defined on $\mathbb{R}^n$, let $r_U u(\mathbf{x}) \equiv u(\mathbf{x})$ for $\mathbf{x} \in \bar{U}$. Then if $p > n$, $r_U : W^{1,p}(\mathbb{R}^n) \rightarrow C(\bar{U})$ is continuous and compact.

Proof: First suppose $u_k \rightarrow 0$ in $W^{1,p}(\mathbb{R}^n)$. Then if $r_U u_k$ does not converge to 0, it follows there exists a sequence, still denoted by k and $\varepsilon > 0$ such that $u_k \rightarrow 0$ in $W^{1,p}(\mathbb{R}^n)$ but $\|r_U u_k\|_\infty \geq \varepsilon$. Selecting a further subsequence which is still denoted by k , you can also assume $u_k(\mathbf{x}) \rightarrow 0$ a.e. Pick such an $\mathbf{x}_0 \in U$ where this convergence takes place. Then from 38.1.13, for all $\mathbf{x} \in \overline{U}$,

$$|u_k(\mathbf{x})| \leq |u_k(\mathbf{x}_0)| + C \|u_k\|_{1,p,\mathbb{R}^n} \text{diam}(U)$$

showing that u_k converges uniformly to 0 on $\overline{U}$ contrary to $\|r_U u_k\|_\infty \geq \varepsilon$. Therefore, r_U is continuous as claimed.

Next let S be a bounded subset of $W^{1,p}(\mathbb{R}^n)$ with $\|u\|_{1,p} < M$ for all $u \in S$. Then for $u \in S$

$$r^p m_n([|u| > r] \cap U) \leq \int_{[|u| > r] \cap U} |u|^p dm_n \leq M^p$$

and so

$$m_n([|u| > r] \cap U) \leq \frac{M^p}{r^p}.$$

Now choosing r large enough, $M^p/r^p < m_n(U)$ and so, for such r , there exists $\mathbf{x}_u \in U$ such that $|u(\mathbf{x}_u)| \leq r$. Therefore from 38.1.13, whenever $\mathbf{x} \in U$,

$$\begin{aligned} |u(\mathbf{x})| &\leq |u(\mathbf{x}_u)| + CM \text{diam}(U)^{1-n/p} \\ &\leq r + CM \text{diam}(U)^{1-n/p} \end{aligned}$$

showing that $\{r_U u : u \in S\}$ is uniformly bounded. But also, for $\mathbf{x}, \mathbf{y} \in \overline{U}$, 38.1.13 implies

$$|u(\mathbf{x}) - u(\mathbf{y})| \leq CM |\mathbf{x} - \mathbf{y}|^{1-\frac{n}{p}}$$

showing that $\{r_U u : u \in S\}$ is equicontinuous. By the Ascoli Arzela theorem, it follows $r_U(S)$ is precompact and so r_U is compact.

Definition 38.1.4 Let $\alpha \in (0, 1]$ and K a compact subset of $\mathbb{R}^n$

$$C^\alpha(K) \equiv \{f \in C(K) : \rho_\alpha(f) + \|f\| \equiv \|f\|_\alpha < \infty\}$$

where

$$\|f\| \equiv \|f\|_\infty \equiv \sup\{|f(\mathbf{x})| : \mathbf{x} \in K\}$$

and

$$\rho_\alpha(f) \equiv \sup \left\{ \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha} : \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y} \right\}.$$

Then $(C^\alpha(K), \|\cdot\|_\alpha)$ is a complete normed linear space called a Holder space.

The verification that this is a complete normed linear space is routine and is left for you. More generally, one considers the following class of Holder spaces.

Definition 38.1.5 Let K be a compact subset of $\mathbb{R}^n$ and let $\lambda \in (0, 1]$. $C^{m, \lambda}(K)$ denotes the set of functions, u which are restrictions of functions defined on $\mathbb{R}^n$ to K such that for $|\alpha| \leq m$,

$$D^\alpha u \in C(K)$$

and if $|\alpha| = m$,

$$D^\alpha u \in C^\lambda(K).$$

Thus $C^{0, \lambda}(K) = C^\lambda(K)$. The norm of a function in $C^{m, \lambda}(K)$ is given by

$$\|u\|_{m, \lambda} \equiv \sup_{|\alpha|=m} \rho_\lambda(D^\alpha u) + \sum_{|\alpha| \leq m} \|D^\alpha u\|_\infty.$$

Lemma 38.1.6 Let m be a positive integer, K a compact subset of $\mathbb{R}^n$, and let $0 < \beta < \lambda \leq 1$. Then the identity map from $C^{m, \lambda}(K)$ into $C^{m, \beta}(K)$ is compact.

Proof: First note that the containment is obvious because for any function, f , if

$$\rho_\lambda(f) \equiv \sup \left\{ \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\lambda} : \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y} \right\} < \infty,$$

Then

$$\begin{aligned} \rho_\beta(f) &\equiv \sup \left\{ \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\beta} : \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y} \right\} \\ &= \sup \left\{ \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\lambda} |\mathbf{x} - \mathbf{y}|^{\lambda-\beta} : \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y} \right\} \\ &\leq \sup \left\{ \frac{|f(\mathbf{x}) - f(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\lambda} \text{diam}(K)^{\lambda-\beta} : \mathbf{x}, \mathbf{y} \in K, \mathbf{x} \neq \mathbf{y} \right\} < \infty. \end{aligned}$$

Suppose the identity map, id , is not compact. Then there exists $\varepsilon > 0$ and a sequence, $\{f_k\}_{k=1}^\infty \subseteq C^{m, \lambda}(K)$ such that $\|f_k\|_{m, \lambda} < M$ for all k but $\|f_k - f_l\|_\beta \geq \varepsilon$ whenever $k \neq l$. By the Ascoli Arzela theorem, there exists a subsequence of this, still denoted by f_k such that $\sum_{|\alpha| \leq m} \|D^\alpha(f_l - f_k)\|_\infty < \delta$ where δ satisfies

$$0 < \delta < \min \left(\frac{\varepsilon}{2}, \left(\frac{\varepsilon}{8} \right) \left(\frac{\varepsilon}{8M} \right)^{\beta/(\lambda-\beta)} \right). \quad (38.1.14)$$

Therefore, $\sup_{|\alpha|=m} \rho_\beta(D^\alpha(f_k - f_l)) \geq \varepsilon - \delta$ for all $k \neq l$. It follows that there exist pairs of points and a multi index, α with $|\alpha| = m$, $\{\mathbf{x}_{kl}, \mathbf{y}_{kl}, \alpha\}$ such that

$$\frac{\varepsilon - \delta}{2} < \frac{|(D^\alpha f_k - D^\alpha f_l)(\mathbf{x}_{kl}) - ((D^\alpha f_k - D^\alpha f_l)(\mathbf{y}_{kl}))|}{|\mathbf{x}_{kl} - \mathbf{y}_{kl}|^\beta} \leq 2M |\mathbf{x}_{kl} - \mathbf{y}_{kl}|^{\lambda-\beta} \quad (38.1.15)$$

and so considering the ends of the above inequality,

$$\left(\frac{\varepsilon - \delta}{4M} \right)^{1/(\lambda-\beta)} < |\mathbf{x}_{kl} - \mathbf{y}_{kl}|.$$

Now also, since $\sum_{|\alpha| \leq m} \|D^\alpha(f_l - f_k)\|_\infty < \delta$, it follows from the first inequality in 38.1.15 that

$$\frac{\varepsilon - \delta}{2} < \frac{2\delta}{\left(\frac{\varepsilon - \delta}{4M}\right)^{\beta/(\lambda - \beta)}}.$$

Since $\delta < \varepsilon/2$, this implies

$$\frac{\varepsilon}{4} < \frac{2\delta}{\left(\frac{\varepsilon}{8M}\right)^{\beta/(\lambda - \beta)}}$$

and so

$$\left(\frac{\varepsilon}{8}\right) \left(\frac{\varepsilon}{8M}\right)^{\beta/(\lambda - \beta)} < \delta$$

contrary to 38.1.14. This proves the lemma.

Corollary 38.1.7 *Let $p > n$, U and r_U be as in Theorem 38.1.3 and let m be a nonnegative integer. Then $r_U : W^{m+1,p}(\mathbb{R}^n) \rightarrow C^{m,\lambda}(\overline{U})$ is continuous as a map into $C^{m,\lambda}(\overline{U})$ for all $\lambda \in [0, 1 - \frac{n}{p}]$ and r_U is compact if $\lambda < 1 - \frac{n}{p}$.*

Proof: Suppose $u_k \rightarrow 0$ in $W^{m+1,p}(\mathbb{R}^n)$. Then from 38.1.13, if $\lambda \leq 1 - \frac{n}{p}$ and $|\alpha| = m$

$$\rho_\lambda(D^\alpha u_k) \leq C \|D^\alpha u_k\|_{1,p} \text{diam}(U)^{1 - \frac{n}{p} - \lambda}.$$

Therefore, $\rho_\lambda(D^\alpha u_k) \rightarrow 0$. From Theorem 38.1.3 it follows that for $|\alpha| \leq m$,

$$\|D^\alpha u_k\|_\infty \rightarrow 0$$

and so $\|u_k\|_{m,\lambda} \rightarrow 0$. This proves the claim about continuity. The claim about compactness for $\lambda < 1 - \frac{n}{p}$ follows from Lemma 38.1.6 and this.

(Bounded in $W^{m,p}(\mathbb{R}^n) \xrightarrow{r_U}$ Bounded in $C^{m,1-\frac{n}{p}}(\overline{U}) \xrightarrow{\text{id}}$ Compact in $C^{m,\lambda}(\overline{U})$.)

It is just as important to consider the case where $p < n$. To do this case the following lemma due to Gagliardo [27] will be of interest. See also [1].

Lemma 38.1.8 *Suppose $n \geq 2$ and w_j does not depend on the j^{th} component of $\mathbf{x}$, x_j . Then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^n |w_j(\mathbf{x})| dm_n \leq \prod_{i=1}^n \left(\int_{\mathbb{R}^{n-1}} |w_j(\mathbf{x})|^{n-1} dm_{n-1} \right)^{1/(n-1)}.$$

In this inequality, assume all the functions are continuous so there can be no measurability questions.

Proof: First note that for $n = 2$ the inequality reduces to the statement

$$\int \int |w_1(x_2)| |w_2(x_1)| dx_1 dx_2 \leq \int |w_1(x_2)| dx_2 \int |w_2(x_1)| dx_1$$

which is obviously true. Suppose then that the inequality is valid for some n . Using Fubini's theorem, Holder's inequality, and the induction hypothesis,

$$\begin{aligned}
 & \int_{\mathbb{R}^{n+1}} \prod_{j=1}^{n+1} |w_j(\mathbf{x})| dm_{n+1} \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \prod_{j=1}^n |w_j(\mathbf{x})| dm_n dx_{n+1} \\
 &= \int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \int_{\mathbb{R}} \prod_{j=1}^n |w_j(\mathbf{x})| dx_{n+1} dm_n \\
 &= \int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \left(\prod_{j=1}^n \int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/n} dm_n \\
 &= \int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})| \prod_{j=1}^n \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/n} dm_n \\
 &\leq \left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n dm_n \right)^{1/n} \\
 &\quad \left(\int_{\mathbb{R}^n} \left(\prod_{j=1}^n \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/n} \right)^{n/(n-1)} dm_n \right)^{(n-1)/n} \\
 &= \left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n dm_n \right)^{1/n} \\
 &\quad \left(\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right)^{1/(n-1)} dm_n \right)^{(n-1)/n} \\
 &\leq \left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n dm_n \right)^{1/n} \\
 &\quad \left(\prod_{j=1}^n \left(\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} |w_j(\mathbf{x})|^n dx_{n+1} \right) dm_{n-1} \right)^{1/(n-1)} \right)^{(n-1)/n} \\
 &= \left(\int_{\mathbb{R}^n} |w_{n+1}(\mathbf{x})|^n dm_n \right)^{1/n} \prod_{j=1}^n \left(\int_{\mathbb{R}^n} |w_j(\mathbf{x})|^n dm_n \right)^{1/n} \\
 &= \prod_{j=1}^{n+1} \left(\int_{\mathbb{R}^n} |w_j(\mathbf{x})|^n dm_n \right)^{1/n}
 \end{aligned}$$

This proves the lemma.

Lemma 38.1.9 *If $\phi \in C_c^\infty(\mathbb{R}^n)$ and $n \geq 1$, then*

$$\|\phi\|_{n/(n-1)} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n \left\| \frac{\partial \phi}{\partial x_j} \right\|_1.$$

Proof: The case where $n = 1$ is obvious if $n/(n-1)$ is interpreted as ∞ . Assume then that $n > 1$ and note that for $a_i \geq 0$,

$$n \prod_{i=1}^n a_i \leq \left(\sum_{j=1}^n a_j \right)^n$$

In fact, the term on the left is one of many terms of the expression on the right. Therefore, taking n^{th} roots

$$\prod_{i=1}^n a_i^{1/n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n a_j.$$

Then observe that for each $j = 1, 2, \dots, n$,

$$|\phi(\mathbf{x})| \leq \int_{-\infty}^{\infty} |\phi_{,j}(\mathbf{x})| dx_j$$

so

$$\begin{aligned} \|\phi\|_{n/(n-1)}^{n/(n-1)} &\equiv \int_{\mathbb{R}^n} |\phi(\mathbf{x})|^{n/(n-1)} dm_n \\ &\leq \int_{\mathbb{R}^n} \prod_{j=1}^n \left(\int_{-\infty}^{\infty} |\phi_{,j}(\mathbf{x})| dx_j \right)^{1/(n-1)} dm_n \end{aligned}$$

and from Lemma 38.1.8 this is dominated by

$$\leq \prod_{j=1}^n \left(\int_{\mathbb{R}^n} |\phi_{,j}(\mathbf{x})| dm_n \right)^{1/(n-1)}.$$

Hence $\prod_{i=1}^n a_i^{1/n} \leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n a_j$

$$\begin{aligned} \|\phi\|_{n/(n-1)} &\leq \prod_{j=1}^n \left(\int_{\mathbb{R}^n} |\phi_{,j}(\mathbf{x})| dm_n \right)^{1/n} \\ &\leq \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n \int_{\mathbb{R}^n} |\phi_{,j}(\mathbf{x})| dm_n \\ &= \frac{1}{\sqrt[n]{n}} \sum_{j=1}^n \|\phi_{,j}\|_1 \end{aligned}$$

and this proves the lemma.

The above lemma is due to Gagliardo and Nirenberg.

With this lemma, it is possible to prove a major embedding theorem which follows.

Theorem 38.1.10 Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$. Then if $f \in W^{1,p}(\mathbb{R}^n)$,

$$\|f\|_q \leq \frac{1}{\sqrt[n]{n}} \frac{(n-1)p}{n-p} \|f\|_{1,p,\mathbb{R}^n}.$$

Proof: From the definition of $W^{1,p}(\mathbb{R}^n)$, $C_c^1(\mathbb{R}^n)$ is dense in $W^{1,p}$. Here $C_c^1(\mathbb{R}^n)$ is the space of continuous functions having continuous derivatives which have compact support. The desired inequality will be established for such ϕ and then the density of this set in $W^{1,p}(\mathbb{R}^n)$ will be exploited to obtain the inequality for all $f \in W^{1,p}(\mathbb{R}^n)$. First note that the case where $p = 1$ follows immediately from the above lemma and so it is only necessary to consider the case where $p > 1$.

Let $\phi \in C_c^1(\mathbb{R}^n)$ and consider $|\phi|^r$ where $r > 1$. Then a short computation shows $|\phi|^r \in C_c^1(\mathbb{R}^n)$ and

$$|\phi|_{,i}^r = r |\phi|^{r-1} |\phi_{,i}|.$$

Therefore, from Lemma 38.1.9,

$$\begin{aligned} & \left(\int |\phi|^{\frac{rn}{n-1}} dm_n \right)^{(n-1)/n} \\ & \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^n \int |\phi|^{r-1} |\phi_{,i}| dm_n \\ & \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^n \left(\int |\phi_{,i}|^p \right)^{1/p} \left(\int (|\phi|^{r-1})^{p/(p-1)} dm_n \right)^{(p-1)/p}. \end{aligned}$$

Now choose r such that

$$\frac{(r-1)p}{p-1} = \frac{rn}{n-1}.$$

That is, let $r = \frac{p(n-1)}{n-p} > 1$ and so $\frac{rn}{n-1} = \frac{np}{n-p}$. Then this reduces to

$$\left(\int |\phi|^{\frac{np}{n-p}} dm_n \right)^{(n-1)/n} \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^n \left(\int |\phi_{,i}|^p \right)^{1/p} \left(\int |\phi|^{\frac{np}{n-p}} dm_n \right)^{(p-1)/p}.$$

Also, $\frac{n-1}{n} - \frac{p-1}{p} = \frac{n-p}{np}$ and so, dividing both sides by the last term yields

$$\left(\int |\phi|^{\frac{np}{n-p}} dm_n \right)^{\frac{n-p}{np}} \leq \frac{r}{\sqrt[n]{n}} \sum_{i=1}^n \left(\int |\phi_{,i}|^p \right)^{1/p} \leq \frac{r}{\sqrt[n]{n}} \|\phi\|_{1,p,\mathbb{R}^n}.$$

Letting $q = \frac{np}{n-p}$, it follows $\frac{1}{q} = \frac{n-p}{np} = \frac{1}{p} - \frac{1}{n}$ and

$$\|\phi\|_q \leq \frac{r}{\sqrt[n]{n}} \|\phi\|_{1,p,\mathbb{R}^n}.$$

Now let $f \in W^{m,p}(\mathbb{R}^n)$ and let $\|\phi_k - f\|_{1,p,\mathbb{R}^n} \rightarrow 0$ as $k \rightarrow \infty$. Taking another subsequence, if necessary, you can also assume $\phi_k(\mathbf{x}) \rightarrow f(\mathbf{x})$ a.e. Therefore, by Fatou's lemma,

$$\begin{aligned} \|f\|_q &\leq \liminf_{k \rightarrow \infty} \left(\int_{\mathbb{R}^n} |\phi_k(\mathbf{x})|^q dm_n \right)^{1/q} \\ &\leq \liminf_{k \rightarrow \infty} \frac{r}{\sqrt[n]{n}} \|\phi_k\|_{1,p,\mathbb{R}^n} = \|f\|_{1,p,\mathbb{R}^n}. \end{aligned}$$

This proves the theorem.

Corollary 38.1.11 *Suppose $mp < n$. Then $W^{m,p}(\mathbb{R}^n) \subseteq L^q(\mathbb{R}^n)$ where $q = \frac{np}{n-mp}$ and the identity map, $\text{id} : W^{m,p}(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$ is continuous.*

Proof: This is true if $m = 1$ according to Theorem 38.1.10. Suppose it is true for $m - 1$ where $m > 1$. If $u \in W^{m,p}(\mathbb{R}^n)$ and $|\alpha| \leq 1$, then $D^\alpha u \in W^{m-1,p}(\mathbb{R}^n)$ so by induction, for all such α ,

$$D^\alpha u \in L^{\frac{np}{n-(m-1)p}}(\mathbb{R}^n).$$

Thus $u \in W^{1,q_1}(\mathbb{R}^n)$ where

$$q_1 = \frac{np}{n - (m-1)p}$$

By Theorem 38.1.10, it follows that $u \in L^q(\mathbb{R}^n)$ where

$$\frac{1}{q} = \frac{n - (m-1)p}{np} - \frac{1}{n} = \frac{n-mp}{np}.$$

This proves the corollary.

There is another similar corollary of the same sort which is interesting and useful.

Corollary 38.1.12 *Suppose $m \geq 1$ and j is a nonnegative integer satisfying $jp < n$. Then*

$$W^{m+j,p}(\mathbb{R}^n) \subseteq W^{m,q}(\mathbb{R}^n)$$

for

$$q \equiv \frac{np}{n - jp} \quad (38.1.16)$$

and the identity map is continuous.

Proof: If $|\alpha| \leq m$, then $D^\alpha u \in W^{j,p}(\mathbb{R}^n)$ and so by Corollary 38.1.11, $D^\alpha u \in L^q(\mathbb{R}^n)$ where q is given above. This means $u \in W^{m,q}(\mathbb{R}^n)$.

The above corollaries imply yet another interesting corollary which involves embeddings in the Holder spaces.

Corollary 38.1.13 *Suppose $jp < n < (j+1)p$ and let m be a positive integer. Let U be any bounded open set in $\mathbb{R}^n$. Then letting r_U denote the restriction to $\overline{U}$, $r_U : W^{m+j,p}(\mathbb{R}^n) \rightarrow C^{m-1,\lambda}(\overline{U})$ is continuous for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{p}$ and if $\lambda < (j+1) - \frac{n}{p}$, then r_U is compact.*

Proof: From Corollary 38.1.12 $W^{m+j,p}(\mathbb{R}^n) \subseteq W^{m,q}(\mathbb{R}^n)$ where q is given by 38.1.16. Therefore,

$$\frac{np}{n-jp} > n$$

and so by Corollary 38.1.7, $W^{m,q}(\mathbb{R}^n) \subseteq C^{m-1,\lambda}(\overline{U})$ for all λ satisfying

$$0 < \lambda < 1 - \frac{(n-jp)n}{np} = \frac{p(j+1)-n}{p} = (j+1) - \frac{n}{p}.$$

The assertion about compactness follows from the compactness of the embedding of $C^{m-1,\lambda_0}(\overline{U})$ into $C^{m-1,\lambda}(\overline{U})$ for $\lambda < \lambda_0$. See Lemma 38.1.6.

There are other embeddings of this sort available. You should see Adams [1] for a more complete listing of these. Next are some theorems about compact embeddings. This requires some consideration of which subsets of $L^p(U)$ are compact. The main theorem is the following. See [1].

Theorem 38.1.14 *Let K be a bounded subset of $L^p(U)$ and suppose that for all $\varepsilon > 0$, there exist a $\delta > 0$ such that if $|\mathbf{h}| < \delta$, then*

$$\int_{\mathbb{R}^n} |\tilde{u}(\mathbf{x} + \mathbf{h}) - \tilde{u}(\mathbf{x})|^p dx < \varepsilon^p \quad (38.1.17)$$

Suppose also that for each $\varepsilon > 0$ there exists an open set, $G \subseteq U$ such that $\overline{G}$ is compact and for all $u \in K$,

$$\int_{U \setminus \overline{G}} |u(\mathbf{x})|^p dx < \varepsilon^p \quad (38.1.18)$$

Then K is precompact in $L^p(\mathbb{R}^n)$.

Proof: To save fussing first consider the case where $U = \mathbb{R}^n$ so that $\tilde{u} = u$. Suppose the two conditions hold and let ϕ_k be a mollifier of the form $\phi_k(\mathbf{x}) = k^n \phi(k\mathbf{x})$ where $\text{spt}(\phi) \subseteq B(\mathbf{0}, 1)$. Consider

$$K_k \equiv \{u * \phi_k : u \in K\}.$$

and verify the conditions for the Ascoli Arzela theorem for these functions defined on $\overline{G}$. Say $\|u\|_p \leq M$ for all $u \in K$.

First of all, for $u \in K$ and $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned}
 |u * \phi_k(\mathbf{x})|^p &\leq \left(\int |u(\mathbf{x} - \mathbf{y}) \phi_k(\mathbf{y})| dy \right)^p \\
 &= \left(\int |u(\mathbf{y}) \phi_k(\mathbf{x} - \mathbf{y})| dy \right)^p \\
 &\leq \int |u(\mathbf{y})|^p \phi_k(\mathbf{x} - \mathbf{y}) dy \\
 &\leq \left(\sup_{\mathbf{z} \in \mathbb{R}^n} \phi_k(\mathbf{z}) \right) \int |u(\mathbf{y})| dy \leq M \left(\sup_{\mathbf{z} \in \mathbb{R}^n} \phi_k(\mathbf{z}) \right)
 \end{aligned}$$

showing the functions in K_k are uniformly bounded.

Next suppose $\mathbf{x}, \mathbf{x}_1 \in K_k$ and consider

$$\begin{aligned}
 &|u * \phi_k(\mathbf{x}) - u * \phi_k(\mathbf{x}_1)| \\
 &\leq \int |u(\mathbf{x} - \mathbf{y}) - u(\mathbf{x}_1 - \mathbf{y})| \phi_k(\mathbf{y}) dy \\
 &\leq \left(\int |u(\mathbf{x} - \mathbf{y}) - u(\mathbf{x}_1 - \mathbf{y})|^p dy \right)^{1/p} \left(\int \phi_k(\mathbf{y})^q dy \right)^q
 \end{aligned}$$

which by assumption 38.1.17 is small independent of the choice of u whenever $|\mathbf{x} - \mathbf{x}_1|$ is small enough. Note that k is fixed in the above. Therefore, the set, K_k is precompact in $C(\bar{G})$ thanks to the Ascoli Arzela theorem. Next consider how well $u \in K$ is approximated by $u * \phi_k$ in $L^p(\mathbb{R}^n)$. By Minkowski's inequality,

$$\begin{aligned}
 &\left(\int |u(\mathbf{x}) - u * \phi_k(\mathbf{x})|^p dx \right)^{1/p} \\
 &\leq \left(\int \left(\int |u(\mathbf{x}) - u(\mathbf{x} - \mathbf{y})| \phi_k(\mathbf{y}) dy \right)^p dx \right)^{1/p} \\
 &\leq \int_{B(\mathbf{0}, \frac{1}{k})} \phi_k(\mathbf{y}) \left(\int |u(\mathbf{x}) - u(\mathbf{x} - \mathbf{y})|^p dx \right)^{1/p} dy.
 \end{aligned}$$

Now let $\eta > 0$ be given. From 38.1.17 there exists k large enough that for all $u \in K$,

$$\int_{B(\mathbf{0}, \frac{1}{k})} \phi_k(\mathbf{y}) \left(\int |u(\mathbf{x}) - u(\mathbf{x} - \mathbf{y})|^p dx \right)^{1/p} dy \leq \int_{B(\mathbf{0}, \frac{1}{k})} \phi_k(\mathbf{y}) \eta dy = \eta.$$

Now let $\varepsilon > 0$ be given and let δ and G correspond to ε as given in the hypotheses and let $1/k < \delta$ and also k is large enough that for all $u \in K$,

$$\|u - u * \phi_k\|_p < \varepsilon$$

as in the above inequality. By the Ascoli Arzela theorem there exists an

$$\left(\frac{\varepsilon}{m(\bar{G} + B(\mathbf{0}, 1))} \right)^{1/p}$$

net for K_k in $C(\overline{G})$. That is, there exist $\{u_i\}_{i=1}^m \subseteq K$ such that for any $u \in K$,

$$\|u * \phi_k - u_j * \phi_k\|_\infty < \left(\frac{\varepsilon}{m(\overline{G} + B(\mathbf{0}, 1))} \right)^{1/p}$$

for some j . Letting $u \in K$ be given, let $u_j \in \{u_i\}_{i=1}^m \subseteq K$ be such that the above inequality holds. Then

$$\begin{aligned} \|u - u_j\|_p &\leq \|u - u * \phi_k\|_p + \|u * \phi_k - u_j * \phi_k\|_p + \|u_j * \phi_k - u_j\|_p \\ &< 2\varepsilon + \|u * \phi_k - u_j * \phi_k\|_p \\ &\leq 2\varepsilon + \left(\int_{\overline{G} + B(\mathbf{0}, 1)} |u * \phi_k - u_j * \phi_k|^p dx \right)^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^n \setminus (\overline{G} + B(\mathbf{0}, 1))} |u * \phi_k - u_j * \phi_k|^p dx \right)^{1/p} \\ &\leq 2\varepsilon + \varepsilon^{1/p} \\ &\quad + \left(\int_{\mathbb{R}^n \setminus (\overline{G} + B(\mathbf{0}, 1))} \left(\int |u(\mathbf{x} - \mathbf{y}) - u_j(\mathbf{x} - \mathbf{y})| \phi_k(\mathbf{y}) dy \right)^p dx \right)^{1/p} \\ &\leq 2\varepsilon + \varepsilon^{1/p} \\ &\quad + \int \phi_k(\mathbf{y}) \left(\int_{\mathbb{R}^n \setminus (\overline{G} + B(\mathbf{0}, 1))} (|u(\mathbf{x} - \mathbf{y})| + |u_j(\mathbf{x} - \mathbf{y})|)^p dx \right)^{1/p} dy \\ &\leq 2\varepsilon + \varepsilon^{1/p} + \int \phi_k(\mathbf{y}) \left(\int_{\mathbb{R}^n \setminus \overline{G}} (|u(\mathbf{x})| + |u_j(\mathbf{x})|)^p dx \right)^{1/p} dy \\ &\leq 2\varepsilon + \varepsilon^{1/p} + 2^{p-1} \int \phi_k(\mathbf{y}) \left(\int_{\mathbb{R}^n \setminus \overline{G}} (|u(\mathbf{x})|^p + |u_j(\mathbf{x})|^p) dx \right)^{1/p} dy \\ &\leq 2\varepsilon + \varepsilon^{1/p} + 2^{p-1} 2^{1/p} \varepsilon \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, this shows that K is totally bounded and is therefore precompact.

Now for an arbitrary open set, U and K given in the hypotheses of the theorem, let $\tilde{K} \equiv \{\tilde{u} : u \in K\}$ and observe that $\tilde{K}$ is precompact in $L^p(\mathbb{R}^n)$. But this is the same as saying that K is precompact in $L^p(U)$. This proves the theorem.

Actually the converse of the above theorem is also true [1] but this will not be needed so I have left it as an exercise for anyone interested.

Lemma 38.1.15 *Let $u \in W^{1,1}(U)$ for U an open set and let $\phi \in C_c^\infty(U)$. Then there exists a constant,*

$$C(\phi, \|u\|_{1,1,U}),$$

depending only on the indicated quantities such that whenever $\mathbf{v} \in \mathbb{R}^n$ with $|\mathbf{v}| < \text{dist}(\text{spt}(\phi), U^C)$, it follows that

$$\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right| dx \leq C \left(\phi, \|u\|_{1,1,U} \right) |\mathbf{v}|.$$

Proof: First suppose $u \in C^\infty(\overline{U})$. Then for any $\mathbf{x} \in \text{spt}(\phi) \cup (\text{spt}(\phi) - \mathbf{v}) \equiv G_{\mathbf{v}}$, the chain rule implies

$$\begin{aligned} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})| &\leq \int_0^1 \sum_{i=1}^n \left| (\phi u)_{,i}(\mathbf{x} + t\mathbf{v}) v_i \right| dt \\ &\leq \int_0^1 \sum_{i=1}^n \left| (\phi_{,i} u + u_{,i} \phi)(\mathbf{x} + t\mathbf{v}) \right| dt |\mathbf{v}|. \end{aligned}$$

Therefore, for such u ,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right| dx \\ &= \int_{G_{\mathbf{v}}} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})| dx \\ &\leq \int_{G_{\mathbf{v}}} \int_0^1 \sum_{i=1}^n \left| (\phi_{,i} u + u_{,i} \phi)(\mathbf{x} + t\mathbf{v}) \right| dt dx |\mathbf{v}| \\ &\leq \int_0^1 \int_{G_{\mathbf{v}}} \sum_{i=1}^n \left| (\phi_{,i} u + u_{,i} \phi)(\mathbf{x} + t\mathbf{v}) \right| dx dt |\mathbf{v}| \\ &\leq C \left(\phi, \|u\|_{1,1,U} \right) |\mathbf{v}| \end{aligned}$$

where C is a continuous function of $\|u\|_{1,1,U}$. Now for general $u \in W^{1,1}(U)$, let $u_k \rightarrow u$ in $W^{1,1}(U)$ where $u_k \in C^\infty(\overline{U})$. Then for $|\mathbf{v}| < \text{dist}(\text{spt}(\phi), U^C)$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right| dx \\ &= \int_{G_{\mathbf{v}}} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})| dx \\ &= \lim_{k \rightarrow \infty} \int_{G_{\mathbf{v}}} |\phi u_k(\mathbf{x} + \mathbf{v}) - \phi u_k(\mathbf{x})| dx \\ &\leq \lim_{k \rightarrow \infty} C \left(\phi, \|u_k\|_{1,1,U} \right) |\mathbf{v}| \\ &= C \left(\phi, \|u\|_{1,1,U} \right) |\mathbf{v}|. \end{aligned}$$

This proves the lemma.

Lemma 38.1.16 *Let U be a bounded open set and define for $p > 1$*

$$S \equiv \left\{ u \in W^{1,1}(U) \cap L^p(U) : \|u\|_{1,1,U} + \|u\|_{L^p(U)} \leq M \right\} \quad (38.1.19)$$

and let $\phi \in C_c^\infty(U)$ and

$$S_1 \equiv \{u\phi : u \in S\}. \quad (38.1.20)$$

Then S_1 is precompact in $L^q(U)$ where $1 \leq q < p$.

Proof: This depends on Theorem 38.1.14. The second condition is satisfied by taking $G \equiv \text{spt}(\phi)$. Thus, for $w \in S_1$,

$$\int_{U \setminus \overline{G}} |w(\mathbf{x})|^q dx = 0 < \varepsilon^p.$$

It remains to satisfy the first condition. It is necessary to verify there exists $\delta > 0$ such that if $|\mathbf{v}| < \delta$, then

$$\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right|^q dx < \varepsilon^p. \quad (38.1.21)$$

Let $\text{spt}(\phi) \cup (\text{spt}(\phi) - \mathbf{v}) \equiv G_{\mathbf{v}}$. Now if h is any measurable function, and if $\theta \in (0, 1)$ is chosen small enough that $\theta q < 1$,

$$\begin{aligned} \int_{G_{\mathbf{v}}} |h|^q dx &= \int_{G_{\mathbf{v}}} |h|^{\theta q} |h|^{(1-\theta)q} dx \\ &\leq \left(\int_{G_{\mathbf{v}}} |h| dx \right)^{\theta q} \left(\int_{G_{\mathbf{v}}} \left(|h|^{(1-\theta)q} \right)^{\frac{1}{1-\theta q}} \right)^{1-\theta q} \\ &= \left(\int_{G_{\mathbf{v}}} |h| dx \right)^{\theta q} \left(\int_{G_{\mathbf{v}}} |h|^{\frac{(1-\theta)q}{1-\theta q}} \right)^{1-\theta q}. \end{aligned} \quad (38.1.22)$$

Now let θ also be small enough that there exists $r > 1$ such that

$$r \frac{(1-\theta)q}{1-\theta q} = p$$

and use Holder's inequality in the last factor of the right side of 38.1.22. Then 38.1.22 is dominated by

$$\begin{aligned} &\left(\int_{G_{\mathbf{v}}} |h| dx \right)^{\theta q} \left(\int_{G_{\mathbf{v}}} |h|^p \right)^{\frac{1-\theta q}{r}} \left(\int_{G_{\mathbf{v}}} 1 dx \right)^{1/r'} \\ &= C \left(\|h\|_{L^p(G_{\mathbf{v}})}, m_n(G_{\mathbf{v}}) \right) \left(\int_{G_{\mathbf{v}}} |h| dx \right)^{\theta q}. \end{aligned}$$

Therefore, for $u \in S$,

$$\int_{\mathbb{R}^n} \left| \widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x}) \right|^q dx = \int_{G_{\mathbf{v}}} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})|^q dx \leq$$

$$\begin{aligned}
& C \left(\|\phi u(\cdot + \mathbf{v}) - \phi u(\cdot)\|_{L^p(G_{\mathbf{v}})}, m_n(G_{\mathbf{v}}) \right) \left(\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})| dx \right)^{\theta q} \\
& \leq C \left(2 \|\phi u(\cdot)\|_{L^p(U)}, m_n(U) \right) \left(\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})| dx \right)^{\theta q} \\
& \leq C(\phi, M, m_n(U)) \left(\int_{G_{\mathbf{v}}} |\phi u(\mathbf{x} + \mathbf{v}) - \phi u(\mathbf{x})| dx \right)^{\theta q} \\
& = C(\phi, M, m_n(U)) \left(\int_{\mathbb{R}^n} |\widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x})| dx \right)^{\theta q}. \quad (38.1.23)
\end{aligned}$$

Now by Lemma 38.1.15,

$$\int_{\mathbb{R}^n} |\widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x})| dx \leq C(\phi, \|u\|_{1,1,U}) |\mathbf{v}| \quad (38.1.24)$$

and so from 38.1.23 and 38.1.24, and adjusting the constants

$$\begin{aligned}
\int_{\mathbb{R}^n} |\widetilde{\phi u}(\mathbf{x} + \mathbf{v}) - \widetilde{\phi u}(\mathbf{x})|^q dx & \leq C(\phi, M, m_n(U)) \left(C(\phi, \|u\|_{1,1,U}) |\mathbf{v}| \right)^{\theta q} \\
& = C(\phi, M, m_n(U)) |\mathbf{v}|^{\theta q}
\end{aligned}$$

which verifies 38.1.21 whenever $|\mathbf{v}|$ is sufficiently small. This proves the lemma because the conditions of Theorem 38.1.14 are satisfied.

Theorem 38.1.17 *Let U be a bounded open set and define for $p > 1$*

$$S \equiv \left\{ u \in W^{1,1}(U) \cap L^p(U) : \|u\|_{1,1,U} + \|u\|_{L^p(U)} \leq M \right\} \quad (38.1.25)$$

Then S is precompact in $L^q(U)$ where $1 \leq q < p$.

Proof: It suffices to show that every sequence, $\{u_k\}_{k=1}^{\infty} \subseteq S$ has a subsequence which converges in $L^q(U)$. Let $\{K_m\}_{m=1}^{\infty}$ denote a sequence of compact subsets of U with the property that $K_m \subseteq K_{m+1}$ for all m and $\cup_{m=1}^{\infty} K_m = U$. Now let $\phi_m \in C_c^{\infty}(U)$ such that $\phi_m(\mathbf{x}) \in [0, 1]$ and $\phi_m(\mathbf{x}) = 1$ for all $\mathbf{x} \in K_m$. Let $S_m \equiv \{\phi_m u : u \in S\}$. By Lemma 38.1.16 there exists a subsequence of $\{u_k\}_{k=1}^{\infty}$, denoted here by $\{u_{1,k}\}_{k=1}^{\infty}$ such that $\{\phi_1 u_{1,k}\}_{k=1}^{\infty}$ converges in $L^q(U)$. Now S_2 is also precompact in $L^q(U)$ and so there exists a subsequence of $\{u_{1,k}\}_{k=1}^{\infty}$, denoted by $\{u_{2,k}\}_{k=1}^{\infty}$ such that $\{\phi_2 u_{2,k}\}_{k=1}^{\infty}$ converges in $L^2(U)$. Thus it is also the case that $\{\phi_1 u_{2,k}\}_{k=1}^{\infty}$ converges in $L^q(U)$. Continue taking subsequences in this manner such that for all $l \leq m$, $\{\phi_l u_{m,k}\}_{k=1}^{\infty}$ converges in $L^q(U)$. Let $\{w_m\}_{m=1}^{\infty} = \{u_{m,m}\}_{m=1}^{\infty}$ so that $\{w_k\}_{k=m}^{\infty}$ is a subsequence of $\{u_{m,k}\}_{k=1}^{\infty}$. Then it follows for all k , $\{\phi_k w_m\}_{m=1}^{\infty}$

must converge in $L^q(U)$. For $u \in S$,

$$\begin{aligned} \|u - \phi_k u\|_{L^q(U)}^q &= \int_U |u|^q (1 - \phi_k)^q dx \\ &\leq \left(\int_U |u|^p dx \right)^{q/p} \left(\int_U (1 - \phi_k)^{qr} dx \right)^{1/r} \\ &\leq M \left(\int_U (1 - \phi_k)^{qr} dx \right)^{1/r} \end{aligned}$$

where $q/p + 1/r = 1$. Now $\phi_l(\mathbf{x}) \rightarrow \mathcal{X}_U(\mathbf{x})$ and so the integrand in the last integral converges to 0 by the dominated convergence theorem. Therefore, k may be chosen large enough that for all $u \in S$,

$$\|u - \phi_k u\|_{L^q(U)}^q \leq \left(\frac{\varepsilon}{3} \right)^q.$$

Fix such a value of k . Then

$$\begin{aligned} &\|w_q - w_p\|_{L^q(U)} \leq \\ &\|w_q - \phi_k w_q\|_{L^q(U)} + \|\phi_k w_q - \phi_k w_p\|_{L^q(U)} + \|w_p - \phi_k w_p\|_{L^q(U)} \\ &\leq \frac{2\varepsilon}{3} + \|\phi_k w_q - \phi_k w_p\|_{L^q(U)}. \end{aligned}$$

But $\{\phi_k w_m\}_{m=1}^\infty$ converges in $L^q(U)$ and so the last term in the above is less than $\varepsilon/3$ whenever p, q are large enough. Thus $\{w_m\}_{m=1}^\infty$ is a Cauchy sequence and must therefore converge in $L^q(U)$. This proves the theorem.

38.2 An Extension Theorem

Definition 38.2.1 An open subset, U , of $\mathbb{R}^n$ has a Lipschitz boundary if it satisfies the following conditions. For each $p \in \partial U \equiv \overline{U} \setminus U$, there exists an open set, Q , containing p , an open interval (a, b) , a bounded open box $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation R such that

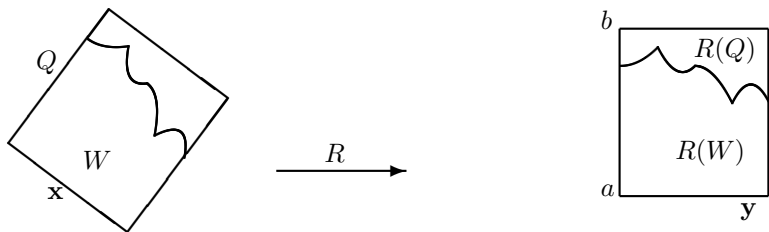
$$RQ = B \times (a, b), \quad (38.2.26)$$

$$R(Q \cap U) = \{\mathbf{y} \in \mathbb{R}^n : \hat{\mathbf{y}} \in B, a < y_n < g(\hat{\mathbf{y}})\} \quad (38.2.27)$$

where g is Lipschitz continuous on $\overline{B}$, $a < \min\{g(\mathbf{x}) : \mathbf{x} \in \overline{B}\}$, and

$$\hat{\mathbf{y}} \equiv (y_1, \dots, y_{n-1}).$$

Letting $W = Q \cap U$ the following picture describes the situation.



The following lemma is important.

Lemma 38.2.2 *If U is an open subset of $\mathbb{R}^n$ which has a Lipschitz boundary, then it satisfies the segment condition and so $X^{m,p}(U) = W^{m,p}(U)$.*

Proof: For $\mathbf{x} \in \partial U$, simply look at a single open set, $Q_{\mathbf{x}}$ described in the above which contains $\mathbf{x}$. Then consider an open set whose intersection with U is of the form $R^T(\{\mathbf{y} : \hat{\mathbf{y}} \in B, g(\hat{\mathbf{y}}) - \varepsilon < y_n < g(\hat{\mathbf{y}})\})$ and a vector of the form $\varepsilon R^T(-\mathbf{e}_n)$ where ε is chosen smaller than $\min\{g(\mathbf{x}) : \mathbf{x} \in \overline{B}\} - a$. There is nothing to prove for points of U .

One way to extend many of the above theorems to more general open sets than $\mathbb{R}^n$ is through the use of an appropriate extension theorem. In this section, a fairly general one will be presented.

Lemma 38.2.3 *Let $B \times (a, b)$ be as described in Definition 38.2.1 and let*

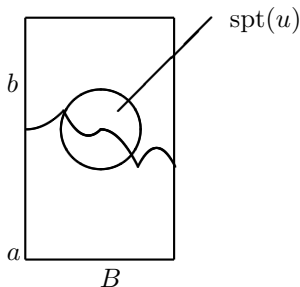
$$V^- \equiv \{(\hat{\mathbf{y}}, y_n) : y_n < g(\hat{\mathbf{y}})\}, \quad V^+ \equiv \{(\hat{\mathbf{y}}, y_n) : y_n > g(\hat{\mathbf{y}})\},$$

for g a Lipschitz function of the sort described in this definition. Suppose u^+ and u^- are Lipschitz functions defined on $\overline{V^+}$ and $\overline{V^-}$ respectively and suppose that $u^+(\hat{\mathbf{y}}, g(\hat{\mathbf{y}})) = u^-(\hat{\mathbf{y}}, g(\hat{\mathbf{y}}))$ for all $\hat{\mathbf{y}} \in B$. Let

$$u(\hat{\mathbf{y}}, y_n) \equiv \begin{cases} u^+(\hat{\mathbf{y}}, y_n) & \text{if } (\hat{\mathbf{y}}, y_n) \in V^+ \\ u^-(\hat{\mathbf{y}}, y_n) & \text{if } (\hat{\mathbf{y}}, y_n) \in V^- \end{cases}$$

and suppose $\text{spt}(u) \subseteq B \times (a, b)$. Then extending u to be 0 off of $B \times (a, b)$, u is continuous and the weak partial derivatives, $u_{,i}$, are all in $L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $p > 1$ and $u_{,i} = (u^+)_{,i}$ on V^+ and $u_{,i} = (u^-)_{,i}$ on V^- .

Proof: Consider the following picture which is descriptive of the situation.



Note first that u is Lipschitz continuous. To see this, consider $|u(\mathbf{y}_1) - u(\mathbf{y}_2)|$ where $(\hat{\mathbf{y}}_i, y_n^i) = \mathbf{y}_i$. There are various cases to consider depending on whether y_n^i is above $g(\hat{\mathbf{y}}_i)$. Suppose $y_n^1 < g(\hat{\mathbf{y}}_1)$ and $y_n^2 > g(\hat{\mathbf{y}}_2)$. Then letting $K \geq \max(\text{Lip}(u^+), \text{Lip}(u^-), \text{Lip}(g))$,

$$\begin{aligned}
 & |u(\hat{\mathbf{y}}_1, y_n^1) - u(\hat{\mathbf{y}}_2, y_n^2)| \leq \\
 & |u(\hat{\mathbf{y}}_1, y_n^1) - u(\hat{\mathbf{y}}_2, y_n^1)| + |u(\hat{\mathbf{y}}_2, y_n^1) - u(\hat{\mathbf{y}}_2, g(\hat{\mathbf{y}}_2))| \\
 & + |u(\hat{\mathbf{y}}_2, g(\hat{\mathbf{y}}_2)) - u(\hat{\mathbf{y}}_2, y_n^2)| \\
 & \leq K|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2| + K[|g(\hat{\mathbf{y}}_2) - g(\hat{\mathbf{y}}_1)| + g(\hat{\mathbf{y}}_1) - y_n^1 + y_n^2 - g(\hat{\mathbf{y}}_2)] \\
 & \leq (2K + K^2)|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2| + K|y_n^1 - y_n^2| \\
 & = (2K + K^2)(|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_2| + |y_n^1 - y_n^2|) \leq (2K + K^2)\sqrt{2}|\mathbf{y}_1 - \mathbf{y}_2|
 \end{aligned}$$

The other cases are similar. Thus u is a Lipschitz continuous function which has compact support. By Corollary 35.5.4 on Page 1054 it follows that $u_{,i} \in L^\infty(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for all $p > 1$. It remains to verify $u_{,i} = (u^+)_{,i}$ on V^+ and $u_{,i} = (u^-)_{,i}$ on V^- . The last claim is obvious from the definition of weak derivatives.

Lemma 38.2.4 *In the situation of Lemma 38.2.3 let $u \in C^1(\overline{V^-}) \cap C_c^1(B \times (a, b))^3$ and define*

$$w(\hat{\mathbf{y}}, y_n) \equiv \begin{cases} u(\hat{\mathbf{y}}, y_n) & \text{if } \hat{\mathbf{y}} \in B \text{ and } y_n \leq g(\hat{\mathbf{y}}), \\ u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n), & \text{if } \hat{\mathbf{y}} \in B \text{ and } y_n > g(\hat{\mathbf{y}}) \\ 0 & \text{if } \hat{\mathbf{y}} \notin B. \end{cases}$$

Then $w \in W^{1,p}(\mathbb{R}^n)$ and there exists a constant, C depending only on $\text{Lip}(g)$ and dimension such that

$$\|w\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(V^-)}.$$

Denote w by $E_0 u$. Thus $E_0(u)(\mathbf{y}) = u(\mathbf{y})$ for all $\mathbf{y} \in V^-$ but $E_0 u = w$ is defined on all of $\mathbb{R}^n$. Also, E_0 is a linear mapping.

³This means that $\text{spt}(u) \subseteq B \times (a, b)$ and $u \in C^1(\overline{V^-})$.

Proof: As in the previous lemma, w is Lipschitz continuous and has compact support so it is clear $w \in W^{1,p}(\mathbb{R}^n)$. The main task is to find $w_{,i}$ for $\hat{\mathbf{y}} \in B$ and $y_n > g(\hat{\mathbf{y}})$ and then to extract an estimate of the right sort. Denote by U the set of points of $\mathbb{R}^n$ with the property that $(\hat{\mathbf{y}}, y_n) \in U$ if and only if $\hat{\mathbf{y}} \notin B$ or $\hat{\mathbf{y}} \in B$ and $y_n > g(\hat{\mathbf{y}})$. Then letting $\phi \in C_c^\infty(U)$, suppose first that $i < n$. Then

$$\begin{aligned} & \int_U w(\hat{\mathbf{y}}, y_n) \phi_{,i}(\mathbf{y}) dy \\ & \equiv \lim_{h \rightarrow 0} \int_U \phi(\mathbf{y}) \frac{u(\hat{\mathbf{y}} - h\mathbf{e}_i^{n-1}, 2g(\hat{\mathbf{y}} - h\mathbf{e}_i^{n-1}) - y_n) - u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n)}{h} dy \end{aligned} \quad (38.2.28)$$

$$\begin{aligned} & = \lim_{h \rightarrow 0} \left\{ \frac{-1}{h} \int_U \phi(\mathbf{y}) [D_1 u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n)(h\mathbf{e}_i^{n-1}) \right. \\ & \quad \left. + 2D_2 u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n)(g(\hat{\mathbf{y}} - h\mathbf{e}_i^{n-1}) - g(\hat{\mathbf{y}}))] dy \right. \\ & \quad \left. + \frac{-1}{h} \int_U \phi(\mathbf{y}) [o(g(\hat{\mathbf{y}} - h\mathbf{e}_i^{n-1}) - g(\hat{\mathbf{y}})) + o(h)] dy \right\} \end{aligned}$$

where $\mathbf{e}_i^{n-1}$ is the unit vector in $\mathbb{R}^{n-1}$ having all zeros except for a 1 in the i^{th} position. Now by Rademacher's theorem, $Dg(\hat{\mathbf{y}})$ exists for a.e. $\hat{\mathbf{y}}$ and so except for a set of measure zero, the expression, $o(g(\hat{\mathbf{y}} - h\mathbf{e}_i^{n-1}) - g(\hat{\mathbf{y}}))$ is $o(h)$ and also for $\hat{\mathbf{y}}$ not in the exceptional set,

$$g(\hat{\mathbf{y}} - h\mathbf{e}_i^{n-1}) - g(\hat{\mathbf{y}}) = -hDg(\hat{\mathbf{y}})\mathbf{e}_i^{n-1} + o(h).$$

Therefore, since the integrand in 38.2.28 has compact support and because of the Lipschitz continuity of all the functions, the dominated convergence theorem may be applied to obtain

$$\begin{aligned} & \int_U w(\hat{\mathbf{y}}, y_n) \phi_{,i}(\mathbf{y}) dy = \\ & \int_U \phi(\mathbf{y}) [-D_1 u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n)(\mathbf{e}_i^{n-1}) + 2D_2 u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n)(Dg(\hat{\mathbf{y}})\mathbf{e}_i^{n-1})] dy \\ & = \int_U \phi(\mathbf{y}) \left[-\frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) + 2\frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \frac{\partial g(\hat{\mathbf{y}})}{\partial y_i} \right] dy \end{aligned}$$

and so

$$w_{,i}(\mathbf{y}) = \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) - 2\frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \frac{\partial g(\hat{\mathbf{y}})}{\partial y_i} \quad (38.2.29)$$

whenever $i < n$ which is what you would expect from a formal application of the chain rule. Next suppose $i = n$.

$$\int_U w(\hat{\mathbf{y}}, y_n) \phi_{,n}(\mathbf{y}) dy$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} - \int_U \frac{u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - (y_n + h)) - u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n)}{h} \phi(\mathbf{y}) dy \\
&= \lim_{h \rightarrow 0} \int_U \frac{D_2 u(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) h + o(h)}{h} \phi(\mathbf{y}) dy \\
&= \int_U \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \phi(\mathbf{y}) dy
\end{aligned}$$

showing that

$$w_{,n}(\mathbf{y}) = \frac{-\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \quad (38.2.30)$$

which is also expected.

From the definition, for $\mathbf{y} \in \mathbb{R}^n \setminus U \equiv \{(\hat{\mathbf{y}}, y_n) : y_n \leq g(\hat{\mathbf{y}})\}$ it follows $w_{,i} = u_{,i}$ and on U , $w_{,i}$ is given by 38.2.29 and 38.2.30. Consider $\|w_{,i}\|_{L^p(U)}^p$ for $i < n$. From 38.2.29

$$\begin{aligned}
\|w_{,i}\|_{L^p(U)}^p &= \int_U \left| \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) - 2 \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \frac{\partial g(\hat{\mathbf{y}})}{\partial y_i} \right|^p dy \\
&\leq 2^{p-1} \int_U \left| \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p \\
&\quad + 2^p \left| \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p \text{Lip}(g)^p dy \\
&\leq 4^p (1 + \text{Lip}(g)^p) \int_U \left| \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p \\
&\quad + \left| \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p dy \\
&= 4^p (1 + \text{Lip}(g)^p) \int_B \int_{g(\hat{\mathbf{y}})}^\infty \left| \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p \\
&\quad + \left| \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p dy_n d\hat{\mathbf{y}} \\
&= 4^p (1 + \text{Lip}(g)^p) \int_B \int_{-\infty}^{g(\hat{\mathbf{y}})} \left| \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, z_n) \right|^p + \left| \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, z_n) \right|^p dz_n d\hat{\mathbf{y}} \\
&= 4^p (1 + \text{Lip}(g)^p) \int_B \int_a^{g(\hat{\mathbf{y}})} \left| \frac{\partial u}{\partial y_i}(\hat{\mathbf{y}}, z_n) \right|^p \\
&\quad + \left| \frac{\partial u}{\partial y_n}(\hat{\mathbf{y}}, z_n) \right|^p dz_n d\hat{\mathbf{y}} \leq 4^p (1 + \text{Lip}(g)^p) \|u\|_{1,p,V-}^p
\end{aligned}$$

Now by similar reasoning,

$$\begin{aligned}
 \|w_n\|_{L^p(U)}^p &= \int_U \left| \frac{-\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p dy \\
 &= \int_B \int_{g(\hat{\mathbf{y}})}^\infty \left| \frac{-\partial u}{\partial y_n}(\hat{\mathbf{y}}, 2g(\hat{\mathbf{y}}) - y_n) \right|^p dy_n d\hat{\mathbf{y}} \\
 &= \int_B \int_a^{g(\hat{\mathbf{y}})} \left| \frac{-\partial u}{\partial y_n}(\hat{\mathbf{y}}, z_n) \right|^p dz_n d\hat{\mathbf{y}} = \|u_n\|_{1,p,V^-}^p.
 \end{aligned}$$

It follows

$$\begin{aligned}
 \|w\|_{1,p,\mathbb{R}^n}^p &= \|w\|_{1,p,U}^p + \|u\|_{1,p,V^-}^p \\
 &\leq 4^p n (1 + \text{Lip}(g)^p) \|u\|_{1,p,V^-}^p + \|u\|_{1,p,V^-}^p
 \end{aligned}$$

and so

$$\|w\|_{1,p,\mathbb{R}^n}^p \leq 4^p n (2 + \text{Lip}(g)^p) \|u\|_{1,p,V^-}^p$$

which implies

$$\|w\|_{1,p,\mathbb{R}^n} \leq 4n^{1/p} (2 + \text{Lip}(g)^p)^{1/p} \|u\|_{1,p,V^-}$$

It is obvious that E_0 is a continuous linear mapping. This proves the lemma.

Now recall Definition 38.2.1, listed here for convenience.

Definition 38.2.5 An open subset, U , of $\mathbb{R}^n$ has a Lipschitz boundary if it satisfies the following conditions. For each $p \in \partial U \equiv \bar{U} \setminus U$, there exists an open set, Q , containing p , an open interval (a, b) , a bounded open box $B \subseteq \mathbb{R}^{n-1}$, and an orthogonal transformation R such that

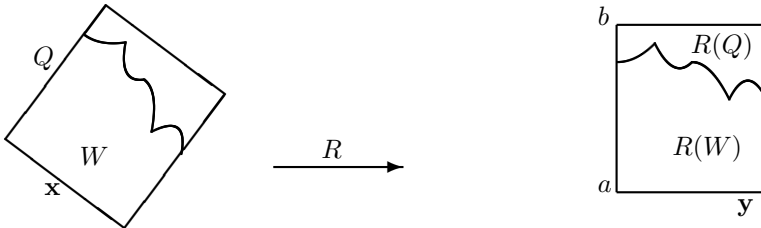
$$RQ = B \times (a, b), \quad (38.2.31)$$

$$R(Q \cap U) = \{\mathbf{y} \in \mathbb{R}^n : \hat{\mathbf{y}} \in B, a < y_n < g(\hat{\mathbf{y}})\} \quad (38.2.32)$$

where g is Lipschitz continuous on $\bar{B}$, $a < \min\{g(\mathbf{x}) : \mathbf{x} \in \bar{B}\}$, and

$$\hat{\mathbf{y}} \equiv (y_1, \dots, y_{n-1}).$$

Letting $W = Q \cap U$ the following picture describes the situation.



Lemma 38.2.6 *In the situation of Definition 38.2.1 let $u \in C^1(\overline{U}) \cap C_c^1(Q)$ and define*

$$Eu \equiv R^* E_0 (R^T)^* u.$$

where $(R^T)^*$ maps $W^{1,p}(U \cap Q)$ to $W^{1,p}(R(W))$. Then E is linear and satisfies

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(Q \cap U)}, \quad Eu(\mathbf{x}) = u(\mathbf{x}) \text{ for } \mathbf{x} \in Q \cap U.$$

where C depends only on the dimension and $\text{Lip}(g)$.

Proof: This follows from Theorem 38.0.16 and Lemma 38.2.4.

The following theorem is a general extension theorem for Sobolev spaces.

Theorem 38.2.7 *Let U be a bounded open set which has Lipschitz boundary. Then for each $p \geq 1$, there exists $E \in \mathcal{L}(W^{1,p}(U), W^{1,p}(\mathbb{R}^n))$ such that $Eu(\mathbf{x}) = u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.*

Proof: Let $\partial U \subseteq \cup_{i=1}^p Q_i$ Where the Q_i are as described in Definition 38.2.5. Also let R_i be the orthogonal transformation and g_i the Lipschitz functions associated with Q_i as in this definition. Now let $Q_0 \subseteq \overline{Q_0} \subseteq U$ be such that $\overline{U} \subseteq \cup_{i=0}^p Q_i$, and let $\psi_i \in C_c^\infty(Q_i)$ with $\psi_i(\mathbf{x}) \in [0, 1]$ and $\sum_{i=0}^p \psi_i(\mathbf{x}) = 1$ on $\overline{U}$. For $u \in C^\infty(\overline{U})$, let $E^0(\psi_0 u) \equiv \psi_0 u$ on Q_0 and 0 off Q_0 . Thus

$$\|E^0(\psi_0 u)\|_{1,p,\mathbb{R}^n} = \|\psi_0 u\|_{1,p,U}.$$

For $i \geq 1$, let

$$E^i(\psi_i u) \equiv R_i^* E_0 (R^T)^* (\psi_i u).$$

Thus, by Lemma 38.2.6

$$\|E^1(\psi_i u)\|_{1,p,\mathbb{R}^n} \leq C \|\psi_i u\|_{1,p,Q_i \cap U}$$

where the constant depends on $\text{Lip}(g_i)$ but is independent of $u \in C^\infty(\overline{U})$. Now define E as follows.

$$Eu \equiv \sum_{i=0}^p E^i(\psi_i u).$$

Thus for $u \in C^\infty(\overline{U})$, it follows $Eu(\mathbf{x}) = u(\mathbf{x})$ for all $\mathbf{x} \in U$. Also,

$$\begin{aligned} \|Eu\|_{1,p,\mathbb{R}^n} &\leq \sum_{i=0}^p \|E^i(\psi_i u)\| \leq \sum_{i=0}^p C_i \|\psi_i u\|_{1,p,Q_i \cap U} \\ &= \sum_{i=0}^p C_i \|\psi_i u\|_{1,p,U} \leq \sum_{i=0}^p C_i \|u\|_{1,p,U} \\ &\leq (p+1) \sum_{i=0}^p C_i \|u\|_{1,p,U} \equiv C \|u\|_{1,p,U}. \end{aligned} \quad (38.2.33)$$

where C depends on the ψ_i and the g_i but is independent of $u \in C^\infty(\overline{U})$. Therefore, by density of $C^\infty(\overline{U})$ in $W^{1,p}(U)$, E has a unique continuous extension to $W^{1,p}(U)$ still denoted by E satisfying the inequality determined by the ends of 38.2.33. It remains to verify that $Eu(\mathbf{x}) = u(\mathbf{x})$ a.e. for $\mathbf{x} \in U$.

Let $u_k \rightarrow u$ in $W^{1,p}(U)$ where $u_k \in C^\infty(\overline{U})$. Therefore, by 38.2.33, $Eu_k \rightarrow Eu$ in $W^{1,p}(\mathbb{R}^n)$. Since $Eu_k(\mathbf{x}) = u_k(\mathbf{x})$ for each k ,

$$\begin{aligned} \|u - Eu\|_{L^p(U)} &= \lim_{k \rightarrow \infty} \|u_k - Eu_k\|_{L^p(U)} \\ &= \lim_{k \rightarrow \infty} \|Eu_k - Eu_k\|_{L^p(U)} = 0 \end{aligned}$$

which shows $u(\mathbf{x}) = Eu(\mathbf{x})$ for a.e. $\mathbf{x} \in U$ as claimed. This proves the theorem.

Definition 38.2.8 Let U be an open set. Then $W_0^{m,p}(U)$ is the closure of the set, $C_c^\infty(U)$ in $W^{m,p}(U)$.

Corollary 38.2.9 Let U be a bounded open set which has Lipschitz boundary and let W be an open set containing $\overline{U}$. Then for each $p \geq 1$, there exists $E_W \in \mathcal{L}(W^{1,p}(U), W_0^{1,p}(W))$ such that $E_W u(\mathbf{x}) = u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.

Proof: Let $\psi \in C_c^\infty(W)$ and $\psi = 1$ on U . Then let $E_W u \equiv \psi Eu$ where E is the extension operator of Theorem 38.2.7.

Extension operators of the above sort exist for many open sets, U , not just for bounded ones. In particular, the above discussion would apply to an open set, U , not necessarily bounded, if you relax the condition that the Q_i must be bounded but require the existence of a finite partition of unity $\{\psi_i\}_{i=1}^p$ having the property that ψ_i and $\psi_{i,j}$ are uniformly bounded for all i, j . The proof would be identical to the above. My main interest is in bounded open sets so the above theorem will suffice. Such an extension operator will be referred to as a $(1, p)$ extension operator.

38.3 General Embedding Theorems

With the extension theorem it is possible to give a useful theory of embeddings.

Theorem 38.3.1 Let $1 \leq p < n$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ and let U be any open set for which there exists a $(1, p)$ extension operator. Then if $u \in W^{1,p}(U)$, there exists a constant independent of u such that

$$\|u\|_{L^q(U)} \leq C \|u\|_{1,p,U}.$$

If U is bounded and $r < q$, then $\text{id} : W^{1,p}(U) \rightarrow L^r(U)$ is also compact.

Proof: Let E be the $(1, p)$ extension operator. Then by Theorem 38.1.10 on Page 1112

$$\begin{aligned} \|u\|_{L^q(U)} &\leq \|Eu\|_{L^q(\mathbb{R}^n)} \leq \frac{1}{\sqrt[n]{n}} \frac{(n-1)p}{(n-p)} \|Eu\|_{1,p,\mathbb{R}^n} \\ &\leq C \|u\|_{1,p,U}. \end{aligned}$$

It remains to prove the assertion about compactness. If $S \subseteq W^{1,p}(U)$ is bounded then

$$\sup_{u \in S} \left\{ \|u\|_{1,1,U} + \|u\|_{L^q(U)} \right\} < \infty$$

and so by Theorem 38.1.17 on Page 1119, it follows S is precompact in $L^r(U)$. This proves the theorem.

Corollary 38.3.2 *Suppose $mp < n$ and U is an open set satisfying the segment condition which has a $(1, p)$ extension operator for all p . Then $\text{id} \in \mathcal{L}(W^{m,p}(U), L^q(U))$ where $q = \frac{np}{n-mp}$.*

Proof: This is true if $m = 1$ according to Theorem 38.3.1. Suppose it is true for $m - 1$ where $m > 1$. If $u \in W^{m,p}(U)$ and $|\alpha| \leq 1$, then $D^\alpha u \in W^{m-1,p}(U)$ so by induction, for all such α ,

$$D^\alpha u \in L^{\frac{np}{n-(m-1)p}}(U).$$

Thus, since U has the segment condition, $u \in W^{1,q_1}(U)$ where

$$q_1 = \frac{np}{n-(m-1)p}$$

By Theorem 38.3.1, it follows $u \in L^q(\mathbb{R}^n)$ where

$$\frac{1}{q} = \frac{n-(m-1)p}{np} - \frac{1}{n} = \frac{n-mp}{np}.$$

This proves the corollary.

There is another similar corollary of the same sort which is interesting and useful.

Corollary 38.3.3 *Suppose $m \geq 1$ and j is a nonnegative integer satisfying $jp < n$. Also suppose U has a $(1, p)$ extension operator for all $p \geq 1$ and satisfies the segment condition. Then*

$$\text{id} \in \mathcal{L}(W^{m+j,p}(U), W^{m,q}(U))$$

where

$$q \equiv \frac{np}{n-jp}. \quad (38.3.34)$$

If, in addition to the above, U is bounded and $1 \leq r < q$, then

$$\text{id} \in \mathcal{L}(W^{m+j,p}(U), W^{m,r}(U))$$

and is compact.

Proof: If $|\alpha| \leq m$, then $D^\alpha u \in W^{j,p}(U)$ and so by Corollary 38.3.2, $D^\alpha u \in L^q(U)$ where q is given above. Since U has the segment property, this means $u \in W^{m,q}(U)$. It remains to verify the assertion about compactness of id .

Let S be bounded in $W^{m+j,p}(U)$. Then S is bounded in $W^{m,q}(U)$ by the first part. Now let $\{u_k\}_{k=1}^\infty$ be any sequence in S . The corollary will be proved

if it is shown that any such sequence has a convergent subsequence in $W^{m,r}(U)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_h\}$ denote the indices satisfying $|\alpha| \leq m$. Then for each of these indices, α ,

$$\sup_{u \in S} \left\{ \|D^\alpha u\|_{1,1,U} + \|D^\alpha u\|_{L^q(U)} \right\} < \infty$$

and so for each such α , satisfying $|\alpha| \leq m$, it follows from Lemma 38.1.16 on Page 1117 that $\{D^\alpha u : u \in S\}$ is precompact in $L^r(U)$. Therefore, there exists a subsequence, still denoted by u_k such that $D^{\alpha_1} u_k$ converges in $L^r(U)$. Applying the same lemma, there exists a subsequence of this subsequence such that both $D^{\alpha_1} u_k$ and $D^{\alpha_2} u_k$ converge in $L^r(U)$. Continue taking subsequences until you obtain a subsequence, $\{u_k\}_{k=1}^\infty$ for which $\{D^\alpha u_k\}_{k=1}^\infty$ converges in $L^r(U)$ for all $|\alpha| \leq m$. But this must be a convergent subsequence in $W^{m,r}(U)$ and this proves the corollary.

Theorem 38.3.4 *Let U be a bounded open set having a $(1,p)$ extension operator and let $p > n$. Then $\text{id} : W^{1,p}(U) \rightarrow C(\bar{U})$ is continuous and compact.*

Proof: Theorem 38.1.3 on Page 38.1.3 implies $r_U : W^{1,p}(\mathbb{R}^n) \rightarrow C(\bar{U})$ is continuous and compact. Thus

$$\|u\|_{\infty,U} = \|Eu\|_{\infty,U} \leq C \|Eu\|_{1,p,\mathbb{R}^n} \leq C \|u\|_{1,p,U}.$$

This proves continuity. If S is a bounded set in $W^{1,p}(U)$, then define $S_1 \equiv \{Eu : u \in S\}$. Then S_1 is a bounded set in $W^{1,p}(\mathbb{R}^n)$ and so by Theorem 38.1.3 the set of restrictions to U , is precompact. However, the restrictions to U are just the functions of S . Therefore, id is compact as well as continuous.

Corollary 38.3.5 *Let $p > n$, let U be a bounded open set having a $(1,p)$ extension operator which also satisfies the segment condition, and let m be a nonnegative integer. Then $\text{id} : W^{m+1,p}(U) \rightarrow C^{m,\lambda}(\bar{U})$ is continuous for all $\lambda \in [0, 1 - \frac{n}{p}]$ and id is compact if $\lambda < 1 - \frac{n}{p}$.*

Proof: Let $u_k \rightarrow 0$ in $W^{m+1,p}(U)$. Then it follows that for each $|\alpha| \leq m$, $D^\alpha u_k \rightarrow 0$ in $W^{1,p}(U)$. Therefore,

$$E(D^\alpha u_k) \rightarrow 0 \text{ in } W^{1,p}(\mathbb{R}^n).$$

Then from Morrey's inequality, 38.1.13 on Page 1106, if $\lambda \leq 1 - \frac{n}{p}$ and $|\alpha| = m$

$$\rho_\lambda(E(D^\alpha u_k)) \leq C \|E(D^\alpha u_k)\|_{1,p,\mathbb{R}^n} \text{diam}(U)^{1-\frac{n}{p}-\lambda}.$$

Therefore, $\rho_\lambda(E(D^\alpha u_k)) = \rho_\lambda(D^\alpha u_k) \rightarrow 0$. From Theorem 38.3.4 it follows that for $|\alpha| \leq m$, $\|D^\alpha u_k\|_\infty \rightarrow 0$ and so $\|u_k\|_{m,\lambda} \rightarrow 0$. This proves the claim about continuity. The claim about compactness for $\lambda < 1 - \frac{n}{p}$ follows from Lemma 38.1.6 on Page 1108 and this. (Bounded in $W^{m,p}(U) \xrightarrow{\text{id}}$ Bounded in $C^{m,1-\frac{n}{p}}(\bar{U}) \xrightarrow{\text{id}}$ Compact in $C^{m,\lambda}(\bar{U})$.)

Theorem 38.3.6 Suppose $jp < n < (j+1)p$ and let m be a positive integer. Let U be any bounded open set in $\mathbb{R}^n$ which has a $(1, p)$ extension operator for each $p \geq 1$ and the segment property. Then $\text{id} \in \mathcal{L}(W^{m+j,p}(U), C^{m-1,\lambda}(\overline{U}))$ for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{p}$ and if $\lambda < (j+1) - \frac{n}{p}$, id is compact.

Proof: From Corollary 38.3.3 $W^{m+j,p}(U) \subseteq W^{m,q}(U)$ where q is given by 38.3.34. Therefore,

$$\frac{np}{n-jp} > n$$

and so by Corollary 38.3.5, $W^{m,q}(U) \subseteq C^{m-1,\lambda}(\overline{U})$ for all λ satisfying

$$0 < \lambda < 1 - \frac{(n-jp)n}{np} = \frac{p(j+1)-n}{p} = (j+1) - \frac{n}{p}.$$

The assertion about compactness follows from the compactness of the embedding of $C^{m-1,\lambda_0}(\overline{U})$ into $C^{m-1,\lambda}(\overline{U})$ for $\lambda < \lambda_0$, Lemma 38.1.6 on Page 1108.

38.4 More Extension Theorems

The theorem about the existence of a $(1, p)$ extension is all that is needed to obtain general embedding theorems for Sobolev spaces. However, a more general theory is needed in order to tie the theory of Sobolev spaces presented thus far to a very appealing description using Fourier transforms. First the problem of extending $W^{k,p}(H)$ to $W^{k,p}(\mathbb{R}^n)$ is considered for H^- a half space

$$H^- \equiv \{\mathbf{y} \in \mathbb{R}^n : y_n < 0\}. \quad (38.4.35)$$

I am following Adams [1].

Lemma 38.4.1 Let H^- be a half space as in 38.4.35. Let H^+ be the half space in which $y_n < 0$ is replaced with $y_n > 0$. Also let $(\mathbf{y}', y_n) = \mathbf{y}$

$$u(\mathbf{y}', y_n) \equiv \begin{cases} u^+(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^+ \\ u^-(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^- \end{cases},$$

suppose $u^+ \in C^\infty(\overline{H^+})$ and $u^- \in C^\infty(\overline{H^-})$, and that for $l \leq k-1$,

$$D^{le_n} \mathbf{u}^+(\mathbf{y}', 0) = D^{le_n} \mathbf{u}^-(\mathbf{y}', 0).$$

Then $u \in W^{k,p}(\mathbb{R}^n)$. Furthermore,

$$D^\alpha u(\mathbf{y}', y_n) \equiv \begin{cases} D^\alpha u^+(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^+ \\ D^\alpha u^-(\mathbf{y}', y_n) & \text{if } \mathbf{y} \in H^- \end{cases}$$

Proof: Consider the following for $\phi \in C_c^\infty(\mathbb{R}^n)$ and $|\alpha| \leq k$.

$$(-1)^{|\alpha|} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty u^+ D^\alpha \phi dy_n dy' + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 u^- D^\alpha \phi dy_n dy' \right).$$

Integrating by parts, this yields

$$\begin{aligned} & (-1)^{|\alpha|} (-1)^{|\beta|} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty D^\beta u^+ D^{\alpha_n \mathbf{e}_n} \phi dy_n dy' \right. \\ & \left. + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 D^\beta u^- D^{\alpha_n \mathbf{e}_n} \phi dy_n dy' \right) \end{aligned}$$

where $\beta \equiv (\alpha_1, \alpha_2, \dots, \alpha_{n-1}, 0)$. Do integration by parts on the inside integral and by assumption, the boundary terms will cancel and the whole thing reduces to

$$\begin{aligned} & (-1)^{|\alpha|} (-1)^{|\beta|} (-1)^{\alpha_n} \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty D^\alpha u^+ \phi dy_n dy' \right. \\ & \left. + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 D^\alpha u^- \phi dy_n dy' \right) \\ & = \left(\int_{\mathbb{R}^{n-1}} \int_0^\infty D^\alpha u^+ \phi dy_n dy' + \int_{\mathbb{R}^{n-1}} \int_{-\infty}^0 D^\alpha u^- \phi dy_n dy' \right) \end{aligned}$$

which proves the lemma.

Lemma 38.4.2 *Let H^- be the half space in 38.4.35 and let $u \in C^\infty(\overline{H^-})$. Then there exists a mapping,*

$$E : C^\infty(\overline{H^-}) \rightarrow W^{k,p}(\mathbb{R}^n)$$

and a constant, C which is independent of $u \in C^\infty(\overline{H^-})$ such that E is linear and for all $l \leq k$,

$$\|Eu\|_{l,p,\mathbb{R}^n} \leq C \|u\|_{l,p,H^-}. \quad (38.4.36)$$

Proof: Define

$$Eu(\mathbf{x}', x_n) \equiv \begin{cases} u(\mathbf{x}', x_n) & \text{if } x_n < 0 \\ \sum_{j=1}^k \lambda_j u(\mathbf{x}', -jx_n) & \text{if } x_n \geq 0 \end{cases}$$

where the λ_j are chosen in such a way that for $l \leq k-1$,

$$D^{l\mathbf{e}_n} u(\mathbf{x}', 0) - D^{l\mathbf{e}_n} \left(\sum_{j=1}^k \lambda_j u \right) (\mathbf{x}', 0) = 0$$

so that Lemma 38.4.1 may be applied. Do there exist such λ_j ? It is necessary to have the following hold for each $r = 0, 1, \dots, k-1$.

$$\sum_{j=1}^k (-j)^r \lambda_j D^{r\mathbf{e}_n} u(\mathbf{x}', 0) = D^{r\mathbf{e}_n} u(\mathbf{x}', 0).$$

This is satisfied if

$$\sum_{j=1}^k (-j)^r \lambda_j = 1$$

for $r = 0, 1, \dots, k-1$. This is a system of k equations for the k variables, the λ_j . The matrix of coefficients is of the form

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & -2 & -3 & \cdots & -k \\ 1 & 4 & 9 & \cdots & k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ (-1)^k & (-2)^k & (-3)^k & \cdots & (-k)^k \end{pmatrix}$$

This matrix has an inverse because its determinant is nonzero.

Now from Lemma 38.4.1, it follows from the above description of E that for $|\alpha| \leq k$,

$$D^\alpha (Eu)(\mathbf{x}', x_n) \equiv \begin{cases} D^\alpha u(\mathbf{x}', x_n) & \text{if } x_n < 0 \\ \sum_{j=1}^k \lambda_j (-j)^{\alpha_n} (D^\alpha u)(\mathbf{x}', -jx_n) & \text{if } x_n \geq 0 \end{cases}$$

It follows that E is linear and there exists a constant, C independent of u such that 38.4.36 holds. This proves the lemma.

Corollary 38.4.3 *Let H^- be the half space of 38.4.35. There exists E with the property that $E : W^{l,p}(H^-) \rightarrow W^{l,p}(\mathbb{R}^n)$ and is linear and continuous for each $l \leq k$.*

Proof: This immediate from the density of $C_c^\infty(\overline{H^-})$ in $W^{k,p}(\overline{H^-})$ and Lemma 38.4.2.

There is nothing sacred about a half space or this particular half space. It is clear that everything works as well for a half space of the form

$$H_k^- \equiv \{\mathbf{x} : x_k < 0\}.$$

Thus the half space featured in the above discussion is H_n^- .

Corollary 38.4.4 *Let $\{k_1, \dots, k_r\} \subseteq \{1, \dots, n\}$ where the k_i are distinct and let*

$$H_{k_1 \dots k_r}^- \equiv H_{k_1}^- \cap H_{k_2}^- \cap \cdots \cap H_{k_r}^-. \quad (38.4.37)$$

Then there exists $E : W^{k,p}(H_{k_1 \dots k_r}^-) \rightarrow W^{k,p}(\mathbb{R}^n)$ such that E is linear and continuous.

Proof: Follow the above argument with minor modifications to first extend from $H_{k_1 \dots k_r}^-$ to $H_{k_1 \dots k_{r-1}}^-$ and then from $H_{k_1 \dots k_{r-1}}^-$ to $H_{k_1 \dots k_{r-2}}^-$ etc.

This easily implies the ability to extend off bounded open sets which near their boundaries look locally like an intersection of half spaces.

Theorem 38.4.5 *Let U be a bounded open set and suppose $U_0, U_1, \dots, U_m$ are open sets with the property that $\bar{U} \subseteq \cup_{k=0}^m U_k$, $\bar{U}_0 \subseteq U$, and $\partial U \subseteq \cup_{k=1}^m U_k$. Suppose also there exist one to one and onto functions, $\mathbf{h}_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mathbf{h}_k(U_k \cap U) = W_k$ where W_k equals the intersection of a bounded open set with a finite intersection of half spaces, $H_{k_1 \dots k_r}^-$, as in 38.4.37 such that $\mathbf{h}_k(\partial U \cap U_k) \subseteq \partial H_{k_1 \dots k_r}^-$. Suppose also that for all $|\alpha| \leq k-1$,*

$$D^\alpha \mathbf{h}_k \text{ and } D^\alpha \mathbf{h}_k^{-1}$$

exist and are Lipschitz continuous. Then there letting W be an open set which contains $\bar{U}$, there exists $E : W^{k,p}(U) \rightarrow W^{k,p}(W)$ such that E is a linear continuous map from $W^{l,p}(U)$ to $W^{l,p}(W)$ for each $l \leq k$.

Proof: Let $\psi_j \in C_c^\infty(U_j)$, $\psi_j(\mathbf{x}) \in [0, 1]$ for all $\mathbf{x} \in \mathbb{R}^n$, and $\sum_{j=0}^m \psi_j(\mathbf{x}) = 1$ on $\bar{U}$. This is a C^∞ partition of unity on $\bar{U}$. By Theorem 38.0.16 $(\mathbf{h}_j^{-1})^* u \psi_j \in W^{k,p}(W_j)$. By the assumption that $\mathbf{h}_j(\partial U \cap U_j) \subseteq \partial H_{k_1 \dots k_r}^-$, the zero extension of $(\mathbf{h}_j^{-1})^* u \psi_j$ to the rest of $H_{k_1 \dots k_r}^-$ results in an element of $W^{k,p}(H_{k_1 \dots k_r}^-)$. Apply Corollary 38.4.4 to conclude there exists $E_j : W^{k,p}(H_{k_1 \dots k_r}^-) \rightarrow W^{k,p}(\mathbb{R}^n)$ which is continuous and linear. Abusing notation slightly, by using $(\mathbf{h}_j^{-1})^* u \psi_j$ as the above zero extension, it follows $E_j \left((\mathbf{h}_j^{-1})^* u \psi_j \right) \in W^{k,p}(\mathbb{R}^n)$. Now let η be a function in $C_c^\infty(\mathbf{h}(W))$ such that $\eta(\mathbf{y}) = 1$ on $\mathbf{h}(\bar{U})$. Then Define

$$Eu \equiv \sum_{j=0}^m \mathbf{h}_j^* \eta E_j \left((\mathbf{h}_j^{-1})^* (u \psi_j) \right).$$

Clearly $Eu(\mathbf{x}) = u(\mathbf{x})$ if $\mathbf{x} \in U$. It is also clear that E is linear. It only remains to verify E is continuous. In what follows, C_j will denote a constant which is

independent of u which may change from line to line. By Theorem 38.0.16,

$$\begin{aligned}
 \|Eu\|_{k,p,W} &\leq \sum_{j=0}^m \left\| \mathbf{h}_j^* \eta E_j \left((\mathbf{h}_j^{-1})^* (u\psi_j) \right) \right\|_{k,p,W} \\
 &\leq \sum_{j=0}^m C_j \left\| \eta E_j \left((\mathbf{h}_j^{-1})^* (u\psi_j) \right) \right\|_{k,p,\mathbf{h}(W)} \\
 &= \sum_{j=0}^m C_j \left\| \eta E_j \left((\mathbf{h}_j^{-1})^* (u\psi_j) \right) \right\|_{k,p,\mathbb{R}^n} \\
 &\leq \sum_{j=0}^m C_j \left\| E_j \left((\mathbf{h}_j^{-1})^* (u\psi_j) \right) \right\|_{k,p,\mathbb{R}^n} \\
 &\leq \sum_{j=0}^m C_j \left\| (\mathbf{h}_j^{-1})^* (u\psi_j) \right\|_{k,p,\mathbf{h}_j(U \cap U_j)} \\
 &\leq \sum_{j=0}^m C_j \|u\psi_j\|_{k,p,U \cap U_k} \\
 &\leq \sum_{j=0}^m C_j \|u\|_{k,p,U \cap U_k} \leq \left(\sum_{j=0}^m C_j \right) \|u\|_{k,p,U}.
 \end{aligned}$$

Similarly $E : W^{l,p}(U) \rightarrow W^{l,p}(U)$ for $l \leq k$. This proves the theorem.

Definition 38.4.6 When E is a linear continuous map from $W^{l,p}(U)$ to $W^{l,p}(\mathbb{R}^n)$ for each $l \leq k$. it is called a strong (k, p) extension map.

There is also a very easy sort of extension theorem for the space, $W_0^{m,p}(U)$ which does not require any assumptions on the boundary of U other than $m_n(\partial U) = 0$. First here is the definition of $W_0^{m,p}(U)$.

Definition 38.4.7 Denote by $W_0^{m,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{m,p}(U)$.

Theorem 38.4.8 For $u \in W_0^{m,p}(U)$, define

$$Eu(\mathbf{x}) \equiv \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in U \\ 0 & \text{if } \mathbf{x} \notin U \end{cases}$$

Then E is a strong (k, p) extension map.

Proof: Letting $l \leq m$, it is clear that for $|\alpha| \leq l$,

$$D^\alpha Eu = \begin{cases} D^\alpha u & \text{for } \mathbf{x} \in U \\ 0 & \text{for } \mathbf{x} \notin U \end{cases}.$$

This follows because, since $m_n(\partial U) = 0$ it suffices to consider $\phi \in C_c^\infty(U)$ and $\phi \in C_c^\infty(\overline{U}^C)$. Therefore, $\|Eu\|_{l,p,\mathbb{R}^n} = \|u\|_{l,p,U}$.

There are many other extension theorems and if you are interested in pursuing this further, consult Adams [1]. One of the most famous which is discussed in this reference is due to Calderon and depends on the theory of singular integrals.

Sobolev Spaces Based On L^2

39.1 Fourier Transform Techniques

Much insight can be obtained easily through the use of Fourier transform methods. This technique will be developed in this chapter. When this is done, it is necessary to use Sobolev spaces of the form $W^{k,2}(U)$, those Sobolev spaces which are based on $L^2(U)$. It is true there are generalizations which use Fourier transform methods in the context of L^p but the spaces so considered are called Bessel potential spaces. They are not really Sobolev spaces. Furthermore, it is Mihlin's theorem rather than the Plancherel theorem which is the main tool of the analysis. This is a hard theorem.

It is convenient to consider the Schwartz class of functions, $\mathfrak{S}$. These are functions which have infinitely many derivatives and vanish quickly together with their derivatives as $|\mathbf{x}| \rightarrow \infty$. In particular, $C_c^\infty(\mathbb{R}^n)$ is contained in $\mathfrak{S}$ which is not true of the functions, $\mathcal{G}$ used earlier in defining the Fourier transforms which are a subspace of $\mathfrak{S}$. Recall the following definition.

Definition 39.1.1 $f \in \mathfrak{S}$, the Schwartz class, if $f \in C^\infty(\mathbb{R}^n)$ and for all positive integers N ,

$$\rho_N(f) < \infty$$

where

$$\rho_N(f) = \sup\{(1 + |\mathbf{x}|^2)^N |D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n, |\alpha| \leq N\}.$$

Thus $f \in \mathfrak{S}$ if and only if $f \in C^\infty(\mathbb{R}^n)$ and

$$\sup\{|\mathbf{x}^\beta D^\alpha f(\mathbf{x})| : \mathbf{x} \in \mathbb{R}^n\} < \infty \quad (39.1.1)$$

for all multi indices α and β .

Thus all partial derivatives of a function in $\mathfrak{S}$ are in $L^p(\mathbb{R}^n)$ for all $p \geq 1$. Therefore, for $f \in \mathfrak{S}$, the Fourier and inverse Fourier transforms are given in the usual way,

$$Ff(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{t} \cdot \mathbf{x}} d\mathbf{x}, \quad F^{-1}f(\mathbf{t}) = \left(\frac{1}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{i\mathbf{t} \cdot \mathbf{x}} d\mathbf{x}.$$

Also recall that the Fourier transform and its inverse are one to one and onto maps from $\mathfrak{S}$ to $\mathfrak{S}$.

To tie the Fourier transform technique in with what has been done so far, it is necessary to make the following assumption on the set, U . This assumption is made so that it is possible to consider elements of $W^{k,2}(U)$ as restrictions of elements of $W^{k,2}(\mathbb{R}^n)$.

Assumption 39.1.2 *Assume U satisfies the segment condition and that for any m of interest, there exists $E \in \mathcal{L}(W^{m,p}(U), W^{m,p}(\mathbb{R}^n))$ such that for each $k \leq m$, $E \in \mathcal{L}(W^{k,p}(U), W^{k,p}(\mathbb{R}^n))$. That is, there exists a strong (m,p) extension operator.*

Lemma 39.1.3 *The Schwartz class, $\mathfrak{S}$, is dense in $W^{m,p}(\mathbb{R}^n)$.*

Proof: The set, $\mathbb{R}^n$ satisfies the segment condition and so $C_c^\infty(\mathbb{R}^n)$ is dense in $W^{m,p}(\mathbb{R}^n)$. However, $C_c^\infty(\mathbb{R}^n) \subseteq \mathfrak{S}$. This proves the lemma.

Recall now Plancherel's theorem which states that $\|f\|_{0,2,\mathbb{R}^n} = \|Ff\|_{0,2,\mathbb{R}^n}$ whenever $f \in L^2(\mathbb{R}^n)$. Also it is routine to verify from the definition of the Fourier transform that for $u \in \mathfrak{S}$,

$$F\partial_k u = ix_k F u.$$

From this it follows that

$$\|D^\alpha u\|_{0,2,\mathbb{R}^n} = \|\mathbf{x}^\alpha F u\|_{0,2,\mathbb{R}^n}.$$

Here $\mathbf{x}^\alpha$ denotes the function $\mathbf{x} \rightarrow \mathbf{x}^\alpha$. Therefore,

$$\|u\|_{m,2,\mathbb{R}^n} = \left(\int_{\mathbb{R}^n} \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} |F u(\mathbf{x})|^2 dx \right)^{1/2}.$$

Also, it is not hard to verify that

$$\sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} \leq \left(1 + \sum_{j=1}^n x_j^2 \right)^m \leq C(n, m) \sum_{|\alpha| \leq m} x_1^{2\alpha_1} \cdots x_n^{2\alpha_n}$$

where $C(n, m)$ is the largest of the multinomial coefficients obtained in the expansion,

$$\left(1 + \sum_{j=1}^n x_j^2 \right)^m.$$

Therefore, for all $u \in \mathfrak{S}$,

$$\|u\|_{m,2,\mathbb{R}^n} \leq \left(\int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2 \right)^m |F u(\mathbf{x})|^2 dx \right)^{1/2} \leq C(n, m) \|u\|_{m,2,\mathbb{R}^n}. \quad (39.1.2)$$

This motivates the following definition.

Definition 39.1.4 Let $H^m(\mathbb{R}^n) \equiv$

$$\left\{ u \in L^2(\mathbb{R}^n) : \|u\|_{H^m(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} (1 + |\mathbf{x}|^2)^m |Fu(\mathbf{x})|^2 dx \right)^{1/2} < \infty \right\}. \quad (39.1.3)$$

Lemma 39.1.5 $\mathfrak{S}$ is dense in $H^m(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n) = W^{2,m}(\mathbb{R}^n)$. Furthermore, the norms are equivalent.

Proof: First it is shown that $\mathfrak{S}$ is dense in $H^m(\mathbb{R}^n)$. Let $u \in H^m(\mathbb{R}^n)$. Let $\mu(E) \equiv \int_E (1 + |\mathbf{x}|^2)^m dx$. Thus μ is a regular measure and $u \in H^m(\mathbb{R}^n)$ just means that $Fu \in L^2(\mu)$, the space of functions which are in $L^2(\mathbb{R}^n)$ with respect to this measure, μ . Therefore, from the regularity of the measure, μ , there exists $u_k \in C_c(\mathbb{R}^n)$ such that

$$\|u_k - Fu\|_{L^2(\mu)} \rightarrow 0.$$

Now let ψ_ε be a mollifier and pick ε_k small enough that

$$\|u_k * \psi_{\varepsilon_k} - u_k\|_{L^2(\mu)} < \frac{1}{2^k}.$$

Then $u_k * \psi_{\varepsilon_k} \in C_c^\infty(\mathbb{R}^n) \subseteq \mathfrak{S}$. Therefore, there exists $w_k \in \mathcal{G}$ such that $Fw_k = u_k * \psi_{\varepsilon_k}$. It follows

$$\|Fw_k - Fu\|_{L^2(\mu)} \leq \|Fw_k - u_k\|_{L^2(\mu)} + \|u_k - Fu\|_{L^2(\mu)}$$

and these last two terms converge to 0 as $k \rightarrow \infty$. Therefore, $w_k \rightarrow u$ in $H^m(\mathbb{R}^n)$ and this proves the first part of this lemma.

Now let $u \in H^m(\mathbb{R}^n)$. By what was just shown, there exists a sequence, $u_k \rightarrow u$ in $H^m(\mathbb{R}^n)$ where $u_k \in \mathfrak{S}$. It follows from 39.1.2 that

$$\|u_k - u_l\|_{H^m} \geq \|u_k - u_l\|_{m,2,\mathbb{R}^n}$$

and so $\{u_k\}$ is a Cauchy sequence in $W^{m,2}(\mathbb{R}^n)$. Therefore, there exists $w \in W^{m,2}(\mathbb{R}^n)$ such that

$$\|u_k - w\|_{m,2,\mathbb{R}^n} \rightarrow 0.$$

But this implies

$$0 = \lim_{k \rightarrow \infty} \|u_k - w\|_{0,2,\mathbb{R}^n} = \lim_{k \rightarrow \infty} \|u_k - u\|_{0,2,\mathbb{R}^n}$$

showing $u = w$ which verifies $H^m(\mathbb{R}^n) \subseteq W^{2,m}(\mathbb{R}^n)$. The opposite inclusion is proved the same way, using density of $\mathfrak{S}$ and the fact that the norms in both spaces are larger than the norms in $L^2(\mathbb{R}^n)$. The equivalence of the norms follows from the density of $\mathfrak{S}$ and the equivalence of the norms on $\mathfrak{S}$. This proves the lemma.

The conclusion of this lemma with the density of $\mathfrak{S}$ and 39.1.2 implies you can use either norm, $\|u\|_{H^m(\mathbb{R}^n)}$ or $\|u\|_{m,2,\mathbb{R}^n}$ when working with these Sobolev spaces.

What of open sets satisfying Assumption 39.1.2? How does $W^{m,2}(U)$ relate to the Fourier transform?

Definition 39.1.6 Let U be an open set in $\mathbb{R}^n$. Then

$$H^m(U) \equiv \{u : u = v|_U \text{ for some } v \in H^m(\mathbb{R}^n)\} \quad (39.1.4)$$

Here the notation, $v|_U$ means v restricted to U . Define the norm in this space by

$$\|u\|_{H^m(U)} \equiv \inf \left\{ \|v\|_{H^m(\mathbb{R}^n)} : v|_U = u \right\}. \quad (39.1.5)$$

Lemma 39.1.7 $H^m(U)$ is a Banach space.

Proof: First it is necessary to verify that the given norm really is a norm. Suppose then that $u = 0$. Is $\|u\|_{H^m(U)} = 0$? Of course it is. Just take $v \equiv 0$. Then $v|_U = u$ and $\|v\|_{H^m} = 0$. Next suppose $\|u\|_{H^m(U)} = 0$. Does it follow that $u = 0$? Letting $\varepsilon > 0$ be given, there exists $v \in H^m(\mathbb{R}^n)$ such that $v|_U = u$ and $\|v\|_{H^m(\mathbb{R}^n)} < \varepsilon$. Therefore,

$$\|u\|_{0,U} \leq \|v\|_{0,\mathbb{R}^n} \leq \|v\|_{H^m(U)} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows $u = 0$ a.e. Next suppose $u_i \in H^m(U)$ for $i = 1, 2$. There exists $v_i \in H^m(\mathbb{R}^n)$ such that

$$\|v_i\|_{H^m(\mathbb{R}^n)} < \|u_i\|_{H^m(U)} + \varepsilon.$$

Therefore,

$$\begin{aligned} \|u_1 + u_2\|_{H^m(U)} &\leq \|v_1 + v_2\|_{H^m(\mathbb{R}^n)} \leq \|v_1\|_{H^m(\mathbb{R}^n)} + \|v_2\|_{H^m(\mathbb{R}^n)} \\ &\leq \|u_1\|_{H^m(U)} + \|u_2\|_{H^m(U)} + 2\varepsilon \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, this shows the triangle inequality.

The interesting question is the one about completeness. Suppose then $\{u_k\}$ is a Cauchy sequence in $H^m(U)$. There exists N_k such that if $k, l \geq N_k$, it follows $\|u_k - u_l\|_{H^m(U)} < \frac{1}{2^k}$ and the numbers, N_k can be taken to be strictly increasing in k . Thus for $l \geq N_k$, $\|u_l - u_{N_k}\|_{H^m(U)} < 1/2^l$. Therefore, there exists $w_l \in H^m(\mathbb{R}^n)$ such that

$$w_l|_U = u_l - u_{N_k}, \quad \|w_l\|_{H^m(\mathbb{R}^n)} < \frac{1}{2^l}.$$

Also let $v_{N_k}|_U = u_{N_k}$ with $v_{N_k} \in H^m(\mathbb{R}^n)$ and

$$\|v_{N_k}\|_{H^m(\mathbb{R}^n)} < \|u_{N_k}\|_{H^m(U)} + \frac{1}{2^k}.$$

Now for $l > N_k$, define v_l by $v_l - v_{N_k} = w_{N_k}$ so that $\|v_l - v_{N_k}\|_{H^m(\mathbb{R}^n)} < 1/2^k$. In particular,

$$\|v_{N_{k+1}} - v_{N_k}\|_{H^m(\mathbb{R}^n)} < 1/2^k$$

which shows that $\{v_{N_k}\}_{k=1}^\infty$ is a Cauchy sequence. Consequently it must converge to $v \in H^m(\mathbb{R}^n)$. Let $u = v|_U$. Then

$$\|u - u_{N_k}\|_{H^m(U)} \leq \|v - v_{N_k}\|_{H^m(\mathbb{R}^n)}$$

which shows the subsequence, $\{u_{N_k}\}_k$ converges to u . Since $\{u_k\}$ is a Cauchy sequence, it follows it too must converge to u . This proves the lemma.

The main result is next.

Theorem 39.1.8 *Suppose U satisfies Assumption 39.1.2. Then for m a nonnegative integer, $H^m(U) = W^{m,2}(U)$ and the two norms are equivalent.*

Proof: Let $u \in H^m(U)$. Then there exists $v \in H^m(\mathbb{R}^n)$ such that $v|_U = u$. Hence $v \in W^{k,2}(\mathbb{R}^n)$ and so all its weak derivatives up to order m are in $L^2(\mathbb{R}^n)$. Therefore, the restrictions of these weak derivatives are in $L^2(U)$. Since U satisfies the segment condition, it follows $u \in W^{m,2}(U)$ which shows $H^m(U) \subseteq W^{m,2}(U)$.

Next take $u \in W^{m,2}(U)$. Then $Eu \in W^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$ and this shows $u \in H^m(U)$. This has shown the two spaces are the same. It remains to verify their norms are equivalent. Let $u \in H^m(U)$ and let $v|_U = u$ where $v \in H^m(\mathbb{R}^n)$ and

$$\|u\|_{H^m(U)} + \varepsilon > \|v\|_{H^m(\mathbb{R}^n)}.$$

Then recalling that $\|\cdot\|_{H^m(\mathbb{R}^n)}$ and $\|\cdot\|_{m,2,\mathbb{R}^n}$ are equivalent norms for $H^m(\mathbb{R}^n)$, there exists a constant, C such that

$$\|u\|_{H^m(U)} + \varepsilon > \|v\|_{H^m(\mathbb{R}^n)} \geq C\|v\|_{m,2,\mathbb{R}^n} \geq C\|u\|_{m,2,U}$$

Now consider the two Banach spaces,

$$\left(H^m(U), \|\cdot\|_{H^m(U)}\right), \left(W^{m,2}(U), \|\cdot\|_{m,2,U}\right).$$

The above inequality shows since $\varepsilon > 0$ is arbitrary that $\text{id} : \left(H^m(U), \|\cdot\|_{H^m(U)}\right) \rightarrow \left(W^{m,2}(U), \|\cdot\|_{m,2,U}\right)$ is continuous. By the open mapping theorem, it follows id is continuous in the other direction. Thus there exists a constant, K such that $\|u\|_{H^m(U)} \leq K\|u\|_{m,2,U}$. Hence the two norms are equivalent as claimed.

Specializing Corollary 38.3.3 and Theorem 38.3.6 starting on Page 1128 to the case of $p = 2$ while also assuming more on U yields the following embedding theorems.

Theorem 39.1.9 *Suppose $m \geq 0$ and j is a nonnegative integer satisfying $2j < n$. Also suppose U is an open set which satisfies Assumption 39.1.2. Then $\text{id} \in \mathcal{L}(H^{m+j}(U), W^{m,q}(U))$ where*

$$q \equiv \frac{2n}{n - 2j}. \quad (39.1.6)$$

If, in addition to the above, U is bounded and $1 \leq r < q$, then

$$\text{id} \in \mathcal{L}(H^{m+j}(U), W^{m,r}(U))$$

and is compact.

Theorem 39.1.10 *Suppose for j a nonnegative integer, $2j < n < 2(j+1)$ and let m be a positive integer. Let U be any bounded open set in $\mathbb{R}^n$ which satisfies Assumption 39.1.2. Then $\text{id} \in \mathcal{L}(H^{m+j}(U), C^{m-1,\lambda}(\bar{U}))$ for every $\lambda \leq \lambda_0 \equiv (j+1) - \frac{n}{2}$ and if $\lambda < (j+1) - \frac{n}{2}$, id is compact.*

39.2 Fractional Order Spaces

What has been gained by all this? The main thing is that $H^{m+s}(U)$ makes sense for any $s \in (0, 1)$ and m an integer. You simply replace m with $m + s$ in the above for $s \in (0, 1)$. This gives what is meant by $H^{m+s}(\mathbb{R}^n)$

Definition 39.2.1 For m an integer and $s \in (0, 1)$, let $H^{m+s}(\mathbb{R}^n) \equiv$

$$\left\{ u \in L^2(\mathbb{R}^n) : \|u\|_{H^{m+s}(\mathbb{R}^n)} \equiv \left(\int_{\mathbb{R}^n} (1 + |\mathbf{x}|^2)^{m+s} |Fu(\mathbf{x})|^2 dx \right)^{1/2} < \infty \right\}. \quad (39.2.7)$$

You could also simply refer to $H^t(\mathbb{R}^n)$ where t is a real number replacing the $m + s$ in the above formula with t but I want to emphasize the notion that $t = m + s$ where m is a nonnegative integer. Therefore, I will often write $m + s$. Let U be an open set in $\mathbb{R}^n$. Then

$$H^{m+s}(U) \equiv \{u : u = v|_U \text{ for some } v \in H^{m+s}(\mathbb{R}^n)\}. \quad (39.2.8)$$

Define the norm in this space by

$$\|u\|_{H^{m+s}(U)} \equiv \inf \left\{ \|v\|_{H^{m+s}(\mathbb{R}^n)} : v|_U = u \right\}. \quad (39.2.9)$$

Lemma 39.2.2 $H^{m+s}(U)$ is a Banach space.

Proof: Just repeat the proof of Lemma 39.1.7.

The theorem about density of $\mathfrak{S}$ also remains true in $H^{m+s}(\mathbb{R}^n)$. Just repeat the proof of that part of Lemma 39.1.5 replacing the integer, m , with the symbol, $m + s$.

Lemma 39.2.3 $\mathfrak{S}$ is dense in $H^{m+s}(\mathbb{R}^n)$.

In fact, more can be said.

Corollary 39.2.4 Let U be an open set and let $\mathfrak{S}|_U$ denote the restrictions of functions of $\mathfrak{S}$ to U . Then $\mathfrak{S}|_U$ is dense in $H^t(U)$.

Proof: Let $u \in H^t(U)$ and let $v \in H^t(\mathbb{R}^n)$ such that $v|_U = u$ a.e. Then since $\mathfrak{S}$ is dense in $H^t(\mathbb{R}^n)$, there exists $w \in \mathfrak{S}$ such that

$$\|w - v\|_{H^t(\mathbb{R}^n)} < \varepsilon.$$

It follows that

$$\begin{aligned} \|u - w\|_{H^t(U)} &\leq \|u - v\|_{H^t(U)} + \|v - w\|_{H^t(U)} \\ &\leq 0 + \|v - w\|_{H^t(\mathbb{R}^n)} < \varepsilon. \end{aligned}$$

These fractional order spaces are important when trying to understand the trace on the boundary. The Fourier transform description also makes it very easy to establish interesting inequalities such as interpolation inequalities.

Lemma 39.2.5 *Let $0 \leq r < s < t$. Then if $u \in H^t(\mathbb{R}^n)$,*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq \|u\|_{H^r(\mathbb{R}^n)}^\theta \|u\|_{H^t(\mathbb{R}^n)}^{1-\theta}$$

where θ is a positive number such that $\theta r + (1 - \theta)t = s$.

Proof: This follows from Holder's inequality applied to the measure μ given by

$$\mu(E) = \int_E |Fu|^2 dx$$

Thus

$$\begin{aligned} & \int (1 + |\mathbf{x}|^2)^s |Fu|^2 dx \\ &= \int (1 + |\mathbf{x}|^2)^{r\theta} (1 + |\mathbf{x}|^2)^{(1-\theta)t} |Fu|^2 dx \\ &\leq \left(\int (1 + |\mathbf{x}|^2)^r |Fu|^2 dx \right)^\theta \left(\int (1 + |\mathbf{x}|^2)^{(1-\theta)t} |Fu|^2 dx \right)^{1-\theta} \\ &= \|u\|_{H^r(\mathbb{R}^n)}^{2\theta} \|u\|_{H^t(\mathbb{R}^n)}^{2(1-\theta)}. \end{aligned}$$

Taking square roots yields the desired inequality.

Corollary 39.2.6 *Let U be an open set satisfying Assumption 39.1.2 and let $p < q$ where p, q are two nonnegative integers. Also let $t \in (p, q)$. Then exists a constant, C independent of $u \in H^q(U)$ such that for all $u \in H^q(U)$,*

$$\|u\|_{H^t(U)} \leq C \|u\|_{H^p(U)}^\theta \|u\|_{H^q(U)}^{1-\theta}$$

where θ is such that $t = \theta p + (1 - \theta)q$.

Proof: Let $E \in \mathcal{L}(H^q(U), H^q(\mathbb{R}^n))$ such that for all positive integers, l less than or equal to q , $E \in \mathcal{L}(H^l(U), H^l(\mathbb{R}^n))$. Then $Eu|_U = u$ and $Eu \in H^t(\mathbb{R}^n)$. Therefore, by Lemma 39.2.5,

$$\begin{aligned} \|u\|_{H^t(U)} &\leq \|Eu\|_{H^t(\mathbb{R}^n)} \leq \|Eu\|_{H^p(\mathbb{R}^n)}^\theta \|Eu\|_{H^q(\mathbb{R}^n)}^{1-\theta} \\ &\leq C \|u\|_{H^p(U)}^\theta \|u\|_{H^q(U)}^{1-\theta}. \end{aligned}$$

Now recall the very important Theorem 38.0.16 on Page 1105 which is listed here for convenience.

Theorem 39.2.7 *Let $\mathbf{h} : U \rightarrow V$ be one to one and onto where U and V are two open sets. Also suppose that $D^\alpha \mathbf{h}$ and $D^\alpha (\mathbf{h}^{-1})$ exist and are Lipschitz continuous if $|\alpha| \leq m - 1$ for m a positive integer. Then*

$$\mathbf{h}^* : W^{m,p}(V) \rightarrow W^{m,p}(U)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $(\mathbf{h}^{-1})^*$.

39.3 An Intrinsic Norm

Is there something like this for the fractional order spaces? Yes there is. However, in order to prove it, it is convenient to use an equivalent norm for $H^{m+s}(\mathbb{R}^n)$ which does not depend explicitly on the Fourier transform. The following theorem is similar to one in [33]. It describes the norm in $H^{m+s}(\mathbb{R}^n)$ in terms which are free of the Fourier transform. This is also called an intrinsic norm [1].

Theorem 39.3.1 *Let $s \in (0, 1)$ and let m be a nonnegative integer. Then an equivalent norm for $H^{m+s}(\mathbb{R}^n)$ is*

$$|||u|||_{m+s}^2 \equiv ||u||_{m,2,\mathbb{R}^n}^2 + \sum_{|\alpha|=m} \int \int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy.$$

Also if $|\beta| \leq m$, there are constants, $m(s)$ and $M(s)$ such that

$$\begin{aligned} m(s) \int |Fu(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz &\leq \int \int |D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &\leq M(s) \int |Fu(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz \end{aligned} \quad (39.3.10)$$

Proof: Let $u \in \mathfrak{S}$ which is dense in $H^{m+s}(\mathbb{R}^n)$. The Fourier transform of the function, $\mathbf{y} \rightarrow D^\alpha u(\mathbf{x} + \mathbf{y}) - D^\alpha u(\mathbf{y})$ equals

$$(e^{i\mathbf{x} \cdot \mathbf{z}} - 1) F D^\alpha u(\mathbf{z}).$$

Now by Fubini's theorem and Plancherel's theorem along with the above, taking $|\alpha| = m$,

$$\begin{aligned} &\int \int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &= \int \int |D^\alpha u(\mathbf{y} + \mathbf{t}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{t}|^{-n-2s} dt dy \\ &= \int |\mathbf{t}|^{-n-2s} \int |D^\alpha u(\mathbf{y} + \mathbf{t}) - D^\alpha u(\mathbf{y})|^2 dy dt \\ &= \int |\mathbf{t}|^{-n-2s} \int |(e^{i\mathbf{t} \cdot \mathbf{z}} - 1) F D^\alpha u(\mathbf{z})|^2 dz dt \\ &= \int |F D^\alpha u(\mathbf{z})|^2 \left(\int |\mathbf{t}|^{-n-2s} |(e^{i\mathbf{t} \cdot \mathbf{z}} - 1)|^2 dt \right) dz. \end{aligned} \quad (39.3.11)$$

Consider the inside integral, the one taken with respect to $\mathbf{t}$.

$$G(\mathbf{z}) \equiv \left(\int |\mathbf{t}|^{-n-2s} |(e^{i\mathbf{t} \cdot \mathbf{z}} - 1)|^2 dt \right).$$

The essential thing to notice about this function of $\mathbf{z}$ is that it is a positive real number whenever $\mathbf{z} \neq \mathbf{0}$. This is because for small $|\mathbf{t}|$, the integrand is dominated by $C|\mathbf{t}|^{-n+2(1-s)}$. Changing to polar coordinates, you see that

$$\int_{[|\mathbf{t}| \leq 1]} |\mathbf{t}|^{-n-2s} |(e^{i\mathbf{t} \cdot \mathbf{z}} - 1)|^2 dt < \infty$$

Next, for $|\mathbf{t}| > 1$, the integrand is bounded by $4|\mathbf{t}|^{-n-2s}$, and changing to polar coordinates shows

$$\int_{[|\mathbf{t}| > 1]} |\mathbf{t}|^{-n-2s} |(e^{i\mathbf{t} \cdot \mathbf{z}} - 1)|^2 dt \leq 4 \int_{[|\mathbf{t}| > 1]} |\mathbf{t}|^{-n-2s} dt < \infty.$$

Now for $\alpha > 0$,

$$\begin{aligned} G(\alpha \mathbf{z}) &= \int |\mathbf{t}|^{-n-2s} |(e^{i\mathbf{t} \cdot \alpha \mathbf{z}} - 1)|^2 dt \\ &= \int |\mathbf{t}|^{-n-2s} |(e^{i\alpha \mathbf{t} \cdot \mathbf{z}} - 1)|^2 dt \\ &= \int \left| \frac{\mathbf{r}}{\alpha} \right|^{-n-2s} |(e^{i\mathbf{r} \cdot \mathbf{z}} - 1)|^2 \frac{1}{\alpha^n} dr \\ &= \alpha^{2s} \int |\mathbf{r}|^{-n-2s} |(e^{i\mathbf{r} \cdot \mathbf{z}} - 1)|^2 dr = \alpha^{2s} G(\mathbf{z}). \end{aligned}$$

Also G is continuous and strictly positive. Letting

$$0 < m(s) = \min \{G(\mathbf{w}) : |\mathbf{w}| = 1\}$$

and

$$M(s) = \max \{G(\mathbf{w}) : |\mathbf{w}| = 1\},$$

it follows from this, and letting $\alpha = |\mathbf{z}|$, $\mathbf{w} \equiv \mathbf{z}/|\mathbf{z}|$, that

$$G(\mathbf{z}) \in \left(m(s) |\mathbf{z}|^{2s}, M(s) |\mathbf{z}|^{2s} \right).$$

More can be said but this will suffice. Also observe that for $s \in (0, 1)$ and $b > 0$,

$$(1+b)^s \leq 1+b^s, \quad 2^{1-s}(1+b)^s \geq 1+b^s.$$

In what follows, $C(s)$ will denote a constant which depends on the indicated quantities which may be different on different lines of the argument. Then from 39.3.11,

$$\begin{aligned} &\int \int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &\leq M(s) \int |FD^\alpha u(\mathbf{z})|^2 |\mathbf{z}|^{2s} dz \\ &= M(s) \int |Fu(\mathbf{z})|^2 |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s} dz. \end{aligned}$$

No reference was made to $|\alpha| = m$ and so this establishes the top half of 39.3.10. Therefore,

$$\begin{aligned} |||u|||_{m+s}^2 &\equiv ||u||_{m,2,\mathbb{R}^n}^2 + \sum_{|\alpha|=m} \int \int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &\leq C \int (1 + |\mathbf{z}|^2)^m |Fu(\mathbf{z})|^2 dz + M(s) \int |Fu(\mathbf{z})|^2 \sum_{|\alpha|=m} |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s} dz \end{aligned}$$

Recall that

$$\sum_{|\alpha| \leq m} z_1^{2\alpha_1} \cdots z_n^{2\alpha_n} \leq \left(1 + \sum_{j=1}^n z_j^2\right)^m \leq C(n, m) \sum_{|\alpha| \leq m} z_1^{2\alpha_1} \cdots z_n^{2\alpha_n}. \quad (39.3.12)$$

Therefore, where $C(n, m)$ is the largest of the multinomial coefficients obtained in the expansion,

$$\left(1 + \sum_{j=1}^n z_j^2\right)^m.$$

Therefore,

$$\begin{aligned} &|||u|||_{m+s}^2 \\ &\leq C \int (1 + |\mathbf{z}|^2)^m |Fu(\mathbf{z})|^2 dz + M(s) \int |Fu(\mathbf{z})|^2 \sum_{|\alpha|=m} |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s} dz \\ &\leq C \int (1 + |\mathbf{z}|^2)^{m+s} |Fu(\mathbf{z})|^2 dz + M(s) \int |Fu(\mathbf{z})|^2 (1 + |\mathbf{z}|^2)^m |\mathbf{z}|^{2s} dz \\ &\leq C \int (1 + |\mathbf{z}|^2)^{m+s} |Fu(\mathbf{z})|^2 dz = C ||u||_{H^{m+s}(\mathbb{R}^n)}^2. \end{aligned}$$

It remains to show the other inequality. From 39.3.11,

$$\begin{aligned} &\int \int |D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &\geq m(s) \int |FD^\alpha u(\mathbf{z})|^2 |\mathbf{z}|^{2s} dz \\ &= m(s) \int |Fu(\mathbf{z})|^2 |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s} dz. \end{aligned}$$

No reference was made to $|\alpha| = m$ and so this establishes the bottom half of 39.3.10. Therefore, from 39.3.12,

$$\begin{aligned}
 & |||u|||_{m+s}^2 \\
 & \geq C \int (1 + |\mathbf{z}|^2)^m |Fu(\mathbf{z})|^2 dz + m(s) \int |Fu(\mathbf{z})|^2 \sum_{|\alpha|=m} |\mathbf{z}^\alpha|^2 |\mathbf{z}|^{2s} dz \\
 & \geq C \int (1 + |\mathbf{z}|^2)^m |Fu(\mathbf{z})|^2 dz + C \int |Fu(\mathbf{z})|^2 (1 + |\mathbf{z}|^2)^m |\mathbf{z}|^{2s} dz \\
 & = C \int (1 + |\mathbf{z}|^2)^m (1 + |\mathbf{z}|^{2s}) |Fu(\mathbf{z})|^2 dz \\
 & \geq C \int (1 + |\mathbf{z}|^2)^m (1 + |\mathbf{z}|^2)^s |Fu(\mathbf{z})|^2 dz \\
 & = C \int (1 + |\mathbf{z}|^2)^{m+s} |Fu(\mathbf{z})|^2 dz = ||u||_{H^{m+s}(\mathbb{R}^n)}^2.
 \end{aligned}$$

This proves the theorem.

With the above intrinsic norm, it becomes possible to prove the following version of Theorem 39.2.7.

Lemma 39.3.2 *Let $\mathbf{h} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be one to one and onto. Also suppose that $D^\alpha \mathbf{h}$ and $D^\alpha (\mathbf{h}^{-1})$ exist and are Lipschitz continuous if $|\alpha| \leq m$ for m a positive integer. Then*

$$\mathbf{h}^* : H^{m+s}(\mathbb{R}^n) \rightarrow H^{m+s}(\mathbb{R}^n)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $(\mathbf{h}^{-1})^*$.

Proof: Let $u \in \mathfrak{S}$. From Theorem 39.2.7 and the equivalence of the norms in $W^{m,2}(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$,

$$\begin{aligned}
 & ||\mathbf{h}^* u||_{H^m(\mathbb{R}^n)}^2 + \int \int \sum_{|\alpha|=m} |D^\alpha \mathbf{h}^* u(\mathbf{x}) - D^\alpha \mathbf{h}^* u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\
 & \leq C ||u||_{H^m(\mathbb{R}^n)}^2 + \int \int \sum_{|\alpha|=m} |D^\alpha \mathbf{h}^* u(\mathbf{x}) - D^\alpha \mathbf{h}^* u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\
 & = C ||u||_{H^m(\mathbb{R}^n)}^2 + \int \int \sum_{|\alpha|=m} \left| \sum_{|\beta(\alpha)| \leq m} \mathbf{h}^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(\mathbf{x}) \right. \\
 & \quad \left. - \mathbf{h}^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(\mathbf{y}) \right|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\
 & \leq C ||u||_{H^m(\mathbb{R}^n)}^2 + C \int \int \sum_{|\alpha|=m} \sum_{|\beta(\alpha)| \leq m} \left| \mathbf{h}^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(\mathbf{x}) \right. \\
 & \quad \left. - \mathbf{h}^* (D^{\beta(\alpha)} u) g_{\beta(\alpha)}(\mathbf{y}) \right|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy
 \end{aligned} \tag{39.3.13}$$

A single term in the last sum corresponding to a given α is then of the form,

$$\int \int \left| \mathbf{h}^* (D^\beta u) g_\beta(\mathbf{x}) - \mathbf{h}^* (D^\beta u) g_\beta(\mathbf{y}) \right|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \tag{39.3.14}$$

$$\begin{aligned}
&\leq \left[\int \int |\mathbf{h}^* (D^\beta u) (\mathbf{x}) g_\beta (\mathbf{x}) - \mathbf{h}^* (D^\beta u) (\mathbf{y}) g_\beta (\mathbf{x})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy + \right. \\
&\quad \left. \int \int |\mathbf{h}^* (D^\beta u) (\mathbf{y}) g_\beta (\mathbf{x}) - \mathbf{h}^* (D^\beta u) (\mathbf{y}) g_\beta (\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \right] \\
&\leq \left[C(\mathbf{h}) \int \int |\mathbf{h}^* (D^\beta u) (\mathbf{x}) - \mathbf{h}^* (D^\beta u) (\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy + \right. \\
&\quad \left. \int \int |\mathbf{h}^* (D^\beta u) (\mathbf{y})|^2 |g_\beta (\mathbf{x}) - g_\beta (\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \right].
\end{aligned}$$

Changing variables, and then using the names of the old variables to simplify the notation,

$$\begin{aligned}
&\leq \left[C(\mathbf{h}, \mathbf{h}^{-1}) \int \int |(D^\beta u) (\mathbf{x}) - (D^\beta u) (\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy + \right. \\
&\quad \left. \int \int |\mathbf{h}^* (D^\beta u) (\mathbf{y})|^2 |g_\beta (\mathbf{x}) - g_\beta (\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \right].
\end{aligned}$$

By 39.3.10,

$$\begin{aligned}
&\leq C(\mathbf{h}) \int |F(u)(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz \\
&\quad + \int \int |\mathbf{h}^* (D^\beta u) (\mathbf{y})|^2 |g_\beta (\mathbf{x}) - g_\beta (\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy.
\end{aligned}$$

In the second term, let $\mathbf{t} = \mathbf{x} - \mathbf{y}$. Then this term is of the form

$$\int |\mathbf{h}^* (D^\beta u) (\mathbf{y})|^2 \int |g_\beta (\mathbf{y} + \mathbf{t}) - g_\beta (\mathbf{y})|^2 |\mathbf{t}|^{-n-2s} dt dy \quad (39.3.15)$$

$$\leq C \int |\mathbf{h}^* (D^\beta u) (\mathbf{y})|^2 dy \leq C \|u\|_{H^m(\mathbb{R}^n)}^2. \quad (39.3.16)$$

because the inside integral equals a constant which depends on the Lipschitz constants and bounds of the function, g_β and these things depend only on $\mathbf{h}$. The reason this integral is finite is that for $|\mathbf{t}| \leq 1$,

$$|g_\beta (\mathbf{y} + \mathbf{t}) - g_\beta (\mathbf{y})|^2 |\mathbf{t}|^{-n-2s} \leq K |\mathbf{t}|^2 |\mathbf{t}|^{-n-2s}$$

and using polar coordinates, you see

$$\int_{[|\mathbf{t}| \leq 1]} |g_\beta (\mathbf{y} + \mathbf{t}) - g_\beta (\mathbf{y})|^2 |\mathbf{t}|^{-n-2s} dt < \infty.$$

Now for $|\mathbf{t}| > 1$, the integrand in 39.3.15 is dominated by $4 |\mathbf{t}|^{-n-2s}$ and using polar coordinates, this yields

$$\int_{[|\mathbf{t}| > 1]} |g_\beta (\mathbf{y} + \mathbf{t}) - g_\beta (\mathbf{y})|^2 |\mathbf{t}|^{-n-2s} dt \leq 4 \int_{[|\mathbf{t}| > 1]} |\mathbf{t}|^{-n-2s} dt < \infty.$$

It follows 39.3.14 is dominated by an expression of the form

$$C(\mathbf{h}) \int |F(u)(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz + C \|u\|_{H^m(\mathbb{R}^n)}^2$$

and so the sum in 39.3.13 is dominated by

$$\begin{aligned} & C(m, \mathbf{h}) \int |F(u)(\mathbf{z})|^2 |\mathbf{z}|^{2s} \sum_{|\beta| \leq m} |\mathbf{z}^\beta|^2 dz + C \|u\|_{H^m(\mathbb{R}^n)}^2 \\ & \leq C(m, \mathbf{h}) \int |F(u)(\mathbf{z})|^2 \left(1 + |\mathbf{z}|^2\right)^s \left(1 + |\mathbf{z}|^2\right)^m dz + C \|u\|_{H^m(\mathbb{R}^n)}^2 \\ & \leq C \|u\|_{H^{m+s}(\mathbb{R}^n)}^2. \end{aligned}$$

This proves the theorem because the assertion about $\mathbf{h}^{-1}$ is obvious. Just replace $\mathbf{h}$ with $\mathbf{h}^{-1}$ in the above argument.

Next consider the case where U is an open set.

Lemma 39.3.3 *Let $\mathbf{h}(U) \subseteq V$ where U and V are open subsets of $\mathbb{R}^n$ and suppose that $\mathbf{h}, \mathbf{h}^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are both functions in $C^{m,1}(\mathbb{R}^n)$. Recall this means $D^\alpha \mathbf{h}$ and $D^\alpha \mathbf{h}^{-1}$ exist and are Lipschitz continuous for all $|\alpha| \leq m$. Then $\mathbf{h}^* \in \mathcal{L}(H^{m+s}(V), H^{m+s}(U))$.*

Proof: Let $u \in H^{m+s}(V)$ and let $v \in H^{m+s}(\mathbb{R}^n)$ such that $v|_V = u$. Then from the above, $\mathbf{h}^*v \in H^{m+s}(\mathbb{R}^n)$ and so $\mathbf{h}^*u \in H^{m+s}(U)$ because $\mathbf{h}^*u = \mathbf{h}^*v|_U$. Then by Lemma 39.3.2,

$$\|\mathbf{h}^*u\|_{H^{m+s}(U)} \leq \|\mathbf{h}^*v\|_{H^{m+s}(\mathbb{R}^n)} \leq C \|v\|_{H^{m+s}(\mathbb{R}^n)}$$

Since this is true for all $v \in H^{m+s}(\mathbb{R}^n)$, it follows that

$$\|\mathbf{h}^*u\|_{H^{m+s}(U)} \leq C \|u\|_{H^{m+s}(V)}.$$

With harder work, you don't need to have $\mathbf{h}, \mathbf{h}^{-1}$ defined on all of $\mathbb{R}^n$ but I don't feel like including the details so this lemma will suffice.

Another interesting application of the intrinsic norm is the following.

Lemma 39.3.4 *Let $\phi \in C^{m,1}(\mathbb{R}^n)$ and suppose $\text{spt}(\phi)$ is compact. Then there exists a constant, C_ϕ such that whenever $u \in H^{m+s}(\mathbb{R}^n)$,*

$$\|\phi u\|_{H^{m+s}(\mathbb{R}^n)} \leq C_\phi \|u\|_{H^{m+s}(\mathbb{R}^n)}.$$

Proof: It is a routine exercise in the product rule to verify that $\|\phi u\|_{H^m(\mathbb{R}^n)} \leq C_\phi \|u\|_{H^m(\mathbb{R}^n)}$. It only remains to consider the term involving the integral. A typical term is

$$\iint |D^\alpha \phi u(\mathbf{x}) - D^\alpha \phi u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy.$$

This is a finite sum of terms of the form

$$\int \int |D^\gamma \phi(\mathbf{x}) D^\beta u(\mathbf{x}) - D^\gamma \phi(\mathbf{y}) D^\beta u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy$$

where $|\gamma|$ and $|\beta| \leq m$.

$$\begin{aligned} &\leq 2 \int \int |D^\gamma \phi(\mathbf{x})|^2 |D^\beta u(\mathbf{x}) - D^\beta u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &\quad + 2 \int \int |D^\beta u(\mathbf{y})|^2 |D^\gamma \phi(\mathbf{x}) - D^\gamma \phi(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \end{aligned}$$

By 39.3.10 and the Lipschitz continuity of all the derivatives of ϕ , this is dominated by

$$\begin{aligned} &CM(s) \int |Fu(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz \\ &+ K \int \int |D^\beta u(\mathbf{y})|^2 |\mathbf{x} - \mathbf{y}|^2 |\mathbf{x} - \mathbf{y}|^{-n-2s} dx dy \\ &= CM(s) \int |Fu(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz \\ &\quad + K \int |D^\beta u(\mathbf{y})|^2 \int |\mathbf{t}|^{-n+2(1-s)} dt dy \\ &\leq C(s) \left(\int |Fu(\mathbf{z})|^2 |\mathbf{z}^\beta|^2 |\mathbf{z}|^{2s} dz + K \int |D^\beta u(\mathbf{y})|^2 dy \right) \\ &\leq C(s) \int (1 + |\mathbf{y}|^2)^{m+s} |Fu(\mathbf{y})|^2 dy. \end{aligned}$$

Since there are only finitely many such terms, this proves the lemma.

Corollary 39.3.5 *Let $t = m + s$ for $s \in [0, 1)$ and let U, V be open sets. Let $\phi \in C_c^{m,1}(V)$. This means $\text{spt}(\phi) \subseteq V$ and $\phi \in C^{m,1}(\mathbb{R}^n)$. Then if $u \in H^t(U)$ it follows that $u\phi \in H^t(U \cap V)$ and $\|u\phi\|_{H^t(U \cap V)} \leq C_\phi \|u\|_{H^t(U)}$.*

Proof: Let $v|_U = u$ a.e. where $v \in H^t(\mathbb{R}^n)$. Then by Lemma 39.3.4, $\phi v \in H^t(\mathbb{R}^n)$ and $\phi v|_{U \cap V} = \phi u$ a.e. Therefore, $\phi u \in H^t(U \cap V)$ and

$$\|\phi u\|_{H^t(U \cap V)} \leq \|\phi v\|_{H^t(\mathbb{R}^n)} \leq C_\phi \|v\|_{H^t(\mathbb{R}^n)}.$$

Taking the infimum for all such v whose restrictions equal u , this yields

$$\|\phi u\|_{H^t(U \cap V)} \leq C_\phi \|u\|_{H^t(U)}.$$

This proves the corollary.

39.4 Embedding Theorems

The Fourier transform description of Sobolev spaces makes possible fairly easy proofs of various embedding theorems.

Definition 39.4.1 Let $C_b^m(\mathbb{R}^n)$ denote the functions which are m times continuously differentiable and for which

$$\sup_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n} |D^\alpha u(\mathbf{x})| \equiv \|u\|_{C_b^m(\mathbb{R}^n)} < \infty.$$

For U an open set, $C^m(\overline{U})$ denotes the functions which are restrictions of $C_b^m(\mathbb{R}^n)$ to U .

It is clear this is a Banach space, the proof being a simple exercise in the use of the fundamental theorem of calculus along with standard results about uniform convergence.

Lemma 39.4.2 Let $u \in \mathfrak{S}$ and let $\frac{n}{2} + m < t$. Then there exists C independent of u such that

$$\|u\|_{C_b^m(\mathbb{R}^n)} \leq C \|u\|_{H^t(\mathbb{R}^n)}.$$

Proof: Using the fact that the Fourier transform maps $\mathfrak{S}$ to $\mathfrak{S}$ and the definition of the Fourier transform,

$$\begin{aligned} |D^\alpha u(\mathbf{x})| &\leq C \|FD^\alpha u\|_{L^1(\mathbb{R}^n)} \\ &= C \int |\mathbf{x}^\alpha| |Fu(\mathbf{x})| dx \\ &\leq C \int (1 + |\mathbf{x}|^2)^{|\alpha|/2} |Fu(\mathbf{x})| dx \\ &\leq C \int (1 + |\mathbf{x}|^2)^{m/2} (1 + |\mathbf{x}|^2)^{-t/2} (1 + |\mathbf{x}|^2)^{t/2} |Fu(\mathbf{x})| dx \\ &\leq C \left(\int (1 + |\mathbf{x}|^2)^{m-t} dx \right)^{1/2} \left(\int (1 + |\mathbf{x}|^2)^t |Fu(\mathbf{x})|^2 dx \right)^{1/2} \\ &\leq C \|u\|_{H^t(\mathbb{R}^n)} \end{aligned}$$

because for the given values of t and m the first integral is finite. This follows from a use of polar coordinates. Taking sup over all $\mathbf{x} \in \mathbb{R}^n$ and $|\alpha| \leq m$, this proves the lemma.

Corollary 39.4.3 Let $u \in H^t(\mathbb{R}^n)$ where $t > m + \frac{n}{2}$. Then u is a.e. equal to a function of $C_b^m(\mathbb{R}^n)$ still denoted by u . Furthermore, there exists a constant, C independent of u such that

$$\|u\|_{C_b^m(\mathbb{R}^n)} \leq C \|u\|_{H^t(\mathbb{R}^n)}.$$

Proof: This follows from the above lemma. Let $\{u_k\}$ be a sequence of functions of $\mathfrak{S}$ which converges to u in H^t and a.e. Then by the inequality of the above lemma, this sequence is also Cauchy in $C_b^m(\mathbb{R}^n)$ and taking the limit,

$$\|u\|_{C_b^m(\mathbb{R}^n)} = \lim_{k \rightarrow \infty} \|u_k\|_{C_b^m(\mathbb{R}^n)} \leq C \lim_{k \rightarrow \infty} \|u_k\|_{H^t(\mathbb{R}^n)} = C \|u\|_{H^t(\mathbb{R}^n)}.$$

What about open sets, U ?

Corollary 39.4.4 *Let $t > m + \frac{n}{2}$ and let U be an open set with $u \in H^t(U)$. Then u is a.e. equal to a function of $C^m(\overline{U})$ still denoted by u . Furthermore, there exists a constant, C independent of u such that*

$$\|u\|_{C^m(\overline{U})} \leq C \|u\|_{H^t(U)}.$$

Proof: Let $u \in H^t(U)$ and let $v \in H^t(\mathbb{R}^n)$ such that $v|_U = u$. Then

$$\|u\|_{C^m(\overline{U})} \leq \|v\|_{C_b^m(\mathbb{R}^n)} \leq C \|v\|_{H^t(\mathbb{R}^n)}.$$

Now taking the inf for all such v yields

$$\|u\|_{C^m(\overline{U})} \leq C \|u\|_{H^t(U)}.$$

39.5 The Trace On The Boundary Of A Half Space

It is important to consider the restriction of functions in a Sobolev space onto a smaller dimensional set such as the boundary of an open set.

Definition 39.5.1 *For $u \in \mathfrak{S}$, define γu a function defined on $\mathbb{R}^{n-1}$ by $\gamma u(\mathbf{x}') \equiv u(\mathbf{x}', 0)$ where $\mathbf{x}' \in \mathbb{R}^{n-1}$ is defined by $\mathbf{x} = (\mathbf{x}', x_n)$.*

The following elementary lemma featuring trig. substitutions is the basis for the proof of some of the arguments which follow.

Lemma 39.5.2 *Consider the integral,*

$$\int_{\mathbb{R}} (a^2 + x^2)^{-t} dx.$$

for $a > 0$ and $t > 1/2$. Then this integral is of the form $C_t a^{-2t+1}$ where C_t is some constant which depends on t .

Proof: Letting $x = a \tan \theta$,

$$\int_{\mathbb{R}} (a^2 + x^2)^{-t} dx = a^{-2t+1} \int_{-\pi/2}^{\pi/2} \cos^{2t-2}(\theta) d\theta$$

and since $t > 1/2$ the last integral is finite. This yields the desired conclusion and proves the lemma.

Lemma 39.5.3 *Let $u \in \mathfrak{S}$. Then there exists a constant, C_n , depending on n but independent of $u \in \mathfrak{S}$ such that*

$$F\gamma u(\mathbf{x}') = C_n \int_{\mathbb{R}} F u(\mathbf{x}', x_n) dx_n.$$

Proof: Using the dominated convergence theorem,

$$\begin{aligned} \int_{\mathbb{R}} F u(\mathbf{x}', x_n) dx_n &\equiv \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} F u(\mathbf{x}', x_n) dx_n \\ &\equiv \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} \left(\frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} e^{-i(\mathbf{x}' \cdot \mathbf{y}' + x_n y_n)} u(\mathbf{y}', y_n) dy' dy_n dx_n \\ &= \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} u(\mathbf{y}', y_n) e^{-i\mathbf{x}' \cdot \mathbf{y}'} \int_{\mathbb{R}} e^{-(\varepsilon x_n)^2} e^{-ix_n y_n} dx_n dy' dy_n. \end{aligned}$$

Now $-(\varepsilon x_n)^2 - ix_n y_n = -\varepsilon^2 \left(x_n + \frac{iy_n}{2} \right)^2 - \varepsilon^2 \frac{y_n^2}{4}$ and so the above reduces to an expression of the form

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} K_n \int_{\mathbb{R}} \frac{1}{\varepsilon} e^{-\varepsilon^2 \frac{y_n^2}{4}} \int_{\mathbb{R}^{n-1}} u(\mathbf{y}', y_n) e^{-i\mathbf{x}' \cdot \mathbf{y}'} dy' dy_n &= K_n \int_{\mathbb{R}^n} u(\mathbf{y}', 0) e^{-i\mathbf{x}' \cdot \mathbf{y}'} dy' \\ &= K_n F\gamma u(\mathbf{x}') \end{aligned}$$

and this proves the lemma with $C_n \equiv K_n^{-1}$.

Earlier $H^t(\mathbb{R}^n)$ was defined and then for U an open subset of $\mathbb{R}^n$, $H^t(U)$ was defined to be the space of restrictions of functions of $H^t(\mathbb{R}^n)$ to U and a norm was given which made $H^t(U)$ into a Banach space. The next task is to consider $\mathbb{R}^{n-1} \times \{0\}$, a smaller dimensional subspace of $\mathbb{R}^n$ and examine the functions defined on this set, denoted by $\mathbb{R}^{n-1}$ for short which are restrictions of functions in $H^t(\mathbb{R}^n)$. You note this is somewhat different because heuristically, the dimension of the domain of the function is changing. An open set in $\mathbb{R}^n$ is considered an n dimensional thing but $\mathbb{R}^{n-1}$ is only $n-1$ dimensional. I realize this is vague because the standard definition of dimension requires a vector space and an open set is not a vector space. However, think in terms of fatness. An open set is fat in n directions whereas $\mathbb{R}^{n-1}$ is only fat in $n-1$ directions. Therefore, something interesting is likely to happen.

Let $\mathfrak{S}$ denote the Schwartz class of functions on $\mathbb{R}^n$ and $\mathfrak{S}'$ the Schwartz class of functions on $\mathbb{R}^{n-1}$. Also, $\mathbf{y}' \in \mathbb{R}^{n-1}$ while $\mathbf{y} \in \mathbb{R}^n$. Let $u \in \mathfrak{S}$. Then from Lemma 39.5.3 and $s > 0$,

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2 \right)^s |F\gamma u(\mathbf{y}')|^2 dy' \\ &= C_n \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2 \right)^s \left| \int_{\mathbb{R}} F u(\mathbf{y}', y_n) dy_n \right|^2 dy' \\ &= C_n \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2 \right)^s \left| \int_{\mathbb{R}} F u(\mathbf{y}', y_n) \left(1 + |\mathbf{y}|^2 \right)^{t/2} \left(1 + |\mathbf{y}|^2 \right)^{-t/2} dy_n \right|^2 dy' \end{aligned}$$

Then by the Cauchy Schwarz inequality,

$$\leq C_n \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^s \int_{\mathbb{R}} |Fu(\mathbf{y}', y_n)|^2 \left(1 + |\mathbf{y}|^2\right)^t dy_n \int_{\mathbb{R}} \left(1 + |\mathbf{y}|^2\right)^{-t} dy_n dy'. \quad (39.5.17)$$

Consider

$$\int_{\mathbb{R}} \left(1 + |\mathbf{y}|^2\right)^{-t} dy_n = \int_{\mathbb{R}} \left(1 + |\mathbf{y}'|^2 + y_n^2\right)^{-t} dy_n$$

by Lemma 39.5.2 and taking $a = \left(1 + |\mathbf{y}'|^2\right)^{1/2}$, this equals

$$C_t \left(\left(1 + |\mathbf{y}'|^2\right)^{1/2} \right)^{-2t+1} = C_t \left(1 + |\mathbf{y}'|^2\right)^{(-2t+1)/2}.$$

Now using this in 39.5.17,

$$\begin{aligned} & \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^s |F\gamma u(\mathbf{y}')|^2 dy' \\ & \leq C_{n,t} \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^s \int_{\mathbb{R}} |Fu(\mathbf{y}', y_n)|^2 \left(1 + |\mathbf{y}|^2\right)^t dy_n \cdot \\ & \quad \left(1 + |\mathbf{y}'|^2\right)^{(-2t+1)/2} dy' \\ & = C_{n,t} \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{y}'|^2\right)^{s+(-2t+1)/2} \int_{\mathbb{R}} |Fu(\mathbf{y}', y_n)|^2 \left(1 + |\mathbf{y}|^2\right)^t dy_n dy'. \end{aligned}$$

What is the correct choice of t so that the above reduces to $\|u\|_{H^t(\mathbb{R}^n)}^2$? It is clearly the one for which

$$s + (-2t + 1)/2 = 0$$

which occurs when $t = s + \frac{1}{2}$. Then for this choice of t , the following inequality is obtained for any $u \in \mathfrak{S}$.

$$\|\gamma u\|_{H^{t-1/2}(\mathbb{R}^{n-1})} \leq C_{n,t} \|u\|_{H^t(\mathbb{R}^n)}. \quad (39.5.18)$$

This has proved part of the following theorem.

Theorem 39.5.4 *For each $t > 1/2$ there exists a unique mapping*

$$\gamma \in \mathcal{L} \left(H^t(\mathbb{R}^n), H^{t-1/2}(\mathbb{R}^{n-1}) \right)$$

which has the property that for $u \in \mathfrak{S}$, $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$. In addition to this, γ is onto. In fact, there exists a continuous map, $\zeta \in \mathcal{L} \left(H^{t-1/2}(\mathbb{R}^{n-1}), H^t(\mathbb{R}^n) \right)$ such that $\gamma \circ \zeta = \text{id}$.

Proof: It only remains to verify that γ is onto and that the continuous map, ζ exists. Now define

$$\phi(\mathbf{y}) \equiv \phi(\mathbf{y}', y_n) \equiv \frac{\left(1 + |\mathbf{y}'|^2\right)^{t-1/2}}{\left(1 + |\mathbf{y}|^2\right)^t}.$$

Then for $u \in \mathfrak{S}'$, let

$$\begin{aligned} \zeta u(\mathbf{x}) &\equiv CF^{-1}(\phi Fu)(\mathbf{x}) = \\ C \int_{\mathbb{R}^n} e^{i\mathbf{y} \cdot \mathbf{x}} \frac{(1 + |\mathbf{y}'|^2)^{t-1/2}}{(1 + |\mathbf{y}|^2)^t} Fu(\mathbf{y}') dy \end{aligned} \quad (39.5.19)$$

Here the inside Fourier transform is taken with respect to $\mathbb{R}^{n-1}$ because u is only defined on $\mathbb{R}^{n-1}$ and C will be chosen in such a way that $\gamma \circ \zeta = \text{id}$. First the existence of C such that $\gamma \circ \zeta = \text{id}$ will be shown. Since $u \in \mathfrak{S}'$ it follows

$$\mathbf{y} \rightarrow \frac{(1 + |\mathbf{y}'|^2)^{t-1/2}}{(1 + |\mathbf{y}|^2)^t} Fu(\mathbf{y}')$$

is in $\mathfrak{S}$. Hence the inverse Fourier transform of this function is also in $\mathfrak{S}$ and so for $u \in \mathfrak{S}'$, it follows $\zeta u \in \mathfrak{S}$. Therefore, to check $\gamma \circ \zeta = \text{id}$ it suffices to plug in $x_n = 0$. From Lemma 39.5.2 this yields

$$\begin{aligned} &\gamma(\zeta u)(\mathbf{x}', 0) \\ &= C \int_{\mathbb{R}^n} e^{i\mathbf{y}' \cdot \mathbf{x}'} \frac{(1 + |\mathbf{y}'|^2)^{t-1/2}}{(1 + |\mathbf{y}|^2)^t} Fu(\mathbf{y}') dy \\ &= C \int_{\mathbb{R}^{n-1}} (1 + |\mathbf{y}'|^2)^{t-1/2} e^{i\mathbf{y}' \cdot \mathbf{x}'} Fu(\mathbf{y}') \int_{\mathbb{R}} \frac{1}{(1 + |\mathbf{y}|^2)^t} dy_n dy' \\ &= CC_t \int_{\mathbb{R}^{n-1}} (1 + |\mathbf{y}'|^2)^{t-1/2} e^{i\mathbf{y}' \cdot \mathbf{x}'} Fu(\mathbf{y}') (1 + |\mathbf{y}'|^2)^{\frac{-2t+1}{2}} dy' \\ &= CC_t \int_{\mathbb{R}^{n-1}} e^{i\mathbf{y}' \cdot \mathbf{x}'} Fu(\mathbf{y}') dy' = CC_t (2\pi)^{n/2} F^{-1}(Fu)(\mathbf{x}') \end{aligned}$$

and so the correct value of C is $(C_t (2\pi)^{n/2})^{-1}$ to obtain $\gamma \circ \zeta = \text{id}$. It only remains to verify that ζ is continuous. From 39.5.19, and Lemma 39.5.2,

$$\begin{aligned} &\|\zeta u\|_{H^t(\mathbb{R}^n)}^2 \\ &= \int_{\mathbb{R}^n} (1 + |\mathbf{x}|^2)^t |F\zeta u(\mathbf{x})|^2 dx \\ &= C^2 \int_{\mathbb{R}^n} (1 + |\mathbf{x}|^2)^t |F(F^{-1}(\phi Fu)(\mathbf{x}))|^2 dx \end{aligned}$$

$$\begin{aligned}
&= C^2 \int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^t |\phi(\mathbf{x}) Fu(\mathbf{x}')|^2 dx \\
&= C^2 \int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^t \left| \frac{\left(1 + |\mathbf{x}'|^2\right)^{t-1/2}}{\left(1 + |\mathbf{x}|^2\right)^t} Fu(\mathbf{x}') \right|^2 dx \\
&= C^2 \int_{\mathbb{R}^n} \left(1 + |\mathbf{x}|^2\right)^{-t} \left| \left(1 + |\mathbf{x}'|^2\right)^{t-1/2} Fu(\mathbf{x}') \right|^2 dx \\
&= C^2 \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{x}'|^2\right)^{2t-1} |Fu(\mathbf{x}')|^2 \int_{\mathbb{R}} \left(1 + |\mathbf{x}|^2\right)^{-t} dx_n dx' \\
&= C^2 C_t \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{x}'|^2\right)^{2t-1} |Fu(\mathbf{x}')|^2 \left(1 + |\mathbf{y}'|^2\right)^{\frac{-2t+1}{2}} dx' \\
&= C^2 C_t \int_{\mathbb{R}^{n-1}} \left(1 + |\mathbf{x}'|^2\right)^{t-1/2} |Fu(\mathbf{x}')|^2 dx' = C^2 C_t \|u\|_{H^{t-1/2}(\mathbb{R}^{n-1})}^2.
\end{aligned}$$

This proves the theorem because $\mathfrak{S}$ is dense in $\mathbb{R}^n$.

Actually, the assertion that $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$ holds for more functions, u than just $u \in \mathfrak{S}$. I will make no effort to obtain the most general description of such functions but the following is a useful lemma which will be needed when the trace on the boundary of an open set is considered.

Lemma 39.5.5 *Suppose u is continuous and $u \in H^1(\mathbb{R}^n)$. Then there exists a set of m_1 measure zero, N such that if $x_n \notin N$, then for every $\phi \in L^2(\mathbb{R}^{n-1})$*

$$(\gamma u, \phi)_H + \int_0^{x_n} (u_n(\cdot, t), \phi)_H dt = (u(\cdot, x_n), \phi)_H$$

where here

$$(f, g)_H \equiv \int_{\mathbb{R}^{n-1}} f \bar{g} dx',$$

just the inner product in $L^2(\mathbb{R}^{n-1})$. Furthermore,

$$u(\cdot, 0) = \gamma u \text{ a.e. } \mathbf{x}'.$$

Proof: Let $\{u_k\}$ be a sequence of functions from $\mathfrak{S}$ which converges to u in $H^1(\mathbb{R}^n)$ and let $\{\phi_k\}$ denote a countable dense subset of $L^2(\mathbb{R}^{n-1})$. Then

$$(\gamma u_k, \phi_j)_H + \int_0^{x_n} (u_{k,n}(\cdot, t), \phi_j)_H dt = (u_k(\cdot, x_n), \phi_j)_H. \quad (39.5.20)$$

Now

$$\begin{aligned}
& \left(\int_0^\infty \left| (u_k(\cdot, x_n), \phi_j)_H - (u(\cdot, x_n), \phi_j)_H \right|^2 dx_n \right)^{1/2} \\
&= \left(\int_0^\infty \left| (u_k(\cdot, x_n) - u(\cdot, x_n), \phi_j)_H \right|^2 dx_n \right)^{1/2} \\
&\leq \left(\int_0^\infty |u_k(\cdot, x_n) - u(\cdot, x_n)|_H^2 |\phi_j|_H^2 dx_n \right)^{1/2} \\
&= |\phi_j|_H^2 \left(\int_0^\infty |u_k(\cdot, x_n) - u(\cdot, x_n)|_H^2 dx_n \right)^{1/2} \\
&= |\phi_j|_H^2 \left(\int_0^\infty \int_{\mathbb{R}^{n-1}} |u_k(\mathbf{x}', x_n) - u(\mathbf{x}', x_n)|^2 dx' dx_n \right)^{1/2}
\end{aligned}$$

which converges to zero. Therefore, there exists a set of measure zero, N_j and a subsequence, still denoted by k such that if $x_n \notin N_j$, then

$$(u_k(\cdot, x_n), \phi_j)_H \rightarrow (u(\cdot, x_n), \phi_j)_H.$$

Now by Theorem 39.5.4, $\gamma u_k \rightarrow \gamma u$ in $H = L^2(\mathbb{R}^{n-1})$. It only remains to consider the term of 39.5.20 which involves an integral.

$$\begin{aligned}
& \left| \int_0^{x_n} (u_{k,n}(\cdot, t), \phi_j)_H dt - \int_0^{x_n} (u_n(\cdot, t), \phi_j)_H dt \right| \\
&\leq \int_0^{x_n} \left| (u_{k,n}(\cdot, t) - u_n(\cdot, t), \phi_j)_H \right| dt \\
&\leq \int_0^{x_n} |u_{k,n}(\cdot, t) - u_n(\cdot, t)|_H |\phi_j|_H dt \\
&\leq \left(\int_0^{x_n} |u_{k,n}(\cdot, t) - u_n(\cdot, t)|_H^2 dt \right)^{1/2} \left(\int_0^{x_n} |\phi_j|_H^2 dt \right)^{1/2} \\
&= x_n^{1/2} |\phi_j|_H \left(\int_0^{x_n} \int_{\mathbb{R}^{n-1}} |u_{k,n}(\mathbf{x}', t) - u_n(\mathbf{x}', t)|^2 dx' \right)^{1/2} dt
\end{aligned}$$

and this converges to zero as $k \rightarrow \infty$. Therefore, using a diagonal sequence argument, there exists a subsequence, still denoted by k and a set of measure zero, $N \equiv \bigcup_{j=1}^\infty N_j$ such that for $\mathbf{x}' \notin N$, you can pass to the limit in 39.5.20 and obtain that for all ϕ_j ,

$$(\gamma u, \phi_j)_H + \int_0^{x_n} (u_n(\cdot, t), \phi_j)_H dt = (u(\cdot, x_n), \phi_j)_H.$$

By density of $\{\phi_j\}$, this equality holds for all $\phi \in L^2(\mathbb{R}^{n-1})$. In particular, the equality holds for every $\phi \in C_c(\mathbb{R}^{n-1})$. Since u is uniformly continuous on the compact set, $\text{spt}(\phi) \times [0, 1]$, there exists a sequence, $(x_n)_k \rightarrow 0$ such that the above

equality holds for x_n replaced with $(x_n)_k$ and ϕ in place of ϕ_j . Now taking $k \rightarrow \infty$, this uniform continuity implies

$$(\gamma u, \phi)_H = (u(\cdot, 0), \phi)_H$$

This implies since $C_c(\mathbb{R}^{n-1})$ is dense in $L^2(\mathbb{R}^{n-1})$ that $\gamma u = u(\cdot, 0)$ a.e. and this proves the lemma.

Lemma 39.5.6 *Suppose U is an open subset of $\mathbb{R}^n$ of the form*

$$U \equiv \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi(\mathbf{u}')\}$$

where U' is an open subset of $\mathbb{R}^{n-1}$ and $\phi(\mathbf{u}')$ is a positive function such that $\phi(\mathbf{u}') \leq \infty$ and

$$\inf \{\phi(\mathbf{u}') : \mathbf{u}' \in U'\} = \delta > 0$$

Suppose $v \in H^t(\mathbb{R}^n)$ such that $v = 0$ a.e. on U . Then $\gamma v = 0$ m_{n-1} a.e. point of U' . Also, if $v \in H^t(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$, then $\gamma v \gamma \phi = \gamma(\phi v)$.

Proof: First consider the second claim. Let $v \in H^t(\mathbb{R}^n)$ and let $v_k \rightarrow v$ in $H^t(\mathbb{R}^n)$ where $v_k \in \mathfrak{S}$. Then from Lemma 39.3.4 and Theorem 39.5.4

$$\|\gamma(\phi v) - \gamma \phi \gamma v\|_{H^{t-1/2}(\mathbb{R}^{n-1})} = \lim_{k \rightarrow \infty} \|\gamma(\phi v_k) - \gamma \phi \gamma v_k\|_{H^{t-1/2}(\mathbb{R}^{n-1})} = 0$$

because each term in the sequence equals zero due to the observation that for $v_k \in \mathfrak{S}$ and $\phi \in C_c^\infty(U)$, $\gamma(\phi v_k) = \gamma v_k \gamma \phi$.

Now suppose $v = 0$ a.e. on U . Define for $0 < r < \delta$, $v_r(\mathbf{x}) \equiv v(\mathbf{x}', x_n + r)$.

Claim: If $u \in H^t(\mathbb{R}^n)$, then

$$\lim_{r \rightarrow 0} \|v_r - v\|_{H^t(\mathbb{R}^n)} = 0.$$

Proof of claim: First of all, let $v \in \mathfrak{S}$. Then $v \in H^m(\mathbb{R}^n)$ for all m and so by Lemma 39.2.5,

$$\|v_r - v\|_{H^t(\mathbb{R}^n)} \leq \|v_r - v\|_{H^m(\mathbb{R}^n)}^\theta \|v_r - v\|_{H^{m+1}(\mathbb{R}^n)}^{1-\theta}$$

where $t \in [m, m+1]$. It follows from continuity of translation in $L^p(\mathbb{R}^n)$ that

$$\lim_{r \rightarrow 0} \|v_r - v\|_{H^m(\mathbb{R}^n)}^\theta \|v_r - v\|_{H^{m+1}(\mathbb{R}^n)}^{1-\theta} = 0$$

and so the claim is proved if $v \in \mathfrak{S}$. Now suppose $u \in H^t(\mathbb{R}^n)$ is arbitrary. By density of $\mathfrak{S}$ in $H^t(\mathbb{R}^n)$, there exists $v \in \mathfrak{S}$ such that

$$\|u - v\|_{H^t(\mathbb{R}^n)} < \varepsilon/3.$$

Therefore,

$$\begin{aligned} \|u_r - u\|_{H^t(\mathbb{R}^n)} &\leq \|u_r - v_r\|_{H^t(\mathbb{R}^n)} + \|v_r - v\|_{H^t(\mathbb{R}^n)} + \|v - u\|_{H^t(\mathbb{R}^n)} \\ &= 2\varepsilon/3 + \|v_r - v\|_{H^t(\mathbb{R}^n)}. \end{aligned}$$

Now using what was just shown, it follows that for r small enough, $\|u_r - u\|_{H^t(\mathbb{R}^n)} < \varepsilon$ and this proves the claim.

Now suppose $v \in H^t(\mathbb{R}^n)$. By the claim,

$$\|v_r - v\|_{H^t(\mathbb{R}^n)} \rightarrow 0$$

and so by continuity of γ ,

$$\gamma v_r \rightarrow \gamma v \text{ in } H^{t-1/2}(\mathbb{R}^{n-1}). \quad (39.5.21)$$

Note $v_r = 0$ a.e. on

$$U_r \equiv \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } -r < u_n < \phi(\mathbf{u}') - r\}$$

Let $\phi \in C_c^\infty(U_r)$ and consider ϕv_r . Then it follows $\phi v_r = 0$ a.e. on $\mathbb{R}^n$. Let $w \equiv 0$. Then $w \in \mathfrak{S}$ and so $\gamma w = 0 = \gamma(\phi v_r) = \gamma\phi\gamma v_r$ in $H^{t-1/2}(\mathbb{R}^{n-1})$. It follows that for m_{n-1} a.e. $\mathbf{x}' \in [\phi \neq 0] \cap \mathbb{R}^{n-1}$, $\gamma v_r(\mathbf{x}') = 0$. Now let $U' = \cup_{k=1}^\infty K_k$ where the K_k are compact sets such that $K_k \subseteq K_{k+1}$ and let $\phi_k \in C_c^\infty(U)$ such that ϕ_k has values in $[0, 1]$ and $\phi_k(\mathbf{x}') = 1$ if $\mathbf{x}' \in K_k$. Then from what was just shown, $\gamma v_r = 0$ for a.e. point of K_k . Therefore, $\gamma v_r = 0$ for m_{n-1} a.e. point in U' . Therefore, since each $\gamma v_r = 0$, it follows from 39.5.21 that $\gamma v = 0$ also. This proves the lemma.

Theorem 39.5.7 *Let $t > 1/2$ and let U be of the form*

$$\{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi(\mathbf{u}')\}$$

where U' is an open subset of $\mathbb{R}^{n-1}$ and $\phi(\mathbf{u}')$ is a positive function such that $\phi(\mathbf{u}') \leq \infty$ and

$$\inf \{\phi(\mathbf{u}') : \mathbf{u}' \in U'\} = \delta > 0.$$

Then there exists a unique

$$\gamma \in \mathcal{L}(H^t(U), H^{t-1/2}(U'))$$

which has the property that if $u = v|_U$ where v is continuous and also a function of $H^1(\mathbb{R}^n)$, then $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$ for a.e. $\mathbf{x}' \in U'$.

Proof: Let $u \in H^t(U)$. Then $u = v|_U$ for some $v \in H^t(\mathbb{R}^n)$. Define

$$\gamma u \equiv \gamma v|_{U'}$$

Is this well defined? The answer is yes because if $v_i|_U = u$ a.e., then $\gamma(v_1 - v_2) = 0$ a.e. on U' which implies $\gamma v_1 = \gamma v_2$ a.e. and so the two different versions of γu differ only on a set of measure zero.

If $u = v|_U$ where v is continuous and also a function of $H^1(\mathbb{R}^n)$, then for a.e. $\mathbf{x}' \in \mathbb{R}^{n-1}$, it follows from Lemma 39.5.5 on Page 1156 that $\gamma v(\mathbf{x}') = v(\mathbf{x}', 0)$. Hence, it follows that for a.e. $\mathbf{x}' \in U'$, $\gamma u(\mathbf{x}') \equiv u(\mathbf{x}', 0)$.

In particular, γ is determined by $\gamma u(\mathbf{x}') = u(\mathbf{x}', 0)$ on $\mathfrak{S}|_U$ and the density of $\mathfrak{S}|_U$ and continuity of γ shows γ is unique.

It only remains to show γ is continuous. Let $u \in H^t(U)$. Thus there exists $v \in H^t(\mathbb{R}^n)$ such that $u = v|_U$. Then

$$\|\gamma u\|_{H^{t-1/2}(U')} \leq \|\gamma v\|_{H^{t-1/2}(\mathbb{R}^{n-1})} \leq C \|v\|_{H^t(\mathbb{R}^n)}$$

for C independent of v . Then taking the inf for all such $v \in H^t(\mathbb{R}^n)$ which are equal to u a.e. on U , it follows

$$\|\gamma u\|_{H^{t-1/2}(U')} \leq C \|u\|_{H^t(\mathbb{R}^n)}$$

and this proves γ is continuous.

39.6 Sobolev Spaces On Manifolds

39.6.1 General Theory

The type of manifold, Γ for which Sobolev spaces will be defined on is:

Definition 39.6.1 1. Γ is a closed subset of $\mathbb{R}^p$ where $p \geq n$.

2. $\Gamma = \bigcup_{i=1}^{\infty} \Gamma_i$ where $\Gamma_i = \Gamma \cap W_i$ for W_i a bounded open set.

3. $\{W_i\}_{i=1}^{\infty}$ is locally finite.

4. There are open bounded sets, U_i and functions $\mathbf{h}_i : U_i \rightarrow \Gamma_i$ which are one to one, onto, and in $C^{m,1}(U_i)$. There exists a constant, C , such that $C \geq \text{Lip } \mathbf{h}_i$ for all i .

5. There exist functions, $\mathbf{g}_i : W_i \rightarrow U_i$ such that $\mathbf{g}_i$ is $C^{m,1}(W_i)$, and $\mathbf{g}_i \circ \mathbf{h}_i = \text{id}$ on U_i while $\mathbf{h}_i \circ \mathbf{g}_i = \text{id}$ on Γ_i .

This will be referred to as a $C^{m,1}$ manifold.

Lemma 39.6.2 Let $\mathbf{g}_i$, $\mathbf{h}_i$, U_i , W_i , and Γ_i be as defined above. Then

$$\mathbf{g}_i \circ \mathbf{h}_k : U_k \cap \mathbf{h}_k^{-1}(\Gamma_i) \rightarrow U_i \cap \mathbf{h}_i^{-1}(\Gamma_k)$$

is $C^{m,1}$. Furthermore, the inverse of this map is $\mathbf{g}_k \circ \mathbf{h}_i$.

Proof: First it is well to show it does indeed map the given open sets. Let $\mathbf{x} \in U_k \cap \mathbf{h}_k^{-1}(\Gamma_i)$. Then $\mathbf{h}_k(\mathbf{x}) \in \Gamma_k \cap \Gamma_i$ and so $\mathbf{g}_i(\mathbf{h}_k(\mathbf{x})) \in U_i$ because $\mathbf{h}_k(\mathbf{x}) \in \Gamma_i$. Now since $\mathbf{h}_k(\mathbf{x}) \in \Gamma_k$, $\mathbf{g}_i(\mathbf{h}_k(\mathbf{x})) \in \mathbf{h}_i^{-1}(\Gamma_k)$ also and this proves the mappings do what they should in terms of mapping the two open sets. That $\mathbf{g}_i \circ \mathbf{h}_k$ is $C^{m,1}$ follows immediately from the chain rule and the assumptions that the functions $\mathbf{g}_i$ and $\mathbf{h}_k$ are $C^{m,1}$. The claim about the inverse follows immediately from the definitions of the functions.

Let $\{\psi_i\}_{i=1}^{\infty}$ be a partition of unity subordinate to the open cover $\{W_i\}$ satisfying $\psi_i \in C_c^{\infty}(W_i)$. Then the following definition provides a norm for $H^{m+s}(\Gamma)$.

Definition 39.6.3 Let $s \in (0, 1)$ and m is a nonnegative integer. Also let μ denote the surface measure for Γ defined in the last section. A μ measurable function, u is in $H^{m+s}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{g}_i\}_{i=1}^\infty$ is described above, $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$ and

$$\|u\|_{H^{m+s}(\Gamma)} \equiv \left(\sum_{i=1}^{\infty} \|\mathbf{h}_i^*(u\psi_i)\|_{H^{m+s}(U_i)}^2 \right)^{1/2} < \infty.$$

Are there functions which are in $H^{m+s}(\Gamma)$? The answer is yes. Just take the restriction to Γ of any function, $u \in C_c^\infty(\mathbb{R}^m)$. Then each $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$ and the sum is finite because $\text{spt } u$ has nonempty intersection with only finitely many W_i .

It is not at all obvious this norm is well defined. What if $\{W'_i, \psi'_i, \Gamma'_i, U_i, \mathbf{h}'_i, \mathbf{g}'_i\}_{i=1}^\infty$ is as described above? Would the two norms be equivalent? If they aren't, then this is not a good way to define $H^{m+s}(\Gamma)$ because it would depend on the choice of partition of unity and functions, $\mathbf{h}_i$ and choice of the open sets, U_i . To begin with pick a particular choice for $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{g}_i\}_{i=1}^\infty$.

Lemma 39.6.4 $H^{m+s}(\Gamma)$ as just described, is a Banach space.

Proof: Let $\{u_j\}_{j=1}^\infty$ be a Cauchy sequence in $H^{m+s}(\Gamma)$. Then $\{\mathbf{h}_i^*(u_j\psi_i)\}_{j=1}^\infty$ is a Cauchy sequence in $H^{m+s}(U_i)$ for each i . Therefore, for each i , there exists $w_i \in H^{m+s}(U_i)$ such that

$$\lim_{j \rightarrow \infty} \mathbf{h}_i^*(u_j\psi_i) = w_i \text{ in } H^{m+s}(U_i). \quad (39.6.22)$$

It is required to show there exists $u \in H^{m+s}(\Gamma)$ such that $w_i = \mathbf{h}_i^*(u\psi_i)$ for each i .

Now from Corollary 37.2.5 it follows easily by approximating with simple functions that for ever nonnegative μ measurable function, f ,

$$\int_{\Gamma} f d\mu = \sum_{r=1}^{\infty} \int_{\mathbf{g}_r \Gamma_r} \psi_r f(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du.$$

Therefore,

$$\begin{aligned} \int_{\Gamma} |u_j - u_k|^2 d\mu &= \sum_{r=1}^{\infty} \int_{\mathbf{g}_r \Gamma_r} \psi_r |u_j - u_k|^2(\mathbf{h}_r(\mathbf{u})) J_r(\mathbf{u}) du \\ &\leq C \sum_{r=1}^{\infty} \int_{\mathbf{g}_r \Gamma_r} \psi_r |u_j - u_k|^2(\mathbf{h}_r(\mathbf{u})) du \\ &= C \sum_{r=1}^{\infty} \|\mathbf{h}_r^*(\psi_r |u_j - u_k|)\|_{0,2,U_r}^2 \\ &\leq C \|u_j - u_k\|_{H^{m+s}(\Gamma)}^2 \end{aligned}$$

and it follows there exists $u \in L^2(\Gamma)$ such that

$$\|u_j - u\|_{0,2,\Gamma} \rightarrow 0.$$

and a subsequence, still denoted by u_j such that $u_j(\mathbf{x}) \rightarrow u(\mathbf{x})$ for μ a.e. $\mathbf{x} \in \Gamma$. It is required to show that $u \in H^{m+s}(\Gamma)$ such that $w_i = \mathbf{h}_i^*(u\psi_i)$ for each i . First of all, u is measurable because it is the limit of measurable functions. The pointwise convergence just established and the fact that sets of measure zero on Γ_i correspond to sets of measure zero on U_i which was discussed in the claim found in the proof of Theorem 37.2.4 on Page 1091 shows that

$$\mathbf{h}_i^*(u_j\psi_i)(\mathbf{x}) \rightarrow \mathbf{h}_i^*(u\psi_i)(\mathbf{x})$$

a.e. $\mathbf{x}$. Therefore,

$$\mathbf{h}_i^*(u\psi_i) = w_i$$

and this shows that $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$. It remains to verify that $u \in H^{m+s}(\Gamma)$. This follows from Fatou's lemma. From 39.6.22,

$$\|\mathbf{h}_i^*(u_j\psi_i)\|_{H^{m+s}(U_i)}^2 \rightarrow \|\mathbf{h}_i^*(u\psi_i)\|_{H^{m+s}(U_i)}^2$$

and so

$$\begin{aligned} \sum_{i=1}^{\infty} \|\mathbf{h}_i^*(u\psi_i)\|_{H^{m+s}(U_i)}^2 &\leq \liminf_{j \rightarrow \infty} \sum_{i=1}^{\infty} \|\mathbf{h}_i^*(u_j\psi_i)\|_{H^{m+s}(U_i)}^2 \\ &= \liminf_{j \rightarrow \infty} \|u_j\|_{H^{m+s}(\Gamma)}^2 < \infty. \end{aligned}$$

This proves the lemma.

In fact any two such norms are equivalent. This follows from the open mapping theorem. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are two such norms and consider the norm $\|\cdot\|_3 \equiv \max(\|\cdot\|_1, \|\cdot\|_2)$. Then $(H^{m+s}(\Gamma), \|\cdot\|_3)$ is also a Banach space and the identity map from this Banach space to $(H^{m+s}(\Gamma), \|\cdot\|_i)$ for $i = 1, 2$ is continuous. Therefore, by the open mapping theorem, there exist constants, C, C' such that for all $u \in H^{m+s}(\Gamma)$,

$$\|u\|_1 \leq \|u\|_3 \leq C\|u\|_2 \leq C\|u\|_3 \leq CC'\|u\|_1$$

Therefore,

$$\|u\|_1 \leq C\|u\|_2, \|u\|_2 \leq C'\|u\|_1.$$

This proves the following theorem.

Theorem 39.6.5 *Let Γ be described above. Defining $H^t(\Gamma)$ as in Definition 39.6.3, any two norms like those given in this definition are equivalent.*

Suppose $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ are as defined above where $\mathbf{h}_i, \mathbf{g}_i$ are $C^{m,1}$ functions. Take W , an open set in $\mathbb{R}^p$ and define $\Gamma' \equiv W \cap \Gamma$. Then letting

$$W'_i \equiv W \cap W_i, \Gamma'_i \equiv W'_i \cap \Gamma,$$

and

$$U'_i \equiv \mathbf{g}_i(\Gamma'_i) = \mathbf{h}_i^{-1}(W'_i \cap \Gamma),$$

it follows that U'_i is an open set because $\mathbf{h}_i$ is continuous and $(\Gamma', W'_i, U'_i, \Gamma'_i, \mathbf{h}'_i, \mathbf{g}'_i)$ is also a $C^{m,1}$ manifold if you define $\mathbf{h}'_i$ to be the restriction of $\mathbf{h}_i$ to U'_i and $\mathbf{g}'_i$ to be the restriction of $\mathbf{g}_i$ to W'_i .

As a case of this, consider a $C^{m,1}$ manifold, Γ where $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ are as described in Definition 39.6.1 and the submanifold consisting of Γ_i . The next lemma shows there is a simple way to define a norm on $H^t(\Gamma_i)$ which does not depend on dragging in a partition of unity.

Lemma 39.6.6 *Suppose Γ is a $C^{m,1}$ manifold and $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ are as described in Definition 39.6.1. Then for $t \in [m, m+s)$, it follows that if $u \in H^t(\Gamma)$, then $u \in H^t(\Gamma_k)$ and the restriction map is continuous. Also an equivalent norm for $H^t(\Gamma_k)$ is given by*

$$|||u|||_t \equiv ||\mathbf{h}_k^* u||_{H^t(U_k)}.$$

Proof: Let $u \in H^t(\Gamma)$ and let $(\Gamma_k, W'_i, U'_i, \Gamma'_i, \mathbf{h}'_i, \mathbf{g}'_i)$ be the sets and functions which define what is meant by Γ_k being a $C^{m,1}$ manifold as described in Definition 39.6.1. Also let $(\Gamma, W_i, U_i, \Gamma_i, \mathbf{h}_i, \mathbf{g}_i)$ be pertain to Γ in the same way and let $\{\phi_j\}$ be a C^∞ partition of unity for the $\{W_j\}$. Since the $\{W'_i\}$ are locally finite, only finitely many can intersect Γ_k , say $\{W'_1, \dots, W'_s\}$. Also only finitely many of the W_i can intersect Γ_k , say $\{W_1, \dots, W_q\}$. Then letting $\{\psi'_i\}$ be a C^∞ partition of unity subordinate to the $\{W'_i\}$.

$$\begin{aligned} & \sum_{i=1}^{\infty} ||\mathbf{h}'_i{}^*(u\psi'_i)||_{H^t(U'_i)} \\ &= \sum_{i=1}^s \left\| \mathbf{h}'_i{}^* \left(\sum_{j=1}^q \phi_j u \psi'_i \right) \right\|_{H^t(U'_i)} \\ &\leq \sum_{i=1}^s \sum_{j=1}^q ||\mathbf{h}'_i{}^* \phi_j u \psi'_i||_{H^t(U'_i)} \\ &= \sum_{j=1}^q \sum_{i=1}^s ||\mathbf{h}'_i{}^* \phi_j u \psi'_i||_{H^t(U'_i)} \\ &= \sum_{j=1}^q \sum_{i=1}^s ||(\mathbf{g}_j \circ \mathbf{h}'_i)^* \mathbf{h}_j^* \phi_j u \psi'_i||_{H^t(U'_i)}. \end{aligned}$$

By Lemma 39.3.3 on page 1149, there exists a single constant, C such that the above is dominated by

$$C \sum_{j=1}^q \sum_{i=1}^s ||\mathbf{h}_j^* \phi_j u \psi'_i||_{H^t(U_j)}.$$

Now by Corollary 39.3.5 on Page 1150, this is no larger than

$$\begin{aligned} C \sum_{j=1}^q \sum_{i=1}^s C_{\psi'_i} \|\mathbf{h}_j^* \phi_j u\|_{H^t(U_j)} &\leq C \sum_{j=1}^q \sum_{i=1}^s \|\mathbf{h}_j^* \phi_j u\|_{H^t(U_j)} \\ &\leq C \sum_{j=1}^q \|\mathbf{h}_j^* \phi_j u\|_{H^t(U_j)} < \infty. \end{aligned}$$

This shows that u restricted to Γ_k is in $H^t(\Gamma_k)$. It also shows that the restriction map of $H^t(\Gamma)$ to $H^t(\Gamma_k)$ is continuous.

Now consider the norm $|||\cdot|||_t$. For $u \in H^t(\Gamma_k)$, let $(\Gamma_k, W'_i, U'_i, \Gamma'_i, \mathbf{h}'_i, \mathbf{g}'_i)$ be sets and functions which define an atlas for Γ_k . Since the $\{W'_i\}$ are locally finite, only finitely many can have nonempty intersection with Γ_k , say $\{W_1, \dots, W_s\}$. Thus $i \leq s$ for some finite s . The problem is to compare $|||\cdot|||_t$ with $\|\cdot\|_{H^t(\Gamma_k)}$. As above, let $\{\psi'_i\}$ denote a C^∞ partition of unity subordinate to the $\{W'_i\}$. Then

$$\begin{aligned} |||u|||_t &\equiv \|\mathbf{h}_k^* u\|_{H^t(U_k)} = \left\| \mathbf{h}_k^* \sum_{j=1}^s \psi'_j u \right\|_{H^t(U_k)} \\ &\leq \sum_{j=1}^s \|\mathbf{h}_k^* (\psi'_j u)\|_{H^t(U_k)} \\ &= \sum_{j=1}^s \left\| (\mathbf{g}'_j \circ \mathbf{h}_k)^* \mathbf{h}'_j{}^* (\psi'_j u) \right\|_{H^t(U_k)} \\ &\leq C \sum_{j=1}^s \|\mathbf{h}'_j{}^* (\psi'_j u)\|_{H^t(U'_j)} \\ &\leq C \left(\sum_{j=1}^s \|\mathbf{h}'_j{}^* (\psi'_j u)\|_{H^t(U'_j)}^2 \right)^{1/2} = \|u\|_{H^t(\Gamma_k)}. \end{aligned}$$

where Lemma 39.3.3 on page 1149 was used in the last step. Now also, from Lemma 39.3.3 on page 1149

$$\begin{aligned}
 \|u\|_{H^t(\Gamma_k)} &= \left(\sum_{j=1}^s \|\mathbf{h}'_j{}^* (\psi'_j u)\|_{H^t(U'_j)}^2 \right)^{1/2} \\
 &= \left(\sum_{j=1}^s \left\| (\mathbf{g}_k \circ \mathbf{h}'_j)^* \mathbf{h}_k^* (\psi'_j u) \right\|_{H^t(U'_j)}^2 \right)^{1/2} \\
 &\leq C \left(\sum_{j=1}^s \|\mathbf{h}_k^* (\psi'_j u)\|_{H^t(U_k)}^2 \right)^{1/2} \\
 &\leq C \left(\sum_{j=1}^s \|\mathbf{h}_k^* u\|_{H^t(U_k)}^2 \right)^{1/2} = C_s \|\mathbf{h}_k^* u\|_{H^t(U_k)} = \|u\|_t.
 \end{aligned}$$

This proves the lemma.

39.6.2 The Trace On The Boundary

Definition 39.6.7 A bounded open subset, Ω , of $\mathbb{R}^n$ has a $C^{m,1}$ boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \overline{\Omega} \setminus \Omega$, there exists an open set, W , containing p , an open interval $(0, b)$, a bounded open box $U' \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, R_W consisting of a distance preserving linear transformation followed by a translation such that

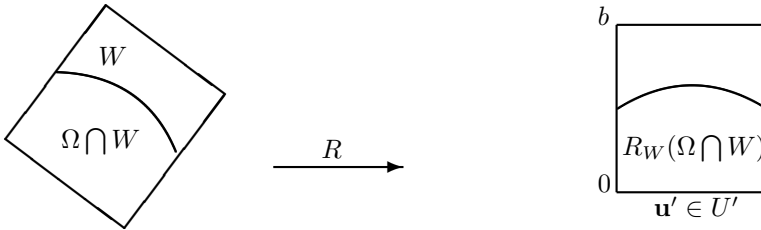
$$R_W W = U' \times (0, b), \quad (39.6.23)$$

$$R_W (W \cap \Omega) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U', 0 < u_n < \phi_W(\mathbf{u}')\} \equiv U_W \quad (39.6.24)$$

where $\phi_W \in C^{m,1}(\overline{U'})$ meaning ϕ_W is the restriction to U' of a function, still denoted by ϕ_W which is in $C^{m,1}(\mathbb{R}^{n-1})$ and

$$\inf \{\phi_W(\mathbf{u}') : \mathbf{u}' \in U'\} > 0$$

The following picture depicts the situation.



For the situation described in the above definition, let $\mathbf{h}_W : U' \rightarrow \Gamma \cap W$ be defined by

$$\mathbf{h}_W(\mathbf{u}') \equiv R_W^{-1}(\mathbf{u}', \phi_W(\mathbf{u}')), \quad \mathbf{g}_W(\mathbf{x}) \equiv (R_W \mathbf{x})', \quad \mathbf{H}_W(\mathbf{u}) \equiv R_W^{-1}(\mathbf{u}', \phi_W(\mathbf{u}') - u_n).$$

where $\mathbf{x}' \equiv (x_1, \dots, x_{n-1})$ for $\mathbf{x} = (x_1, \dots, x_n)$. Thus $\mathbf{g}_W \circ \mathbf{h}_W = \text{id}$ on U' and $\mathbf{h}_W \circ \mathbf{g}_W = \text{id}$ on $\Gamma \cap W$. Also note that $\mathbf{H}_W$ is defined on all of $\mathbb{R}^n$ is $C^{m,1}$, and has an inverse with the same properties. To see this, let $\mathbf{G}_W(\mathbf{u}) = (\mathbf{u}', \phi_W(\mathbf{u}') - u_n)$. Then $\mathbf{H}_W = R_W^{-1} \circ \mathbf{G}_W$ and $\mathbf{G}_W^{-1} = (\mathbf{u}', \phi_W(\mathbf{u}') - u_n)$ and so $\mathbf{H}_W^{-1} = \mathbf{G}_W^{-1} \circ R_W$. Note also that as indicated in the picture,

$$R_W(W \cap \Omega) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi_W(\mathbf{u}')\}.$$

Since $\Gamma = \partial\Omega$ is compact, there exist finitely many of these open sets, W , denoted by $\{W_i\}_{i=1}^q$ such that $\Gamma \subseteq \bigcup_{i=1}^q W_i$. Let the corresponding sets, U' be denoted by U'_i and let the functions, ϕ be denoted by ϕ_i . Also let $\mathbf{h}_i = \mathbf{h}_{W_i}$ etc. Now let $\{\psi_i\}_{i=1}^q$ be a C^∞ partition of unity subordinate to the $\{W_i\}_{i=1}^q$. If $u \in H^t(\Omega)$, then by Corollary 39.3.5 on Page 1150 it follows that $u\psi_i \in H^t(W_i \cap \Omega)$. Now

$$\mathbf{H}_i : U_i \equiv \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U'_i, 0 < u_n < \phi_i(\mathbf{u}')\} \rightarrow W_i \cap \Omega$$

and $\mathbf{H}_i$ and its inverse are defined on $\mathbb{R}^n$ and are in $C^{m,1}(\mathbb{R}^n)$. Therefore, by Lemma 39.3.3 on Page 1149,

$$\mathbf{H}_i^* \in \mathcal{L}(H^t(W_i \cap \Omega), H^t(U_i)).$$

Provide $t = m + s$ where $s > 0$.

Now it is possible to define the trace on $\Gamma \equiv \partial\Omega$. For $u \in H^t(\Omega)$,

$$\gamma u \equiv \sum_{i=1}^q \mathbf{g}_i^*(\gamma \mathbf{H}_i^*(u\psi_i)). \quad (39.6.25)$$

I must show it satisfies what it should. Recall the definition of what it means for a function to be in $H^{t-1/2}(\Gamma)$ where $t = m + s$.

Definition 39.6.8 Let $s \in (0, 1)$ and m is a nonnegative integer. Also let μ denote the surface measure for Γ . A μ measurable function, u is in $H^{m+s}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{g}_i\}_{i=1}^\infty$ is described above, $\mathbf{h}_i^*(u\psi_i) \in H^{m+s}(U_i)$ and

$$\|u\|_{H^{m+s}(\Gamma)} \equiv \left(\sum_{i=1}^\infty \|\mathbf{h}_i^*(u\psi_i)\|_{H^{m+s}(U_i)}^2 \right)^{1/2} < \infty.$$

Recall that all these norms which are obtained from various partitions of unity and functions, $\mathbf{h}_i$ and $\mathbf{g}_i$, are equivalent. Here there are only finitely many W_i so the sum is a finite sum. The theorem is the following.

Theorem 39.6.9 *Let Ω be a bounded open set having $C^{m,1}$ boundary as discussed above in Definition 39.6.7. Then for $t \leq m+1$, there exists a unique*

$$\gamma \in \mathcal{L} \left(H^t(\Omega), H^{t-1/2}(\Gamma) \right)$$

which has the property that for μ the measure on the boundary,

$$\gamma u(\mathbf{x}) = u(\mathbf{x}) \text{ for } \mu \text{ a.e. } \mathbf{x} \in \Gamma \text{ whenever } u \in \mathfrak{S}|_{\Omega}. \quad (39.6.26)$$

Proof: First consider the claim that $\gamma \in \mathcal{L} \left(H^t(\Omega), H^{t-1/2}(\Gamma) \right)$. This involves first showing that for $u \in H^t(\Omega)$, $\gamma u \in H^{t-1/2}(\Gamma)$. To do this, use the above definition.

$$\begin{aligned} \mathbf{h}_j^* (\psi_j (\gamma u)) &= \sum_{i=1}^q \mathbf{h}_j^* (\psi_j \mathbf{g}_i^* (\gamma \mathbf{H}_i^* (u\psi_i))) \\ &= \sum_{i=1}^q (\mathbf{h}_j^* \psi_j) (\mathbf{h}_j^* (\mathbf{g}_i^* (\gamma \mathbf{H}_i^* (u\psi_i)))) \\ &= \sum_{i=1}^q (\mathbf{h}_j^* \psi_j) (\mathbf{g}_i \circ \mathbf{h}_j)^* (\gamma \mathbf{H}_i^* (u\psi_i)) \end{aligned} \quad (39.6.27)$$

First note that

$$\gamma \mathbf{H}_i^* (u\psi_i) \in H^{t-1/2}(U'_i)$$

Now $\mathbf{g}_i \circ \mathbf{h}_j$ and its inverse, $\mathbf{g}_j \circ \mathbf{h}_i$ are both functions in $C^{m,1}(\mathbb{R}^{n-1})$ and

$$\mathbf{g}_i \circ \mathbf{h}_j : U'_j \rightarrow U'_i.$$

Therefore, by Lemma 39.3.3 on Page 1149,

$$(\mathbf{g}_i \circ \mathbf{h}_j)^* (\gamma \mathbf{H}_i^* (u\psi_i)) \in H^{t-1/2}(U'_j)$$

and

$$\|(\mathbf{g}_i \circ \mathbf{h}_j)^* (\gamma \mathbf{H}_i^* (u\psi_i))\|_{H^{t-1/2}(U'_j)} \leq C_{ij} \|\gamma \mathbf{H}_i^* (u\psi_i)\|_{H^{t-1/2}(U'_i)}.$$

Also $\mathbf{h}_j^* \psi_j \in C^{m,1}(U'_j)$ and has compact support in U'_j and so by Corollary 39.3.5 on Page 1150

$$(\mathbf{h}_j^* \psi_j) (\mathbf{g}_i \circ \mathbf{h}_j)^* (\gamma \mathbf{H}_i^* (u\psi_i)) \in H^{t-1/2}(U'_j)$$

and

$$\begin{aligned} &\|(\mathbf{h}_j^* \psi_j) (\mathbf{g}_i \circ \mathbf{h}_j)^* (\gamma \mathbf{H}_i^* (u\psi_i))\|_{H^{t-1/2}(U'_j)} \\ &\leq C_{ij} \|(\mathbf{g}_i \circ \mathbf{h}_j)^* (\gamma \mathbf{H}_i^* (u\psi_i))\|_{H^{t-1/2}(U'_j)} \end{aligned} \quad (39.6.28)$$

$$\leq C_{ij} \|\gamma \mathbf{H}_i^* (u\psi_i)\|_{H^{t-1/2}(U'_i)}. \quad (39.6.29)$$

This shows $\gamma u \in H^{t-1/2}(\Gamma)$ because each $\mathbf{h}_j^*(\psi_j(\gamma u)) \in H^{t-1/2}(U'_j)$. Also from 39.6.29 and 39.6.27

$$\begin{aligned}
 \|\gamma u\|_{H^{t-1/2}(\Gamma)}^2 &\leq \sum_{j=1}^q \|\mathbf{h}_j^*(\psi_j(\gamma u))\|_{H^{t-1/2}(U'_j)}^2 \\
 &= \sum_{j=1}^q \|\mathbf{h}_j^*(\psi_j(\gamma u))\|_{H^{t-1/2}(U'_j)}^2 \\
 &= \sum_{j=1}^q \left\| \sum_{i=1}^q (\mathbf{h}_j^* \psi_j)(\mathbf{g}_i \circ \mathbf{h}_j)^*(\gamma \mathbf{H}_i^*(u\psi_i)) \right\|_{H^{t-1/2}(U'_j)}^2 \\
 &\leq C_q \sum_{j=1}^q \sum_{i=1}^q \|(\mathbf{h}_j^* \psi_j)(\mathbf{g}_i \circ \mathbf{h}_j)^*(\gamma \mathbf{H}_i^*(u\psi_i))\|_{H^{t-1/2}(U'_j)}^2 \\
 &\leq C_q \sum_{j=1}^q \sum_{i=1}^q C_{ij} \|(\gamma \mathbf{H}_i^*(u\psi_i))\|_{H^{t-1/2}(U'_i)}^2 \\
 &\leq C_q \sum_{i=1}^q \|(\gamma \mathbf{H}_i^*(u\psi_i))\|_{H^{t-1/2}(U'_i)}^2 \\
 &\leq C_q \sum_{i=1}^q \|\mathbf{H}_i^*(u\psi_i)\|_{H^t(R_i(W_i \cap \Omega))}^2 \\
 &\leq C_q \sum_{i=1}^q \|u\psi_i\|_{H^t(W_i \cap \Omega)}^2 \leq C_q \|u\|_{H^t(\Omega)}^2.
 \end{aligned}$$

Does γ satisfy 39.6.26? Let $\mathbf{x} \in \Gamma$ and $u \in \mathfrak{S}|_\Omega$. Let

$$I_{\mathbf{x}} \equiv \{i \in \{1, 2, \dots, q\} : \mathbf{x} = \mathbf{h}_i(\mathbf{u}'_i) \text{ for some } \mathbf{u}'_i \in U'_i\}.$$

Then

$$\begin{aligned}
 \gamma u(\mathbf{x}) &= \sum_{i \in I_{\mathbf{x}}} (\gamma \mathbf{H}_i^*(u\psi_i))(\mathbf{g}_i(\mathbf{x})) \\
 &= \sum_{i \in I_{\mathbf{x}}} (\gamma \mathbf{H}_i^*(u\psi_i))(\mathbf{g}_i(\mathbf{h}_i(\mathbf{u}'_i))) \\
 &= \sum_{i \in I_{\mathbf{x}}} (\gamma \mathbf{H}_i^*(u\psi_i))(\mathbf{u}'_i).
 \end{aligned}$$

Now because $\mathbf{H}_i$ is Lipschitz continuous and $u\psi \in \mathfrak{S}$, it follows that $\mathbf{H}_i^*(u\psi_i) \in H^1(\mathbb{R}^n)$ and is continuous and so by Theorem 39.5.7 on Page 1159 for a.e. $\mathbf{u}'_i$,

$$\begin{aligned}
 &= \sum_{i \in I_{\mathbf{x}}} \mathbf{H}_i^*(u\psi_i)(\mathbf{u}'_i, 0) \\
 &= \sum_{i \in I_{\mathbf{x}}} \mathbf{h}_i^*(u\psi_i)(\mathbf{u}'_i) \\
 &= \sum_{i \in I_{\mathbf{x}}} (u\psi_i)(\mathbf{h}_i(\mathbf{u}'_i)) = u(\mathbf{x}) \text{ for } \mu \text{ a.e. } \mathbf{x}.
 \end{aligned} \tag{39.6.30}$$

This verifies 39.6.26 and completes the proof of the theorem.

Weak Solutions

40.1 The Lax Milgram Theorem

The Lax Milgram theorem is a fundamental result which is useful for obtaining weak solutions to many types of partial differential equations. It is really a general theorem in functional analysis.

Definition 40.1.1 *Let $A \in \mathcal{L}(V, V')$ where V is a Hilbert space. Then A is said to be coercive if*

$$A(v)(v) \geq \delta \|v\|^2$$

for some $\delta > 0$.

Theorem 40.1.2 *(Lax Milgram) Let $A \in \mathcal{L}(V, V')$ be coercive. Then A maps one to one and onto.*

Proof: The proof that A is onto involves showing $A(V)$ is both dense and closed.

Consider first the claim that $A(V)$ is closed. Let $Ax_n \rightarrow y^* \in V'$. Then

$$\delta \|x_n - x_m\|_V^2 \leq \|Ax_n - Ax_m\|_{V'} \|x_n - x_m\|_V.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in V . It follows $x_n \rightarrow x \in V$ and since A is continuous, $Ax_n \rightarrow Ax$. This shows $A(V)$ is closed.

Now let $R : V \rightarrow V'$ denote the Riesz map defined by $Rx(y) = (y, x)$. Recall that the Riesz map is one to one, onto, and preserves norms. Therefore, $R^{-1}(A(V))$ is a closed subspace of V . If there $R^{-1}(A(V)) \neq V$, then $(R^{-1}(A(V)))^\perp \neq \{0\}$. Let $x \in (R^{-1}(A(V)))^\perp$ and $x \neq 0$. Then in particular,

$$0 = (x, R^{-1}Ax) = R(R^{-1}(A(x)))(x) = A(x)(x) \geq \delta \|x\|_V^2,$$

a contradiction to $x \neq 0$. Therefore, $R^{-1}(A(V)) = V$ and so $A(V) = R(V) = V'$.

Since $A(V)$ is both closed and dense, $A(V) = V'$. This shows A is onto.

If $Ax = Ay$, then $0 = A(x - y)(x - y) \geq \delta \|x - y\|_V^2$, and this shows A is one to one. This proves the theorem.

Here is a simple example which illustrates the use of the above theorem. In the example the repeated index summation convention is being used. That is, you sum over the repeated indices.

Example 40.1.3 Let U be an open subset of $\mathbb{R}^n$ and let V be a closed subspace of $H^1(U)$. Let $\alpha^{ij} \in L^\infty(U)$ for $i, j = 1, 2, \dots, n$. Now define $A : V \rightarrow V'$ by

$$A(u)(v) \equiv \int_U (\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) + u(\mathbf{x}) v(\mathbf{x})) dx.$$

Suppose also that

$$\alpha^{ij} v_i v_j \geq \delta |\mathbf{v}|^2$$

whenever $\mathbf{v} \in \mathbb{R}^n$. Then A maps V to V' one to one and onto.

Here is why. It is obvious that A is in $\mathcal{L}(V, V')$. It only remains to verify that it is coercive.

$$\begin{aligned} A(u)(u) &\equiv \int_U (\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) u_{,j}(\mathbf{x}) + u(\mathbf{x}) u(\mathbf{x})) dx \\ &\geq \int_U \delta |\nabla u(\mathbf{x})|^2 + |u(\mathbf{x})|^2 dx \\ &\geq \delta \|u\|_{H^1(U)}^2 \end{aligned}$$

This proves coercivity and verifies the claim.

What has been obtained in the above example? This depends on how you choose V . In Example 40.1.3 suppose U is a bounded open set with $C^{0,1}$ boundary and $V = H_0^1(U)$ where

$$H_0^1(U) \equiv \{u \in H^1(U) : \gamma u = 0\}.$$

Also suppose $f \in L^2(U)$. Then you can consider $F \in V'$ by defining

$$F(v) \equiv \int_U f(\mathbf{x}) v(\mathbf{x}) dx.$$

According to the Lax Milgram theorem and the verification of its conditions in Example 40.1.3, there exists a unique solution to the problem of finding $u \in H_0^1(U)$ such that for all $v \in H_0^1(U)$,

$$\int_U (\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) + u(\mathbf{x}) v(\mathbf{x})) dx = \int_U f(\mathbf{x}) v(\mathbf{x}) dx \quad (40.1.1)$$

In particular, this holds for all $v \in C_c^\infty(U)$. Thus for all such v ,

$$\int_U \left(-(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}))_{,j} + u(\mathbf{x}) - f(\mathbf{x}) \right) v(\mathbf{x}) dx = 0.$$

Therefore, in terms of weak derivatives,

$$-(\alpha^{ij} u_{,i})_{,j} + u = f$$

and since $u \in H_0^1(U)$, it must be the case that $\gamma u = 0$ on ∂U . This is why the solution to 40.1.1 is referred to as a weak solution to the boundary value problem

$$-(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}))_{,j} + u(\mathbf{x}) = f(\mathbf{x}), \quad u = 0 \text{ on } \partial U.$$

Of course you then begin to ask the important question whether u really has two derivatives. It is not immediately clear that just because $-(\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}))_{,j} \in L^2(U)$ it follows that the second derivatives of u exist. Actually this will often be true and is discussed somewhat in the next section.

Next suppose you choose $V = H^1(U)$ and let $g \in H^{1/2}(\partial U)$. Define $F \in V'$ by

$$F(v) \equiv \int_U f(\mathbf{x}) v(\mathbf{x}) dx + \int_{\partial U} g(\mathbf{x}) \gamma v(\mathbf{x}) d\mu.$$

Everything works the same way and you get the existence of a unique $u \in H^1(U)$ such that for all $v \in H^1(U)$,

$$\int_U (\alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) + u(\mathbf{x}) v(\mathbf{x})) dx = \int_U f(\mathbf{x}) v(\mathbf{x}) dx + \int_{\partial U} g(\mathbf{x}) \gamma v(\mathbf{x}) d\mu \quad (40.1.2)$$

is satisfied. If you pretend u has all second order derivatives in $L^2(U)$ and apply the divergence theorem, you find that you have obtained a weak solution to

$$-(\alpha^{ij} u_{,i})_{,j} + u = f, \quad \alpha^{ij} u_{,i} n_j = g \text{ on } \partial U$$

where n_j is the j^{th} component of $\mathbf{n}$, the unit outer normal. Therefore, u is a weak solution to the above boundary value problem.

The conclusion is that the Lax Milgram theorem gives a way to obtain existence and uniqueness of weak solutions to various boundary value problems. The following theorem is often very useful in establishing coercivity. To prove this theorem, here is a definition.

Definition 40.1.4 Let U be an open set and $\delta > 0$. Then

$$U_\delta \equiv \{\mathbf{x} \in U : \text{dist}(\mathbf{x}, U^C) > \delta\}.$$

Theorem 40.1.5 Let U be a connected bounded open set having $C^{0,1}$ boundary such that for some sequence, $\eta_k \downarrow 0$,

$$U = \cup_{k=1}^{\infty} U_{\eta_k} \quad (40.1.3)$$

and U_{η_k} is a connected open set. Suppose $\Gamma \subseteq \partial U$ has positive surface measure and that

$$V \equiv \{u \in H^1(U) : \gamma u = 0 \text{ a.e. on } \Gamma\}.$$

Then the norm $|||\cdot|||$ given by

$$|||u||| \equiv \left(\int_U |\nabla u|^2 dx \right)^{1/2}$$

is equivalent to the usual norm on V .

Proof: First it is necessary to verify this is actually a norm. It clearly satisfies all the usual axioms of a norm except for the condition that $|||u||| = 0$ if and only if $u = 0$. Suppose then that $|||u||| = 0$. Let $\delta_0 = \eta_k$ for one of those η_k mentioned above and define

$$u_\delta(\mathbf{x}) \equiv \int_{B(\mathbf{0}, \delta)} u(\mathbf{x} - \mathbf{y}) \phi_\delta(\mathbf{y}) d\mathbf{y}$$

where ϕ_δ is a mollifier having support in $B(\mathbf{0}, \delta)$. Then changing the variables, it follows that for $\mathbf{x} \in U_{\delta_0}$

$$u_\delta(\mathbf{x}) = \int_{B(\mathbf{x}, \delta)} u(\mathbf{t}) \phi_\delta(\mathbf{x} - \mathbf{t}) d\mathbf{t} = \int_U u(\mathbf{t}) \phi_\delta(\mathbf{x} - \mathbf{t}) d\mathbf{t}$$

and so $u_\delta \in C^\infty(U_{\delta_0})$ and

$$\nabla u_\delta(\mathbf{x}) = \int_U u(\mathbf{t}) \nabla \phi_\delta(\mathbf{x} - \mathbf{t}) d\mathbf{t} = \int_{B(\mathbf{0}, \delta)} \nabla u(\mathbf{x} - \mathbf{y}) \phi_\delta(\mathbf{y}) d\mathbf{y} = 0.$$

Therefore, u_δ equals a constant on U_{δ_0} because U_{δ_0} is a connected open set and u_δ is a smooth function defined on this set which has its gradient equal to $\mathbf{0}$. By Minkowski's inequality,

$$\begin{aligned} & \left(\int_{U_{\delta_0}} |u(\mathbf{x}) - u_\delta(\mathbf{x})|^2 d\mathbf{x} \right)^{1/2} \\ & \leq \int_{B(\mathbf{0}, \delta)} \phi_\delta(\mathbf{y}) \left(\int_{U_{\delta_0}} |u(\mathbf{x}) - u(\mathbf{x} - \mathbf{y})|^2 d\mathbf{x} \right)^{1/2} d\mathbf{y} \end{aligned}$$

and this converges to 0 as $\delta \rightarrow 0$ by continuity of translation in L^2 . It follows there exists a sequence of constants, $c_\delta \equiv u_\delta(\mathbf{x})$ such that $\{c_\delta\}$ converges to u in $L^2(U_{\delta_0})$. Consequently, a subsequence, still denoted by u_δ , converges to u a.e. By Eggoroff's theorem there exists a set, N_k having measure no more than $3^{-k} m_n(U_{\delta_0})$ such that u_δ converges to u uniformly on N_k^C . Thus u is constant on N_k^C . Now $\sum_k m_n(N_k) \leq \frac{1}{2} m_n(U_{\delta_0})$ and so there exists $\mathbf{x}_0 \in U_{\delta_0} \setminus \cup_{k=1}^\infty N_k$. Therefore, if $\mathbf{x} \notin N_k$ it follows $u(\mathbf{x}) = u(\mathbf{x}_0)$ and so, if $u(\mathbf{x}) \neq u(\mathbf{x}_0)$ it must be the case that $\mathbf{x} \in \cap_{k=1}^\infty N_k$, a set of measure zero. This shows that u equals a constant a.e. on $U_{\delta_0} = U_{\eta_k}$. Since k is arbitrary, 40.1.3 shows u is a.e. equal to a constant on U . Therefore, u equals the restriction of a function of $\mathfrak{S}$ to U and so γu equals this constant in $L^2(\partial\Omega)$. Since the surface measure of Γ is positive, the constant must equal zero. Therefore, $|||\cdot|||$ is a norm.

It remains to verify that it is equivalent to the usual norm. It is clear that $|||u||| \leq \|u\|_{1,2}$. What about the other direction? Suppose it is not true that for some constant, K , $\|u\|_{1,2} \leq K |||u|||$. Then for every $k \in \mathbb{N}$, there exists $u_k \in V$ such that

$$\|u_k\|_{1,2} > k |||u_k|||.$$

Replacing u_k with $u_k/\|u_k\|_{1,2}$, it can be assumed that $\|u_k\|_{1,2} = 1$ for all k . Therefore, using the compactness of the embedding of $H^1(U)$ into $L^2(U)$, there exists a subsequence, still denoted by u_k such that

$$u_k \rightarrow u \text{ weakly in } V, \quad (40.1.4)$$

$$u_k \rightarrow u \text{ strongly in } L^2(U), \quad (40.1.5)$$

$$\|u_k\| \rightarrow 0, \quad (40.1.6)$$

$$u_k \rightarrow u \text{ weakly in } (V, \|\cdot\|). \quad (40.1.7)$$

From 40.1.6 and 40.1.7, it follows $u = 0$. Therefore, $\|u_k\|_{L^2(U)} \rightarrow 0$. This with 40.1.6 contradicts the fact that $\|u_k\|_{1,2} = 1$ and this proves the equivalence of the two norms.

The proof of the above theorem yields the following interesting corollary.

Corollary 40.1.6 *Let U be a connected open set with the property that for some sequence, $\eta_k \downarrow 0$,*

$$U = \bigcup_{k=1}^{\infty} U_{\eta_k}$$

for U_{η_k} a connected open set and suppose $u \in W^{1,p}(U)$ and $\nabla u = 0$ a.e. Then u equals a constant a.e.

Example 40.1.7 *Let U be a bounded open connected subset of $\mathbb{R}^n$ and let V be a closed subspace of $H^1(U)$ defined by*

$$V \equiv \{u \in H^1(U) : \gamma u = 0 \text{ on } \Gamma\}$$

where the surface measure of Γ is positive.

Let $\alpha^{ij} \in L^\infty(U)$ for $i, j = 1, 2, \dots, n$ and define $A : V \rightarrow V'$ by

$$A(u)(v) \equiv \int_U \alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx.$$

for

$$\alpha^{ij} v_i v_j \geq \delta |\mathbf{v}|^2$$

whenever $\mathbf{v} \in \mathbb{R}^n$. Then A maps V to V' one to one and onto.

This follows from Theorem 40.1.5 using the equivalent norm defined there. Define $F \in V'$ by

$$\int_U f(\mathbf{x}) v(\mathbf{x}) dx + \int_{\partial U \setminus \Gamma} g(\mathbf{x}) \gamma v(\mathbf{x}) dx$$

for $f \in L^2(U)$ and $g \in H^{1/2}(\partial U)$. Then the equation,

$$Au = F \text{ in } V'$$

which is equivalent to $u \in V$ and for all $v \in V$,

$$\int_U \alpha^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx = \int_U f(\mathbf{x}) v(\mathbf{x}) dx + \int_{\partial U \setminus \Gamma} g(\mathbf{x}) \gamma v(\mathbf{x}) d\mu$$

is a weak solution for the boundary value problem,

$$-(\alpha^{ij}u_{,i})_{,j} = f \text{ in } U, \quad \alpha^{ij}u_{,i}n_j = g \text{ on } \partial U \setminus \Gamma, \quad u = 0 \text{ on } \Gamma$$

as you can verify by using the divergence theorem formally.

Korn's Inequality

A fundamental inequality used in elasticity to obtain coercivity and then apply the Lax Milgram theorem or some other theorem is Korn's inequality. The proof given here of this fundamental result follows [53] and [23].

41.1 A Fundamental Inequality

The proof of Korn's inequality depends on a fundamental inequality involving negative Sobolev space norms. The theorem to be proved is the following.

Theorem 41.1.1 *Let $f \in L^2(\Omega)$ where Ω is a bounded Lipschitz domain. Then there exist constants, C_1 and C_2 such that*

$$C_1 \|f\|_{0,2,\Omega} \leq \left(\|f\|_{-1,2,\Omega} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right) \leq C_2 \|f\|_{0,2,\Omega},$$

where here $\|\cdot\|_{0,2,\Omega}$ represents the L^2 norm and $\|\cdot\|_{-1,2,\Omega}$ represents the norm in the dual space of $H_0^1(\Omega)$, denoted by $H^{-1}(\Omega)$.

Similar conventions will apply for any domain in place of Ω . The proof of this theorem will proceed through the use of several lemmas.

Lemma 41.1.2 *Let U^- denote the set,*

$$\{(\mathbf{x}, x_n) \in \mathbb{R}^n : x_n < g(\mathbf{x})\}$$

where $g : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ is Lipschitz and denote by U^+ the set

$$\{(\mathbf{x}, x_n) \in \mathbb{R}^n : x_n > g(\mathbf{x})\}.$$

Let $f \in L^2(U^-)$ and extend f to all of $\mathbb{R}^n$ in the following way.

$$f(\mathbf{x}, x_n) \equiv -3f(\mathbf{x}, 2g(\mathbf{x}) - x_n) + 4f(\mathbf{x}, 3g(\mathbf{x}) - 2x_n).$$

Then there is a constant, C_g , depending on g such that

$$\|f\|_{-1,2,\mathbb{R}^n} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} \leq C_g \left(\|f\|_{-1,2,U^-} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,U^-} \right).$$

Proof: Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then,

$$\begin{aligned} \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_n} dx &= \int_{U^+} \frac{\partial \phi}{\partial x_n} [-3f(\mathbf{x}, 2g(\mathbf{x}) - x_n) + 4f(\mathbf{x}, 3g(\mathbf{x}) - 2x_n)] dx \\ &\quad + \int_{U^-} f \frac{\partial \phi}{\partial x_n} dx. \end{aligned} \quad (41.1.1)$$

Consider the first integral on the right in 41.1.1. Changing the variables, letting $y_n = 2g(\mathbf{x}) - x_n$ in the first term of the integrand and $3g(\mathbf{x}) - 2x_n$ in the next, it equals

$$\begin{aligned} &-3 \int_{U^-} \frac{\partial \phi}{\partial x_n}(\mathbf{x}, 2g(\mathbf{x}) - y_n) f(\mathbf{x}, y_n) dy_n dx \\ &+ 2 \int_{U^-} \frac{\partial \phi}{\partial x_n} \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right) f(\mathbf{x}, y_n) dy_n dx. \end{aligned}$$

For $(\mathbf{x}, y_n) \in U^-$, and defining

$$\psi(\mathbf{x}, y_n) \equiv \phi(\mathbf{x}, y_n) + 3\phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) - 4\phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right),$$

it follows $\psi = 0$ when $y_n = g(\mathbf{x})$ and so

$$\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_n} dx = \int_{U^-} \frac{\partial \psi}{\partial y_n} f(\mathbf{x}, y_n) dx dy_n.$$

Now from the definition of ψ given above,

$$\|\psi\|_{1,2,U^-} \leq C_g \|\phi\|_{1,2,U^-} \leq C_g \|\phi\|_{1,2,\mathbb{R}^n}$$

and so

$$\begin{aligned} &\left\| \frac{\partial f}{\partial x_n} \right\|_{-1,2,\mathbb{R}^n} \equiv \\ &\sup \left\{ \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_n} dx : \phi \in C_c^\infty(\mathbb{R}^n), \|\phi\|_{1,2,\mathbb{R}^n} \leq 1 \right\} \leq \\ &\sup \left\{ \left| \int_{U^-} f \frac{\partial \psi}{\partial x_n} dx dy_n \right| : \psi \in H_0^1(U^-), \|\psi\|_{1,2,U^-} \leq C_g \right\} \\ &= C_g \left\| \frac{\partial f}{\partial x_n} \right\|_{-1,2,U^-} \end{aligned} \quad (41.1.2)$$

It remains to establish a similar inequality for the case where the derivatives are taken with respect to x_i for $i < n$. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} dx = \int_{U^-} f \frac{\partial \phi}{\partial x_i} dx$$

$$\int_{U^+} \frac{\partial \phi}{\partial x_i} [-3f(\mathbf{x}, g(\mathbf{x}) - x_n) + 4f(\mathbf{x}, 3g(\mathbf{x}) - 2x_n)] dx.$$

Changing the variables as before, this last integral equals

$$\begin{aligned} & -3 \int_{U^-} D_i \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) f(\mathbf{x}, y_n) dy_n dx \\ & + 2 \int_{U^-} D_i \phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right) f(\mathbf{x}, y_n) dy_n dx. \end{aligned} \quad (41.1.3)$$

Now let

$$\psi_1(\mathbf{x}, y_n) \equiv \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n), \quad \psi_2(\mathbf{x}, y_n) \equiv \phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right).$$

Then

$$\begin{aligned} \frac{\partial \psi_1}{\partial x_i} &= D_i \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) + D_n \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) 2D_i g(\mathbf{x}), \\ \frac{\partial \psi_2}{\partial x_i} &= D_i \phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right) + D_n \phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right) \frac{3}{2}D_i g(\mathbf{x}). \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial \psi_1}{\partial y_n}(\mathbf{x}, y_n) &= -D_n \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n), \\ \frac{\partial \psi_2}{\partial y_n}(\mathbf{x}, y_n) &= \left(\frac{-1}{2} \right) D_n \phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \psi_1}{\partial x_i}(\mathbf{x}, y_n) &= D_i \phi(\mathbf{x}, 2g(\mathbf{x}) - y_n) - 2 \frac{\partial \psi_1}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}), \\ \frac{\partial \psi_2}{\partial x_i}(\mathbf{x}, y_n) &= D_i \phi \left(\mathbf{x}, \frac{3}{2}g(\mathbf{x}) - \frac{y_n}{2} \right) - 3 \frac{\partial \psi_2}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}). \end{aligned}$$

Using this in 41.1.3, the integrals in this expression equal

$$\begin{aligned} & -3 \int_{U^-} \left[\frac{\partial \psi_1}{\partial x_i}(\mathbf{x}, y_n) + 2 \frac{\partial \psi_1}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}) \right] f(\mathbf{x}, y_n) dy_n dx + \\ & 2 \int_{U^-} \left[\frac{\partial \psi_2}{\partial x_i}(\mathbf{x}, y_n) + 3 \frac{\partial \psi_2}{\partial y_n}(\mathbf{x}, y_n) D_i g(\mathbf{x}) \right] f(\mathbf{x}, y_n) dy_n dx \end{aligned}$$

$$= \int_{U^-} \left[-3 \frac{\partial \psi_1(\mathbf{x}, y)}{\partial x_i} + 2 \frac{\partial \psi_2(\mathbf{x}, y_n)}{\partial x_i} \right] f(\mathbf{x}, y_n) dy_n dx.$$

Therefore,

$$\int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} f dx = \int_{U^-} \left[\frac{\partial \phi}{\partial x_i} - 3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] f dx dy_n$$

and also

$$\begin{aligned} \phi(\mathbf{x}, g(\mathbf{x})) - 3\psi_1(\mathbf{x}, g(\mathbf{x})) + 2\psi_2(\mathbf{x}, g(\mathbf{x})) = \\ \phi(\mathbf{x}, g(\mathbf{x})) - 3\phi(\mathbf{x}, g(\mathbf{x})) + 2\phi(\mathbf{x}, g(\mathbf{x})) = 0 \end{aligned}$$

and so $\phi - 3\psi_1 + 2\psi_2 \in H_0^1(U^-)$. It also follows from the definition of the functions, ψ_i and the assumption that g is Lipschitz, that

$$\|\psi_i\|_{1,2,U^-} \leq C_g \|\phi\|_{1,2,U^-} \leq C_g \|\phi\|_{1,2,\mathbb{R}^n}. \quad (41.1.4)$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} &\equiv \sup \left\{ \left| \int_{\mathbb{R}^n} f \frac{\partial \phi}{\partial x_i} dx \right| : \|\phi\|_{1,2,\mathbb{R}^n} \leq 1 \right\} \\ &= \sup \left\{ \left| \int_{U^-} f \left[\frac{\partial \phi}{\partial x_i} - 3 \frac{\partial \psi_1}{\partial x_i} + 2 \frac{\partial \psi_2}{\partial x_i} \right] dx \right| : \|\phi\|_{1,2,\mathbb{R}^n} \leq 1 \right\} \\ &\leq C_g \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,U^-} \end{aligned}$$

where C_g is a constant which depends on g . This inequality along with 41.1.2 yields

$$\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} \leq C_g \left(\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,U^-} \right).$$

The inequality,

$$\|f\|_{-1,2,\mathbb{R}^n} \leq C_g \|f\|_{-1,2,U^-}$$

follows from 41.1.4 and the equation,

$$\begin{aligned} \int_{\mathbb{R}^n} f \phi dx &= \int_{U^-} f \phi dx - 3 \int_{U^-} f(\mathbf{x}, y_n) \psi_1(\mathbf{x}, y_n) dx dy_n \\ &\quad + 2 \int_{U^-} f(\mathbf{x}, y_n) \psi_2(\mathbf{x}, y_n) dx dy_n \end{aligned}$$

which results in the same way as before by changing variables using the definition of f off U^- . This proves the lemma.

The next lemma is a simple application of Fourier transforms.

Lemma 41.1.3 *If $f \in L^2(\mathbb{R}^n)$, then the following formula holds.*

$$C_n \|f\|_{0,2,\mathbb{R}^n} = \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} + \|f\|_{-1,2,\mathbb{R}^n}$$

Proof: For $\phi \in C_c^\infty(\mathbb{R}^n)$

$$\|\phi\|_{1,2,\mathbb{R}^n} \equiv \left(\int_{\mathbb{R}^n} (1 + |\mathbf{t}|^2) |F\phi|^2 dt \right)^{1/2}$$

is an equivalent norm to the usual Sobolev space norm for $H_0^1(\mathbb{R}^n)$ and is used in the following argument which depends on Plancherel's theorem and the fact that $F\left(\frac{\partial\phi}{\partial x_i}\right) = t_i F(\phi)$.

$$\begin{aligned} \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} &\equiv \sup \left\{ \left| \int_{\mathbb{R}^n} \frac{\partial \phi}{\partial x_i} \bar{f} dx \right| : \|\phi\|_{1,2} \leq 1 \right\} \\ &= C_n \sup \left\{ \left| \int_{\mathbb{R}^n} t_i (F\phi) \overline{(Ff)} dt \right| : \|\phi\|_{1,2} \leq 1 \right\} \\ &= C_n \sup \left\{ \left| \int_{\mathbb{R}^n} \frac{t_i (F\phi) (1 + |\mathbf{t}|^2)^{1/2}}{(1 + |\mathbf{t}|^2)^{1/2}} \overline{(Ff)} dt \right| : \|\phi\|_{1,2} \leq 1 \right\} \\ &= C_n \left(\int \frac{|Ff|^2 t_i^2}{(1 + |\mathbf{t}|^2)} dt \right)^{1/2} \end{aligned} \quad (41.1.5)$$

Also,

$$\begin{aligned} \|f\|_{-1,2} &\equiv \sup \left\{ \left| \int_{\mathbb{R}^n} \phi \bar{f} dx \right| : \|\phi\|_{1,2} \leq 1 \right\} \\ &= C_n \sup \left\{ \left| \int_{\mathbb{R}^n} (F\phi) \overline{(Ff)} dx \right| : \|\phi\|_{1,2} \leq 1 \right\} \\ &= C_n \sup \left\{ \left| \int_{\mathbb{R}^n} \frac{F\phi (1 + |\mathbf{t}|^2)^{1/2}}{(1 + |\mathbf{t}|^2)^{1/2}} \overline{(Ff)} dt \right| : \|\phi\|_{1,2} \leq 1 \right\} \\ &= C_n \left(\int_{\mathbb{R}^n} \frac{|Ff|^2}{(1 + |\mathbf{t}|^2)} dt \right)^{1/2} \end{aligned}$$

This along with 41.1.5 yields the conclusion of the lemma because

$$\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2}^2 + \|f\|_{-1,2}^2 = C_n \int_{\mathbb{R}^n} |Ff|^2 dx = C_n \|f\|_{0,2}^2.$$

Now consider Theorem 41.1.1. First note that by Lemma 41.1.2 and U^- defined there, Lemma 41.1.3 implies that for f extended as in Lemma 41.1.2,

$$\|f\|_{0,2,U^-} \leq \|f\|_{0,2,\mathbb{R}^n} = C_n \left(\|f\|_{-1,2,\mathbb{R}^n} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} \right)$$

$$\leq C_{gn} \left(\|f\|_{-1,2,U^-} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,U^-} \right). \quad (41.1.6)$$

Let Ω be a bounded open set having Lipschitz boundary which lies locally on one side of its boundary. Let $\{Q_i\}_{i=0}^p$ be cubes of the sort used in the proof of the divergence theorem such that $\bar{Q}_0 \subseteq \Omega$ and the other cubes cover the boundary of Ω . Let $\{\psi_i\}$ be a C^∞ partition of unity with $\text{spt}(\psi_i) \subseteq Q_i$ and let $f \in L^2(\Omega)$. Then for $\phi \in C_c^\infty(\Omega)$ and ψ one of these functions in the partition of unity,

$$\left\| \frac{\partial(f\psi)}{\partial x_i} \right\|_{-1,2,\Omega} \leq \sup_{\|\phi\|_{1,2} \leq 1} \left| \int_{\Omega} f \frac{\partial}{\partial x_i} (\psi\phi) dx \right| + \sup_{\|\phi\|_{1,2} \leq 1} \left| \int_{\Omega} f \phi \frac{\partial \psi}{\partial x_i} dx \right|$$

Now if $\|\phi\|_{1,2} \leq 1$, then for a suitable constant, C_ψ ,

$$\|\psi\phi\|_{1,2} \leq C_\psi \|\phi\|_{1,2} \leq C_\psi, \quad \left\| \phi \frac{\partial \psi}{\partial x_i} \right\|_{1,2} \leq C_\psi.$$

Therefore,

$$\begin{aligned} \left\| \frac{\partial(f\psi)}{\partial x_i} \right\|_{-1,2,\Omega} &\leq \sup_{\|\eta\|_{1,2} \leq C_\psi} \left| \int_{\Omega} f \frac{\partial \eta}{\partial x_i} dx \right| + \sup_{\|\eta\|_{1,2} \leq C_\psi} \left| \int_{\Omega} f \eta dx \right| \\ &\leq C_\psi \left(\left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} + \|f\|_{-1,2,\Omega} \right). \end{aligned} \quad (41.1.7)$$

Now using 41.1.7 and 41.1.6

$$\begin{aligned} \|f\psi_j\|_{0,2,\Omega} &\leq C_g \left(\|f\psi_j\|_{-1,2,\Omega} + \sum_{i=1}^n \left\| \frac{\partial(f\psi_j)}{\partial x_i} \right\|_{-1,2,\Omega} \right) \\ &\leq C_{\psi_j} C_g \left(\|f\|_{-1,2,\Omega} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right). \end{aligned}$$

Therefore, letting $C = \sum_{j=1}^p C_{\psi_j} C_g$,

$$\|f\|_{0,2,\Omega} \leq \sum_{j=1}^p \|f\psi_j\|_{0,2,\Omega} \leq C \left(\|f\|_{-1,2,\Omega} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} \right). \quad (41.1.8)$$

This proves the hard half of the inequality of Theorem 41.1.1.

To complete the proof, let $\bar{f}$ denote the zero extension of f off Ω . Then

$$\begin{aligned} \|f\|_{-1,2,\Omega} + \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{-1,2,\Omega} &\leq \|\bar{f}\|_{-1,2,\mathbb{R}^n} + \sum_{i=1}^n \left\| \frac{\partial \bar{f}}{\partial x_i} \right\|_{-1,2,\mathbb{R}^n} \\ &\leq C_n \|\bar{f}\|_{0,2,\mathbb{R}^n} = C_n \|f\|_{0,2,\Omega}. \end{aligned}$$

This along with 41.1.8 proves Theorem 41.1.1.

41.2 Korn's Inequality

The inequality in this section is known as Korn's second inequality. It is also known as coercivity of strains. For $\mathbf{u}$ a vector valued function in $\mathbb{R}^n$, define

$$\varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}(u_{i,j} + u_{j,i})$$

This is known as the strain or small strain. Korn's inequality says that the norm given by,

$$|||\mathbf{u}||| \equiv \left(\sum_{i=1}^n \|u_i\|_{0,2,\Omega}^2 + \sum_{i=1}^n \sum_{j=1}^n \|\varepsilon_{ij}(\mathbf{u})\|_{0,2,\Omega}^2 \right)^{1/2} \quad (41.2.9)$$

is equivalent to the norm,

$$\|\mathbf{u}\| \equiv \left(\sum_{i=1}^n \|u_i\|_{0,2,\Omega}^2 + \sum_{i=1}^n \sum_{j=1}^n \left\| \frac{\partial u_i}{\partial x_j} \right\|_{0,2,\Omega}^2 \right)^{1/2} \quad (41.2.10)$$

It is very significant because it is the strain as just defined which occurs in many of the physical models proposed in continuum mechanics. The inequality is far from obvious because the strains only involve certain combinations of partial derivatives.

Theorem 41.2.1 (*Korn's second inequality*) *Let Ω be any domain for which the conclusion of Theorem 41.1.1 holds. Then the two norms in 41.2.9 and 41.2.10 are equivalent.*

Proof: Let $\mathbf{u}$ be such that $u_i \in H^1(\Omega)$ for each $i = 1, \dots, n$. Note that

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial}{\partial x_j} (\varepsilon_{ik}(\mathbf{u})) + \frac{\partial}{\partial x_k} (\varepsilon_{ij}(\mathbf{u})) - \frac{\partial}{\partial x_i} (\varepsilon_{jk}(\mathbf{u})).$$

Therefore, by Theorem 41.1.1,

$$\begin{aligned} \left\| \frac{\partial u_i}{\partial x_j} \right\|_{0,2,\Omega} &\leq C \left[\left\| \frac{\partial u_i}{\partial x_j} \right\|_{-1,2,\Omega} + \sum_{k=1}^n \left\| \frac{\partial^2 u_i}{\partial x_j \partial x_k} \right\|_{-1,2,\Omega} \right] \\ &\leq C \left[\left\| \frac{\partial u_i}{\partial x_j} \right\|_{-1,2,\Omega} + \sum_{r,s,p} \left\| \frac{\partial \varepsilon_{rs}(\mathbf{u})}{\partial x_p} \right\|_{-1,2,\Omega} \right] \\ &\leq C \left[\left\| \frac{\partial u_i}{\partial x_j} \right\|_{-1,2,\Omega} + \sum_{r,s} \|\varepsilon_{rs}(\mathbf{u})\|_{0,2,\Omega} \right]. \end{aligned}$$

But also by this theorem,

$$\|u_i\|_{-1,2,\Omega} + \sum_p \left\| \frac{\partial u_i}{\partial x_p} \right\|_{-1,2,\Omega} \leq C \|u_i\|_{0,2,\Omega}$$

and so

$$\left\| \frac{\partial u_i}{\partial x_j} \right\|_{0,2,\Omega} \leq C \left[\|u_i\|_{0,2,\Omega} + \sum_{r,s} \|\varepsilon_{rs}(\mathbf{u})\|_{0,2,\Omega} \right]$$

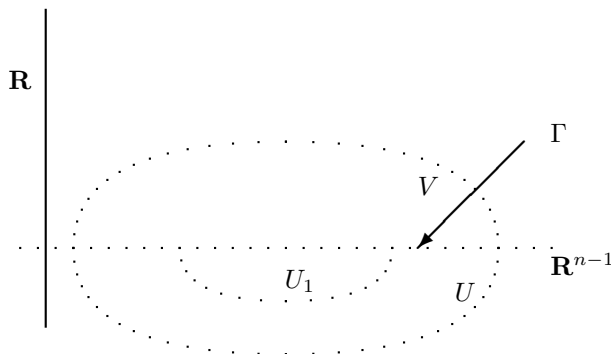
This proves the theorem.

Note that Ω did not need to be bounded. It suffices to be able to conclude the result of Theorem 41.1.1 which would hold whenever the boundary of Ω can be covered with finitely many boxes of the sort to which Lemma 41.1.2 can be applied.

Elliptic Regularity And Nirenberg Differences

42.1 The Case Of A Half Space

Regularity theorems are concerned with obtaining more regularity given a weak solution. This extra regularity is essential in order to obtain error estimates for various problems. In this section a regularity is given for weak solutions to various elliptic boundary value problems. To save on notation, I will use the repeated index summation convention. Thus you sum over repeated indices. Consider the following picture.



Here V is an open set,

$$U \equiv \{ \mathbf{y} \in V : y_n < 0 \}, \Gamma \equiv \{ \mathbf{y} \in V : y_n = 0 \}$$

and U_1 is an open set as shown for which $U_1 \subseteq V \cap U$. Assume also that V is bounded. Suppose

$$\begin{aligned} f &\in L^2(U), \\ \alpha^{rs} &\in C^{0,1}(\overline{U}), \end{aligned} \tag{42.1.1}$$

$$\alpha^{rs}(\mathbf{y}) v_r v_s \geq \delta |\mathbf{v}|^2, \quad \delta > 0. \quad (42.1.2)$$

The following technical lemma gives the essential ideas.

Lemma 42.1.1 *Suppose*

$$w \in H^1(U), \quad (42.1.3)$$

$$\alpha^{rs} \in C^{0,1}(\overline{U}), \quad (42.1.4)$$

$$h_s \in H^1(U), \quad (42.1.5)$$

$$f \in L^2(U). \quad (42.1.6)$$

and

$$\int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy + \int_U h_s(\mathbf{y}) \frac{\partial z}{\partial y^s} dy = \int_U f z dy \quad (42.1.7)$$

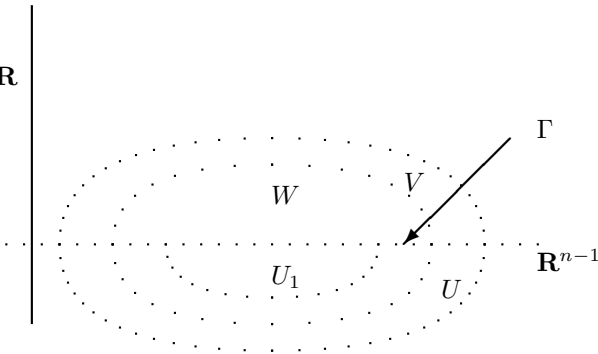
for all $z \in H^1(U)$ having the property that $\text{spt}(z) \subseteq V$. Then $w \in H^2(U_1)$ and for some constant C , independent of f, w , and g , the following estimate holds.

$$\|w\|_{H^2(U_1)}^2 \leq C \left(\|w\|_{H^1(U)}^2 + \|f\|_{L^2(U)}^2 + \sum_s \|h_s\|_{H^1(U)}^2 \right). \quad (42.1.8)$$

Proof: Define for small real h ,

$$D_k^h l(\mathbf{y}) \equiv \frac{1}{h} (l(\mathbf{y} + h\mathbf{e}_k) - l(\mathbf{y})).$$

Let $U_1 \subseteq \overline{U_1} \subseteq W \subseteq \overline{W} \subseteq V$ and let $\eta \in C_c^\infty(W)$ with $\eta(\mathbf{y}) \in [0, 1]$, and $\eta = 1$ on $\overline{U_1}$ as shown in the following picture.



For h small ($3h < \text{dist}(\overline{W}, V^C)$), let

$$z(\mathbf{y}) \equiv \frac{1}{h} \left\{ \eta^2(\mathbf{y} - h\mathbf{e}_k) \left[\frac{w(\mathbf{y}) - w(\mathbf{y} - h\mathbf{e}_k)}{h} \right] \right\}$$

$$-\eta^2(\mathbf{y}) \left[\frac{w(\mathbf{y} + h\mathbf{e}_k) - w(\mathbf{y})}{h} \right] \Bigg\} \quad (42.1.9)$$

$$\equiv -D_k^{-h}(\eta^2 D_k^h w), \quad (42.1.10)$$

where here $k < n$. Thus z can be used in equation 42.1.7. Begin by estimating the left side of 42.1.7.

$$\begin{aligned} & \int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy \\ &= \frac{1}{h} \int_U \alpha^{rs}(\mathbf{y} + h\mathbf{e}_k) \frac{\partial w}{\partial y^r}(\mathbf{y} + h\mathbf{e}_k) \frac{\partial(\eta^2 D_k^h w)}{\partial y^s} dy \\ & \quad - \frac{1}{h} \int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r} \frac{\partial(\eta^2 D_k^h w)}{\partial y^s} dy \\ &= \int_U \alpha^{rs}(\mathbf{y} + h\mathbf{e}_k) \frac{\partial(D_k^h w)}{\partial y^r} \frac{\partial(\eta^2 D_k^h w)}{\partial y^s} dy + \\ & \quad \frac{1}{h} \int_U (\alpha^{rs}(\mathbf{y} + h\mathbf{e}_k) - \alpha^{rs}(\mathbf{y})) \frac{\partial w}{\partial y^r} \frac{\partial(\eta^2 D_k^h w)}{\partial y^s} dy \end{aligned} \quad (42.1.11)$$

Now

$$\frac{\partial(\eta^2 D_k^h w)}{\partial y^s} = 2\eta \frac{\partial \eta}{\partial y^s} D_k^h w + \eta^2 \frac{\partial(D_k^h w)}{\partial y^s}. \quad (42.1.12)$$

therefore,

$$\begin{aligned} &= \int_U \eta^2 \alpha^{rs}(\mathbf{y} + h\mathbf{e}_k) \frac{\partial(D_k^h w)}{\partial y^r} \frac{\partial(D_k^h w)}{\partial y^s} dy \\ & \quad + \left\{ \int_{W \cap U} \alpha^{rs}(\mathbf{y} + h\mathbf{e}_k) \frac{\partial(D_k^h w)}{\partial y^r} 2\eta \frac{\partial \eta}{\partial y^s} D_k^h w dy \right. \\ & \quad \left. + \frac{1}{h} \int_{W \cap U} (\alpha^{rs}(\mathbf{y} + h\mathbf{e}_k) - \alpha^{rs}(\mathbf{y})) \frac{\partial w}{\partial y^r} \frac{\partial(\eta^2 D_k^h w)}{\partial y^s} dy \right\} \equiv A. + \{B.\}. \end{aligned} \quad (42.1.13)$$

Now consider these two terms. From 42.1.2,

$$A. \geq \delta \int_U \eta^2 |\nabla D_k^h w|^2 dy. \quad (42.1.14)$$

Using the Lipschitz continuity of α^{rs} and 42.1.12,

$$\begin{aligned} B. &\leq C(\eta, \text{Lip}(\alpha), \alpha) \left\{ \|D_k^h w\|_{L^2(W \cap U)} \|\eta \nabla D_k^h w\|_{L^2(W \cap U; \mathbb{R}^n)} + \right. \\ & \quad \|\eta \nabla w\|_{L^2(W \cap U; \mathbb{R}^n)} \|\eta \nabla D_k^h w\|_{L^2(W \cap U; \mathbb{R}^n)} \\ & \quad \left. + \|\eta \nabla w\|_{L^2(W \cap U; \mathbb{R}^n)} \|D_k^h w\|_{L^2(W \cap U)} \right\}. \end{aligned} \quad (42.1.15)$$

$$\begin{aligned}
&\leq C(\eta, \text{Lip}(\alpha), \alpha) C_\varepsilon \left(\|D_k^h w\|_{L^2(W \cap U)}^2 + \|\eta \nabla w\|_{L^2(W \cap U; \mathbb{R}^n)}^2 \right) + \\
&\varepsilon C(\eta, \text{Lip}(\alpha), \alpha) \left(\|\eta \nabla D_k^h w\|_{L^2(W \cap U; \mathbb{R}^n)}^2 + \|D_k^h w\|_{L^2(W \cap U)}^2 \right). \quad (42.1.16)
\end{aligned}$$

Now

$$\|D_k^h w\|_{L^2(W)} \leq \|\nabla w\|_{L^2(U; \mathbb{R}^n)}. \quad (42.1.17)$$

To see this, observe that if w is smooth, then

$$\begin{aligned}
&\left(\int_W \left| \frac{w(\mathbf{y} + h\mathbf{e}_k) - w(\mathbf{y})}{h} \right|^2 dy \right)^{1/2} \\
&\leq \left(\int_W \left| \frac{1}{h} \int_0^h \nabla w(\mathbf{y} + t\mathbf{e}_k) \cdot \mathbf{e}_k dt \right|^2 dy \right)^{1/2} \\
&\leq \left(\int_0^h \left(\int_W |\nabla w(\mathbf{y} + t\mathbf{e}_k) \cdot \mathbf{e}_k|^2 dy \right)^{1/2} \frac{dt}{h} \right) \leq \|\nabla w\|_{L^2(U; \mathbb{R}^n)}
\end{aligned}$$

so by density of such functions in $H^1(U)$, 42.1.17 holds. Therefore, changing ε , yields

$$B. \leq C_\varepsilon(\eta, \text{Lip}(\alpha), \alpha) \|\nabla w\|_{L^2(U; \mathbb{R}^n)}^2 + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(W \cap U; \mathbb{R}^n)}^2. \quad (42.1.18)$$

With 42.1.14 and 42.1.18 established, consider the other terms of 42.1.7.

$$\begin{aligned}
&\left| \int_U f z dy \right| \\
&\leq \left| \int_U f (-D_k^{-h} \eta^2 D_k^h w) dy \right| \\
&\leq \left(\int_U |f|^2 dy \right)^{1/2} \left(\int_U |D_k^{-h} (\eta^2 D_k^h w)|^2 dy \right)^{1/2} \\
&\leq \|f\|_{L^2(U)} \|\nabla (\eta^2 D_k^h w)\|_{L^2(U; \mathbb{R}^n)} \\
&\leq \|f\|_{L^2(U)} \left(\|2\eta \nabla \eta D_k^h w\|_{L^2(U; \mathbb{R}^n)} + \|\eta^2 \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)} \right) \\
&\leq C \|f\|_{L^2(U)} \|\nabla w\|_{L^2(U; \mathbb{R}^n)} + \|f\|_{L^2(U)} \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)} \\
&\leq C_\varepsilon \left(\|f\|_{L^2(U)}^2 + \|\nabla w\|_{L^2(U; \mathbb{R}^n)}^2 \right) + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2 \quad (42.1.19)
\end{aligned}$$

$$\begin{aligned}
& \left| \int_U h_s(\mathbf{y}) \frac{\partial z}{\partial y^s} dy \right| \\
& \leq \left| \int_U h_s(\mathbf{y}) \frac{\partial (-D_k^{-h}(\eta^2 D_k^h w))}{\partial y^s} dy \right| \\
& \leq \left| \int_U D_k^h h_s(\mathbf{y}) \frac{\partial ((\eta^2 D_k^h w))}{\partial y^s} \right| \\
& \leq \int_U \left| D_k^h h_s 2\eta \frac{\partial \eta}{\partial y^s} D_k^h w \right| dy + \int_U \left| (\eta D_k^h h_s) \left(\eta \frac{\partial (D_k^h w)}{\partial y^s} \right) \right| dy \\
& \leq C \sum_s \|h_s\|_{H^1(U)} \left(\|w\|_{H^1(U)} + \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)} \right) \\
& \leq C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 + \|w\|_{H^1(U)}^2 + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2. \quad (42.1.20)
\end{aligned}$$

The following inequalities in 42.1.14, 42.1.18, 42.1.19 and 42.1.20 are summarized here.

$$\begin{aligned}
A. & \geq \delta \int_U \eta^2 |\nabla D_k^h w|^2 dy, \\
B. & \leq C_\varepsilon (\eta, \text{Lip}(\alpha), \alpha) \|\nabla w\|_{L^2(U; \mathbb{R}^n)}^2 + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(W \cap U; \mathbb{R}^n)}^2, \\
\left| \int_U f z dy \right| & \leq C_\varepsilon \left(\|f\|_{L^2(U)}^2 + \|\nabla w\|_{L^2(U; \mathbb{R}^n)}^2 \right) + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2 \\
\left| \int_U h_s(\mathbf{y}) \frac{\partial z}{\partial y^s} dy \right| & \leq C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 \\
& \quad + \|w\|_{H^1(U)}^2 + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \delta \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2 \\
& \leq C_\varepsilon (\eta, \text{Lip}(\alpha), \alpha) \|\nabla w\|_{L^2(U; \mathbb{R}^n)}^2 + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2 \\
& \quad + C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 + \|w\|_{H^1(U)}^2 + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2 \\
& \quad + C_\varepsilon \left(\|f\|_{L^2(U)}^2 + \|\nabla w\|_{L^2(U; \mathbb{R}^n)}^2 \right) + \varepsilon \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2.
\end{aligned}$$

Letting ε be small enough and adjusting constants yields

$$\begin{aligned}
& \|\nabla D_k^h w\|_{L^2(U_1; \mathbb{R}^n)}^2 \leq \|\eta \nabla D_k^h w\|_{L^2(U; \mathbb{R}^n)}^2 \leq \\
& C \left(\|w\|_{H^1(U)}^2 + \|f\|_{L^2(U)}^2 + C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 \right)
\end{aligned}$$

where the constant, C , depends on $\eta, \text{Lip}(\alpha), \alpha, \delta$. Since this holds for all h small enough, it follows $\frac{\partial w}{\partial y^k} \in H^1(U_1)$ and

$$\left\| \nabla \frac{\partial w}{\partial y^k} \right\|_{L^2(U_1; \mathbb{R}^n)}^2 \leq C \left(\|w\|_{H^1(U)}^2 + \|f\|_{L^2(U)}^2 + C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 \right) \quad (42.1.21)$$

for each $k < n$. It remains to estimate $\left\| \frac{\partial^2 w}{\partial y_n^2} \right\|_{L^2(U_1)}^2$. To do this return to 42.1.7 which must hold for all $z \in C_c^\infty(U_1)$. Therefore, using 42.1.7 it follows that for all $z \in C_c^\infty(U_1)$,

$$\int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy = - \int_U \frac{\partial h_s}{\partial y^s} z dy + \int_U f z dy.$$

Now from the Lipschitz assumption on α^{rs} , it follows

$$\begin{aligned} F &\equiv \sum_{r,s \leq n-1} \frac{\partial}{\partial y^s} \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) \\ &\quad + \sum_{s \leq n-1} \frac{\partial}{\partial y^s} \left(\alpha^{ns} \frac{\partial w}{\partial y^n} \right) - \sum_s \frac{\partial h_s}{\partial y^s} + f \\ &\in L^2(U_1) \end{aligned}$$

and

$$\|F\|_{L^2(U_1)} \leq C \left(\|w\|_{H^1(U)}^2 + \|f\|_{L^2(U)}^2 + C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 \right). \quad (42.1.22)$$

Therefore, from density of $C_c^\infty(U_1)$ in $L^2(U_1)$,

$$-\frac{\partial}{\partial y^n} \left(\alpha^{nn}(\mathbf{y}) \frac{\partial w}{\partial y^n} \right) = F, \text{ no sum on } n$$

and so

$$-\frac{\partial \alpha^{nn}}{\partial y^n} \frac{\partial w}{\partial y^n} - \alpha^{nn} \frac{\partial^2 w}{\partial (y^n)^2} = F$$

By 42.1.2 $\alpha^{nn}(\mathbf{y}) \geq \delta$ and so it follows from 42.1.22 that there exists a constant, C depending on δ such that

$$\left| \frac{\partial^2 w}{\partial (y^n)^2} \right|_{L^2(U_1)} \leq C \left(\|F\|_{L^2(U_1)} + \|w\|_{H^1(U)} \right)$$

which with 42.1.21 and 42.1.22 implies the existence of a constant, C depending on δ such that

$$\|w\|_{H^2(U_1)}^2 \leq C \left(\|w\|_{H^1(U)}^2 + \|f\|_{L^2(U)}^2 + C_\varepsilon \sum_s \|h_s\|_{H^1(U)}^2 \right),$$

proving the lemma.

What if more regularity is known for f , h_s, α^{rs} and w ? Could more be said about the regularity of the solution? The answer is yes and is the content of the next corollary.

First here is some notation. For α a multi-index with $|\alpha| = k - 1$, $\alpha = (\alpha_1, \dots, \alpha_n)$ define

$$D_\alpha^h l(\mathbf{y}) \equiv \prod_{k=1}^n (D_k^h)^{\alpha_k} l(\mathbf{y}).$$

Also, for α and τ multi indices, $\tau < \alpha$ means $\tau_i < \alpha_i$ for each i .

Corollary 42.1.2 *Suppose in the context of Lemma 42.1.1 the following for $k \geq 1$.*

$$\begin{aligned} w &\in H^k(U), \\ \alpha^{rs} &\in C^{k-1,1}(\overline{U}), \\ h_s &\in H^k(U), \\ f &\in H^{k-1}(U), \end{aligned}$$

and

$$\int_U \alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r} \frac{\partial z}{\partial y^s} dy + \int_U h_s(\mathbf{y}) \frac{\partial z}{\partial y^s} dy = \int_U f z dy \quad (42.1.23)$$

for all $z \in H^1(U)$ or $H_0^1(U)$ such that $\text{spt}(z) \subseteq V$. Then there exists C independent of w such that

$$\|w\|_{H^{k+1}(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \sum_s \|h_s\|_{H^k(U)} + \|w\|_{H^k(U)} \right). \quad (42.1.24)$$

Proof: The proof involves the following claim which is proved using the conclusion of Lemma 42.1.1 on Page 1186.

Claim : If $\alpha = (\alpha', 0)$ where $|\alpha'| \leq k - 1$, then there exists a constant independent of w such that

$$\|D^\alpha w\|_{H^2(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \sum_s \|h_s\|_{H^k(U)} + \|w\|_{H^k(U)} \right). \quad (42.1.25)$$

Proof of claim: First note that if $|\alpha| = 0$, then 42.1.25 follows from Lemma 42.1.1 on Page 1186. Now suppose the conclusion of the claim holds for all $|\alpha| \leq j - 1$ where $j < k$. Let $|\alpha| = j$ and $\alpha = (\alpha', 0)$. Then for $z \in H^1(U)$ having compact support in V , it follows that for h small enough,

$$D_\alpha^{-h} z \in H^1(U), \quad \text{spt}(D_\alpha^h z) \subseteq V.$$

Therefore, you can replace z in 42.1.23 with $D_\alpha^{-h} z$. Now note that you can apply the following manipulation.

$$\int_U p(\mathbf{y}) D_\alpha^{-h} z(\mathbf{y}) dy = \int_U D_\alpha^h p(\mathbf{y}) z(\mathbf{y}) dy$$

and obtain

$$\int_U \left(D_\alpha^h \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) \frac{\partial z}{\partial y^s} + D_\alpha^h (h_s) \frac{\partial z}{\partial y^s} \right) dy = \int_U ((D_\alpha^h f) z) dy. \quad (42.1.26)$$

Letting $h \rightarrow 0$, this gives

$$\int_U \left(D^\alpha \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) \frac{\partial z}{\partial y^s} + D^\alpha (h_s) \frac{\partial z}{\partial y^s} \right) dy = \int_U ((D^\alpha f) z) dy.$$

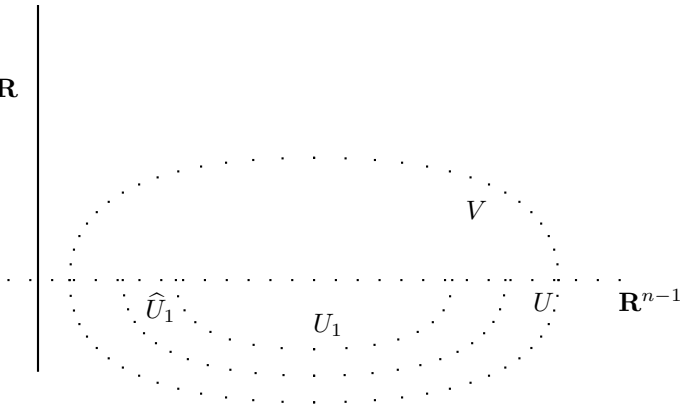
Now

$$D^\alpha \left(\alpha^{rs} \frac{\partial w}{\partial y^r} \right) = \alpha^{rs} \frac{\partial (D^\alpha w)}{\partial y^r} + \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r}$$

where $C(\tau)$ is some coefficient. Therefore, from 42.1.26,

$$\begin{aligned} \int_U \alpha^{rs} \frac{\partial (D^\alpha w)}{\partial y^r} \frac{\partial z}{\partial y^s} dy + \int_U \left(\sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial (D^\tau w)}{\partial y^r} + D^\alpha (h_s) \right) \frac{\partial z}{\partial y^s} dy \\ = \int_U (D^\alpha f) z dy. \end{aligned} \quad (42.1.27)$$

Let $\widehat{U}_1$ be as indicated in the following picture



Now apply the induction hypothesis to $\widehat{U}_1$ in order to write

$$\left\| \frac{\partial (D^\tau w)}{\partial y^r} \right\|_{H^1(\widehat{U}_1)} \leq \|D^\tau w\|_{H^2(\widehat{U}_1)}$$

$$\leq C \left(\|f\|_{H^{k-1}(U)} + \sum_s \|h_s\|_{H^k(U)} + \|w\|_{H^k(U)} \right).$$

Since $\alpha^{rs} \in C^{k-1,1}(\overline{U})$, it follows that each term from the sum in 42.1.27 satisfies an inequality of the form

$$\left\| C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial(D^\tau w)}{\partial y^r} \right\|_{H^1(\widehat{U}_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \sum_s \|h_s\|_{H^k(U)} + \|w\|_{H^k(U)} \right)$$

and consequently,

$$\left\| \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial(D^\tau w)}{\partial y^r} + D^\alpha(h_s) \right\|_{H^1(\widehat{U}_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \sum_s \|h_s\|_{H^k(U)} + \|w\|_{H^k(U)} \right). \quad (42.1.28)$$

Now consider 42.1.27. The equation remains true if you replace U with $\widehat{U}_1$ and require that $\text{spt}(z) \subseteq \widehat{U}_1$. Therefore, by Lemma 42.1.1 on Page 1186 there exists a constant, C independent of w such that

$$\|D^\alpha w\|_{H^2(U_1)} \leq C \left(\|D^\alpha f\|_{L^2(\widehat{U}_1)} + \|D^\alpha w\|_{H^1(\widehat{U}_1)} + \sum_s \left\| \sum_{\tau < \alpha} C(\tau) D^{\alpha-\tau} (\alpha^{rs}) \frac{\partial(D^\tau w)}{\partial y^r} + D^\alpha(h_s) \right\|_{H^1(\widehat{U}_1)} \right)$$

and by 42.1.28, this implies

$$\|D^\alpha w\|_{H^2(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right)$$

which proves the Claim.

To establish 42.1.24 it only remains to verify that if $|\alpha| \leq k+1$, then

$$\|D^\alpha w\|_{L^2(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right). \quad (42.1.29)$$

If $|\alpha| < k+1$, there is nothing to show because it is given that $w \in H^k(U)$. Therefore, assume $|\alpha| = k+1$. If α_n equals 0 the conclusion follows from the claim

because in this case, you can subtract 1 from a pair of positive α_i and obtain a new multi index, β such that $|\beta| = k - 1$ and $\beta_n = 0$ and then from the claim,

$$\|D^\alpha w\|_{L^2(U_1)} \leq \|D^\beta w\|_{H^2(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right).$$

If $\alpha_n = 1$, then subtract 1 from some positive α_i and consider

$$\beta = (\alpha_1, \dots, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_{n-1}, 0)$$

Then from the claim,

$$\|D^\alpha w\|_{L^2(U_1)} \leq \|D^\beta w\|_{H^2(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right).$$

Suppose 42.1.29 holds for $\alpha_n \leq j - 1$ where $j - 1 \geq 1$ and consider α for which $|\alpha| = k + 1$ and $\alpha_n = j$. Let

$$\beta \equiv (\alpha_1, \dots, \alpha_{n-1}, \alpha_n - 2).$$

Thus $D^\alpha = D^\beta D_n^2$. Restricting 42.1.23 to $z \in C_c^\infty(U_1)$ and using the density of this set of functions in $L^2(U_1)$, it follows that

$$-\frac{\partial}{\partial y^s} \left(\alpha^{rs}(\mathbf{y}) \frac{\partial w}{\partial y^r} \right) - \frac{\partial h_s}{\partial y^s} = f.$$

Therefore, from the product rule,

$$\frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \frac{\partial h_s}{\partial y^s} = -f$$

and so

$$\begin{aligned} \alpha^{nn} D_n^2 w &= - \left(\frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \right. \\ &\quad \left. \sum_s \alpha^{ns} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn} \frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f \right). \end{aligned}$$

As noted earlier, the condition, 42.1.2 implies $\alpha^{nn}(\mathbf{y}) \geq \delta > 0$ and so

$$\begin{aligned} D_n^2 w &= -\frac{1}{\alpha^{nn}} \left(\frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \right. \\ &\quad \left. \sum_s \alpha^{ns} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn} \frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f \right). \end{aligned}$$

It follows from $D^\alpha = D^\beta D_n^2$ that

$$D^\alpha w = D^\beta \left[-\frac{1}{\alpha^{nn}} \left(\frac{\partial \alpha^{rs}}{\partial y^s} \frac{\partial w}{\partial y^r} + \sum_{r \leq n-1} \sum_{s \leq n-1} \alpha^{rs} \frac{\partial^2 w}{\partial y^s \partial y^r} + \sum_s \alpha^{ns} \frac{\partial^2 w}{\partial y^s \partial y^n} + \sum_r \alpha^{rn} \frac{\partial^2 w}{\partial y^n \partial y^r} + \frac{\partial h_s}{\partial y^s} + f \right) \right].$$

Now you note that terms like $D^\beta \left(\frac{\partial^2 w}{\partial y^s \partial y^n} \right)$ have $\alpha_n = j - 1$ and so, from the induction hypothesis along with the assumptions on the given functions,

$$\|D^\alpha w\|_{L^2(U_1)} \leq C \left(\|f\|_{H^{k-1}(U)} + \|w\|_{H^k(U)} + \sum_s \|h_s\|_{H^k(U)} \right).$$

This proves the corollary.

42.2 The Case Of Bounded Open Sets

The main interest in all this is in the application to bounded open sets. Recall the following definition.

Definition 42.2.1 A bounded open subset, Ω , of $\mathbb{R}^n$ has a $C^{m,1}$ boundary if it satisfies the following conditions. For each $p \in \Gamma \equiv \overline{\Omega} \setminus \Omega$, there exists an open set, W , containing p , an open interval $(0, b)$, a bounded open box $U' \subseteq \mathbb{R}^{n-1}$, and an affine orthogonal transformation, R_W consisting of a distance preserving linear transformation followed by a translation such that

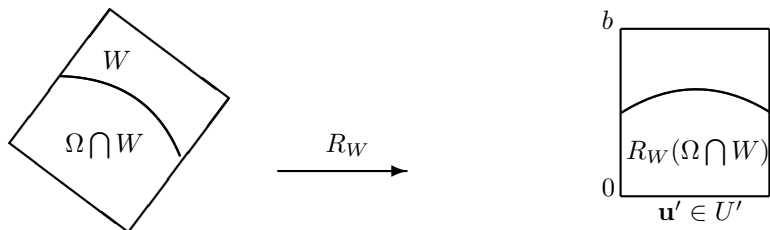
$$R_W W = U' \times (0, b), \quad (42.2.30)$$

$$R_W (W \cap \Omega) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U', 0 < u_n < \phi_W(\mathbf{u}')\} \quad (42.2.31)$$

where $\phi_W \in C^{m,1}(\overline{U'})$ meaning ϕ_W is the restriction to U' of a function, still denoted by ϕ_W which is in $C^{m,1}(\mathbb{R}^{n-1})$ and

$$\inf \{\phi_W(\mathbf{u}') : \mathbf{u}' \in U'\} > 0$$

The following picture depicts the situation.



For the situation described in the above definition, let $\mathbf{h}_W : U' \rightarrow \Gamma \cap W$ be defined by

$$\mathbf{h}_W(\mathbf{u}') \equiv R_W^{-1}(\mathbf{u}', \phi_W(\mathbf{u}')), \quad \mathbf{g}_W(\mathbf{x}) \equiv (R_W \mathbf{x})', \quad \mathbf{H}_W(\mathbf{u}) \equiv R_W^{-1}(\mathbf{u}', \phi_W(\mathbf{u}') - u_n).$$

where $\mathbf{x}' \equiv (x_1, \dots, x_{n-1})$ for $\mathbf{x} = (x_1, \dots, x_n)$. Thus $\mathbf{g}_W \circ \mathbf{h}_W = \text{id}$ on U' and $\mathbf{h}_W \circ \mathbf{g}_W = \text{id}$ on $\Gamma \cap W$. Also note that $\mathbf{H}_W$ is defined on all of $\mathbb{R}^n$ is $C^{m,1}$, and has an inverse with the same properties. To see this, let $\mathbf{G}_W(\mathbf{u}) = (\mathbf{u}', \phi_W(\mathbf{u}') - u_n)$. Then $\mathbf{H}_W = R_W^{-1} \circ \mathbf{G}_W$ and $\mathbf{G}_W^{-1} = (\mathbf{u}', \phi_W(\mathbf{u}') - u_n)$ and so $\mathbf{H}_W^{-1} = \mathbf{G}_W^{-1} \circ R_W$. Note also that as indicated in the picture,

$$R_W(W \cap \Omega) = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u}' \in U' \text{ and } 0 < u_n < \phi_W(\mathbf{u}')\}.$$

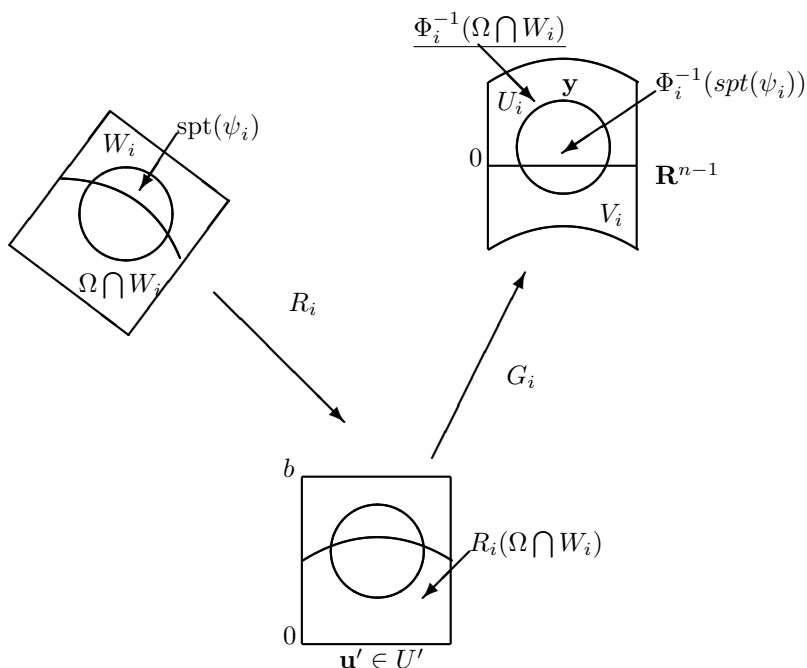
Since $\Gamma = \partial\Omega$ is compact, there exist finitely many of these open sets, W , denoted by $\{W_i\}_{i=1}^q$ such that $\Gamma \subseteq \cup_{i=1}^q W_i$. Let the corresponding sets, U' be denoted by U'_i and let the functions, ϕ be denoted by ϕ_i . Also let $\mathbf{h}_i = \mathbf{h}_{W_i}$, $G_{W_i} = G_i$ etc. Now let

$$\Phi_i : \mathbf{G}_i R_i(\Omega \cap W) \equiv V_i \rightarrow \Omega \cap W_i$$

be defined by

$$\Phi_i(\mathbf{y}) \equiv R_i^{-1} \circ \mathbf{G}_i^{-1}(\mathbf{y}).$$

Thus $\Phi_i, \Phi_i^{-1} \in C^{m,1}(\mathbb{R}^n)$. The following picture might be helpful.



Therefore, by Lemma 39.3.3 on Page 1149, it follows that for $t \in [m, m+1)$,

$$\Phi_i^* \in \mathcal{L}(H^t(W_i \cap \Omega), H^t(V_i)).$$

Assume

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2. \quad (42.2.32)$$

Lemma 42.2.2 *Let W be one of the sets described in the above definition and let $m \geq 1$. Let $W_1 \subseteq \overline{W_1} \subseteq W$ where W_1 is an open set. Suppose also that*

$$\begin{aligned} u &\in H^1(\Omega), \\ \alpha^{rs} &\in C^{0,1}(\overline{\Omega}), \\ f &\in L^2(\Omega), \\ h_k &\in H^1(\Omega), \end{aligned}$$

and that for all $v \in H^1(\Omega \cap W)$ such that $\text{spt}(v) \subseteq \Omega \cap W$,

$$\int_{\Omega} a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_{\Omega} h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx. \quad (42.2.33)$$

Then there exists a constant, C , independent of f, u , and g such that

$$\|u\|_{H^2(\Omega \cap W_1)}^2 \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right). \quad (42.2.34)$$

Proof: Let

$$E \equiv \{v \in H^1(\Omega \cap W) : \text{spt}(v) \subseteq W\}$$

u restricted to $W \cap \Omega$ is in $H^1(\Omega \cap W)$ and

$$\int_{\Omega \cap W} a^{ij}(\mathbf{x}) u_{,i} v_{,j} dx + \int_{\Omega} h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx \text{ for all } v \in E. \quad (42.2.35)$$

Now let $\Phi_i(\mathbf{y}) = \mathbf{x}$. For this particular W , denote Φ_i more simply by Φ , $U_i \equiv \Phi_i(\Omega \cap W_i)$ by U , and V_i by V . Denoting the coordinates of V by $\mathbf{y}$, and letting $u(\mathbf{x}) \equiv w(\mathbf{y})$ and $v(\mathbf{x}) \equiv z(\mathbf{y})$, it follows that in terms of the new coordinates, 42.2.35 takes the form

$$\begin{aligned} & \int_U a^{ij}(\Phi(\mathbf{y})) \frac{\partial w}{\partial y^r} \frac{\partial y^r}{\partial x^i} \frac{\partial z}{\partial y^s} \frac{\partial y^s}{\partial x^j} |\det D\Phi(\mathbf{y})| dy \\ & + \int_U h_k(\Phi(\mathbf{y})) \frac{\partial z}{\partial y^l} \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})| dx \\ & = \int_U f(\Phi(\mathbf{y})) z(\mathbf{y}) |\det D\Phi(\mathbf{y})| dy \end{aligned}$$

Let

$$\alpha^{rs}(\mathbf{y}) \equiv a^{ij}(\Phi(\mathbf{y})) \frac{\partial y^r}{\partial x^i} \frac{\partial y^s}{\partial x^j} |\det D\Phi(\mathbf{y})|, \quad (42.2.36)$$

$$\tilde{h}_l(\mathbf{y}) \equiv h_k(\Phi(\mathbf{y})) \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})|, \quad (42.2.37)$$

and

$$\tilde{f}(\mathbf{y}) \equiv \Phi^* f |\det D\Phi|(\mathbf{y}) \equiv f(\Phi(\mathbf{y})) |\det D\Phi(\mathbf{y})|. \quad (42.2.38)$$

Now the function on the right in 42.2.36 is in $C^{0,1}(\overline{U})$. This is because of the assumption that $m \geq 1$ in the statement of the lemma. This function is therefore a finite product of bounded functions in $C^{0,1}(\overline{U})$.

The function $\tilde{h}_l$ defined in 42.2.37 is in $H^1(U)$ and

$$\left\| \tilde{h}_l \right\|_{H^1(U)} \leq C \sum_k \|h_k\|_{H^1(\Omega \cap W)}$$

again because $m \geq 1$.

Finally, the right side of 42.2.38 is a function in $L^2(U)$ by Lemma 39.3.3 on Page 1149 and the observation that $|\det D\Phi(\cdot)| \in C^{0,1}(\overline{U})$ which follows from the assumption of the lemma that $m \geq 1$ so $\Phi \in C^{1,1}(\mathbb{R}^n)$. Also

$$\left\| \tilde{f} \right\|_{L^2(U)} \leq C \|f\|_{L^2(\Omega \cap W)}.$$

Therefore, 42.2.35 is of the form

$$\int_U \alpha^{rs}(\mathbf{y}) w_{,r} z_{,s} dy + \int_U \tilde{h}_l z_{,l} dy = \int_U \tilde{f} z dy, \quad (42.2.39)$$

for all z in $H^1(U)$ having support in V .

Claim: There exists $r > 0$ independent of $\mathbf{y} \in \overline{U}$ such that for all $\mathbf{y} \in \overline{U}$,

$$\alpha^{rs}(\mathbf{y}) v_r v_s \geq r |\mathbf{v}|^2.$$

Proof of the claim: If this is not so, there exist vectors, $\mathbf{v}^n, |\mathbf{v}^n| = 1$, and $\mathbf{y}_n \in \overline{U}$ such that $\alpha^{rs}(\mathbf{y}_n) v_r^n v_s^n \leq \frac{1}{n}$. Taking a subsequence, there exists $\mathbf{y} \in \overline{U}$ and $|\mathbf{v}| = 1$ such that $\alpha^{rs}(\mathbf{y}) v_r v_s = 0$ contradicting 42.2.32.

Therefore, by Lemma 42.1.1, there exists a constant, C , independent of f, g , and w such that

$$\|w\|_{H^2(\Phi^{-1}(W_1 \cap \Omega))}^2 \leq C \left(\left\| \tilde{f} \right\|_{L^2(U)}^2 + \|w\|_{H^1(U)}^2 + \sum_l \left\| \tilde{h}_l \right\|_{H^1(U)}^2 \right).$$

Therefore,

$$\begin{aligned} \|u\|_{H^2(W_1 \cap \Omega)}^2 &\leq C \left(\|f\|_{L^2(W \cap \Omega)}^2 + \|w\|_{H^1(W \cap \Omega)}^2 + \sum_k \|h_k\|_{H^1(W \cap \Omega)}^2 \right) \\ &\leq C \left(\|f\|_{L^2(\Omega)}^2 + \|w\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right). \end{aligned}$$

which proves the lemma.

With this lemma here is the main result.

Theorem 42.2.3 *Let Ω be a bounded open set with $C^{1,1}$ boundary as in Definition 42.2.1, let $f \in L^2(\Omega)$, $h_k \in H^1(\Omega)$, and suppose that for all $\mathbf{x} \in \overline{\Omega}$,*

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2.$$

Suppose also that $u \in H^1(\Omega)$ and

$$\int_{\Omega} a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_{\Omega} h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx$$

for all $v \in H^1(\Omega)$. Then $u \in H^2(\Omega)$ and for some C independent of f, g , and u ,

$$\|u\|_{H^2(\Omega)}^2 \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right).$$

Proof: Let the W_i for $i = 1, \dots, l$ be as described in Definition 42.2.1. Thus $\partial\Omega \subseteq \cup_{j=1}^l W_j$. Then let $C_1 \equiv \partial\Omega \setminus \cup_{i=2}^l W_i$, a closed subset of W_1 . Let D_1 be an open set satisfying

$$C_1 \subseteq D_1 \subseteq \overline{D_1} \subseteq W_1.$$

Then $D_1, W_2, \dots, W_l$ cover $\partial\Omega$. Let $C_2 = \partial\Omega \setminus (D_1 \cup (\cup_{i=3}^l W_i))$. Then C_2 is a closed subset of W_2 . Choose an open set, D_2 such that

$$C_2 \subseteq D_2 \subseteq \overline{D_2} \subseteq W_2.$$

Thus $D_1, D_2, W_3, \dots, W_l$ covers $\partial\Omega$. Continue in this way to get $\overline{D_i} \subseteq W_i$, and $\partial\Omega \subseteq \cup_{i=1}^l D_i$, and D_i is an open set. Now let

$$D_0 \equiv \Omega \setminus \cup_{i=1}^l \overline{D_i}.$$

Also, let $\overline{D_i} \subseteq V_i \subseteq \overline{V_i} \subseteq W_i$. Therefore, $D_0, V_1, \dots, V_l$ covers Ω . Then the same estimation process used above yields

$$\|u\|_{H^2(D_0)} \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right).$$

From Lemma 42.2.2

$$\|u\|_{H^2(V_i \cap \Omega)} \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right)$$

also. This proves the theorem since

$$\|u\|_{H^2(\Omega)} \leq \sum_{i=1}^l \|u\|_{H^2(V_i \cap \Omega)} + \|u\|_{H^2(D_0)}.$$

What about the Dirichlet problem? The same differencing procedure as above yields the following.

Theorem 42.2.4 *Let Ω be a bounded open set with $C^{1,1}$ boundary, let $f \in L^2(\Omega)$, $h_k \in H^1(\Omega)$, and suppose that for all $\mathbf{x} \in \overline{\Omega}$,*

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2.$$

Suppose also that $u \in H_0^1(\Omega)$ and

$$\int_{\Omega} a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_{\Omega} h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx$$

for all $v \in H_0^1(\Omega)$. Then $u \in H^2(\Omega)$ and for some C independent of f, g , and u ,

$$\|u\|_{H^2(\Omega)}^2 \leq C \left(\|f\|_{L^2(\Omega)}^2 + \|u\|_{H^1(\Omega)}^2 + \sum_k \|h_k\|_{H^1(\Omega)}^2 \right).$$

What about higher regularity?

Lemma 42.2.5 *Let W be one of the sets described in Definition 42.2.1 and let $m \geq k$. Let $W_1 \subseteq \overline{W_1} \subseteq W$ where W_1 is an open set. Suppose also that*

$$\begin{aligned} u &\in H^k(\Omega), \\ \alpha^{rs} &\in C^{k-1,1}(\overline{\Omega}), \\ f &\in H^{k-1}(\Omega), \\ h_s &\in H^k(\Omega), \end{aligned}$$

and that for all $v \in H^1(\Omega \cap W)$ such that $\text{spt}(v) \subseteq \Omega \cap W$,

$$\int_{\Omega} a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_{\Omega} h_s(\mathbf{x}) v_{,s}(\mathbf{x}) dx = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx. \quad (42.2.40)$$

Then there exists a constant, C , independent of f, u , and g such that

$$\|u\|_{H^{k+1}(\Omega \cap W_1)}^2 \leq C \left(\|f\|_{H^{k-1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_s \|h_s\|_{H^k(\Omega)}^2 \right). \quad (42.2.41)$$

Proof: Let

$$E \equiv \{v \in H^k(\Omega \cap W) : \text{spt}(v) \subseteq W\}$$

u restricted to $W \cap \Omega$ is in $H^k(\Omega \cap W)$ and

$$\begin{aligned} &\int_{\Omega \cap W} a^{ij}(\mathbf{x}) u_{,i} v_{,j} dx + \int_{\Omega} h_s(\mathbf{x}) v_{,s}(\mathbf{x}) dx \\ &= \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx \text{ for all } v \in E. \end{aligned} \quad (42.2.42)$$

Now let $\Phi_i(\mathbf{y}) = \mathbf{x}$. For this particular W , denote Φ_i more simply by Φ , $U_i \equiv \Phi_i(\Omega \cap W_i)$ by U , and V_i by V . Denoting the coordinates of V by $\mathbf{y}$, and letting

$u(\mathbf{x}) \equiv w(\mathbf{y})$ and $v(\mathbf{x}) \equiv z(\mathbf{y})$, it follows that in terms of the new coordinates, 42.2.35 takes the form

$$\begin{aligned} & \int_U a^{ij}(\Phi(\mathbf{y})) \frac{\partial w}{\partial y^r} \frac{\partial y^r}{\partial x^i} \frac{\partial z}{\partial y^s} \frac{\partial y^s}{\partial x^j} |\det D\Phi(\mathbf{y})| dy \\ & + \int_U h_k(\Phi(\mathbf{y})) \frac{\partial z}{\partial y^l} \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})| dx \\ & = \int_U f(\Phi(\mathbf{y})) z(\mathbf{y}) |\det D\Phi(\mathbf{y})| dy \end{aligned}$$

Let

$$\alpha^{rs}(\mathbf{y}) \equiv a^{ij}(\Phi(\mathbf{y})) \frac{\partial y^r}{\partial x^i} \frac{\partial y^s}{\partial x^j} |\det D\Phi(\mathbf{y})|, \quad (42.2.43)$$

$$\tilde{h}_l(\mathbf{y}) \equiv h_k(\Phi(\mathbf{y})) \frac{\partial y^l}{\partial x^k} |\det D\Phi(\mathbf{y})|, \quad (42.2.44)$$

and

$$\tilde{f}(\mathbf{y}) \equiv \Phi^* f |\det D\Phi|(\mathbf{y}) \equiv f(\Phi(\mathbf{y})) |\det D\Phi(\mathbf{y})|. \quad (42.2.45)$$

Now the function on the right in 42.2.43 is in $C^{k,1}(\overline{U})$. This is because of the assumption that $m \geq k$ in the statement of the lemma. This function is therefore a finite product of bounded functions in $C^{k,1}(\overline{U})$.

The function $\tilde{h}_l$ defined in 42.2.44 is in $H^k(U)$ and

$$\left\| \tilde{h}_l \right\|_{H^k(U)} \leq C \sum_s \|h_s\|_{H^k(\Omega \cap W)}$$

again because $m \geq k$.

Finally, the right side of 42.2.45 is a function in $H^{k-1}(U)$ by Lemma 39.3.3 on Page 1149 and the observation that $|\det D\Phi(\cdot)| \in C^{k-1,1}(\overline{U})$ which follows from the assumption of the lemma that $m \geq k$ so $\Phi \in C^{k-1,1}(\mathbb{R}^n)$. Also

$$\left\| \tilde{f} \right\|_{H^{k-1}(U)} \leq C \|f\|_{H^{k-1}(\Omega \cap W)}.$$

Therefore, 42.2.42 is of the form

$$\int_U \alpha^{rs}(\mathbf{y}) w_{,r} z_{,s} dy + \int_U \tilde{h}_l z_{,l} dy = \int_U \tilde{f} z dy, \quad (42.2.46)$$

for all z in $H^1(U)$ having support in V .

Claim: There exists $r > 0$ independent of $\mathbf{y} \in \overline{U}$ such that for all $\mathbf{y} \in \overline{U}$,

$$\alpha^{rs}(\mathbf{y}) v_r v_s \geq r |\mathbf{v}|^2.$$

Proof of the claim: If this is not so, there exist vectors, $\mathbf{v}^n, |\mathbf{v}^n| = 1$, and $\mathbf{y}_n \in \overline{U}$ such that $\alpha^{rs}(\mathbf{y}_n) v_r^n v_s^n \leq \frac{1}{n}$. Taking a subsequence, there exists $\mathbf{y} \in \overline{U}$ and $|\mathbf{v}| = 1$ such that $\alpha^{rs}(\mathbf{y}) v_r v_s = 0$ contradicting 42.2.32.

Therefore, by Corollary 42.1.2, there exists a constant, C , independent of f, g , and w such that

$$\|w\|_{H^{k+1}(\Phi^{-1}(W_1 \cap \Omega))}^2 \leq C \left(\|\tilde{f}\|_{H^{k-1}(U)}^2 + \|w\|_{H^k(U)}^2 + \sum_l \|\tilde{h}_l\|_{H^k(U)}^2 \right).$$

Therefore,

$$\begin{aligned} \|u\|_{H^{k+1}(W_1 \cap \Omega)}^2 &\leq C \left(\|f\|_{H^{k-1}(W \cap \Omega)}^2 + \|w\|_{H^k(W \cap \Omega)}^2 + \sum_s \|h_s\|_{H^k(W \cap \Omega)}^2 \right) \\ &\leq C \left(\|f\|_{H^{k-1}(\Omega)}^2 + \|w\|_{H^k(\Omega)}^2 + \sum_s \|h_s\|_{H^k(\Omega)}^2 \right). \end{aligned}$$

which proves the lemma.

Now here is a theorem which generalizes the one above in the case where more regularity is known.

Theorem 42.2.6 *Let Ω be a bounded open set with $C^{k,1}$ boundary as in Definition 42.2.1, let $f \in H^{k-1}(\Omega)$, $h_s \in H^k(\Omega)$, and suppose that for all $\mathbf{x} \in \bar{\Omega}$,*

$$a^{ij}(\mathbf{x}) v_i v_j \geq \delta |\mathbf{v}|^2.$$

Suppose also that $u \in H^k(\Omega)$ and

$$\int_{\Omega} a^{ij}(\mathbf{x}) u_{,i}(\mathbf{x}) v_{,j}(\mathbf{x}) dx + \int_{\Omega} h_k(\mathbf{x}) v_{,k}(\mathbf{x}) dx = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) dx$$

for all $v \in H^k(\Omega)$. Then $u \in H^{k+1}(\Omega)$ and for some C independent of f, g , and u ,

$$\|u\|_{H^{k+1}(\Omega)}^2 \leq C \left(\|f\|_{H^{k-1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_s \|h_s\|_{H^k(\Omega)}^2 \right).$$

Proof: Let the W_i for $i = 1, \dots, l$ be as described in Definition 42.2.1. Thus $\partial\Omega \subseteq \cup_{j=1}^l W_j$. Then let $C_1 \equiv \partial\Omega \setminus \cup_{i=2}^l W_i$, a closed subset of W_1 . Let D_1 be an open set satisfying

$$C_1 \subseteq D_1 \subseteq \overline{D_1} \subseteq W_1.$$

Then $D_1, W_2, \dots, W_l$ cover $\partial\Omega$. Let $C_2 = \partial\Omega \setminus (D_1 \cup (\cup_{i=3}^l W_i))$. Then C_2 is a closed subset of W_2 . Choose an open set, D_2 such that

$$C_2 \subseteq D_2 \subseteq \overline{D_2} \subseteq W_2.$$

Thus $D_1, D_2, W_3, \dots, W_l$ covers $\partial\Omega$. Continue in this way to get $\overline{D_i} \subseteq W_i$, and $\partial\Omega \subseteq \cup_{i=1}^l D_i$, and D_i is an open set. Now let

$$D_0 \equiv \Omega \setminus \cup_{i=1}^l \overline{D_i}.$$

Also, let $\overline{D_i} \subseteq V_i \subseteq \overline{V_i} \subseteq W_i$. Therefore, $D_0, V_1, \dots, V_l$ covers Ω . Then the same estimation process used above yields

$$\|u\|_{H^{k+1}(D_0)} \leq C \left(\|f\|_{H^{k-1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_k \|h_k\|_{H^k(\Omega)}^2 \right).$$

From Lemma 42.2.5

$$\|u\|_{H^{k+1}(V_i \cap \Omega)} \leq C \left(\|f\|_{H^{k-1}(\Omega)}^2 + \|u\|_{H^k(\Omega)}^2 + \sum_k \|h_k\|_{H^k(\Omega)}^2 \right)$$

also. This proves the theorem since

$$\|u\|_{H^{k+1}(\Omega)} \leq \sum_{i=1}^l \|u\|_{H^{k+1}(V_i \cap \Omega)} + \|u\|_{H^{k+1}(D_0)}.$$

Interpolation In Banach Space

43.1 An Assortment Of Important Theorems

43.1.1 Weak Vector Valued Derivatives

In this section, several significant theorems are presented. Unless indicated otherwise, the measure will be Lebesgue measure. First here is a lemma.

Lemma 43.1.1 *Suppose $g \in L^1([a, b]; X)$ where X is a Banach space. Then if $\int_a^b g(t) \phi(t) dt = 0$ for all $\phi \in C_c^\infty(a, b)$, then $g(t) = 0$ a.e.*

Proof: Let E be a measurable subset of (a, b) and let $K \subseteq E \subseteq V \subseteq (a, b)$ where K is compact, V is open and $m(V \setminus K) < \varepsilon$. Let $K \prec h \prec V$ as in the proof of the Riesz representation theorem for positive linear functionals. Enlarging K slightly and convolving with a mollifier, it can be assumed $h \in C_c^\infty(a, b)$. Then

$$\begin{aligned} \left| \int_a^b \chi_E(t) g(t) dt \right| &= \left| \int_a^b (\chi_E(t) - h(t)) g(t) dt \right| \\ &\leq \int_a^b |\chi_E(t) - h(t)| \|g(t)\| dt \\ &\leq \int_{V \setminus K} \|g(t)\| dt. \end{aligned}$$

Now let $K_n \subseteq E \subseteq V_n$ with $m(V_n \setminus K_n) < 2^{-n}$. Then from the above,

$$\left| \int_a^b \chi_E(t) g(t) dt \right| \leq \int_a^b \chi_{V_n \setminus K_n}(t) \|g(t)\| dt$$

and the integrand of the last integral converges to 0 a.e. as $n \rightarrow \infty$ because $\sum_n m(V_n \setminus K_n) < \infty$. By the dominated convergence theorem, this last integral

converges to 0. Therefore, whenever $E \subseteq (a, b)$,

$$\int_a^b \chi_E(t) g(t) dt = 0.$$

Since the endpoints have measure zero, it also follows that for any measurable E , the above equation holds.

Now $g \in L^1([a, b]; X)$ and so it is measurable. Therefore, $g([a, b])$ is separable. Let D be a countable dense subset and let E denote the set of linear combinations of the form $\sum_i a_i d_i$ where a_i is a rational point of $\mathbb{F}$ and $d_i \in D$. Thus E is countable. Denote by Y the closure of E in X . Thus Y is a separable closed subspace of X which contains all the values of g .

Now let $S_n \equiv g^{-1}(B(y_n, \|y_n\|/2))$ where $E = \{y_n\}_{n=1}^\infty$. Therefore, $\cup_n S_n = g^{-1}(X \setminus \{0\})$. This follows because if $x \in Y$ and $x \neq 0$, then in $B(x, \frac{\|x\|}{4})$ there is a point of E, y_n . Therefore, $\|y_n\| > \frac{3}{4}\|x\|$ and so $\frac{\|y_n\|}{2} > \frac{3\|x\|}{8} > \frac{\|x\|}{4}$ so $x \in B(y_n, \|y_n\|/2)$. It follows that if each S_n has measure zero, then $g(t) = 0$ for a.e. t . Suppose then that for some n , the set, S_n has positive measure. Then from what was shown above,

$$\begin{aligned} \|y_n\| &= \left\| \frac{1}{m(S_n)} \int_{S_n} g(t) dt - y_n \right\| = \left\| \frac{1}{m(S_n)} \int_{S_n} g(t) - y_n dt \right\| \\ &\leq \frac{1}{m(S_n)} \int_{S_n} \|g(t) - y_n\| dt \leq \frac{1}{m(S_n)} \int_{S_n} \|y_n\|/2 dt = \|y_n\|/2 \end{aligned}$$

and so $y_n = 0$ which implies $S_n = \emptyset$, a contradiction to $m(S_n) > 0$. This contradiction shows each S_n has measure zero and so as just explained, $g(t) = 0$ a.e.

Definition 43.1.2 For $f \in L^1(a, b; X)$, define an extension, $\bar{f}$ defined on

$$[2a - b, 2b - a] = [a - (b - a), b + (b - a)]$$

as follows.

$$\bar{f}(t) \equiv \begin{cases} f(t) & \text{if } t \in [a, b] \\ f(2a - t) & \text{if } t \in [2a - b, a] \\ f(2b - t) & \text{if } t \in [b, 2b - a] \end{cases}$$

Definition 43.1.3 Also if $f \in L^p(a, b; X)$ and $h > 0$, define for $t \in [a, b]$, $f_h(t) \equiv \bar{f}(t - h)$ for all $h < b - a$. Thus the map $f \rightarrow f_h$ is continuous and linear on $L^p(a, b; X)$. It is continuous because

$$\begin{aligned} \int_a^b \|f_h(t)\|^p dt &= \int_a^{a+h} \|f(2a - t + h)\|^p dt + \int_a^{b-h} \|f(t)\|^p dt \\ &= \int_a^{a+h} \|f(t)\|^p dt + \int_a^{b-h} \|f(t)\|^p dt \leq 2\|f\|_p^p. \end{aligned}$$

The following lemma is on continuity of translation in $L^p(a, b; X)$.

Lemma 43.1.4 *Let $\bar{f}$ be as defined in Definition 43.1.2. Then for $f \in L^p(a, b; X)$ for $p \in [1, \infty)$,*

$$\lim_{\delta \rightarrow 0} \int_a^b \|\bar{f}(t - \delta) - f(t)\|_X^p dt = 0.$$

Proof: Regarding the measure space as (a, b) with Lebesgue measure, by Lemma 23.5.9 there exists $g \in C_c(a, b; X)$ such that $\|f - g\|_p < \varepsilon$. Here the norm is the norm in $L^p(a, b; X)$. Therefore,

$$\begin{aligned} \|f_h - f\|_p &\leq \|f_h - g_h\|_p + \|g_h - g\|_p + \|g - f\|_p \\ &\leq \left(2^{1/p} + 1\right) \|f - g\|_p + \|g_h - g\|_p \\ &< \left(2^{1/p} + 1\right) \varepsilon + \varepsilon \end{aligned}$$

whenever h is sufficiently small. This is because of the uniform continuity of g . Therefore, since $\varepsilon > 0$ is arbitrary, this proves the lemma.

Definition 43.1.5 *Let $f \in L^1(a, b; X)$. Then the distributional derivative in the sense of X valued distributions is given by*

$$f'(\phi) \equiv - \int_a^b f(t) \phi'(t) dt$$

Then $f' \in L^1(a, b; X)$ if there exists $h \in L^1(a, b; X)$ such that for all $\phi \in C_c^\infty(a, b)$,

$$f'(\phi) = \int_a^b h(t) \phi(t) dt.$$

Then f' is defined to equal h . Here f and f' are considered as vector valued distributions in the same way as was done for scalar valued functions.

Lemma 43.1.6 *The above definition is well defined.*

Proof: Suppose both h and g work in the definition. Then for all $\phi \in C_c^\infty(a, b)$,

$$\int_a^b (h(t) - g(t)) \phi(t) dt = 0.$$

Therefore, by Lemma 43.1.1, $h(t) - g(t) = 0$ a.e.

The other thing to notice about this is the following lemma. It follows immediately from the definition.

Lemma 43.1.7 *Suppose $f, f' \in L^1(a, b; X)$. Then if $[c, d] \subseteq [a, b]$, it follows that $(f|_{[c, d]})' = f'|_{[c, d]}$. This notation means the restriction to $[c, d]$.*

Recall that in the case of scalar valued functions, if you had both f and its weak derivative, f' in $L^1(a, b)$, then you were able to conclude that f is almost everywhere equal to a continuous function, still denoted by f and

$$f(t) = f(a) + \int_a^t f'(s) ds.$$

In particular, you can define $f(a)$ to be the initial value of this continuous function. It turns out that an identical theorem holds in this case. To begin with here is the same sort of lemma which was used earlier for the case of scalar valued functions. It says that if $f' = 0$ where the derivative is taken in the sense of X valued distributions, then f equals a constant.

Lemma 43.1.8 *Suppose $f \in L^1(a, b; X)$ and for all $\phi \in C_c^\infty(a, b)$,*

$$\int_a^b f(t) \phi'(t) dt = 0.$$

Then there exists a constant, $a \in X$ such that $f(t) = a$ a.e.

Proof: Let $\phi_0 \in C_c^\infty(a, b)$, $\int_a^b \phi_0(x) dx = 1$ and define for $\phi \in C_c^\infty(a, b)$

$$\psi_\phi(x) \equiv \int_a^x [\phi(t) - \left(\int_a^b \phi(y) dy \right) \phi_0(t)] dt$$

Then $\psi_\phi \in C_c^\infty(a, b)$ and $\psi'_\phi = \phi - \left(\int_a^b \phi(y) dy \right) \phi_0$. Then

$$\begin{aligned} \int_a^b f(t) (\phi(t)) dt &= \int_a^b f(t) \left(\psi'_\phi(t) + \left(\int_a^b \phi(y) dy \right) \phi_0(t) \right) dt \\ &= \int_a^b f(t) \psi'_\phi(t) dt + \left(\int_a^b \phi(y) dy \right) \int_a^b f(t) \phi_0(t) dt \\ &= \left(\int_a^b \left(\int_a^b f(t) \phi_0(t) dt \right) \phi(y) dy \right). \end{aligned}$$

It follows that for all $\phi \in C_c^\infty(a, b)$,

$$\int_a^b \left(f(t) - \left(\int_a^b f(t) \phi_0(t) dt \right) \right) \phi(t) dt$$

and so by Lemma 43.1.1,

$$f(t) - \left(\int_a^b f(t) \phi_0(t) dt \right) = 0 \text{ a.e.}$$

This proves the lemma.

Theorem 43.1.9 *Suppose f, f' both are in $L^1(a, b; X)$ where the derivative is taken in the sense of X valued distributions. Then there exists a unique point of X , denoted by $f(a)$ such that the following formula holds a.e. t .*

$$f(t) = f(a) + \int_a^t f'(s) ds$$

Proof:

$$\int_a^b \left(f(t) - \int_a^t f'(s) ds \right) \phi'(t) dt = \int_a^b f(t) \phi'(t) dt - \int_a^b \int_a^t f'(s) \phi'(t) ds dt.$$

Now consider $\int_a^b \int_a^t f'(s) \phi'(t) ds dt$. Let $\Lambda \in X'$. Then it is routine from approximating f' with simple functions to verify

$$\Lambda \left(\int_a^b \int_a^t f'(s) \phi'(t) ds dt \right) = \int_a^b \int_a^t \Lambda(f'(s)) \phi'(t) ds dt.$$

Now the ordinary Fubini theorem can be applied to obtain

$$\begin{aligned} &= \int_a^b \int_s^b \Lambda(f'(s)) \phi'(t) dt ds \\ &= \Lambda \left(\int_a^b \int_s^b f'(s) \phi'(t) dt ds \right). \end{aligned}$$

Since X' separates the points of X , it follows

$$\int_a^b \int_a^t f'(s) \phi'(t) ds dt = \int_a^b \int_s^b f'(s) \phi'(t) dt ds.$$

Therefore,

$$\begin{aligned} &\int_a^b \left(f(t) - \int_a^t f'(s) ds \right) \phi'(t) dt \\ &= \int_a^b f(t) \phi'(t) dt - \int_a^b \int_s^b f'(s) \phi'(t) dt ds \\ &= \int_a^b f(t) \phi'(t) dt - \int_a^b f'(s) \int_s^b \phi'(t) dt ds \\ &= \int_a^b f(t) \phi'(t) dt + \int_a^b f'(s) \phi(s) ds = 0. \end{aligned}$$

Therefore, by Lemma 43.1.8, there exists a constant, denoted as $f(a)$ such that

$$f(t) - \int_a^t f'(s) ds = f(a)$$

and this proves the theorem.

The integration by parts formula is also important.

Corollary 43.1.10 Suppose $f, f' \in L^1(a, b; X)$ and suppose $\phi \in C^1([a, b])$. Then the following integration by parts formula holds.

$$\int_a^b f(t) \phi'(t) dt = f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) dt.$$

Proof: From Theorem 43.1.9

$$\begin{aligned} & \int_a^b f(t) \phi'(t) dt \\ &= \int_a^b \left(f(a) + \int_a^t f'(s) ds \right) \phi'(t) dt \\ &= f(a) (\phi(b) - \phi(a)) + \int_a^b \int_a^t f'(s) ds \phi'(t) dt \\ &= f(a) (\phi(b) - \phi(a)) + \int_a^b f'(s) \int_s^b \phi'(t) dt ds \\ &= f(a) (\phi(b) - \phi(a)) + \int_a^b f'(s) (\phi(b) - \phi(s)) ds \\ &= f(a) (\phi(b) - \phi(a)) - \int_a^b f'(s) \phi(s) ds + (f(b) - f(a)) \phi(b) \\ &= f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(s) \phi(s) ds. \end{aligned}$$

The interchange in order of integration is justified as in the proof of Theorem 43.1.9.

With this integration by parts formula, the following interesting lemma is obtained. This lemma shows why it was appropriate to define $\bar{f}$ as in Definition 43.1.2.

Lemma 43.1.11 Let $\bar{f}$ be given in Definition 43.1.2 and suppose $f, f' \in L^1(a, b; X)$. Then $\bar{f}, \bar{f}' \in L^1(2a - b, 2b - a; X)$ also and

$$\bar{f}'(t) \equiv \begin{cases} f'(t) & \text{if } t \in [a, b] \\ -f(2a - t) & \text{if } t \in [2a - b, a] \\ -f(2b - t) & \text{if } t \in [b, 2b - a] \end{cases} \quad (43.1.1)$$

Proof: It is clear from the definition of $\bar{f}$ that $\bar{f} \in L^1(2a - b, 2b - a; X)$ and that in fact

$$\|\bar{f}\|_{L^1(2a-b, 2b-a; X)} \leq 3 \|f\|_{L^1(a, b; X)}. \quad (43.1.2)$$

Let $\phi \in C_c^\infty(2a - b, 2b - a)$. Then from the integration by parts formula,

$$\begin{aligned}
 & \int_{2a-b}^{2b-a} \bar{f}(t) \phi'(t) dt \\
 = & \int_a^b f(t) \phi'(t) dt + \int_b^{2b-a} f(2b-t) \phi'(t) dt + \int_{2a-b}^a f(2a-t) \phi'(t) dt \\
 = & \int_a^b f(t) \phi'(t) dt + \int_a^b f(u) \phi'(2b-u) du + \int_a^b f(u) \phi'(2a-u) du \\
 = & f(b) \phi(b) - f(a) \phi(a) - \int_a^b f'(t) \phi(t) dt - f(b) \phi(b) + f(a) \phi(2b-a) \\
 & + \int_a^b f'(u) \phi(2b-u) du - f(b) \phi(2a-b) \\
 & + f(a) \phi(a) + \int_a^b f'(u) \phi(2a-u) du \\
 = & - \int_a^b f'(t) \phi(t) dt + \int_a^b f'(u) \phi(2b-u) du + \int_a^b f'(u) \phi(2a-u) du \\
 = & - \int_a^b f'(t) \phi(t) dt - \int_b^{2b-a} -f'(2b-t) \phi(t) dt - \int_{2a-b}^a -f'(2a-t) \phi(t) dt \\
 = & - \int_{2a-b}^{2b-a} \bar{f}'(t) \phi(t) dt
 \end{aligned}$$

where $\bar{f}'(t)$ is given in 43.1.1. This proves the lemma.

Definition 43.1.12 Let V be a Banach space and let H be a Hilbert space. (Typically $H = L^2(\Omega)$) Suppose $V \subseteq H$ is dense in H meaning that the closure in H of V gives H . Then it is often the case that H is identified with its dual space, and then because of the density of V in H , it is possible to write

$$V \subseteq H = H' \subseteq V'$$

When this is done, H is called a pivot space. Another notation which is often used is $\langle f, g \rangle$ to denote $f(g)$ for $f \in V'$ and $g \in V$. This may also be written as $\langle f, g \rangle_{V', V}$.

The next theorem is an example of a trace theorem. In this theorem, $f \in L^p(0, T; V)$ while $f' \in L^p(0, T; V')$. It makes no sense to consider the initial values of f in V because it is not even continuous with values in V . However, because of the derivative of f it will turn out that f is continuous with values in a larger space and so it makes sense to consider initial values of f in this other space. This other space is called a trace space.

Theorem 43.1.13 Let V and H be a Banach space and Hilbert space as described in Definition 43.1.12. Suppose $f \in L^p(0, T; V)$ and $f' \in L^p(0, T; V')$. Then f is

a.e. equal to a continuous function mapping $[0, T]$ to H . Furthermore, there exists $f(0) \in H$ such that

$$\frac{1}{2} |f(t)|_H^2 - \frac{1}{2} |f(0)|_H^2 = \int_0^t \langle f'(s), f(s) \rangle ds, \quad (43.1.3)$$

and for all $t \in [0, T]$,

$$\int_0^t f'(s) ds \in H, \quad (43.1.4)$$

and for a.e. $t \in [0, T]$,

$$f(t) = f(0) + \int_0^t f'(s) ds \text{ in } H, \quad (43.1.5)$$

Here f' is being taken in the sense of V' valued distributions and $\frac{1}{p} + \frac{1}{p'} = 1$ and $p \geq 2$.

Proof: Let $\Psi \in C_c^\infty(-T, 2T)$ satisfy $\Psi(t) = 1$ if $t \in [-T/2, 3T/2]$ and $\Psi(t) \geq 0$. For $t \in \mathbb{R}$, define

$$\widehat{f}(t) \equiv \begin{cases} \bar{f}(t) \Psi(t) & \text{if } t \in [-T, 2T] \\ 0 & \text{if } t \notin [-T, 2T] \end{cases}$$

and

$$f_n(t) \equiv \int_{-1/n}^{1/n} \widehat{f}(t-s) \phi_n(s) ds \quad (43.1.6)$$

where ϕ_n is a mollifier having support in $(-1/n, 1/n)$. Then by Minkowski's inequality

$$\begin{aligned} \|f_n - \widehat{f}\|_{L^p(\mathbb{R}; V)} &= \left(\int_{\mathbb{R}} \left\| \widehat{f}(t) - \int_{-1/n}^{1/n} \widehat{f}(t-s) \phi_n(s) ds \right\|_V^p dt \right)^{1/p} \\ &= \left(\int_{\mathbb{R}} \left\| \int_{-1/n}^{1/n} (\widehat{f}(t) - \widehat{f}(t-s)) \phi_n(s) ds \right\|_V^p dt \right)^{1/p} \\ &\leq \left(\int_{\mathbb{R}} \left(\int_{-1/n}^{1/n} \|\widehat{f}(t) - \widehat{f}(t-s)\|_V \phi_n(s) ds \right)^p dt \right)^{1/p} \\ &\leq \int_{-1/n}^{1/n} \phi_n(s) \left(\int_{\mathbb{R}} \|\widehat{f}(t) - \widehat{f}(t-s)\|_V^p dt \right)^{1/p} ds \\ &\leq \int_{-1/n}^{1/n} \phi_n(s) \varepsilon ds = \varepsilon \end{aligned}$$

provided n is large enough. This follows from Lemma 43.1.4 about continuity of translation. Since $\varepsilon > 0$ is arbitrary, it follows $f_n \rightarrow \widehat{f}$ in $L^p(\mathbb{R}; V)$. Similarly,

$f_n \rightarrow f$ in $L^2(\mathbb{R}; H)$. This follows because $p \geq 2$ and the norm in V and norm in H are related by $|x|_H \leq C \|x\|_V$ for some constant, C . Now

$$\widehat{f}(t) = \begin{cases} \Psi(t) f(t) & \text{if } t \in [0, T], \\ \Psi(t) f(2T - t) & \text{if } t \in [T, 2T], \\ \Psi(t) f(-t) & \text{if } t \in [-T, 0], \\ 0 & \text{if } t \notin [-T, 2T]. \end{cases}$$

An easy modification of the argument of Lemma 43.1.11 yields

$$\widehat{f'}(t) = \begin{cases} \Psi'(t) f(t) + \Psi(t) f'(t) & \text{if } t \in [0, T], \\ \Psi'(t) f(2T - t) - \Psi(t) f'(2T - t) & \text{if } t \in [T, 2T], \\ \Psi'(t) f(-t) - \Psi(t) f'(-t) & \text{if } t \in [-T, 0], \\ 0 & \text{if } t \notin [-T, 2T]. \end{cases}.$$

Recall

$$\begin{aligned} f_n(t) &= \int_{-1/n}^{1/n} \widehat{f}(t-s) \phi_n(s) ds = \int_{\mathbb{R}} \widehat{f}(t-s) \phi_n(s) ds \\ &= \int_{\mathbb{R}} \widehat{f}(s) \phi_n(t-s) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} f'_n(t) &= \int_{\mathbb{R}} \widehat{f}(s) \phi'_n(t-s) ds = \int_{-T-\frac{1}{n}}^{2T+\frac{1}{n}} \widehat{f}(s) \phi'_n(t-s) ds \\ &= \int_{-T-\frac{1}{n}}^{2T+\frac{1}{n}} \widehat{f'}(s) \phi_n(t-s) ds = \int_{\mathbb{R}} \widehat{f'}(s) \phi_n(t-s) ds \\ &= \int_{\mathbb{R}} \widehat{f'}(t-s) \phi_n(s) ds = \int_{-1/n}^{1/n} \widehat{f'}(t-s) \phi_n(s) ds \end{aligned}$$

and it follows from the first line above that f'_n is continuous with values in V for all $t \in \mathbb{R}$. Also note that both f'_n and f_n equal zero if $t \notin [-T, 2T]$ whenever n is large enough. Exactly similar reasoning to the above shows that $f'_n \rightarrow \widehat{f'}$ in $L^{p'}(\mathbb{R}; V')$.

Now let $\phi \in C_c^\infty(0, T)$.

$$\begin{aligned} \int_{\mathbb{R}} |f_n(t)|_H^2 \phi'(t) dt &= \int_{\mathbb{R}} (f_n(t), f_n(t))_H \phi'(t) dt \quad (43.1.7) \\ &= - \int_{\mathbb{R}} 2 \langle f'_n(t), f_n(t) \rangle \phi(t) dt = - \int_{\mathbb{R}} 2 \langle f'_n(t), f_n(t) \rangle \phi(t) dt \end{aligned}$$

Now

$$\begin{aligned} &\left| \int_{\mathbb{R}} \langle f'_n(t), f_n(t) \rangle \phi(t) dt - \int_{\mathbb{R}} \langle f'(t), f(t) \rangle \phi(t) dt \right| \\ &\leq \int_{\mathbb{R}} (|\langle f'_n(t) - f'(t), f_n(t) \rangle| + |\langle f'(t), f_n(t) - f(t) \rangle|) \phi(t) dt. \end{aligned}$$

From the first part of this proof which showed that $f_n \rightarrow \hat{f}$ in $L^p(\mathbb{R}; V)$ and $f'_n \rightarrow \hat{f}'$ in $L^{p'}(\mathbb{R}; V')$, an application of Holder's inequality shows the above converges to 0 as $n \rightarrow \infty$. Therefore, passing to the limit as $n \rightarrow \infty$ in the 43.1.8,

$$\int_{\mathbb{R}} \left| \hat{f}(t) \right|_H^2 \phi'(t) dt = - \int_{\mathbb{R}} 2 \langle \hat{f}'(t), \hat{f}(t) \rangle \phi(t) dt$$

which shows $t \rightarrow \left| \hat{f}(t) \right|_H^2$ equals a continuous function a.e. and it also has a weak derivative equal to $2 \langle \hat{f}', \hat{f} \rangle$.

It remains to verify that $\hat{f}$ is continuous on $[0, T]$. Of course $\hat{f} = f$ on this interval. Let N be large enough that $f_n(-T) = 0$ for all $n > N$. Then for $m, n > N$ and $t \in [-T, 2T]$

$$\begin{aligned} |f_n(t) - f_m(t)|_H^2 &= 2 \int_{-T}^t (f'_n(s) - f'_m(s), f_n(s) - f_m(s)) ds \\ &= 2 \int_{-T}^t \langle f'_n(s) - f'_m(s), f_n(s) - f_m(s) \rangle_{V', V} ds \\ &\leq 2 \int_{\mathbb{R}} \|f'_n(s) - f'_m(s)\|_{V'} \|f_n(s) - f_m(s)\|_V ds \\ &\leq 2 \|f_n - f_m\|_{L^{p'}(\mathbb{R}; V')} \|f_n - f_m\|_{L^p(\mathbb{R}; V)} \end{aligned}$$

which shows from the above that $\{f_n\}$ is uniformly Cauchy on $[-T, 2T]$ with values in H . Therefore, there exists g a continuous function defined on $[-T, 2T]$ having values in H such that

$$\lim_{n \rightarrow \infty} \max \{|f_n(t) - g(t)|_H; t \in [-T, 2T]\} = 0.$$

However, $g = \hat{f}$ a.e. because f_n converges to f in $L^p(0, T; V)$. Therefore, taking a subsequence, the convergence is a.e. It follows from the fact that $V \subseteq H = H' \subseteq V'$ and Theorem 43.1.9 there exists $f(0) \in V'$ such that for a.e. t ,

$$f(t) = f(0) + \int_0^t f'(s) ds \text{ in } V'$$

Now $g = f$ a.e. and g is continuous with values in H hence continuous with values in V' and so

$$g(t) = f(0) + \int_0^t f'(s) ds \text{ in } V'$$

for all t . Since g is continuous with values in H it is continuous with values in V' . Taking the limit as $t \downarrow 0$ in the above, $g(a) = \lim_{t \rightarrow 0+} g(t) = f(0)$, showing that $f(0) \in H$. Therefore, for a.e. t ,

$$f(t) = f(0) + \int_0^t f'(s) ds \text{ in } H, \quad \int_0^t f'(s) ds \in H.$$

This proves the theorem.

Note that if $f \in L^p(0, T; V)$ and $f' \in L^{p'}(0, T; V')$, then you can consider the initial value of f and it will be in H . What if you start with something in H ? Is it an initial condition for a function $f \in L^p(0, T; V)$ such that $f' \in L^{p'}(0, T; V')$? This is worth thinking about. If it is not so, what is the space of initial values? How can you give this space a norm? What are its properties? It turns out that if V is a closed subspace of the Sobolev space, $W^{1,p}(\Omega)$ which contains $W_0^{1,p}(\Omega)$ for $p \geq 2$ and $H = L^2(\Omega)$ the answer to the above question is yes. Not surprisingly, there are many generalizations of the above ideas.

43.1.2 Some Imbedding Theorems

The next theorem is very useful in getting estimates in partial differential equations. It is called Erling's lemma.

Definition 43.1.14 Let E, W be Banach spaces such that $E \subseteq W$ and the injection map from E into W is continuous. The injection map is said to be compact if every bounded set in E has compact closure in W . In other words, if a sequence is bounded in E it has a convergent subsequence converging in W . This is also referred to by saying that bounded sets in E are precompact in W .

Theorem 43.1.15 Let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W . Then for every $\varepsilon > 0$ there exists a constant, C_ε such that for all $u \in E$,

$$\|u\|_W \leq \varepsilon \|u\|_E + C_\varepsilon \|u\|_X$$

Proof: Suppose not. Then there exists $\varepsilon > 0$ and for each $n \in \mathbb{N}$, u_n such that

$$\|u_n\|_W > \varepsilon \|u_n\|_E + n \|u_n\|_X$$

Now let $v_n = u_n / \|u_n\|_E$. Therefore, $\|v_n\|_E = 1$ and

$$\|v_n\|_W > \varepsilon + n \|v_n\|_X$$

It follows there exists a subsequence, still denoted by v_n such that v_n converges to v in W . However, the above inequality shows that $\|v_n\|_X \rightarrow 0$. Therefore, $v = 0$. But then the above inequality would imply that $\|v_n\| > \varepsilon$ and passing to the limit yields $0 > \varepsilon$, a contradiction.

Definition 43.1.16 Define $C([a, b]; X)$ the space of functions continuous at every point of $[a, b]$ having values in X .

You should verify that this is a Banach space with norm

$$\|u\|_{\infty, X} = \max \{ \|u_{n_k}(t) - u(t)\|_X : t \in [a, b] \}.$$

The following theorem is an infinite dimensional version of the Ascoli Arzela theorem.

Theorem 43.1.17 Let $q > 1$ and let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W . Let S be defined by

$$\left\{ u \text{ such that } \|u(t)\|_E + \|u'\|_{L^q([a,b];X)} \leq R \text{ for all } t \in [a,b] \right\}.$$

Then $S \subseteq C([a,b];W)$ and if $\{u_n\} \subseteq S$, there exists a subsequence, $\{u_{n_k}\}$ which converges to a function $u \in C([a,b];W)$ in the following way.

$$\lim_{k \rightarrow \infty} \|u_{n_k} - u\|_{\infty, W} = 0.$$

Proof: First consider the issue of S being a subset of $C([a,b];W)$. By Theorem 43.1.9 on Page 1209 the following holds in X for $u \in S$.

$$u(t) - u(s) = \int_s^t u'(r) dr.$$

Thus $S \subseteq C([a,b];X)$. Let $\varepsilon > 0$ be given. Then by Theorem 43.1.15 there exists a constant, C_ε such that for all $u \in W$

$$\|u\|_W \leq \frac{\varepsilon}{4R} \|u\|_E + C_\varepsilon \|u\|_X.$$

Therefore, for all $u \in S$,

$$\begin{aligned} \|u(t) - u(s)\|_W &\leq \frac{\varepsilon}{6R} \|u(t) - u(s)\|_E + C_\varepsilon \|u(t) - u(s)\|_X \\ &\leq \frac{\varepsilon}{3} + C_\varepsilon \left\| \int_s^t u'(r) dr \right\|_X \\ &\leq \frac{\varepsilon}{3} + C_\varepsilon \int_s^t \|u'(r)\|_X dr \leq \frac{\varepsilon}{3} + C_\varepsilon R |t - s|^{1/q}. \end{aligned} \quad (43.1.8)$$

Since ε is arbitrary, it follows $u \in C([a,b];W)$.

Let $D = \mathbb{Q} \cap [a,b]$ so D is a countable dense subset of $[a,b]$. Let $D = \{t_n\}_{n=1}^\infty$. By compactness of the embedding of E into W , there exists a subsequence $u_{(n,1)}$ such that as $n \rightarrow \infty$, $u_{(n,1)}(t_1)$ converges to a point in W . Now take a subsequence of this, called $(n,2)$ such that as $n \rightarrow \infty$, $u_{(n,2)}(t_2)$ converges to a point in W . It follows that $u_{(n,2)}(t_1)$ also converges to a point of W . Continue this way. Now consider the diagonal sequence, $u_k \equiv u_{(k,k)}$. This sequence is a subsequence of $u_{(n,l)}$ whenever $k > l$. Therefore, $u_k(t_j)$ converges for all $t_j \in D$.

Claim: Let $\{u_k\}$ be as just defined, converging at every point of $[a,b]$. Then $\{u_k\}$ converges at every point of $[a,b]$.

Proof of claim: Let $\varepsilon > 0$ be given. Let $t \in [a,b]$. Pick $t_m \in D \cap [a,b]$ such that in 43.1.8 $C_\varepsilon R |t - t_m| < \varepsilon/3$. Then there exists N such that if $l, n > N$, then $\|u_l(t_m) - u_n(t_m)\|_X < \varepsilon/3$. It follows that for $l, n > N$,

$$\begin{aligned} \|u_l(t) - u_n(t)\|_X &\leq \|u_l(t) - u_l(t_m)\| + \|u_l(t_m) - u_n(t_m)\| \\ &\quad + \|u_n(t_m) - u_n(t)\| \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} < 2\varepsilon \end{aligned}$$

Since ε was arbitrary, this shows $\{u_k(t)\}_{k=1}^\infty$ is a Cauchy sequence. Since W is complete, this shows this sequence converges.

Now for $t \in [a, b]$, it was just shown that if $\varepsilon > 0$ there exists N_t such that if $n, m > N_t$, then

$$\|u_n(t) - u_m(t)\| < \frac{\varepsilon}{3}.$$

Now let $s \neq t$. Then

$$\|u_n(s) - u_m(s)\| \leq \|u_n(s) - u_n(t)\| + \|u_n(t) - u_m(t)\| + \|u_m(t) - u_m(s)\|$$

From 43.1.8

$$\|u_n(s) - u_m(s)\| \leq 2 \left(\frac{\varepsilon}{3} + C_\varepsilon R |t - s|^{1/q} \right) + \|u_n(t) - u_m(t)\|$$

and so it follows that if δ is sufficiently small and $s \in B(t, \delta)$, then when $n, m > N_t$

$$\|u_n(s) - u_m(s)\| < \varepsilon.$$

Since $[a, b]$ is compact, there are finitely many of these balls, $\{B(t_i, \delta)\}_{i=1}^p$, such that for $s \in B(t_i, \delta)$ and $n, m > N_{t_i}$, the above inequality holds. Let $N > \max\{N_{t_1}, \dots, N_{t_p}\}$. Then if $m, n > N$ and $s \in [a, b]$ is arbitrary, it follows the above inequality must hold. Therefore, this has shown the following claim.

Claim: Let $\varepsilon > 0$ be given. Then there exists N such that if $m, n > N$, then $\|u_n - u_m\|_{\infty, W} < \varepsilon$.

Now let $u(t) = \lim_{k \rightarrow \infty} u_k(t)$.

$$\|u(t) - u(s)\|_W \leq \|u(t) - u_n(t)\|_W + \|u_n(t) - u_n(s)\|_W + \|u_n(s) - u(s)\|_W \quad (43.1.9)$$

Let N be in the above claim and fix $n > N$. Then

$$\|u(t) - u_n(t)\|_W = \lim_{m \rightarrow \infty} \|u_m(t) - u_n(t)\|_W \leq \varepsilon$$

and similarly, $\|u_n(s) - u(s)\|_W \leq \varepsilon$. Then if $|t - s|$ is small enough, 43.1.8 shows the middle term in 43.1.9 is also smaller than ε . Therefore, if $|t - s|$ is small enough,

$$\|u(t) - u(s)\|_W < 3\varepsilon.$$

Thus u is continuous. Finally, let N be as in the above claim. Then letting $m, n > N$, it follows that for all $t \in [a, b]$,

$$\|u_m(t) - u_n(t)\| < \varepsilon.$$

Therefore, letting $m \rightarrow \infty$, it follows that for all $t \in [a, b]$,

$$\|u(t) - u_n(t)\| \leq \varepsilon.$$

and so $\|u - u_n\|_{\infty, W} \leq \varepsilon$. Since ε is arbitrary, this proves the theorem.

The next theorem is another such imbedding theorem. It is often used in partial differential equations.

Theorem 43.1.18 *Let $E \subseteq W \subseteq X$ where the injection map is continuous from W to X and compact from E to W . Let $p \geq 1$, let $q > 1$, and define*

$$S \equiv \{u \in L^p([a, b]; E) : u' \in L^q([a, b]; X)$$

$$\text{and } \|u\|_{L^p([a, b]; E)} + \|u'\|_{L^q([a, b]; X)} \leq R\}.$$

Then S is precompact in $L^p([a, b]; W)$. This means that if $\{u_n\}_{n=1}^\infty \subseteq S$, it has a subsequence $\{u_{n_k}\}$ which converges in $L^p([a, b]; W)$.

Proof: By Proposition 6.2.5 on Page 142 it suffices to show S has an η net in $L^p([a, b]; W)$ for each $\eta > 0$.

If not, there exists $\eta > 0$ and a sequence $\{u_n\} \subseteq S$, such that

$$\|u_n - u_m\| \geq \eta \quad (43.1.10)$$

for all $n \neq m$ and the norm refers to $L^p([a, b]; W)$. Let

$$a = t_0 < t_1 < \cdots < t_n = b, \quad t_k - t_{k-1} = T/k.$$

Now define

$$\bar{u}_n(t) \equiv \sum_{i=1}^k \bar{u}_{n_i} \mathcal{X}_{[t_{i-1}, t_i)}(t), \quad \bar{u}_{n_i} \equiv \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} u_n(s) ds.$$

The idea is to show that $\bar{u}_n$ approximates u_n well and then to argue that a subsequence of the $\{\bar{u}_n\}$ is a Cauchy sequence yielding a contradiction to 43.1.10.

Therefore,

$$u_n(t) - \bar{u}_n(t) = \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - \bar{u}_n(s)) ds \mathcal{X}_{[t_{i-1}, t_i)}(t).$$

It follows from Jensen's inequality that

$$\begin{aligned} & \|u_n(t) - \bar{u}_n(t)\|_W^p \\ &= \sum_{i=1}^k \left\| \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} (u_n(t) - u_n(s)) ds \right\|_W^p \mathcal{X}_{[t_{i-1}, t_i)}(t) \\ &\leq \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \|u_n(t) - u_n(s)\|_W^p ds \mathcal{X}_{[t_{i-1}, t_i)}(t) \end{aligned}$$

and so

$$\begin{aligned} & \int_a^b \|u_n(t) - \bar{u}_n(s)\|_W^p ds \\ &\leq \int_a^b \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \|u_n(t) - u_n(s)\|_W^p ds \mathcal{X}_{[t_{i-1}, t_i)}(t) dt \\ &= \sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \|u_n(t) - u_n(s)\|_W^p ds dt. \end{aligned} \quad (43.1.11)$$

From Theorems 43.1.15 and 43.1.9, if $\varepsilon > 0$, there exists C_ε such that

$$\begin{aligned}
 \|u_n(t) - u_n(s)\|_W^p &\leq \varepsilon \|u_n(t) - u_n(s)\|_E^p + C_\varepsilon \|u_n(t) - u_n(s)\|_X^p \\
 &\leq 2^{p-1}\varepsilon (\|u_n(t)\|^p + \|u_n(s)\|^p) + C_\varepsilon \left\| \int_s^t u'_n(r) dr \right\|_X^p \\
 &\leq 2^{p-1}\varepsilon (\|u_n(t)\|^p + \|u_n(s)\|^p) + C_\varepsilon \left(\int_s^t \|u'_n(r)\|_X dr \right)^p \\
 &\leq 2^{p-1}\varepsilon (\|u_n(t)\|^p + \|u_n(s)\|^p) \\
 &\quad + C_\varepsilon \left(\left(\int_s^t \|u'_n(r)\|_X^q dr \right)^{1/q} |t-s|^{1/q'} \right)^p \\
 &= 2^{p-1}\varepsilon (\|u_n(t)\|^p + \|u_n(s)\|^p) + C_\varepsilon R^{p/q} |t-s|^{p/q'}.
 \end{aligned}$$

This is substituted in to 43.1.11 to obtain

$$\begin{aligned}
 &\int_a^b \|(u_n(t) - \bar{u}_n(s))\|_W^p ds \leq \\
 &\sum_{i=1}^k \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} (2^{p-1}\varepsilon (\|u_n(t)\|^p + \|u_n(s)\|^p) \\
 &\quad + C_\varepsilon R^{p/q} |t-s|^{p/q'}) ds dt \\
 &= \sum_{i=1}^k 2^p \varepsilon \int_{t_{i-1}}^{t_i} \|u_n(t)\|_W^p dt + C_\varepsilon R^{p/q} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} |t-s|^{p/q'} ds dt \\
 &= 2^p \varepsilon \int_a^b \|u_n(t)\|^p dt + C_\varepsilon R^{p/q} \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})} (t_i - t_{i-1})^{p/q'} \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} ds dt \\
 &= 2^p \varepsilon \int_a^b \|u_n(t)\|^p dt + C_\varepsilon R^{p/q} \sum_{i=1}^k \frac{1}{(t_i - t_{i-1})} (t_i - t_{i-1})^{p/q'} (t_i - t_{i-1})^2 \\
 &\leq 2^p \varepsilon R^p + C_\varepsilon R^{p/q} \sum_{i=1}^k (t_i - t_{i-1})^{1+p/q'} = 2^p \varepsilon R^p + C_\varepsilon R^{p/q} k \left(\frac{T}{k} \right)^{1+p/q'}.
 \end{aligned}$$

Taking ε so small that $2^p \varepsilon R^p < \eta^p/8^p$ and then choosing k sufficiently large, it follows

$$\|u_n - \bar{u}_n\|_{L^p([a,b];W)} < \frac{\eta}{4}.$$

Now use compactness of the embedding of E into W to obtain a subsequence such that $\{\bar{u}_n\}$ is Cauchy in $L^p(a, b; W)$ and use this to contradict 43.1.10. Suppose $\bar{u}_n(t) = \sum_{i=1}^k u_i^n \mathcal{X}_{[t_{i-1}, t_i)}(t)$. Thus

$$\|\bar{u}_n(t)\|_E = \sum_{i=1}^k \|u_i^n\|_E \mathcal{X}_{[t_{i-1}, t_i)}(t)$$

and so

$$R \geq \int_a^b \|\bar{u}_n(t)\|_E^p dt = \frac{T}{k} \sum_{i=1}^k \|u_i^n\|_E^p$$

Therefore, the $\{u_i^n\}$ are all bounded. It follows that after taking subsequences k times there exists a subsequence $\{u_{n_k}\}$ such that u_{n_k} is a Cauchy sequence in $L^p(a, b; W)$. You simply get a subsequence such that $u_i^{n_k}$ is a Cauchy sequence in W for each i . Then denoting this subsequence by n ,

$$\begin{aligned} \|u_n - u_m\|_{L^p(a,b;W)} &\leq \|u_n - \bar{u}_n\|_{L^p(a,b;W)} \\ &\quad + \|\bar{u}_n - \bar{u}_m\|_{L^p(a,b;W)} + \|\bar{u}_m - u_m\|_{L^p(a,b;W)} \\ &\leq \frac{\eta}{4} + \|\bar{u}_n - \bar{u}_m\|_{L^p(a,b;W)} + \frac{\eta}{4} < \eta \end{aligned}$$

provided m, n are large enough, contradicting 43.1.10. This proves the theorem.

43.2 The K Method

This considers the problem of interpolating Banach spaces. The idea is to build a Banach space between two others in a systematic way, thus constructing a new Banach space from old ones. The first method of defining intermediate Banach spaces is called the K method. For more on this topic as well as the other topics on interpolation see [10] which is what I am following. See also [64]. There is far more on these subjects in these books than what I am presenting here! My goal is to present only enough to give an introduction to the topic and to use it in presenting more theory of Sobolev spaces.

In what follows a topological vector space is a vector space in which vector addition and scalar multiplication are continuous. That is $\cdot : \mathbb{F} \times X \rightarrow X$ is continuous and $+$: $X \times X \rightarrow X$ is also continuous.

A common example of a topological vector space is the dual space, X' of a Banach space, X with the weak $*$ topology. For $S \subseteq X$ a finite set, define

$$B_S(x^*, r) \equiv \{y^* \in X' : |y^*(x) - x^*(x)| < r \text{ for all } x \in S\}$$

Then the $B_S(x^*, r)$ for S a finite subset of X and $r > 0$ form a basis for the topology on X' called the weak $*$ topology. You can check that the vector space operations are continuous.

Definition 43.2.1 *Let A_0 and A_1 be two Banach spaces with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, also written as $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ and let X be a topological vector space such that $A_i \subseteq X$ for $i = 1, 2$, and the identity map from A_i to X is continuous. For each $t > 0$, define a norm on $A_0 + A_1$ by*

$$K(t, a) \equiv \|a\|_t \equiv \inf \{\|a_0\|_0 + t\|a_1\|_1 : a_0 + a_1 = a\}.$$

This is short for $K(t, a, A_0, A_1)$. Thus $K(t, a, A_1, A_0)$ will mean

$$K(t, a, A_1, A_0) \equiv \inf \{\|a_1\|_{A_1} + t\|a_0\|_{A_0} : a_0 + a_1 = a\}$$

but the default is $K(t, a, A_0, A_1)$ if $K(t, a)$ is written.

The following lemma is an interesting exercise.

Lemma 43.2.2 $(A_0 + A_1, K(t, \cdot))$ is a Banach space and all the norms, $K(t, \cdot)$ are equivalent.

Proof: First, why is $K(t, \cdot)$ a norm? It is clear that $K(t, a) \geq 0$ and that if $a = 0$ then $K(t, a) = 0$. Is this the only way this can happen? Suppose $K(t, a) = 0$. Then there exist $a_{0n} \in A_0$ and $a_{1n} \in A_1$ such that $\|a_{0n}\|_0 \rightarrow 0$, $\|a_{1n}\|_1 \rightarrow 0$, and $a = a_{0n} + a_{1n}$. Since the embedding of A_i into X is continuous and since X is a topological vector space¹, it follows

$$a = a_{0n} + a_{1n} \rightarrow 0$$

and so $a = 0$.

Let α be a nonzero scalar. Then

$$\begin{aligned} K(t, \alpha a) &= \inf \{ \|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = \alpha a \} \\ &= \inf \left\{ |\alpha| \left\| \frac{a_0}{\alpha} \right\|_0 + t |\alpha| \left\| \frac{a_1}{\alpha} \right\|_1 : \frac{a_0}{\alpha} + \frac{a_1}{\alpha} = a \right\} \\ &= |\alpha| \inf \left\{ \left\| \frac{a_0}{\alpha} \right\|_0 + t \left\| \frac{a_1}{\alpha} \right\|_1 : \frac{a_0}{\alpha} + \frac{a_1}{\alpha} = a \right\} \\ &= |\alpha| \inf \{ \|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = a \} = |\alpha| K(t, a). \end{aligned}$$

It remains to verify the triangle inequality. Let $\varepsilon > 0$ be given. Then there exist a_0, a_1, b_0 , and b_1 in A_0, A_1, A_0 , and A_1 respectively such that $a_0 + a_1 = a, b_0 + b_1 = b$ and

$$\begin{aligned} \varepsilon + K(t, a) + K(t, b) &> \|a_0\|_0 + t \|a_1\|_1 + \|b_0\|_0 + t \|b_1\|_1 \\ &\geq \|a_0 + b_0\|_0 + t \|b_1 + a_1\|_1 \geq K(t, a + b). \end{aligned}$$

This has shown that $K(t, \cdot)$ is at least a norm. Are all these norms equivalent? If $0 < s < t$ then it is clear that $K(t, a) \geq K(s, a)$. To show there exists a constant, C such that $CK(s, a) \geq K(t, a)$ for all a ,

$$\begin{aligned} \frac{t}{s} K(s, a) &\equiv \frac{t}{s} \inf \{ \|a_0\|_0 + s \|a_1\|_1 : a_0 + a_1 = a \} \\ &= \inf \left\{ \frac{t}{s} \|a_0\|_0 + s \frac{t}{s} \|a_1\|_1 : a_0 + a_1 = a \right\} \\ &= \inf \left\{ \frac{t}{s} \|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = a \right\} \\ &\geq \inf \{ \|a_0\|_0 + t \|a_1\|_1 : a_0 + a_1 = a \} = K(t, a). \end{aligned}$$

Therefore, the two norms are equivalent as hoped.

¹Vector addition is continuous is the property which is used here.

Finally, it is required to verify that $(A_0 + A_1, K(t, \cdot))$ is a Banach space. Since all these norms are equivalent, it suffices to only consider the norm, $K(1, \cdot)$. Let $\{a_{0n} + a_{1n}\}_{n=1}^\infty$ be a Cauchy sequence in $A_0 + A_1$. Then for m, n large enough,

$$K(1, a_{0n} + a_{1n} - (a_{0m} + a_{1m})) < \varepsilon.$$

It follows there exist $x_n \in A_0$ and $y_n \in A_1$ such that $x_n + y_n = 0$ for every n and whenever m, n are large enough,

$$\|a_{0n} + x_n - (a_{0m} + x_m)\|_0 + \|a_{1n} + y_n - (a_{1m} + y_m)\|_1 < \varepsilon$$

Hence $\{a_{1n} + y_n\}$ is a Cauchy sequence in A_1 and $\{a_{0n} + x_n\}$ is a Cauchy sequence in A_0 . Let

$$\begin{aligned} a_{0n} + x_n &\rightarrow a_0 \in A_0 \\ a_{1n} + y_n &\rightarrow a_1 \in A_1. \end{aligned}$$

Then

$$\begin{aligned} K(1, a_{0n} + a_{1n} - (a_0 + a_1)) &= K(1, a_{0n} + x_n + a_{1n} + y_n - (a_0 + a_1)) \\ &\leq \|a_{0n} + x_n - a_0\|_0 + \|a_{1n} + y_n - a_1\|_1 \end{aligned}$$

which converges to 0. Thus $A_0 + A_1$ is a Banach space as claimed.

With this, there exists a method for constructing a Banach space which lies between $A_0 \cap A_1$ and $A_0 + A_1$.

Definition 43.2.3 Let $1 \leq q < \infty, 0 < \theta < 1$. Define $(A_0, A_1)_{\theta, q}$ to be those elements of $A_0 + A_1$, a , such that

$$\|a\|_{\theta, q} \equiv \left[\int_0^\infty (t^{-\theta} K(t, a, A_0, A_1))^q \frac{dt}{t} \right]^{1/q} < \infty.$$

Theorem 43.2.4 $(A_0, A_1)_{\theta, q}$ is a normed linear space satisfying

$$A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta, q} \subseteq A_0 + A_1, \quad (43.2.12)$$

with the inclusion maps continuous, and

$$\left((A_0, A_1)_{\theta, q}, \|\cdot\|_{\theta, q} \right) \text{ is a Banach space.} \quad (43.2.13)$$

If $a \in A_0 \cap A_1$, then

$$\|a\|_{\theta, q} \leq \left(\frac{1}{q\theta(1-\theta)} \right)^{1/q} \|a\|_1^\theta \|a\|_0^{1-\theta}. \quad (43.2.14)$$

If $A_0 \subseteq A_1$ with $\|\cdot\|_0 \geq \|\cdot\|_1$, then

$$A_0 \cap A_1 = A_0 \subseteq (A_0, A_1)_{\theta, q} \subseteq A_1 = A_0 + A_1.$$

Also, if bounded sets in A_0 have compact closures in A_1 then the same is true if A_1 is replaced with $(A_0, A_1)_{\theta, q}$. Finally, if

$$T \in \mathcal{L}(A_0, B_0), T \in \mathcal{L}(A_1, B_1), \quad (43.2.15)$$

and T is a linear map from $A_0 + A_1$ to $B_0 + B_1$ where the A_i and B_i are Banach spaces with the properties described above, then it follows

$$T \in \mathcal{L}\left((A_0, A_1)_{\theta, q}, (B_0, B_1)_{\theta, q}\right) \quad (43.2.16)$$

and if M is its norm, and M_0 and M_1 are the norms of T as a map in $\mathcal{L}(A_0, B_0)$ and $\mathcal{L}(A_1, B_1)$ respectively, then

$$M \leq M_0^{1-\theta} M_1^\theta. \quad (43.2.17)$$

Proof: Suppose first $a \in A_0 \cap A_1$. Then

$$\|a\|_{\theta, q}^q \equiv \int_0^r (t^{-\theta} K(t, a))^q \frac{dt}{t} + \int_r^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \quad (43.2.18)$$

$$\begin{aligned} &\leq \int_0^r (t^{-\theta} \|a\|_1 t)^q \frac{dt}{t} + \int_r^\infty (t^{-\theta} \|a\|_0)^q \frac{dt}{t} \\ &= \|a\|_1^q \int_0^r t^{q(1-\theta)-1} dt + \|a\|_0^q \int_r^\infty t^{-1-\theta q} dt \\ &= \|a\|_1^q \frac{r^{q-q\theta}}{q-q\theta} + \|a\|_0^q \frac{r^{-\theta q}}{\theta q} < \infty \end{aligned} \quad (43.2.19)$$

Which shows the first inclusion of 43.2.12. The above holds for all $r > 0$ and in particular for the value of r which minimizes the expression on the right in 43.2.19, $r = \|a\|_0 / \|a\|_1$. Therefore, doing some calculus,

$$\|a\|_{\theta, q}^q \leq \frac{1}{\theta q (1 + \theta)} \|a\|_0^{q(1-\theta)} \|a\|_1^{q\theta}$$

which shows 43.2.14. This also verifies that the first inclusion map is continuous in 43.2.12 because if $a_n \rightarrow 0$ in $A_0 \cap A_1$, then $a_n \rightarrow 0$ in A_0 and in A_1 and so the above shows $a_n \rightarrow 0$ in $(A_0, A_1)_{\theta, q}$.

Now consider the second inclusion in 43.2.12. The inclusion is obvious because $(A_0, A_1)_{\theta, q}$ is given to be a subset of $A_0 + A_1$ defined by

$$\left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} < \infty$$

It remains to verify the inclusion map is continuous. Therefore, suppose $a_n \rightarrow 0$ in $(A_0, A_1)_{\theta, q}$. Since $a_n \rightarrow 0$ in $(A_0, A_1)_{\theta, q}$, it follows the function, $t \rightarrow t^{-\theta} K(t, a_n)$ converges to zero in $L^q(0, \infty)$ with respect to the measure, dt/t . Therefore, taking another subsequence, still denoted as a_n , you can assume this function converges

to 0 a.e. Pick such a t where this convergence takes place. Then $K(t, a_n) \rightarrow 0$ as $n \rightarrow \infty$ and so $a_n \rightarrow 0$ in $A_0 + A_1$. (Recall all these norms $K(t, \cdot)$ are equivalent.) This shows that if $a_n \rightarrow 0$ in $(A_0, A_1)_{\theta, q}$, then there exists a subsequence $\{a_{n_k}\}$ such that $a_{n_k} \rightarrow 0$ in $A_0 + A_1$. It follows that if $a_n \rightarrow 0$ in $(A_0, A_1)_{\theta, q}$, then $a_n \rightarrow 0$ in $A_0 + A_1$. This proves the continuity of the embedding.

What about 43.2.13? Suppose $\{a_n\}$ is a Cauchy sequence in $(A_0, A_1)_{\theta, q}$. Then from what was just shown this is a Cauchy sequence in $A_0 + A_1$ and so there exists $a \in A_0 + A_1$ such that $a_n \rightarrow a$ in $A_0 + A_1$ because $A_0 + A_1$ is a Banach space. Thus, $K(t, a_n) \rightarrow K(t, a)$ for all $t > 0$. (Recall all these norms $K(t, \cdot)$ are equivalent.) Therefore, by Fatou's lemma,

$$\begin{aligned} \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} &\leq \liminf_{n \rightarrow \infty} \left(\int_0^\infty (t^{-\theta} K(t, a_n))^q \frac{dt}{t} \right)^{1/q} \\ &\leq \max \left\{ \|a_n\|_{\theta, q} : n \in \mathbb{N} \right\} < \infty \end{aligned}$$

and so $a \in (A_0, A_1)_{\theta, q}$. Now

$$\begin{aligned} \|a - a_n\|_{\theta, q} &\leq \liminf_{m \rightarrow \infty} \left(\int_0^\infty (t^{-\theta} K(t, a_n - a_m))^q \frac{dt}{t} \right)^{1/q} \\ &= \liminf_{m \rightarrow \infty} \|a_n - a_m\|_{\theta, q} < \varepsilon \end{aligned}$$

whenever n is large enough. Thus $(A_0, A_1)_{\theta, q}$ is complete as claimed.

Next suppose $A_0 \subseteq A_1$ and the inclusion map is compact. In this case, $A_0 \cap A_1 = A_0$ and so it has been shown above that $A_0 \subseteq (A_0, A_1)_{\theta, q}$. It remains to show that every bounded subset, S , contained in A_0 has an η net in $(A_0, A_1)_{\theta, q}$. Recall the inequality, 43.2.14

$$\begin{aligned} \|a\|_{\theta, q} &\leq \left(\frac{1}{q\theta(1-\theta)} \right)^{1/q} \|a\|_1^\theta \|a\|_0^{1-\theta} \\ &= \frac{C}{\varepsilon} \|a\|_1^\theta \varepsilon \|a\|_0^{1-\theta}. \end{aligned}$$

Now this implies

$$\|a\|_{\theta, q} \leq \left(\frac{C}{\varepsilon} \right)^{1/\theta} \theta \|a\|_1 + \varepsilon^{1/(1-\theta)} (1-\theta) \|a\|_0$$

By compactness of the embedding of A_0 into A_1 , it follows there exists an $\varepsilon^{(1+\theta)/\theta}$ net for S in A_1 , $\{a_1, \dots, a_p\}$. Then for $a \in S$, there exists k such that $\|a - a_k\|_1 < \varepsilon^{(1+\theta)/\theta}$. It follows

$$\begin{aligned} \|a - a_k\|_{\theta, q} &\leq \left(\frac{C}{\varepsilon} \right)^{1/\theta} \theta \|a - a_k\|_1 + \varepsilon^{1/(1-\theta)} (1-\theta) \|a - a_k\|_0 \\ &\leq \left(\frac{C}{\varepsilon} \right)^{1/\theta} \theta \varepsilon^{(1+\theta)/\theta} + \varepsilon^{1/(1-\theta)} (1-\theta) 2M \\ &= C^{1/\theta} \theta \varepsilon + \varepsilon^{1/(1-\theta)} (1-\theta) 2M \end{aligned}$$

where M is large enough that $\|a\|_0 \leq M$ for all $a \in S$. Since ε is arbitrary, this shows the existence of a η net and proves the compactness of the embedding into $(A_0, A_1)_{\theta, q}$.

It remains to verify the assertions 43.2.15-43.2.17. Let $T \in \mathcal{L}(A_0, B_0), T \in \mathcal{L}(A_1, B_1)$ with T a linear map from $A_0 + A_1$ to $B_0 + B_1$. Let $a \in (A_0, A_1)_{\theta, q} \subseteq A_0 + A_1$ and consider $Ta \in B_0 + B_1$. Denote by $K(t, \cdot)$ the norm described above for both $A_0 + A_1$ and $B_0 + B_1$ since this will cause no confusion. Then

$$\|Ta\|_{\theta, q} \equiv \left(\int_0^\infty (t^{-\theta} K(t, Ta))^q \frac{dt}{t} \right)^{1/q}. \quad (43.2.20)$$

Now let $a_0 + a_1 = a$ and so $Ta_0 + Ta_1 = Ta$

$$\begin{aligned} K(t, Ta) &\leq \|Ta_0\|_0 + t\|Ta_1\|_1 \leq M_0\|a_0\|_0 + M_1t\|a_1\|_1 \\ &\leq M_0 \left(\|a_0\|_0 + t \left(\frac{M_1}{M_0} \right) \|a_1\|_1 \right) \end{aligned}$$

and so, taking inf for all $a_0 + a_1 = a$, yields

$$K(t, Ta) \leq M_0 K \left(t \left(\frac{M_1}{M_0} \right), a \right)$$

It follows from 43.2.20 that

$$\begin{aligned} \|Ta\|_{\theta, q} &\equiv \left(\int_0^\infty (t^{-\theta} K(t, Ta))^q \frac{dt}{t} \right)^{1/q} \\ &\leq \left(\int_0^\infty \left(t^{-\theta} M_0 K \left(t \left(\frac{M_1}{M_0} \right), a \right) \right)^q \frac{dt}{t} \right)^{1/q} \\ &= M_0 \left(\int_0^\infty \left(t^{-\theta} K \left(t \left(\frac{M_1}{M_0} \right), a \right) \right)^q \frac{dt}{t} \right)^{1/q} \\ &= M_0 \left(\int_0^\infty \left(\left(\frac{M_0}{M_1} s \right)^{-\theta} K(s, a) \right)^q \frac{ds}{s} \right)^{1/q} \\ &= M_1^\theta M_0^{(1-\theta)} \left(\int_0^\infty (s^{-\theta} K(s, a))^q \frac{ds}{s} \right)^{1/q} = M_1^\theta M_0^{(1-\theta)} \|a\|_{\theta, q}. \end{aligned}$$

This shows $T \in \mathcal{L}((A_0, A_1)_{\theta, q}, (B_0, B_1)_{\theta, q})$ and if M is the norm of $T, M \leq M_0^{1-\theta} M_1^\theta$ as claimed. This proves the theorem.

43.3 The J Method

There is another method known as the J method. Instead of

$$K(t, a) \equiv \inf \{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} : a_0 + a_1 = a \}$$

for $a \in A_0 + A_1$, this method considers $a \in A_0 \cap A_1$ and $J(t, a)$ defined below gives a norm on $A_0 \cap A_1$.

Definition 43.3.1 For A_0 and A_1 Banach spaces as described above, and $a \in A_0 \cap A_1$,

$$J(t, a) \equiv \max (||a||_{A_0}, t ||a||_{A_1}). \quad (43.3.21)$$

this is short for $J(t, a, A_0, A_1)$. Thus

$$J(t, a, A_1, A_0) \equiv \max (||a||_{A_1}, t ||a||_{A_0})$$

but unless indicated otherwise, A_0 will come first. Now for $\theta \in (0, 1)$ and $q \geq 1$, define a space, $(A_0, A_1)_{\theta, q, J}$ as follows. The space, $(A_0, A_1)_{\theta, q, J}$ will consist of those elements, a , of $A_0 + A_1$ which can be written in the form

$$a = \int_0^\infty u(t) \frac{dt}{t} \equiv \lim_{\varepsilon \rightarrow 0+} \int_\varepsilon^1 u(t) \frac{dt}{t} + \lim_{r \rightarrow \infty} \int_1^r u(t) \frac{dt}{t} \quad (43.3.22)$$

the limits taking place in $A_0 + A_1$ with the norm

$$K(1, a) \equiv \inf_{a=a_0+a_1} (||a_0||_{A_0} + ||a_1||_{A_1}),$$

where $u(t)$ is strongly measurable with values in $A_0 \cap A_1$ and bounded on every compact subset of $(0, \infty)$ such that

$$\left(\int_0^\infty (t^{-\theta} J(t, u(t), A_0, A_1))^q \frac{dt}{t} \right)^{1/q} < \infty. \quad (43.3.23)$$

For such $a \in A_0 + A_1$, define

$$||a||_{\theta, q, J} \equiv \inf_u \left\{ \left(\int_0^\infty (t^{-\theta} J(t, u(t), A_0, A_1))^q \frac{dt}{t} \right)^{1/q} \right\} \quad (43.3.24)$$

where the infimum is taken over all u satisfying 43.3.22 and 43.3.23.

Note that a norm on $A_0 \times A_1$ would be

$$||(a_0, a_1)|| \equiv \max (||a_0||_{A_0}, ||a_1||_{A_1})$$

and so $J(t, \cdot)$ is the restriction of this norm to the subspace of $A_0 \times A_1$ defined by $\{(a, a) : a \in A_0 \cap A_1\}$. Also for each $t > 0$ $J(t, \cdot)$ is a norm on $A_0 \cap A_1$ and furthermore, any two of these norms are equivalent. In fact, for $0 < t < s$,

$$\begin{aligned} J(t, a) &= \max (||a||_{A_0}, t ||a||_{A_1}) \\ &\geq \max (||a||_{A_0}, s ||a||_{A_1}) \\ &= J(s, a) \\ &\geq \max \left(\frac{s}{t} ||a||_{A_0}, s ||a||_{A_1} \right) \\ &= \frac{s}{t} \max (||a||_{A_0}, t ||a||_{A_1}) \\ &\geq \frac{s}{t} J(t, a). \end{aligned}$$

The following lemma is significant and follows immediately from the above definition.

Lemma 43.3.2 Suppose $a \in (A_0, A_1)_{\theta, q, J}$ and $a = \int_0^\infty u(t) \frac{dt}{t}$ where u is described above. Then letting $r > 1$,

$$u_r(t) \equiv \begin{cases} u(t) & \text{if } t \in (\frac{1}{r}, r) \\ 0 & \text{otherwise} \end{cases}.$$

it follows that

$$\int_0^\infty u_r(t) \frac{dt}{t} \in A_0 \cap A_1.$$

Proof: The integral equals $\int_{1/r}^r u(t) \frac{dt}{t} \cdot \int_{1/r}^r \frac{1}{t} dt = 2 \ln r < \infty$. Now u_r is measurable in $A_0 \cap A_1$ and bounded. Therefore, there exists a sequence of measurable simple functions, $\{s_n\}$ having values in $A_0 \cap A_1$ which converges pointwise and uniformly to u_r . It can also be assumed $J(r, s_n(t)) \leq J(r, u_r(t))$ for all $t \in [1/r, r]$. Therefore,

$$\lim_{n, m \rightarrow \infty} \int_{1/r}^r J(r, s_m - s_n) \frac{dt}{t} = 0.$$

It follows from the definition of the Bochner integral that

$$\lim_{n \rightarrow \infty} \int_{1/r}^r s_n \frac{dt}{t} = \int_{1/r}^r u_r \frac{dt}{t} \in A_0 \cap A_1.$$

This proves the lemma.

The remarkable thing is that the two spaces, $(A_0, A_1)_{\theta, q}$ and $(A_0, A_1)_{\theta, q, J}$ coincide and have equivalent norms. The following important lemma, called the fundamental lemma of interpolation theory in [10] is used to prove this. This lemma is really incredible.

Lemma 43.3.3 Suppose for $a \in A_0 + A_1$, $\lim_{t \rightarrow 0+} K(t, a) = 0$ and $\lim_{t \rightarrow \infty} \frac{K(t, a)}{t} = 0$. Then for any $\varepsilon > 0$, there is a representation,

$$a = \sum_{i=-\infty}^{\infty} u_i = \lim_{n, m \rightarrow \infty} \sum_{i=-m}^n u_i, \quad u_i \in A_0 \cap A_1, \quad (43.3.25)$$

the convergences taking place in $A_0 + A_1$, such that

$$J(2^i, u_i) \leq 3(1 + \varepsilon) K(2^i, a). \quad (43.3.26)$$

Proof: For each i , there exist $a_{0,i} \in A_0$ and $a_{1,i} \in A_1$ such that

$$a = a_{0,i} + a_{1,i},$$

and

$$(1 + \varepsilon) K(2^i, a) \geq \|a_{0,i}\|_{A_0} + 2^i \|a_{1,i}\|_{A_1}. \quad (43.3.27)$$

This follows directly from the definition of $K(t, a)$. From the assumed limit conditions on $K(t, a)$,

$$\lim_{i \rightarrow \infty} \|a_{1,i}\|_{A_1} = 0, \quad \lim_{i \rightarrow -\infty} \|a_{0,i}\|_{A_0} = 0. \quad (43.3.28)$$

Then let $u_i \equiv a_{0,i} - a_{0,i-1} = a_{1,i-1} - a_{1,i}$. The reason these are equal is $a = a_{0,i} + a_{1,i} = a_{0,i-1} + a_{1,i-1}$. Then

$$\sum_{i=-m}^n u_i = a_{0,n} - a_{0,-(m+1)} = a_{1,-(m+1)} - a_{1,n}.$$

It follows $a - \sum_{i=-m}^n u_i = a - (a_{0,n} - a_{0,-(m+1)}) = a_{0,-(m+1)} + a_{1,n}$, and both terms converge to zero as m and n converge to ∞ by 43.3.28. Therefore,

$$K\left(1, a - \sum_{i=-m}^n u_i\right) \leq \|a_{0,-(m+1)}\| + \|a_{1,n}\|$$

and so this shows $a = \sum_{i=-\infty}^{\infty} u_i$ which is one of the claims of the lemma. Also

$$\begin{aligned} J(2^i, u_i) &\equiv \max(\|u_i\|_{A_0}, 2^i \|u_i\|_{A_1}) \leq \|u_i\|_{A_0} + 2^i \|u_i\|_{A_1} \\ &\leq 2(\|a_{0,i-1}\|_{A_0} + 2^{i-1} \|a_{1,i-1}\|_{A_1}) \\ &\leq \|a_{0,i}\|_{A_0} + 2^i \|a_{1,i}\|_{A_1} + \overbrace{\|a_{0,i-1}\|_{A_0} + 2^{i-1} \|a_{1,i-1}\|_{A_1}} \\ &\leq (1 + \varepsilon) K(2^i, a) + 2(1 + \varepsilon) K(2^{i-1}, a) \leq 3(1 + \varepsilon) K(2^i, a) \end{aligned}$$

because $t \rightarrow K(t, a)$ is nondecreasing. This proves the lemma.

Lemma 43.3.4 *If $a \in A_0 \cap A_1$, then $K(t, a) \leq \min(1, \frac{t}{s}) J(s, a)$.*

Proof: If $s \geq t$, then $\min(1, \frac{t}{s}) = \frac{t}{s}$ and so

$$\begin{aligned} \min\left(1, \frac{t}{s}\right) J(s, a) &= \frac{t}{s} \max(\|a\|_{A_0}, s \|a\|_{A_1}) \geq \left(\frac{t}{s}\right) s \|a\|_{A_1} \\ &= t \|a\|_{A_1} \geq K(t, a). \end{aligned}$$

Now in case $s < t$, then $\min(1, \frac{t}{s}) = 1$ and so

$$\begin{aligned} \min\left(1, \frac{t}{s}\right) J(s, a) &= \max(\|a\|_{A_0}, s \|a\|_{A_1}) \geq \|a\|_{A_0} \\ &\geq K(t, a). \end{aligned}$$

This proves the lemma.

Theorem 43.3.5 *Let A_0, A_1, K and J be as described above. Then for all $q \geq 1$ and $\theta \in (0, 1)$,*

$$(A_0, A_1)_{\theta, q} = (A_0, A_1)_{\theta, q, J}$$

and furthermore, the norms are equivalent.

Proof: Begin with $a \in (A_0, A_1)_{\theta, q}$. Thus

$$\|a\|_{\theta, q}^q = \int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} < \infty \quad (43.3.29)$$

and it is necessary to produce $u(t)$ as described above,

$$a = \int_0^\infty u(t) \frac{dt}{t} \text{ where } \int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} < \infty.$$

From 43.3.29, $\lim_{t \rightarrow 0+} K(t, a) = 0$ since $t \rightarrow K(t, a)$ is nondecreasing and so if its limit is positive, the integrand would have a non integrable singularity like $t^{-\theta q - 1}$. Next consider what happens to $\frac{K(t, a)}{t}$ as $t \rightarrow \infty$.

Claim: $t \rightarrow \frac{K(t, a)}{t}$ is decreasing.

Proof of the claim: Choose $a_0 \in A_0$ and $a_1 \in A_1$ such that $a_0 + a_1 = a$ and

$$K(t, a) + \varepsilon t > \|a_0\|_{A_0} + t \|a_1\|_{A_1}$$

let $s > t$. Then

$$\frac{K(t, a) + t\varepsilon}{t} \geq \frac{\|a_0\|_{A_0} + t \|a_1\|_{A_1}}{t} \geq \frac{\|a_0\|_{A_0} + s \|a_1\|_{A_1}}{s} \geq \frac{K(s, a)}{s}.$$

Since ε is arbitrary, this proves the claim.

Let $r \equiv \lim_{t \rightarrow \infty} \frac{K(t, a)}{t}$. Is $r = 0$? Suppose to the contrary that $r > 0$. Then the integrand of 43.3.29, is at least as large as

$$\begin{aligned} t^{-\theta q} K(t, a)^{q-1} \frac{K(t, a)}{t} &\geq t^{-\theta q} K(t, a)^{q-1} r \\ &\geq t^{-\theta q} (tr)^{q-1} r \geq r^q t^{q(1-\theta)-1} \end{aligned}$$

whose integral is infinite. Therefore, $r = 0$.

Lemma 43.3.3, implies there exist $u_i \in A_0 \cap A_1$ such that $a = \sum_{i=-\infty}^\infty u_i$, the convergence taking place in $A_0 + A_1$ with the inequality of that Lemma holding,

$$J(2^i, u_i) \leq 3(1 + \varepsilon) K(2^i, a).$$

For i an integer and $t \in [2^{i-1}, 2^i]$, let

$$u(t) \equiv u_i / \ln 2.$$

Then

$$a = \sum_{i=-\infty}^\infty u_i = \int_0^\infty u(t) \frac{dt}{t}. \quad (43.3.30)$$

Now

$$\begin{aligned}
 \|a\|_{\theta,q,J}^q &\leq \int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} \\
 &= \sum_{i=-\infty}^\infty \int_{2^{i-1}}^{2^i} \left(t^{-\theta} J\left(t, \frac{u_i}{\ln 2}\right)\right)^q \frac{dt}{t} \\
 &\leq \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^\infty \int_{2^{i-1}}^{2^i} (t^{-\theta} J(2^i, u_i))^q \frac{dt}{t} \\
 &\leq \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^\infty \int_{2^{i-1}}^{2^i} (t^{-\theta} 3(1+\varepsilon) K(2^i, a))^q \frac{dt}{t}
 \end{aligned}$$

Using the above claim, $\frac{K(2^i, a)}{2^i} \leq \frac{K(2^{i-1}, a)}{2^{i-1}}$ and so $K(2^i, a) \leq 2K(2^{i-1}, a)$. Therefore, the above is no larger than

$$\begin{aligned}
 &\leq 2 \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^\infty \int_{2^{i-1}}^{2^i} (t^{-\theta} 3(1+\varepsilon) K(2^{i-1}, a))^q \frac{dt}{t} \\
 &\leq 2 \left(\frac{1}{\ln 2}\right)^q \sum_{i=-\infty}^\infty \int_{2^{i-1}}^{2^i} (t^{-\theta} 3(1+\varepsilon) K(t, a))^q \frac{dt}{t} \\
 &= 2 \left(\frac{3(1+\varepsilon)}{\ln 2}\right)^q \int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \equiv 2 \left(\frac{3(1+\varepsilon)}{\ln 2}\right)^q \|a\|_{\theta,q}^q \quad (43.3.31)
 \end{aligned}$$

This has shown that if $a \in (A_0, A_1)_{\theta,q}$, then by 43.3.30 and 43.3.31, $a \in (A_0, A_1)_{\theta,q,J}$ and

$$\|a\|_{\theta,q,J}^q \leq 2 \left(\frac{3(1+\varepsilon)}{\ln 2}\right)^q \|a\|_{\theta,q}^q. \quad (43.3.32)$$

It remains to prove the other inclusion and norm inequality, both of which are much easier to obtain. Thus, let $a \in (A_0, A_1)_{\theta,q,J}$ with

$$a = \int_0^\infty u(t) \frac{dt}{t} \quad (43.3.33)$$

where u is a strongly measurable function having values in $A_0 \cap A_1$ and for which

$$\int_0^\infty (t^{-\theta} J(t, u(t)))^q dt < \infty. \quad (43.3.34)$$

$$K(t, a) = K\left(t, \int_0^\infty u(s) \frac{ds}{s}\right) \leq \int_0^\infty K(t, u(s)) \frac{ds}{s}. \quad (43.3.35)$$

Now by Lemma 43.3.4, this is dominated by an expression of the form

$$\leq \int_0^\infty \min\left(1, \frac{t}{s}\right) J(s, u(s)) \frac{ds}{s} = \int_0^\infty \min\left(1, \frac{1}{s}\right) J(ts, u(ts)) \frac{ds}{s} \quad (43.3.36)$$

where the equation follows from a change of variable. From Minkowski's inequality and 43.3.36,

$$\begin{aligned} \|a\|_{\theta,q} &\equiv \left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t} \right)^{1/q} \\ &\leq \left(\int_0^\infty \left(t^{-\theta} \int_0^\infty \min\left(1, \frac{1}{s}\right) J(ts, u(ts)) \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q} \\ &\leq \int_0^\infty \left(\int_0^\infty \left(t^{-\theta} \min\left(1, \frac{1}{s}\right) J(ts, u(ts)) \right)^q \frac{dt}{t} \right)^{1/q} \frac{ds}{s}. \end{aligned}$$

Now change the variable in the inside integral to obtain, letting $t = \tau s$,

$$\begin{aligned} &\leq \int_0^\infty \min\left(1, \frac{1}{s}\right) \left(\int_0^\infty (t^{-\theta} J(ts, u(ts)))^q \frac{dt}{t} \right)^{1/q} \frac{ds}{s} \\ &= \int_0^\infty \min\left(1, \frac{1}{s}\right) s^\theta \frac{ds}{s} \left(\int_0^\infty (\tau^{-\theta} J(\tau, u(\tau)))^q \frac{d\tau}{\tau} \right)^{1/q} \\ &= \left(\frac{1}{(1-\theta)\theta} \right) \left(\int_0^\infty (\tau^{-\theta} J(\tau, u(\tau)))^q \frac{d\tau}{\tau} \right)^{1/q}. \end{aligned}$$

This has shown that

$$\|a\|_{\theta,q} \leq \left(\frac{1}{(1-\theta)\theta} \right) \left(\int_0^\infty (\tau^{-\theta} J(\tau, u(\tau)))^q \frac{d\tau}{\tau} \right)^{1/q} < \infty$$

for all u satisfying 43.3.33 and 43.3.34. Therefore, taking the infimum it follows $a \in (A_0, A_1)_{\theta,q}$ and

$$\|a\|_{\theta,q} \leq \left(\frac{1}{(1-\theta)\theta} \right) \|a\|_{\theta,q,J}.$$

This proves the theorem.

43.4 Duality And Interpolation

In this section it will be assumed that $A_0 \cap A_1$ is dense in A_i for $i = 0, 1$. This is done so that $A'_i \subseteq (A_0 \cap A_1)'$ and the inclusion map is continuous. Thus it makes sense to add something in A'_0 to something in A'_1 .

What is the dual space of $(A_0, A_1)_{\theta,q}$? The answer is based on the following

lemma, [10]. Remember that

$$J(t, a) = \max(\|a\|_{A_0}, t\|a\|_{A_1})$$

and this is a norm on $A_0 \cap A_1$ and

$$K(t, a) = \inf \{ \|a_0\|_{A_0} + t \|a_1\|_{A_1} : a = a_0 + a_1 \}.$$

As mentioned above, $A'_0 + A'_1 \subseteq (A_0 \cap A_1)'$. In fact these two are equal. This is the first part of the following lemma.

Lemma 43.4.1 *Suppose $A_0 \cap A_1$ is dense in A_i , $i = 0, 1$. Then*

$$A'_0 + A'_1 = (A_0 \cap A_1)', \quad (43.4.37)$$

and for $a' \in A'_0 + A'_1 = (A_0 \cap A_1)'$,

$$K(t, a') = \sup_{a \in A_0 \cap A_1} \frac{|a'(a)|}{J(t^{-1}, a)}. \quad (43.4.38)$$

Thus $K(t, \cdot)$ is an equivalent norm to the usual operator norm on $(A_0 \cap A_1)'$ taken with respect to $J(t^{-1}, \cdot)$. If, in addition to this, A_i is reflexive, then for $a' \in A'_0 \cap A'_1$, and $a \in A_0 \cap A_1$,

$$J(t, a') K(t^{-1}, a) \geq |a'(a)|. \quad (43.4.39)$$

Proof: First consider the claim that $A'_0 + A'_1 = (A_0 \cap A_1)'$. As noted above, $\subseteq$ is clear. Define a norm on $A_0 \times A_1$ as follows.

$$\|(a_0, a_1)\|_{A_0 \times A_1} \equiv \max(\|a_0\|_{A_0}, t^{-1} \|a_1\|_{A_1}). \quad (43.4.40)$$

Let $a' \in (A_0 \cap A_1)'$. Let

$$E \equiv \{(a, a) : a \in A_0 \cap A_1\}$$

with the norm $J(t^{-1}, a) \equiv \max(\|a\|_{A_0}, t^{-1} \|a\|_{A_1})$. Now define λ on E , the subspace of $A_0 \times A_1$ by

$$\lambda((a, a)) \equiv a'(a).$$

Thus λ is a continuous linear map on E and in fact,

$$|\lambda((a, a))| = |a'(a)| \leq \|a'\| J(t^{-1}, a).$$

By the Hahn Banach theorem there exists an extension of λ to all of $A_0 \times A_1$. This extension is of the form $(a'_0, a'_1) \in A'_0 \times A'_1$. Thus

$$(a'_0, a'_1)((a, a)) = a'_0(a) + a'_1(a) = a'(a)$$

and therefore, $a'_0 + a'_1 = a'$ provided $a'_0 + a'_1$ is continuous. But

$$\begin{aligned} |(a'_0 + a'_1)(a)| &= |a'_0(a) + a'_1(a)| \leq |a'_0(a)| + |a'_1(a)| \\ &\leq \|a'_0\| \|a\|_{A_0} + \|a'_1\| \|a\|_{A_1} \\ &\leq \|a'_0\| \|a\|_{A_0} + t \|a'_1\| t^{-1} \|a\|_{A_1} \\ &\leq (\|a'_0\| + t \|a'_1\|) J(t^{-1}, a) \end{aligned}$$

which shows that $a'_0 + a'_1$ is continuous and in fact

$$\|a'_0 + a'_1\|_{(A_0 \cap A_1)'} \leq (\|a'_0\| + t\|a'_1\|).$$

This proves the first part of the lemma.

Claim: With this definition of the norm in 43.4.40, the operator norm of $(a'_0, a'_1) \in (A_0 \times A_1)' = A'_0 \times A'_1$ is

$$\|(a'_0, a'_1)\|_{(A_0 \times A_1)'} = \|a'_0\|_{A'_0} + t\|a'_1\|_{A'_1}. \quad (43.4.41)$$

Proof of the claim: $|(a'_0, a'_1)(a_0, a_1)| \leq \|a'_0\| \|a_0\| + \|a'_1\| \|a_1\|$. Now suppose that $\|a_0\| = \max(\|a_0\|, t^{-1}\|a_1\|)$. Then this is no larger than

$$(\|a'_0\| + t\|a'_1\|)\|a_0\| = (\|a'_0\| + t\|a'_1\|)\max(\|a_0\|, t^{-1}\|a_1\|).$$

The other case is that $t^{-1}\|a_1\| = \max(\|a_0\|, t^{-1}\|a_1\|)$. In this case,

$$\begin{aligned} |(a'_0, a'_1)(a_0, a_1)| &\leq \|a'_0\| \|a_0\| + \|a'_1\| \|a_1\| \\ &\leq \|a'_0\| t^{-1}\|a_1\| + \|a'_1\| \|a_1\| \\ &= (\|a'_0\| + t\|a'_1\|) t^{-1}\|a_1\| \\ &= (\|a'_0\| + t\|a'_1\|)\max(\|a_0\|, t^{-1}\|a_1\|). \end{aligned}$$

This shows $\|(a'_0, a'_1)\|_{(A_0 \times A_1)'} \leq (\|a'_0\| + t\|a'_1\|)$. Is equality achieved? Let a_{0n} and a_{1n} be points of A_0 and A_1 respectively such that $\|a_{0n}\|, \|a_{1n}\| \leq 1$ and $\lim_{n \rightarrow \infty} a'_i(a_{in}) = \|a'_i\|$. Then

$$(a'_0, a'_1)(a_{0n}, ta_{1n}) \rightarrow \|a'_0\| + t\|a'_1\|$$

and also, $\|(a_{0n}, ta_{1n})\|_{A_0 \times A_1} = \max(\|a_{0n}\|, t^{-1}t\|a_{1n}\|_{A_1}) \leq 1$. Therefore, equality is indeed achieved and this proves the claim.

Consider 43.4.38. Take $a' \in A'_0 + A'_1 = (A_0 \cap A_1)'$ and let

$$E \equiv \{(a, a) \in A_0 \times A_1 : a \in A_0 \cap A_1\}.$$

Now define a linear map, λ on E as before.

$$\lambda((a, a)) \equiv a'(a).$$

If $a' = \tilde{a}'_0 + \tilde{a}'_1$,

$$\begin{aligned} |\lambda((a, a))| &\leq \|\tilde{a}'_0\|_{A'_0} \|a\|_{A_0} + \|\tilde{a}'_1\|_{A'_1} \|a\|_{A_1} \\ &= \|\tilde{a}'_0\|_{A'_0} \|a\|_{A_0} + t\|\tilde{a}'_1\|_{A'_1} t^{-1}\|a\|_{A_1} \\ &\leq (\|\tilde{a}'_0\| + t\|\tilde{a}'_1\|)\|(a, a)\|_{A_0 \times A_1} \end{aligned}$$

so λ is continuous on the subspace, E of $A_0 \times A_1$ and

$$\|\lambda\|_{E'} \leq \|\tilde{a}'_0\| + t\|\tilde{a}'_1\|. \quad (43.4.42)$$

By the Hahn Banach theorem, there exists an extension of λ defined on all of $A_0 \times A_1$ with the same norm. Thus, from 43.4.41, there exists $(a'_0, a'_1) \in (A_0 \times A_1)'$ which is an extension of λ such that

$$\|(a'_0, a'_1)\|_{(A_0 \times A_1)'} = \|a'_0\|_{A'_0} + t \|a'_1\|_{A'_1} = \|\lambda\|_{E'}$$

and for all $a \in A_0 \cap A_1$,

$$a'_0(a) + a'_1(a) = \lambda((a, a)) = a'(a).$$

It follows that $a'_0 + a'_1 = a'$ in $(A_0 \cap A_1)'$. Therefore, from 43.4.42,

$$\|\lambda\|_{E'} \leq \inf \{ \|\tilde{a}'_0\|_{A_0} + t \|\tilde{a}'_1\|_{A_1} : a' = \tilde{a}'_0 + \tilde{a}'_1 \} \equiv K(t, a') \quad (43.4.43)$$

$$\leq \|a'_0\|_{A'_0} + t \|a'_1\|_{A'_1} = \|\lambda\|_{E'} \equiv \sup_{a \in A_0 \cap A_1} \frac{|a'(a)|}{J(t^{-1}, a)} \quad (43.4.44)$$

because on E , $J(t^{-1}, a) = \|(a, a)\|_{A_0 \times A_1}$ which proves 43.4.38.

To obtain 43.4.39 in the case that A_i is reflexive, apply 43.4.38 to the case where A'_i plays the role of A_i in 43.4.38. Thus, for $a'' \in A'_0 + A'_1$,

$$K(t, a'') = \sup_{a' \in A'_0 \cap A'_1} \frac{|a''(a')|}{J(t^{-1}, a')}.$$

Now $a'' = a'_1 + a'_0 = \eta_1 a_1 + \eta_0 a_0$ where η_i is the map from A_i to A'_i which is onto and preserves norms, given by $\eta a(a') \equiv a'(a)$. Therefore, letting $a_1 + a_0 = a$

$$\begin{aligned} K(t, a) &= K(t, a'') = \sup_{a' \in A'_0 \cap A'_1} \frac{|a''(a')|}{J(t^{-1}, a')} \\ &= \sup_{a' \in A'_0 \cap A'_1} \frac{|(\eta_1 a_1 + \eta_0 a_0)(a')|}{J(t^{-1}, a')} = \sup_{a' \in A'_0 \cap A'_1} \frac{|a'(a_1 + a_0)|}{J(t^{-1}, a')} \end{aligned}$$

and so

$$K(t, a) = \sup_{a' \in A'_0 \cap A'_1} \frac{|a'(a)|}{J(t^{-1}, a')}$$

Changing $t \rightarrow t^{-1}$,

$$K(t^{-1}, a) J(t, a') \geq |a'(a)|.$$

which proves the lemma.

Consider $(A_0, A_1)'_{\theta, q}$.

Definition 43.4.2 Let $q \geq 1$. Then $\lambda^{\theta, q}$ will denote the sequences, $\{\alpha_i\}_{i=-\infty}^{\infty}$ such that

$$\sum_{i=-\infty}^{\infty} (|\alpha_i| 2^{-i\theta})^q < \infty.$$

For $\alpha \in \lambda^{\theta,q}$,

$$\|\alpha\|_{\lambda^{\theta,q}} \equiv \left(\sum_{i=-\infty}^{\infty} (|\alpha_i| 2^{-i\theta})^q \right)^{1/q}.$$

Thus $\alpha \in \lambda^{\theta,q}$ means $\{\alpha_i 2^{-i\theta}\} \in l_q$.

Lemma 43.4.3 *Let $f(t) \geq 0$, and let $f(t) = \alpha_i$ for $t \in [2^i, 2^{i+1})$ where $\alpha \in \lambda^{\theta,q}$. Then there exists a constant, C , such that*

$$\|t^{-\theta} f\|_{L^q(0, \infty; \frac{dt}{t})} \leq C \|\alpha\|_{\lambda^{\theta,q}}. \quad (43.4.45)$$

Also, if whenever $\alpha \in \lambda^{\theta,q}$, and $\alpha_i \geq 0$ for all i ,

$$\sum_i f(2^i) 2^{-i} \alpha_i \leq C \|\alpha\|_{\lambda^{\theta,q}}, \quad (43.4.46)$$

then

$$\left\| \{f(2^i)\}_{i=-\infty}^{\infty} \right\|_{\lambda^{1-\theta,q'}} \leq C. \quad (43.4.47)$$

Proof: Consider 43.4.45.

$$\begin{aligned} \int_0^{\infty} (t^{-\theta} f(t))^q \frac{dt}{t} &= \sum_i \int_{2^i}^{2^{i+1}} t^{-\theta q} \alpha_i^q \frac{dt}{t} \\ &\leq \sum_i \int_{2^i}^{2^{i+1}} (2^{-i\theta} \alpha_i)^q \frac{dt}{t} = \ln 2 \sum_i (2^{-i\theta} \alpha_i)^q = \ln 2 \|\alpha\|_{\lambda^{\theta,q}}^q. \end{aligned}$$

43.4.47 is next. By 43.4.46, whenever $\alpha \in \lambda^{\theta,q}$,

$$\left| \sum_i \left(f(2^i) 2^{-(1-\theta)i} \right) 2^{-\theta i} \alpha_i \right| \leq C \|\{2^{-\theta i} |\alpha_i|\}\|_{l_q}.$$

It follows from the Riesz representation theorem that $\{f(2^i) 2^{-(1-\theta)i}\}$ is in $l_{q'}$ and

$$\left\| \{f(2^i) 2^{-(1-\theta)i}\} \right\|_{l_{q'}} = \|\{f(2^i)\}\|_{\lambda^{1-\theta,q'}} \leq C.$$

This proves the lemma.

The dual space of $(A_0, A_1)_{\theta,q,J}$ is discussed next.

Lemma 43.4.4 *Let $\theta \in (0, 1)$ and let $q \geq 1$. Then,*

$$(A_0, A_1)'_{\theta,q,J} \subseteq (A'_1, A'_0)_{1-\theta,q'}$$

and the inclusion map is continuous.

Proof: Let $a' \in (A_0, A_1)'_{\theta, q, J}$. Now

$$A_0 \cap A_1 \subseteq (A_0, A_1)_{\theta, q, J}$$

and if

$$a \in (A_0, A_1)_{\theta, q, J},$$

then a has a representation of the form

$$a = \int_0^\infty u(t) \frac{dt}{t}$$

where

$$\int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} < \infty$$

where

$$J(t, u(t)) = \max(\|u(t)\|_{A_0}, t\|u(t)\|_{A_1})$$

for $u(t) \in A_0 \cap A_1$. Now let

$$u_r(t) \equiv \begin{cases} u(t) & \text{if } t \in (\frac{1}{r}, r) \\ 0 & \text{otherwise} \end{cases}.$$

Then $\int_0^\infty (t^{-\theta} J(t, u_r(t)))^q \frac{dt}{t} < \infty$ and

$$a_r \equiv \int_0^\infty u_r(t) \frac{dt}{t} \in A_0 \cap A_1$$

by Lemma 43.3.2. Also

$$\begin{aligned} \|a - a_r\|_{\theta, q, J}^q &\leq \int_0^{\frac{1}{r}} (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} + \\ &\quad \int_r^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} \end{aligned}$$

which is small whenever r is large enough thanks to the dominated convergence theorem. Therefore, $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta, q, J}$ and so

$$(A_0, A_1)'_{\theta, q, J} \subseteq (A_0 \cap A_1)' = A'_0 + A'_1,$$

the equality following from Lemma 43.4.1.

It follows $a' \in A'_0 + A'_1$ and so, by Lemma 43.4.1, there exists $b_i \in A_0 \cap A_1$ such that

$$K(2^{-i}, a', A'_0, A'_1) - \varepsilon \min(1, 2^{-i}) \leq \frac{a'(b_i)}{J(2^i, b_i, A_0, A_1)}.$$

Now let $\alpha \in \lambda^{\theta, q}$ with $\alpha_i \geq 0$ for all i and let

$$a_\infty \equiv \sum_i J(2^i, b_i, A_0, A_1)^{-1} b_i \alpha_i. \quad (43.4.48)$$

Consider first whether a_∞ makes sense before proceeding further.

$$a_\infty \equiv \sum_i \frac{b_i 2^{i\theta}}{\max(\|b_i\|_{A_0}, 2^i \|b_i\|_{A_1})} 2^{-i\theta} \alpha_i.$$

Now

$$\left\| \frac{b_i 2^{i\theta}}{\max(\|b_i\|_{A_0}, 2^i \|b_i\|_{A_1})} \right\|_{A_0+A_1} \leq \begin{cases} 2^{i\theta} & \text{if } i < 0 \\ 2^{-i(1-\theta)} & \text{if } i \geq 0 \end{cases} \quad (43.4.49)$$

This is fairly routine to verify. Consider the case where $i \geq 0$. Then

$$\left\| \frac{b_i 2^{i\theta}}{\max(\|b_i\|_{A_0}, 2^i \|b_i\|_{A_1})} \right\|_{A_0+A_1} \leq \left\| \frac{b_i 2^{i\theta}}{2^i \|b_i\|_{A_1}} \right\|_{A_0+A_1} \leq 2^{-i(1-\theta)}$$

because $\|b_i\|_{A_1} \geq \|b_i\|_{A_0+A_1}$. Therefore,

$$\begin{aligned} \sum_{i=0}^M \left\| \frac{b_i 2^{i\theta}}{\max(\|b_i\|_{A_0}, 2^i \|b_i\|_{A_1})} 2^{-i\theta} \alpha_i \right\|_{A_0+A_1} &\leq \\ \sum_{i=0}^M 2^{-i(1-\theta)} 2^{-i\theta} \alpha_i &\leq \left(\sum_{i=0}^{\infty} 2^{-i(1-\theta)q'} \right)^{1/q'} \left(\sum_{i=0}^{\infty} 2^{-iq\theta} \alpha_i^q \right)^{1/q} < \infty \end{aligned}$$

and similarly,

$$\sum_{i=-\infty}^0 \left\| \frac{b_i 2^{i\theta}}{\max(\|b_i\|_{A_0}, 2^i \|b_i\|_{A_1})} 2^{-i\theta} \alpha_i \right\|_{A_0+A_1}$$

converges. Therefore, a_∞ makes sense in $A_0 + A_1$ and also from 43.4.49, we see that

$$\left\{ \frac{\|b_i\|_{A_0+A_1} 2^{i\theta}}{J(2^i, b_i)} \right\} \in \lambda^{(1-\theta)q'}$$

Now let

$$u(t) \equiv \frac{\alpha_i b_i}{J(2^i, b_i) \ln 2} \text{ on } [2^{i-1}, 2^i).$$

Then

$$\begin{aligned} \int_0^\infty u(t) \frac{dt}{t} &= \sum_i \int_{2^{i-1}}^{2^i} \frac{\alpha_i b_i}{J(2^i, b_i) \ln 2} \frac{dt}{t} \\ &= \sum_i \frac{\alpha_i b_i}{J(2^i, b_i)} = a_\infty. \end{aligned}$$

Also

$$\int_0^\infty (t^{-\theta} J(t, u(t)))^q \frac{dt}{t} \leq \sum_i \int_{2^{i-1}}^{2^i} \left(2^{(1-i)\theta} J(2^i, u(2^{i-1})) \right) \frac{dt}{t}$$

$$\begin{aligned}
&\leq \sum_i \left[2^{-(i-1)\theta} J(2^i, u(2^{i-1})) \right]^q \ln 2 \\
&= \sum_i \left[2^{-(i-1)\theta} \frac{J(2^i, b_i) \alpha_i}{J(2^i, b_i) \ln 2} \right]^q \ln 2 \\
&= C \sum_i (2^{-i\theta} |\alpha_i|)^q < \infty
\end{aligned} \tag{43.4.50}$$

and so $\|a_\infty\|_{\theta, q, J} < \infty$. Now for a' as above, $a' \in (A_0, A_1)'_{\theta, q, J} \subseteq (A_0 + A_1)'$, and so since the sum for a_∞ converges in $A_0 + A_1$, we have

$$a'(a_\infty) = \sum_i J(2^i, b_i)^{-1} \alpha_i a'(b_i).$$

Therefore,

$$\begin{aligned}
a'(a_\infty) &\geq \sum_i [K(2^{-i}, a') - \varepsilon \min(1, 2^{-i})] \alpha_i \\
&= \sum_i K(2^{-i}, a') \alpha_i - \sum_i \varepsilon \min(1, 2^{-i}) \alpha_i \\
&= \sum_i K(2^{-i}, a') \alpha_i - O(\varepsilon)
\end{aligned} \tag{43.4.51}$$

The reason for this is that $\alpha \in \lambda^{\theta, q}$ so $\{\alpha_i 2^{-i\theta}\} \in l_q$. Therefore,

$$\begin{aligned}
&\sum_i \varepsilon \min(1, 2^{-i}) \alpha_i \\
&= \varepsilon \left\{ \sum_{i=0}^{\infty} 2^{-i} \alpha_i + \sum_{i=-\infty}^{-1} \alpha_i \right\} \\
&= \varepsilon \left\{ \sum_{i=0}^{\infty} 2^{-i\theta} 2^{(\theta-1)i} \alpha_i + \sum_{i=-\infty}^{-1} \alpha_i 2^{-i\theta} 2^{i\theta} \right\} \\
&\leq \varepsilon \left\{ \left(\sum_i |\alpha_i 2^{-i\theta}|^q \right)^{1/q} \left(\sum_{i=0}^{\infty} (2^{(\theta-1)i})^{q'} \right)^{1/q'} \right. \\
&\quad \left. + \left(\sum_i |\alpha_i 2^{-i\theta}|^q \right)^{1/q} \left(\sum_{i=0}^{\infty} (2^{\theta i})^{q'} \right)^{1/q'} \right\} \\
&< C\varepsilon.
\end{aligned}$$

Also

$$|a'(a_\infty)| \leq \|a'\|_{(A_0, A_1)'_{\theta, q, J}} \|a_\infty\|_{(A_0, A_1)_{\theta, q, J}}.$$

Now from the definition of K ,

$$K(2^{-i}, a', A'_0, A'_1) = 2^{-i} K(2^i, a', A'_1, A'_0)$$

and so from 43.4.51

$$\begin{aligned} \sum_i 2^{-i} K(2^i, a', A'_1, A'_0) \alpha_i - O(\varepsilon) &\leq a'(a_\infty) \\ &\leq \|a'\|_{(A_0, A_1)'_{\theta, q, J}} C_\theta \|\alpha\|_{\lambda^{\theta, q}}. \end{aligned}$$

Since ε is arbitrary, it follows that whenever, $\alpha \in \lambda^{\theta, q}$, $\alpha_i \geq 0$,

$$\sum_i 2^{-i} K(2^i, a', A'_1, A'_0) \alpha_i \leq \|a'\|_{(A_0, A_1)'_{\theta, q, J}} C_\theta \|\alpha\|_{\lambda^{\theta, q}}.$$

By Lemma 43.4.3, $\{K(2^i, a', A'_1, A'_0)\} \in \lambda^{1-\theta, q'}$ and

$$\|\{K(2^i, a', A'_1, A'_0)\}\|_{\lambda^{1-\theta, q'}} \leq \|a'\|_{(A_0, A_1)'_{\theta, q, J}} C_\theta.$$

Therefore,

$$\begin{aligned} &\left(\frac{1}{\ln 2} \int_0^\infty \left(K(t, a', A'_1, A'_0) t^{-(1-\theta)} \right)^{q'} \frac{dt}{t} \right)^{1/q'} \\ &= \left(\sum_i \frac{1}{\ln 2} \int_{2^i}^{2^{i+1}} \left(K(t, a', A'_1, A'_0) t^{-(1-\theta)} \right)^{q'} \frac{dt}{t} \right)^{1/q'} \\ &\leq \left(\sum_i \left(2^{-i(1-\theta)} K(2^i, a', A'_1, A'_0) \right)^{q'} \right)^{1/q'} \\ &\leq \|a'\|_{(A_0, A_1)'_{\theta, q, J}} C_\theta. \end{aligned}$$

Thus

$$\begin{aligned} \|a'\|_{(A'_1, A'_0)_{1-\theta, q'}} &\equiv \left\| t^{-(1-\theta)} K(t, a', A'_1, A'_0) \right\|_{L^{q'}(0, \infty, \frac{dt}{t})} \\ &\leq C \|a'\|_{(A_0, A_1)'_{\theta, q, J}} \end{aligned}$$

which shows that $(A_0, A_1)'_{\theta, q, J} \subseteq (A'_1, A'_0)_{1-\theta, q'}$ with the inclusion map continuous. This proves the lemma.

Lemma 43.4.5 *If A_i is reflexive for $i = 0, 1$ and if $A_0 \cap A_1$ is dense in A_i , then*

$$(A'_1, A'_0)_{1-\theta, q', J} \subseteq (A_0, A_1)'_{\theta, q}$$

and the inclusion map is continuous.

Proof: Let $a' \in (A'_1, A'_0)_{1-\theta, q', J}$. Thus, there exists u^* bounded on compact subsets of $(0, \infty)$ and measurable with values in $A_0 \cap A_1$ and

$$a' = \int_0^\infty u^*(t) \frac{dt}{t}, \quad (43.4.52)$$

$$\int_0^\infty \left(t^{-(1-\theta)} J(t, u^*(t)) \right)^{q'} \frac{dt}{t} < \infty.$$

Then

$$a' = \sum_{i=-\infty}^\infty \int_{2^i}^{2^{i+1}} u^*(t) \frac{dt}{t} \equiv \sum_{i=-\infty}^\infty a'_i$$

where $a'_i \in A'_1 \cap A'_0$, the convergence taking place in $A'_1 + A'_0$. Now let $a \in A_0 \cap A_1$. From Lemma 43.4.1

$$\begin{aligned} |a'(a)| &\leq \sum_{i=-\infty}^\infty |a'_i(a)| \\ &\leq \sum_{i=-\infty}^\infty J(2^{-i}, a'_i, A'_0, A'_1) K(2^i, a, A_0, A_1) \\ &= \sum_{i=-\infty}^\infty 2^{-i} J(2^i, a'_i, A'_1, A'_0) K(2^i, a, A_0, A_1) \\ &\leq \left(\sum_i \left(2^{-(1-\theta)i} J(2^i, a'_i, A'_1, A'_0) \right)^{q'} \right)^{1/q'} \\ &\quad \left(\sum_i \left(2^{-\theta i} K(2^i, a, A_0, A_1) \right)^q \right)^{1/q} \\ &\leq C \left[\int_0^\infty \left(t^{-(1-\theta)} J(t, u^*(t), A'_1, A'_0) \right)^{q'} \frac{dt}{t} \right]^{1/q'} \\ &\quad \left[\int_0^\infty \left(t^{-\theta} K(t, a, A_0, A_1) \right)^q \frac{dt}{t} \right]^{1/q}. \end{aligned}$$

In going from the sums to the integrals, express the first sum as a sum of integrals on $[2^i, 2^{i+1})$ and the second sum as a sum of integrals on $(2^{i-1}, 2^i]$.

Taking the infimum over all u^* representing a' ,

$$|a'(a)| \leq C \|a'\|_{(A'_1, A'_0)_{1-\theta, q', J}} \|a\|_{\theta, q}.$$

It follows $a' \in (A_0, A_1)'_{\theta, q}$ and $\|a'\|_{(A_0, A_1)'_{\theta, q}} \leq C \|a'\|_{(A'_1, A'_0)_{1-\theta, q', J}}$ which proves the lemma.

With these two lemmas the main result follows.

Theorem 43.4.6 *Suppose $A_0 \cap A_1$ is dense in A_i and A_i is reflexive. Then*

$$(A'_1, A'_0)_{1-\theta, q'} = (A_0, A_1)'_{\theta, q}$$

and the norms are equivalent.

Proof: By Theorem 43.3.5, and the last two lemmas,

$$\begin{aligned} (A_0, A_1)'_{\theta, q} &= (A_0, A_1)'_{\theta, q, J} \subseteq (A'_1, A'_0)_{1-\theta, q'} \\ &= (A'_1, A'_0)_{1-\theta, q', J} \subseteq (A_0, A_1)'_{\theta, q}. \end{aligned}$$

This proves the theorem.

Trace Spaces

44.1 Definition And Basic Theory Of Trace Spaces

Another approach to these sorts of problems is to use trace spaces. This allows the consideration of fractional order Sobolev spaces. In so far as the subject of Sobolev spaces is concerned, I will present this material in a manner which is essentially independent of the previous material on interpolation spaces.

As in the case of interpolation spaces, suppose A_0 and A_1 are two Banach spaces which are continuously embedded in some topological vector space, X .

Definition 44.1.1 Define a norm on $A_0 + A_1$ as follows.

$$\|a\|_{A_0+A_1} \equiv \inf \{ \|a_0\|_{A_0} + \|a_1\|_{A_1} : a_0 + a_1 = a \} \quad (44.1.1)$$

Lemma 44.1.2 $A_0 + A_1$ with the norm just described is a Banach space.

Proof: This was already explained in the treatment of the K method of interpolation. It is just $K(1, a)$.

Definition 44.1.3 Take f' in the sense of distributions for any

$$f \in L^1_{loc}(0, \infty; A_0 + A_1)$$

as follows.

$$f'(\phi) \equiv \int_0^\infty -f(t) \phi'(t) dt$$

whenever $\phi \in C_c^\infty(0, \infty)$. Define a Banach space, $W(A_0, A_1, p, \theta) = W$ where $p \geq 1, \theta \in (0, 1)$. Let

$$\|f\|_W \equiv \max \left(\|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} \right) \quad (44.1.2)$$

and let W consist of $f \in L^1_{loc}(0, \infty; A_0 + A_1)$ such that $\|f\|_W < \infty$.

Note that to be in W , $f(t) \in A_0$ and $f'(t) \in A_1$.

Lemma 44.1.4 *If $f \in W$, then*

$$\text{Trace}(f) \equiv f(0) \equiv \lim_{t \rightarrow 0} f(t)$$

exists in $A_0 + A_1$. Also $Z \equiv \{f \in W : f(0) = 0\}$ is a closed subspace of W . In addition to this, for every $f \in W$ and $\varepsilon > 0$ there exists a $g \in W$ such that $\|f - g\|_W < \varepsilon$ and $g \in C^\infty(0, \infty; A_0)$ while $g' \in C^\infty(0, \infty; A_1)$.

Proof: Let $0 < s < t$. Let $\nu + \frac{1}{p} = \theta$. Then for a generic g ,

$$\int_0^\infty \|\tau^\nu g(\tau)\|^p d\tau = \int_0^\infty \|\tau^\theta g(\tau)\|^p \frac{d\tau}{\tau}$$

so that $t^\nu f' \in L^p(0, \infty; A_1)$, the measure in this case being usual Lebesgue measure. Then

$$f(t) - f(s) = \int_s^t f'(\tau) d\tau = \int_s^t \tau^\nu f'(\tau) \tau^{-\nu} d\tau.$$

For $\frac{1}{p} + \frac{1}{p'} = 1$, $\nu p' = \left(\theta - \frac{1}{p}\right)p' < 1$ because $\theta < 1 = \frac{1}{p'} + \frac{1}{p}$. Therefore,

$$\begin{aligned} & \|f(t) - f(s)\|_{A_0+A_1} \\ & \leq \int_s^t \|f'(\tau)\|_{A_0+A_1} d\tau \\ & \leq \int_s^t \|f'(\tau)\|_{A_1} d\tau = \int_s^t \|\tau^\nu f'(\tau)\|_{A_1} \tau^{-\nu} d\tau \\ & \leq \left(\int_s^t \|\tau^\nu f'(\tau)\|_{A_1}^p d\tau \right)^{1/p} \left(\int_s^t \tau^{-\nu p'} d\tau \right)^{1/p'} \\ & \leq \|f\|_W \left(\frac{t^{1-\nu p'}}{1-\nu p'} - \frac{s^{1-\nu p'}}{1-\nu p'} \right) \\ & \leq \|f\|_W \frac{t^{1-\nu p'}}{1-\nu p'}. \end{aligned} \tag{44.1.3}$$

which converges to 0 as $t \rightarrow 0$. This shows that $\lim_{t \rightarrow 0+} f(t)$ exists in $A_0 + A_1$.

Clearly Z is a subspace. Let $f_n \rightarrow f$ in W and suppose $f_n \in Z$. Then since $f \in W$, 44.1.3 implies f is continuous. Using 44.1.3 and replacing f with $f_n - f_m$ and then taking a limit as $s \rightarrow 0$,

$$\|f_n(t) - f_m(t)\|_{A_0+A_1} \leq \|f_n - f_m\|_W C_\nu t^{1-\nu p'}$$

Taking a subsequence, it can be assumed $f_n(t)$ converges to $f(t)$ a.e. But the above inequality shows that $f_n(t)$ is a Cauchy sequence in $C([0, \beta]; A_0 + A_1)$ for all $\beta < \infty$. Therefore, $f_n(t) \rightarrow f(t)$ for all t . Also,

$$\|f_n(t)\|_{A_0+A_1} \leq C_\nu \|f_n\|_W t^{1-\nu p'} \leq K t^{1-\nu p'}$$

for some K depending on $\max \{\|f_n\| : n \geq 1\}$ and so

$$\|f(t)\|_{A_0+A_1} \leq K t^{1-\nu_{p'}}$$

which implies $f(0) = 0$. Thus Z is closed.

Consider the last claim. For a generic $t^\theta g \in L^p(0, \infty, \frac{dt}{t}; A)$, changing variables $t = e^\tau$,

$$\int_0^\infty t^{\theta p} |g(t)|^p \frac{dt}{t} = \int_{-\infty}^\infty e^{\tau \theta p} |g(e^\tau)|^p d\tau$$

Let $\tilde{g}(\tau) \equiv g(e^\tau)$. Thus $\tau \rightarrow e^{\tau \theta} \tilde{g}(\tau)$ is $L^p(\mathbb{R}; A)$ and $\tilde{g} \in L^1_{loc}(\mathbb{R})$. Now let ψ_δ be a mollifier and consider

$$\begin{aligned} e^{\theta \tau} \tilde{g}_\delta(\tau) &\equiv \int_{-\infty}^\infty e^{\theta \sigma} \tilde{g}(\sigma) \psi_\delta(\tau - \sigma) d\sigma \\ &= \int_{-\infty}^\infty e^{\theta(\tau - \sigma)} \tilde{g}(\tau - \sigma) \psi_\delta(\sigma) d\sigma \end{aligned}$$

so that

$$\begin{aligned} \tilde{g}_\delta(\tau) &= \int_{-\infty}^\infty e^{-\theta \sigma} \tilde{g}(\tau - \sigma) \psi_\delta(\sigma) d\sigma \\ &= \int_{-\infty}^\infty e^{-\theta(\tau - \sigma)} \tilde{g}(\sigma) \psi_\delta(\tau - \sigma) d\sigma \end{aligned}$$

Thus $\tilde{g}_\delta \in C^\infty(\mathbb{R}; A)$ and using Minkowski's inequality,

$$\begin{aligned} &\left(\int_{-\infty}^\infty \|e^{\theta \tau} \tilde{g}_\delta(\tau) - e^{\theta \tau} \tilde{g}(\tau)\|^p d\tau \right)^{1/p} = \\ &\left(\int_{-\infty}^\infty \left\| \int_{-\infty}^\infty \left(e^{\theta(\tau - \sigma)} \tilde{g}(\tau - \sigma) - e^{\theta \tau} \tilde{g}(\tau) \right) \psi_\delta(\sigma) d\sigma \right\|^p d\tau \right)^{1/p} \\ &\leq \int_{-\delta}^\delta \psi_\delta(\sigma) \left(\int_{-\infty}^\infty \|e^{\theta(\tau - \sigma)} \tilde{g}(\tau - \sigma) - e^{\theta \tau} \tilde{g}(\tau)\|^p d\tau \right)^{1/p} d\sigma \\ &\leq \varepsilon \int_{-\infty}^\infty \psi_\delta(\sigma) d\sigma = \varepsilon \end{aligned} \tag{44.1.4}$$

provided δ is small enough due to continuity of translation in L^p . Thus changing variables in 44.1.4, letting $\tau = \ln(t)$ and $g_\delta(t) \equiv \tilde{g}_\delta(\ln(t))$, it follows $g_\delta \in C^\infty(0, \infty; A)$ and this integral equals

$$\left(\int_0^\infty t^{\theta p} \|g_\delta(t) - g(t)\|^p \frac{dt}{t} \right)^{1/p}$$

This result applied to f and f' with $A = A_0$ and then $A = A_1$ shows the last claim. This proves the lemma.

Definition 44.1.5 Let W be a Banach space and let Z be a closed subspace. Then the quotient space, denoted by W/Z consists of the set of equivalence classes $[x]$ where the equivalence relation is defined by $x \sim y$ means $x - y \in Z$. Then W/Z is a vector space if the operations are defined by $\alpha[x] \equiv [\alpha x]$ and $[x] + [y] \equiv [x + y]$ and these vector space operations are well defined. The norm on the quotient space is defined as $||[x]|| \equiv \inf \{||x + z|| : z \in Z\}$.

The verification of the algebraic claims made in the above definition is left to the reader. It is routine. What is not as routine is the following lemma. However, it is similar to some topics in the presentation of the K method of interpolation.

Lemma 44.1.6 Let W be a Banach space and let Z be a closed subspace of W . Then W/Z with the norm described above is a Banach space.

Proof: That W/Z is a vector space is left to the reader. Why is $||\cdot||$ a norm? Suppose $\alpha \neq 0$. Then

$$\begin{aligned} ||\alpha[x]|| &= ||[\alpha x]|| \equiv \inf \{||\alpha x + z|| : z \in Z\} \\ &= \inf \{||\alpha x + \alpha z|| : z \in Z\} \\ &= |\alpha| \inf \{||x + z|| : z \in Z\} = |\alpha| ||[x]||. \end{aligned}$$

Now let $||[x]|| \geq ||x + z_1|| - \varepsilon$ and let $||[y]|| \geq ||y + z_2|| - \varepsilon$ where $z_i \in Z$. Then

$$\begin{aligned} ||[x] + [y]|| &\equiv ||[x + y]|| \leq ||x + y + z_1 + z_2|| \\ &\leq ||x + z_1|| + ||y + z_2|| \leq ||[x]|| + ||[y]|| + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, this shows the triangle inequality. Clearly, $||[x]|| \geq 0$. It remains to show that the only way $||[x]|| = 0$ is for $x \in Z$. Suppose then that $||[x]|| = 0$. This means there exist $z_n \in Z$ such that $||x + z_n|| \rightarrow 0$. Therefore, $-x$ is a limit of a sequence of points of Z and since Z is closed, this requires $-x \in Z$. Hence $x \in Z$ also because Z is a subspace. This shows $||\cdot||$ is a norm on W/Z . It remains to verify that W/Z is a Banach space.

Suppose $\{[x_n]\}$ is a Cauchy sequence in W/Z and suppose $||[x_n] - [x_{n+1}]|| < \frac{1}{2^{n+1}}$. Let $x'_1 = x_1$. If x'_n has been chosen let $x'_{n+1} = x_{n+1} + z_{n+1}$ where $z_{n+1} \in Z$ be such that

$$\begin{aligned} ||x'_{n+1} - x'_n|| &\leq ||[x_{n+1} - x_n]|| + \frac{1}{2^{(n+1)}} \\ &= ||[x_{n+1}] - [x_n]|| + \frac{1}{2^{(n+1)}} < \frac{1}{2^n}. \end{aligned}$$

It follows $\{x'_n\}$ is a Cauchy sequence in W and so it must converge to some $x \in W$. Now

$$||[x] - [x_n]|| = ||[x - x_n]|| = ||[x - x'_n]|| \leq ||x - x'_n||$$

which converges to 0. Now if $\{[x_n]\}$ is just a Cauchy sequence, there exists a subsequence satisfying $||[x_{n_k}] - [x_{n_{k+1}}]|| < \frac{1}{2^{k+1}}$ and so from the first part, the subsequence converges to some $[x] \in W/Z$ and so the original Cauchy sequence also converges. therefore, W/Z is a Banach space as claimed.

Definition 44.1.7 Define $T(A_0, A_1, p, \theta) = T$, to consist of

$$\left\{ a \in A_0 + A_1 : a = \lim_{t \rightarrow 0+} f(t) \text{ for some } f \in W(A_0, A_1, p, \theta) \right\},$$

the limit taking place in $A_0 + A_1$. Let γf be defined for $f \in W$ by $\gamma f \equiv \lim_{t \rightarrow 0+} f(t)$. Thus $T = \gamma(W)$. As above $Z \equiv \{f \in W : \gamma f = 0\} = \ker(\gamma)$.

Lemma 44.1.8 T is a Banach space with norm given by

$$\|a\|_T \equiv \inf \{\|f\|_W : f(0) = a\}. \quad (44.1.5)$$

Proof: Define a mapping, $\psi : W/Z \rightarrow T$ by

$$\psi([f]) \equiv \gamma f.$$

Then ψ is one to one and onto. Also

$$\|[f]\| \equiv \inf \{\|f + g\| : g \in Z\} = \inf \{\|h\|_W : \gamma h = \gamma f\} = \|\gamma(f)\|_T.$$

Therefore, the Banach space, W/Z and T are isometric and so T must be a Banach space since W/Z is.

The following is an important interpolation inequality.

Theorem 44.1.9 If $a \in T$, then

$$\|a\|_T = \inf \left\{ \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}^\theta \right\} \quad (44.1.6)$$

where the infimum is taken over all $f \in W$ such that $a = f(0)$. Also, if $a \in A_0 \cap A_1$, then $a \in T$ and

$$\|a\|_T \leq K \|a\|_{A_1}^{1-\theta} \|a\|_{A_0}^\theta \quad (44.1.7)$$

for some constant K . Also

$$A_0 \cap A_1 \subseteq T(A_0, A_1, p, \theta) \subseteq A_0 + A_1 \quad (44.1.8)$$

and the inclusion maps are continuous.

Proof: First suppose $f(0) = a$ where $f \in W$. Then letting $f_\lambda(t) \equiv f(\lambda t)$, it follows that $f_\lambda(0) = a$ also and so

$$\begin{aligned} \|a\|_T &\leq \max \left(\|t^\theta f_\lambda\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, \|t^\theta (f_\lambda)'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} \right) \\ &= \max \left(\lambda^{-\theta} \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, \lambda^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} \right) \\ &\equiv \max \left(\lambda^{-\theta} R, \lambda^{1-\theta} S \right). \end{aligned}$$

Now choose $\lambda = R/S$ to obtain

$$\|a\|_T \leq R^{1-\theta} S^\theta = \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}^\theta.$$

Thus

$$\|a\|_T \leq \inf \left\{ \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}^\theta \right\}.$$

Next choose $f \in W$ such that $f(0) = a$ and $\|f\|_W \approx \|a\|_T$. More precisely, pick $f \in W$ such that $f(0) = a$ and $\|a\|_T > -\varepsilon + \|f\|_W$. Also let

$$R \equiv \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, S \equiv \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}.$$

Then as before,

$$\|t^\theta f_\lambda\|_{L^p(0, \infty, \frac{dt}{t}; A_0)} = \lambda^{-\theta} R, \quad \|t^\theta (f_\lambda)'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} = \lambda^{1-\theta} S. \quad (44.1.9)$$

so that $\|f\|_W = \max(R, S)$. Then, changing the variables, letting $\lambda = R/S$,

$$\|t^\theta f_\lambda\|_{L^p(0, \infty, \frac{dt}{t}; A_0)} = \|t^\theta (f_\lambda)'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} = R^{1-\theta} S^\theta \quad (44.1.10)$$

Since $f_\lambda(0) = a$, $f_\lambda \in W$, and it is always the case that for positive R, S , $R^{1-\theta} S^\theta \leq \max(R, S)$, this shows that

$$\begin{aligned} \|a\|_T &\leq \max \left(\|t^\theta f_\lambda\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, \|t^\theta (f_\lambda)'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)} \right) \\ &= R^{1-\theta} S^\theta \leq \max(R, S) = \|f\|_W < \|a\|_T + \varepsilon, \end{aligned}$$

the first inequality holding because $\|a\|_T$ is the infimum of such things on the right. This shows 44.1.6.

It remains to verify 44.1.7. To do this, let $\psi \in C^\infty([0, \infty))$, with $\psi(0) = 1$ and $\psi(t) = 0$ for all $t > 1$. Then consider the special $f \in W$ which is given by $f(t) \equiv a\psi(t)$ where $a \in A_0 \cap A_1$. Thus $f \in W$ and $f(0) = a$ so $a \in T(A_0, A_1, p, \theta)$. From the first part, there exists a constant, K such that

$$\begin{aligned} \|a\|_T &\leq \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}^\theta \\ &\leq K \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^\theta \end{aligned}$$

This shows 44.1.7 and the first inclusion in 44.1.8. From the inequality just obtained,

$$\begin{aligned} \|a\|_T &\leq K((1-\theta)\|a\|_{A_0} + \theta\|a\|_{A_1}) \\ &\leq K\|a\|_{A_0 \cap A_1}. \end{aligned}$$

This shows the first inclusion map of 44.1.8 is continuous.

Now take $a \in T$. Let $f \in W$ be such that $a = f(0)$ and

$$\|a\|_T + \varepsilon > \|f\|_W \geq \|a\|_T.$$

By 44.1.3,

$$\|a - f(t)\|_{A_0 + A_1} \leq C_\nu t^{1-\nu p'} \|f\|_W$$

where $\frac{1}{p} + \nu = \theta$, and so

$$\|a\|_{A_0+A_1} \leq \|f(t)\|_{A_0+A_1} + C_\nu t^{1-\nu p'} \|f\|_W.$$

Now $\|f(t)\|_{A_0+A_1} \leq \|f(t)\|_{A_0}$.

$$\begin{aligned} \|a\|_{A_0+A_1} &\leq t^\nu \|f(t)\|_{A_0+A_1} t^{-\nu} + C_\nu t^{1-\nu p'} \|f\|_W \\ &\leq t^\nu \|f(t)\|_{A_0} t^{-\nu} + C_\nu t^{1-\nu p'} \|f\|_W \end{aligned}$$

Therefore, recalling that $\nu p' < 1$, and integrating both sides from 0 to 1,

$$\|a\|_{A_0+A_1} \leq C_\nu \|f\|_W \leq C_\nu (\|a\|_T + \varepsilon).$$

To see this,

$$\begin{aligned} \int_0^1 t^\nu \|f(t)\|_{A_0} t^{-\nu} dt &\leq \left(\int_0^1 (t^\nu \|f(t)\|_{A_0})^p dt \right)^{1/p} \left(\int_0^1 t^{-\nu p'} dt \right)^{1/p'} \\ &\leq C \|f\|_W. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this verifies the second inclusion and continuity of the inclusion map completing the proof of the theorem.

The interpolation inequality, 44.1.7 is very significant. The next result concerns bounded linear transformations.

Theorem 44.1.10 *Now suppose A_0, A_1 and B_0, B_1 are pairs of Banach spaces such that A_i embeds continuously into a topological vector space, X and B_i embeds continuously into a topological vector space, Y . Suppose also that $L \in \mathcal{L}(A_0, B_0)$ and $L \in \mathcal{L}(A_1, B_1)$ where the operator norm of L in these spaces is $K_i, i = 0, 1$. Then*

$$L \in \mathcal{L}(A_0 + A_1, B_0 + B_1) \quad (44.1.11)$$

with

$$\|La\|_{B_0+B_1} \leq \max(K_0, K_1) \|a\|_{A_0+A_1} \quad (44.1.12)$$

and

$$L \in \mathcal{L}(T(A_0, A_1, p, \theta), T(B_0, B_1, p, \theta)) \quad (44.1.13)$$

and for K the operator norm,

$$K \leq K_0^{1-\theta} K_1^\theta. \quad (44.1.14)$$

Proof: To verify 44.1.11, let $a \in A_0 + A_1$ and pick $a_0 \in A_0$ and $a_1 \in A_1$ such that

$$\|a\|_{A_0+A_1} + \varepsilon > \|a_0\|_{A_0} + \|a_1\|_{A_1}.$$

Then

$$\|L(a)\|_{B_0+B_1} = \|La_0 + La_1\|_{B_0+B_1} \leq \|La_0\|_{B_0} + \|La_1\|_{B_1}$$

$$\leq K_0 \|a_0\|_{A_0} + K_1 \|a_1\|_{A_1} \leq \max(K_0, K_1) (\|a\|_{A_0+A_1} + \varepsilon).$$

This establishes 44.1.12. Now consider the other assertions.

Let $a \in T(A_0, A_1, p, \theta)$ and pick $f \in W(A_0, A_1, p, \theta)$ such that $\gamma f = a$ and

$$\|a\|_{T(A_0, A_1, p, \theta)} + \varepsilon > \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}, A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}, A_1)}^\theta.$$

Then consider Lf . Since L is continuous on $A_0 + A_1$,

$$Lf(0) = La$$

and $Lf \in W(B_0, B_1, p, \theta)$. Therefore, by Theorem 44.1.9,

$$\begin{aligned} \|La\|_{T(B_0, B_1, p, \theta)} &\leq \|t^\theta Lf\|_{L^p(0, \infty, \frac{dt}{t}, B_0)}^{1-\theta} \|t^\theta Lf'\|_{L^p(0, \infty, \frac{dt}{t}, B_1)}^\theta \\ &\leq K_0^{1-\theta} K_1^\theta \|t^\theta f\|_{L^p(0, \infty, \frac{dt}{t}, A_0)}^{1-\theta} \|t^\theta f'\|_{L^p(0, \infty, \frac{dt}{t}, A_1)}^\theta \\ &\leq K_0^{1-\theta} K_1^\theta (\|a\|_{T(A_0, A_1, p, \theta)} + \varepsilon). \end{aligned}$$

and since $\varepsilon > 0$ is arbitrary, this proves the theorem.

44.2 Trace And Interpolation Spaces

Trace spaces are equivalent to interpolation spaces. In showing this, a more general sort of trace space than that presented earlier will be used.

Definition 44.2.1 Define for m a positive integer, $V^m = V^m(A_0, A_1, p, \theta)$ to be the set of functions, u such that

$$t \rightarrow t^\theta u(t) \in L^p\left(0, \infty, \frac{dt}{t}; A_0\right) \quad (44.2.15)$$

and

$$t \rightarrow t^{\theta+m-1} u^{(m)}(t) \in L^p\left(0, \infty, \frac{dt}{t}; A_1\right). \quad (44.2.16)$$

V^m is a Banach space with the norm

$$\|u\|_{V^m} \equiv \max\left(\|t^\theta u(t)\|_{L^p(0, \infty, \frac{dt}{t}; A_0)}, \|t^{\theta+m-1} u^{(m)}(t)\|_{L^p(0, \infty, \frac{dt}{t}; A_1)}\right).$$

Thus V^m equals W in the case when $m = 1$. More generally, as in [10] different exponents are used for the two L^p spaces, p_0 in place of p for the space corresponding to A_0 and p_1 in place of p for the space corresponding to A_1 .

Definition 44.2.2 Denote by $T^m(A_0, A_1, p, \theta)$ the set of all $a \in A_0 + A_1$ such that for some $u \in V^m$,

$$a = \lim_{t \rightarrow 0+} u(t) \equiv \text{trace}(u), \quad (44.2.17)$$

the limit holding in $A_0 + A_1$. For the norm

$$\|a\|_{T^m} \equiv \inf\{\|u\|_{V^m} : \text{trace}(u) = a\}. \quad (44.2.18)$$

The case when $m = 1$ was discussed in Section 44.1. Note it is not known at this point whether $\lim_{t \rightarrow 0+} u(t)$ even exists for every $u \in V^m$. Of course, if $m = 1$ this was shown earlier but it has not been shown for $m > 1$. The following theorem is absolutely amazing. Note the lack of dependence on m of the right side!

Theorem 44.2.3 *The following hold.*

$$T^m(A_0, A_1, p, \theta) = (A_0, A_1)_{\theta, p, J} = (A_0, A_1)_{\theta, p}. \quad (44.2.19)$$

Proof: It is enough to show the first equality because of Theorem 43.3.5 which identifies $(A_0, A_1)_{\theta, p, J}$ and $(A_0, A_1)_{\theta, p}$. Let $a \in T^m$. Then there exists $u \in V^m$ such that

$$a = \lim_{t \rightarrow 0+} u(t) \text{ in } A_0 + A_1.$$

The first task is to modify this $u(t)$ to get a better one which is more usable in order to show $a \in (A_0, A_1)_{\theta, p, J}$. Remember, it is required to find $w(t) \in A_0 \cap A_1$ for all $t \in (0, \infty)$ and $a = \int_0^\infty w(t) \frac{dt}{t}$, a representation which is not known at this time. To get such a thing, let

$$\phi \in C_c^\infty(0, \infty), \text{ spt}(\phi) \subseteq [\alpha, \beta] \quad (44.2.20)$$

with $\phi \geq 0$ and

$$\int_0^\infty \phi(t) \frac{dt}{t} = 1. \quad (44.2.21)$$

Then define

$$\tilde{u}(t) \equiv \int_0^\infty \phi\left(\frac{t}{\tau}\right) u(\tau) \frac{d\tau}{\tau} = \int_0^\infty \phi(s) u\left(\frac{t}{s}\right) \frac{ds}{s}. \quad (44.2.22)$$

Claim: $\lim_{t \rightarrow 0+} \tilde{u}(t) = a$ and $\lim_{t \rightarrow \infty} \tilde{u}^{(k)}(t) = 0$ in $A_0 + A_1$ for all $k \leq m$.

Proof of the claim: From 44.2.22 and 44.2.21 it follows that for $\|\cdot\|$ referring to $\|\cdot\|_{A_0 + A_1}$,

$$\begin{aligned} \|\tilde{u}(t) - a\| &\leq \int_0^\infty \left\| u\left(\frac{t}{s}\right) - a \right\| \phi(s) \frac{ds}{s} \\ &= \int_0^\infty \|u(\tau) - a\| \phi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \\ &= \int_{t/\beta}^{t/\alpha} \|u(\tau) - a\| \phi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} \\ &\leq \int_{t/\beta}^{t/\alpha} \varepsilon \phi\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau} = \varepsilon \int_\alpha^\beta \phi(s) \frac{ds}{s} = \varepsilon \end{aligned}$$

whenever t is small enough due to the convergence of $u(t)$ to a in $A_0 + A_1$.

Now consider what occurs when $t \rightarrow \infty$. For $\|\cdot\|$ referring to the norm in A_0 ,

$$\tilde{u}^{(k)}(t) = \int_0^\infty \phi^{(k)}\left(\frac{t}{\tau}\right) \frac{1}{\tau^k} u(\tau) \frac{d\tau}{\tau}$$

and so

$$\begin{aligned} \left\| \tilde{u}^{(k)}(t) \right\|_{A_0} &\leq C_k \int_{t/\beta}^{t/\alpha} \|u(\tau)\|_{A_0} \frac{d\tau}{\tau} \\ &\leq C \left(\int_{t/\beta}^{t/\alpha} \frac{d\tau}{\tau} \right)^{1/p'} \left(\int_{t/\beta}^{t/\alpha} \|u(\tau)\|_{A_0}^p \frac{d\tau}{\tau} \right)^{1/p}. \end{aligned}$$

Now $\left(\frac{\beta}{t}\right)^\theta \tau^\theta \geq 1$ for $\tau \geq t/\beta$ and so the above expression

$$\leq C \left(\ln \frac{\beta}{\alpha} \right)^{1/p'} \left(\frac{\beta}{t} \right)^\theta \left(\int_{t/\beta}^\infty (\tau^\theta \|u(\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p}$$

and so $\lim_{t \rightarrow \infty} \|\tilde{u}^{(k)}(t)\|_{A_0} = 0$ and therefore, this also holds in $A_0 + A_1$. This proves the claim.

Thus $\tilde{u}$ has the same properties as u in terms of having a as its trace. $\tilde{u}$ is used to build the desired w , representing a as an integral. Define

$$\begin{aligned} v(t) &\equiv \frac{(-1)^m t^m}{(m-1)!} \tilde{u}^{(m)}(t) = \frac{(-1)^m}{(m-1)!} \int_0^\infty \frac{t^m}{\tau^m} \phi^{(m)}\left(\frac{t}{\tau}\right) u(\tau) \frac{d\tau}{\tau} \\ &= \frac{(-1)^m}{(m-1)!} \int_0^\infty s^m \phi^{(m)}(s) u\left(\frac{t}{s}\right) \frac{ds}{s}. \end{aligned} \quad (44.2.23)$$

Then from the claim, and integration by parts in the last step,

$$\int_0^\infty v\left(\frac{1}{t}\right) \frac{dt}{t} = \int_0^\infty v(t) \frac{dt}{t} = \frac{(-1)^m}{(m-1)!} \int_0^\infty t^{m-1} \tilde{u}^{(m)}(t) dt = a. \quad (44.2.24)$$

Thus $v\left(\frac{1}{t}\right)$ represents a in the way desired for $(A_0, A_1)_{\theta, p, J}$ if it is also true that $v\left(\frac{1}{t}\right) \in A_0 \cap A_1$ and $t \rightarrow t^{-\theta} v\left(\frac{1}{t}\right)$ is in $L^p(0, \infty, \frac{dt}{t}; A_0)$ and $t \rightarrow t^{1-\theta} v\left(\frac{1}{t}\right)$ is in $L^p(0, \infty, \frac{dt}{t}; A_1)$. First consider whether $v(t) \in A_0 \cap A_1$. $v(t) \in A_0$ for each t from 44.2.23 and the assumption that $u \in L^p(0, \infty, \frac{dt}{t}; A_0)$. To verify $v(t) \in A_1$, integrate by parts in 44.2.23 to obtain

$$\begin{aligned} v(t) &= \frac{(-1)^m}{(m-1)!} \int_0^\infty \phi^{(m)}(s) \left(s^{m-1} u\left(\frac{t}{s}\right) \right) ds \\ &= \frac{1}{(m-1)!} \int_0^\infty \phi(s) \frac{d^m}{ds^m} \left(s^{m-1} u\left(\frac{t}{s}\right) \right) ds \\ &= \frac{(-1)^m}{(m-1)!} \int_0^\infty \phi(s) \frac{t^m}{s^{m+1}} u^{(m)}\left(\frac{t}{s}\right) ds \in A_1 \end{aligned} \quad (44.2.25)$$

The last step may look very mysterious. If so, consider the case where $m = 2$.

$$\begin{aligned}
 & \phi(s) \left(su \left(\frac{t}{s} \right) \right)'' \\
 = & \phi(s) \left(-\frac{t}{s} u' \left(\frac{t}{s} \right) + u \left(\frac{t}{s} \right) \right)' \\
 = & \phi(s) \left(\left(-\frac{t}{s} \right) u'' \left(\frac{t}{s} \right) \left(-\frac{t}{s^2} \right) + \frac{t}{s^2} u' \left(\frac{t}{s} \right) - \frac{t}{s^2} u' \left(\frac{t}{s} \right) \right) \\
 = & \phi(s) \frac{t^2}{s^3} u'' \left(\frac{t}{s} \right).
 \end{aligned}$$

You can see the same pattern will take place for other values of m .

Now

$$\begin{aligned}
 \|a\|_{\theta,p,J} & \leq \left(\int_0^\infty \left(t^{-\theta} J \left(t, v \left(\frac{1}{t} \right) \right) \right)^p \frac{dt}{t} \right)^{1/p} \\
 & \leq C_p \left\{ \int_0^\infty \left[\left(t^{-\theta} \left\| v \left(\frac{1}{t} \right) \right\|_{A_0} \right) + \left(t^{1-\theta} \left\| v \left(\frac{1}{t} \right) \right\|_{A_1} \right) \right]^p \frac{dt}{t} \right\}^{1/p} \\
 & \leq C_p \left\{ \left(\int_0^\infty \left(t^{-\theta} \left\| v \left(\frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \right. \\
 & \quad \left. + \left(\int_0^\infty \left(t^{1-\theta} \left\| v \left(\frac{1}{t} \right) \right\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \right\}. \tag{44.2.26}
 \end{aligned}$$

The first term equals

$$\begin{aligned}
 & \left(\int_0^\infty \left(t^{-\theta} \left\| v \left(\frac{1}{t} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \\
 = & \left(\int_0^\infty \left(t^\theta \|v(t)\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \\
 = & \left(\int_0^\infty \left(t^\theta \left\| \int_0^\infty s^m \phi^{(m)}(s) u \left(\frac{t}{s} \right) \frac{ds}{s} \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \\
 \leq & \int_0^\infty \left(\int_0^\infty \left(t^\theta s^m |\phi^{(m)}(s)| \left\| u \left(\frac{t}{s} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \frac{ds}{s} \\
 \leq & \int_0^\infty s^m |\phi^{(m)}(s)| \left(\int_0^\infty \left(t^\theta \left\| u \left(\frac{t}{s} \right) \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \frac{ds}{s}
 \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty s^{\theta+m} \left| \phi^{(m)}(s) \right| \frac{ds}{s} \left(\int_0^\infty (\tau^\theta \|u(\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p} \\
&= C \left(\int_0^\infty (\tau^\theta \|u(\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p}.
\end{aligned} \tag{44.2.27}$$

The second term equals

$$\begin{aligned}
&\left(\int_0^\infty \left(t^{1-\theta} \left\| v\left(\frac{1}{t}\right) \right\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} = \left(\int_0^\infty (t^{\theta-1} \|v(t)\|_{A_1})^p \frac{dt}{t} \right)^{1/p} \\
&= \left(\int_0^\infty \left(t^{\theta-1} \left\| \frac{1}{(m-1)!} \int_0^\infty \phi(s) \frac{t^m}{s^m} u^{(m)}\left(\frac{t}{s}\right) \frac{ds}{s} \right\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \\
&\leq \int_0^\infty \left(\int_0^\infty \left(\left(\frac{t^{\theta+m-1}}{s^m} \right) |\phi(s)| \left\| u^{(m)}\left(\frac{t}{s}\right) \right\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \frac{ds}{s} \\
&\leq \int_0^\infty \frac{|\phi(s)|}{s^m} \left(\int_0^\infty \left(t^{\theta+m-1} \left\| u^{(m)}\left(\frac{t}{s}\right) \right\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \frac{ds}{s} \\
&= \int_0^\infty \frac{|\phi(s)|}{s^m} s^{\theta+m-1} \left(\int_0^\infty \left(\tau^{\theta+m-1} \left\| u^{(m)}(\tau) \right\|_{A_1} \right)^p \frac{d\tau}{\tau} \right)^{1/p} \frac{ds}{s} \\
&= C \left(\int_0^\infty \left(\tau^{\theta+m-1} \left\| u^{(m)}(\tau) \right\|_{A_1} \right)^p \frac{d\tau}{\tau} \right)^{1/p}.
\end{aligned} \tag{44.2.28}$$

Now from the estimates on the two terms in 44.2.26 found in 44.2.27 and 44.2.28, and the simple estimate,

$$2 \max(\alpha, \beta) \geq \alpha + \beta,$$

it follows

$$\|a\|_{\theta,p,J} \tag{44.2.29}$$

$$\leq C \max \left(\left(\int_0^\infty (\tau^\theta \|u(\tau)\|_{A_0})^p \frac{d\tau}{\tau} \right)^{1/p} \right. \tag{44.2.30}$$

$$\left. , \left(\int_0^\infty \left(\tau^{\theta+m-1} \left\| u^{(m)}(\tau) \right\|_{A_1} \right)^p \frac{d\tau}{\tau} \right)^{1/p} \right) \tag{44.2.31}$$

which shows that after taking the infimum over all u whose trace is a , it follows $a \in (A_0, A_1)_{\theta,p,J}$.

$$\|a\|_{\theta,p,J} \leq C \|a\|_{T^m} \tag{44.2.32}$$

Thus $T^m(A_0, A_1, \theta, p) \subseteq (A_0, A_1)_{\theta,p,J}$.

Is $(A_0, A_1)_{\theta, p, J} \subseteq T^m(A_0, A_1, \theta, p)$? Let $a \in (A_0, A_1)_{\theta, p, J}$. There exists u having values in $A_0 \cap A_1$ and such that

$$a = \int_0^\infty u(t) \frac{dt}{t} = \int_0^\infty u\left(\frac{1}{t}\right) \frac{dt}{t},$$

in $A_0 + A_1$ such that

$$\int_0^\infty (t^{-\theta} J(t, u(t)))^p dt < \infty, \text{ where } J(t, a) = \max(\|a\|_{A_0}, t\|a\|_{A_1}).$$

Then let

$$w(t) \equiv \int_t^\infty \left(1 - \frac{t}{\tau}\right)^{m-1} u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau} = \quad (44.2.33)$$

$$\int_0^{1/t} (1 - st)^{m-1} u(s) \frac{ds}{s} = \int_0^1 (1 - \tau)^{m-1} u\left(\frac{\tau}{t}\right) \frac{d\tau}{\tau}. \quad (44.2.34)$$

It is routine to verify from 44.2.33 that

$$w^{(m)}(t) = (m-1)!(-1)^m \frac{u\left(\frac{1}{t}\right)}{t^m}. \quad (44.2.35)$$

For example, consider the case where $m = 2$.

$$\begin{aligned} \left(\int_t^\infty \left(1 - \frac{t}{\tau}\right) u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau}\right)'' &= \left(0 + \int_t^\infty \left(-\frac{1}{\tau}\right) u\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau}\right)' \\ &= \frac{1}{t^2} u\left(\frac{1}{t}\right). \end{aligned}$$

Also from 44.2.33, it follows that $\text{trace}(w) = a$. It remains to verify $w \in V^m$. From 44.2.35,

$$\begin{aligned} &\left(\int_0^\infty \left(t^{\theta+m-1} \|w^{(m)}(t)\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} = \\ C_m &\left(\int_0^\infty \left(t^{\theta-1} \left\|u\left(\frac{1}{t}\right)\right\|_{A_1}\right)^p \frac{dt}{t}\right)^{1/p} = C_m \left(\int_0^\infty (t^{1-\theta} \|u(t)\|_{A_1})^p \frac{dt}{t}\right)^{1/p} \\ &\leq C_m \left(\int_0^\infty (t^{-\theta} J(t, u(t)))^p \frac{dt}{t}\right)^{1/p} < \infty. \end{aligned} \quad (44.2.36)$$

It remains to consider $\left(\int_0^\infty (t^\theta \|w(t)\|_{A_0})^p \frac{dt}{t}\right)^{1/p}$. From 44.2.34,

$$\begin{aligned} &\left(\int_0^\infty (t^\theta \|w(t)\|_{A_0})^p \frac{dt}{t}\right)^{1/p} \\ &= \left(\int_0^\infty \left(t^\theta \left\|\int_0^1 (1 - \tau)^{m-1} u\left(\frac{\tau}{t}\right) \frac{d\tau}{\tau}\right\|_{A_0}\right)^p \frac{dt}{t}\right)^{1/p} \end{aligned}$$

$$\begin{aligned}
&= \left(\int_0^\infty \left(t^{-\theta} \left\| \int_0^1 (1-\tau)^{m-1} u(\tau t) \frac{d\tau}{\tau} \right\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \\
&\leq \int_0^1 \left(\int_0^\infty \left(t^{-\theta} (1-\tau)^{m-1} \|u(\tau t)\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p} \frac{d\tau}{\tau} \\
&= \int_0^1 \tau^\theta (1-\tau)^{m-1} \left(\int_0^\infty \left(s^{-\theta} \|u(s)\|_{A_0} \right)^p \frac{ds}{s} \right)^{1/p} \frac{d\tau}{\tau} \\
&= \left(\int_0^1 \tau^{\theta-1} (1-\tau)^{m-1} d\tau \right) \left(\int_0^\infty \left(s^{-\theta} \|u(s)\|_{A_0} \right)^p \frac{ds}{s} \right)^{1/p} \\
&\leq C \left(\int_0^\infty \left(s^{-\theta} \|u(s)\|_{A_0} \right)^p \frac{ds}{s} \right)^{1/p} \\
&\leq C \left(\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^p dt \right)^{1/p} < \infty. \tag{44.2.37}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|w\|_{V^m} \equiv \\
&\max \left(\left(\int_0^\infty \left(t^\theta \|w(t)\|_{A_0} \right)^p \frac{dt}{t} \right)^{1/p}, \left(\int_0^\infty \left(t^{\theta+m-1} \|w^{(m)}(t)\|_{A_1} \right)^p \frac{dt}{t} \right)^{1/p} \right) \\
&\leq C \left(\int_0^\infty \left(t^{-\theta} J(t, u(t)) \right)^p dt \right)^{1/p} < \infty
\end{aligned}$$

which shows that $a \in T^m(A_0, A_1, \theta, p)$. Taking the infimum,

$$\|a\|_{T^m} \leq C \|a\|_{\theta, p, J}.$$

This together with 44.2.32 proves the theorem.

By Theorem 44.2.3 and Theorem 43.4.6, we obtain the following important corollary describing the dual space of a trace space.

Corollary 44.2.4 *Let $A_0 \cap A_1$ be dense in A_i for $i = 0, 1$ and suppose that A_i is reflexive for $i = 0, 1$. Then for $\infty > p \geq 1$,*

$$T^m(A_0, A_1, \theta, p)' = T^m(A_1', A_0', 1 - \theta, p')$$

Traces Of Sobolev Spaces And Fractional Order Spaces

45.1 Traces Of Sobolev Spaces On The Boundary Of A Half Space

In this section consider the trace of $W^{m,p}(\mathbb{R}_+^n)$ onto a Sobolev space of functions defined on $\mathbb{R}^{n-1}$. This latter Sobolev space will be defined in terms of the following theory in such a way that the trace map is continuous. The trace map is continuous as a map from $W^{m,p}(\mathbb{R}_+^n)$ to $W^{m-1,p}(\mathbb{R}^{n-1})$ but here I will give a better conclusion using the above theory.

Definition 45.1.1 *Let $\theta \in (0, 1)$ and let Ω be an open subset of $\mathbb{R}^m$. We define*

$$W^{\theta,p}(\Omega) \equiv T(W^{1,p}(\Omega), L^p(\Omega), p, 1 - \theta).$$

Thus, from the above general theory, $W^{1,p}(\Omega) \hookrightarrow W^{\theta,p}(\Omega) \hookrightarrow L^p(\Omega) = L^p(\Omega) + W^{1,p}(\Omega)$. Now we consider the trace map for Sobolev space.

Lemma 45.1.2 *Let $\phi \in C^\infty(\overline{\mathbb{R}_+^n})$. Then $\gamma\phi(\mathbf{x}') \equiv \phi(\mathbf{x}', 0)$. Then $\gamma : C^\infty(\overline{\mathbb{R}_+^n}) \rightarrow L^p(\mathbb{R}^{n-1})$ is continuous as a map from $W^{1,p}(\mathbb{R}_+^n)$ to $L^p(\mathbb{R}^{n-1})$.*

Proof: We know

$$\phi(\mathbf{x}', x_n) = \gamma\phi(\mathbf{x}') + \int_0^{x_n} \frac{\partial\phi(\mathbf{x}', t)}{\partial t} dt$$

Then by Jensen's inequality,

$$\begin{aligned}
 & \int_{\mathbb{R}^{n-1}} |\gamma\phi(\mathbf{x}')|^p dx' \\
 = & \int_0^1 \int_{\mathbb{R}^{n-1}} |\gamma\phi(\mathbf{x}')|^p dx' dx_n \\
 \leq & C \int_0^1 \int_{\mathbb{R}^{n-1}} |\phi(\mathbf{x}', x_n)|^p dx' dx_n \\
 & + C \int_0^1 \int_{\mathbb{R}^{n-1}} \left| \int_0^{x_n} \frac{\partial\phi(\mathbf{x}', t)}{\partial t} dt \right|^p dx' dx_n \\
 \leq & C \|\phi\|_{0,p,\mathbb{R}_+^n}^p + C \int_0^1 x_n^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^{x_n} \left| \frac{\partial\phi(\mathbf{x}', t)}{\partial t} \right|^p dt dx' dx_n \\
 \leq & C \|\phi\|_{0,p,\mathbb{R}_+^n}^p + C \int_0^1 x_n^{p-1} \int_{\mathbb{R}^{n-1}} \int_0^\infty \left| \frac{\partial\phi(\mathbf{x}', t)}{\partial t} \right|^p dt dx' dx_n \\
 \leq & C \|\phi\|_{0,p,\mathbb{R}_+^n}^p + \frac{C}{p} \int_{\mathbb{R}^{n-1}} \int_0^\infty \left| \frac{\partial\phi(\mathbf{x}', t)}{\partial t} \right|^p dt dx' \\
 \leq & C \|\phi\|_{1,p,\mathbb{R}_+^n}^p
 \end{aligned}$$

This proves the lemma.

Definition 45.1.3 We define the trace, $\gamma : W^{1,p}(\mathbb{R}_+^n) \rightarrow L^p(\mathbb{R}^{n-1})$ as follows. $\gamma\phi(\mathbf{x}') \equiv \phi(\mathbf{x}', 0)$ whenever $\phi \in C^\infty(\overline{\mathbb{R}_+^n})$. For $u \in W^{1,p}(\mathbb{R}_+^n)$, we define $\gamma u \equiv \lim_{k \rightarrow \infty} \gamma\phi_k$ in $L^p(\mathbb{R}^{n-1})$ where $\phi_k \rightarrow u$ in $W^{1,p}(\mathbb{R}_+^n)$. Then the above lemma shows this is well defined.

Also from this lemma we obtain a constant, C such that

$$\|\phi\|_{0,p,\mathbb{R}^{n-1}} \leq C \|\phi\|_{1,p,\mathbb{R}_+^n}$$

and the same constant holds for all $u \in W^{1,p}(\mathbb{R}_+^n)$.

From the definition of the norm in the trace space, if $f \in C^\infty(\overline{\mathbb{R}_+^n})$, and letting $\theta = 1 - \frac{1}{p}$, it follows from the definition

$$\begin{aligned}
 & \|\gamma f\|_{1-\frac{1}{p},p,\mathbb{R}^{n-1}} \\
 \leq & \max \left(\left(\int_0^\infty \left(t^{1/p} \|f(t)\|_{1,p,\mathbb{R}^{n-1}} \right)^p \frac{dt}{t} \right)^{1/p} \right. \\
 & \left. , \left(\int_0^\infty \left(t^{1/p} \|f'(t)\|_{0,p,\mathbb{R}^{n-1}} \right)^p \frac{dt}{t} \right)^{1/p} \right) \\
 \leq & C \|f\|_{1,p,\mathbb{R}_+^n}.
 \end{aligned}$$

Thus, if $f \in W^{1,p}(\mathbb{R}_+^n)$, define $\gamma f \in W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ according to the rule,

$$\gamma f = \lim_{k \rightarrow \infty} \gamma\phi_k,$$

where $\phi_k \rightarrow f$ in $W^{1,p}(\mathbb{R}_+^n)$ and $\phi_k \in C^\infty(\overline{\mathbb{R}_+^n})$. This shows the continuity part of the following lemma.

Lemma 45.1.4 *The trace map, γ , is a continuous map from $W^{1,p}(\mathbb{R}_+^n)$ onto*

$$W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}).$$

Furthermore, for $f \in W^{1,p}(\mathbb{R}_+^n)$,

$$\gamma f = f(0) = \lim_{t \rightarrow 0+} f(t)$$

the limit taking place in $L^p(\mathbb{R}^{n-1})$.

Proof: It remains to verify γ is onto along with the displayed equation. But by definition, things in $W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ are of the form $\lim_{t \rightarrow 0+} f(t)$ where $f \in L^p(0, \infty; W^{1,p}(\mathbb{R}^{n-1}))$, and $f' \in L^p(0, \infty; L^p(\mathbb{R}^{n-1}))$, the limit taking place in

$$W^{1,p}(\mathbb{R}^{n-1}) + L^p(\mathbb{R}^{n-1}) = L^p(\mathbb{R}^{n-1}),$$

and

$$\left(\int_0^\infty \|f(t)\|_{1,p,\mathbb{R}^{n-1}}^p dt \right)^{1/p} + \left(\int_0^\infty \|f'(t)\|_{0,p}^p dt \right)^{1/p} < \infty.$$

Then taking a measurable representative, we see $f \in W^{1,p}(\mathbb{R}_+^n)$ and $f_{,x_n} = f'$. Also, as an equation in $L^p(\mathbb{R}^{n-1})$, the following holds for all $t > 0$.

$$f(\cdot, t) = f(0) + \int_0^t f_{,x_n}(\cdot, s) ds$$

But also, for a.e. $\mathbf{x}'$, the following equation holds for a.e. $t > 0$.

$$f(\mathbf{x}', t) = \gamma f(\mathbf{x}') + \int_0^t f_{,x_n}(\mathbf{x}', s) ds, \quad (45.1.1)$$

showing that

$$\gamma f = f(0) \in W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \equiv T\left(W^{1,p}(\Omega), L^p(\Omega), p, \frac{1}{p}\right).$$

To see that 45.1.1 holds, approximate f with a sequence from $C^\infty(\overline{\mathbb{R}_+^n})$ and finally obtain an equation of the form

$$\int_{\mathbb{R}^{n-1}} \int_0^\infty \left[f(\mathbf{x}', t) - \gamma f(\mathbf{x}') - \int_0^t f_{,x_n}(\mathbf{x}', s) ds \right] \psi(\mathbf{x}', t) dt dx' = 0,$$

which holds for all $\psi \in C_c^\infty(\mathbb{R}_+^n)$. This proves the lemma.

Thus taking the trace on the boundary loses exactly $\frac{1}{p}$ derivatives.

45.2 A Right Inverse For The Trace For A Half Space

It is also important to show there is a continuous linear function,

$$R : W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}) \rightarrow W^{1,p}(\mathbb{R}_+^n)$$

which has the property that $\gamma(Rg) = g$. Define this function as follows.

$$Rg(\mathbf{x}', x_n) \equiv \int_{\mathbb{R}^{n-1}} g(\mathbf{y}') \phi\left(\frac{\mathbf{x}' - \mathbf{y}'}{x_n}\right) \frac{1}{x_n^{n-1}} d\mathbf{y}' \quad (45.2.2)$$

where ϕ is a mollifier having support in $B(\mathbf{0}, 1)$.

Lemma 45.2.1 *Let R be defined in 45.2.2. Then $Rg \in W^{1,p}(\mathbb{R}_+^n)$ and is a continuous linear map from $W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$ to $W^{1,p}(\mathbb{R}_+^n)$ with the property that $\gamma Rg = g$.*

Proof: Let $f \in W^{1,p}(\mathbb{R}_+^n)$ be such that $\gamma f = g$. Let $\psi(x_n) \equiv (1 - x_n)_+$ and assume f is Borel measurable by taking a Borel measurable representative. Then for a.e. $\mathbf{x}'$ we have the following formula holding for a.e. x_n .

$$\begin{aligned} & Rg(\mathbf{x}', x_n) \\ &= \int_{\mathbb{R}^{n-1}} \left[\psi(x_n) f(\mathbf{y}', \psi(x_n)) - \int_0^{\psi(x_n)} (\psi f)_{,n}(\mathbf{y}', t) dt \right] \phi\left(\frac{\mathbf{x}' - \mathbf{y}'}{x_n}\right) x_n^{1-n} dy'. \end{aligned}$$

Using the repeated index summation convention to save space, we obtain that in terms of weak derivatives,

$$\begin{aligned} & Rg_{,n}(\mathbf{x}', x_n) \\ &= \int_{\mathbb{R}^{n-1}} \left[\psi(x_n) f(\mathbf{y}', \psi(x_n)) - \int_0^{\psi(x_n)} (\psi f)_{,n}(\mathbf{y}', t) dt \right] \cdot \\ & \quad \left[\phi_{,k}\left(\frac{\mathbf{x}' - \mathbf{y}'}{x_n}\right) \left(\frac{y_k - x_k}{x_n^n}\right) + \phi\left(\frac{\mathbf{x}' - \mathbf{y}'}{x_n}\right) \frac{(1-n)}{x_n^n} \right] dy' \\ &= \int_{\mathbb{R}^{n-1}} \left[\psi(x_n) f(\mathbf{x}' - x_n \mathbf{z}', \psi(x_n)) - \int_0^{\psi(x_n)} (\psi f)_{,n}(\mathbf{x}' - x_n \mathbf{z}', t) dt \right] \cdot \\ & \quad \left[\phi_{,k}(\mathbf{z}') \left(\frac{y_k - x_k}{x_n^n}\right) z_k + \phi(\mathbf{z}') \frac{(1-n)}{x_n^n} \right] x_n^n dz' \end{aligned}$$

and so

$$\begin{aligned} |Rg_{,n}(\mathbf{x}', x_n)| &\leq C(\phi) \left| \int_{B(\mathbf{0},1)} [\psi(x_n) f(\mathbf{x}' - x_n \mathbf{z}', \psi(x_n)) \right. \\ & \quad \left. - \int_0^{\psi(x_n)} (\psi f)_{,n}(\mathbf{x}' - x_n \mathbf{z}', t) dt] \right| \end{aligned}$$

$$\leq \frac{C(\phi)}{x_n^{n-1}} \left\{ \int_{B(\mathbf{0}, x_n)} |\psi(x_n) f(\mathbf{x}' + \mathbf{y}', \psi(x_n))| dy' \right. \\ \left. + \int_{B(\mathbf{0}, x_n)} \int_0^{\psi(x_n)} |(\psi f)_{,n}(\mathbf{x}' + \mathbf{y}', t)| dt dy' \right\}$$

Therefore,

$$\left(\int_0^\infty \int_{\mathbb{R}^{n-1}} |Rg_{,n}(\mathbf{x}', x_n)|^p dx' dx_n \right)^{1/p} \leq \\ C(\phi) \left(\int_0^\infty \int_{\mathbb{R}^{n-1}} \left(\frac{1}{x_n^{n-1}} \int_{B(\mathbf{0}, x_n)} |\psi(x_n) f(\mathbf{x}' + \mathbf{y}', \psi(x_n))| dy' \right)^p dx' dx_n \right)^{1/p} \\ + C(\phi) \left(\int_0^\infty \int_{\mathbb{R}^{n-1}} \left(\frac{1}{x_n^{n-1}} \int_{B(\mathbf{0}, x_n)} \int_0^{\psi(x_n)} |(\psi f)_{,n}(\mathbf{x}' + \mathbf{y}', t)| dt dy' \right)^p dx' dx_n \right)^{1/p} \quad (45.2.3)$$

Consider the first term on the right. We change variables, letting $\mathbf{y}' = \mathbf{z}' x_n$. Then this term becomes

$$C(\phi) \left(\int_0^1 \int_{\mathbb{R}^{n-1}} \left(\int_{B(\mathbf{0}, 1)} |\psi(x_n) f(\mathbf{x}' + x_n \mathbf{z}', \psi(x_n))| dz' \right)^p dx' dx_n \right)^{1/p} \\ \leq C(\phi) \int_{B(\mathbf{0}, 1)} \left(\int_0^1 \int_{\mathbb{R}^{n-1}} |\psi(x_n) f(\mathbf{x}' + x_n \mathbf{z}', \psi(x_n))|^p dx' dx_n \right)^{1/p} dz'$$

Now we change variables, letting $t = \psi(x_n)$. This yields

$$= C(\phi) \int_{B(\mathbf{0}, 1)} \left(\int_0^1 \int_{\mathbb{R}^{n-1}} |t f(\mathbf{x}' + x_n \mathbf{z}', t)|^p dx' dt \right)^{1/p} dz' \leq C(\phi) \|f\|_{0,p,\mathbb{R}_+^n} \cdot \quad (45.2.4)$$

Now we consider the second term on the right in 45.2.3. Using the same arguments which were used on the first term involving Minkowski's inequality and changing the variables, we obtain the second term

$$\leq C(\phi) \int_{B(\mathbf{0}, 1)} \int_0^1 \left(\int_0^1 \int_{\mathbb{R}^{n-1}} |(\psi f)_{,n}(\mathbf{x}' + x_n \mathbf{z}', t)|^p d\mathbf{x}' dx_n \right)^{1/p} dt dy' \\ \leq C(\phi) \|f\|_{1,p,\mathbb{R}_+^n} \cdot \quad (45.2.5)$$

It is somewhat easier to verify that

$$\|Rg_{,j}\|_{0,p,\mathbb{R}_+^n} \leq C(\phi) \|f\|_{1,p,\mathbb{R}_+^n} \cdot$$

Therefore, we have shown that whenever $\gamma f = f(0) = g$,

$$\|Rg\|_{1,p,\mathbb{R}_+^n} \leq C(\phi) \|f\|_{1,p,\mathbb{R}_+^n} \cdot$$

Taking the infimum over all such f and using the definition of the norm in

$$W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1}),$$

it follows

$$\|Rg\|_{1,p,\mathbb{R}_+^n} \leq C(\phi) \|g\|_{1-\frac{1}{p},p,\mathbb{R}^{n-1}},$$

showing that this map, R , is continuous as claimed. It is obvious that

$$\lim_{x_n \rightarrow 0} Rg(x_n) = g,$$

the convergence taking place in $L^p(\mathbb{R}^{n-1})$ because of general results about convolution with mollifiers. This proves the lemma.

45.3 Intrinsic Norms

The above presentation is very abstract, involving the trace of a function in

$$W(A_0, A_1, p, \theta)$$

and a norm which was the infimum of norms of functions in W which have trace equal to the given function. It is very useful to have a description of the norm in these fractional order spaces which is defined in terms of the function itself rather than functions which have the given function as trace. This leads to something called an intrinsic norm. I am following Adams [1].

The following interesting lemma is called Young's inequality. It holds more generally than stated.

Lemma 45.3.1 *Let $g = f * h$ where $f \in L^1(\mathbb{R})$, $h \in L^p(\mathbb{R})$, and f, h are all Borel measurable, $p \geq 1$. Then $g \in L^p(\mathbb{R})$ and*

$$\|g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|h\|_{L^p(\mathbb{R})}$$

Proof: First of all it is good to show g is well defined. Using Minkowski's inequality

$$\begin{aligned} & \left(\int \left(\int |h(t-s)f(s)| ds \right)^p dt \right)^{1/p} \\ & \leq \int \left(\int |h(t-s)|^p |f(s)|^p dt \right)^{1/p} ds \\ & = \int |f(s)| \left(\int |h(t-s)|^p dt \right)^{1/p} ds \\ & = \|f\|_{L^1} \|h\|_{L^p} \end{aligned}$$

Therefore, for a.e. t ,

$$\int |h(t-s)f(s)|ds = \int |h(s)f(t-s)|ds < \infty$$

and so for all such t the convolution $f * h(t)$ makes sense. The above also shows

$$\|g\|_{L^p} \equiv \left(\int \left| \int f(t-s)h(s)ds \right|^p dt \right)^{1/p} \leq \|f\|_{L^1} \|h\|_{L^p}$$

and this proves the lemma.

The following is a very interesting inequality of Hardy Littlewood and Pólya.

Lemma 45.3.2 *Let f be a real valued function defined a.e. on $[0, \infty)$ and let $\alpha \in (-\infty, 1)$ and*

$$g(t) = \frac{1}{t} \int_0^t f(\xi) d\xi \quad (45.3.6)$$

For $1 \leq p < \infty$

$$\int_0^\infty t^{\alpha p} |g(t)|^p \frac{dt}{t} \leq \frac{1}{(1-\alpha)^p} \int_0^\infty t^{\alpha p} |f(t)|^p \frac{dt}{t} \quad (45.3.7)$$

Proof: First it can be assumed the right side of 45.3.7 is finite since otherwise there is nothing to show. Changing the variables letting $t = e^\tau$, the above inequality takes the form

$$\int_{-\infty}^\infty e^{\tau p \alpha} |g(e^\tau)|^p d\tau \leq \frac{1}{(1-\alpha)^p} \int_{-\infty}^\infty e^{\tau p \alpha} |f(e^\tau)|^p d\tau$$

Now from the definition of g it follows

$$\begin{aligned} g(e^\tau) &= e^{-\tau} \int_{-\infty}^{e^\tau} f(\xi) d\xi \\ &= e^{-\tau} \int_{-\infty}^\tau f(e^\sigma) e^\sigma d\sigma \end{aligned}$$

and so the left side equals

$$\begin{aligned} & \int_{-\infty}^\infty e^{\tau p(\alpha-1)} \left| \int_{-\infty}^\tau f(e^\sigma) e^\sigma d\sigma \right|^p d\tau \\ &= \int_{-\infty}^\infty \left| \int_{-\infty}^\tau e^{\tau \alpha} f(e^\sigma) e^{-(\tau-\sigma)} d\sigma \right|^p d\tau \\ &= \int_{-\infty}^\infty \left| \int_{-\infty}^\tau e^{(\tau-\sigma)\alpha} e^{-(\tau-\sigma)} e^{\sigma \alpha} f(e^\sigma) d\sigma \right|^p d\tau \end{aligned}$$

$$= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \mathcal{X}_{(-\infty, 0)}(\tau - \sigma) e^{(\tau - \sigma)(\alpha - 1)} e^{\sigma \alpha} f(e^{\sigma}) d\sigma \right|^p d\tau$$

and by Lemma 45.3.1,

$$\begin{aligned} &\leq \left(\int_{-\infty}^0 e^{(\alpha - 1)u} du \right)^p \int_{-\infty}^{\infty} e^{p\sigma \alpha} |f(e^{\sigma})|^p d\sigma \\ &= \left(\frac{1}{1 - \alpha} \right)^p \int_{-\infty}^{\infty} e^{p\sigma \alpha} |f(e^{\sigma})|^p d\sigma \end{aligned}$$

which was to be shown. This proves the lemma.

Next consider the case where $G(t)$, $t > 0$ is a continuous semigroup on A_1 and $A_0 \equiv D(\Lambda)$ where Λ is the generator of this semigroup. Recall that from Proposition 15.12.5 on Page 469 Λ is a closed densely defined operator and so A_0 is a Banach space if the norm is given by

$$\|u\|_{A_0} \equiv \|u\|_{A_1} + \|\Lambda u\|_{A_1}$$

Also assume $\|G(t)\|$ is uniformly bounded for $t \in [0, \infty)$. I have in mind the case where $A_1 = L^p(\mathbb{R}^n)$ and $G(t)u(\mathbf{x}) = u(\mathbf{x} + t\mathbf{e}_i)$ but it is notationally easier to discuss this in the general case. First here is a simple lemma.

Lemma 45.3.3 *Let $A_0 = D(\Lambda)$ as just described. Then for $u \in A_1$*

$$\|u\|_{A_1 + A_0} = \|u\|_{A_1}$$

Proof: $D(\Lambda) \subseteq A_1$. Now let $u \in A_1$.

$$\|u\|_{A_0 + A_1} \equiv \inf \{ \|u_0\|_{A_1} + \|\Lambda u_0\|_{A_1} + \|u_1\|_{A_1} : u = u_0 + u_1 \}$$

To make this as small as possible you should clearly take $u_1 = u$ because

$$\begin{aligned} \|u_0\|_{A_1} + \|\Lambda u_0\|_{A_1} + \|u_1\|_{A_1} &\geq \|u_0 + u_1\|_{A_1} + \|\Lambda u_0\|_{A_1} \\ &= \|u\|_{A_1} + \|\Lambda u_0\|_{A_1} \end{aligned}$$

Therefore, the result of the lemma follows.

Lemma 45.3.4 *Let Λ be the generator of $G(t)$ and let $t \rightarrow g(t)$ be in $C^1(0, \infty; A_1)$. Then there exists a unique solution to the initial value problem*

$$y' - \Lambda y = g, \quad y(0) = y_0 \in D(\Lambda)$$

and it is given by

$$y(t) = G(t)y_0 + \int_0^t G(t-s)g(s)ds. \quad (45.3.8)$$

This solution is continuous having continuous derivative and has values in $D(\Lambda)$.

Proof: First I show the following claim.

Claim: $\int_0^t G(t-s)g(s)ds \in D(\Lambda)$ and

$$\Lambda \left(\int_0^t G(t-s)g(s)ds \right) = G(t)g(0) - g(t) + \int_0^t G(t-s)g'(s)ds$$

Proof of the claim:

$$\begin{aligned} & \frac{1}{h} \left(G(h) \int_0^t G(t-s)g(s)ds - \int_0^t G(t-s)g(s)ds \right) \\ &= \frac{1}{h} \left(\int_0^t G(t-s+h)g(s)ds - \int_0^t G(t-s)g(s)ds \right) \\ &= \frac{1}{h} \left(\int_{-h}^{t-h} G(t-s)g(s+h)ds - \int_0^t G(t-s)g(s)ds \right) \\ &= \frac{1}{h} \int_{-h}^0 G(t-s)g(s+h)ds + \int_0^{t-h} G(t-s) \frac{g(s+h) - g(s)}{h} ds \\ &\quad - \frac{1}{h} \int_{t-h}^t G(t-s)g(s)ds \end{aligned}$$

Using the estimate in Theorem 15.12.3 on Page 468 and the dominated convergence theorem the limit as $h \rightarrow 0$ of the above equals

$$G(t)g(0) - g(t) + \int_0^t G(t-s)g'(s)ds$$

which proves the claim.

Since $y_0 \in D(\Lambda)$,

$$\begin{aligned} G(t)\Lambda y_0 &= G(t) \lim_{h \rightarrow 0} \frac{G(h)y_0 - y_0}{h} \\ &= \lim_{h \rightarrow 0} \frac{G(t+h) - G(t)}{h} y_0 \\ &= \lim_{h \rightarrow 0} \frac{G(h)G(t)y_0 - G(t)y_0}{h} \end{aligned} \tag{45.3.9}$$

Since this limit exists, the last limit in the above exists and equals

$$\Lambda G(t)y_0 \tag{45.3.10}$$

and $G(t)y_0 \in D(\Lambda)$. Now consider 45.3.8.

$$\frac{y(t+h) - y(t)}{h} = \frac{G(t+h) - G(t)}{h} y_0 +$$

$$\begin{aligned}
& \frac{1}{h} \left(\int_0^{t+h} G(t-s+h) g(s) ds - \int_0^t G(t-s) g(s) ds \right) \\
&= \frac{G(t+h) - G(t)}{h} y_0 + \frac{1}{h} \int_t^{t+h} G(t-s+h) g(s) ds \\
& \quad + \frac{1}{h} \left(G(h) \int_0^t G(t-s) g(s) ds - \int_0^t G(t-s) g(s) ds \right)
\end{aligned}$$

From the claim and 45.3.9, 45.3.10 the limit of the right side is

$$\begin{aligned}
& \Lambda G(t) y_0 + g(t) + \Lambda \left(\int_0^t G(t-s) g(s) ds \right) \\
&= \Lambda \left(G(t) y_0 + \int_0^t G(t-s) g(s) ds \right) + g(t)
\end{aligned}$$

Hence

$$y'(t) = \Lambda y(t) + g(t)$$

and from the formula, y' is continuous since by the claim and 45.3.10 it also equals

$$G(t) \Lambda y_0 + g(t) + G(t) g(0) - g(t) + \int_0^t G(t-s) g'(s) ds$$

which is continuous. The claim and 45.3.10 also shows $y(t) \in D(\Lambda)$. This proves the existence part of the lemma.

It remains to prove the uniqueness part. It suffices to show that if

$$y' - \Lambda y = 0, \quad y(0) = 0$$

and y is C^1 having values in $D(\Lambda)$, then $y = 0$. Suppose then that y is this way. Letting $0 < s < t$,

$$\begin{aligned}
& \frac{d}{ds} (G(t-s) y(s)) \\
&\equiv \lim_{h \rightarrow 0} G(t-s-h) \frac{y(s+h) - y(s)}{h} \\
& \quad - \frac{G(t-s) y(s) - G(t-s-h) y(s)}{h}
\end{aligned}$$

provided the limit exists. Since y' exists and $y(s) \in D(\Lambda)$, this equals

$$G(t-s) y'(s) - G(t-s) \Lambda y(s) = 0.$$

Let $y^* \in A'_1$. This has shown that on the open interval $(0, t)$ the function $s \rightarrow y^*(G(t-s) y(s))$ has a derivative equal to 0. Also from continuity of G and y , this function is continuous on $[0, t]$. Therefore, it is constant on $[0, t]$ by the mean value theorem. At $s = 0$, this function equals 0. Therefore, it equals 0 on $[0, t]$. Thus for fixed $s > 0$ and letting $t > s$, $y^*(G(t-s) y(s)) = 0$. Now let t decrease toward s . Then $y^*(y(s)) = 0$ and since y^* was arbitrary, it follows $y(s) = 0$. This proves uniqueness.

Definition 45.3.5 Let $G(t)$ be a uniformly bounded continuous semigroup defined on A_1 and let Λ be its generator. Let the norm on $D(\Lambda)$ be given by

$$\|u\|_{D(\Lambda)} \equiv \|u\|_{A_1} + \|\Lambda u\|_{A_1}$$

so that by Lemma 45.3.3 the norm on $A_1 + D(\Lambda)$ is just $\|\cdot\|_{A_1}$. Let

$$T_0 \equiv \left\{ u \in A_1 : \|u\|_{A_1}^p + \int_0^\infty t^{\theta p} \left\| \frac{G(t)u - u}{t} \right\|_{A_1}^p \frac{dt}{t} \equiv \|u\|_{T_0}^p < \infty \right\}$$

Theorem 45.3.6 $T_0 = T(D(\Lambda), A_1, p, \theta) \equiv T$ and the two norms are equivalent.

Proof: Take $u \in T(D(\Lambda), A_1, p, \theta)$. I will show $\|u\|_{T_0} \leq C(\theta, p) \|u\|_T$. By the definition of the norm in T , there exists $f \in W(D(\Lambda), A_1, p, \theta)$ such that

$$\|u\|_T^p + \delta > \|f\|_W^p, \quad f(0) = u.$$

Now by Lemma 44.1.4 there exists $g_r \in W$ such that $\|g_r - f\|_W < r$ and $g_r \in C^\infty(0, \infty; D(\Lambda))$ and $g'_r \in C^\infty(0, \infty; A_1)$. Thus for each $\varepsilon > 0$, $g_r(\varepsilon) \in D(\Lambda)$ although possibly $g_r(0) \notin D(\Lambda)$. Then letting $h_r(t)$ be defined by

$$g'_r(t) - \Lambda g_r(t) = h_r(t)$$

it follows $h_r \in C^1(0, \infty; A_1)$ and applying Lemma 45.3.4 on $[\varepsilon, \infty)$ it follows

$$g_r(t) = G(t - \varepsilon)g_r(\varepsilon) + \int_\varepsilon^t G(t - s)h_r(s)ds. \quad (45.3.11)$$

By Lemma 44.1.4 again, $g_r(\varepsilon)$ converges to $g_r(0)$ in A_1 . Thus

$$\int_\varepsilon^t \|G(t - s)h_r(s)\|_{A_1} ds \leq C$$

for some constant independent of ε . Thus $s \rightarrow G(t - s)h_r(s)$ is in $L^1(0, t; A_1)$ and it is possible to pass to the limit in 45.3.11 as $\varepsilon \rightarrow 0$ to conclude

$$g_r(t) = G(t)g_r(0) + \int_0^t G(t - s)h_r(s)ds$$

Now

$$\frac{G(t)g_r(0) - g_r(0)}{t} = \frac{1}{t} \int_0^t g'_r(s)ds - \frac{1}{t} \int_0^t G(t - s)h_r(s)ds$$

and so using the assumption that $G(t)$ is uniformly bounded,

$$\left\| \frac{G(t)g_r(0) - g_r(0)}{t} \right\| \leq \frac{1}{t} \int_0^t \|g'_r\|_{A_1} + M \|h_r\|_{A_1}$$

$$\begin{aligned}
&\leq \frac{1}{t} \int_0^t \|g'_r\| (M+1) + M \|\Lambda g_r\| ds \\
&\leq \frac{M+1}{t} \int_0^t \|g'_r\|_{A_1} + \|g_r\|_{D(\Lambda)} ds
\end{aligned}$$

Therefore, from Lemma 45.3.2

$$\begin{aligned}
&\int_0^\infty t^{p\theta-p} \|G(t) g_r(0) - g_r(0)\|_{A_1}^p \frac{dt}{t} \\
&= \int_0^\infty t^{p\theta} \left\| \frac{G(t) g_r(0) - g_r(0)}{t} \right\|_{A_1}^p \frac{dt}{t} \\
&\leq \int_0^\infty t^{p\theta} \left| \frac{M+1}{t} \int_0^t \|g'_r\|_{A_1} + \|g_r\|_{D(\Lambda)} ds \right|^p \frac{dt}{t} \\
&\leq (M+1)^p 2^{p-1} \left(\frac{1}{1-\theta} \right)^p \int_0^\infty t^{p\theta} \left(\|g'_r\|_{A_1}^p + \|g_r\|_{D(\Lambda)}^p \right)
\end{aligned}$$

Now since $g_r \rightarrow f$ in W , it follows from Lemma 44.1.8 that $g_r(0) \rightarrow u$ in T and hence by Theorem 44.1.9 this also in A_1 . Therefore, using Fatou's lemma in the above along with the convergence of g_r to f ,

$$\begin{aligned}
&\int_0^\infty t^{p\theta-p} \|G(t) u - u\|_{A_1}^p \frac{dt}{t} \\
&\leq (M+1)^p 2^{p-1} \left(\frac{1}{1-\theta} \right)^p \int_0^\infty t^{p\theta} \left(\|f'\|_{A_1}^p + \|f\|_{D(\Lambda)}^p \right) \\
&\leq (M+1)^p 2^{p-1} \left(\frac{1}{1-\theta} \right)^p (\|u\|_T^p + \delta)
\end{aligned}$$

Since $u \in T$, Theorem 44.1.9 implies $u \in A_1$ and $\|u\|_{A_1} \leq C \|u\|_T$. Therefore, since δ was arbitrary, this has shown that $u \in T_0$ and

$$\|u\|_{T_0} \leq C(\theta, p) \|u\|_T.$$

This shows $T \subseteq T_0$ with continuous inclusion.

Now it is necessary to take $u \in T_0$ and show it is in T . Since $u \in T_0$

$$\infty > \|u\|_{A_1}^p + \int_0^\infty t^{\theta p} \left\| \frac{G(t) u - u}{t} \right\|_{A_1}^p \frac{dt}{t} \equiv \|u\|_{T_0}^p$$

Let ϕ be a nonnegative decreasing infinitely differentiable function such that $\phi(0) = 1$ and $\phi(t) = 0$ for all $t > 1$. Then define

$$f(t) \equiv \phi(t) \frac{1}{t} \int_0^t G(\tau) u d\tau.$$

It is easy to see that $f(t) \in D(\Lambda)$. In fact, changing variables as needed,

$$\begin{aligned} & \frac{1}{h} \left(G(h) \int_0^t G(\tau) u d\tau - \int_0^t G(\tau) u d\tau \right) \\ &= \frac{1}{h} \int_h^{t+h} G(\tau) u d\tau - \frac{1}{h} \int_0^t G(\tau) u d\tau \\ &= \frac{1}{h} \int_t^{t+h} G(\tau) u d\tau - \frac{1}{h} \int_0^h G(\tau) u d\tau \end{aligned}$$

which converges to $G(t)u - u$ and so

$$\Lambda \int_0^t G(\tau) u d\tau = G(t)u - u. \quad (45.3.12)$$

Thus

$$\begin{aligned} \int_0^\infty t^{p\theta} \|\Lambda f\|_{A_1}^p \frac{dt}{t} &\leq \int_0^\infty t^{p\theta} \left\| \frac{G(t)u - u}{t} \right\|_{A_1}^p \frac{dt}{t} \\ &\leq \|u\|_{T_0}^p \end{aligned}$$

Next it is necessary to consider

$$\begin{aligned} & \int_0^\infty t^{p\theta} \|f'\|_{A_1}^p \frac{dt}{t} \\ & f'(t) = \phi'(t) \frac{1}{t} \int_0^t G(\tau) u d\tau + \\ & \phi(t) \left(-\frac{1}{t^2} \int_0^t G(\tau) u d\tau + \frac{1}{t} G(t)u \right) \\ &= \phi'(t) \frac{1}{t} \int_0^t G(\tau) u d\tau + \phi(t) \left(\frac{1}{t^2} \int_0^t (G(t)u - G(\tau)u) d\tau \right) \end{aligned}$$

and so there is a constant C depending on ϕ and the uniform bound on $\|G(t)\|$ such that

$$\begin{aligned} \|f'(t)\|_{A_1} &\leq C \mathcal{X}_{[0,1]}(t) \left(\|u\|_{A_1} + \frac{1}{t^2} \int_0^t \|G(t-\tau)u - u\| d\tau \right) \\ &= C \mathcal{X}_{[0,1]}(t) \left(\|u\|_{A_1} + \frac{1}{t^2} \int_0^t \|G(\tau)u - u\| d\tau \right) \end{aligned}$$

Now

$$\int_0^\infty \mathcal{X}_{[0,1]}(t) t^{p\theta} \|u\|_{A_1}^p dt/t \leq C(p, \theta) \|u\|_{T_0}^p$$

and using Lemma 45.3.2,

$$\begin{aligned}
 & \int_0^\infty \mathcal{X}_{[0,1]}(t) \left| \frac{1}{t^2} \int_0^t \|G(\tau)u - u\| d\tau \right|^p t^{p\theta} \frac{dt}{t} \\
 & \leq \int_0^\infty \left| \frac{1}{t} \int_0^t \|G(\tau)u - u\| d\tau \right|^p t^{p(\theta-1)} \frac{dt}{t} \\
 & \leq \frac{1}{(1-(\theta-1))^p} \int_0^\infty \|G(\tau)u - u\|^p t^{p(\theta-1)} \frac{dt}{t} \\
 & = \frac{1}{(2-\theta)^p} \int_0^\infty \left\| \frac{G(\tau)u - u}{t} \right\|^p t^{p\theta} \frac{dt}{t} \leq C(\theta, p) \|u\|_{T_0}^p
 \end{aligned}$$

This proves the theorem.

Of course the case of most interest here is where $A_1 = L^p(\mathbb{R}^n)$ and

$$G(t)u(\mathbf{x}) \equiv u(\mathbf{x} + t\mathbf{e}_i)$$

Thus $\Lambda u = \partial u / \partial x_i$, the weak derivative. The trace space $T(D(\Lambda), L^p(\mathbb{R}^n), p, 1-\theta)$ then is a space of functions in $L^p(\mathbb{R}^n)$ which have a fractional order partial derivative with respect to x_i .

Recall from Definition 45.1.1 that for $\theta \in (0, 1)$,

$$W^{\theta,p}(\mathbb{R}^n) \equiv T(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n), p, 1-\theta)$$

Let $f \in W(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n), p, 1-\theta)$. Then

$$\|f\|_W \equiv \max \left(\int_0^\infty t^{(1-\theta)p} \|f(t)\|_{W^{1,p}}^p \frac{dt}{t}, \int_0^\infty t^{(1-\theta)p} \|f'(t)\|_{L^p}^p \frac{dt}{t} \right)$$

Letting $G_i(t)u(\mathbf{x}) \equiv u(\mathbf{x} + t\mathbf{e}_i)$ and Λ_i its generator,

$$W^{1,p}(\Omega) = \cap_{i=1}^n D\Lambda_i \cap L^p(\mathbb{R}^n)$$

with the norm given by

$$\|u\|^p = \|u\|_{L^p}^p + \sum_{i=1}^n \|\Lambda_i u\|_{L^p}^p$$

which is equivalent to the norm

$$\|u\|^p = \sum_{i=1}^n \|u\|_{D(\Lambda_i)}^p.$$

Then by considering each of the G_i and repeating the above argument in Theorem 45.3.6, it follows an equivalent intrinsic norm is

$$\|u\|_{W^{\theta,p}(\mathbb{R}^n)}^p = \|u\|_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left\| \frac{G_i(t)u - u}{t} \right\|_{L^p}^p \frac{dt}{t}$$

$$= \|u\|_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left\| \frac{u(\cdot + t\mathbf{e}_i) - u(\cdot)}{t} \right\|_{L^p}^p \frac{dt}{t} \quad (45.3.13)$$

and $u \in W^{\theta,p}(\mathbb{R}^n)$ when this norm is finite. The only new detail is that in showing that for $u \in T_0$ it follows it is in T , you use the function

$$f(t) \equiv \phi(t) \frac{1}{t^n} \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u d\tau_1 \cdots d\tau_n$$

and the fact that these semigroups commute. To get this started, note that

$$g(t) \equiv \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u d\tau_1 \cdots d\tau_n \in D(\Lambda_i)$$

for each i . This follows from writing it as

$$\int_0^t G_i(\tau_i)(w_i) d\tau_i$$

for $w_i \in L^p$ coming from the other integrals and then repeating the earlier argument to get

$$\Lambda_i g(t) = G_i(t) w_i - w_i$$

and then

$$\begin{aligned} & \int_0^\infty t^{p(1-\theta)} \|\Lambda_i f\|_{L^p}^p \frac{dt}{t} \\ & \leq \int_0^\infty t^{p(1-\theta)} \left\| \frac{G_i(t) w_i - w_i}{t} \right\|_{L^p}^p \frac{dt}{t} \\ & \leq C \int_0^\infty t^{p(1-\theta)} \left\| \frac{G_i(t) u - u}{t} \right\|_{L^p}^p \frac{dt}{t} \leq C \|u\|_{T_0}^p \end{aligned}$$

Thus all is well as far as f is concerned and the proof will work as it did earlier in Theorem 45.3.6. What about f' ? As before, the only term which is problematic is

$$\phi(t) \left(\frac{1}{t^n} \int_0^t \cdots \int_0^t G_1(\tau_1) G_2(\tau_2) \cdots G_n(\tau_n) u d\tau_1 \cdots d\tau_n \right)'$$

After enough massaging, it becomes

$$\sum_{i=1}^n \prod_{j \neq i} \frac{1}{t} \int_0^t G_j(\tau_j) d\tau_j \frac{1}{t^2} \int_0^t (G_i(t) u - G_i(\tau_i) u) d\tau_i$$

where the operator $\sum_{i=1}^n \prod_{j \neq i} \frac{1}{t} \int_0^t G_j(\tau_j) d\tau_j$ is bounded. Thus similar arguments to those of Theorem 45.3.6 will work, the only difference being a sum.

Theorem 45.3.7 *An equivalent norm for $W^{\theta,p}(\mathbb{R}^n)$ for $\theta \in (0, 1)$ is*

$$\begin{aligned} \|u\|_{W^{\theta,p}(\mathbb{R}^n)}^p &= \\ & \|u\|_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left\| \frac{G_i(t)u - u}{t} \right\|_{L^p}^p \frac{dt}{t} \\ &= \|u\|_{L^p(\mathbb{R}^n)}^p + \sum_{i=1}^n \int_0^\infty t^{(1-\theta)p} \left\| \frac{u(\cdot + t\mathbf{e}_i) - u(\cdot)}{t} \right\|_{L^p}^p \frac{dt}{t} \quad (45.3.14) \end{aligned}$$

Note it is obvious from 45.3.13 that a Lipschitz map takes $W^{\theta,p}(\mathbb{R}^n)$ to $W^{\theta,p}(\mathbb{R}^n)$ and is continuous.

The above description in Theorem 45.3.7 also makes possible the following corollary.

Corollary 45.3.8 *$W^{\theta,p}(\mathbb{R}^n)$ is reflexive.*

Proof: Let $u \in W^{\theta,p}(\mathbb{R}^n)$. For each $i = 1, 2, \dots, n$, define for $t > 0$,

$$\Delta_i u(t)(\mathbf{x}) \equiv \frac{u(\mathbf{x} + t\mathbf{e}_i) - u(\mathbf{x})}{t}$$

Then by Theorem 45.3.7,

$$\Delta_i u \in L^p((0, \infty); L^p(\mathbb{R}^n), \mu) \equiv Y$$

where

$$\mu(E) \equiv \int_E t^{(1-\theta)p} t^{-1} dt.$$

Clearly the measure space is σ finite and so Y is reflexive by Corollary 23.8.9 on Page 745. Also Δ_i is a closed operator whose domain is $W^{\theta,p}(\mathbb{R}^n)$. To see this, suppose $u_n \in W^{\theta,p}(\mathbb{R}^n)$ and $u_n \rightarrow u$ in $L^p(\mathbb{R}^n)$ while $\Delta_i u_n \rightarrow g$ in Y . Then in particular $\|\Delta_i u_n\|_Y$ is bounded. Now by Fatou's lemma,

$$\begin{aligned} & \int_0^\infty t^{(1-\theta)p} \left\| \frac{u(\cdot + t\mathbf{e}_i) - u(\cdot)}{t} \right\|_{L^p(\mathbb{R}^n)}^p \frac{dt}{t} \leq \\ & \liminf_{n \rightarrow \infty} \int_0^\infty t^{(1-\theta)p} \left\| \frac{u_n(\cdot + t\mathbf{e}_i) - u_n(\cdot)}{t} \right\|_{L^p(\mathbb{R}^n)}^p \frac{dt}{t} < \infty. \end{aligned}$$

Letting $\vec{\Delta} \equiv (\Delta_1, \Delta_2, \dots, \Delta_n)$, it follows from similar reasoning that $\vec{\Delta}$ is a closed operator mapping $W^{\theta,p}(\mathbb{R}^n)$ to Y^n . Therefore

$$(\text{id}, \vec{\Delta})(W^{\theta,p}(\mathbb{R}^n)) \subseteq L^p(\mathbb{R}^n) \times Y^n$$

and is a closed subspace of the reflexive space $L^p(\mathbb{R}^n) \times Y^n$. With the norm in $L^p(\mathbb{R}^n) \times Y^n$ given as the sum of the norms of the components, it follows the

mapping $(\text{id}, \vec{\Delta})$ is a norm preserving isomorphism between $W^{\theta,p}(\mathbb{R}^n)$ and this closed subspace of $L^p(\mathbb{R}^n) \times Y^n$. Since $L^p(\mathbb{R}^n)$ and Y are reflexive, their product is reflexive. By Lemma 23.2.7 on Page 710 it follows $(\text{id}, \vec{\Delta})(W^{\theta,p}(\mathbb{R}^n))$ and hence $W^{\theta,p}(\mathbb{R}^n)$ is reflexive. This proves the theorem.

One can generalize this to find an intrinsic norm for $W^{\theta,p}(\Omega)$. The version given above will not do because it requires the function to be defined on all of $\mathbb{R}^n$ in order to make sense of the shift operators G_i . However, you can give a different version of this intrinsic norm which will make sense for $\Omega \neq \mathbb{R}^n$.

Lemma 45.3.9 *Let $t \neq 0$ be a number. Then there is a constant $C(n, \theta, p)$ depending on the indicated quantities such that*

$$\int_{\mathbb{R}^{n-1}} \frac{1}{(t^2 + |\mathbf{s}|^2)^{\frac{1}{2}(n+p\theta)}} ds = \frac{C(n, \theta, p)}{|t|^{1+p\theta}}$$

Proof: Change the integral to polar coordinates. Thus the integral equals

$$\int_{S^{n-1}} \int_0^\infty \frac{\rho^{n-2}}{(t^2 + \rho^2)^{\frac{1}{2}(n+p\theta)}} d\rho d\sigma$$

Now change the variables, $\rho = |t|u$. Then the above integral becomes

$$\begin{aligned} & C_n \int_0^\infty \frac{|t|^{n-2} u^{n-2} |t|}{|t|^{n+p\theta} (1+u^2)^{\frac{1}{2}(n+p\theta)}} du \\ &= C_n \frac{1}{|t|^{1+p\theta}} \int_0^\infty \frac{u^{n-2}}{(1+u^2)^{\frac{1}{2}(n+p\theta)}} du \equiv \frac{C(n, \theta, p)}{|t|^{1+p\theta}}. \end{aligned}$$

This proves the lemma.

Now let $u \in W^{\theta,p}(\mathbb{R}^n)$. This means the norm of $\|u\|_{W^{\theta,p}}^p$ can be taken as

$$\begin{aligned} & \|u\|_{L^p}^p + \sum_{i=1}^n \int_0^\infty |t|^{(1-\theta)p} \int_{\mathbb{R}^n} \left| \frac{u(\mathbf{x} + t\mathbf{e}_i) - u(\mathbf{x})}{t} \right|^p dx \frac{dt}{|t|} \\ &= \|u\|_{L^p}^p + \frac{1}{2} \sum_{i=1}^n \int_{-\infty}^\infty |t|^{(1-\theta)p} \int_{\mathbb{R}^n} \left| \frac{u(\mathbf{x} + t\mathbf{e}_i) - u(\mathbf{x})}{t} \right|^p dx \frac{dt}{|t|} \end{aligned}$$

That integral over $\mathbb{R}^n$ can be massaged and one obtains the above equal to

$$\|u\|_{L^p}^p +$$

$$\begin{aligned} & \frac{1}{2} \sum_{i=1}^n \int_{-\infty}^\infty \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^i} \frac{1}{t^{1+p\theta}} |u(x_1, \dots, x_i + t, y_{i+1}, \dots, y_n) \\ & - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p dx_i dy_{n-i} dt \end{aligned}$$

where dx_i refers to the first i entries and dy_{n-i} refers to the remaining entries. From Lemma 45.3.9, the complicated expression above equals

$$\frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^n \int_{-\infty}^{\infty} \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^i} \int_{\mathbb{R}^{n-1}} \frac{1}{\left(t^2 + |\mathbf{s}|^2\right)^{\frac{1}{2}(n+p\theta)}} \\ |u(x_1, \dots, x_{i-1}, x_i + t, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p \\ ds dx_i dy_{n-i} dt$$

Now Fubini this to get

$$\frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^i} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{\left(t^2 + |\mathbf{s}|^2\right)^{\frac{1}{2}(n+p\theta)}} \\ |u(x_1, \dots, x_{i-1}, x_i + t, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p \\ dt ds dx_i dy_{n-i}$$

Changing the variable in the inside integral to $t = y_i - x_i$, this equals

$$\frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^i} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{\left((y_i - x_i)^2 + |\mathbf{s}|^2\right)^{\frac{1}{2}(n+p\theta)}} \\ |u(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p \\ dy_i ds dx_i dy_{n-i}$$

Next let

$$(s_1, \dots, s_{n-1}) \\ \equiv (y_1 - x_1, \dots, y_{i-1} - x_{i-1}, x_{i+1} - y_{i+1}, \dots, x_n - y_n)$$

where the new variables of integration in the integral corresponding to ds are $y_1, \dots, y_{i-1}$ and $x_{i+1}, \dots, x_n$. Then changing the variables, the above reduces to

$$\frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^{n-i}} \int_{\mathbb{R}^i} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \frac{1}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} \\ |u(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p \\ dy_i dy_1 \dots dy_{i-1} dx_{i+1} \dots dx_n dx_1 \dots dx_i dy_{i+1} \dots dy_n$$

Then if you Fubini again, it reduces to the expression

$$\frac{1}{C(n, \theta, p)} \frac{1}{2} \sum_{i=1}^n \\ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy$$

Now taking the sum inside and adjusting the constants yields

$$\geq C(n, \theta, p) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy$$

Thus there exists a constant $C(n, \theta, p)$ such that

$$\|u\|_{W^{\theta,p}(\mathbb{R}^n)} \geq C(n, \theta, p) \left(\|u\|_{L^p}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \right)^{1/p}.$$

Next start with the right side of the above. It suffices to consider only the complicated term. First note that for $\mathbf{a}$ a vector,

$$\left(\sum_{i=1}^n a_i^2 \right)^{p/2} \geq |a_i|^p$$

and so

$$\sum_{i=1}^n |a_i|^p \leq n \left(\sum_{i=1}^n a_i^2 \right)^{p/2} = n |\mathbf{a}|^p$$

from which it follows

$$\frac{1}{n} \sum_{i=1}^n |a_i|^p \leq |\mathbf{a}|^p$$

Then it follows

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy &\geq \frac{1}{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \\ \frac{\sum_{i=1}^n |u(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \end{aligned} \quad (45.3.15)$$

Consider the i^{th} term. By Fubini's theorem it equals

$$\begin{aligned} &\frac{1}{n} \int \dots \int \frac{1}{\left((y_i - x_i)^2 + \sum_{j \neq i} (y_j - x_j)^2 \right)^{\frac{1}{2}(n+p\theta)}} \\ &|u(x_1, \dots, x_{i-1}, y_i, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p \\ &dy_i dx_1 \dots dx_{i-1} dy_{i+1} \dots dy_n dy_1 \dots dy_{i-1} dx_i \dots dx_n \end{aligned}$$

Let $t = y_i - x_i$. Then it reduces to

$$\begin{aligned} &\frac{1}{n} \int \dots \int \frac{1}{\left(t^2 + \sum_{j \neq i} (y_j - x_j)^2 \right)^{\frac{1}{2}(n+p\theta)}} \\ &|u(x_1, \dots, x_{i-1}, x_i + t, y_{i+1}, \dots, y_n) - u(x_1, \dots, x_i, y_{i+1}, \dots, y_n)|^p \\ &dt dx_1 \dots dx_{i-1} dy_{i+1} \dots dy_n dy_1 \dots dy_{i-1} dx_i \dots dx_n \end{aligned}$$

Now let

$$(s_1, \dots, s_{n-1}) = (x_1 - y_1, \dots, x_{i-1} - y_{i-1}, y_{i+1} - x_{i+1}, \dots, y_n - x_n)$$

on the next $n-1$ iterated integrals. Then using Fubini's theorem again and changing the variables, it equals

$$\begin{aligned} & \frac{1}{n} \int \dots \int \frac{1}{(t^2 + |\mathbf{s}|^2)^{\frac{1}{2}(n+p\theta)}} \cdot \\ & |u(s_1 + y_1, \dots, y_{i-1} + s_{i-1}, x_i + t, x_{i+1} + s_i, \dots, x_n + s_{n-1}) \\ & - u(s_1 + y_1, \dots, y_{i-1} + s_{i-1}, x_i, x_{i+1} + s_i, \dots, x_n + s_{n-1})|^p \\ & dy_1 \dots dy_{i-1} dx_i \dots dx_n ds_1 \dots ds_{i-1} ds_i \dots ds_{n-1} dt \end{aligned}$$

By translation invariance of the measure, the inside integrals corresponding to

$$dy_1 \dots dy_{i-1} dx_i \dots dx_n$$

simplify and the expression can be written as

$$\begin{aligned} & \frac{1}{n} \int \dots \int \frac{1}{(t^2 + |\mathbf{s}|^2)^{\frac{1}{2}(n+p\theta)}} \cdot \\ & |u(x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n) - u(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)|^p \\ & dx_1 \dots dx_{i-1} dx_i \dots dx_n ds_1 \dots ds_{i-1} ds_i \dots ds_{n-1} dt \end{aligned}$$

where I just renamed the variables. Use Fubini's theorem again to get

$$\begin{aligned} & \frac{1}{n} \int \dots \int \frac{1}{(t^2 + |\mathbf{s}|^2)^{\frac{1}{2}(n+p\theta)}} \cdot \\ & |u(x_1, \dots, x_{i-1}, x_i + t, x_{i+1}, \dots, x_n) - u(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)|^p \\ & ds_1 \dots ds_{i-1} ds_i \dots ds_{n-1} dx_1 \dots dx_{i-1} dx_i \dots dx_n dt \end{aligned}$$

Now from Lemma 45.3.9, the inside $n-1$ integrals corresponding to $ds_1 \dots ds_{i-1} ds_i \dots ds_{n-1}$ can be replaced with

$$\frac{C(n, \theta, p)}{|t|^{1+p\theta}}$$

and this yields

$$\begin{aligned} & C(n, \theta, p) \int_{\mathbb{R}} \frac{1}{|t|^{p\theta}} \int_{\mathbb{R}^n} |u(\mathbf{x} + t\mathbf{e}_i) - u(\mathbf{x})|^p d\mathbf{x} \frac{dt}{t} \\ & = \frac{1}{2} C(n, \theta, p) \int_0^\infty t^{p(1-\theta)} \frac{\|u(\cdot + t\mathbf{e}_i) - u(\cdot)\|_{L^p(\mathbb{R}^n)}^p}{t^p} \frac{dt}{t} \end{aligned}$$

Applying this to each term of the sum in 45.3.15 and adjusting the constant, it follows

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \geq C(n, \theta, p) \sum_{i=1}^n \int_0^\infty t^{p(1-\theta)} \frac{\|u(\cdot + t\mathbf{e}_i) - u(\cdot)\|_{L^p(\mathbb{R}^n)}^p}{t^p} \frac{dt}{t}$$

Therefore,

$$\begin{aligned} & \|u\|_{W^{\theta,p}(\mathbb{R}^n)} \\ & \geq C(n, \theta, p) \left(\|u\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \right)^{1/p} \end{aligned}$$

This has proved most of the following theorem about the intrinsic norm.

Theorem 45.3.10 *An equivalent norm for $W^{\theta,p}(\mathbb{R}^n)$ is*

$$\begin{aligned} & \|u\| = \\ & \left(\|u\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \right)^{1/p}. \end{aligned}$$

Also for any open subset of $\mathbb{R}^n$

$$\begin{aligned} & \|u\| = \\ & \left(\|u\|_{L^p(\Omega)}^p + \int_{\Omega} \int_{\Omega} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \right)^{1/p}. \end{aligned} \quad (45.3.16)$$

is a norm.

Proof: It only remains to verify this is a norm. Recall the l_p norm on $\mathbb{R}^2$ given by

$$|(x, y)|_{l_p} \equiv (|x|^p + |y|^p)^{1/p}$$

For $u, v \in W^{\theta,p}$ denote by $\rho(u)$ the expression

$$\left(\int_{\Omega} \int_{\Omega} \frac{|u(\mathbf{y}) - u(\mathbf{x})|^p}{|\mathbf{x} - \mathbf{y}|^{(n+p\theta)}} dx dy \right)^{1/p}$$

a similar definition holding for v . Then it follows from the usual Minkowski inequality that $\rho(u+v) \leq \rho(u) + \rho(v)$. Then from 45.3.16

$$\begin{aligned} \|u+v\| &= (\|u+v\|_{L^p}^p + \rho(u+v)^p)^{1/p} \\ &\leq ((\|u\|_{L^p} + \|v\|_{L^p})^p + (\rho(u) + \rho(v))^p)^{1/p} \end{aligned}$$

$$\begin{aligned}
&= (||u||_{L^p}, \rho(u)) + (||v||_{L^p}, \rho(v))|_{l_p} \\
&\leq (||u||_{L^p}, \rho(u))|_{l_p} + (||v||_{L^p}, \rho(v))|_{l_p} \\
&= (||u||_{L^p}^p + \rho(u)^p)^{1/p} + (||v||_{L^p}^p + \rho(v)^p)^{1/p} \\
&= ||u|| + ||v||
\end{aligned}$$

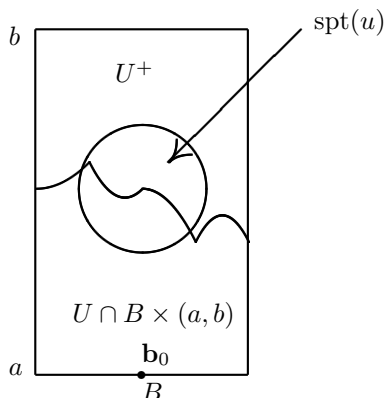
The other properties of a norm are obvious. This proves the theorem.

As pointed out in the above theorem, this is a norm in 45.3.16. One could define a set of functions for which this norm is finite. In the case where $\Omega = \mathbb{R}^n$ the conclusion of Theorem 45.3.10 is that this space of functions is the same as $W^{\theta,p}(\mathbb{R}^n)$ and the norms are equivalent. Does this happen for other open subsets of $\mathbb{R}^n$?

Definition 45.3.11 Denote by $\widetilde{W^{\theta,p}(U)}$ the functions in $L^p(U)$ for which the norm of Theorem 45.3.10 is finite. Here $\theta \in (0, 1)$.

Proposition 45.3.12 Let U be a bounded open set which has Lipschitz boundary and $\theta \in (0, 1)$. Then for each $p \geq 1$, there exists $E \in \mathcal{L}(\widetilde{W^{\theta,p}(U)}, W^{\theta,p}(\mathbb{R}^n))$ such that $Eu(\mathbf{x}) = u(\mathbf{x})$ a.e. $\mathbf{x} \in U$.

Proof: In proving this, I will use the equivalent norm of Theorem 45.3.10 as the norm of $W^{\theta,p}(\mathbb{R}^n)$. Consider the following picture.



The wavy line signifies a part of the boundary of U and $\text{spt}(u)$ is contained in the circle as shown. It is drawn as a circle but this is not important. Denote by U^+ the region above the part of the boundary which is shown. Also let the boundary be given by $x_n = g(\widehat{\mathbf{x}})$ for $\widehat{\mathbf{x}} \in B \equiv B(\widehat{\mathbf{y}}_0, r) \subseteq \mathbb{R}^{n-1}$. Of course u is only defined on U so actually the support of u is contained in the intersection of the circle with $\overline{U}$. Let the Lipschitz constant for g be very small and denote it by K . In fact, assume $8K^2 < 1$. I will first show how to extend when this condition holds and then I will remove it with a simple trick. Define

$$Eu(\widehat{\mathbf{x}}, x_n) \equiv \begin{cases} u(\widehat{\mathbf{x}}, x_n) & \text{if } x_n \leq g(\widehat{\mathbf{x}}) \\ u(\widehat{\mathbf{x}}, 2g(\widehat{\mathbf{x}}) - x_n) & \text{if } x_n > g(\widehat{\mathbf{x}}) \\ 0 & \text{if } \widehat{\mathbf{x}} \notin B \end{cases}$$

I will write U instead of $U \cap B \times (a, b)$ to save space but this does not matter because u is assumed to be zero outside the indicated region. Then

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|Eu(\widehat{\mathbf{x}}, x_n) - Eu(\widehat{\mathbf{y}}, y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx dy \\ &= \int_U \int_U \frac{|u(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx dy + \\ & \int_{U^+} \int_U \frac{|u(\widehat{\mathbf{x}}, x_n) - Eu(\widehat{\mathbf{y}}, y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx dy + \end{aligned} \quad (45.3.17)$$

$$\int_U \int_{U^+} \frac{|Eu(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx dy + \quad (45.3.18)$$

$$\int_{U^+} \int_{U^+} \frac{|Eu(\widehat{\mathbf{x}}, x_n) - Eu(\widehat{\mathbf{y}}, y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx dy \quad (45.3.19)$$

Consider the second of the integrals on the right of the equal sign. Using Fubini's theorem, it equals

$$\begin{aligned} & \int_U \int_{U^+} \frac{|u(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dy dx \\ &= \int_U \int_B \int_{g(\widehat{\mathbf{y}})}^\infty \frac{|u(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dy_n d\widehat{\mathbf{y}} dx \\ &= \int_U \int_B \int_{-\infty}^{g(\widehat{\mathbf{y}})} \frac{|u(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, z_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - (2g(\widehat{\mathbf{y}}) - z_n))^2 \right|^{(1/2)(n+p\theta)}} dz_n d\widehat{\mathbf{y}} dx \end{aligned}$$

I need to estimate $|x_n - z_n|$.

$$\begin{aligned} |x_n - z_n| &\leq |x_n - g(\widehat{\mathbf{x}})| + |g(\widehat{\mathbf{x}}) - g(\widehat{\mathbf{y}})| + |g(\widehat{\mathbf{y}}) - z_n| \\ &\leq |g(\widehat{\mathbf{x}}) - x_n| + K|\widehat{\mathbf{x}} - \widehat{\mathbf{y}}| + |y_n - g(\widehat{\mathbf{y}})| \\ &\leq |y_n - x_n| + 2K|\widehat{\mathbf{x}} - \widehat{\mathbf{y}}| \end{aligned}$$

and so

$$\begin{aligned} |x_n - z_n|^2 &\leq 8K^2 |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + 2|y_n - x_n|^2 \\ &\leq |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + 2|y_n - x_n|^2 \end{aligned}$$

Thus,

$$(y_n - x_n)^2 \geq \frac{1}{2} |x_n - z_n|^2 - \frac{1}{2} |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2$$

and so, the above change of variables results in an expression which is dominated by

$$\int_U \int_U \frac{|u(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, z_n)|^p}{\left| \frac{1}{2} |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + \frac{1}{2} (x_n - z_n)^2 \right|^{(1/2)(n+p\theta)}} dy dx$$

where y refers to $(\widehat{\mathbf{y}}, z_n)$ in the above formula. Hence there is a constant $C(n, \theta)$ such that 45.3.17 is dominated by $C(n, \theta) \|u\|_{\widetilde{W^{\theta, p}(U)}}^p$. A similar inequality holds for the third term. Finally consider 45.3.19. This equals

$$\int_{U^+} \int_{U^+} \frac{|u(\widehat{\mathbf{x}}, 2g(\widehat{\mathbf{x}}) - x_n) - u(\widehat{\mathbf{y}}, 2g(\widehat{\mathbf{y}}) - y_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx dy$$

Changing variables, $x'_n = 2g(\widehat{\mathbf{x}}) - x_n, y'_n = 2g(\widehat{\mathbf{y}}) - y_n$, it equals

$$\int_{U^+} \int_{U^+} \frac{|u(\widehat{\mathbf{x}}, x'_n) - u(\widehat{\mathbf{y}}, y'_n)|^p}{\left| |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2 \right|^{(1/2)(n+p\theta)}} dx' dy', \quad (45.3.20)$$

each of x_n, y_n being a function of x'_n, y'_n where an estimate needs to be obtained on $|x'_n - y'_n|$ in terms of $|x_n - y_n|$.

$$\begin{aligned} (x'_n - y'_n)^2 &= (2(g(\widehat{\mathbf{x}}) - g(\widehat{\mathbf{y}})) + y_n - x_n)^2 \\ &= (y_n - x_n)^2 + 4(g(\widehat{\mathbf{x}}) - g(\widehat{\mathbf{y}}))(y_n - x_n) \\ &\quad + 4(g(\widehat{\mathbf{x}}) - g(\widehat{\mathbf{y}}))^2 \\ &\leq (y_n - x_n)^2 + 2(g(\widehat{\mathbf{x}}) - g(\widehat{\mathbf{y}}))^2 \\ &\quad + 2(y_n - x_n)^2 + 4(g(\widehat{\mathbf{x}}) - g(\widehat{\mathbf{y}}))^2 \end{aligned}$$

and so

$$(x'_n - y'_n)^2 \leq 3(y_n - x_n)^2 + 6K^2 |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2$$

which implies

$$(y_n - x_n)^2 \geq \frac{1}{3} (x'_n - y'_n)^2 - 2K^2 |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2.$$

Then substituting this in to 45.3.20, a short computation shows 45.3.19 is dominated by an expression of the form $C(n, \theta) \|u\|_{\widetilde{W^{\theta, p}(U)}}^p$ and this proves the existence of an extension operator provided the Lipschitz constant is small enough. It is clear E is linear where E is defined above.

Now this assumption on the smallness of K needs to be removed. For $(\widehat{\mathbf{x}}, x_n) \in U$ define

$$U' \equiv \left\{ \widehat{\mathbf{x}}' = \lambda \left(\widehat{\mathbf{x}} - \widehat{\mathbf{b}}_0 \right) : \widehat{\mathbf{x}} \in U \right\}$$

here this is centering at $\mathbf{0}$ and stretching B since λ will be large. Let $\mathbf{h}$ be the name of this mapping. Thus

$$\begin{aligned}\mathbf{h}(\widehat{\mathbf{x}}) &\equiv \lambda(\widehat{\mathbf{x}} - \widehat{\mathbf{b}}_0), \\ \mathbf{k}(\widehat{\mathbf{x}}) &\equiv \mathbf{h}^{-1}(\widehat{\mathbf{x}}) = \frac{1}{\lambda}\widehat{\mathbf{x}} + \widehat{\mathbf{b}}_0\end{aligned}$$

These mappings are defined on all of $\mathbb{R}^n$. Now let u' be defined on U' as follows.

$$u'(\widehat{\mathbf{x}}, x_n) \equiv \mathbf{k}^* u(\widehat{\mathbf{x}}, x_n).$$

Also let

$$g'(\widehat{\mathbf{x}}) \equiv \mathbf{k}^* g(\widehat{\mathbf{x}}).$$

Thus $g'(\widehat{\mathbf{x}}) \equiv g\left(\frac{1}{\lambda}\widehat{\mathbf{x}} + \widehat{\mathbf{b}}_0\right) = g(\mathbf{k}(\widehat{\mathbf{x}}))$. Then choosing λ large enough the Lipschitz condition for g' is as small as desired. Always assume λ has been chosen this large and also $\lambda \geq 1$. Furthermore, $g'(\widehat{\mathbf{x}}) = x'_n$ describes the boundary in the same way as $x_n = g(\widehat{\mathbf{x}})$. Now I need to consider whether $u' \in \widetilde{W^{\theta,p}(U')}$. Consider

$$\begin{aligned}& \int_{U'} \int_{U'} \frac{|u'(\widehat{\mathbf{x}}', x_n) - u'(\widehat{\mathbf{y}}', y_n)|^p}{\left(|\widehat{\mathbf{x}}' - \widehat{\mathbf{y}}'|^2 + (x_n - y_n)^2\right)^{p+n\theta}} dx' dy' \\ &= \int_{U'} \int_{U'} \frac{|\mathbf{k}^* u(\widehat{\mathbf{x}}', x_n) - \mathbf{k}^* u(\widehat{\mathbf{y}}', y_n)|^p}{\left(|\widehat{\mathbf{x}}' - \widehat{\mathbf{y}}'|^2 + (x_n - y_n)^2\right)^{p+n\theta}} dx' dy'\end{aligned}$$

Then change the variables $\widehat{\mathbf{x}}' = \lambda(\widehat{\mathbf{x}} - \widehat{\mathbf{b}}_0) = \mathbf{h}(\widehat{\mathbf{x}})$ with a similar change for $\widehat{\mathbf{y}}'$, the above expression equals

$$(\lambda^{n-1})^2 \int_U \int_U \frac{|u(\widehat{\mathbf{x}}, x_n) - u(\widehat{\mathbf{y}}, y_n)|^p}{\left(\lambda^2 |\widehat{\mathbf{x}} - \widehat{\mathbf{y}}|^2 + (x_n - y_n)^2\right)^{p+n\theta}} dx dy$$

Thus

$$\|u'\|_{\widetilde{W^{\theta,p}(U')}} \leq \lambda^{n-1} \|u\|_{\widetilde{W^{\theta,p}(U)}} < \infty \quad (45.3.21)$$

and $\mathbf{k}^* : \widetilde{W^{\theta,p}(U)}$ to $\widetilde{W^{\theta,p}(U')}$ is continuous and linear. Similar reasoning shows that $\mathbf{h}^*$ is continuous and linear mapping $\widetilde{W^{\theta,p}(\mathbb{R}^n)}$ to $\widetilde{W^{\theta,p}(\mathbb{R}^n)}$. By the first part of the argument there exists a continuous linear map

$$E' : \widetilde{W^{\theta,p}(U')} \rightarrow \widetilde{W^{\theta,p}(\mathbb{R}^n)}$$

Now define

$$Eu \equiv \mathbf{h}^* E' (\mathbf{k}^* u)$$

Say $x_n \leq g(\hat{\mathbf{x}})$. Then $(\hat{\mathbf{x}}, x_n) \in U$ and so $(\mathbf{h}(\hat{\mathbf{x}}), x_n) \in U'$. Thus

$$\begin{aligned} Eu(\hat{\mathbf{x}}, x_n) &\equiv \mathbf{h}^* E' (\mathbf{k}^* u)(\hat{\mathbf{x}}, x_n) \\ &\equiv E' (\mathbf{k}^* u)(\mathbf{h}(\hat{\mathbf{x}}), x_n) \end{aligned} \quad (45.3.22)$$

Now $g(\hat{\mathbf{x}}) \geq x_n$ and so $g'(\mathbf{h}(\hat{\mathbf{x}})) \geq x_n$ because

$$g'(\mathbf{h}(\hat{\mathbf{x}})) \equiv \mathbf{k}^* g(\mathbf{h}(\hat{\mathbf{x}})) = g(\hat{\mathbf{x}})$$

and so 45.3.22 equals

$$= (\mathbf{k}^* u)(\mathbf{h}(\hat{\mathbf{x}}), x_n) = u(\hat{\mathbf{x}}, x_n)$$

Thus E leaves u unchanged at points $(\hat{\mathbf{x}}, x_n)$ where $x_n \leq g(\hat{\mathbf{x}})$. Also

$$\begin{aligned} \|Eu\|_{\widetilde{W^{\theta,p}(\mathbb{R}^n)}} &= \|\mathbf{h}^* E' \mathbf{k}^* u\|_{\widetilde{W^{\theta,p}(\mathbb{R}^n)}} \leq \|\mathbf{h}^*\| \|E' \mathbf{k}^* u\|_{\widetilde{W^{\theta,p}(\mathbb{R}^n)}} \\ &\leq \|\mathbf{h}^*\| \|E'\| \|\mathbf{k}^*\| \|u\|_{\widetilde{W^{\theta,p}(U)}} = C \|u\|_{\widetilde{W^{\theta,p}(U)}} \end{aligned}$$

To complete the proof, cover U with finitely many sets of this sort oriented with respect to one of the coordinate axes as this one was along with an open set whose closure is contained in U and then use a smooth partition of unity to localize the function to the situation of the sort just discussed and one whose support is contained in U . Extend that one to equal zero off its support and treat the others as above. This proves the proposition.

Recall Theorem 38.2.7 which provides an extension operator from $W^{1,p}(U)$ to $W^{1,p}(\mathbb{R}^n)$ also denoted by E . Now it is not hard to see that $\widetilde{W^{\theta,p}(U)} = W^{\theta,p}(U)$ and the two norms are equivalent.

Theorem 45.3.13 *Let U be a bounded open set which has Lipschitz boundary and $\theta \in (0, 1)$. Then $\widetilde{W^{\theta,p}(U)} = W^{\theta,p}(U)$ and the two norms are equivalent.*

Proof: Let $u \in \widetilde{W^{\theta,p}(U)}$. Letting E be the extension operator of Lemma 45.3.12, there is a constant C such that

$$C \|u\|_{\widetilde{W^{\theta,p}(U)}} \geq \|Eu\|_{\widetilde{W^{\theta,p}(\mathbb{R}^n)}} = \|Eu\|_{W^{\theta,p}(\mathbb{R}^n)} \geq \|u\|_{W^{\theta,p}(U)}.$$

Thus if $u \in \widetilde{W^{\theta,p}(U)}$, then $u \in W^{\theta,p}(U)$ and the inclusion map is continuous.

Next suppose $u \in W^{\theta,p}(U)$ and let $\gamma f = u$ for

$$f \in W(W^{1,p}(U), L^p(U), p, 1 - \theta) \equiv W_U$$

Then from Theorem 38.2.7 and the definition of the norm in W_U

$$Ef \in W(W^{1,p}(\mathbb{R}^n), L^p(\mathbb{R}^n), p, 1 - \theta) \equiv W$$

where this E pertains to extending $W^{1,p}(U)$.

$$C_1 \|f\|_{W_U} \geq \|Ef\|_W \geq \|Eu\|_{\widetilde{W^{\theta,p}(\mathbb{R}^n)}} \geq \|u\|_{\widetilde{W^{\theta,p}(U)}}$$

Since this is true for every $f \in W_U$, it follows $u \in \widetilde{W^{\theta,p}(U)}$ and

$$\|u\|_{\widetilde{W^{\theta,p}(U)}} \leq C_1 \|u\|_{W^{\theta,p}(U)}.$$

This proves the theorem.

Corollary 45.3.14 *Let U be a bounded open set with Lipschitz boundary. Then $W^{\theta,p}(U)$ is reflexive.*

Proof: From Proposition 45.3.12 and Theorem 45.3.13, there exists an extension operator $E : W^{\theta,p}(U) \rightarrow W^{\theta,p}(\mathbb{R}^n)$ which is continuous. This operator is one to one and continuous. Furthermore, $\|Eu\|_{W^{\theta,p}(\mathbb{R}^n)} \geq \|u\|_{W^{\theta,p}(U)}$ and so $E(W^{\theta,p}(U))$ is closed. Therefore, by Corollary 23.2.8 on Page 711 and the fact $W^{\theta,p}(\mathbb{R}^n)$ is reflexive which was shown in Corollary 45.3.8, it follows $W^{\theta,p}(U)$ is reflexive. This proves the corollary.

There may be other sets U for which the intrinsic norm is an equivalent norm for $W^{\theta,p}(U)$ but this much will suffice. It should be routine to verify that this works for U a half space for example and the extension argument should be much easier than that presented above. More generally, the assumption that U was bounded in the above extension argument of Proposition 45.3.12 was never needed except for giving finitely many of those special sets covering the boundary. If you just assumed this at the outset instead of an assumption the set is bounded, the same sort of extension would work.

45.4 Fractional Order Sobolev Spaces

Now it is time to define fractional order Sobolev spaces between $W^{m,p}$ and $W^{m+1,p}$.

Definition 45.4.1 *Let m be a nonnegative integer and let $s = m + \sigma$ where $\sigma \in (0, 1)$. Then $W^{s,p}(\Omega)$ will consist of those elements of $W^{m,p}(\Omega)$ for which $D^\alpha u \in W^{\sigma,p}(\Omega)$ for all $|\alpha| = m$. The norm is given by the following.*

$$\|u\|_{s,p,\Omega} \equiv \left(\|u\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \|D^\alpha u\|_{\sigma,p,\Omega}^p \right)^{1/p}.$$

Corollary 45.4.2 *The space, $W^{s,p}(\Omega)$ is a reflexive Banach space whenever $p > 1$.*

Proof: From the theory of interpolation spaces, $W^{\sigma,p}(\Omega)$ is reflexive. This is because it is an interpolation space for the two reflexive spaces, $L^p(\Omega)$ and $W^{1,p}(\Omega)$. (Alternatively, you could use Corollary 45.3.14 in the case where Ω is a bounded

open set with Lipschitz boundary or you could use Corollary 45.3.8 in case $\Omega = \mathbb{R}^n$. In addition, the same ideas would work if Ω were any space for which there was a continuous extension map from $W^{\sigma,p}(\Omega)$ to $W^{\sigma,p}(\mathbb{R}^n)$.) Now the formula for the norm of an element in $W^{s,p}(\Omega)$ shows this space is isometric to a closed subspace of $W^{m,p}(\Omega) \times W^{\sigma,p}(\Omega)^k$ for suitable k . Therefore, from Corollary 23.2.8 on Page 711, $W^{s,p}(\Omega)$ is also reflexive.

Theorem 45.4.3 *The trace map, $\gamma : W^{m,p}(\mathbb{R}_+^n) \rightarrow W^{m-\frac{1}{p},p}(\mathbb{R}^{n-1})$ is continuous.*

Proof: Let $f \in \mathfrak{S}$, the Schwartz class. Let $\sigma = 1 - \frac{1}{p}$ so that $m - \left(\frac{1}{p}\right) = m - 1 + \sigma$. Then from the definition and using $f \in \mathfrak{S}$,

$$\begin{aligned} \|\gamma f\|_{m-\frac{1}{p},p,\mathbb{R}^{n-1}} &= \left(\|\gamma f\|_{m-1,p,\mathbb{R}^{n-1}}^p + \sum_{|\alpha|=m-1} \|D^\alpha \gamma f\|_{1-\frac{1}{p},p,\mathbb{R}^{n-1}}^p \right)^{1/p} \\ &= \left(\|\gamma f\|_{m-1,p,\mathbb{R}^{n-1}}^p + \sum_{|\alpha|=m-1} \|\gamma D^\alpha f\|_{1-\frac{1}{p},p,\mathbb{R}^{n-1}}^p \right)^{1/p} \end{aligned}$$

and from Lemma 45.1.4, and the fact that the trace is continuous as a map from $W^{m,p}(\mathbb{R}_+^n)$ to $W^{m-1,p}(\mathbb{R}^{n-1})$,

$$\begin{aligned} \|\gamma f\|_{m-\frac{1}{p},p,\mathbb{R}^{n-1}} &\leq \left(C_1 \|f\|_{m,p,\mathbb{R}_+^n}^p + C_2 \sum_{|\alpha|=m-1} \|D^\alpha f\|_{1,p,\mathbb{R}^n}^p \right)^{1/p} \\ &\leq C \|f\|_{m,p,\mathbb{R}^{n+p}}. \end{aligned}$$

Then using density of $\mathfrak{S}$ this implies the desired result.

With the definition of $W^{s,p}(\Omega)$ for s not an integer, here is a generalization of an earlier theorem.

Theorem 45.4.4 *Let $\mathbf{h} : U \rightarrow V$ where U and V are two open sets and suppose $\mathbf{h}$ is bilipschitz and that $D^\alpha \mathbf{h}$ and $D^\alpha \mathbf{h}^{-1}$ exist and are Lipschitz continuous if $|\alpha| \leq m$ where $m = 0, 1, \dots$ and $s = m + \sigma$ where $\sigma \in (0, 1)$. Then*

$$\mathbf{h}^* : W^{s,p}(V) \rightarrow W^{s,p}(U)$$

is continuous, linear, one to one, and has an inverse with the same properties, the inverse being $(\mathbf{h}^{-1})^*$.

Proof: In case $m = 0$, the conclusion of the theorem is immediate from the general theory of trace spaces. Therefore, assume $m \geq 1$. It follows from the definition that

$$\|\mathbf{h}^* u\|_{m+\sigma,p,U} \equiv \left[\|\mathbf{h}^* u\|_{m,p,U}^p + \sum_{|\alpha|=m} \|D^\alpha (\mathbf{h}^* u)\|_{\sigma,p,U}^p \right]^{1/p}$$

Consider the case when $m = 1$. Then it is routine to verify that

$$D_j \mathbf{h}^* u(\mathbf{x}) = u_{,k}(\mathbf{h}(\mathbf{x})) h_{k,j}(\mathbf{x}).$$

Let $L_k : W^{1,p}(V) \rightarrow W^{1,p}(U)$ be defined by

$$L_k v = \mathbf{h}^*(v) h_{k,j}.$$

Then L_k is continuous as a map from $W^{1,p}(V)$ to $W^{1,p}(U)$ and as a map from $L^p(V)$ to $L^p(U)$ and therefore, it follows that L_k is continuous as a map from $W^{\sigma,p}(V)$ to $W^{\sigma,p}(U)$. Therefore,

$$\|L_k(v)\|_{\sigma,p,U} \leq C_k \|v\|_{\sigma,p,U}$$

and so

$$\begin{aligned} \|D_j(\mathbf{h}^* u)\|_{\sigma,p,U} &\leq \sum_k \|L_k(u_{,k})\|_{\sigma,p,U} \\ &\leq \sum_k C_k \|D_k u\|_{\sigma,p,V} \\ &\leq C \left(\sum_k \|D_k u\|_{\sigma,p,V}^p \right)^{1/p}. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} \|\mathbf{h}^* u\|_{1+\sigma,p,U} &\leq \left[\|\mathbf{h}^* u\|_{1,p,U}^p + \sum_j C^p \sum_k \|D_k u\|_{\sigma,p,V}^p \right]^{1/p} \\ &\leq C \left[\|u\|_{1,p,V}^p + \sum_k \|D_k u\|_{\sigma,p,V}^p \right]^{1/p} = C \|u\|_{1+\sigma,p,V}. \end{aligned}$$

The general case is similar except for the use of a more complicated linear operator in place of L_k . This proves the theorem.

It is interesting to prove this theorem using Theorem 45.3.13 and the intrinsic norm.

Now we prove an important interpolation inequality for Sobolev spaces.

Theorem 45.4.5 *Let Ω be an open set in $\mathbb{R}^n$ which has the segment property and let $f \in W^{m+1,p}(\Omega)$ and $\sigma \in (0, 1)$. Then for some constant, C , independent of f ,*

$$\|f\|_{m+\sigma,p,\Omega} \leq C \|f\|_{m+1,p,\Omega}^{1-\sigma} \|f\|_{m,p,\Omega}^{\sigma}.$$

Also, if $L \in \mathcal{L}(W^{m,p}(\Omega), W^{m,p}(\Omega))$ for all $m = 0, 1, \dots$, and $L \circ D^\alpha = D^\alpha \circ L$ on $C^\infty(\bar{\Omega})$, then $L \in \mathcal{L}(W^{m+\sigma,p}(\Omega), W^{m+\sigma,p}(\Omega))$ for any $m = 0, 1, \dots$.

Proof: Recall from above, $W^{1-\theta,p}(\Omega) \equiv T(W^{1,p}(\Omega), L^p(\Omega), p, \theta)$. Therefore, from Theorem 44.1.9, if $f \in W^{1,p}(\Omega)$,

$$\|f\|_{1-\theta,p,\Omega} \leq K \|f\|_{1,p,\Omega}^\theta \|f\|_{0,p,\Omega}^{1-\theta}$$

Therefore,

$$\begin{aligned} \|f\|_{m+\sigma,p,\Omega} &\leq \left(\|f\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} K \left(\|D^\alpha f\|_{1,p,\Omega}^{1-\sigma} \|D^\alpha f\|_{0,p,\Omega}^\sigma \right)^p \right)^{1/p} \\ &\leq C \left[\|f\|_{m,p,\Omega}^p + \left(\|f\|_{m+1,p,\Omega}^{1-\sigma} \|f\|_{m,p,\Omega}^\sigma \right)^p \right]^{1/p} \\ &\leq C \left[\left(\|f\|_{m+1,p,\Omega}^{1-\sigma} \|f\|_{m,p,\Omega}^\sigma \right)^p + \left(\|f\|_{m+1,p,\Omega}^{1-\sigma} \|f\|_{m,p,\Omega}^\sigma \right)^p \right]^{1/p} \\ &\leq C \|f\|_{m+1,p,\Omega}^{1-\sigma} \|f\|_{m,p,\Omega}^\sigma. \end{aligned}$$

This proves the first part. Now consider the second. Let $\phi \in C^\infty(\overline{\Omega})$

$$\begin{aligned} \|L\phi\|_{m+\sigma,p,\Omega} &= \left(\|L\phi\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \|D^\alpha L\phi\|_{\sigma,p,\Omega}^p \right)^{1/p} \\ &= \left(\|L\phi\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \|LD^\alpha \phi\|_{T(W^{1,p}, L^p, p, 1-\sigma)}^p \right)^{1/p} \\ &= \left(\|L\phi\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \left[\inf \left(\|t^{1-\sigma} Lf_\alpha\|_1^\sigma \|t^{1-\sigma} Lf'_\alpha\|_2^{1-\sigma} \right) \right]^p \right)^{1/p} \quad (45.4.23) \end{aligned}$$

where

$$\inf \left(\|t^{1-\sigma} Lf_\alpha\|_1^\sigma \|t^{1-\sigma} Lf'_\alpha\|_2^{1-\sigma} \right) =$$

$$\inf \left(\|t^{1-\sigma} Lf_\alpha\|_{L^p(0,\infty;\frac{dt}{t};W^{1,p}(\Omega))}^\sigma \|t^{1-\sigma} Lf'_\alpha\|_{L^p(0,\infty;\frac{dt}{t};L^p(\Omega))}^{1-\sigma} \right),$$

$f_\alpha(0) \equiv \lim_{t \rightarrow 0} f_\alpha(t) = D^\alpha \phi$ in $W^{1,p}(\Omega) + L^p(\Omega)$, and the infimum is taken over all such functions. Therefore, from 45.4.23, and letting $\|L\|_1$ denote the operator

norm of L in $W^{1,p}(\Omega)$ and $\|L\|_2$ denote the operator norm of L in $L^p(\Omega)$,

$$\begin{aligned}
 & \|L\phi\|_{m+\sigma,p,\Omega} \\
 & \leq \left(\|L\phi\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \left[\inf \left(\|L\|_1^\sigma \|L\|_2^{1-\sigma} \|t^{1-\sigma} f_\alpha\|_1^\sigma \|t^{1-\sigma} f'_\alpha\|_2^{1-\sigma} \right) \right]^p \right)^{1/p} \\
 & \leq \left(\|L\phi\|_{m,p,\Omega}^p + \left(\|L\|_1^\sigma \|L\|_2^{1-\sigma} \right)^p \sum_{|\alpha|=m} \left[\inf \left(\|t^{1-\sigma} f_\alpha\|_1^\sigma \|t^{1-\sigma} f'_\alpha\|_2^{1-\sigma} \right) \right]^p \right)^{1/p} \\
 & \leq C \left(\|\phi\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \left[\|D^\alpha \phi\|_{\sigma,p,\Omega} \right]^p \right)^{1/p} = C \|\phi\|_{m+\sigma,p,\Omega}.
 \end{aligned}$$

Since $C^\infty(\overline{\Omega})$ is dense in all the Sobolev spaces, this inequality establishes the desired result.

Definition 45.4.6 Define for $s \geq 0$, $W^{-s,p'}(\mathbb{R}^n)$ to be the dual space of

$$W^{s,p}(\mathbb{R}^n).$$

Here $\frac{1}{p} + \frac{1}{p'} = 1$.

Note that in the case of $m = 0$ this is consistent with the Riesz representation theorem for the L^p spaces.

Sobolev Spaces On Manifolds

46.1 Basic Definitions

Consider the following situation. There exists a set, $\Gamma \subseteq \mathbb{R}^m$ where $m > n$, mappings, $\mathbf{h}_i : U_i \rightarrow \Gamma_i = \Gamma \cap W_i$ for W_i an open set in $\mathbb{R}^m$ with $\Gamma \subseteq \cup_{i=1}^l W_i$ and U_i is an open subset of $\mathbb{R}^n$ which $\mathbf{h}_i$ one to one and onto. Assume $\mathbf{h}_i$ is of the form

$$\mathbf{h}_i(\mathbf{x}) = \mathbf{H}_i(\mathbf{x}, \mathbf{0}) \quad (46.1.1)$$

where for some open set, O_i , $\mathbf{H}_i : U_i \times O_i \rightarrow W_i$ is bilipschitz having bilipschitz inverse such that for $\mathbf{G} = \mathbf{H}_i$ or $\mathbf{H}_i^{-1}$, $D^\alpha \mathbf{G}$ is Lipschitz for $|\alpha| \leq k$.

For example, let $m = n + 1$ and let

$$\mathbf{H}_i(\mathbf{x}, y) = \begin{pmatrix} \mathbf{x} \\ \phi(\mathbf{x}) + y \end{pmatrix}$$

where ϕ is a Lipschitz function having $D^\alpha \phi$ Lipschitz for all $|\alpha| \leq k$. This is an example of the sort of thing just described if $\mathbf{x} \in U_i \subseteq \mathbb{R}^n$ and $O_i = \mathbb{R}$, because it is obvious the inverse of $\mathbf{H}_i$ is given by

$$\mathbf{H}_i^{-1}(\mathbf{x}, y) = \begin{pmatrix} \mathbf{x} \\ y - \phi(\mathbf{x}) \end{pmatrix}.$$

Also let $\{\psi_i\}_{i=1}^l$ be a partition of unity subordinate to the open cover $\{W_i\}$ satisfying $\psi_i \in C_c^\infty(W_i)$. Then the definition of $W^{s,p}(\Gamma)$ follows.

Definition 46.1.1 $u \in W^{s,p}(\Gamma)$ if whenever $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$ is described above with $\mathbf{h}_i \in C^{k,1}$, $\mathbf{h}_i^*(u\psi_i) \in W^{s,p}(U_i)$. The norm is given by

$$\|u\|_{s,p,\Gamma} \equiv \sum_{i=1}^l \|\mathbf{h}_i^*(u\psi_i)\|_{s,p,U_i}$$

It is not at all obvious this norm is well defined. What if

$$\{W'_i, \phi_i, \Gamma_i, V_i, \mathbf{g}_i, \mathbf{G}_i\}_{i=1}^r$$

is as described above. Would the two norms be equivalent? To begin with consider the following lemma which involves a particular choice for $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$.

Lemma 46.1.2 $W^{s,p}(\Gamma)$ as just described, is a Banach space. If $p > 1$ then it is reflexive.

Proof: Let $L : W^{s,p}(\Gamma) \rightarrow \prod_{i=1}^l W^{s,p}(U_i)$ be defined by $(Lu)_i \equiv \mathbf{h}_i^*(u\psi_i)$. Let $\{u_j\}_{j=1}^\infty$ be a Cauchy sequence in $W^{s,p}(\Gamma)$. Then $\{\mathbf{h}_i^*(u_j\psi_i)\}_{j=1}^\infty$ is a Cauchy sequence in $W^{s,p}(U_i)$ for each i . Therefore, for each i , there exists $w_i \in W^{s,p}(U_i)$ such that

$$\lim_{j \rightarrow \infty} \mathbf{h}_i^*(u_j\psi_i) = w_i \text{ in } W^{s,p}(U_i).$$

But also, there exists a subsequence, still denoted by j such that for each i

$$\{\mathbf{h}_i^*(u_j\psi_i)(\mathbf{x})\}_{j=1}^\infty$$

is a Cauchy sequence for a.e. $\mathbf{x}$. Since $\mathbf{h}_i$ is given to be Lipschitz, it maps sets of measure 0 to sets of n dimensional Hausdorff measure zero. Therefore,

$$\{u_j\psi_i(\mathbf{y})\}_{j=1}^\infty$$

is a Cauchy sequence for μ a.e. $\mathbf{y} \in W_i \cap \Gamma$ where μ denotes the n dimensional Hausdorff measure. It follows that for μ a.e. $\mathbf{y}$, $\{u_j(\mathbf{y})\}_{j=1}^\infty$ is a Cauchy sequence and so it converges to a function denoted as $u(\mathbf{y})$.

$$u_j(\mathbf{y}) \rightarrow u(\mathbf{y}) \quad \mu \text{ a.e.}$$

Therefore, $w_i(\mathbf{x}) = \mathbf{h}_i^*(u\psi_i)(\mathbf{x})$ a.e. and this shows $\mathbf{h}_i^*(u\psi_i) \in W^{s,p}(U_i)$. Thus $u \in W^{s,p}(\Gamma)$ showing completeness. It is clear $\|\cdot\|_{s,p,\Gamma}$ is a norm. Thus L is an isometry of $W^{s,p}(\Gamma)$ and a closed subspace of $\prod_{i=1}^l W^{s,p}(U_i)$. By Corollary 45.4.2, $W^{s,p}(U_i)$ is reflexive which implies the product is reflexive. Closed subspaces of reflexive spaces are reflexive by Lemma 23.2.7 on Page 710 and so $W^{s,p}(\Gamma)$ is also reflexive. This proves the lemma.

I now show that any two such norms are equivalent.

Suppose $\{W'_j, \phi_j, \Gamma_j, V_j, \mathbf{g}_j, \mathbf{G}_j\}_{j=1}^r$ and $\{W_i, \psi_i, \Gamma_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$ both satisfy the conditions described above. Let $\|\cdot\|_{s,p,\Gamma}^1$ denote the norm defined by

$$\begin{aligned} \|u\|_{s,p,\Gamma}^1 &\equiv \sum_{j=1}^r \|\mathbf{g}_j^*(u\phi_j)\|_{s,p,V_j} \\ &\leq \sum_{j=1}^r \left\| \mathbf{g}_j^* \left(\sum_{i=1}^l u\phi_j\psi_i \right) \right\|_{s,p,V_j} \leq \sum_{j,i} \|\mathbf{g}_j^*(u\phi_j\psi_i)\|_{s,p,V_j} \\ &= \sum_{j,i} \|\mathbf{g}_j^*(u\phi_j\psi_i)\|_{s,p,\mathbf{g}_j^{-1}(W_i \cap W'_j)} \end{aligned} \quad (46.1.2)$$

Now define a new norm $\|u\|_{s,p,\Gamma}^{1,\mathbf{g}}$ by the formula 46.1.2. This norm is determined by

$$\{W'_j \cap W_i, \psi_i\phi_j, \Gamma_j \cap \Gamma_i, V_j, \mathbf{g}_{i,j}, \mathbf{G}_{i,j}\}$$

where $\mathbf{g}_{i,j} = \mathbf{g}_j$. Thus the identity map is continuous from $(W^{s,p}(\Gamma), \|\cdot\|_{s,p,\Gamma}^{1,\mathbf{g}})$ to $(W^{s,p}(\Gamma), \|\cdot\|_{s,p,\Gamma}^1)$. It follows the two norms, $\|\cdot\|_{s,p,\Gamma}^{1,\mathbf{g}}$ and $\|\cdot\|_{s,p,\Gamma}^1$, are equivalent by the open mapping theorem. In a similar way, the norms, $\|\cdot\|_{s,p,\Gamma}^{2,\mathbf{h}}$ and $\|\cdot\|_{s,p,\Gamma}^2$ are equivalent where

$$\|u\|_{s,p,\Gamma}^2 \equiv \sum_{j=1}^l \|\mathbf{h}_i^*(u\psi_i)\|_{s,p,U_i}$$

and

$$\|u\|_{s,p,\Gamma}^{2,\mathbf{h}} \equiv \sum_{j,i} \|\mathbf{h}_i^*(u\phi_j\psi_i)\|_{s,p,U_i} = \sum_{j,i} \|\mathbf{h}_i^*(u\phi_j\psi_i)\|_{s,p,\mathbf{h}_i^{-1}(W_i \cap W'_j)}$$

But from the assumptions on $\mathbf{h}$ and $\mathbf{g}$, in particular the assumption that these are restrictions of functions which are defined on open subsets of $\mathbb{R}^m$ which have Lipschitz derivatives up to order k along with their inverses, Theorem 45.4.4 implies, there exist constants C_i , independent of u such that

$$\|\mathbf{h}_i^*(u\phi_j\psi_i)\|_{s,p,\mathbf{h}_i^{-1}(W_i \cap W'_j)} \leq C_1 \|\mathbf{g}_j^*(u\phi_j\psi_i)\|_{s,p,\mathbf{g}_j^{-1}(W_i \cap W'_j)}$$

and

$$\|\mathbf{g}_j^*(u\phi_j\psi_i)\|_{s,p,\mathbf{g}_j^{-1}(W_i \cap W'_j)} \leq C_2 \|\mathbf{h}_i^*(u\phi_j\psi_i)\|_{s,p,\mathbf{h}_i^{-1}(W_i \cap W'_j)}.$$

Therefore, the two norms, $\|\cdot\|_{s,p,\Gamma}^{1,\mathbf{g}}$ and $\|\cdot\|_{s,p,\Gamma}^{2,\mathbf{h}}$ are equivalent. It follows that the norms, $\|\cdot\|_{s,p,\Gamma}^2$ and $\|\cdot\|_{s,p,\Gamma}^1$ are equivalent. This proves the following theorem.

Theorem 46.1.3 *Let Γ be described above. Then any two norms for $W^{s,p}(\Gamma)$ as in Definition 39.6.3 are equivalent.*

46.2 The Trace On The Boundary Of An Open Set

Next is a generalization of earlier theorems about the loss of $\frac{1}{p}$ derivatives on the boundary.

Definition 46.2.1 *Define*

$$\mathbb{R}_k^{n-1} \equiv \{\mathbf{x} \in \mathbb{R}^n : x_k = 0\}, \widehat{\mathbf{x}}_k \equiv (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n).$$

An open set, Ω is $C^{m,1}$ if there exist open sets, $W_i, i = 0, 1, \dots, l$ such that

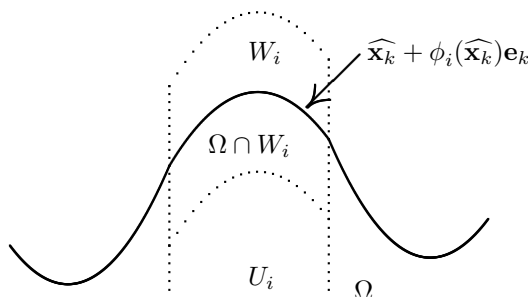
$$\Omega = \cup_{i=0}^l W_i$$

with $\overline{W_0} \subseteq \Omega$, open sets $U_i \subseteq \mathbb{R}_k^{n-1}$ for some k , and open intervals, (a_i, b_i) containing 0 such that for $i \geq 1$,

$$\partial\Omega \cap W_i = \{\widehat{\mathbf{x}}_k + \phi_i(\widehat{\mathbf{x}}_k) \mathbf{e}_k : \widehat{\mathbf{x}}_k \in U_i\},$$

$$\Omega \cap W_i = \{\widehat{\mathbf{x}}_k + (\phi_i(\widehat{\mathbf{x}}_k) + x_k) \mathbf{e}_k : (\widehat{\mathbf{x}}_k, x_k) \in U_i \times I_i\},$$

where ϕ_i is Lipschitz with partial derivatives up to order m also Lipschitz. Here $I_i = (a_i, 0)$ or $(0, b_i)$. The case of $(a_i, 0)$ is shown in the picture.



Assume Ω is $C^{m-1,1}$. Define

$$\mathbf{h}_i(\widehat{\mathbf{x}}_k) = \widehat{\mathbf{x}}_k + \phi_i(\widehat{\mathbf{x}}_k) \mathbf{e}_k, \mathbf{H}_i(\mathbf{x}) \equiv \widehat{\mathbf{x}}_k + (\phi_i(\widehat{\mathbf{x}}_k) + x_k) \mathbf{e}_k,$$

and let $\psi_i \in C_c^\infty(W_i)$ with $\sum_{i=1}^l \psi_i(\mathbf{x}) = 1$ on $\overline{\Omega}$. Thus

$$\{W_i, \psi_i, \partial\Omega \cap W_i, U_i, \mathbf{h}_i, \mathbf{H}_i\}_{i=1}^l$$

satisfies all the conditions for defining $W^{s,p}(\partial\Omega)$ for $s \leq m$. Let $u \in C^\infty(\overline{\Omega})$ and let $\mathbf{h}_i$ be as just described. The trace, denoted by γ is that operator which evaluates functions in $C^\infty(\overline{\Omega})$ on $\partial\Omega$. Thus for $u \in C^\infty(\overline{\Omega})$, and $\mathbf{y} \in \partial\Omega$,

$$u(\mathbf{y}) = \sum_{i=1}^l (u\psi_i)(\mathbf{y})$$

and so using the notation to suppress the reference to $\mathbf{y}$,

$$\gamma u = \sum_{i=1}^l \gamma(u\psi_i)$$

It is necessary to show this is a continuous map. Letting $u \in W^{m,p}(\Omega)$, it follows from Theorem 45.4.3, and Theorem 38.0.16,

$$\begin{aligned} \|\gamma u\|_{m-\frac{1}{p}, p, \partial\Omega} &= \sum_{i=1}^l \|\mathbf{h}_i^*(\gamma(\psi_i u))\|_{m-\frac{1}{p}, p, U_i} \\ &= \sum_{i=1}^l \|\mathbf{h}_i^* \gamma(\psi_i u)\|_{m-\frac{1}{p}, p, \mathbb{R}_k^{n-1}} \leq C \sum_{i=1}^l \|\mathbf{H}_i^*(\psi_i u)\|_{m, p, \mathbb{R}_+^n} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{i=1}^l \|\mathbf{H}_i^*(\psi_i u)\|_{m,p,U_i \times (a_i,0)} \leq C \sum_{i=1}^l \|(\psi_i u)\|_{m,p,W_i \cap \Omega} \\
&\leq C \sum_{i=1}^l \|(\psi_i u)\|_{m,p,\Omega} \leq C \sum_{i=1}^l \|u\|_{m,p,\Omega} \leq Cl \|u\|_{m,p,\Omega}.
\end{aligned}$$

Now use the density of $C^\infty(\overline{\Omega})$ in $W^{m,p}(\Omega)$ to see that γ extends to a continuous linear map defined on $W^{m,p}(\Omega)$ still called γ such that for all $u \in W^{m,p}(\Omega)$,

$$\|\gamma u\|_{m-\frac{1}{p},p,\partial\Omega} \leq Cl \|u\|_{m,p,\Omega}. \quad (46.2.3)$$

Also, it can be shown that γ maps $W^{m,p}(\Omega)$ onto $W^{m-\frac{1}{p}}(\partial\Omega)$. Let $g \in W^{m-\frac{1}{p}}(\partial\Omega)$. By definition, this means

$$\mathbf{h}_i^*(\psi_i g) \in W^{m-\frac{1}{p}}(U_i), \text{ each } i$$

and so, using a cutoff function, there exists $w_i \in W^{m,p}(U_i \times I_i)$ such that

$$\gamma w_i = \mathbf{h}_i^*(\psi_i g) = \mathbf{h}_i^*(\gamma \psi_i g)$$

Thus $(\mathbf{H}_i^{-1})^* w_i \in W^{mp}(\Omega \cap W_i)$. Let

$$w \equiv \sum_{i=1}^l \psi_i (\mathbf{H}_i^{-1})^* w_i \in W^{mp}(\Omega)$$

then

$$\begin{aligned}
\gamma w &= \sum_i \gamma \psi_j \gamma (\mathbf{H}_i^{-1})^* w_i = \sum_i \gamma \psi_j (\mathbf{H}_i^{-1})^* \gamma w_i \\
&= \sum_i \gamma \psi_j (\mathbf{H}_i^{-1})^* \mathbf{h}_i^*(\gamma \psi_i g) = g
\end{aligned}$$

In addition to this, in the case where $m = 1$, Lemma 45.2.1 implies there exists a linear map, R , from $W^{1-\frac{1}{p},p}(\partial\Omega)$ to $W^{1,p}(\Omega)$ which has the property that $\gamma Rg = g$ for every $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$. I show this now. Letting $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$,

$$g = \sum_{i=1}^l \psi_i g.$$

Then also,

$$\mathbf{h}_i^*(\psi_i g) \in W^{1-\frac{1}{p},p}(\mathbb{R}^{n-1})$$

if extended to equal 0 off U_i . From Lemma 45.2.1, there exists an extension of this to $W^{1,p}(\mathbb{R}_+^n)$, $R\mathbf{h}_i^*(\psi_i g)$. Without loss of generality, assume that $R\mathbf{h}_i^*(\psi_i g) \in$

$W^{1,p}(U_i \times (a_i, 0))$. If not so, multiply by a suitable cut off function in the definition of R . Then the extension is

$$Rg = \sum_{i=1}^l (\mathbf{H}_i^{-1})^* R \mathbf{h}_i^* (\psi_i g).$$

This works because from the definition of γ on $C^\infty(\overline{\Omega})$ and continuity of the map established above, γ and $(\mathbf{H}_i^{-1})^*$ commute and so

$$\begin{aligned} \gamma Rg &\equiv \sum_{i=1}^l \gamma (\mathbf{H}_i^{-1})^* R \mathbf{h}_i^* (\psi_i g) \\ &= \sum_{i=1}^l (\mathbf{H}_i^{-1})^* \gamma R \mathbf{h}_i^* (\psi_i g) \\ &= \sum_{i=1}^l (\mathbf{H}_i^{-1})^* \mathbf{h}_i^* (\psi_i g) = g. \end{aligned}$$

This proves the following theorem about the trace.

Theorem 46.2.2 *Let $\Omega \in C^{m-1,1}$. Then there exists a constant, C independent of $u \in W^{m,p}(\Omega)$ and a continuous linear map, $\gamma : W^{m,p}(\Omega) \rightarrow W^{m-\frac{1}{p},p}(\partial\Omega)$ such that 46.2.3 holds. This map satisfies $\gamma u(\mathbf{x}) = u(\mathbf{x})$ for all $u \in C^\infty(\overline{\Omega})$ and γ is onto. In the case where $m = 1$, there exists a continuous linear map, $R : W^{1-\frac{1}{p},p}(\partial\Omega) \rightarrow W^{1,p}(\Omega)$ which has the property that $\gamma Rg = g$ for all $g \in W^{1-\frac{1}{p},p}(\partial\Omega)$.*

Of course more can be proved but this is all to be presented here.

Part V

Topics In Probability

Basic Probability

Caution: This material on probability and stochastic processes may be half baked in places. I have not yet rewritten it several times. This is not to say that nothing else is half baked. However, the probability is higher here.

47.1 Random Variables And Independence

Recall Lemma 11.1.3 on Page 323 which is stated here for convenience.

Lemma 47.1.1 *Let M be a metric space with the closed balls compact and suppose λ is a measure defined on the Borel sets of M which is finite on compact sets. Then there exists a unique Radon measure, $\bar{\lambda}$ which equals λ on the Borel sets. In particular λ must be both inner and outer regular on all Borel sets.*

Also recall from earlier the following fundamental result which is called the Borel Cantelli lemma.

Lemma 47.1.2 *Let $(\Omega, \mathcal{F}, \lambda)$ be a measure space and let $\{A_i\}$ be a sequence of measurable sets satisfying*

$$\sum_{i=1}^{\infty} \lambda(A_i) < \infty.$$

Then letting S denote the set of $\omega \in \Omega$ which are in infinitely many A_i , it follows S is a measurable set and $\lambda(S) = 0$.

Proof: $S = \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} A_m$. Therefore, S is measurable and also

$$\lambda(S) \leq \lambda\left(\bigcup_{m=k}^{\infty} A_m\right) \leq \sum_{m=k}^{\infty} \lambda(A_k)$$

and this converges to 0 as $k \rightarrow \infty$ because of the convergence of the series. This proves the lemma.

Definition 47.1.3 A probability space is a measure space, $(\Omega, \mathcal{F}, P)$ where P is a measure satisfying $P(\Omega) = 1$. A random vector (variable) is a measurable function, $\mathbf{X} : \Omega \rightarrow Z$ where Z is some topological space. It is often the case that Z will equal $\mathbb{R}^p$. Assume Z is a separable Banach space. Define the following σ algebra.

$$\sigma(\mathbf{X}) \equiv \mathcal{H}_{\mathbf{X}} \equiv \{\mathbf{X}^{-1}(E) : E \text{ is Borel in } Z\}$$

Thus $\sigma(\mathbf{X}) = \mathcal{H}_{\mathbf{X}} \subseteq \mathcal{F}$. For E a Borel set in Z define

$$\lambda_{\mathbf{X}}(E) \equiv P(\mathbf{X}^{-1}(E)).$$

This is called the distribution of the random variable, $\mathbf{X}$. If

$$\int_{\Omega} |\mathbf{X}(\omega)| dP < \infty$$

then define

$$E(\mathbf{X}) \equiv \int_{\Omega} \mathbf{X} dP$$

where the integral is defined in the obvious way componentwise.

For random variables having values in a separable Banach space or even more generally for a separable metric space, much can be said about regularity of $\lambda_{\mathbf{X}}$.

Definition 47.1.4 A measure, μ defined on $\mathcal{B}(E)$ will be called inner regular if for all $F \in \mathcal{B}(E)$,

$$\mu(F) = \sup \{\mu(K) : K \subseteq F \text{ and } K \text{ is closed}\}$$

A measure, μ defined on $\mathcal{B}(E)$ will be called outer regular if for all $F \in \mathcal{B}(E)$,

$$\mu(F) = \inf \{\mu(V) : V \supseteq F \text{ and } V \text{ is open}\}$$

When a measure is both inner and outer regular, it is called regular.

For probability measures, the above definition of regularity tends to come free. Note it is a little weaker than the usual definition of regularity because K is only assumed to be closed, not compact.

Lemma 47.1.5 Let μ be a finite measure defined on $\mathcal{B}(E)$ where E is a metric space. Then μ is regular.

Proof: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let V be an open set and let

$$K_k \equiv \{x \in V : \text{dist}(x, V^C) \geq 1/k\}.$$

Then clearly the union of the K_k equals V . Next, for K closed let

$$V_k \equiv \{x \in E : \text{dist}(x, K) < 1/k\}.$$

Clearly the intersection of the V_k equals K . Therefore, letting V denote an open set and K a closed set,

$$\begin{aligned}\mu(V) &= \sup \{ \mu(K) : K \subseteq V \text{ and } K \text{ is closed} \} \\ \mu(K) &= \inf \{ \mu(V) : V \supseteq K \text{ and } V \text{ is open} \}.\end{aligned}$$

Also since V is open and K is closed,

$$\begin{aligned}\mu(V) &= \inf \{ \mu(U) : U \supseteq V \text{ and } U \text{ is open} \} \\ \mu(K) &= \sup \{ \mu(L) : L \subseteq K \text{ and } L \text{ is closed} \}\end{aligned}$$

In words, μ is regular on open and closed sets. Let

$$\mathcal{F} \equiv \{ F \in \mathcal{B}(E) \text{ such that } \mu \text{ is regular on } F \}.$$

Then $\mathcal{F}$ contains the open sets. I want to show $\mathcal{F}$ is a σ algebra and then it will follow $\mathcal{F} = \mathcal{B}(E)$.

First I will show $\mathcal{F}$ is closed with respect to complements. Let $F \in \mathcal{F}$. Then since μ is finite and F is inner regular, there exists $K \subseteq F$ such that $\mu(F \setminus K) < \varepsilon$. But $K^C \setminus F^C = F \setminus K$ and so $\mu(K^C \setminus F^C) < \varepsilon$ showing that F^C is outer regular. I have just approximated the measure of F^C with the measure of K^C , an open set containing F^C . A similar argument works to show F^C is inner regular. You start with $V \supseteq F$ such that $\mu(V \setminus F) < \varepsilon$, note $F^C \setminus V^C = V \setminus F$, and then conclude $\mu(F^C \setminus V^C) < \varepsilon$, thus approximating F^C with the closed subset, V^C .

Next I will show $\mathcal{F}$ is closed with respect to taking countable unions. Let $\{F_k\}$ be a sequence of sets in $\mathcal{F}$. Then μ is inner regular on each of these so there exist $\{K_k\}$ such that $K_k \subseteq F_k$ and $\mu(F_k \setminus K_k) < \varepsilon/2^{k+1}$. First choose m large enough that

$$\mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m F_k)) < \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned}\mu((\cup_{k=1}^m F_k) \setminus (\cup_{k=1}^m K_k)) &\leq \mu(\cup_{k=1}^m F_k \setminus K_k) \\ &\leq \sum_{k=1}^m \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}\end{aligned}$$

and so

$$\begin{aligned}\mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m K_k)) &\leq \mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m F_k)) \\ &\quad + \mu((\cup_{k=1}^m F_k) \setminus (\cup_{k=1}^m K_k)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

showing μ is inner regular on $\cup_{k=1}^{\infty} F_k$. Since μ is outer regular on F_k , there exists V_k such that $\mu(V_k \setminus F_k) < \varepsilon/2^k$. Then

$$\begin{aligned} \mu((\cup_{k=1}^{\infty} V_k) \setminus (\cup_{k=1}^{\infty} F_k)) &\leq \mu(\cup_{k=1}^{\infty} V_k \setminus K_k) \\ &\leq \sum_{k=1}^{\infty} \mu(V_k \setminus F_k) \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon \end{aligned}$$

and this shows μ is outer regular on $\cup_{k=1}^{\infty} F_k$ and this proves the lemma.

Lemma 47.1.6 *Let μ be a finite measure on a σ algebra containing $\mathcal{B}(E)$, the Borel sets of E , a separable complete metric space. Then if C is a closed set,*

$$\mu(C) = \sup \{ \mu(K) : K \subseteq C \text{ and } K \text{ is compact.} \}$$

Proof: Let $\{a_k\}$ be a countable dense subset of C . Thus $\cup_{k=1}^{\infty} B(a_k, \frac{1}{n}) \supseteq C$. Therefore, there exists m_n such that

$$\mu\left(C \setminus \overline{\cup_{k=1}^{m_n} B\left(a_k, \frac{1}{n}\right)}\right) \equiv \mu(C \setminus C_n) < \frac{\varepsilon}{2^n}.$$

Now let $K = C \cap (\cap_{n=1}^{\infty} C_n)$. Then K is a subset of C_n for each n and so for each $\varepsilon > 0$ there exists an ε net for K since C_n has a $1/n$ net, namely $a_1, \dots, a_{m_n}$. Since K is closed, it is complete and so it is also compact since it is complete and totally bounded. Now

$$\mu(C \setminus K) = \mu(\cup_{n=1}^{\infty} (C \setminus C_n)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for K a compact subset of C . This proves the lemma.

Corollary 47.1.7 *Let $\mathbf{X}$ be a random variable with values in a complete metric space, Z . Then $\lambda_{\mathbf{X}}$ is an inner and outer regular measure defined on $\mathcal{B}(Z)$.*

Lemma 47.1.8 *For $\mathbf{X}$ a random vector defined above, $\mathbf{X}$ having values in a complete separable metric space, Z , then $\lambda_{\mathbf{X}}$ is inner and outer regular, Borel, and its completion is a Radon measure and $\lambda_{\mathbf{X}}(Z) = 1$. Furthermore, if h is any bounded Borel measurable function,*

$$\int_{\Omega} h(\mathbf{X}(\omega)) dP = \int_{\mathbb{R}^p} h(\mathbf{x}) d\lambda_{\mathbf{X}}.$$

Proof: The completion is automatically inner and outer regular. To see this, note

$$\overline{\lambda_{\mathbf{X}}}(S) \equiv \inf \{ \lambda_{\mathbf{X}}(F) : F \supseteq S, F \in \mathcal{B}(Z) \}$$

and so $\overline{\lambda_{\mathbf{X}}}$ is automatically outer regular. Letting $F \in \overline{\mathcal{B}(Z)}$ where this symbol denotes the completed σ algebra, use outer regularity to conclude there exists $V \supseteq F^C$ such that V is open and

$$\overline{\lambda_{\mathbf{X}}}(V \setminus F^C) < \varepsilon.$$

Then $V^C \subseteq F$ and

$$\overline{\lambda_{\mathbf{X}}}(F \setminus V^C) = \overline{\lambda_{\mathbf{X}}}(V \setminus F^C) < \varepsilon.$$

Now this shows $\overline{\lambda_{\mathbf{X}}}$ is inner regular with respect to closed sets. By Lemma 47.1.6 $\overline{\lambda_{\mathbf{X}}}$ is inner regular with respect to compact sets.

It remains to verify the formula involving the integrals. Suppose first

$$h(\mathbf{x}) = c\mathcal{X}_E(\mathbf{x})$$

where E is a Borel set in $\mathbb{R}^p$. Then the left side equals

$$\int_{\Omega} c\mathcal{X}_E(\mathbf{X}(\omega)) dP = \int_{[\mathbf{X} \in E]} cdP = c\lambda_{\mathbf{X}}(E)$$

The right side equals

$$\int_{\mathbb{R}^p} c\mathcal{X}_E(\mathbf{x}) d\lambda_{\mathbf{X}} = c\lambda_{\mathbf{X}}(E).$$

Similarly, if h is any Borel simple function, the same result will hold. For an arbitrary bounded Borel function, h , there exists a sequence of Borel simple functions, $\{s_n\}$ converging to h . Hence, by the dominated convergence theorem,

$$\int_{\Omega} h(\mathbf{X}(\omega)) dP = \lim_{n \rightarrow \infty} \int_{\Omega} s_n(\mathbf{X}(\omega)) dP = \lim_{n \rightarrow \infty} \int s_n(\mathbf{x}) d\lambda_{\mathbf{X}} = \int_{\mathbb{R}^p} h(\mathbf{x}) d\lambda_{\mathbf{X}}.$$

This proves the lemma.

Obviously, h could also be vector valued with values in a separable Banach space and Borel measurable and the same argument would work or else you could simply consider the component functions of $\mathbf{h}$ in the case where $\mathbf{h}$ has values in $\mathbb{R}^n$.

47.1.1 Independence

Definition 47.1.9 Let $(\Omega, \mathcal{F}, P)$ be a probability space. The sets in $\mathcal{F}$ are called events. A set of events, $\{A_i\}_{i \in I}$ is called independent if whenever $\{A_{i_k}\}_{k=1}^m$ is a finite subset

$$P(\cap_{k=1}^m A_{i_k}) = \prod_{k=1}^m P(A_{i_k}).$$

Each of these events defines a rather simple σ algebra, $(A_i, A_i^C, \emptyset, \Omega)$ denoted by $\mathcal{F}_i$. Now the following lemma is interesting.

Lemma 47.1.10 *Suppose $B_i \in \mathcal{F}_i$ for $i \in I$. Then for any $m \in \mathbb{N}$*

$$P(\cap_{k=1}^m B_{i_k}) = \prod_{k=1}^m P(B_{i_k}).$$

Proof: The proof is by induction on the number l of the B_{i_k} which are not equal to A_{i_k} . First suppose $l = 0$. Then the above assertion is true by assumption. Suppose it is so for some l and there are $l + 1$ sets not equal to A_{i_k} . If any equals $\emptyset$ there is nothing to show. Both sides equal 0. If any equals Ω , there is also nothing to show. You can ignore that set in both sides and then you have by induction the two sides are equal because you have no more than l sets different than A_{i_k} . The only remaining case is where some $B_{i_k} = A_{i_k}^C$. Say $B_{i_{m+1}} = A_{i_{m+1}}^C$ for simplicity.

$$\begin{aligned} P(\cap_{k=1}^{m+1} B_{i_k}) &= P(A_{i_{m+1}}^C \cap \cap_{k=1}^m B_{i_k}) \\ &= P(\cap_{k=1}^m B_{i_k}) - P(A_{i_{m+1}} \cap \cap_{k=1}^m B_{i_k}) \end{aligned}$$

Then by induction,

$$\begin{aligned} &= \prod_{k=1}^m P(B_{i_k}) - P(A_{i_{m+1}}) \prod_{k=1}^m P(B_{i_k}) = \prod_{k=1}^m P(B_{i_k}) (1 - P(A_{i_{m+1}})) \\ &= P(A_{i_{m+1}}^C) \prod_{k=1}^m P(B_{i_k}) = \prod_{k=1}^{m+1} P(B_{i_k}) \end{aligned}$$

thus proving it for $l + 1$. This proves the Lemma.

This motivates a more general notion of independence in terms of σ algebras.

Definition 47.1.11 *If $\{\mathcal{F}_i\}_{i \in I}$ is any set of σ algebras contained in $\mathcal{F}$, they are said to be independent if whenever $A_{i_k} \in \mathcal{F}_{i_k}$ for $k = 1, 2, \dots, m$, then*

$$P(\cap_{k=1}^m A_{i_k}) = \prod_{k=1}^m P(A_{i_k}).$$

A set of random variables $\{\mathbf{X}_i\}_{i \in I}$ is independent if the σ algebras $\{\sigma(\mathbf{X}_i)\}_{i \in I}$ are independent σ algebras. Here $\sigma(\mathbf{X})$ denotes the smallest σ algebra such that $\mathbf{X}$ is measurable. Thus $\sigma(\mathbf{X}) = \{\mathbf{X}^{-1}(U) : U \text{ is a Borel set}\}$. More generally, $\sigma(\mathbf{X}_i : i \in I)$ is the smallest σ algebra such that each $\mathbf{X}_i$ is measurable.

Note that by Lemma 47.1.10 you can consider independent events in terms of independent σ algebras. That is, a set of independent events can always be considered as events taken from a set of independent σ algebras. This is a more general notion because here the σ algebras might have infinitely many sets in them.

Lemma 47.1.12 *Suppose the set of random variables, $\{\mathbf{X}_i\}_{i \in I}$ is independent. Also suppose $I_1 \subseteq I$ and $j \notin I_1$. Then the σ algebras $\sigma(\mathbf{X}_i : i \in I_1)$, $\sigma(\mathbf{X}_j)$ are independent σ algebras.*

Proof: Let $B \in \sigma(\mathbf{X}_j)$. I want to show that for any $A \in \sigma(\mathbf{X}_i : i \in I_1)$, it follows that $P(A \cap B) = P(A)P(B)$. Let $\mathcal{K}$ consist of finite intersections of sets of the form $\mathbf{X}_k^{-1}(B_k)$ where B_k is a Borel set and $k \in I_1$. Thus $\mathcal{K}$ is a π system and $\sigma(\mathcal{K}) = \sigma(\mathbf{X}_i : i \in I_1)$. Now if you have one of these sets of the form $A = \cap_{k=1}^m \mathbf{X}_k^{-1}(B_k)$ where without loss of generality, it can be assumed the k are distinct since $\mathbf{X}_k^{-1}(B_k) \cap \mathbf{X}_k^{-1}(B'_k) = \mathbf{X}_k^{-1}(B_k \cap B'_k)$, then

$$\begin{aligned} P(A \cap B) &= P\left(\cap_{k=1}^m \mathbf{X}_k^{-1}(B_k) \cap B\right) = P(B) \prod_{k=1}^m P\left(\mathbf{X}_k^{-1}(B_k)\right) \\ &= P(B) P\left(\cap_{k=1}^m \mathbf{X}_k^{-1}(B_k)\right). \end{aligned}$$

Thus $\mathcal{K}$ is contained in

$$\mathcal{G} \equiv \{A \in \sigma(\mathbf{X}_i : i \in I_1) : P(A \cap B) = P(A)P(B)\}.$$

Now $\mathcal{G}$ is closed with respect to complements and countable disjoint unions. Here is why: If each $A_i \in \mathcal{G}$ and the A_i are disjoint,

$$\begin{aligned} P\left(\left(\cup_{i=1}^{\infty} A_i\right) \cap B\right) &= P\left(\cup_{i=1}^{\infty} (A_i \cap B)\right) \\ &= \sum_i P(A_i \cap B) = \sum_i P(A_i)P(B) \\ &= P(B) \sum_i P(A_i) = P(B)P\left(\cup_{i=1}^{\infty} A_i\right) \end{aligned}$$

If $A \in \mathcal{G}$,

$$P(A^C \cap B) + P(A \cap B) = P(B)$$

and so

$$\begin{aligned} P(A^C \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1 - P(A)) = P(B)P(A^C). \end{aligned}$$

Therefore, from the lemma on π systems, Lemma 9.11.3 on Page 275, it follows $\mathcal{G} \supseteq \sigma(\mathcal{K}) = \sigma(\mathbf{X}_i : i \in I_1)$ and this proves the lemma.

Lemma 47.1.13 *If $\{\mathbf{X}_k\}_{k=1}^r$ are independent random variables having values in Z a separable metric space, and if g_k is a Borel measurable function, then $\{g_k(\mathbf{X}_k)\}_{k=1}^n$ is also independent. Furthermore, if the random variables have values in $\mathbb{R}$, and they are all bounded, then*

$$E\left(\prod_{i=1}^r X_i\right) = \prod_{i=1}^r E(X_i).$$

Proof: First consider the claim about $\{g_k(\mathbf{X}_k)\}_{k=1}^r$. Letting O be an open set in Z ,

$$(g_k \circ \mathbf{X}_k)^{-1}(O) = \mathbf{X}_k^{-1}(g_k^{-1}(O)) = \mathbf{X}_k^{-1}(\text{Borel set}) \in \sigma(\mathbf{X}_k).$$

It follows $(g_k \circ \mathbf{X}_k)^{-1}(E)$ is in $\sigma(\mathbf{X}_k)$ whenever E is Borel because the sets whose inverse images are measurable includes the Borel sets. Thus $\sigma(g_k \circ \mathbf{X}_k) \subseteq \sigma(\mathbf{X}_k)$ and this proves the first part of the lemma.

Now by Theorem 23.1.2 there exists $\{s_n^i\}_{n=1}^\infty$ be a bounded sequence of simple functions measurable in $\sigma(\mathbf{X}_i)$ which converges to X_i pointwise. Say

$$s_n^i(\omega) = \sum_{k=1}^{m_n} c_k^{n,i} \chi_{E_k^{n,i}}(\omega)$$

where the E_k are disjoint elements of $\sigma(\mathbf{X}_i)$ and some might be empty. This is for convenience in keeping the same index on the top of the sum. Then since all the random variables are bounded, there is no problem about existence of any of the above. Then from the assumption that the X_i are independent and the dominated convergence theorem coming from the boundedness of the $\mathbf{X}_i$,

$$\begin{aligned} E\left(\prod_{i=1}^r X_i\right) &= \int_{\Omega} \prod_{i=1}^r X_i(\omega) dP = \lim_{n \rightarrow \infty} \int_{\Omega} \prod_{i=1}^r s_n^i(\omega) dP \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \prod_{i=1}^r \sum_{k=1}^{m_n} c_k^{n,i} \chi_{E_k^{n,i}}(\omega) dP \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \sum_{k_1, k_2, \dots, k_r} c_{k_1}^{n,1} c_{k_2}^{n,2} \dots c_{k_r}^{n,r} \chi_{E_{k_1}^{n,1}} \chi_{E_{k_2}^{n,2}} \dots \chi_{E_{k_r}^{n,r}} dP \\ &= \lim_{n \rightarrow \infty} \sum_{k_1, k_2, \dots, k_r} \int_{\Omega} c_{k_1}^{n,1} c_{k_2}^{n,2} \dots c_{k_r}^{n,r} \chi_{E_{k_1}^{n,1}} \chi_{E_{k_2}^{n,2}} \dots \chi_{E_{k_r}^{n,r}} dP \\ &= \lim_{n \rightarrow \infty} \sum_{k_1, k_2, \dots, k_r} c_{k_1}^{n,1} c_{k_2}^{n,2} \dots c_{k_r}^{n,r} \prod_{i=1}^r P(E_{k_i}^{n,i}) \\ &= \lim_{n \rightarrow \infty} \prod_{i=1}^r \int_{\Omega} s_n^i(\omega) dP = \prod_{i=1}^r E(X_i). \end{aligned}$$

This proves the lemma.

47.1.2 Independence For Banach Space Valued Random Variables

Recall that for X a random variable, $\sigma(X)$ is the smallest σ algebra containing all the sets of the form $X^{-1}(F)$ where F is Borel. Since such sets, $X^{-1}(F)$ for F Borel form a σ algebra it follows $\sigma(X) = \{X^{-1}(F) : F \text{ is Borel}\}$.

Next consider the case where you have a set of σ algebras. The following lemma is helpful when you try to verify such a set of σ algebras is independent. It says you only need to check things on π systems contained in the σ algebras.

Lemma 47.1.14 Suppose $\{\mathcal{F}_i\}_{i \in I}$ is a set of σ algebras contained in $\mathcal{F}$ where $\mathcal{F}$ is a σ algebra of sets of Ω . Suppose that $\mathcal{K}_i \subseteq \mathcal{F}_i$ is a π system and $\mathcal{F}_i = \sigma(\mathcal{K}_i)$. Suppose also that whenever J is a finite subset of I and $A_j \in \mathcal{K}_j$ for $j \in J$, it follows

$$P(\cap_{j \in J} A_j) = \prod_{j \in J} P(A_j).$$

Then $\{\mathcal{F}_i\}_{i \in I}$ is independent.

Proof: I need to verify that for all $n \in \mathbb{N}$, if $\{j_1, j_2, \dots, j_n\} \subseteq I$ and $A_{j_k} \subseteq \mathcal{F}_{j_k}$, then

$$P(\cap_{k=1}^n A_{j_k}) = \prod_{k=1}^n P(A_{j_k}).$$

Pick $(A_{j_1}, \dots, A_{j_{n-1}}) \in \mathcal{K}_{j_1} \times \dots, \mathcal{K}_{j_{n-1}}$ and let

$$\mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-1}})} \equiv \left\{ A_{j_n} \in \mathcal{F}_{j_n} : P(\cap_{k=1}^n A_{j_k}) = \prod_{k=1}^n P(A_{j_k}) \right\}$$

Then by hypothesis, $\mathcal{K}_{j_n} \subseteq \mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-1}})}$. If $A_{j_n} \in \mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-1}})}$,

$$\begin{aligned} \prod_{k=1}^{n-1} P(A_{j_k}) &= P(\cap_{k=1}^{n-1} A_{j_k}) \\ &= P((\cap_{k=1}^{n-1} A_{j_k} \cap A_{j_n}^C) \cup (\cap_{k=1}^{n-1} A_{j_k} \cap A_{j_n})) \\ &= P(\cap_{k=1}^{n-1} A_{j_k} \cap A_{j_n}^C) + P(\cap_{k=1}^{n-1} A_{j_k} \cap A_{j_n}) \\ &= P(\cap_{k=1}^{n-1} A_{j_k} \cap A_{j_n}^C) + \prod_{k=1}^n P(A_{j_k}) \end{aligned}$$

and so

$$\begin{aligned} P(\cap_{k=1}^{n-1} A_{j_k} \cap A_{j_n}^C) &= \prod_{k=1}^{n-1} P(A_{j_k}) (1 - P(A_{j_n})) \\ &= \prod_{k=1}^{n-1} P(A_{j_k}) P(A_{j_n}^C) \end{aligned}$$

showing if $A_{j_n} \in \mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-1}})}$, then so is $A_{j_n}^C$. It is clear that $\mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-1}})}$ is closed with respect to disjoint unions also. Therefore, by Lemma 9.11.3 $\mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-1}})} = \mathcal{F}_{j_n}$.

Next fix $A_{j_n} \in \mathcal{F}_{j_n}$ and $(A_{j_1}, \dots, A_{j_{n-2}}) \in \mathcal{K}_{j_1} \times \dots, \mathcal{K}_{j_{n-2}}$. Let

$$\mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-2}})} \equiv \left\{ A_{j_{n-1}} \in \mathcal{F}_{j_{n-1}} : P(\cap_{k=1}^n A_{j_k}) = \prod_{k=1}^n P(A_{j_k}) \right\}$$

It was just shown $\mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-2}})} \supseteq \mathcal{K}_{j_{n-1}}$. Also by similar reasoning to the above, it follows $\mathcal{G}_{(A_{j_1}, \dots, A_{j_{n-2}})}$ satisfies the conditions needed to apply Lemma 9.11.3 on Page 275 and so whenever $A_{j_n}, A_{j_{n-1}}$ are in $\mathcal{F}_{j_n}$ and $\mathcal{F}_{j_{n-1}}$ respectively and

$$(A_{j_1}, \dots, A_{j_{n-2}}) \in \mathcal{K}_{j_1} \times \dots \times \mathcal{K}_{j_{n-2}},$$

it follows $P(\cap_{k=1}^n A_{j_k}) = \prod_{k=1}^n P(A_{j_k})$. Continue this way to obtain the desired result. This proves the lemma.

What is a useful π system for $\mathcal{B}(E)$, the Borel sets of E where E is a Banach space?

Recall the fundamental lemma used to prove the Pettis theorem. It was proved on Page 699 but here I want to show that in addition, the set D' can be taken as a subset of a given dense subspace, M of E' . Thus I will present next a generalization of that important lemma. You might consider whether the following lemma can be generalized even more.

Lemma 47.1.15 *If E is a separable Banach space with B' the closed unit ball in E' , and if M is a dense subspace of E' , then there exists a sequence $\{f_n\}_{n=1}^\infty \equiv D' \subseteq B' \cap M$ with the property that for every $x \in E$,*

$$\|x\| = \sup_{f \in D'} |f(x)|$$

The set, D' also has the property that if $f \in D'$ then $-f \in D'$.

Proof: Let $\{a_n\} \equiv D$ be a dense subset of E . Define a subset of $\mathbb{C}^n$, C_n by

$$\{(f(a_1), \dots, f(a_n)) : f \in B' \cap M\}$$

Then C_n is a bounded subset of $\mathbb{C}^n$ and so it is separable. Note each f delivers a point in $\mathbb{C}^n$. Now let $\{f_k^n\}_{k=1}^\infty \subseteq B' \cap M$ be such that the points

$$\{(f_k^n(a_1), \dots, f_k^n(a_n))\}_{k=1}^\infty$$

are dense in C_n . Then $D' \equiv \cup_{n=1}^\infty \{f_k^n\}_{k=1}^\infty$.

It remains to verify D' works. Pick $x \in E$. I need to show there exists f_k^n such that $\|x\| < |f_k^n(x)| + \varepsilon$. By a standard exercise in the Hahn Banach theorem, there exists $f \in B'$ such that $f(x) = \|x\|$. Next choose $a_n \in D$ such that $\|x - a_n\|_E < \varepsilon/4$. Since M is dense, there exists $g \in M \cap B'$ such that $|g(a_n) - f(a_n)| < \varepsilon/4$. Finally, there exists $f_k^n \in D'$ such that $|f_k^n(a_n) - g(a_n)| < \varepsilon/4$. Then

$$\begin{aligned} \left| \|x\| - |f_k^n(x)| \right| &= \left| f(x) - |f_k^n(x)| \right| \\ &\leq \left| f(x) - |f(a_n)| \right| + \left| |f(a_n)| - |g(a_n)| \right| \\ &\quad + \left| |g(a_n)| - |f_k^n(a_n)| \right| + \left| |f_k^n(a_n)| - |f_k^n(x)| \right| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

It follows $\|x\| < |f_k^n(x)| + \varepsilon$ and this proves the lemma because for every $f \in D'$ you can simply include $-f$.

Lemma 47.1.16 *Let E be a separable real Banach space. Sets of the form*

$$\{x \in E : x_i^*(x) \leq \alpha_i, i = 1, 2, \dots, m\}$$

where $x_i^* \in M$, a dense subspace of E' and $\alpha_i \in [-\infty, \infty)$ are a π system, and denoting this π system by $\mathcal{K}$, it follows $\sigma(\mathcal{K}) = \mathcal{B}(E)$.

Proof: The sets described are obviously a π system. I want to show $\sigma(\mathcal{K})$ contains the closed balls because then $\sigma(\mathcal{K})$ contains the open balls and hence the open sets and the result will follow. Let $D' \subseteq B' \cap M$ be described in Lemma 47.1.15. Then

$$\begin{aligned} & \{x \in E : \|x - a\| \leq r\} \\ &= \left\{ x \in E : \sup_{f \in D'} |f(x - a)| \leq r \right\} \\ &= \left\{ x \in E : \sup_{f \in D'} |f(x) - f(a)| \leq r \right\} \\ &= \cap_{f \in D'} \{x \in E : f(a) - r \leq f(x) \leq f(a) + r\} \\ &= \cap_{f \in D'} \{x \in E : f(x) \leq f(a) + r \text{ and } (-f)(x) \leq r - f(a)\} \end{aligned}$$

which equals a countable intersection of sets of the given π system. Therefore, every closed ball is contained in $\sigma(\mathcal{K})$. It follows easily that every open ball is also contained in $\sigma(\mathcal{K})$ because

$$B(a, r) = \cup_{n=1}^{\infty} \overline{B\left(a, r - \frac{1}{n}\right)}.$$

Since the Banach space is separable, it is completely separable and so every open set is the countable union of balls. This shows the open sets are in $\sigma(\mathcal{K})$ and so $\sigma(\mathcal{K}) \supseteq \mathcal{B}(E)$. However, all the sets in the π system are closed hence Borel because they are inverse images of closed sets. Therefore, $\sigma(\mathcal{K}) \subseteq \mathcal{B}(E)$ and so $\sigma(\mathcal{K}) = \mathcal{B}(E)$. This proves the lemma.

Next suppose you have some random variables having values in a separable Banach space, E , $\{X_i\}_{i \in I}$. How can you tell if they are independent? To show they are independent, you need to verify that

$$P\left(\cap_{k=1}^n X_{i_k}^{-1}(F_{i_k})\right) = \prod_{k=1}^n P\left(X_{i_k}^{-1}(F_{i_k})\right)$$

whenever the F_{i_k} are Borel sets in E . It is desirable to find a way to do this easily.

Lemma 47.1.17 *Let $\mathcal{K}$ be a π system of sets of E , a separable real Banach space and let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \rightarrow E$ be a random variable. Then*

$$X^{-1}(\sigma(\mathcal{K})) = \sigma(X^{-1}(\mathcal{K}))$$

Proof: First note that $X^{-1}(\sigma(\mathcal{K}))$ is a σ algebra which contains $X^{-1}(\mathcal{K})$ and so it contains $\sigma(X^{-1}(\mathcal{K}))$. Now let

$$\mathcal{G} \equiv \{A \in \sigma(\mathcal{K}) : X^{-1}(A) \in \sigma(X^{-1}(\mathcal{K}))\}$$

Then $\mathcal{G} \supseteq \mathcal{K}$. If $A \in \mathcal{G}$, then

$$X^{-1}(A) \in \sigma(X^{-1}(\mathcal{K}))$$

and so

$$X^{-1}(A)^C = X^{-1}(A^C) \in \sigma(X^{-1}(\mathcal{K}))$$

because $\sigma(X^{-1}(\mathcal{K}))$ is a σ algebra. Hence $A^C \in \mathcal{G}$. Finally suppose $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$. Then

$$X^{-1}(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} X^{-1}(A_i) \in \sigma(X^{-1}(\mathcal{K}))$$

again because $\sigma(X^{-1}(\mathcal{K}))$ is a σ algebra. It follows from Lemma 9.11.3 on Page 275 that $\mathcal{G} \supseteq \sigma(\mathcal{K})$ and this shows that whenever $A \in \sigma(\mathcal{K})$, $X^{-1}(A) \in \sigma(X^{-1}(\mathcal{K}))$. Thus $X^{-1}(\sigma(\mathcal{K})) \subseteq \sigma(X^{-1}(\mathcal{K}))$ and this proves the lemma.

With this lemma, here is the desired result about independent random variables. Essentially, you can reduce to the case of random vectors having values in $\mathbb{R}^n$.

Theorem 47.1.18 *The random variables, $\{X_i\}_{i \in I}$ are independent if whenever*

$$\{i_1, \dots, i_n\} \subseteq I,$$

$m_{i_1}, \dots, m_{i_n}$ are positive integers, and $\mathbf{g}_{m_{i_1}}, \dots, \mathbf{g}_{m_{i_n}}$ are respectively in

$$(M)^{m_{i_1}}, \dots, (M)^{m_{i_n}}$$

for M a dense subspace of E' , $\{\mathbf{g}_{m_{i_j}} \circ X_{i_j}\}_{j=1}^n$ are independent random vectors having values in $\mathbb{R}^{m_{i_1}}, \dots, \mathbb{R}^{m_{i_n}}$ respectively.

Proof: Let $\mathcal{K}$ denote sets of the form

$$\{x \in E : x_i^*(x) \leq \alpha_i, i = 1, 2, \dots, m\}$$

as described in Lemma 47.1.16. Then as proved in this lemma, $\sigma(\mathcal{K}) = \mathcal{B}(E)$. Then the random vectors are independent if whenever

$$\{i_1, \dots, i_n\} \subseteq I$$

and $A_{i_1}, \dots, A_{i_n}$ sets of $\sigma(X_{i_1}), \dots, \sigma(X_{i_n})$ respectively,

$$P\left(\cap_{j=1}^n A_{i_j}\right) = \prod_{j=1}^n P(A_{i_j}).$$

By Lemma 9.11.3 on Page 275 if $\mathcal{K}_{i_j}$ is a π system contained in $\sigma(X_{i_j})$ such that $\sigma(\mathcal{K}_{i_j}) = \sigma(X_{i_j})$, then it suffices to check only the case where the A_{i_j} is in $\mathcal{K}_{i_j}$. So what will serve for such a collection of π systems? Let

$$\mathcal{K}_{i_j} \equiv X_{i_j}^{-1}(\mathcal{K}) \equiv \left\{ X_{i_j}^{-1}(A) : A \in \mathcal{K} \right\}.$$

This is clearly a π system contained in $\sigma(X_{i_j})$ and by Lemma 47.1.17

$$\sigma(\mathcal{K}_{i_j}) = \sigma(X_{i_j}^{-1}(\mathcal{K})) = X_{i_j}^{-1}(\sigma(\mathcal{K})) \equiv \sigma(X_{i_j}).$$

Thus it suffices to show that whenever $B_{i_1}, \dots, B_{i_n}$ are sets of $\mathcal{K}$,

$$P\left(\bigcap_{j=1}^n X_{i_j}^{-1}(B_{i_j})\right) = \prod_{j=1}^n P\left(X_{i_j}^{-1}(B_{i_j})\right)$$

Let $B_{i_j} = \{x \in E : \mathbf{g}_{m_{i_j}}(x) \in A_{i_j}\}$ where $A_{i_j} = \prod_{j=1}^{m_j} (-\infty, \alpha_i]$ and $\mathbf{g}_{m_j} \in M^{m_j}$. It follows

$$X_{i_j}^{-1}(B_{i_j}) = \left(\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right)^{-1}(A_{i_j}).$$

Then by the assumption the random vectors $\mathbf{g}_{m_{i_j}} \circ X_{i_j}$ are independent,

$$\begin{aligned} P\left(\bigcap_{j=1}^n X_{i_j}^{-1}(B_{i_j})\right) &= P\left(\bigcap_{j=1}^n \left(\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right)^{-1}(A_{i_j})\right) \\ &= \prod_{j=1}^n P\left(\left(\mathbf{g}_{m_{i_j}} \circ X_{i_j}\right)^{-1}(A_{i_j})\right) \\ &= \prod_{j=1}^n P\left(X_{i_j}^{-1}(B_{i_j})\right) \end{aligned}$$

and this proves the theorem.

The above assertion also goes the other way as you may want to show.

47.1.3 0, 1 Laws

I am following [62] for the proof of many of the following theorems. Recall the set of ω which are in infinitely many of the sets $\{A_n\}$ is

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

This is because ω is in the above set if and only if for every n there exists $m \geq n$ such that it is in A_m .

Theorem 47.1.19 Suppose $A_n \in \mathcal{F}_n$ where the σ algebras $\{\mathcal{F}_n\}_{n=1}^{\infty}$ are independent. Suppose also that

$$\sum_{k=1}^{\infty} P(A_k) = \infty.$$

Then

$$P(\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m) = 1.$$

Proof: It suffices to verify that

$$P(\cup_{n=1}^{\infty} \cap_{m=n}^{\infty} A_m^C) = 0$$

which can be accomplished by showing

$$P(\cap_{m=n}^{\infty} A_m^C) = 0$$

for each n . The sets $\{A_k^C\}$ satisfy $A_k^C \in \mathcal{F}_k$. Therefore, noting that $e^{-x} \geq 1 - x$,

$$\begin{aligned} P(\cap_{m=n}^{\infty} A_m^C) &= \lim_{N \rightarrow \infty} P(\cap_{m=n}^N A_m^C) = \lim_{N \rightarrow \infty} \prod_{m=n}^N P(A_m^C) \\ &= \lim_{N \rightarrow \infty} \prod_{m=n}^N (1 - P(A_m)) \leq \lim_{N \rightarrow \infty} \prod_{m=n}^N e^{-P(A_m)} \\ &= \lim_{N \rightarrow \infty} \exp\left(-\sum_{m=n}^N P(A_m)\right) = 0. \end{aligned}$$

This proves the theorem.

The Kolmogorov zero one law follows next. It has to do with something called a tail event.

Definition 47.1.20 Let $\{\mathcal{F}_n\}$ be a sequence of σ algebras. Then $\mathcal{T}_n \equiv \sigma(\cup_{k=n}^{\infty} \mathcal{F}_k)$ where this means the smallest σ algebra which contains each $\mathcal{F}_k$ for $k \geq n$. Then a tail event is a set which is in the σ algebra, $\mathcal{T} \equiv \cap_{n=1}^{\infty} \mathcal{T}_n$.

As usual, $(\Omega, \mathcal{F}, P)$ is the underlying probability space such that all σ algebras are contained in $\mathcal{F}$.

Theorem 47.1.21 Suppose the σ algebras, $\{\mathcal{F}_n\}_{n=1}^{\infty}$ are independent and suppose A is a tail event. Then $P(A)$ either equals 0 or 1.

Proof: Let $A \in \mathcal{T}$. Let

$$\mathcal{H} \equiv \{B \in \mathcal{F} : P(A \cap B) = P(A)P(B)\}$$

If it can be shown $A \in \mathcal{H}$ then the theorem is proved because then you could let $B = A$ and conclude $P(A \cap A) = P(A)P(A)$ which requires $P(A)$ is either 0 or 1.

Let $\mathcal{S}_n = \sigma(\cup_{k=1}^n \mathcal{F}_k)$, the smallest σ algebra containing $\cup_{k=1}^n \mathcal{F}_k$.

Claim: $\mathcal{S}_n \subseteq \mathcal{H}$

Proof of claim: Let $\mathcal{K}$ denote sets of the form $A_{k_1} \cap \cdots \cap A_{k_m}$ for some m where $A_{k_j} \in \mathcal{F}_{k_j}$ where each $k_j \leq n$. Thus $\mathcal{K}$ is a π system and

$$\sigma(\mathcal{K}) = \sigma(\cup_{k=1}^n \mathcal{F}_k)$$

Let

$$\mathcal{G} \equiv \{B \in \mathcal{T}_{n+1} : B \in \mathcal{H}\}$$

That is $P(A \cap B) = P(A)P(B)$. Thus $\mathcal{K} \subseteq \mathcal{G}$ because

$$P(A_{k_1} \cap \cdots \cap A_{k_m} \cap A) = P(A_{k_1} \cap \cdots \cap A_{k_m})P(A)$$

by Lemma 47.1.12. However, $\mathcal{G}$ is closed with respect to countable disjoint unions and complements. Therefore by the Lemma on π systems Lemma 9.11.3 on Page 275, it follows $\mathcal{G} = \sigma(\mathcal{K}) = \sigma(\cup_{k=1}^n \mathcal{F}_k)$. This proves the claim.

Thus $\cup_{n=1}^\infty \mathcal{S}_n \subseteq \mathcal{H}$. The $\mathcal{S}_n$ are increasing so $\mathcal{K} \equiv \cup_{n=1}^\infty \mathcal{S}_n$ is a π system. It is also clear $\mathcal{H}$ is closed with respect to complements and countable disjoint unions. By the π systems lemma again, it follows

$$\begin{aligned} \mathcal{H} &\supseteq \sigma(\cup_{n=1}^\infty \mathcal{S}_n) = \sigma(\cup_{n=1}^\infty \sigma(\cup_{k=1}^n \mathcal{F}_k)) \\ &\supseteq \sigma(\cup_{n=1}^\infty \mathcal{F}_n) \supseteq \sigma(\cup_{k=n}^\infty \mathcal{F}_k) = \mathcal{T}_n \end{aligned}$$

and so, since $A \in \mathcal{T}_n$ it follows $A \in \mathcal{H}$ which shows

$$P(A \cap A) = P(A) = P(A)^2$$

and so $P(A)$ equals either 0 or 1. This proves the theorem.

What sorts of things are tail events of independent σ algebras?

Theorem 47.1.22 *Let $\{\mathbf{X}_k\}$ be a sequence of independent random variables having values in Z a Banach space. Then*

$$A \equiv \{\omega : \{\mathbf{X}_k(\omega)\} \text{ converges}\}$$

is a tail event of the independent σ algebras $\{\sigma(\mathbf{X}_k)\}$. So is

$$B \equiv \left\{ \omega : \left\{ \sum_{k=1}^{\infty} \mathbf{X}_k(\omega) \right\} \text{ converges} \right\}.$$

Proof: Since Z is complete, A is the same as the set where $\{\mathbf{X}_k(\omega)\}$ is a Cauchy sequence. This set is

$$\cap_{n=1}^\infty \cap_{p=1}^\infty \cup_{m=p}^\infty \cap_{l,k \geq m} \{\omega : \|\mathbf{X}_k(\omega) - \mathbf{X}_l(\omega)\| < 1/n\}$$

Note that

$$\cup_{m=p}^\infty \cap_{l,k \geq m} \{\omega : \|\mathbf{X}_k(\omega) - \mathbf{X}_l(\omega)\| < 1/n\} \in \sigma(\cup_{j=p}^\infty \sigma(\mathbf{X}_j))$$

for every p . Thus the set where ultimately any pair of $\mathbf{X}_k, \mathbf{X}_l$ are closer together than $1/n$,

$$\cap_{p=1}^\infty \cup_{m=p}^\infty \cap_{l,k \geq m} \{\omega : \|\mathbf{X}_k(\omega) - \mathbf{X}_l(\omega)\| < 1/n\}$$

is a tail event. The set where $\{\mathbf{X}_k(\omega)\}$ is a Cauchy sequence is the intersection of all these and is therefore, also a tail event.

Now consider B . This set is the same as the set where the partial sums are Cauchy sequences. Let $\mathbf{S}_n \equiv \sum_{k=1}^n \mathbf{X}_k$. The set where the sum converges is then

$$\cap_{n=1}^{\infty} \cap_{p=2}^{\infty} \cup_{m=p}^{\infty} \cap_{l,k \geq m} \{\omega : \|\mathbf{S}_k(\omega) - \mathbf{S}_l(\omega)\| < 1/n\}$$

Say $k < l$ and consider for $m \geq p$

$$\{\omega : \|\mathbf{S}_k(\omega) - \mathbf{S}_l(\omega)\| < 1/n, k \geq m\}$$

This is the same as

$$\left\{ \omega : \left\| \sum_{j=k-1}^l \mathbf{X}_j(\omega) \right\| < 1/n, k \geq m \right\} \in \sigma \left(\cup_{j=p-1}^{\infty} \sigma(\mathbf{X}_j) \right)$$

Thus

$$\cup_{m=p}^{\infty} \cap_{l,k \geq m} \{\omega : \|\mathbf{S}_k(\omega) - \mathbf{S}_l(\omega)\| < 1/n\} \in \sigma \left(\cup_{j=p-1}^{\infty} \sigma(\mathbf{X}_j) \right)$$

and so the intersection for all p of these is a tail event. Then the intersection over all n of these tail events is a tail event. This proves the theorem.

From this it can be concluded that if you have a sequence of independent random variables, $\{\mathbf{X}_k\}$ the set where it converges is either of probability 1 or probability 0. A similar conclusion holds for the set where the infinite sum of these random variables converges. This is stated in the next corollary.

Corollary 47.1.23 *Let $\{\mathbf{X}_k\}$ be a sequence of random variables having values in a Banach space. Then*

$$\lim_{n \rightarrow \infty} \mathbf{X}_n(\omega)$$

either exists for a.e. ω or the convergence fails to take place for a.e. ω . Also if

$$A \equiv \left\{ \omega : \sum_{k=1}^{\infty} \mathbf{X}_k(\omega) \text{ converges} \right\},$$

then $P(A) = 0$ or 1 .

47.1.4 Kolmogorov's Inequality, Strong Law Of Large Numbers

Kolmogorov's inequality is a very interesting inequality which depends on independence of a set of random vectors. The random vectors have values in $\mathbb{R}^n$ or more generally some real separable Hilbert space.

Lemma 47.1.24 *If $\mathbf{Y}, \mathbf{X}$ are independent random variables having values in a real separable Hilbert space, H with $E(|\mathbf{X}|^2), E(|\mathbf{Y}|^2) < \infty$, then*

$$\int_{\Omega} (\mathbf{X}, \mathbf{Y}) dP = \left(\int_{\Omega} \mathbf{X} dP, \int_{\Omega} \mathbf{Y} dP \right).$$

Proof: Let $\{\mathbf{e}_k\}$ be a complete orthonormal basis. Thus

$$\int_{\Omega} (\mathbf{X}, \mathbf{Y}) dP = \int_{\Omega} \sum_{k=1}^{\infty} (\mathbf{X}, \mathbf{e}_k) (\mathbf{Y}, \mathbf{e}_k) dP$$

Now

$$\begin{aligned} \int_{\Omega} \sum_{k=1}^{\infty} |(\mathbf{X}, \mathbf{e}_k) (\mathbf{Y}, \mathbf{e}_k)| dP &\leq \int_{\Omega} \left(\sum_k |(\mathbf{X}, \mathbf{e}_k)|^2 \right)^{1/2} \left(\sum_k |(\mathbf{Y}, \mathbf{e}_k)|^2 \right)^{1/2} dP \\ &= \int_{\Omega} |\mathbf{X}| |\mathbf{Y}| dP \leq \left(\int_{\Omega} |\mathbf{X}|^2 dP \right)^{1/2} \left(\int_{\Omega} |\mathbf{Y}|^2 dP \right)^{1/2} < \infty \end{aligned}$$

and so by Fubini's theorem,

$$\begin{aligned} \int_{\Omega} (\mathbf{X}, \mathbf{Y}) dP &= \int_{\Omega} \sum_{k=1}^{\infty} (\mathbf{X}, \mathbf{e}_k) (\mathbf{Y}, \mathbf{e}_k) dP = \sum_{k=1}^{\infty} \int_{\Omega} (\mathbf{X}, \mathbf{e}_k) (\mathbf{Y}, \mathbf{e}_k) dP \\ &= \sum_{k=1}^{\infty} \int_{\Omega} (\mathbf{X}, \mathbf{e}_k) dP \int_{\Omega} (\mathbf{Y}, \mathbf{e}_k) dP = \sum_{k=1}^{\infty} \left(\int_{\Omega} \mathbf{X} dP, \mathbf{e}_k \right) \left(\int_{\Omega} \mathbf{Y} dP, \mathbf{e}_k \right) dP \\ &= \left(\int_{\Omega} \mathbf{X} dP, \int_{\Omega} \mathbf{Y} dP \right) \end{aligned}$$

as claimed.

Theorem 47.1.25 Suppose $\{\mathbf{X}_k\}_{k=1}^n$ are independent with $E(|\mathbf{X}_k|) < \infty$, $E(\mathbf{X}_k) = \mathbf{0}$. Then for any $\varepsilon > 0$,

$$P \left(\left[\max_{1 \leq k \leq n} \left| \sum_{j=1}^k \mathbf{X}_j \right| \geq \varepsilon \right] \right) \leq \frac{1}{\varepsilon^2} \sum_{j=1}^n E(|\mathbf{X}_j|^2).$$

Proof: Let

$$A = \left[\max_{1 \leq k \leq n} \left| \sum_{j=1}^k \mathbf{X}_j \right| \geq \varepsilon \right]$$

Now let $A_1 \equiv [|\mathbf{X}_1| \geq \varepsilon]$ and if $A_1, \dots, A_m$ have been chosen,

$$A_{m+1} \equiv \left[\left| \sum_{j=1}^{m+1} \mathbf{X}_j \right| \geq \varepsilon \right] \cap \bigcap_{r=1}^m \left[\left| \sum_{j=1}^r \mathbf{X}_j \right| < \varepsilon \right]$$

Thus the A_k partition A and $\omega \in A_k$ means

$$\left| \sum_{j=1}^k \mathbf{X}_j \right| \geq \varepsilon$$

but this did not happen for $\left| \sum_{j=1}^r \mathbf{X}_j \right|$ for any $r < k$. Note also that $A_k \in \sigma(\mathbf{X}_1, \dots, \mathbf{X}_k)$. Then from algebra,

$$\begin{aligned} \left| \sum_{j=1}^n \mathbf{X}_j \right|^2 &= \left(\sum_{i=1}^k \mathbf{X}_i + \sum_{j=k+1}^n \mathbf{X}_j, \sum_{i=1}^k \mathbf{X}_i + \sum_{j=k+1}^n \mathbf{X}_j \right) \\ &= \left| \sum_{j=1}^k \mathbf{X}_j \right|^2 + \sum_{i \leq k, j > k} (\mathbf{X}_i, \mathbf{X}_j) + \sum_{i \leq k, j > k} (\mathbf{X}_j, \mathbf{X}_i) + \sum_{i > k, j > k} (\mathbf{X}_j, \mathbf{X}_i) \end{aligned}$$

Written more succinctly,

$$\left| \sum_{j=1}^n \mathbf{X}_j \right|^2 = \left| \sum_{j=1}^k \mathbf{X}_j \right|^2 + \sum_{j > k \text{ or } i > k} (\mathbf{X}_i, \mathbf{X}_j)$$

Now multiply both sides by $\mathcal{X}_{A_k}$ and integrate. Suppose $i \leq k$ for one of the terms in the second sum. Then by Lemma 47.1.12 and $A_k \in \sigma(\mathbf{X}_1, \dots, \mathbf{X}_k)$, the two random vectors $\mathcal{X}_{A_k} \mathbf{X}_i, \mathbf{X}_j$ are independent,

$$\int_{\Omega} \mathcal{X}_{A_k} (\mathbf{X}_i, \mathbf{X}_j) dP = \left(\int_{\Omega} \mathcal{X}_{A_k} \mathbf{X}_i dP, \int_{\Omega} \mathbf{X}_j dP \right) = 0$$

the last equality holding because by assumption $E(\mathbf{X}_j) = \mathbf{0}$. Therefore, it can be assumed both i, j are larger than k and

$$\begin{aligned} \int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^n \mathbf{X}_j \right|^2 dP &= \int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^k \mathbf{X}_j \right|^2 dP \\ &\quad + \sum_{j > k, i > k} \int_{\Omega} \mathcal{X}_{A_k} (\mathbf{X}_i, \mathbf{X}_j) dP \end{aligned} \quad (47.1.1)$$

The last term on the right is interesting. Suppose $i > j$. Then $i > k$ because one of these i, j is larger than k . The integral inside the sum is of the form

$$\int_{\Omega} (\mathbf{X}_i, \mathcal{X}_{A_k} \mathbf{X}_j) dP \quad (47.1.2)$$

The second factor in the inner product is in

$$\sigma(\mathbf{X}_1, \dots, \mathbf{X}_k, \mathbf{X}_j)$$

and $\mathbf{X}_i$ is not included in the list of random vectors. Thus by Lemma 47.1.12, the two random vectors $\mathbf{X}_i, \mathcal{X}_{A_k} \mathbf{X}_j$ are independent and so 47.1.2 reduces to

$$\left(\int_{\Omega} \mathbf{X}_i dP, \int_{\Omega} \mathcal{X}_{A_k} \mathbf{X}_j dP \right) = \left(\mathbf{0}, \int_{\Omega} \mathcal{X}_{A_k} \mathbf{X}_j dP \right) = 0.$$

A similar result holds if $j > i$. Thus the mixed terms in the last term of 47.1.1 are all equal to 0. Hence 47.1.1 reduces to

$$\begin{aligned} \int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^n \mathbf{X}_j \right|^2 dP &= \int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^k \mathbf{X}_j \right|^2 dP \\ &\quad + \sum_{i>k} \int_{\Omega} \mathcal{X}_{A_k} |\mathbf{X}_i|^2 dP \end{aligned}$$

and so

$$\int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^n \mathbf{X}_j \right|^2 dP \geq \int_{\Omega} \mathcal{X}_{A_k} \left| \sum_{j=1}^k \mathbf{X}_j \right|^2 dP \geq \varepsilon^2 P(A_k).$$

Now, summing these yields

$$\begin{aligned} \varepsilon^2 P(A) &\leq \int_{\Omega} \mathcal{X}_A \left| \sum_{j=1}^n \mathbf{X}_j \right|^2 dP \leq \int_{\Omega} \left| \sum_{j=1}^n \mathbf{X}_j \right|^2 dP \\ &= \sum_{i,j} \int_{\Omega} (\mathbf{X}_i, \mathbf{X}_j) dP \end{aligned}$$

By independence of the random vectors the mixed terms of the above sum equal zero and so it reduces to

$$\sum_{i=1}^n \int_{\Omega} |\mathbf{X}_i|^2 dP$$

and this proves the theorem.

This theorem implies the following amazing result.

Theorem 47.1.26 *Let $\{\mathbf{X}_k\}_{k=1}^{\infty}$ be independent random vectors having values in a separable real Hilbert space and suppose $E(|\mathbf{X}_k|) < \infty$ for each k and $E(\mathbf{X}_k) = \mathbf{0}$. Suppose also that*

$$\sum_{j=1}^{\infty} E(|\mathbf{X}_j|^2) < \infty.$$

Then

$$\sum_{j=1}^{\infty} \mathbf{X}_j$$

converges a.e.

Proof: Let $\varepsilon > 0$ be given. By Kolmogorov's inequality, Theorem 47.1.25, it follows that for $p \leq m < n$

$$\begin{aligned} P \left(\left[\max_{m \leq k \leq n} \left| \sum_{j=m}^k \mathbf{X}_j \right| \geq \varepsilon \right] \right) &\leq \frac{1}{\varepsilon^2} \sum_{j=p}^n E(|\mathbf{X}_j|^2) \\ &\leq \frac{1}{\varepsilon^2} \sum_{j=p}^{\infty} E(|\mathbf{X}_j|^2). \end{aligned}$$

Therefore, letting $n \rightarrow \infty$ it follows that for all m, n such that $p \leq m \leq n$

$$P \left(\left[\max_{p \leq m \leq n} \left| \sum_{j=m}^n \mathbf{X}_j \right| \geq \varepsilon \right] \right) \leq \frac{1}{\varepsilon^2} \sum_{j=p}^{\infty} E(|\mathbf{X}_j|^2).$$

It follows from the assumption

$$\sum_{j=1}^{\infty} E(|\mathbf{X}_j|^2) < \infty$$

there exists a sequence, $\{p_n\}$ such that if $m \geq p_n$

$$P \left(\left[\max_{k \geq m \geq p_n} \left| \sum_{j=m}^k \mathbf{X}_j \right| \geq 2^{-n} \right] \right) \leq 2^{-n}.$$

By the Borel Cantelli lemma, Lemma 47.1.2, there is a set of measure 0, N such that for $\omega \notin N$, ω is in only finitely many of the sets,

$$\left[\max_{k \geq m \geq p_n} \left| \sum_{j=m}^k \mathbf{X}_j \right| \geq 2^{-n} \right]$$

and so for $\omega \notin N$, it follows that for large enough n ,

$$\left[\max_{k \geq m \geq p_n} \left| \sum_{j=m}^k \mathbf{X}_j(\omega) \right| < 2^{-n} \right]$$

However, this says the partial sums $\left\{ \sum_{j=1}^k \mathbf{X}_j(\omega) \right\}_{k=1}^{\infty}$ are a Cauchy sequence. Therefore, they converge and this proves the theorem.

With this amazing result, there is a simple proof of the strong law of large numbers. In the following lemma, s_k and a_j could have values in any normed linear space.

Lemma 47.1.27 Suppose $s_k \rightarrow s$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_k = s.$$

Also if

$$\sum_{j=1}^{\infty} \frac{a_j}{j}$$

converges, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n a_j = 0.$$

Proof: Consider the first part. Since $s_k \rightarrow s$, it follows there is some constant, C such that $|s_k| < C$ for all k and $|s| < C$ also. Choose K so large that if $k \geq K$, then for $n > K$,

$$\begin{aligned} |s - s_k| &< \varepsilon/2. \\ \left| s - \frac{1}{n} \sum_{k=1}^n s_k \right| &\leq \frac{1}{n} \sum_{k=1}^n |s_k - s| \\ &= \frac{1}{n} \sum_{k=1}^K |s_k - s| + \frac{1}{n} \sum_{k=K}^n |s_k - s| \\ &\leq \frac{2CK}{n} + \frac{\varepsilon}{2} \frac{n-K}{n} < \frac{2CK}{n} + \frac{\varepsilon}{2} \end{aligned}$$

Therefore, whenever n is large enough,

$$\left| s - \frac{1}{n} \sum_{k=1}^n s_k \right| < \varepsilon.$$

Now consider the second claim. Let

$$s_k = \sum_{j=1}^k \frac{a_j}{j}$$

and $s = \lim_{k \rightarrow \infty} s_k$. Then by the first part,

$$\begin{aligned} s &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_k = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \sum_{j=1}^k \frac{a_j}{j} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{a_j}{j} \sum_{k=j}^n 1 = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \frac{a_j}{j} (n-j) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{j=1}^n \frac{a_j}{j} - \frac{1}{n} \sum_{j=1}^n a_j \right) = s - \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n a_j \end{aligned}$$

and this proves the second part.

Now here is the strong law of large numbers.

Theorem 47.1.28 *Suppose $\{\mathbf{X}_k\}$ are independent random variables and $E(|\mathbf{X}_k|) < \infty$ for each k and $E(\mathbf{X}_k) = \mathbf{m}_k$. Suppose also*

$$\sum_{j=1}^{\infty} \frac{1}{j^2} E(|\mathbf{X}_j - \mathbf{m}_j|^2) < \infty. \quad (47.1.3)$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (\mathbf{X}_j - \mathbf{m}_j) = \mathbf{0}$$

Proof: Consider the sum

$$\sum_{j=1}^{\infty} \frac{\mathbf{X}_j - \mathbf{m}_j}{j}.$$

This sum converges *a.e.* because of 47.1.3 and Theorem 47.1.26 applied to the random vectors $\left\{ \frac{\mathbf{X}_j - \mathbf{m}_j}{j} \right\}$. Therefore, from Lemma 47.1.27 it follows that for a.e. ω ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (\mathbf{X}_j(\omega) - \mathbf{m}_j) = \mathbf{0}$$

This proves the theorem.

The next corollary is often called the strong law of large numbers. It follows immediately from the above theorem.

Corollary 47.1.29 *Suppose $\{\mathbf{X}_j\}_{j=1}^{\infty}$ are independent and identically distributed random vectors, $\lambda_{\mathbf{X}_i} = \lambda_{\mathbf{X}_j}$ for all i, j having mean $\mathbf{m}$ and variance equal to*

$$\sigma^2 \equiv \int_{\Omega} |\mathbf{X}_j - \mathbf{m}|^2 dP < \infty.$$

Then for a.e. $\omega \in \Omega$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mathbf{X}_j(\omega) = \mathbf{m}$$

47.2 The Characteristic Function

One of the most important tools in probability is the characteristic function. To begin with, assume the random variables have values in $\mathbb{R}^m$.

Definition 47.2.1 *Let $\mathbf{X}$ be a random variable as above. The characteristic function is*

$$\phi_{\mathbf{X}}(\mathbf{t}) \equiv E(e^{i\mathbf{t} \cdot \mathbf{X}}) \equiv \int_{\Omega} e^{i\mathbf{t} \cdot \mathbf{X}(\omega)} dP = \int_{\mathbb{R}^p} e^{i\mathbf{t} \cdot \mathbf{x}} d\lambda_{\mathbf{X}}$$

the last equation holding by Lemma 47.1.8.

Recall the following fundamental lemma and definition, Lemma 21.3.4 on Page 642.

Definition 47.2.2 For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by

$$FT(\phi) \equiv T(F\phi), \quad F^{-1}T(\phi) \equiv T(F^{-1}\phi)$$

Lemma 47.2.3 F and F^{-1} are both one to one, onto, and are inverses of each other.

The main result on characteristic functions is the following.

Theorem 47.2.4 Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors with values in $\mathbb{R}^p$ and suppose $E(e^{it \cdot \mathbf{X}}) = E(e^{it \cdot \mathbf{Y}})$ for all $\mathbf{t} \in \mathbb{R}^p$. Then $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$.

Proof: For $\psi \in \mathcal{G}$, let $\lambda_{\mathbf{X}}(\psi) \equiv \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}(\psi) \equiv \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{Y}}$. Thus both $\lambda_{\mathbf{X}}$ and $\lambda_{\mathbf{Y}}$ are in $\mathcal{G}^*$. Then letting $\psi \in \mathcal{G}$ and using Fubini's theorem,

$$\begin{aligned} \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot \mathbf{y}} \psi(\mathbf{t}) dt d\lambda_{\mathbf{Y}} &= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot \mathbf{y}} d\lambda_{\mathbf{Y}} \psi(\mathbf{t}) dt \\ &= \int_{\mathbb{R}^p} E(e^{it \cdot \mathbf{Y}}) \psi(\mathbf{t}) dt \\ &= \int_{\mathbb{R}^p} E(e^{it \cdot \mathbf{X}}) \psi(\mathbf{t}) dt \\ &= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot \mathbf{x}} d\lambda_{\mathbf{X}} \psi(\mathbf{t}) dt \\ &= \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{it \cdot \mathbf{x}} \psi(\mathbf{t}) dt d\lambda_{\mathbf{X}}. \end{aligned}$$

Thus $\lambda_{\mathbf{Y}}(F^{-1}\psi) = \lambda_{\mathbf{X}}(F^{-1}\psi)$. Since $\psi \in \mathcal{G}$ is arbitrary and F^{-1} is onto, this implies $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$ in $\mathcal{G}^*$. But $\mathcal{G}$ is dense in $C_0(\mathbb{R}^p)$ and so $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$ as measures. This proves the theorem.

47.3 Conditional Probability

Here I will consider the concept of conditional probability depending on the theory of differentiation of general Radon measures and leading to the Doob Dynkin lemma.

If $\mathbf{X}, \mathbf{Y}$ are two random vectors defined on a probability space having values in $\mathbb{R}^{p_1}$ and $\mathbb{R}^{p_2}$ respectively, and if E is a Borel set in the appropriate space, then $(\mathbf{X}, \mathbf{Y})$ is a random vector with values in $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$ and $\lambda_{(\mathbf{X}, \mathbf{Y})}(E \times \mathbb{R}^{p_2}) = \lambda_{\mathbf{X}}(E)$, $\lambda_{(\mathbf{X}, \mathbf{Y})}(\mathbb{R}^{p_1} \times E) = \lambda_{\mathbf{Y}}(E)$. Thus, by Theorem 20.3.3 on Page 625, there exist probability measures, denoted here by $\lambda_{\mathbf{X}|\mathbf{Y}}$ and $\lambda_{\mathbf{Y}|\mathbf{X}}$, such that whenever E is a Borel set in $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$,

$$\int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} \chi_E d\lambda_{(\mathbf{X}, \mathbf{Y})} = \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \chi_E d\lambda_{\mathbf{Y}|\mathbf{X}} d\lambda_{\mathbf{X}},$$

and

$$\int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} \mathcal{X}_E d\lambda_{(\mathbf{X}, \mathbf{Y})} = \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} \mathcal{X}_E d\lambda_{\mathbf{X}|\mathbf{Y}} d\lambda_{\mathbf{Y}}.$$

Definition 47.3.1 Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors defined on a probability space. The conditional probability measure of $\mathbf{Y}$ given $\mathbf{X}$ is the measure $\lambda_{\mathbf{Y}|\mathbf{X}}$ in the above. Similarly the conditional probability measure of $\mathbf{X}$ given $\mathbf{Y}$ is the measure $\lambda_{\mathbf{X}|\mathbf{Y}}$.

More generally, one can use the theory of slicing measures to consider any finite list of random vectors, $\{\mathbf{X}_i\}$, defined on a probability space with $\mathbf{X}_i \in \mathbb{R}^{p_i}$, and write the following for E a Borel set in $\prod_{i=1}^n \mathbb{R}^{p_i}$.

$$\begin{aligned} \int_{\mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_1, \dots, \mathbf{X}_n)} &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2} \times \dots \times \mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_2, \dots, \mathbf{X}_n)|\mathbf{X}_1} d\lambda_{\mathbf{X}_1} \\ &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_3} \times \dots \times \mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_3, \dots, \mathbf{X}_n)|\mathbf{X}_1 \mathbf{X}_2} d\lambda_{\mathbf{X}_2|\mathbf{X}_1} d\lambda_{\mathbf{X}_1} \\ &\quad \vdots \\ &= \int_{\mathbb{R}^{p_1}} \dots \int_{\mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{\mathbf{X}_n|\mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_{n-1}} d\lambda_{\mathbf{X}_{n-1}|\mathbf{X}_1 \dots \mathbf{X}_{n-2}} \dots d\lambda_{\mathbf{X}_2|\mathbf{X}_1} d\lambda_{\mathbf{X}_1}. \end{aligned} \quad (47.3.4)$$

Obviously, this could have been done in any order in the iterated integrals by simply modifying the “given” variables, those occurring after the symbol $|$, to be those which have been integrated in an outer level of the iterated integral.

Definition 47.3.2 Let $\{\mathbf{X}_1, \dots, \mathbf{X}_n\}$ be random vectors defined on a probability space having values in $\mathbb{F}^{p_1}, \dots, \mathbb{F}^{p_n}$ respectively. The random vectors are independent if for every E a Borel set in $\mathbb{F}^{p_1} \times \dots \times \mathbb{F}^{p_n}$,

$$\begin{aligned} &\int_{\mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{(\mathbf{X}_1, \dots, \mathbf{X}_n)} \\ &= \int_{\mathbb{R}^{p_1}} \dots \int_{\mathbb{R}^{p_n}} \mathcal{X}_E d\lambda_{\mathbf{X}_n} d\lambda_{\mathbf{X}_{n-1}} \dots d\lambda_{\mathbf{X}_2} d\lambda_{\mathbf{X}_1} \end{aligned} \quad (47.3.5)$$

and the iterated integration may be taken in any order. If $\mathcal{A}$ is any set of random vectors defined on a probability space, $\mathcal{A}$ is independent if any finite set of random vectors from $\mathcal{A}$ is independent.

Thus, the random vectors are independent exactly when the dependence on the givens in 47.3.4 can be dropped.

Does this amount to the same thing as discussed earlier? These two random vectors, $\mathbf{X}, \mathbf{Y}$ were independent if whenever $A \in \sigma(\mathbf{X})$ and $B \in \sigma(\mathbf{Y})$, $P(A \cap B) =$

$P(A)P(B)$. Suppose the above definition and A and B as described. Let $A = \mathbf{X}^{-1}(E)$ and $B = \mathbf{Y}^{-1}(F)$. Then

$$\begin{aligned}
 P(A \cap B) &= P((\mathbf{X}, \mathbf{Y}) \in E \times F) \\
 &= \int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) d\lambda_{(\mathbf{X}, \mathbf{Y})} \\
 &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) d\lambda_{\mathbf{Y}|\mathbf{x}} d\lambda_{\mathbf{X}} \\
 &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) d\lambda_{\mathbf{Y}} d\lambda_{\mathbf{X}} \\
 &= \lambda_{\mathbf{X}}(E) \lambda_{\mathbf{Y}}(F) = P(A)P(B)
 \end{aligned}$$

Next suppose $P(A \cap B) = P(A)P(B)$ where $A \in \sigma(\mathbf{X})$ and $B \in \sigma(\mathbf{Y})$, $A = \mathbf{X}^{-1}(E)$ and $B = \mathbf{Y}^{-1}(F)$. Can it be asserted $\lambda_{\mathbf{X}|\mathbf{Y}} = \lambda_{\mathbf{X}}$? In this case, for all E Borel in $\mathbb{R}^{p_1}$ and F Borel in $\mathbb{R}^{p_2}$,

$$\begin{aligned}
 &\int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) d\lambda_{\mathbf{Y}} d\lambda_{\mathbf{X}} \\
 &= P(A)P(B) = P(A \cap B) \\
 &= \int_{\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) d\lambda_{(\mathbf{X}, \mathbf{Y})} \\
 &= \int_{\mathbb{R}^{p_1}} \int_{\mathbb{R}^{p_2}} \mathcal{X}_E(\mathbf{x}) \mathcal{X}_F(\mathbf{y}) d\lambda_{\mathbf{Y}|\mathbf{x}} d\lambda_{\mathbf{X}}
 \end{aligned}$$

and so, by uniqueness of the slicing measures, $d\lambda_{\mathbf{Y}|\mathbf{x}} = d\lambda_{\mathbf{Y}}$. A similar argument shows $d\lambda_{\mathbf{X}|\mathbf{Y}} = d\lambda_{\mathbf{X}}$. The argument clearly extends to any finite set of random variables. Thus this amounts to the same thing discussed earlier.

Proposition 47.3.3 *Equations 47.3.5 and 47.3.4 hold with $\mathcal{X}_E$ replaced by any nonnegative Borel measurable function and for any bounded continuous function.*

Proof: The two equations hold for simple functions in place of $\mathcal{X}_E$ and so an application of the monotone convergence theorem applied to an increasing sequence of simple functions converging pointwise to a given nonnegative Borel measurable function yields the conclusion of the proposition in the case of the nonnegative Borel function. For a bounded continuous function, one can apply the result just established to the positive and negative parts of the real and imaginary parts of the function.

Lemma 47.3.4 *Let $\mathbf{X}_1, \dots, \mathbf{X}_n$ be random vectors with values in $\mathbb{R}^{p_1}, \dots, \mathbb{R}^{p_n}$ respectively and let $\mathbf{g} : \mathbb{R}^{p_1} \times \dots \times \mathbb{R}^{p_n} \rightarrow \mathbb{R}^k$ be Borel measurable. Then $\mathbf{g}(\mathbf{X}_1, \dots, \mathbf{X}_n)$ is a random vector with values in $\mathbb{R}^k$ and if $h : \mathbb{R}^k \rightarrow [0, \infty)$, then*

$$\int_{\mathbb{R}^k} h(\mathbf{y}) d\lambda_{\mathbf{g}(\mathbf{X}_1, \dots, \mathbf{X}_n)}(\mathbf{y}) =$$

$$\int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} h(\mathbf{g}(\mathbf{x}_1, \cdots, \mathbf{x}_n)) d\lambda_{(\mathbf{x}_1, \cdots, \mathbf{x}_n)}. \quad (47.3.6)$$

If $\mathbf{X}_i$ is a random vector with values in $\mathbb{R}^{p_i}$, $i = 1, 2, \cdots$ and if $\mathbf{g}_i : \mathbb{R}^{p_i} \rightarrow \mathbb{R}^{k_i}$, where $\mathbf{g}_i$ is Borel measurable, then the random vectors $\mathbf{g}_i(\mathbf{X}_i)$ are also independent whenever the $\mathbf{X}_i$ are independent.

Proof: First let E be a Borel set in $\mathbb{R}^k$. From the definition,

$$\begin{aligned} \int_{\mathbb{R}^k} \mathcal{X}_E d\lambda_{\mathbf{g}(\mathbf{x}_1, \cdots, \mathbf{x}_n)} &= \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{X}_{\mathbf{g}^{-1}(E)} d\lambda_{(\mathbf{x}_1, \cdots, \mathbf{x}_n)} \\ &= \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{X}_E(\mathbf{g}(\mathbf{x}_1, \cdots, \mathbf{x}_n)) d\lambda_{(\mathbf{x}_1, \cdots, \mathbf{x}_n)}. \end{aligned}$$

This proves 47.3.6 in the case when h is $\mathcal{X}_E$. To prove it in the general case, approximate the nonnegative Borel measurable function with simple functions for which the formula is true, and use the monotone convergence theorem.

It remains to prove the last assertion that functions of independent random vectors are also independent random vectors. Let E be a Borel set in $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}$. Then for

$$\begin{aligned} \pi_i(\mathbf{x}_1, \cdots, \mathbf{x}_n) &\equiv \mathbf{x}_i, \\ \int_{\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_n}} \mathcal{X}_E d\lambda_{(\mathbf{g}_1(\mathbf{x}_1), \cdots, \mathbf{g}_n(\mathbf{x}_n))} \\ &\equiv \int_{\mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_n}} \mathcal{X}_E(\mathbf{g}_1 \circ \pi_1, \cdots, \mathbf{g}_n \circ \pi_n) d\lambda_{(\mathbf{x}_1, \cdots, \mathbf{x}_n)} \\ &= \int_{\mathbb{R}^{p_1}} \cdots \int_{\mathbb{R}^{p_n}} \mathcal{X}_E(\mathbf{g}_1 \circ \pi_1, \cdots, \mathbf{g}_n \circ \pi_n) d\lambda_{\mathbf{x}_n} \cdots d\lambda_{\mathbf{x}_1} \\ &= \int_{\mathbb{R}^{k_1}} \cdots \int_{\mathbb{R}^{k_n}} \mathcal{X}_E d\lambda_{\mathbf{g}_n(\mathbf{x}_n)} \cdots d\lambda_{\mathbf{g}_1(\mathbf{x}_1)} \end{aligned}$$

and this proves the last assertion.

Proposition 47.3.5 Let $\nu_1, \cdots, \nu_n$ be Radon probability measures defined on $\mathbb{R}^p$. Then there exists a probability space and independent random vectors $\{\mathbf{X}_1, \cdots, \mathbf{X}_n\}$ defined on this probability space such that $\lambda_{\mathbf{X}_i} = \nu_i$.

Proof: Let $(\Omega, \mathcal{S}, P) \equiv ((\mathbb{R}^p)^n, \mathcal{S}_1 \times \cdots \times \mathcal{S}_n, \nu_1 \times \cdots \times \nu_n)$ where this is just the product σ algebra and product measure which satisfies the following for measurable rectangles.

$$(\nu_1 \times \cdots \times \nu_n) \left(\prod_{i=1}^n E_i \right) = \prod_{i=1}^n \nu_i(E_i).$$

Now let $\mathbf{X}_i(\mathbf{x}_1, \cdots, \mathbf{x}_i, \cdots, \mathbf{x}_n) = \mathbf{x}_i$. Then from the definition, if E is a Borel set in $\mathbb{R}^p$,

$$\lambda_{\mathbf{X}_i}(E) \equiv P\{\mathbf{X}_i \in E\}$$

$$= (\nu_1 \times \cdots \times \nu_n)(\mathbb{R}^p \times \cdots \times E \times \cdots \times \mathbb{R}^p) = \nu_i(E).$$

Let $\mathcal{M}$ consist of all Borel sets of $(\mathbb{R}^p)^n$ such that

$$\int_{\mathbb{R}^p} \cdots \int_{\mathbb{R}^p} \mathcal{X}_E(\mathbf{x}_1, \dots, \mathbf{x}_n) d\lambda_{\mathbf{x}_1} \cdots d\lambda_{\mathbf{x}_n} = \int_{(\mathbb{R}^p)^n} \mathcal{X}_E d\lambda_{(\mathbf{x}_1, \dots, \mathbf{x}_n)}.$$

From what was just shown and the definition of $(\nu_1 \times \cdots \times \nu_n)$ that $\mathcal{M}$ contains all sets of the form $\prod_{i=1}^n E_i$ where each $E_i \in \text{Borel sets of } \mathbb{R}^p$. Therefore, $\mathcal{M}$ contains the algebra of all finite disjoint unions of such sets. It is also clear that $\mathcal{M}$ is a monotone class and so by the theorem on monotone classes, $\mathcal{M}$ equals the Borel sets. Therefore, the given random vectors are independent and this proves the proposition.

The following Lemma was proved earlier in a different way.

Lemma 47.3.6 *If $\{X_i\}_{i=1}^n$ are independent random variables having values in $\mathbb{R}$,*

$$E\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n E(X_i).$$

Proof: By Lemma 47.3.4 and denoting by P the product, $\prod_{i=1}^n X_i$,

$$\begin{aligned} E\left(\prod_{i=1}^n X_i\right) &= \int_{\mathbb{R}} z d\lambda_P(z) = \int_{\mathbb{R} \times \mathbb{R}} \prod_{i=1}^n x_i d\lambda_{(X_1, \dots, X_n)} \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^n x_i d\lambda_{X_1} \cdots d\lambda_{X_n} = \prod_{i=1}^n E(X_i). \end{aligned}$$

47.4 Characteristic Functions And Independence

There is a way to tell if random vectors are independent by using their characteristic functions.

Proposition 47.4.1 *If $\mathbf{X}_1$ and $\mathbf{X}_2$ are random vectors having values in $\mathbb{R}^{p_1}$ and $\mathbb{R}^{p_2}$ respectively, then the random vectors are independent if and only if*

$$E(e^{iP}) = \prod_{j=1}^2 E(e^{i\mathbf{t}_j \cdot \mathbf{X}_j})$$

where $P \equiv \sum_{j=1}^2 \mathbf{t}_j \cdot \mathbf{X}_j$ for $\mathbf{t}_j \in \mathbb{R}^{p_j}$. More generally, if $\mathbf{X}_i$ is a random vector having values in $\mathbb{R}^{p_i}$ for $i = 1, 2, \dots, n$, and if $P = \sum_{j=1}^n \mathbf{t}_j \cdot \mathbf{X}_j$, then the random vectors are independent if and only if

$$E(e^{iP}) = \prod_{j=1}^n E(e^{i\mathbf{t}_j \cdot \mathbf{X}_j}).$$

The proof of this proposition will depend on the following lemma.

Lemma 47.4.2 *Let $\mathbf{Y}$ be a random vector with values in $\mathbb{R}^p$ and let f be bounded and measurable with respect to the Radon measure, $\lambda_{\mathbf{Y}}$, and satisfy*

$$\int f(\mathbf{y}) e^{i\mathbf{t} \cdot \mathbf{y}} d\lambda_{\mathbf{Y}} = 0$$

for all $\mathbf{t} \in \mathbb{R}^p$. Then $f(\mathbf{y}) = 0$ for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$.

Proof: The proof is just like the proof of Theorem 47.2.4 on Page 1319 applied to the measure, $f(\mathbf{y}) d\lambda_{\mathbf{Y}}$. Thus $\int_E f(\mathbf{y}) d\lambda_{\mathbf{Y}} = 0$ for all E Borel. Hence $f(\mathbf{y}) = 0$ a.e.

Proof of the proposition: If the $\mathbf{X}_j$ are independent, the formula follows from Lemma 47.3.6 and Lemma 47.3.4.

Now suppose the formula holds. Then

$$\begin{aligned} \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1 \cdot \mathbf{x}_1} e^{i\mathbf{t}_2 \cdot \mathbf{x}_2} d\lambda_{\mathbf{X}_1} d\lambda_{\mathbf{X}_2} &= E(e^{iP}) \\ &= \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1 \cdot \mathbf{x}_1} e^{i\mathbf{t}_2 \cdot \mathbf{x}_2} d\lambda_{\mathbf{X}_1|\mathbf{x}_2} d\lambda_{\mathbf{X}_2}. \end{aligned}$$

Now apply Lemma 47.4.2 to conclude that

$$\int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1 \cdot \mathbf{x}_1} d\lambda_{\mathbf{X}_1} = \int_{\mathbb{R}^{p_1}} e^{i\mathbf{t}_1 \cdot \mathbf{x}_1} d\lambda_{\mathbf{X}_1|\mathbf{x}_2} \quad (47.4.7)$$

for $\lambda_{\mathbf{X}_2}$ a.e. $\mathbf{x}_2$, the exceptional set depending on $\mathbf{t}_1$. Therefore, taking the union of all exceptional sets corresponding to $\mathbf{t}_1 \in \mathbb{Q}^{p_1}$, it follows by continuity and the dominated convergence theorem that 47.4.7 holds for all $\mathbf{t}_1$ whenever $\mathbf{x}_2$ is not an element of this exceptional set of measure zero. Therefore, for such $\mathbf{x}_2$, Theorem 47.2.4 applies and it follows $\lambda_{\mathbf{X}_1|\mathbf{x}_2} = \lambda_{\mathbf{X}_1}$ for $\lambda_{\mathbf{X}_2}$ a.e. $\mathbf{x}_2$. Hence, if E is a Borel set in $\mathbb{R}^{p_1} \times \mathbb{R}^{p_2}$, $\int_{\mathbb{R}^{p_1+p_2}} \mathcal{X}_E d\lambda_{(\mathbf{X}_1, \mathbf{X}_2)} = \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} \mathcal{X}_E d\lambda_{\mathbf{X}_1|\mathbf{x}_2} d\lambda_{\mathbf{X}_2} = \int_{\mathbb{R}^{p_2}} \int_{\mathbb{R}^{p_1}} \mathcal{X}_E d\lambda_{\mathbf{X}_1} d\lambda_{\mathbf{X}_2}$. A repeat of the above argument will give the iterated integral in the reverse order or else one could apply Fubini's theorem to obtain this. The proposition also holds if 2 is replaced with n and the argument is a longer version of what was just presented. This proves the proposition.

With this preparation, it is time to present the Doob Dynkin lemma. I am not entirely sure what the Doob Dynkin lemma says actually. What follows is a generalization of what is identified as a special case of this lemma in [54]. I am not sure I have the right generalization. However, it is a very interesting lemma regardless its name.

Lemma 47.4.3 *Suppose $\mathbf{X}, \mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_k$ are random vectors, $\mathbf{X}$ having values in $\mathbb{R}^n$ and $\mathbf{Y}_j$ having values in $\mathbb{R}^{p_j}$ and*

$$\mathbf{X}, \mathbf{Y}_j \in L^1(\Omega).$$

Suppose $\mathbf{X}$ is $\sigma(\mathbf{Y}_1, \dots, \mathbf{Y}_k)$ measurable. Thus

$$\{\mathbf{X}^{-1}(E) : E \text{ Borel}\} \subseteq \left\{ (\mathbf{Y}_1, \dots, \mathbf{Y}_k)^{-1}(F) : F \text{ is Borel in } \prod_{j=1}^k \mathbb{R}^{p_j} \right\}$$

Then there exists a Borel function, $\mathbf{g} : \prod_{j=1}^k \mathbb{R}^{p_j} \rightarrow \mathbb{R}^n$ such that

$$\mathbf{X} = \mathbf{g}(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_k).$$

Proof: For the sake of brevity, denote by $\mathbf{Y}$ the vector $(\mathbf{Y}_1, \dots, \mathbf{Y}_k)$ and by $\mathbf{y}$ the vector $(\mathbf{y}_1, \dots, \mathbf{y}_k)$ and let $\prod_{j=1}^k \mathbb{R}^{p_j} \equiv \mathbb{R}^P$. For E a Borel set of $\mathbb{R}^n$,

$$\begin{aligned} \int_{\mathbf{Y}^{-1}(E)} \mathbf{X} dP &= \int_{\mathbb{R}^n \times \mathbb{R}^P} \mathcal{X}_{\mathbb{R}^n \times E}(\mathbf{x}, \mathbf{y}) \mathbf{x} d\lambda_{(\mathbf{X}, \mathbf{Y})} \\ &= \int_E \int_{\mathbb{R}^n} \mathbf{x} d\lambda_{\mathbf{X}|\mathbf{y}} d\lambda_{\mathbf{Y}}. \end{aligned} \quad (47.4.8)$$

Consider the function

$$\mathbf{y} \rightarrow \int_{\mathbb{R}^n} \mathbf{x} d\lambda_{\mathbf{X}|\mathbf{y}}.$$

Since $d\lambda_{\mathbf{Y}}$ is a Radon measure having inner and outer regularity, it follows the above function is equal to a Borel function for $\lambda_{\mathbf{Y}}$ a.e. $\mathbf{y}$. This function will be denoted by $\mathbf{g}$. Then from 47.4.8

$$\begin{aligned} \int_{\mathbf{Y}^{-1}(E)} \mathbf{X} dP &= \int_E \mathbf{g}(\mathbf{y}) d\lambda_{\mathbf{Y}} = \int_{\mathbb{R}^P} \mathcal{X}_E(\mathbf{y}) \mathbf{g}(\mathbf{y}) d\lambda_{\mathbf{Y}} \\ &= \int_{\Omega} \mathcal{X}_E(\mathbf{Y}(\omega)) \mathbf{g}(\mathbf{Y}(\omega)) dP \\ &= \int_{\mathbf{Y}^{-1}(E)} \mathbf{g}(\mathbf{Y}(\omega)) dP \end{aligned}$$

and since $\mathbf{Y}^{-1}(E)$ is an arbitrary element of $\sigma(\mathbf{Y})$, this shows that since $\mathbf{X}$ is $\sigma(\mathbf{Y})$ measurable,

$$\mathbf{X} = \mathbf{g}(\mathbf{Y}) \text{ } P \text{ a.e.}$$

This proves the lemma.

Later the concept of conditional expectation will be presented. The above argument shows $\mathbf{g}(\mathbf{Y}) = E(\mathbf{X}|\sigma(\mathbf{Y}_1, \dots, \mathbf{Y}_k))$. Then if the symbol,

$$E(\mathbf{X}|\mathbf{y}_1, \dots, \mathbf{y}_k)$$

is defined as the function $\mathbf{g}$ in the above, then for a.e. ω ,

$$E(\mathbf{X}|\sigma(\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_k))(\omega) = E(\mathbf{X}|\mathbf{Y}_1(\omega), \mathbf{Y}_2(\omega), \dots, \mathbf{Y}_k(\omega))$$

which is a fairly attractive formula.

47.5 Characteristic Functions For Measures

Recall the characteristic function for a random variable having values in $\mathbb{R}^n$. I will give a review of this to begin with. Then the concept will be generalized to random variables (vectors) which have values in a real separable Banach space.

Definition 47.5.1 *Let $\mathbf{X}$ be a random variable. The characteristic function is*

$$\phi_{\mathbf{X}}(\mathbf{t}) \equiv E(e^{i\mathbf{t} \cdot \mathbf{X}}) \equiv \int_{\Omega} e^{i\mathbf{t} \cdot \mathbf{X}(\omega)} dP = \int_{\mathbb{R}^p} e^{i\mathbf{t} \cdot \mathbf{x}} d\lambda_{\mathbf{X}}$$

the last equation holding by Lemma 47.1.8 on Page 1300.

Recall the following fundamental lemma and definition, Lemma 21.3.4 on Page 642.

Definition 47.5.2 *For $T \in \mathcal{G}^*$, define $FT, F^{-1}T \in \mathcal{G}^*$ by*

$$FT(\phi) \equiv T(F\phi), \quad F^{-1}T(\phi) \equiv T(F^{-1}\phi)$$

Lemma 47.5.3 *F and F^{-1} are both one to one, onto, and are inverses of each other.*

The main result on characteristic functions is the following is in Theorem 47.2.4 on Page 1319 which is stated here for convenience.

Theorem 47.5.4 *Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors with values in $\mathbb{R}^p$ and suppose $E(e^{i\mathbf{t} \cdot \mathbf{X}}) = E(e^{i\mathbf{t} \cdot \mathbf{Y}})$ for all $\mathbf{t} \in \mathbb{R}^p$. Then $\lambda_{\mathbf{X}} = \lambda_{\mathbf{Y}}$.*

I want to do something similar for random variables which have values in a separable real Banach space, E instead of $\mathbb{R}^p$.

Corollary 47.5.5 *Let $\mathcal{K}$ be a π system of subsets of Ω and suppose two probability measures, μ and ν defined on $\sigma(\mathcal{K})$ are equal on $\mathcal{K}$. Then $\mu = \nu$.*

Proof: This follows from the Lemma 9.11.3 on Page 275. Let

$$\mathcal{G} \equiv \{E \in \sigma(\mathcal{K}) : \mu(E) = \nu(E)\}$$

Then $\mathcal{K} \subseteq \mathcal{G}$, since μ and ν are both probability measures, it follows that if $E \in \mathcal{G}$, then so is E^C . Since these are measures, if $\{A_i\}$ is a sequence of disjoint sets from $\mathcal{G}$ then

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_i \mu(A_i) = \sum_i \nu(A_i) = \nu(\cup_{i=1}^{\infty} A_i)$$

and so from Lemma 9.11.3, $\mathcal{G} = \sigma(\mathcal{K})$. This proves the corollary.

Next recall the following fundamental lemma used to prove Pettis' theorem. It is proved on Page 699 but is stated here for convenience.

Lemma 47.5.6 *If E is a separable Banach space with B' the closed unit ball in E' , then there exists a sequence $\{f_n\}_{n=1}^\infty \equiv D' \subseteq B'$ with the property that for every $x \in E$,*

$$\|x\| = \sup_{f \in D'} |f(x)|$$

Definition 47.5.7 *Let E be a separable real Banach space. A cylindrical set is one which is of the form*

$$\{x \in E : x_i^*(x) \in \Gamma_i, i = 1, 2, \dots, m\}$$

where here $x_i^* \in E'$ and Γ_i is a Borel set in $\mathbb{R}$.

It is obvious that $\emptyset$ is a cylindrical set and that the intersection of two cylindrical sets is another cylindrical set. Thus the cylindrical sets form a π system. What is the smallest σ algebra containing the cylindrical sets? Letting $\{f_n\}_{n=1}^\infty = D'$ be the sequence of Lemma 47.5.6 it follows that

$$\begin{aligned} & \{x \in E : \|x - a\| \leq \delta\} \\ &= \left\{ x \in E : \sup_{f \in D'} |f(x - a)| \leq \delta \right\} \\ &= \left\{ x \in E : \sup_{f \in D'} |f(x) - f(a)| \leq \delta \right\} \\ &= \bigcap_{n=1}^\infty \left\{ x \in E : f_n(x) \in \overline{B(f_n(a), \delta)} \right\} \end{aligned}$$

which yields a countable intersection of cylindrical sets. It follows the smallest σ algebra containing the cylindrical sets contains the closed balls and hence the open balls and consequently the open sets and so it contains the Borel sets. However, each cylindrical set is a Borel set and so in fact this σ algebra equals $\mathcal{B}(E)$.

From Corollary 47.5.5 it follows that two probability measures which are equal on the cylindrical sets are equal on the Borel sets, $\mathcal{B}(E)$.

Definition 47.5.8 *Let μ be a probability measure on $\mathcal{B}(E)$ where E is a real separable Banach space. Then for $x^* \in E'$,*

$$\phi_\mu(x^*) \equiv \int_E e^{ix^*(x)} d\mu(x).$$

ϕ_μ is called the characteristic function for the measure μ .

Note this is a little different than earlier when the symbol $\phi_X(\mathbf{t})$ was used and X was a random variable. Here the focus is more on the measure than a random variable, X such that $\mathcal{L}(X) = \mu$. It might appear this is a more general concept but in fact this is not the case. This is a consequence of Skorokhod's theorem which will be presented later.

Theorem 47.5.9 Let μ and ν be two probability measures on $\mathcal{B}(E)$ where E is a separable real Banach space. Suppose

$$\phi_\mu(x^*) = \phi_\nu(x^*)$$

for all $x^* \in E'$. Then $\mu = \nu$.

Proof: Let $x_1^*, \dots, x_n^*$ be in E' and define for A a Borel set of $\mathbb{R}^n$,

$$\begin{aligned}\tilde{\mu}(A) &\equiv \mu(\{x \in E : (x_1^*(x), \dots, x_n^*(x)) \in A\}), \\ \tilde{\nu}(A) &\equiv \nu(\{x \in E : (x_1^*(x), \dots, x_n^*(x)) \in A\}).\end{aligned}\quad (47.5.9)$$

Note these sets in the parentheses are cylindrical sets. Letting $\lambda \in \mathbb{R}^n$, consider in the definition of the characteristic function, $\lambda_1 x_1^* + \dots + \lambda_n x_n^* \in E'$. Thus

$$\int_E e^{i(\lambda_1 x_1^*(x) + \dots + \lambda_n x_n^*(x))} d\mu = \int_E e^{i(\lambda_1 x_1^*(x) + \dots + \lambda_n x_n^*(x))} d\nu$$

Now if F is a Borel measurable subset of $\mathbb{R}^n$,

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_F(\mathbf{y}) d\tilde{\mu}(y) &= \tilde{\mu}(F) \\ &\equiv \mu(\{x \in E : (x_1^*(x), \dots, x_n^*(x)) \in F\}) \\ &= \int_E \chi_F(x_1^*(x), \dots, x_n^*(x)) d\mu\end{aligned}$$

and using the usual approximations involving simple functions, it follows that for any f bounded and Borel measurable,

$$\int_{\mathbb{R}^n} f(\mathbf{y}) d\tilde{\mu}(y) = \int_E f((x_1^*(x), \dots, x_n^*(x))) d\mu(x).$$

Similarly,

$$\int_{\mathbb{R}^n} f(\mathbf{y}) d\tilde{\nu}(y) = \int_E f((x_1^*(x), \dots, x_n^*(x))) d\nu(x),$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^n} e^{i\lambda \cdot \mathbf{y}} d\tilde{\mu}(y) &= \int_E e^{i(\lambda_1 x_1^*(x) + \dots + \lambda_n x_n^*(x))} d\mu \\ &= \int_E e^{i(\lambda_1 x_1^*(x) + \dots + \lambda_n x_n^*(x))} d\nu \\ &= \int_{\mathbb{R}^n} e^{i\lambda \cdot \mathbf{y}} d\tilde{\nu}(y)\end{aligned}$$

which shows from Theorem 47.5.4 that $\tilde{\nu} = \tilde{\mu}$ on the Borel sets of $\mathbb{R}^n$. However, from the definition of these measures in 47.5.9 this says nothing more than $\mu = \nu$ on any cylindrical set. Hence by Corollary 47.5.5 this shows $\mu = \nu$ on $\mathcal{B}(E)$. This proves the theorem.

47.6 Characteristic Functions And Independence In Banach Space

Finally, I will consider the relation between the characteristic function and independence of random variables. Recall an earlier proposition which relates independence of random vectors with characteristic functions. It is proved starting on Page 1323 in the case of two random variables and concludes with the observation that the general case is entirely similar but more tedious to write down.

Proposition 47.6.1 *Let $\{\mathbf{X}_k\}_{k=1}^n$ be random vectors such that $\mathbf{X}_k$ has values in $\mathbb{R}^{p_k}$. Then the random vectors are independent if and only if*

$$E(e^{iP}) = \prod_{j=1}^n E(e^{it_j \cdot \mathbf{X}_j})$$

where $P \equiv \sum_{j=1}^n \mathbf{t}_j \cdot \mathbf{X}_j$ for $\mathbf{t}_j \in \mathbb{R}^{p_j}$.

It turns out there is a generalization of the above proposition to the case where the random variables have values in a real separable Banach space. Before proving this recall an earlier theorem which had to do with reducing to the case where the random variables had values in $\mathbb{R}^n$, Theorem 47.1.18. It is restated here for convenience.

Theorem 47.6.2 *The random variables, $\{X_i\}_{i \in I}$ are independent if whenever*

$$\{i_1, \dots, i_n\} \subseteq I,$$

$m_{i_1}, \dots, m_{i_n}$ are positive integers, and $\mathbf{g}_{m_{i_1}}, \dots, \mathbf{g}_{m_{i_n}}$ are in

$$(E')^{m_{i_1}}, \dots, (E')^{m_{i_n}}$$

respectively, $\left\{ \mathbf{g}_{m_{i_j}} \circ X_{i_j} \right\}_{j=1}^n$ are independent random vectors having values in

$$\mathbb{R}^{m_{i_1}}, \dots, \mathbb{R}^{m_{i_n}}$$

respectively.

Now here is the theorem about independence and the characteristic functions.

Theorem 47.6.3 *Let $\{X_k\}_{k=1}^n$ be random variables such that X_k has values in E_k , a real separable Banach space. Then the random variables are independent if and only if*

$$E(e^{iP}) = \prod_{j=1}^n E(e^{it_j^*(X_j)})$$

where $P \equiv \sum_{j=1}^n t_j^*(X_j)$ for $t_j^* \in E'_j$.

Proof: If the random variables are independent, then so are the random variables, $t_j^*(X_j)$ and so the equation follows.

The interesting case is when the equation holds.

It suffices to consider only the case where each $E_k = E$. This is because you can consider each X_j to have values in $\prod_{k=1}^n E_k$ by letting X_j take its values in the j^{th} component of the product and 0 in the other components. Can you draw the conclusion the random variables are independent? By Theorem 47.1.18, it suffices to show the random variables $\{\mathbf{g}_{m_k} \circ X_k\}_{k=1}^n$ are independent. This happens if whenever $\mathbf{t}_{m_k} \in \mathbb{R}^{m_k}$ and

$$P = \sum_{k=1}^n \mathbf{t}_{m_k} \cdot (\mathbf{g}_{m_k} \circ X_k),$$

it follows

$$E(e^{iP}) = \prod_{k=1}^n E\left(e^{i\mathbf{t}_{m_k} \cdot (\mathbf{g}_{m_k} \circ X_k)}\right). \quad (47.6.10)$$

Now consider one of these terms in the exponent on the right.

$$\begin{aligned} \mathbf{t}_{m_k} \cdot (\mathbf{g}_{m_k} \circ X_k)(\omega) &= \sum_{j=1}^{m_k} t_j x_j^*(X_k(\omega)) \\ &= y_k^*(X_k(\omega)) \end{aligned}$$

where $y^* \equiv \sum_{j=1}^{m_k} t_j x_j^*$. Therefore, 47.6.10 reduces to

$$E\left(e^{i \sum_{k=1}^n y_k^*(X_k)}\right) = \prod_{k=1}^n E\left(e^{iy_k^*(X_k)}\right)$$

which is assumed to hold. Therefore, the random variables are independent. This proves the theorem.

There is an obvious corollary which is useful.

Corollary 47.6.4 *Let $\{X_k\}_{k=1}^n$ be random variables such that X_k has values in E_k , a real separable Banach space. Then the random variables are independent if and only if*

$$E(e^{iP}) = \prod_{j=1}^n E\left(e^{it_j^*(X_j)}\right)$$

where $P \equiv \sum_{j=1}^n t_j^*(X_j)$ for $t_j^* \in M_j$ where M_j is a dense subset of E'_j .

Proof: The easy direction follows from Theorem 47.6.3. Suppose then the above equation holds for all $t_j^* \in M_j$. Then let $t_j^* \in E'$ and let $\{t_{n_j}^*\}$ be a sequence in M_j such that

$$\lim_{n \rightarrow \infty} t_{n_j}^* = t_j^* \text{ in } E'$$

Then define

$$P \equiv \sum_{j=1}^n t_j^* X_j, \quad P_n \equiv \sum_{j=1}^n t_{nj}^* X_j.$$

It follows

$$\begin{aligned} E(e^{iP}) &= \lim_{n \rightarrow \infty} E(e^{iP_n}) \\ &= \lim_{n \rightarrow \infty} \prod_{j=1}^n E(e^{it_{nj}^* X_j}) \\ &= \prod_{j=1}^n E(e^{it_j^* X_j}) \end{aligned}$$

and this proves the corollary.

47.7 Convolution And Sums

Lemma 47.1.6 on Page 1300 makes possible a definition of convolution of two probability measures defined on $\mathcal{B}(E)$ where E is a separable Banach space as well as some other interesting theorems which held earlier in the context of locally compact spaces. I will first show a little theorem about density of continuous functions in $L^p(E)$ and then define the convolution of two finite measures. First here is a simple technical lemma.

Lemma 47.7.1 *Suppose K is a compact subset of U an open set in E a metric space. Then there exists $\delta > 0$ such that*

$$\text{dist}(x, K) + \text{dist}(x, U^C) \geq \delta \text{ for all } x \in E.$$

Proof: For each $x \in K$, there exists a ball, $B(x, \delta_x)$ such that $B(x, 3\delta_x) \subseteq U$. Finitely many of these balls cover K because K is compact, say $\{B(x_i, \delta_{x_i})\}_{i=1}^m$. Let

$$0 < \delta < \min(\delta_{x_i} : i = 1, 2, \dots, m).$$

Now pick any $x \in K$. Then $x \in B(x_i, \delta_{x_i})$ for some x_i and so $B(x, \delta) \subseteq B(x_i, 2\delta_{x_i}) \subseteq U$. Therefore, for any $x \in K$, $\text{dist}(x, U^C) \geq \delta$. If $x \in B(x_i, 2\delta_{x_i})$ for some x_i , it follows $\text{dist}(x, U^C) \geq \delta$ because then $B(x, \delta) \subseteq B(x_i, 3\delta_{x_i}) \subseteq U$. If $x \notin B(x_i, 2\delta_{x_i})$ for any of the x_i , then $x \notin B(y, \delta)$ for any $y \in K$ because all these sets are contained in some $B(x_i, 2\delta_{x_i})$. Consequently $\text{dist}(x, K) \geq \delta$. This proves the lemma.

From this lemma, there is an easy corollary.

Corollary 47.7.2 *Suppose K is a compact subset of U , an open set in E a metric space. Then there exists a uniformly continuous function f defined on all of E , having values in $[0, 1]$ such that $f(x) = 0$ if $x \notin U$ and $f(x) = 1$ if $x \in K$.*

Proof: Consider

$$f(x) \equiv \frac{\text{dist}(x, U^C)}{\text{dist}(x, U^C) + \text{dist}(x, K)}.$$

Then some algebra yields

$$|f(x) - f(x')| \leq$$

$$\frac{1}{\delta} (|\text{dist}(x, U^C) - \text{dist}(x', U^C)| + |\text{dist}(x, K) - \text{dist}(x', K)|)$$

where δ is the constant of Lemma 47.7.1. Now it is a general fact that

$$|\text{dist}(x, S) - \text{dist}(x', S)| \leq d(x, x').$$

Therefore,

$$|f(x) - f(x')| \leq \frac{2}{\delta} d(x, x')$$

and this proves the corollary.

Now suppose μ is a finite measure defined on the Borel sets of a separable Banach space, E . It was shown above that μ is inner and outer regular. Lemma 47.1.6 on Page 1300 shows that μ is inner regular in the usual sense with respect to compact sets. This makes possible the following theorem.

Theorem 47.7.3 *Let μ be a finite measure on $\mathcal{B}(E)$ where E is a separable Banach space and let $f \in L^p(E; \mu)$. Then for any $\varepsilon > 0$, there exists a uniformly continuous, bounded g defined on E such that*

$$\|f - g\|_{L^p(E)} < \varepsilon.$$

Proof: As usual in such situations, it suffices to consider only $f \geq 0$. Then by Theorem 8.3.9 on Page 197 and an application of the monotone convergence theorem, there exists a simple measurable function,

$$s(x) \equiv \sum_{k=1}^m c_k \chi_{A_k}(x)$$

such that $\|f - s\|_{L^p(E)} < \varepsilon/2$. Now by regularity of μ there exist compact sets, K_k and open sets, V_k such that $2 \sum_{k=1}^m |c_k| \mu(V_k \setminus K_k)^{1/p} < \varepsilon/2$ and by Corollary 47.7.2 there exist uniformly continuous functions g_k having values in $[0, 1]$ such that $g_k = 1$ on K_k and 0 on V_k^C . Then consider

$$g(x) = \sum_{k=1}^m c_k g_k(x).$$

This function is bounded and uniformly continuous. Furthermore,

$$\begin{aligned}
 \|s - g\|_{L^p(E)} &\leq \left(\int_E \left| \sum_{k=1}^m c_k \mathcal{X}_{A_k}(x) - \sum_{k=1}^m c_k g_k(x) \right|^p d\mu \right)^{1/p} \\
 &\leq \left(\int_E \left(\sum_{k=1}^m |c_k| |\mathcal{X}_{A_k}(x) - g_k(x)| \right)^p d\mu \right)^{1/p} \\
 &\leq \sum_{k=1}^m |c_k| \left(\int_E |\mathcal{X}_{A_k}(x) - g_k(x)|^p d\mu \right)^{1/p} \\
 &\leq \sum_{k=1}^m |c_k| \left(\int_{V_k \setminus K_k} 2^p d\mu \right)^{1/p} \\
 &= 2 \sum_{k=1}^m |c_k| \mu(V_k \setminus K)^{1/p} < \varepsilon/2.
 \end{aligned}$$

Therefore,

$$\|f - g\|_{L^p} \leq \|f - s\|_{L^p} + \|s - g\|_{L^p} < \varepsilon/2 + \varepsilon/2.$$

This proves the theorem.

Lemma 47.7.4 *Let $A \in \mathcal{B}(E)$ where μ is a finite measure on $\mathcal{B}(E)$ for E a separable Banach space. Also let $x_i \in E$ for $i = 1, 2, \dots, m$. Then for $\mathbf{x} \in E^m$,*

$$\mathbf{x} \rightarrow \mu \left(A + \sum_{i=1}^m x_i \right), \quad \mathbf{x} \rightarrow \mu \left(A - \sum_{i=1}^m x_i \right)$$

are Borel measurable functions. Furthermore, the above functions are

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

measurable where the above denotes the product measurable sets as described in Theorem 9.11.6 on Page 279.

Proof: First consider the case where $A = U$, an open set. Let

$$\mathbf{y} \in \left\{ \mathbf{x} \in E^m : \mu \left(U + \sum_{i=1}^m x_i \right) > \alpha \right\} \quad (47.7.11)$$

Then from Lemma 47.1.6 on Page 1300 there exists a compact set, $K \subseteq U + \sum_{i=1}^m y_i$ such that $\mu(K) > \alpha$. Then if $\mathbf{y}'$ is close enough to $\mathbf{y}$, it follows $K \subseteq U + \sum_{i=1}^m y'_i$ also. Therefore, for all $\mathbf{y}'$ close enough to $\mathbf{y}$,

$$\mu \left(U + \sum_{i=1}^m y'_i \right) \geq \mu(K) > \alpha.$$

In other words the set described in 47.7.11 is an open set and so $\mathbf{y} \rightarrow \mu(U + \sum_{i=1}^m y_i)$ is Borel measurable whenever U is an open set in E .

Define a π system, $\mathcal{K}$ to consist of all open sets in E . Then define $\mathcal{G}$ as

$$\left\{ A \in \sigma(\mathcal{K}) = \mathcal{B}(E) : \mathbf{y} \rightarrow \mu\left(A + \sum_{i=1}^m y_i\right) \text{ is Borel measurable} \right\}$$

I just showed $\mathcal{G} \supseteq \mathcal{K}$. Now suppose $A \in \mathcal{G}$. Then

$$\mu\left(A^C + \sum_{i=1}^m y_i\right) = \mu(E) - \mu\left(A + \sum_{i=1}^m y_i\right)$$

and so $A^C \in \mathcal{G}$ whenever $A \in \mathcal{G}$. Next suppose $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$. Then

$$\begin{aligned} \mu\left(\left(\bigcup_{i=1}^{\infty} A_i\right) + \sum_{j=1}^m y_j\right) &= \mu\left(\bigcup_{i=1}^{\infty} \left(A_i + \sum_{j=1}^m y_j\right)\right) \\ &= \sum_{i=1}^{\infty} \mu\left(A_i + \sum_{j=1}^m y_j\right) \end{aligned}$$

and so $\bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$ because it is the sum of Borel measurable functions. By the lemma on π systems, Lemma 9.11.3 on Page 275, it follows $\mathcal{G} = \sigma(\mathcal{K}) = \mathcal{B}(E)$. Similarly, $\mathbf{x} \rightarrow \mu\left(A - \sum_{j=1}^m x_j\right)$ is also Borel measurable whenever $A \in \mathcal{B}(E)$. Finally note that

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

contains the open sets of E^m because the separability of E implies the existence of a countable basis for the topology of E^m consisting of sets of the form

$$\prod_{i=1}^m U_i$$

where the U_i come from a countable basis for E . Since every open set is the countable union of sets like the above, each being a measurable box, the open sets are contained in

$$\mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$$

which implies $\mathcal{B}(E^m) \subseteq \mathcal{B}(E) \times \cdots \times \mathcal{B}(E)$ also. This proves the lemma.

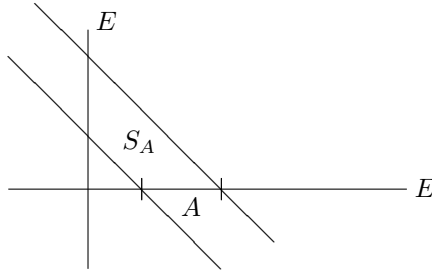
With this lemma, it is possible to define the convolution of two finite measures.

Definition 47.7.5 Let μ and ν be two finite measures on $\mathcal{B}(E)$, for E a separable Banach space. Then define a new measure, $\mu * \nu$ on $\mathcal{B}(E)$ as follows

$$\mu * \nu(A) \equiv \int_E \nu(A - x) d\mu(x).$$

This is well defined because of Lemma 47.7.4 which says that $x \rightarrow \nu(A - x)$ is Borel measurable.

Here is an interesting theorem about convolutions. However, first here is a little lemma. The following picture is descriptive of the set described in the following lemma.



Lemma 47.7.6 For A a Borel set in E , a separable Banach space, define

$$S_A \equiv \{(x, y) \in E \times E : x + y \in A\}$$

Then $S_A \in \mathcal{B}(E) \times \mathcal{B}(E)$, the σ algebra of product measurable sets, the smallest σ algebra which contains all the sets of the form $A \times B$ where A and B are Borel.

Proof: Let $\mathcal{K}$ denote the open sets in E . Then $\mathcal{K}$ is a π system. Let

$$\mathcal{G} \equiv \{A \in \sigma(\mathcal{K}) = \mathcal{B}(E) : S_A \in \mathcal{B}(E) \times \mathcal{B}(E)\}.$$

Then $\mathcal{K} \subseteq \mathcal{G}$ because if $U \in \mathcal{K}$ then S_U is an open set in $E \times E$ and all open sets are in $\mathcal{B}(E) \times \mathcal{B}(E)$ because a countable basis for the topology of $E \times E$ are sets of the form $B \times C$ where B and C come from a countable basis for E . Therefore, $\mathcal{K} \subseteq \mathcal{G}$. Now let $A \in \mathcal{G}$. For $(x, y) \in E \times E$, either $x + y \in A$ or $x + y \notin A$. Hence $E \times E = S_A \cup S_{A^C}$ which shows that if $A \in \mathcal{G}$ then so is A^C . Finally if $\{A_i\}$ is a sequence of disjoint sets of $\mathcal{G}$

$$S_{\cup_{i=1}^{\infty} A_i} = \cup_{i=1}^{\infty} S_{A_i}$$

and this shows that $\mathcal{G}$ is also closed with respect to countable unions of disjoint sets. Therefore, by the lemma on π systems, Lemma 9.11.3 on Page 275 it follows $\mathcal{G} = \sigma(\mathcal{K}) = \mathcal{B}(E)$. This proves the lemma.

Theorem 47.7.7 Let μ , ν , and λ be finite measures on $\mathcal{B}(E)$ for E a separable Banach space. Then

$$\mu * \nu = \nu * \mu \tag{47.7.12}$$

$$(\mu * \nu) * \lambda = \mu * (\nu * \lambda) \tag{47.7.13}$$

If μ is the distribution for an E valued random variable, X and if ν is the distribution for an E valued random variable, Y , and X and Y are independent, then $\mu * \nu$ is the distribution for the random variable, $X + Y$. Also the characteristic function of a convolution equals the product of the characteristic functions.

Proof: First consider 47.7.12. Letting $A \in \mathcal{B}(E)$, the following computation holds from Fubini's theorem and Lemma 47.7.6

$$\begin{aligned}\mu * \nu(A) &\equiv \int_E \nu(A - x) d\mu(x) = \int_E \int_E \mathcal{X}_{S_A}(x, y) d\nu(y) d\mu(x) \\ &= \int_E \int_E \mathcal{X}_{S_A}(x, y) d\mu(x) d\nu(y) = \nu * \mu(A).\end{aligned}$$

Next consider 47.7.13. Using 47.7.12 whenever convenient,

$$\begin{aligned}(\mu * \nu) * \lambda(A) &\equiv \int_E (\mu * \nu)(A - x) d\lambda(x) \\ &= \int_E \int_E \nu(A - x - y) d\mu(y) d\lambda(x)\end{aligned}$$

while

$$\begin{aligned}\mu * (\nu * \lambda)(A) &\equiv \int_E (\nu * \lambda)(A - y) d\mu(y) \\ &= \int_E \int_E \nu(A - y - x) d\lambda(x) d\mu(y) \\ &= \int_E \int_E \nu(A - y - x) d\mu(y) d\lambda(x).\end{aligned}$$

The necessary product measurability comes from Lemma 47.7.4.

Recall

$$(\mu * \nu)(A) \equiv \int_E \nu(A - x) d\mu(x).$$

Therefore, if s is a simple function, $s(x) = \sum_{k=1}^n c_k \mathcal{X}_{A_k}(x)$,

$$\begin{aligned}\int_E s d(\mu * \nu) &= \sum_{k=1}^n c_k \int_E \nu(A_k - x) d\mu(x) \\ &= \int_E \sum_{k=1}^n c_k \nu(A_k - x) d\mu(x) \\ &= \int_E \int_E s(x + y) d\nu(x) d\mu(y)\end{aligned}$$

Approximating with simple functions it follows that whenever f is bounded and measurable or nonnegative and measurable,

$$\int_E f d(\mu * \nu) = \int_E \int_E f(x + y) d\nu(y) d\mu(x) \quad (47.7.14)$$

Therefore, letting $Z = X + Y$, and λ the distribution of Z , it follows from independence of X and Y that for $t^* \in E'$,

$$\phi_\lambda(t^*) \equiv E\left(e^{it^*(Z)}\right) = E\left(e^{it^*(X+Y)}\right) = E\left(e^{it^*(X)}\right) E\left(e^{it^*(Y)}\right)$$

But also, it follows from 47.7.14

$$\begin{aligned}
 \phi_{(\mu * \nu)}(t^*) &= \int_E e^{it^*(z)} d(\mu * \nu)(z) \\
 &= \int_E \int_E e^{it^*(x+y)} d\nu(y) d\mu(x) \\
 &= \int_E \int_E e^{it^*(x)} e^{it^*(y)} d\nu(y) d\mu(x) \\
 &= \left(\int_E e^{it^*(y)} d\nu(y) \right) \left(\int_E e^{it^*(x)} d\mu(x) \right) \\
 &= E(e^{it^*(X)}) E(e^{it^*(Y)})
 \end{aligned}$$

Since $\phi_\lambda(t^*) = \phi_{(\mu * \nu)}(t^*)$, it follows $\lambda = \mu * \nu$.

Note the last part of this argument shows the characteristic function of a convolution equals the product of the characteristic functions. This proves the theorem.

47.8 The Convergence Of Sums Of Symmetric Random Variables

It turns out that when random variables have symmetric distributions, some remarkable things can be said about infinite sums of these random variables. Conditions are given here that enable one to conclude the convergence of the sequence of partial sums from the convergence of some subsequence of partial sums.

The following lemma is like an earlier result but does not depend on a function being bounded.

Definition 47.8.1 Let $\mathbf{X}$ be a random variable. $\mathcal{L}(\mathbf{X}) = \mu$ means $\lambda_{\mathbf{X}} = \mu$. This is called the law of $\mathbf{X}$. It is the same as saying the distribution measure of $\mathbf{X}$ is μ .

Lemma 47.8.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathbf{X} : \Omega \rightarrow E$ be a random variable, where E is a real separable Banach space. Also let $\mathcal{L}(\mathbf{X}) = \mu$, a probability measure defined on $\mathcal{B}(E)$, the Borel sets of E . Suppose $h : E \rightarrow \mathbb{R}$ is continuous and also suppose $h \circ \mathbf{X}$ is in $L^1(\Omega)$. Then

$$\int_{\Omega} (h \circ \mathbf{X}) dP = \int_E h(\mathbf{x}) d\mu.$$

Proof: Let $\{a_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathbb{R}$. Let $B_i^n \equiv B(a_i, \frac{1}{n}) \subseteq \mathbb{R}$ and define Borel sets, $A_i^n \subseteq E$ as follows:

$$A_1^n = h^{-1}(B_1^n), \quad A_{k+1}^n \equiv h^{-1}(B_{k+1}^n) \setminus (\cup_{i=1}^k A_i^n).$$

Thus $\{A_i^n\}_{i=1}^{\infty}$ are disjoint Borel sets, with $h(A_i^n) \subseteq B_i^n$. Also let b_i^n denote the endpoint of $B(a_i, \frac{1}{n})$ which is closer to 0.

$$h_i^n \equiv \begin{cases} b_i^n & \text{if } h^{-1}(B_i^n) \neq \emptyset \\ 0 & \text{if } h^{-1}(B_i^n) = \emptyset \end{cases}$$

Then define

$$h^n(\mathbf{x}) \equiv \sum_{i=1}^{\infty} h_i^n \mathcal{X}_{A_i^n}(\mathbf{x})$$

Thus $|h^n(\mathbf{x})| \leq |h(\mathbf{x})|$ for all $\mathbf{x} \in E$ and $|h^n(\mathbf{x}) - h(\mathbf{x})| \leq 1/n$ for all $\mathbf{x} \in E$. Then $h^n \circ \mathbf{X}$ is in $L^1(\Omega)$ and for all n ,

$$|h^n \circ \mathbf{X}(\omega)| \leq |h \circ \mathbf{X}(\omega)|.$$

Let

$$h_k^n(\mathbf{x}) \equiv \sum_{i=1}^k h_i^n \mathcal{X}_{A_i^n}(\mathbf{x}).$$

Then from the construction in which the $\{A_i^n\}_{i=1}^{\infty}$ are disjoint,

$$|h_k^n(\mathbf{X}(\omega))| \leq |h(\mathbf{X}(\omega))|, \quad |h_k^n(\mathbf{x})| \leq |h(\mathbf{x})|$$

and $|h_k^n(\mathbf{x})|$ is increasing in k .

$$\begin{aligned} \int_{\Omega} |h_k^n(\mathbf{X}(\omega))| dP &= \int_{\Omega} \sum_{i=1}^k |h_i^n| \mathcal{X}_{A_i^n}(\mathbf{X}(\omega)) dP \\ &= \int_{\Omega} \sum_{i=1}^k |h_i^n| \mathcal{X}_{\mathbf{X}^{-1}(A_i^n)}(\omega) dP \\ &= \sum_{i=1}^k |h_i^n| P(\mathbf{X}^{-1}(A_i^n)) \\ &= \sum_{i=1}^k |h_i^n| \mu(A_i^n) \\ &= \int_E |h_k^n(\mathbf{x})| d\mu. \end{aligned}$$

By the monotone convergence theorem, and letting $k \rightarrow \infty$,

$$\int_{\Omega} |h^n(\mathbf{X}(\omega))| dP = \int_E |h^n(\mathbf{x})| d\mu.$$

Now by the uniform convergence in the construction, you can let $n \rightarrow \infty$ and obtain

$$\int_{\Omega} |h(\mathbf{X}(\omega))| dP = \int_E |h(\mathbf{x})| d\mu.$$

Thus $h \in L^1(E, \mu)$. It is obviously Borel measurable, being the limit of a sequence of Borel measurable functions. Now similar reasoning to the above and using the dominated convergence theorem when necessary yields

$$\int_{\Omega} h^n(\mathbf{X}(\omega)) dP = \int_E h^n(\mathbf{x}) d\mu$$

Now another application of the dominated convergence theorem yields

$$\int_{\Omega} h(\mathbf{X}(\omega)) dP = \int_E h(\mathbf{x}) d\mu.$$

This proves the lemma.

First is a simple definition and lemma about random variables whose distribution is symmetric.

Definition 47.8.3 *Let X be a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ having values in a Banach space, E . Then it has a symmetric distribution if whenever A is a Borel set,*

$$P([X \in A]) = P([X \in -A])$$

In terms of the distribution,

$$\lambda_X = \lambda_{-X}.$$

It is good to observe that if X, Y are independent random variables defined on a probability space, $(\Omega, \mathcal{F}, P)$ such that each has symmetric distribution, then $X + Y$ also has symmetric distribution. Here is why. Let A be a Borel set in E . Then by Theorem 47.7.7 on Page 1335,

$$\begin{aligned} \lambda_{X+Y}(A) &= \int_E \lambda_X(A - z) d\lambda_Y(z) \\ &= \int_E \lambda_{-X}(A - z) d\lambda_{-Y}(z) \\ &= \lambda_{-(X+Y)}(A) = \lambda_{X+Y}(-A) \end{aligned}$$

By induction, it follows that if you have n independent random variables each having symmetric distribution, then their sum has symmetric distribution.

Here is a simple lemma about random variables having symmetric distributions. It will depend on Lemma 47.8.2 on Page 1337.

Lemma 47.8.4 *Let $\mathbf{X} \equiv (X_1, \dots, X_n)$ and Y be random variables defined on a probability space, $(\Omega, \mathcal{F}, P)$ such that $X_i, i = 1, 2, \dots, n$ and Y have values in E a separable Banach space. Thus $\mathbf{X}$ has values in E^n . Suppose also that $\{X_1, \dots, X_n, Y\}$ are independent and that Y has symmetric distribution. Then if $A \in \mathcal{B}(E^n)$, it follows*

$$\begin{aligned} &P\left([\mathbf{X} \in A] \cap \left[\left\|\sum_{i=1}^n X_i + Y\right\| < r\right]\right) \\ &= P\left([\mathbf{X} \in A] \cap \left[\left\|\sum_{i=1}^n X_i - Y\right\| < r\right]\right) \end{aligned}$$

You can also change the inequalities in the obvious way, $<$ to $\leq$, $>$ or $\geq$.

Proof: Denote by $\lambda_{\mathbf{X}}$ and λ_Y the distribution measures for $\mathbf{X}$ and Y respectively. Since the random variables are independent, the distribution for the random variable, $(\mathbf{X}, Y)$ mapping into E^{n+1} is $\lambda_{\mathbf{X}} \times \lambda_Y$ where this denotes product measure. Since the Banach space is separable, the Borel sets are contained in the product measurable sets. Then by symmetry of the distribution of Y

$$\begin{aligned}
 & P \left([\mathbf{X} \in A] \cap \left[\left\| \sum_{i=1}^n X_i + Y \right\| < r \right] \right) \\
 &= \int_{E^n \times E} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)} \left(\sum_{i=1}^n x_i + y \right) d(\lambda_{\mathbf{X}} \times \lambda_Y)(\mathbf{x}, y) \\
 &= \int_E \int_{E^n} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)} \left(\sum_{i=1}^n x_i + y \right) d\lambda_{\mathbf{X}} d\lambda_Y \\
 &= \int_E \int_{E^n} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)} \left(\sum_{i=1}^n x_i + y \right) d\lambda_{\mathbf{X}} d\lambda_{-Y} \\
 &= \int_{E^n \times E} \mathcal{X}_A(\mathbf{x}) \mathcal{X}_{B(0,r)} \left(\sum_{i=1}^n x_i + y \right) d(\lambda_{\mathbf{X}} \times \lambda_{-Y})(\mathbf{x}, y) \\
 &= P \left([\mathbf{X} \in A] \cap \left[\left\| \sum_{i=1}^n X_i + (-Y) \right\| < r \right] \right)
 \end{aligned}$$

This proves the lemma. Other cases are similar.

Now here is a really interesting lemma.

Lemma 47.8.5 *Let E be a real separable Banach space. Assume $\xi_1, \dots, \xi_N$ are independent random variables having values in E , a separable Banach space which have symmetric distributions. Also let $S_k = \sum_{i=1}^k \xi_i$. Then for any $r > 0$,*

$$P \left(\left[\sup_{k \leq N} \|S_k\| > r \right] \right) \leq 2P(\|S_N\| > r).$$

Proof: First of all,

$$\begin{aligned}
 & P \left(\left[\sup_{k \leq N} \|S_k\| > r \right] \right) \\
 &= P \left(\left[\sup_{k \leq N} \|S_k\| > r \text{ and } \|S_N\| > r \right] \right) \\
 &\quad + P \left(\left[\sup_{k \leq N-1} \|S_k\| > r \text{ and } \|S_N\| \leq r \right] \right) \\
 &\leq P(\|S_N\| > r) + P \left(\left[\sup_{k \leq N-1} \|S_k\| > r \text{ and } \|S_N\| \leq r \right] \right). \quad (47.8.15)
 \end{aligned}$$

I need to estimate the second of these terms. Let

$$A_1 \equiv [\|S_1\| > r], \dots, A_k \equiv [\|S_k\| > r, \|S_j\| \leq r \text{ for } j < k].$$

Thus A_k consists of those ω where $\|S_k(\omega)\| > r$ for the first time at k . Thus

$$\left[\sup_{k \leq N-1} \|S_k\| > r \text{ and } \|S_N\| \leq r \right] = \cup_{j=1}^{N-1} A_j \cap [\|S_N\| \leq r]$$

and the sets in the above union are disjoint. Consider $A_j \cap [\|S_N\| \leq r]$. For ω in this set,

$$\|S_j(\omega)\| > r, \|S_i(\omega)\| \leq r \text{ if } i < j.$$

Since $\|S_N(\omega)\| \leq r$ in this set, it follows

$$\|S_N(\omega)\| = \left\| S_j(\omega) + \sum_{i=j+1}^N \xi_i(\omega) \right\| \leq r$$

Thus

$$P(A_j \cap [\|S_N\| \leq r]) \quad (47.8.16)$$

$$= P\left(\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap \left[\left\| S_j + \sum_{i=j+1}^N \xi_i \right\| \leq r\right]\right) \quad (47.8.17)$$

Now $\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r]$ is of the form

$$[(\xi_1, \dots, \xi_j) \in A]$$

for some Borel set, A . Then letting $Y = \sum_{i=j+1}^N \xi_i$ in Lemma 47.8.4 and $X_i = \xi_i$, 47.8.17 equals

$$\begin{aligned} & P\left(\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap \left[\left\| S_j - \sum_{i=j+1}^N \xi_i \right\| \leq r\right]\right) \\ &= P\left(\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap [\|S_j - (S_N - S_j)\| \leq r]\right) \\ &= P\left(\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap [\|2S_j - S_N\| \leq r]\right) \end{aligned}$$

Now since $\|S_j(\omega)\| > r$,

$$\begin{aligned} [\|2S_j - S_N\| \leq r] &\subseteq [2\|S_j\| - \|S_N\| \leq r] \\ &\subseteq [2r - \|S_N\| < r] \\ &= [\|S_N\| > r] \end{aligned}$$

and so, referring to 47.8.16, this has shown

$$P(A_j \cap [\|S_N\| \leq r])$$

$$\begin{aligned}
&= P\left(\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap [\|2S_j - S_N\| \leq r]\right) \\
&\leq P\left(\cap_{i=1}^{j-1} [\|S_i\| \leq r] \cap [\|S_j\| > r] \cap [\|S_N\| > r]\right) \\
&= P(A_j \cap [\|S_N\| > r]).
\end{aligned}$$

It follows that

$$\begin{aligned}
P\left(\left[\sup_{k \leq N-1} \|S_k\| > r \text{ and } \|S_N\| \leq r\right]\right) &= \sum_{i=1}^{N-1} P(A_i \cap [\|S_N\| \leq r]) \\
&\leq \sum_{i=1}^{N-1} P(A_i \cap [\|S_N\| > r]) \\
&\leq P([\|S_N\| > r])
\end{aligned}$$

and using 47.8.15, this proves the lemma.

This interesting lemma will now be used to prove the following which concludes a sequence of partial sums converges given a subsequence of the sequence of partial sums converges.

Lemma 47.8.6 *Let $\{\zeta_k\}$ be a sequence of independent random variables having values in a separable real Banach space, E whose distributions are symmetric. Letting $S_k \equiv \sum_{i=1}^k \zeta_i$, suppose $\{S_{n_k}\}$ converges a.e. Also suppose that for every $m > n_k$,*

$$P([\|S_m - S_{n_k}\|_E > 2^{-k}]) < 2^{-k}. \quad (47.8.18)$$

Then in fact,

$$S_k(\omega) \rightarrow S(\omega) \text{ a.e. } \omega \quad (47.8.19)$$

and off a set of measure zero, the convergence of S_k to S is uniform.

Proof: Let $n_k \leq l \leq m$. Then by Lemma 47.8.5

$$P\left(\left[\sup_{n_k < l \leq m} \|S_l - S_{n_k}\| > 2^{-k}\right]\right) \leq 2P([\|S_m - S_{n_k}\| > 2^{-k}])$$

In using this lemma, you could renumber the ζ_i so that the sum

$$\sum_{j=n_k+1}^l \zeta_j$$

corresponds to

$$\sum_{j=1}^{l-n_k} \xi_j$$

where $\xi_j = \zeta_{j+n_k}$.

Then using 47.8.18,

$$P\left(\left[\sup_{n_k < l \leq m} \|S_l - S_{n_k}\| > 2^{-k}\right]\right) \leq 2P\left(\|S_m - S_{n_k}\| > 2^{-k}\right) < 2^{-(k-1)}$$

If $S_l(\omega)$ fails to converge then ω must be in infinitely many of the sets,

$$\left[\sup_{n_k < l} \|S_l - S_{n_k}\| > 2^{-k}\right]$$

each of which has measure no more than $2^{-(k-1)}$. Thus ω must be in a set of measure zero. This proves the lemma.

47.9 The Multivariate Normal Distribution

Definition 47.9.1 A random vector, $\mathbf{X}$, with values in $\mathbb{R}^p$ has a multivariate normal distribution written as $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ if for all Borel $E \subseteq \mathbb{R}^p$,

$$\lambda_{\mathbf{X}}(E) = \int_{\mathbb{R}^p} \chi_E(\mathbf{x}) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx$$

for μ a given vector and Σ a given positive definite symmetric matrix.

Theorem 47.9.2 For $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$, $\mathbf{m} = E(\mathbf{X})$ and

$$\Sigma = E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*).$$

Proof: Let R be an orthogonal transformation such that

$$R\Sigma R^* = D = \text{diag}(\sigma_1^2, \dots, \sigma_p^2).$$

Changing the variable by $\mathbf{x} - \mathbf{m} = R^* \mathbf{y}$,

$$\begin{aligned} E(\mathbf{X}) &\equiv \int_{\mathbb{R}^p} \mathbf{x} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx \left(\frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} \right) \\ &= \int_{\mathbb{R}^p} (R^* \mathbf{y} + \mathbf{m}) e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) \\ &= \mathbf{m} \int_{\mathbb{R}^p} e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) = \mathbf{m} \end{aligned}$$

by Fubini's theorem and the easy to establish formula

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy = 1.$$

Next let $M \equiv E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*)$. Thus, changing the variable as above by $\mathbf{x} - \mathbf{m} = R^* \mathbf{y}$

$$\begin{aligned} M &= \int_{\mathbb{R}^p} (\mathbf{x} - \mathbf{m})(\mathbf{x} - \mathbf{m})^* e^{-\frac{1}{2}(\mathbf{x} - \mathbf{m})^* \Sigma^{-1}(\mathbf{x} - \mathbf{m})} dx \left(\frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} \right) \\ &= R^* \int_{\mathbb{R}^p} \mathbf{y} \mathbf{y}^* e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) R \end{aligned}$$

Therefore,

$$(RMR^*)_{ij} = \int_{\mathbb{R}^p} y_i y_j e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right) = 0,$$

so; RMR^* is a diagonal matrix.

$$(RMR^*)_{ii} = \int_{\mathbb{R}^p} y_i^2 e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} dy \left(\frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \right).$$

Using Fubini's theorem and the easy to establish equations,

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-\frac{y^2}{2\sigma^2}} dy = 1, \quad \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} y^2 e^{-\frac{y^2}{2\sigma^2}} dy = \sigma^2,$$

it follows $(RMR^*)_{ii} = \sigma_i^2$. Hence $RMR^* = D$ and so $M = R^* DR = \Sigma$. This proves the theorem.

Theorem 47.9.3 Suppose $\mathbf{X}_1 \sim N_p(\mathbf{m}_1, \Sigma_1)$, $\mathbf{X}_2 \sim N_p(\mathbf{m}_2, \Sigma_2)$ and the two random vectors are independent. Then

$$\mathbf{X}_1 + \mathbf{X}_2 \sim N_p(\mathbf{m}_1 + \mathbf{m}_2, \Sigma_1 + \Sigma_2). \quad (47.9.20)$$

Also, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then $-\mathbf{X} \sim N_p(-\mathbf{m}, \Sigma)$. Furthermore, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then

$$E(e^{it \cdot \mathbf{X}}) = e^{it \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^* \Sigma \mathbf{t}} \quad (47.9.21)$$

Also if a is a constant and $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then $a\mathbf{X} \sim N_p(a\mathbf{m}, a^2\Sigma)$.

Proof: Consider $E(e^{it \cdot \mathbf{X}})$ for $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$.

$$E(e^{it \cdot \mathbf{X}}) = \frac{1}{(2\pi)^{p/2} (\det \Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{it \cdot \mathbf{x}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{m})^* \Sigma^{-1}(\mathbf{x} - \mathbf{m})} dx.$$

Let R be an orthogonal transformation such that

$$R\Sigma R^* = D = \text{diag}(\sigma_1^2, \dots, \sigma_p^2).$$

Then let $R(\mathbf{x} - \mathbf{m}) = \mathbf{y}$. Then

$$E(e^{i\mathbf{t} \cdot \mathbf{X}}) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \int_{\mathbb{R}^p} e^{i\mathbf{t} \cdot (R^* \mathbf{y} + \mathbf{m})} e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} d\mathbf{x}.$$

Therefore

$$E(e^{i\mathbf{t} \cdot \mathbf{X}}) = \frac{1}{(2\pi)^{p/2} \prod_{i=1}^p \sigma_i} \int_{\mathbb{R}^p} e^{i\mathbf{s} \cdot (\mathbf{y} + R\mathbf{m})} e^{-\frac{1}{2} \mathbf{y}^* D^{-1} \mathbf{y}} d\mathbf{y}$$

where $\mathbf{s} = R\mathbf{t}$. This equals

$$\begin{aligned} & e^{i\mathbf{t} \cdot \mathbf{m}} \prod_{i=1}^p \left(\int_{\mathbb{R}} e^{is_i y_i} e^{-\frac{1}{2\sigma_i^2} y_i^2} dy_i \right) \frac{1}{\sqrt{2\pi}\sigma_i} \\ &= e^{i\mathbf{t} \cdot \mathbf{m}} \prod_{i=1}^p \left(\int_{\mathbb{R}} e^{is_i \sigma_i u} e^{-\frac{1}{2} u^2} du \right) \frac{1}{\sqrt{2\pi}} \\ &= e^{i\mathbf{t} \cdot \mathbf{m}} \prod_{i=1}^p e^{-\frac{1}{2} s_i^2 \sigma_i^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2} (u - is_i \sigma_i)^2} du \\ &= e^{i\mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \sum_{i=1}^p s_i^2 \sigma_i^2} = e^{i\mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^* \Sigma \mathbf{t}} \end{aligned}$$

This proves 47.9.21.

Since $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent, $e^{i\mathbf{t} \cdot \mathbf{X}_1}$ and $e^{i\mathbf{t} \cdot \mathbf{X}_2}$ are also independent. Hence

$$E(e^{i\mathbf{t} \cdot \mathbf{X}_1 + \mathbf{X}_2}) = E(e^{i\mathbf{t} \cdot \mathbf{X}_1}) E(e^{i\mathbf{t} \cdot \mathbf{X}_2}).$$

Thus,

$$\begin{aligned} E(e^{i\mathbf{t} \cdot \mathbf{X}_1 + \mathbf{X}_2}) &= E(e^{i\mathbf{t} \cdot \mathbf{X}_1}) E(e^{i\mathbf{t} \cdot \mathbf{X}_2}) \\ &= e^{i\mathbf{t} \cdot \mathbf{m}_1} e^{-\frac{1}{2} \mathbf{t}^* \Sigma_1 \mathbf{t}} e^{i\mathbf{t} \cdot \mathbf{m}_2} e^{-\frac{1}{2} \mathbf{t}^* \Sigma_2 \mathbf{t}} \\ &= e^{i\mathbf{t} \cdot (\mathbf{m}_1 + \mathbf{m}_2)} e^{-\frac{1}{2} \mathbf{t}^* (\Sigma_1 + \Sigma_2) \mathbf{t}} \end{aligned}$$

which is the characteristic function of a random vector distributed as

$$N_p(\mathbf{m}_1 + \mathbf{m}_2, \Sigma_1 + \Sigma_2).$$

Now it follows that $\mathbf{X}_1 + \mathbf{X}_2 \sim N_p(\mathbf{m}_1 + \mathbf{m}_2, \Sigma_1 + \Sigma_2)$ by Theorem 47.2.4. This proves 47.9.20.

The assertion about $-\mathbf{X}$ is also easy to see because

$$\begin{aligned} E(e^{i\mathbf{t} \cdot (-\mathbf{X})}) &= E(e^{i(-\mathbf{t}) \cdot \mathbf{X}}) \\ &= \frac{1}{(2\pi)^{p/2} (\det \Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{i(-\mathbf{t}) \cdot \mathbf{x}} e^{-\frac{1}{2} (\mathbf{x} - \mathbf{m})^* \Sigma^{-1} (\mathbf{x} - \mathbf{m})} d\mathbf{x} \\ &= \frac{1}{(2\pi)^{p/2} (\det \Sigma)^{1/2}} \int_{\mathbb{R}^p} e^{i\mathbf{t} \cdot \mathbf{x}} e^{-\frac{1}{2} (\mathbf{x} + \mathbf{m})^* \Sigma^{-1} (\mathbf{x} + \mathbf{m})} d\mathbf{x} \end{aligned}$$

which is the characteristic function of a random variable which is $N(-\mathbf{m}, \Sigma)$. Theorem 47.2.4 again implies $-\mathbf{X} \sim N(-\mathbf{m}, \Sigma)$. Finally consider the last claim. You apply what is known about $\mathbf{X}$ with $\mathbf{t}$ replaced with $a\mathbf{t}$ and then massage things. This gives the characteristic function for $a\mathbf{X}$ is given by

$$E(\exp(it \cdot a\mathbf{X})) = \exp(it \cdot a\mathbf{m}) \exp\left(-\frac{1}{2} \mathbf{t}^* \Sigma a^2 \mathbf{t}\right)$$

which is the characteristic function of a normal random vector having mean $a\mathbf{m}$ and covariance $a^2\Sigma$. This proves the theorem.

Following [54] a random vector has a generalized normal distribution if its characteristic function is given as

$$e^{it \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^* \Sigma \mathbf{t}} \quad (47.9.22)$$

where Σ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $\mathbf{m}$ is scalar and so is Σ so the characteristic function of such a generalized normally distributed random variable is

$$e^{it\mu} e^{-\frac{1}{2} t^2 \sigma^2} \quad (47.9.23)$$

These generalized normal distributions do not require Σ to be invertible, only that the eigenvalues be nonnegative. In one dimension this would correspond the characteristic function of a dirac measure having point mass 1 at μ . In higher dimensions, it could be a mixture of such things with more familiar things. I won't try very hard to distinguish between generalized normal distributions and normal distributions in which the covariance matrix has all positive eigenvalues.

Here are some other interesting results about normal distributions found in [54]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense.

Theorem 47.9.4 *Let $\mathbf{X} = (X_1, \dots, X_p)$ where each X_i is a real valued random variable. Then $\mathbf{X}$ is normally distributed in the above generalized sense if and only if every linear combination, $\sum_{j=1}^p a_j X_j$ is normally distributed. In this case the mean of $\mathbf{X}$ is*

$$\mathbf{m} = (E(X_1), \dots, E(X_p))$$

and the covariance matrix for $\mathbf{X}$ is

$$\Sigma_{jk} = E((X_j - m_j)(X_k - m_k)^*).$$

Proof: Suppose first $\mathbf{X}$ is normally distributed. Then its characteristic function is of the form

$$\phi_{\mathbf{X}}(\mathbf{t}) = E(e^{it \cdot \mathbf{X}}) = e^{it \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{t}^* \Sigma \mathbf{t}}.$$

Then letting $\mathbf{a} = (a_1, \dots, a_p)$

$$E\left(e^{it \sum_{j=1}^p a_j X_j}\right) = E(e^{it \mathbf{a} \cdot \mathbf{X}}) = e^{it \mathbf{a} \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{a}^* \Sigma \mathbf{a} t^2}$$

which is the characteristic function of a normally distributed random variable with mean $\mathbf{a} \cdot \mathbf{m}$ and variance $\sigma^2 = \mathbf{a}^* \Sigma \mathbf{a}$. This proves half of the theorem.

Next suppose $\sum_{j=1}^p a_j X_j = \mathbf{a} \cdot \mathbf{X}$ is normally distributed with mean μ and variance σ^2 so that its characteristic function is given in 47.9.23. I will now relate μ and σ^2 to various quantities involving the X_j . Letting $m_j = E(X_j)$, $\mathbf{m} = (m_1, \dots, m_p)^*$

$$\begin{aligned} \mu &= \sum_{j=1}^p a_j E(X_j) = \sum_{j=1}^p a_j m_j, \quad \sigma^2 = E \left(\left(\sum_{j=1}^p a_j X_j - \sum_{j=1}^p a_j m_j \right)^2 \right) \\ &= E \left(\left(\sum_{j=1}^p a_j (X_j - m_j) \right)^2 \right) = \sum_{j,k} a_j a_k E((X_j - m_j)(X_k - m_k)) \end{aligned}$$

It follows the mean of the normally distributed random variable, $\mathbf{a} \cdot \mathbf{X}$ is

$$\mu = \sum_j a_j m_j = \mathbf{a} \cdot \mathbf{m}$$

and its variance is

$$\sigma^2 = \mathbf{a}^* E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*) \mathbf{a}$$

Therefore,

$$\begin{aligned} E(e^{it\mathbf{a} \cdot \mathbf{X}}) &= e^{it\mu} e^{-\frac{1}{2}t^2\sigma^2} \\ &= e^{it\mathbf{a} \cdot \mathbf{m}} e^{-\frac{1}{2}t^2\mathbf{a}^* E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*) \mathbf{a}}. \end{aligned}$$

Then letting $\mathbf{s} = t\mathbf{a}$ this shows

$$\begin{aligned} E(e^{i\mathbf{s} \cdot \mathbf{X}}) &= e^{i\mathbf{s} \cdot \mathbf{m}} e^{-\frac{1}{2}\mathbf{s}^* E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*) \mathbf{s}} \\ &= e^{i\mathbf{s} \cdot \mathbf{m}} e^{-\frac{1}{2}\mathbf{s}^* \Sigma \mathbf{s}} \end{aligned}$$

which is the characteristic function of a normally distributed random variable with $\mathbf{m}$ given above and Σ given by

$$\Sigma_{jk} = E((X_j - m_j)(X_k - m_k)).$$

This proves the theorem.

Corollary 47.9.5 *Let $\mathbf{X} = (X_1, \dots, X_p)$, $\mathbf{Y} = (Y_1, \dots, Y_p)$ where each X_i, Y_i is a real valued random variable. Suppose also that for every $\mathbf{a} \in \mathbb{R}^p$, $\mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both normally distributed with the same mean and variance. Then $\mathbf{X}$ and $\mathbf{Y}$ are both multivariate normal random vectors with the same mean and variance.*

Proof: In the Proof of Theorem 47.9.4 the proof implies that the characteristic functions of $\mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both of the form

$$e^{itm} e^{-\frac{1}{2}\sigma^2 t^2}.$$

Then as in the proof of that theorem, it must be the case that

$$m = \sum_{j=1}^p a_j m_j$$

where $E(X_i) = m_i = E(Y_i)$ and

$$\begin{aligned} \sigma^2 &= \mathbf{a}^* E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*) \mathbf{a} \\ &= \mathbf{a}^* E((\mathbf{Y} - \mathbf{m})(\mathbf{Y} - \mathbf{m})^*) \mathbf{a} \end{aligned}$$

and this last equation must hold for every $\mathbf{a}$. Therefore,

$$E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*) = E((\mathbf{Y} - \mathbf{m})(\mathbf{Y} - \mathbf{m})^*) \equiv \Sigma$$

and so the characteristic function of both $\mathbf{X}$ and $\mathbf{Y}$ is $e^{is \cdot \mathbf{m}} e^{-\frac{1}{2} \mathbf{s}^* \Sigma \mathbf{s}}$ as in the proof of Theorem 47.9.4. This proves the corollary.

Theorem 47.9.6 Suppose $\mathbf{X} = (X_1, \dots, X_p)$ is normally distributed with mean $\mathbf{m}$ and covariance Σ . Then if X_1 is uncorrelated with any of the X_i , meaning

$$E((X_1 - m_1)(X_j - m_j)) = 0 \text{ for } j > 1,$$

then X_1 and $(X_2, \dots, X_p)$ are both normally distributed and the two random vectors are independent. Here $m_j \equiv E(X_j)$. More generally, if the covariance matrix is a diagonal matrix, the random variables, $\{X_1, \dots, X_p\}$ are linearly independent.

Proof: From Theorem 47.9.2

$$\Sigma = E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*).$$

Then by assumption,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \mathbf{0} \\ \mathbf{0} & \Sigma_{p-1} \end{pmatrix}. \quad (47.9.24)$$

I need to verify that if $E \in \mathcal{H}_{X_1}(\sigma(X_1))$ and $F \in \mathcal{H}_{(X_2, \dots, X_p)}(\sigma(X_2, \dots, X_p))$, then

$$P(E \cap F) = P(E)P(F).$$

Let $E = X_1^{-1}(A)$ and

$$F = (X_2, \dots, X_p)^{-1}(B)$$

where A and B are Borel sets in $\mathbb{R}$ and $\mathbb{R}^{p-1}$ respectively. Thus I need to verify that

$$\begin{aligned} P([(X_1, (X_2, \dots, X_p)) \in (A, B)]) &= \\ \mu_{(X_1, (X_2, \dots, X_p))}(A \times B) &= \mu_{X_1}(A) \mu_{(X_2, \dots, X_p)}(B). \end{aligned} \quad (47.9.25)$$

Using 47.9.24, Fubini's theorem, and definitions,

$$\mu_{(X_1, (X_2, \dots, X_p))}(A \times B) =$$

$$\begin{aligned}
& \int_{\mathbb{R}^p} \mathcal{X}_{A \times B}(\mathbf{x}) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{\frac{-1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})} dx \\
&= \int_{\mathbb{R}} \mathcal{X}_A(x_1) \int_{\mathbb{R}^{p-1}} \mathcal{X}_B(X_2, \dots, X_p) \cdot \\
&\quad \frac{1}{(2\pi)^{(p-1)/2} \sqrt{2\pi} (\sigma_1^2)^{1/2} \det(\Sigma_{p-1})^{1/2}} e^{\frac{-(x_1-m_1)^2}{2\sigma_1^2}} \cdot \\
&\quad e^{\frac{-1}{2}(\mathbf{x}'-\mathbf{m}')^* \Sigma_{p-1}^{-1}(\mathbf{x}'-\mathbf{m}')} dx' dx_1
\end{aligned}$$

where $\mathbf{x}' = (x_2, \dots, x_p)$ and $\mathbf{m}' = (m_2, \dots, m_p)$. Now this equals

$$\int_{\mathbb{R}} \mathcal{X}_A(x_1) \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{-(x_1-m_1)^2}{2\sigma_1^2}} \int_B \frac{1}{(2\pi)^{(p-1)/2} \det(\Sigma_{p-1})^{1/2}} \cdot (47.9.26)$$

$$e^{\frac{-1}{2}(\mathbf{x}'-\mathbf{m}')^* \Sigma_{p-1}^{-1}(\mathbf{x}'-\mathbf{m}')} dx' dx_1. \quad (47.9.27)$$

In case $B = \mathbb{R}^{p-1}$, the inside integral equals 1 and

$$\begin{aligned}
\lambda_{X_1}(A) &= \lambda_{(X_1, (X_2, \dots, X_p))}(A \times \mathbb{R}^{p-1}) \\
&= \int_{\mathbb{R}} \mathcal{X}_A(x_1) \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\frac{-(x_1-m_1)^2}{2\sigma_1^2}} dx_1
\end{aligned}$$

which shows X_1 is normally distributed as claimed. Similarly, letting $A = \mathbb{R}$,

$$\begin{aligned}
& \lambda_{(X_2, \dots, X_p)}(B) \\
&= \lambda_{(X_1, (X_2, \dots, X_p))}(\mathbb{R} \times B) \\
&= \int_B \frac{1}{(2\pi)^{(p-1)/2} \det(\Sigma_{p-1})^{1/2}} e^{\frac{-1}{2}(\mathbf{x}'-\mathbf{m}')^* \Sigma_{p-1}^{-1}(\mathbf{x}'-\mathbf{m}')} dx'
\end{aligned}$$

and $(X_2, \dots, X_p)$ is also normally distributed with mean $\mathbf{m}'$ and covariance Σ_{p-1} . Now from 47.9.26, 47.9.25 follows. In case the covariance matrix is diagonal, the above reasoning extends in an obvious way to prove the random variables, $\{X_1, \dots, X_p\}$ are independent.

However, another way to prove this is to use Proposition 47.4.1 on Page 1323 and consider the characteristic function. Let $E(X_j) = m_j$ and

$$P = \sum_{j=1}^p t_j X_j.$$

Then since $\mathbf{X}$ is normally distributed and the covariance is a diagonal,

$$D \equiv \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_p^2 \end{pmatrix}$$

,

$$\begin{aligned}
 E(e^{iP}) &= E(e^{i\mathbf{t} \cdot (\mathbf{X} - \mathbf{m})}) = e^{i\mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^* \Sigma \mathbf{t}} \\
 &= \exp\left(\sum_{j=1}^p it_j m_j - \frac{1}{2} t_j^2 \sigma_j^2\right) \\
 &= \prod_{j=1}^p \exp\left(it_j m_j - \frac{1}{2} t_j^2 \sigma_j^2\right)
 \end{aligned} \tag{47.9.28}$$

Also,

$$\begin{aligned}
 E(e^{it_j X_j}) &= E\left(\exp\left(it_j X_j + \sum_{k \neq j} i0 X_k\right)\right) \\
 &= \exp\left(it_j m_j - \frac{1}{2} t_j^2 \sigma_j^2\right)
 \end{aligned}$$

With 47.9.28, this shows

$$E(e^{iP}) = \prod_{j=1}^p E(e^{it_j X_j})$$

which shows by Proposition 47.4.1 that the random variables,

$$\{X_1, \dots, X_p\}$$

are independent. This proves the theorem.

47.10 Use Of Characteristic Functions To Find Moments

Let X be a random variable with characteristic function

$$\phi_X(t) \equiv E(\exp(itX))$$

Then this can be used to find moments of the random variable assuming they exist. The k^{th} moment is defined as

$$E(X^k).$$

This can be done by using the dominated convergence theorem to differentiate the characteristic function with respect to t and then plugging in $t = 0$. For example,

$$\phi'_X(t) = E(iX \exp(itX))$$

and now plugging in $t = 0$ you get $iE(X)$. Doing another differentiation you obtain

$$\phi''_X(t) = E(-X^2 \exp(itX))$$

and plugging in $t = 0$ you get $-E(X^2)$ and so forth.

An important case is where X is normally distributed with mean 0 and variance σ^2 . In this case, as shown above, the characteristic function is

$$e^{-\frac{1}{2}t^2\sigma^2}$$

Also all moments exist when X is normally distributed. So what are these moments?

$$D_t \left(e^{-\frac{1}{2}t^2\sigma^2} \right) = -t\sigma^2 e^{-\frac{1}{2}t^2\sigma^2}$$

and plugging in $t = 0$ you find the mean equals 0 as expected.

$$D_t \left(-t\sigma^2 e^{-\frac{1}{2}t^2\sigma^2} \right) = -\sigma^2 e^{-\frac{1}{2}t^2\sigma^2} + t^2\sigma^4 e^{-\frac{1}{2}t^2\sigma^2}$$

and plugging in $t = 0$ you find the second moment is σ^2 . Then do it again.

$$D_t \left(-\sigma^2 e^{-\frac{1}{2}t^2\sigma^2} + t^2\sigma^4 e^{-\frac{1}{2}t^2\sigma^2} \right) = 3\sigma^4 t e^{-\frac{1}{2}t^2\sigma^2} - t^3\sigma^6 e^{-\frac{1}{2}t^2\sigma^2}$$

Then $E(X^3) = 0$.

$$\begin{aligned} D_t \left(3\sigma^4 t e^{-\frac{1}{2}t^2\sigma^2} - t^3\sigma^6 e^{-\frac{1}{2}t^2\sigma^2} \right) \\ = 3\sigma^4 e^{-\frac{1}{2}t^2\sigma^2} - 6\sigma^6 t^2 e^{-\frac{1}{2}t^2\sigma^2} + t^4\sigma^8 e^{-\frac{1}{2}t^2\sigma^2} \end{aligned}$$

and so $E(X^4) = 3\sigma^4$. By now you can see the pattern. If you continue this way, you find the odd moments are all 0 and

$$E(X^{2m}) = C_m (\sigma^2)^m. \quad (47.10.29)$$

This is an important observation.

47.11 The Central Limit Theorem

The central limit theorem is one of the most marvelous theorems in mathematics. It can be proved through the use of characteristic functions. Recall for $\mathbf{x} \in \mathbb{R}^p$,

$$\|\mathbf{x}\|_\infty \equiv \max \{|x_j|, j = 1, \dots, p\}.$$

Also recall the definition of the distribution function for a random vector, $\mathbf{X}$.

$$F_{\mathbf{X}}(\mathbf{x}) \equiv P(X_j \leq x_j, j = 1, \dots, p).$$

Definition 47.11.1 Let $\{\mathbf{X}_n\}$ be random vectors with values in $\mathbb{R}^p$. Then $\{\lambda_{\mathbf{X}_n}\}_{n=1}^\infty$ is called “tight” if for all $\varepsilon > 0$ there exists a compact set, K_ε such that

$$\lambda_{\mathbf{X}_n}([\mathbf{x} \notin K_\varepsilon]) < \varepsilon$$

for all $\lambda_{\mathbf{X}_n}$. Similarly, if $\{\mu_n\}$ is a sequence of probability measures defined on the Borel sets of $\mathbb{R}^p$, then this sequence is “tight” if for each $\varepsilon > 0$ there exists a compact set, K_ε such that

$$\mu_n([\mathbf{x} \notin K_\varepsilon]) < \varepsilon$$

for all μ_n .

Lemma 47.11.2 If $\{\mathbf{X}_n\}$ is a sequence of random vectors with values in $\mathbb{R}^p$ such that

$$\lim_{n \rightarrow \infty} \phi_{\mathbf{X}_n}(\mathbf{t}) = \psi(\mathbf{t})$$

for all $\mathbf{t}$, where $\psi(\mathbf{0}) = 1$ and ψ is continuous at $\mathbf{0}$, then $\{\lambda_{\mathbf{X}_n}\}_{n=1}^\infty$ is tight.

Proof: Let $\mathbf{e}_j$ be the j^{th} standard unit basis vector.

$$\begin{aligned} & \left| \frac{1}{u} \int_{-u}^u (1 - \phi_{\mathbf{X}_n}(t\mathbf{e}_j)) dt \right| \\ &= \left| \frac{1}{u} \int_{-u}^u \left(1 - \int_{\mathbb{R}^p} e^{itx_j} d\lambda_{\mathbf{X}_n} \right) dt \right| \\ &= \left| \frac{1}{u} \int_{-u}^u \left(\int_{\mathbb{R}^p} (1 - e^{itx_j}) d\lambda_{\mathbf{X}_n} \right) dt \right| \\ &= \left| \int_{\mathbb{R}^p} \frac{1}{u} \int_{-u}^u (1 - e^{itx_j}) dt d\lambda_{\mathbf{X}_n}(x) \right| \\ &= \left| 2 \int_{\mathbb{R}^p} \left(1 - \frac{\sin(ux_j)}{ux_j} \right) d\lambda_{\mathbf{X}_n}(x) \right| \\ &\geq 2 \int_{[|x_j| \geq \frac{2}{u}]} \left(1 - \frac{1}{|ux_j|} \right) d\lambda_{\mathbf{X}_n}(x) \\ &\geq 2 \int_{[|x_j| \geq \frac{2}{u}]} \left(1 - \frac{1}{|u|(2/u)} \right) d\lambda_{\mathbf{X}_n}(x) \\ &= \int_{[|x_j| \geq \frac{2}{u}]} 1 d\lambda_{\mathbf{X}_n}(x) \\ &= \lambda_{\mathbf{X}_n} \left(\left[\mathbf{x} : |x_j| \geq \frac{2}{u} \right] \right). \end{aligned}$$

If $\varepsilon > 0$ is given, there exists $r > 0$ such that if $u \leq r$,

$$\frac{1}{u} \int_{-u}^u (1 - \psi(t\mathbf{e}_j)) dt < \varepsilon/p$$

for all $j = 1, \dots, p$ and so, by the dominated convergence theorem, the same is true with $\phi_{\mathbf{X}_n}$ in place of ψ provided n is large enough, say $n \geq N(u)$. Thus, if $u \leq r$, and $n \geq N(u)$,

$$\lambda_{\mathbf{X}_n} \left(\left[\mathbf{x} : |x_j| \geq \frac{2}{u} \right] \right) < \varepsilon/p$$

for all $j \in \{1, \dots, p\}$. It follows that for $u \leq r$ and $n \geq N(u)$,

$$\lambda_{\mathbf{X}_n} \left(\left[\mathbf{x} : \|\mathbf{x}\|_\infty \geq \frac{2}{u} \right] \right) < \varepsilon.$$

because

$$\left[\mathbf{x} : \|\mathbf{x}\|_\infty \geq \frac{2}{u} \right] \subseteq \bigcup_{j=1}^p \left[\mathbf{x} : |x_j| \geq \frac{2}{u} \right]$$

This proves the lemma because there are only finitely many measures, $\lambda_{\mathbf{X}_n}$ for $n < N(u)$ and the compact set can be enlarged finitely many times to obtain a single compact set, K_ε such that for all n , $\lambda_{\mathbf{X}_n}([\mathbf{x} \notin K_\varepsilon]) < \varepsilon$. This proves the lemma.

Lemma 47.11.3 *If $\phi_{\mathbf{X}_n}(\mathbf{t}) \rightarrow \phi_{\mathbf{X}}(\mathbf{t})$ for all $\mathbf{t}$, then whenever $\psi \in \mathfrak{S}$,*

$$\lambda_{\mathbf{X}_n}(\psi) \equiv \int_{\mathbb{R}^p} \psi(\mathbf{y}) d\lambda_{\mathbf{X}_n}(\mathbf{y}) \rightarrow \int_{\mathbb{R}^p} \psi(\mathbf{y}) d\lambda_{\mathbf{X}}(\mathbf{y}) \equiv \lambda_{\mathbf{X}}(\psi)$$

as $n \rightarrow \infty$.

Proof: Recall that if $\mathbf{X}$ is any random vector, its characteristic function is given by

$$\phi_{\mathbf{X}}(\mathbf{y}) \equiv \int_{\mathbb{R}^p} e^{i\mathbf{y} \cdot \mathbf{x}} d\lambda_{\mathbf{X}}(\mathbf{x}).$$

Also remember the inverse Fourier transform. Letting $\psi \in \mathfrak{S}$, the Schwartz class,

$$\begin{aligned} F^{-1}(\lambda_{\mathbf{X}})(\psi) &\equiv \lambda_{\mathbf{X}}(F^{-1}\psi) \equiv \int_{\mathbb{R}^p} F^{-1}\psi d\lambda_{\mathbf{X}} \\ &= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \int_{\mathbb{R}^p} e^{i\mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x} d\lambda_{\mathbf{X}}(\mathbf{y}) \\ &= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(\mathbf{x}) \int_{\mathbb{R}^p} e^{i\mathbf{y} \cdot \mathbf{x}} d\lambda_{\mathbf{X}}(\mathbf{y}) d\mathbf{x} \\ &= \frac{1}{(2\pi)^{p/2}} \int_{\mathbb{R}^p} \psi(\mathbf{x}) \phi_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \end{aligned}$$

and so, considered as elements of $\mathfrak{S}^*$,

$$F^{-1}(\lambda_{\mathbf{X}}) = \phi_{\mathbf{X}}(\cdot) (2\pi)^{-(p/2)} \in L^\infty.$$

By the dominated convergence theorem

$$\begin{aligned} (2\pi)^{p/2} F^{-1}(\lambda_{\mathbf{X}_n})(\psi) &\equiv \int_{\mathbb{R}^p} \phi_{\mathbf{X}_n}(\mathbf{t}) \psi(\mathbf{t}) d\mathbf{t} \\ &\rightarrow \int_{\mathbb{R}^p} \phi_{\mathbf{X}}(\mathbf{t}) \psi(\mathbf{t}) d\mathbf{t} \\ &= (2\pi)^{p/2} F^{-1}(\lambda_{\mathbf{X}})(\psi) \end{aligned}$$

whenever $\psi \in \mathfrak{S}$. Thus

$$\begin{aligned}\lambda_{\mathbf{X}_n}(\psi) &= FF^{-1}\lambda_{\mathbf{X}_n}(\psi) \equiv F^{-1}\lambda_{\mathbf{X}_n}(F\psi) \rightarrow F^{-1}\lambda_{\mathbf{X}}(F\psi) \\ &\equiv FF^{-1}\lambda_{\mathbf{X}}(\psi) = \lambda_{\mathbf{X}}(\psi).\end{aligned}$$

This proves the lemma.

Lemma 47.11.4 *If $\phi_{\mathbf{X}_n}(\mathbf{t}) \rightarrow \phi_{\mathbf{X}}(\mathbf{t})$, then if ψ is any bounded uniformly continuous function,*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} = \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}}.$$

Proof: Let $\varepsilon > 0$ be given, let ψ be a bounded function in $C^\infty(\mathbb{R}^p)$. Now let $\eta \in C_c^\infty(Q_r)$ where $Q_r \equiv [-r, r]^p$ satisfy the additional requirement that $\eta = 1$ on $Q_{r/2}$ and $\eta(\mathbf{x}) \in [0, 1]$ for all $\mathbf{x}$. By Lemma 47.11.2 the set, $\{\lambda_{\mathbf{X}_n}\}_{n=1}^\infty$, is tight and so if $\varepsilon > 0$ is given, there exists r sufficiently large such that for all n ,

$$\int_{[\mathbf{x} \notin Q_{r/2}]} |1 - \eta| |\psi| d\lambda_{\mathbf{X}_n} < \frac{\varepsilon}{3},$$

and

$$\int_{[\mathbf{x} \notin Q_{r/2}]} |1 - \eta| |\psi| d\lambda_{\mathbf{X}} < \frac{\varepsilon}{3}.$$

Thus,

$$\begin{aligned}\left| \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}} \right| &\leq \left| \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi \eta d\lambda_{\mathbf{X}_n} \right| + \\ &\quad \left| \int_{\mathbb{R}^p} \psi \eta d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi \eta d\lambda_{\mathbf{X}} \right| + \left| \int_{\mathbb{R}^p} \psi \eta d\lambda_{\mathbf{X}} - \int_{\mathbb{R}^p} \psi d\lambda_{\mathbf{X}} \right| \\ &\leq \frac{2\varepsilon}{3} + \left| \int_{\mathbb{R}^p} \psi \eta d\lambda_{\mathbf{X}_n} - \int_{\mathbb{R}^p} \psi \eta d\lambda_{\mathbf{X}} \right| < \varepsilon\end{aligned}$$

whenever n is large enough by Lemma 47.11.3 because $\psi\eta \in \mathfrak{S}$. This establishes the conclusion of the lemma in the case where ψ is also infinitely differentiable. To consider the general case, let ψ only be uniformly continuous and let $\psi_k = \psi * \phi_k$ where ϕ_k is a mollifier whose support is in $(-(1/k), (1/k))^p$. Then ψ_k converges uniformly to ψ and so the desired conclusion follows for ψ after a routine estimate.

Definition 47.11.5 *Let μ be a Radon measure on $\mathbb{R}^p$. A Borel set, A , is a μ continuity set if $\mu(\partial A) = 0$ where $\partial A \equiv \overline{A} \setminus \text{interior}(A)$.*

The main result is the following continuity theorem. More can be said about the equivalence of various criteria [11].

Theorem 47.11.6 *If $\phi_{\mathbf{X}_n}(\mathbf{t}) \rightarrow \phi_{\mathbf{X}}(\mathbf{t})$ then $\lambda_{\mathbf{X}_n}(A) \rightarrow \lambda_{\mathbf{X}}(A)$ whenever A is a $\lambda_{\mathbf{X}}$ continuity set.*

Proof: First suppose K is a closed set and let

$$\psi_k(\mathbf{x}) \equiv (1 - k \operatorname{dist}(\mathbf{x}, K))^+.$$

Thus, since K is closed $\lim_{k \rightarrow \infty} \psi_k(\mathbf{x}) = \chi_K(\mathbf{x})$. Choose k large enough that

$$\int_{\mathbb{R}^p} \psi_k d\lambda_{\mathbf{X}} \leq \lambda_{\mathbf{X}}(K) + \varepsilon.$$

Then by Lemma 47.11.4, applied to the bounded uniformly continuous function ψ_k ,

$$\limsup_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(K) \leq \limsup_{n \rightarrow \infty} \int \psi_k d\lambda_{\mathbf{X}_n} = \int \psi_k d\lambda_{\mathbf{X}} \leq \lambda_{\mathbf{X}}(K) + \varepsilon.$$

Since ε is arbitrary, this shows

$$\limsup_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(K) \leq \lambda_{\mathbf{X}}(K)$$

for all K closed.

Next suppose V is open and let

$$\psi_k(\mathbf{x}) = 1 - \left(1 - k \operatorname{dist}(\mathbf{x}, V^C)\right)^+.$$

Thus $\psi_k(\mathbf{x}) \in [0, 1]$, $\psi_k = 1$ if $\operatorname{dist}(\mathbf{x}, V^C) \geq 1/k$, and $\psi_k = 0$ on V^C . Since V is open, it follows

$$\lim_{k \rightarrow \infty} \psi_k(\mathbf{x}) = \chi_V(\mathbf{x}).$$

Choose k large enough that

$$\int \psi_k d\lambda_{\mathbf{X}} \geq \lambda_{\mathbf{X}}(V) - \varepsilon.$$

Then by Lemma 47.11.4,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(V) &\geq \liminf_{n \rightarrow \infty} \int \psi_k(\mathbf{x}) d\lambda_{\mathbf{X}_n} = \\ &= \int \psi_k(\mathbf{x}) d\lambda_{\mathbf{X}} \geq \lambda_{\mathbf{X}}(V) - \varepsilon \end{aligned}$$

and since ε is arbitrary,

$$\liminf_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(V) \geq \lambda_{\mathbf{X}}(V).$$

Now let $\lambda_{\mathbf{X}}(\partial A) = 0$ for A a Borel set.

$$\begin{aligned} \lambda_{\mathbf{X}}(\operatorname{interior}(A)) &\leq \liminf_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(\operatorname{interior}(A)) \leq \liminf_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(A) \leq \\ \limsup_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(A) &\leq \limsup_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(\overline{A}) \leq \lambda_{\mathbf{X}}(\overline{A}). \end{aligned}$$

But $\lambda_{\mathbf{X}}(\text{interior}(A)) = \lambda_{\mathbf{X}}(\overline{A})$ by assumption and so $\lim_{n \rightarrow \infty} \lambda_{\mathbf{X}_n}(A) = \lambda_{\mathbf{X}}(A)$ as claimed. This proves the theorem.

As an application of this theorem the following is a version of the central limit theorem in the situation in which the limit distribution is multivariate normal. It concerns a sequence of random vectors, $\{\mathbf{X}_k\}_{k=1}^{\infty}$, which are identically distributed, have finite mean $\mathbf{m}$, and satisfy

$$E(|\mathbf{X}_k|^2) < \infty. \quad (47.11.30)$$

Theorem 47.11.7 *Let $\{\mathbf{X}_k\}_{k=1}^{\infty}$ be random vectors satisfying 47.11.30, which are independent and identically distributed with mean $\mathbf{m}$ and positive definite covariance $\Sigma \equiv E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*)$. Let*

$$\mathbf{Z}_n \equiv \sum_{j=1}^n \frac{\mathbf{X}_j - \mathbf{m}}{\sqrt{n}}. \quad (47.11.31)$$

Then for $\mathbf{Z} \sim N_p(\mathbf{0}, \Sigma)$,

$$\lim_{n \rightarrow \infty} F_{\mathbf{Z}_n}(\mathbf{x}) = F_{\mathbf{Z}}(\mathbf{x}) \quad (47.11.32)$$

for all $\mathbf{x}$.

Proof: The characteristic function of $\mathbf{Z}_n$ is given by

$$\phi_{\mathbf{Z}_n}(\mathbf{t}) = E\left(e^{i\mathbf{t} \cdot \sum_{j=1}^n \frac{\mathbf{X}_j - \mathbf{m}}{\sqrt{n}}}\right) = \prod_{j=1}^n E\left(e^{i\mathbf{t} \cdot \left(\frac{\mathbf{X}_j - \mathbf{m}}{\sqrt{n}}\right)}\right).$$

By Taylor's theorem applied to real and imaginary parts of e^{ix} , it follows

$$e^{ix} = 1 + ix - f(x) \frac{x^2}{2}$$

where $|f(x)| < 2$ and

$$\lim_{x \rightarrow 0} f(x) = 1.$$

Denoting $\mathbf{X}_j$ as $\mathbf{X}$, this implies

$$e^{i\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}}\right)} = 1 + i\mathbf{t} \cdot \frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} - f\left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}}\right)\right) \frac{(\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2}{2n}$$

Thus

$$\begin{aligned} e^{i\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}}\right)} &= 1 + i\mathbf{t} \cdot \frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} - \frac{(\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2}{2n} \\ &+ \left(1 - f\left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}}\right)\right)\right) \frac{(\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2}{2n}. \end{aligned}$$

Thus

$$\begin{aligned}
 \phi_{\mathbf{Z}_n}(\mathbf{t}) &= \prod_{j=1}^n \left[1 - E \left(\frac{(\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2}{2n} \right) \right. \\
 &\quad \left. + E \left(\left(1 - f \left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} \right) \right) \right) \frac{(\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2}{2n} \right) \right] \\
 &= \prod_{j=1}^n \left[1 - \frac{1}{2n} \mathbf{t}^* \Sigma \mathbf{t} + \right. \\
 &\quad \left. \frac{1}{2n} E \left(\left(1 - f \left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} \right) \right) \right) (\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2 \right) \right]. \tag{47.11.33}
 \end{aligned}$$

(Note $(\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2 = \mathbf{t}^* (\mathbf{X} - \mathbf{m}) (\mathbf{X} - \mathbf{m})^* \mathbf{t}$.) Now here is a simple inequality for complex numbers whose moduli are no larger than one. I will give a proof of this at the end. It follows easily by induction.

$$|z_1 \cdots z_n - w_1 \cdots w_n| \leq \sum_{k=1}^n |z_k - w_k|. \tag{47.11.34}$$

Also for each $\mathbf{t}$, and all n large enough,

$$\left| \frac{1}{2n} E \left(\left(1 - f \left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} \right) \right) \right) (\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2 \right) \right| < 1.$$

Applying 47.11.34 to 47.11.33,

$$\begin{aligned}
 \phi_{\mathbf{Z}_n}(\mathbf{t}) &= \left(\prod_{j=1}^n \left(1 - \frac{1}{2n} \mathbf{t}^* \Sigma \mathbf{t} \right) \right) + e_n \\
 &= \left(1 - \frac{1}{2n} \mathbf{t}^* \Sigma \mathbf{t} \right)^n + e_n
 \end{aligned}$$

where

$$\begin{aligned}
 |e_n| &\leq \sum_{j=1}^n \left| \frac{1}{2n} E \left(\left(1 - f \left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} \right) \right) \right) (\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2 \right) \right| \\
 &= \frac{1}{2} \left| E \left(\left(1 - f \left(\mathbf{t} \cdot \left(\frac{\mathbf{X} - \mathbf{m}}{\sqrt{n}} \right) \right) \right) (\mathbf{t} \cdot (\mathbf{X} - \mathbf{m}))^2 \right) \right|
 \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$ by the Dominated Convergence theorem. Therefore,

$$\lim_{n \rightarrow \infty} \left| \phi_{\mathbf{Z}_n}(\mathbf{t}) - \left(1 - \frac{\mathbf{t}^* \Sigma \mathbf{t}}{2n} \right)^n \right| = 0$$

and so

$$\lim_{n \rightarrow \infty} \phi_{\mathbf{Z}_n}(\mathbf{t}) = e^{-\frac{1}{2} \mathbf{t}^* \Sigma \mathbf{t}} = \phi_{\mathbf{Z}}(\mathbf{t})$$

where $\mathbf{Z} \sim N_p(\mathbf{0}, \Sigma)$. Therefore, $F_{\mathbf{Z}_n}(\mathbf{x}) \rightarrow F_{\mathbf{Z}}(\mathbf{x})$ for all $\mathbf{x}$ because $R_{\mathbf{x}} \equiv \prod_{k=1}^p (-\infty, x_k]$ is a set of $\lambda_{\mathbf{Z}}$ continuity due to the assumption that $\lambda_{\mathbf{Z}} \ll m_p$ which is implied by $\mathbf{Z} \sim N_p(\mathbf{0}, \Sigma)$. This proves the theorem.

Here is the proof of the little inequality used above. The inequality is obviously true if $n = 1$. Assume it is true for n . Then since all the numbers have absolute value no larger than one,

$$\begin{aligned} \left| \prod_{i=1}^{n+1} z_i - \prod_{i=1}^{n+1} w_i \right| &\leq \left| \prod_{i=1}^{n+1} z_i - z_{n+1} \prod_{i=1}^n w_i \right| \\ &\quad + \left| z_{n+1} \prod_{i=1}^n w_i - \prod_{i=1}^{n+1} w_i \right| \\ &\leq \left| \prod_{i=1}^n z_i - \prod_{i=1}^n w_i \right| + |z_{n+1} - w_{n+1}| \\ &\leq \sum_{k=1}^{n+1} |z_k - w_k| \end{aligned}$$

by induction.

Suppose $\mathbf{X}$ is a random vector with covariance Σ and mean $\mathbf{m}$, and suppose also that Σ^{-1} exists. Consider $\Sigma^{-(1/2)}(\mathbf{X} - \mathbf{m}) \equiv \mathbf{Y}$. Then $E(\mathbf{Y}) = 0$ and

$$\begin{aligned} E(\mathbf{Y}\mathbf{Y}^*) &= E\left(\Sigma^{-(1/2)}(\mathbf{X} - \mathbf{m})(\mathbf{X}^* - \mu)\Sigma^{-(1/2)}\right) \\ &= \Sigma^{-(1/2)}E((\mathbf{X} - \mathbf{m})(\mathbf{X}^* - \mu))\Sigma^{-(1/2)} = I. \end{aligned}$$

Thus $\mathbf{Y}$ has zero mean and covariance I . This implies the following corollary to Theorem 47.11.7.

Corollary 47.11.8 *Let independent identically distributed random variables,*

$$\{\mathbf{X}_j\}_{j=1}^{\infty}$$

have mean $\mathbf{m}$ and positive definite covariance Σ where Σ^{-1} exists. Then if

$$\mathbf{Z}_n \equiv \sum_{j=1}^n \Sigma^{-(1/2)} \frac{(\mathbf{X}_j - \mu)}{\sqrt{n}},$$

it follows that for $\mathbf{Z} \sim N_p(\mathbf{0}, I)$,

$$F_{\mathbf{Z}_n}(\mathbf{x}) \rightarrow F_{\mathbf{Z}}(\mathbf{x})$$

for all $\mathbf{x}$.

47.12 Characteristic Functions Of Probability Measures

Recall one can define the characteristic function of a probability measure. In a sense it is more natural.

Definition 47.12.1 *Let μ be a probability measure defined on the Borel sets of $\mathbb{R}^n$. Then*

$$\phi_{\mu}(\mathbf{t}) \equiv \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} d\mu.$$

Then there is a version of Lemma 47.11.2 whose proof is identical to the proof of that lemma.

Lemma 47.12.2 *If $\{\mu_n\}$ is a sequence of Borel probability measures defined on the Borel sets of $\mathbb{R}^p$ such that*

$$\lim_{n \rightarrow \infty} \phi_{\mu_n}(\mathbf{t}) = \psi(\mathbf{t})$$

for all $\mathbf{t}$, where $\psi(\mathbf{0}) = 1$ and ψ is continuous at $\mathbf{0}$, then $\{\mu_n\}_{n=1}^{\infty}$ is tight.

As before, there are simple modifications of Lemmas 47.11.3 and 47.11.4. The version of Lemma 47.11.4 being the following.

Lemma 47.12.3 *If $\phi_{\mu_n}(\mathbf{t}) \rightarrow \phi_{\mu}(\mathbf{t})$ where $\{\mu_n\}$ and μ are probability measures defined on the Borel sets of $\mathbb{R}^p$, then if ψ is any bounded uniformly continuous function,*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^p} \psi d\mu_n = \int_{\mathbb{R}^p} \psi d\mu.$$

The next theorem is really important. It gives the existence of a measure based on an assumption that a set of measures is tight.

Theorem 47.12.4 *Let $\Lambda = \{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability measures defined on the Borel sets of $\mathbb{R}^p$. If Λ is tight then there exists a probability measure, λ and a subsequence of $\{\mu_n\}_{n=1}^{\infty}$, still denoted by $\{\mu_n\}_{n=1}^{\infty}$ such that whenever ϕ is a continuous bounded complex valued function defined on E ,*

$$\lim_{n \rightarrow \infty} \int \phi d\mu_n = \int \phi d\lambda.$$

Proof: By tightness, there exists an increasing sequence of compact sets, $\{K_n\}$ such that

$$\mu(K_n) > 1 - \frac{1}{n}$$

for all $\mu \in \Lambda$. Now letting $\mu \in \Lambda$ and $\phi \in C(K_n)$ such that $\|\phi\|_{\infty} \leq 1$, it follows

$$\left| \int_{K_n} \phi d\mu \right| \leq \mu(K_n) \leq 1$$

and so the restrictions of the measures of Λ to K_n are contained in the unit ball of $C(K_n)'$. Recall from the Riesz representation theorem, the dual space of $C(K_n)$ is a space of complex Borel measures. Theorem 13.4.5 on Page 384 implies the unit ball of $C(K_n)'$ is weak * sequentially compact. This follows from the observation that $C(K_n)$ is separable which follows easily from the Weierstrass approximation theorem. Thus the unit ball in $C(K_n)'$ is actually metrizable by Theorem 13.4.5 on Page 384. Therefore, there exists a subsequence of Λ , $\{\mu_{1k}\}$ such that their restrictions to K_1 converge weak * to a measure, $\lambda_1 \in C(K_1)'$. That is, for every $\phi \in C(K_1)$,

$$\lim_{k \rightarrow \infty} \int_{K_1} \phi d\mu_{1k} = \int_{K_1} \phi d\lambda_1$$

By the same reasoning, there exists a further subsequence $\{\mu_{2k}\}$ such that the restrictions of these measures to K_2 converge weak * to a measure $\lambda_2 \in C(K_2)'$ etc. Continuing this way,

$$\begin{aligned} \mu_{11}, \mu_{12}, \mu_{13}, \dots &\rightarrow \text{Weak * in } C(K_1)' \\ \mu_{21}, \mu_{22}, \mu_{23}, \dots &\rightarrow \text{Weak * in } C(K_2)' \\ \mu_{31}, \mu_{32}, \mu_{33}, \dots &\rightarrow \text{Weak * in } C(K_3)' \\ &\vdots \end{aligned}$$

Here the j^{th} sequence is a subsequence of the $(j-1)^{th}$. Let λ_n denote the measure in $C(K_n)'$ to which the sequence $\{\mu_{nk}\}_{k=1}^{\infty}$ converges weak *. Let $\{\mu_n\} \equiv \{\mu_{nn}\}$, the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so μ_n converges weak * in $C(K_m)'$ to λ_m for each m .

Claim: For $p > n$, the restriction of λ_p to the Borel sets of K_n equals λ_n .

Proof of claim: Let H be a compact subset of K_n . Then there are sets, V_l open in K_n which are decreasing and whose intersection equals H . This follows because this is a metric space. Then let $H \prec \phi_l \prec V_l$. It follows

$$\begin{aligned} \lambda_n(V_l) &\geq \int_{K_n} \phi_l d\lambda_n = \lim_{k \rightarrow \infty} \int_{K_n} \phi_l d\mu_k \\ &= \lim_{k \rightarrow \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \geq \lambda_p(H). \end{aligned}$$

Now considering the ends of this inequality, let $l \rightarrow \infty$ and pass to the limit to conclude

$$\lambda_n(H) \geq \lambda_p(H).$$

Similarly,

$$\begin{aligned} \lambda_n(H) &\leq \int_{K_n} \phi_l d\lambda_n = \lim_{k \rightarrow \infty} \int_{K_n} \phi_l d\mu_k \\ &= \lim_{k \rightarrow \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \leq \lambda_p(V_l). \end{aligned}$$

Then passing to the limit as $l \rightarrow \infty$, it follows

$$\lambda_n(H) \leq \lambda_p(H).$$

Thus the restriction of $\lambda_p, \lambda_p|_{K_n}$ to the compact sets of K_n equals λ_n . Then by inner regularity it follows the two measures, $\lambda_p|_{K_n}$, and λ_n are equal on all Borel sets of K_n . Recall that for finite measures on the Borel sets of separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that $\lambda_m(F) \geq 0$ for all F a Borel subset of K_m . (Whenever $\phi \geq 0$, $\int_{K_m} \phi d\lambda_m \geq 0$ because $\int_{K_m} \phi d\mu_k \geq 0$. Now you can approximate $\mathcal{X}_F$ with a suitable nonnegative ϕ using regularity of the measure.) Also, letting $\phi \equiv 1$,

$$1 \geq \lambda_m(K_m) \geq 1 - \frac{1}{m}. \quad (47.12.35)$$

Define for F a Borel set,

$$\lambda(F) \equiv \lim_{n \rightarrow \infty} \lambda_n(F \cap K_n).$$

The limit exists because the sequence on the right is increasing due to the above observation that $\lambda_n = \lambda_m$ on the Borel subsets of K_m whenever $n > m$. Thus for $n > m$

$$\lambda_n(F \cap K_n) \geq \lambda_n(F \cap K_m) = \lambda_m(F \cap K_m).$$

Now let $\{F_k\}$ be a sequence of disjoint Borel sets. Then

$$\begin{aligned} \lambda(\cup_{k=1}^{\infty} F_k) &\equiv \lim_{n \rightarrow \infty} \lambda_n(\cup_{k=1}^{\infty} F_k \cap K_n) = \lim_{n \rightarrow \infty} \lambda_n(\cup_{k=1}^{\infty} (F_k \cap K_n)) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_n(F_k \cap K_n) = \sum_{k=1}^{\infty} \lambda(F_k) \end{aligned}$$

the last equation holding by the monotone convergence theorem.

It remains to verify

$$\lim_{k \rightarrow \infty} \int \phi d\mu_k = \int \phi d\lambda$$

for every ϕ bounded and continuous. This is where tightness is used again. Suppose $\|\phi\|_{\infty} < M$. Then as noted above,

$$\lambda_n(K_n) = \lambda(K_n)$$

because for $p > n$, $\lambda_p(K_n) = \lambda_n(K_n)$ and so letting $p \rightarrow \infty$, the above is obtained. Also, from 47.12.35,

$$\begin{aligned} \lambda(K_n^C) &= \lim_{p \rightarrow \infty} \lambda_p(K_n^C \cap K_p) \\ &\leq \limsup_{p \rightarrow \infty} (\lambda_p(K_p) - \lambda_p(K_n)) \\ &\leq \limsup_{p \rightarrow \infty} (\lambda_p(K_p) - \lambda_n(K_n)) \\ &\leq \limsup_{p \rightarrow \infty} \left(1 - \left(1 - \frac{1}{n} \right) \right) = \frac{1}{n} \end{aligned}$$

Consequently,

$$\begin{aligned}
 \left| \int \phi d\mu_k - \int \phi d\lambda \right| &\leq \left| \int_{K_n^C} \phi d\mu_k + \int_{K_n} \phi d\mu_k - \left(\int_{K_n} \phi d\lambda + \int_{K_n^C} \phi d\lambda \right) \right| \\
 &\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n \right| + \left| \int_{K_n^C} \phi d\mu_k - \int_{K_n^C} \phi d\lambda \right| \\
 &\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n \right| + \left| \int_{K_n^C} \phi d\mu_k \right| + \left| \int_{K_n^C} \phi d\lambda \right| \\
 &\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n \right| + \frac{M}{n} + \frac{M}{n}
 \end{aligned}$$

First let n be so large that $2M/n < \varepsilon/2$ and then pick k large enough that the above expression is less than ε . This proves the theorem.

Definition 47.12.5 Let $\mu, \{\mu_n\}$ be probability measures defined on the Borel sets of $\mathbb{R}^p$ and let the sequence of probability measures, $\{\mu_n\}$ satisfy

$$\lim_{n \rightarrow \infty} \int \phi d\mu_n = \int \phi d\mu.$$

for every ϕ a bounded continuous function. Then μ_n is said to converge weakly to μ .

With the above, it is possible to prove the following amazing theorem of Levy.

Theorem 47.12.6 Suppose $\{\mu_n\}$ is a sequence of probability measures defined on the Borel sets of $\mathbb{R}^p$ and let $\{\phi_{\mu_n}\}$ denote the corresponding sequence of characteristic functions. If there exists ψ which is continuous at $\mathbf{0}$, $\psi(\mathbf{0}) = 1$, and for all $\mathbf{t}$,

$$\phi_{\mu_n}(\mathbf{t}) \rightarrow \psi(\mathbf{t}),$$

then there exists a probability measure, λ defined on the Borel sets of $\mathbb{R}^p$ and

$$\phi_\lambda(\mathbf{t}) = \psi(\mathbf{t}).$$

That is, ψ is a characteristic function of a probability measure. Also, $\{\mu_n\}$ converges weakly to λ .

Proof: By Lemma 47.12.2 $\{\mu_n\}$ is tight. Therefore, there exists a subsequence $\{\mu_{n_k}\}$ converging weakly to a probability measure, λ . In particular,

$$\begin{aligned}
 \phi_\lambda(\mathbf{t}) &= \int e^{i\mathbf{t} \cdot \mathbf{x}} d\lambda(x) = \lim_{n \rightarrow \infty} \int e^{i\mathbf{t} \cdot \mathbf{x}} d\mu_{n_k}(x) \\
 &= \lim_{n \rightarrow \infty} \phi_{\mu_{n_k}}(\mathbf{t}) = \psi(\mathbf{t})
 \end{aligned}$$

The last claim follows from this and Lemma 47.12.3. This proves the theorem.

Note how it was only necessary to assume $\psi(\mathbf{0}) = 1$ and ψ is continuous at $\mathbf{0}$ in order to conclude that ψ is a characteristic function. Thus you find that $|\psi(\mathbf{t})| \leq 1$ for free. This helps to see why Prokhorov's and Levy's theorems are so amazing.

47.13 Positive Definite Functions, Bochner's Theorem

First here is a nice little lemma about matrices.

Lemma 47.13.1 *Suppose M is an $n \times n$ matrix. Suppose also that*

$$\alpha^* M \alpha = 0$$

for all $\alpha \in \mathbb{C}^n$. Then $M = 0$.

Proof: Suppose λ is an eigenvalue for M and let α be an associated eigenvector.

$$0 = \alpha^* M \alpha = \alpha^* \lambda \alpha = \lambda \alpha^* \alpha = \lambda |\alpha|^2$$

and so all the eigenvalues of M equal zero. By Schur's theorem there is a unitary matrix U such that

$$M = U \begin{pmatrix} 0 & & *_1 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} U^* \quad (47.13.36)$$

where the matrix in the middle has zeros down the main diagonal and zeros below the main diagonal. Thus

$$M^* = U \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ *_2 & & 0 \end{pmatrix} U^*$$

where M^* has zeros down the main diagonal and zeros above the main diagonal. Also taking the adjoint of the given equation for M , it follows that for all α ,

$$\alpha^* M^* \alpha = 0$$

Therefore, $M + M^*$ is Hermitian and has the property that

$$\alpha^* (M + M^*) \alpha = 0.$$

Thus $M + M^* = 0$ because it is unitarily similar to a diagonal matrix and the above equation can only hold for all α if $M + M^*$ has all zero eigenvalues which implies the diagonal matrix has zeros down the main diagonal. Therefore, from the formulas for M, M^* ,

$$0 = U \left(\begin{pmatrix} 0 & & 0 \\ & \ddots & \\ *_2 & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & *_1 \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right) U^*$$

and so the sum of the two matrices in the middle must also equal 0. Hence the entries of the matrix in the middle in 47.13.36 are all equal to zero. Thus $M = 0$ as claimed.

Definition 47.13.2 A Borel measurable function, $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is called positive definite if whenever $\{\mathbf{t}_k\}_{k=1}^p \subseteq \mathbb{R}^n$, $\alpha \in \mathbb{C}^p$

$$\sum_{k,j} f(\mathbf{t}_j - \mathbf{t}_k) \alpha_j \overline{\alpha_k} \geq 0 \quad (47.13.37)$$

The first thing to notice about a positive definite function is the following which implies these functions are automatically bounded.

Lemma 47.13.3 If f is positive definite then whenever $\{\mathbf{t}_k\}_{k=1}^p$ are p points in $\mathbb{R}^n$, $|f(\mathbf{t}_j - \mathbf{t}_k)| \leq f(\mathbf{0})$. In particular, for all $\mathbf{t}$, $|f(\mathbf{t})| \leq f(\mathbf{0})$.

Proof: Let F be the $p \times p$ matrix such that

$$F_{kj} = f(\mathbf{t}_j - \mathbf{t}_k).$$

Then 47.13.37 is of the form

$$\alpha^* F \alpha = (F \alpha, \alpha) \geq 0 \quad (47.13.38)$$

where this is the inner product in $\mathbb{C}^p$. Letting $[\alpha, \beta] \equiv (F \alpha, \beta) \equiv \beta^* F \alpha$, it is obvious that $[\alpha, \beta]$ satisfies

$$[\alpha, \alpha] \geq 0, [a\alpha + b\beta, \gamma] = a[\alpha, \gamma] + b[\beta, \gamma].$$

I claim it also satisfies

$$[\alpha, \beta] = \overline{[\beta, \alpha]}.$$

To verify this last claim, note that since $\alpha^* F \alpha$ is real,

$$\alpha^* F^* \alpha = \alpha^* F \alpha \geq 0$$

and so for all $\alpha \in \mathbb{C}^p$,

$$\alpha^* (F^* - F) \alpha = 0$$

which from Lemma 47.13.1 implies $F^* = F$. Hence F is self adjoint and it follows

$$[\alpha, \beta] \equiv \beta^* F \alpha = \beta^* F^* \alpha = \alpha^T F^{*T} \bar{\beta} = \overline{\alpha^* F \beta} = \overline{[\beta, \alpha]}.$$

Therefore, the Cauchy Schwarz inequality holds for $[\cdot, \cdot]$ and it follows

$$|[\alpha, \beta]| = |(F \alpha, \beta)| \leq (F \alpha, \alpha)^{1/2} (F \beta, \beta)^{1/2}.$$

Letting $\alpha = \mathbf{e}_k$ and $\beta = \mathbf{e}_j$, it follows $F_{ss} \geq 0$ for all s and

$$|F_{kj}| \leq F_{kk}^{1/2} F_{jj}^{1/2}$$

which says nothing more than

$$|f(\mathbf{t}_j - \mathbf{t}_k)| \leq f(\mathbf{0})^{1/2} f(\mathbf{0})^{1/2} = f(\mathbf{0}).$$

This proves the lemma.

With this information, here is another useful lemma involving positive definite functions. It is interesting because it looks like the formula which defines what it means for the function to be positive definite.

Lemma 47.13.4 *Let f be a positive definite function as defined above and let μ be a finite Borel measure. Then*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) d\mu(x) d\mu(y) \geq 0. \quad (47.13.39)$$

If μ also has the property that it is symmetric, $\mu(F) = \mu(-F)$ for all F Borel, then

$$\int_{\mathbb{R}^n} f(\mathbf{x}) d(\mu * \mu)(x) \geq 0. \quad (47.13.40)$$

Proof: By definition if $\{t_j\}_{j=1}^p \subseteq \mathbb{R}^n$, and letting $\alpha = (1, \dots, 1)^T \in \mathbb{R}^n$,

$$\sum_{j,k} f(\mathbf{t}_j - \mathbf{t}_k) \geq 0.$$

Therefore, integrating over each of the variables,

$$0 \leq \sum_{j=1}^p \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{t}_j - \mathbf{t}_j) d\mu(t_j) d\mu(t_j) + \sum_{j \neq k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{t}_j - \mathbf{t}_k) d\mu(t_j) d\mu(t_k)$$

and so

$$0 \leq f(0) \mu(\mathbb{R}^n)^2 p + p(p-1) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) d\mu(x) d\mu(y).$$

Dividing both sides by $p(p-1)$ and letting $p \rightarrow \infty$, it follows

$$0 \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) d\mu(x) d\mu(y)$$

which shows 47.13.39.

To verify 47.13.40, use 47.7.14.

$$\int_{\mathbb{R}^n} f d(\mu * \mu) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x} + \mathbf{y}) d\mu(x) d\mu(y)$$

and since μ is symmetric, this equals

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(\mathbf{x} - \mathbf{y}) d\mu(x) d\mu(y) \geq 0$$

by the first part of the lemma. This proves the lemma.

Lemma 47.13.5 *Let μ_t be the measure defined on $\mathcal{B}(\mathbb{R}^n)$ by*

$$\mu_t(F) \equiv \int_F \frac{1}{(\sqrt{2\pi t})^n} e^{-\frac{1}{2t}|\mathbf{x}|^2} dx$$

for $t > 0$. Then $\mu_t * \mu_t = \mu_{2t}$ and each μ_t is a probability measure.

Proof: By Theorem 47.7.7,

$$\phi_{\mu_t * \mu_t}(\mathbf{s}) = \phi_{\mu_t}(\mathbf{s}) \phi_{\mu_t}(\mathbf{s}) = \left(e^{-\frac{1}{2}t|\mathbf{s}|^2} \right)^2 = e^{-\frac{1}{2}(2t)|\mathbf{s}|^2} = \phi_{\mu_{2t}}(\mathbf{s}).$$

Each μ_t is a probability measure because it is the distribution of a normally distributed random variable of mean $\mathbf{0}$ and covariance tI .

Now let μ be a probability measure on $\mathcal{B}(\mathbb{R}^n)$.

$$\phi_\mu(\mathbf{t}) \equiv \int e^{i\mathbf{t} \cdot \mathbf{y}} d\mu(y)$$

and so by the dominated convergence theorem, ϕ_μ is continuous and also $\phi_\mu(\mathbf{0}) = 1$. I claim ϕ_μ is also positive definite. Let $\alpha \in \mathbb{C}^p$ and $\{\mathbf{t}_k\}_{k=1}^p$ a sequence of points of $\mathbb{R}^n$. Then

$$\begin{aligned} \sum_{k,j} \phi_\mu(\mathbf{t}_k - \mathbf{t}_j) \alpha_k \overline{\alpha_j} &= \sum_{k,j} \int e^{i\mathbf{t}_k \cdot \mathbf{y}} \alpha_k e^{-i\mathbf{t}_j \cdot \mathbf{y}} \overline{\alpha_j} d\mu(y) \\ &= \int \sum_{k,j} e^{i\mathbf{t}_k \cdot \mathbf{y}} \alpha_k \overline{e^{i\mathbf{t}_j \cdot \mathbf{y}} \alpha_j} d\mu(y). \end{aligned}$$

Now let $\beta(\mathbf{y}) \equiv (e^{i\mathbf{t}_1 \cdot \mathbf{y}} \alpha_1, \dots, e^{i\mathbf{t}_p \cdot \mathbf{y}} \alpha_p)^T$. Then the above equals

$$\int (1, \dots, 1) \beta(\mathbf{y}) \beta^*(\mathbf{y}) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} d\mu$$

The integrand is of the form

$$\overline{\left(\beta^* \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right)} \left(\beta^* \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right) \geq 0$$

because it is just a complex number times its conjugate.

Thus every characteristic function is continuous, equals 1 at $\mathbf{0}$, and is positive definite. Bochner's theorem goes the other direction.

To begin with, suppose μ is a finite measure on $\mathcal{B}(\mathbb{R}^n)$. Then for $\mathfrak{S}$ the Schwartz class, μ can be considered to be in the space of linear transformations defined on $\mathfrak{S}$, $\mathfrak{S}^*$ as follows.

$$\mu(f) \equiv \int f d\mu.$$

Recall $F^{-1}(\mu)$ is defined as

$$F^{-1}(\mu)(f) \equiv \mu(F^{-1}f) = \int_{\mathbb{R}^n} F^{-1}f d\mu$$

$$\begin{aligned}
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \mathbf{y}} f(\mathbf{y}) d\mathbf{y} d\mu \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \mathbf{y}} d\mu \right) f(\mathbf{y}) d\mathbf{y}
\end{aligned}$$

and so $F^{-1}(\mu)$ is the bounded continuous function

$$\mathbf{y} \rightarrow \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \mathbf{y}} d\mu.$$

Now the following lemma has the main ideas for Bochner's theorem.

Lemma 47.13.6 *Suppose $\psi(\mathbf{t})$ is positive definite, $\mathbf{t} \rightarrow \psi(\mathbf{t})$ is in $L^1(\mathbb{R}^n, m_n)$ where m_n is Lebesgue measure, $\psi(\mathbf{0}) = 1$, and ψ is continuous at $\mathbf{0}$. Then there exists a unique probability measure, μ defined on the Borel sets of $\mathbb{R}^n$ such that*

$$\phi_\mu(\mathbf{t}) = \psi(\mathbf{t}).$$

Proof: If the conclusion is true, then

$$\psi(\mathbf{t}) = \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} d\mu(x) = (2\pi)^{n/2} F^{-1}(\mu)(\mathbf{t}).$$

Recall that $\mu \in \mathfrak{S}^*$, the algebraic dual of $\mathfrak{S}$. Therefore, in $\mathfrak{S}^*$,

$$\frac{1}{(2\pi)^{n/2}} F(\psi) = \mu.$$

That is, for all $f \in \mathfrak{S}$,

$$\begin{aligned}
\int_{\mathbb{R}^n} f(\mathbf{y}) d\mu(\mathbf{y}) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(\psi)(\mathbf{y}) f(\mathbf{y}) d\mathbf{y} \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\mathbb{R}^n} e^{-i\mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x} \right) d\mathbf{y}. \quad (47.13.41)
\end{aligned}$$

I will show

$$f \rightarrow \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\mathbb{R}^n} e^{-i\mathbf{y} \cdot \mathbf{x}} \psi(\mathbf{x}) d\mathbf{x} \right) d\mathbf{y}$$

is a positive linear functional and then it will follow from 47.13.41 that μ is unique. Thus it is needed to show the inside integral in 47.13.41 is nonnegative. First note that the integrand is a positive definite function of $\mathbf{x}$ for each fixed $\mathbf{y}$. This follows from

$$\begin{aligned}
&\sum_{k,j} e^{-i\mathbf{y} \cdot (\mathbf{x}_k - \mathbf{x}_j)} \psi(\mathbf{x}_k - \mathbf{x}_j) \alpha_k \overline{\alpha_j} \\
&= \sum_{k,j} \psi(\mathbf{x}_k - \mathbf{x}_j) \left(e^{-i\mathbf{y} \cdot (\mathbf{x}_k)} \alpha_k \right) \overline{e^{-i\mathbf{y} \cdot (\mathbf{x}_j)} \alpha_j} \geq 0.
\end{aligned}$$

Let $t > 0$ and

$$h_{2t}(\mathbf{x}) \equiv \frac{1}{(4\pi t)^{1/2}} e^{-\frac{1}{4t}|\mathbf{x}|^2}.$$

Then by dominated convergence theorem,

$$\int_{\mathbb{R}^n} e^{-i\mathbf{y}\cdot\mathbf{x}} \psi(\mathbf{x}) d\mathbf{x} = \lim_{t \rightarrow \infty} \int_{\mathbb{R}^n} e^{-i\mathbf{y}\cdot\mathbf{x}} \psi(\mathbf{x}) h_{2t}(\mathbf{x}) d\mathbf{x}$$

Letting $d\eta_{2t} = h_{2t}(\mathbf{x}) d\mathbf{x}$, it follows from Lemma 47.13.5 $\eta_{2t} = \eta_t * \eta_t$ and since these are symmetric measures, it follows from Lemma 47.13.4 the above equals

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^n} e^{-i\mathbf{y}\cdot\mathbf{x}} \psi(\mathbf{x}) d(\eta_t * \eta_t) \geq 0$$

Thus the above functional is a positive linear functional and so there exists a unique Radon measure, μ satisfying

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{y}) d\mu(\mathbf{y}) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} F(\psi)(\mathbf{y}) f(\mathbf{y}) d\mathbf{y} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(\mathbf{y}) \left(\int_{\mathbb{R}^n} e^{-i\mathbf{y}\cdot\mathbf{x}} \psi(\mathbf{x}) d\mathbf{x} \right) d\mathbf{y} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \psi(\mathbf{x}) \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\mathbf{y}) e^{-i\mathbf{y}\cdot\mathbf{x}} d\mathbf{y} \right) d\mathbf{x} \end{aligned}$$

for all $f \in C_c(\mathbb{R}^n)$. Thus from the dominated convergence theorem, the above holds for all $f \in \mathfrak{S}$ also. Hence for all $f \in \mathfrak{S}$ and considering μ as an element of $\mathfrak{S}^*$,

$$\begin{aligned} F^{-1}\mu(Ff) &= \mu(f) = \int_{\mathbb{R}^n} f(\mathbf{y}) d\mu(\mathbf{y}) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \psi(\mathbf{x}) F(f)(\mathbf{x}) d\mathbf{x} \\ &= \frac{1}{(2\pi)^{n/2}} F(\psi)(f) \equiv \frac{1}{(2\pi)^{n/2}} \psi(Ff). \end{aligned}$$

It follows that in $\mathfrak{S}^*$,

$$\psi = (2\pi)^{n/2} F^{-1}\mu$$

Thus

$$\psi(\mathbf{t}) = \int_{\mathbb{R}^n} e^{i\mathbf{t}\cdot\mathbf{x}} d\mu$$

in L^1 . Since the right side is continuous and the left is given continuous at $\mathbf{t} = \mathbf{0}$ and equal to 1 there, it follows

$$1 = \psi(\mathbf{0}) = \int_{\mathbb{R}^n} e^{i\mathbf{0}\cdot\mathbf{x}} d\mu = \mu(\mathbb{R}^n)$$

and so μ is a probability measure as claimed. This proves the lemma.

The following is Bochner's theorem.

Theorem 47.13.7 *Let ψ be positive definite, continuous at $\mathbf{0}$, and $\psi(\mathbf{0}) = 1$. Then there exists a unique Radon probability measure μ such that $\psi = \phi_\mu$.*

Proof: If $\psi \in L^1(\mathbb{R}^n, m_n)$, then the result follows from Lemma 47.13.6. By Lemma 47.13.3 ψ is bounded. Consider

$$\psi_t(\mathbf{x}) \equiv \psi(\mathbf{x}) \frac{1}{(2\pi t)^{n/2}} e^{-\frac{1}{2t}|\mathbf{x}|^2}.$$

Then $\psi_t(\mathbf{0}) = 1$, $\mathbf{x} \mapsto \psi_t(\mathbf{x})$ is continuous at $\mathbf{0}$, and $\psi_t \in L^1(\mathbb{R}^n, m_n)$. Therefore, by Lemma 47.13.6 there exists a unique Radon probability measure μ_t such that

$$\psi_t(\mathbf{x}) = \int_{\mathbb{R}^n} e^{i\mathbf{x} \cdot \mathbf{y}} d\mu_t(y) = \phi_{\mu_t}(\mathbf{x})$$

Now letting $t \rightarrow \infty$,

$$\lim_{t \rightarrow \infty} \psi_t(\mathbf{x}) = \lim_{t \rightarrow \infty} \phi_{\mu_t}(\mathbf{x}) = \psi(\mathbf{x}).$$

By Levy's theorem, Theorem 47.12.6 it follows there exists μ , a probability measure on $\mathcal{B}(\mathbb{R}^n)$ such that $\psi(\mathbf{x}) = \phi_\mu(\mathbf{x})$. The measure is unique because the characteristic functions are uniquely determined by the measure. This proves the theorem.

Conditional Expectation And Martingales

48.1 Conditional Expectation

Definition 48.1.1 Let $(\Omega, \mathcal{M}, P)$ be a probability space and let $\mathcal{S} \subseteq \mathcal{F}$ be two σ algebras contained in $\mathcal{M}$. Let f be $\mathcal{F}$ measurable and in $L^1(\Omega)$. Then $E(f|\mathcal{S})$, called the conditional expectation of f with respect to $\mathcal{S}$ is defined as follows:

$$E(f|\mathcal{S}) \text{ is } \mathcal{S} \text{ measurable}$$

For all $E \in \mathcal{S}$,

$$\int_E E(f|\mathcal{S}) dP = \int_E f dP$$

Lemma 48.1.2 The above is well defined. Also, if $\mathcal{S} \subseteq \mathcal{F}$ then

$$E(X|\mathcal{S}) = E(E(X|\mathcal{F})|\mathcal{S}). \quad (48.1.1)$$

If Z is bounded and measurable in $\mathcal{S}$ then

$$ZE(X|\mathcal{S}) = E(ZX|\mathcal{S}). \quad (48.1.2)$$

Proof: Let a finite measure on $\mathcal{S}$, μ be given by

$$\mu(E) \equiv \int_E f dP.$$

Then $\mu \ll P$ and so by the Radon Nikodym theorem, there exists a unique $\mathcal{S}$ measurable function, $E(f|\mathcal{S})$ such that

$$\int_E f dP \equiv \mu(E) = \int_E E(f|\mathcal{S}) dP$$

for all $E \in \mathcal{S}$.

Let $F \in \mathcal{S}$. Then

$$\begin{aligned} \int_F E(E(X|\mathcal{F})|\mathcal{S}) dP &\equiv \int_F E(X|\mathcal{F}) dP \\ &\equiv \int_F X dP \equiv \int_F E(X|\mathcal{S}) dP \end{aligned}$$

and so, by uniqueness, $E(E(X|\mathcal{F})|\mathcal{S}) = E(X|\mathcal{S})$. This shows 48.1.1.

To establish 48.1.2, note that if $Z = \mathcal{X}_F$ where $F \in \mathcal{S}$,

$$\int \mathcal{X}_F E(X|\mathcal{S}) dP = \int \mathcal{X}_F X dP = \int E(\mathcal{X}_F X|\mathcal{S}) dP$$

which shows 48.1.2 in the case where Z is the characteristic function of a set in $\mathcal{S}$. It follows this also holds for simple functions. Let $\{s_n\}$ be a sequence of simple functions which converges uniformly to Z and let $F \in \mathcal{S}$. Then by what was just shown,

$$\int_F s_n E(X|\mathcal{S}) dP = \int_F s_n X dP.$$

Then passing to the limit using the dominated convergence theorem, yields

$$\int_F Z E(X|\mathcal{S}) dP = \int_F Z X dP \equiv \int_F E(ZX|\mathcal{S}) dP.$$

Since this holds for every $F \in \mathcal{S}$, this shows 48.1.2.

The next major result is a generalization of Jensen's inequality whose proof depends on the following lemma about convex functions.

Lemma 48.1.3 *Let I be an open interval on $\mathbb{R}$ and let ϕ be a convex function defined on I . Then there exists a sequence $\{(a_n, b_n)\}$ such that*

$$\phi(x) = \sup \{a_n x + b_n, n = 1, \dots\}.$$

Proof: Let $x \in I$ and let $t > x$. Then by convexity of ϕ ,

$$\begin{aligned} \frac{\phi(x + \lambda(t-x)) - \phi(x)}{\lambda(t-x)} &\leq \frac{\phi(x)(1-\lambda) + \lambda\phi(t) - \phi(x)}{\lambda(t-x)} \\ &= \frac{\phi(t) - \phi(x)}{t-x}. \end{aligned}$$

Therefore $t \rightarrow \frac{\phi(t) - \phi(x)}{t-x}$ is increasing if $t > x$. If $t < x$

$$\begin{aligned} \frac{\phi(x + \lambda(t-x)) - \phi(x)}{\lambda(t-x)} &\geq \frac{\phi(x)(1-\lambda) + \lambda\phi(t) - \phi(x)}{\lambda(t-x)} \\ &= \frac{\phi(t) - \phi(x)}{t-x} \end{aligned}$$

and so $t \rightarrow \frac{\phi(t) - \phi(x)}{t - x}$ is increasing for $t \neq x$. Let

$$a_x \equiv \inf \left\{ \frac{\phi(t) - \phi(x)}{t - x} : t > x \right\}.$$

Then if $t_1 < x$, and $t > x$,

$$\frac{\phi(t_1) - \phi(x)}{t_1 - x} \leq a_x \leq \frac{\phi(t) - \phi(x)}{t - x}.$$

Thus for all $t \in I$,

$$\phi(t) \geq a_x(t - x) + \phi(x). \quad (48.1.3)$$

Pick $t_2 > x$. Then for all $t \in (x, t_2)$

$$a_x \leq \frac{\phi(t) - \phi(x)}{t - x} \leq \frac{\phi(t_2) - \phi(x)}{t_2 - x}$$

and so

$$a_x(t - x) + \phi(x) \leq \phi(t) \leq \left(\frac{\phi(t_2) - \phi(x)}{t_2 - x} \right) (t - x) + \phi(x). \quad (48.1.4)$$

Pick $t_3 < x$. Then for $t_3 < t < x$

$$a_x \geq \frac{\phi(t) - \phi(x)}{t - x} \geq \frac{\phi(t_3) - \phi(x)}{t_3 - x}$$

and so

$$a_x(t - x) + \phi(x) \leq \phi(t) \leq \left(\frac{\phi(t_3) - \phi(x)}{t_3 - x} \right) (t - x) + \phi(x). \quad (48.1.5)$$

48.1.4 and 48.1.5 imply ϕ is continuous. Let

$$\psi(x) \equiv \sup \{a_r(x - r) + \phi(r) : r \in \mathbb{Q} \cap I\}.$$

Then ψ is convex on I so ψ is continuous. Also $\psi(r) \geq \phi(r)$ so by 48.1.3,

$$\psi(x) \geq \phi(x) \geq \sup \{a_r(x - r) + \phi(r)\} \equiv \psi(x).$$

Thus $\psi(x) = \phi(x)$ and letting $\mathbb{Q} \cap I = \{r_n\}$, $a_n = a_{r_n}$ and $b_n = a_{r_n}r_n + \phi(r_n)$. This proves the lemma.

Lemma 48.1.4 *If $X \leq Y$, then $E(X|\mathcal{S}) \leq E(Y|\mathcal{S})$ a.e. Also*

$$X \rightarrow E(X|\mathcal{S})$$

is linear.

Proof: Let $A \in \mathcal{S}$.

$$\begin{aligned} \int_A E(X|\mathcal{S}) dP &\equiv \int_A X dP \\ &\leq \int_A Y dP \equiv \int_A E(Y|\mathcal{S}) dP. \end{aligned}$$

Hence $E(X|\mathcal{S}) \leq E(Y|\mathcal{S})$ a.e. as claimed. It is obvious $X \rightarrow E(X|\mathcal{S})$ is linear.

Theorem 48.1.5 (*Jensen's inequality*) Let $X(\omega) \in I$ and let $\phi: I \rightarrow \mathbb{R}$ be convex. Suppose

$$E(|X|), E(|\phi(X)|) < \infty.$$

Then

$$\phi(E(X|\mathcal{S})) \leq E(\phi(X)|\mathcal{S}).$$

Proof: Let $\phi(x) = \sup \{a_n x + b_n\}$. Letting $A \in \mathcal{S}$,

$$\frac{1}{P(A)} \int_A E(X|\mathcal{S}) dP = \frac{1}{P(A)} \int_A X dP \in I \text{ a.e.}$$

whenever $P(A) \neq 0$. Hence $E(X|\mathcal{S})(\omega) \in I$ a.e. and so it makes sense to consider $\phi(E(X|\mathcal{S}))$. Now

$$a_n E(X|\mathcal{S}) + b_n = E(a_n X + b_n|\mathcal{S}) \leq E(\phi(X)|\mathcal{S}).$$

Thus

$$\begin{aligned} &\sup \{a_n E(X|\mathcal{S}) + b_n\} \\ &= \phi(E(X|\mathcal{S})) \leq E(\phi(X)|\mathcal{S}) \text{ a.e.} \end{aligned}$$

which proves the theorem.

48.2 Discrete Martingales

Definition 48.2.1 Let $\mathcal{S}_k$ be an increasing sequence of σ algebras which are subsets of $\mathcal{S}$ and X_k be a sequence of real-valued random variables with $E(|X_k|) < \infty$ such that X_k is $\mathcal{S}_k$ measurable. Then this sequence is called a martingale if

$$E(X_{k+1}|\mathcal{S}_k) = X_k,$$

a submartingale if

$$E(X_{k+1}|\mathcal{S}_k) \geq X_k,$$

and a supermartingale if

$$E(X_{k+1}|\mathcal{S}_k) \leq X_k.$$

Saying that X_k is $\mathcal{S}_k$ measurable is referred to by saying $\{X_k\}$ is adapted to $\mathcal{S}_k$.

An upcrossing occurs when a sequence goes from a up to b . Thus it crosses the interval, $[a, b]$ in the up direction, hence upcrossing. More precisely,

Definition 48.2.2 Let $\{x_i\}_{i=1}^I$ be any sequence of real numbers, $I \leq \infty$. Define an increasing sequence of integers $\{m_k\}$ as follows. m_1 is the first integer ≥ 1 such that $x_{m_1} \leq a$, m_2 is the first integer larger than m_1 such that $x_{m_2} \geq b$, m_3 is the first integer larger than m_2 such that $x_{m_3} \leq a$, etc. Then each sequence, $\{x_{m_{2k-1}}, \dots, x_{m_{2k}}\}$, is called an upcrossing of $[a, b]$.

Proposition 48.2.3 Let $\{X_i\}_{i=1}^n$ be a finite sequence of real random variables defined on Ω where $(\Omega, \mathcal{S}, P)$ is a probability space. Let $U_{[a,b]}(\omega)$ denote the number of upcrossings of $X_i(\omega)$ of the interval $[a, b]$. Then $U_{[a,b]}$ is a random variable.

Proof: Let $X_0(\omega) \equiv a+1$, let $Y_0(\omega) \equiv 0$, and let $Y_k(\omega)$ remain 0 for $k = 0, \dots, l$ until $X_l(\omega) \leq a$. When this happens (if ever), $Y_{l+1}(\omega) \equiv 1$. Then let $Y_i(\omega)$ remain 1 for $i = l+1, \dots, r$ until $X_r(\omega) \geq b$ when $Y_{r+1}(\omega) \equiv 0$. Let $Y_k(\omega)$ remain 0 for $k \geq r+1$ until $X_k(\omega) \leq a$ when $Y_k(\omega) \equiv 1$ and continue in this way. Thus the upcrossings of $X_i(\omega)$ are identified as unbroken strings of ones for Y_k with a zero at each end, with the possible exception of the last string of ones which may be missing the zero at the upper end and may or may not be an upcrossing.

Note also that Y_0 is measurable because it is identically equal to 0 and that if Y_k is measurable, then Y_{k+1} is measurable because the only change in going from k to $k+1$ is a change from 0 to 1 or from 1 to 0 on a measurable set determined by X_k . Now let

$$Z_k(\omega) = \begin{cases} 1 & \text{if } Y_k(\omega) = 1 \text{ and } Y_{k+1}(\omega) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

if $k < n$ and

$$Z_n(\omega) = \begin{cases} 1 & \text{if } Y_n(\omega) = 1 \text{ and } X_n(\omega) \geq b, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $Z_k(\omega) = 1$ exactly when an upcrossing has been completed and each Z_i is a random variable.

$$U_{[a,b]}(\omega) = \sum_{k=1}^n Z_k(\omega)$$

so $U_{[a,b]}$ is a random variable as claimed.

The following corollary collects some key observations found in the above construction.

Corollary 48.2.4 $U_{[a,b]}(\omega) \leq$ the number of unbroken strings of ones in the sequence, $\{Y_k(\omega)\}$ there being at most one unbroken string of ones which produces no upcrossing. Also

$$Y_i(\omega) = \psi_i\left(\{X_j(\omega)\}_{j=1}^{i-1}\right), \quad (48.2.6)$$

where ψ_i is some function of the past values of $X_j(\omega)$.

Lemma 48.2.5 Let ϕ be a convex and increasing function and suppose

$$\{(X_n, \mathcal{S}_n)\}$$

is a submartingale. Then if $E(|\phi(X_n)|) < \infty$, it follows

$$\{(\phi(X_n), \mathcal{S}_n)\}$$

is also a submartingale.

Proof: It is given that $E(X_{n+1}, \mathcal{S}_n) \geq X_n$ and so

$$\phi(X_n) \leq \phi(E(X_{n+1}, \mathcal{S}_n)) \leq E(\phi(X_{n+1}) | \mathcal{S}_n)$$

by Jensen's inequality.

The following is called the upcrossing lemma.

48.2.1 Upcrossings

Lemma 48.2.6 (*upcrossing lemma*) Let $\{(X_i, \mathcal{S}_i)\}_{i=1}^n$ be a submartingale and let $U_{[a,b]}(\omega)$ be the number of upcrossings of $[a, b]$. Then

$$E(U_{[a,b]}) \leq \frac{E(|X_n|) + |a|}{b - a}.$$

Proof: Let $\phi(x) \equiv a + (x - a)^+$ so that ϕ is an increasing convex function always at least as large as a . By Lemma 48.2.5 it follows that $\{(\phi(X_k), \mathcal{S}_k)\}$ is also a submartingale.

$$\begin{aligned} \phi(X_{k+r}) - \phi(X_k) &= \sum_{i=k+1}^{k+r} \phi(X_i) - \phi(X_{i-1}) \\ &= \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) Y_i + \sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) (1 - Y_i). \end{aligned}$$

Observe that Y_i is $\mathcal{S}_{i-1}$ measurable from its construction in Proposition 48.2.3, Y_i depending only on X_j for $j < i$. Therefore, letting

$$A_i \equiv \{\omega : Y_i(\omega) = 0\},$$

$$\begin{aligned} &E \left(\sum_{i=k+1}^{k+r} (\phi(X_i) - \phi(X_{i-1})) (1 - Y_i) \right) \\ &= \sum_{i=k+1}^{k+r} \int_{\Omega} (\phi(X_i) - \phi(X_{i-1})) (1 - Y_i) dP \\ &= \sum_{i=k+1}^{k+r} \int_{A_i} (\phi(X_i) - \phi(X_{i-1})) dP \end{aligned}$$

because if $Y_i = 1$, $(\phi(X_i) - \phi(X_{i-1}))(1 - Y_i) = 0$. Continuing this chain of formulas, and using the fact that A_i is $\mathcal{S}_{i-1}$ measurable, it follows from the definition of conditional expectation and the definition of a submartingale,

$$\begin{aligned} &= \sum_{i=k+1}^{k+r} \int_{A_i} E(\phi(X_i), \mathcal{S}_{i-1}) dP - \int_{A_i} \phi(X_{i-1}) dP \\ &\geq \sum_{i=k+1}^{k+r} \int_{A_i} \phi(X_{i-1}) dP - \int_{A_i} \phi(X_{i-1}) dP = 0. \end{aligned} \quad (48.2.7)$$

Now let the unbroken strings of ones for $\{Y_i(\omega)\}$ be

$$\{k_1, \dots, k_1 + r_1\}, \{k_2, \dots, k_2 + r_2\}, \dots, \{k_m, \dots, k_m + r_m\} \quad (48.2.8)$$

where $m = V(\omega) \equiv$ the number of unbroken strings of ones in the sequence $\{Y_i(\omega)\}$. By Corollary 48.2.4 $V(\omega) \geq U_{[a,b]}(\omega)$.

$$\begin{aligned} &\phi(X_n(\omega)) - \phi(X_1(\omega)) \\ &= \sum_{k=1}^n (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) Y_k(\omega) \\ &\quad + \sum_{k=1}^n (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) (1 - Y_k(\omega)). \end{aligned}$$

The first sum in the above reduces to summing over the unbroken strings of ones because the terms in which $Y_i(\omega) = 0$ contribute nothing. implies

$$\begin{aligned} &\phi(X_n(\omega)) - \phi(X_1(\omega)) \\ &\geq U_{[a,b]}(\omega) (b - a) + 0 + \\ &\quad \sum_{k=1}^n (\phi(X_k(\omega)) - \phi(X_{k-1}(\omega))) (1 - Y_k(\omega)) \end{aligned} \quad (48.2.9)$$

where the zero on the right side results from a string of ones which does not produce an upcrossing. It is here that it is important that $\phi(x) \geq a$. Such a string begins with $\phi(X_k(\omega)) = a$ and results in an expression of the form $\phi(X_{k+m}(\omega)) - \phi(X_k(\omega)) \geq 0$ since $\phi(X_{k+m}(\omega)) \geq a$. If X_k had not been replaced with $\phi(X_k)$, it would have been possible for $\phi(X_{k+m}(\omega))$ to be less than a and the zero in the above could have been a negative number. This would have been inconvenient.

Next take the expected value of both sides in 48.2.9. Using 48.2.7, this results in

$$\begin{aligned} E(\phi(X_n) - \phi(X_1)) &\geq (b - a) E(U_{[a,b]}) \\ &\quad + E\left(\sum_{k=1}^n (\phi(X_k) - \phi(X_{k-1})) (1 - Y_k)\right) \\ &\geq (b - a) E(U_{[a,b]}) \end{aligned}$$

and this proves the lemma.

The reason for this lemma is to prove the amazing submartingale convergence theorem.

48.2.2 The Submartingale Convergence Theorem

Theorem 48.2.7 (*submartingale convergence theorem*) *Let*

$$\{(X_i, \mathcal{S}_i)\}_{i=1}^{\infty}$$

be a submartingale with $K \equiv \sup E(|X_n|) < \infty$. Then there exists a random variable, X , such that $E(|X|) \leq K$ and

$$\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \text{ a.e.}$$

Proof: Let $a, b \in \mathbb{Q}$ and let $a < b$. Let $U_{[a,b]}^n(\omega)$ be the number of upcrossings of $\{X_i(\omega)\}_{i=1}^n$. Then let

$$U_{[a,b]}(\omega) \equiv \lim_{n \rightarrow \infty} U_{[a,b]}^n(\omega) = \text{number of upcrossings of } \{X_i\}.$$

By the upcrossing lemma,

$$E(U_{[a,b]}^n) \leq \frac{E(|X_n|) + |a|}{b - a} \leq \frac{K + |a|}{b - a}$$

and so by the monotone convergence theorem,

$$E(U_{[a,b]}) \leq \frac{K + |a|}{b - a} < \infty$$

which shows $U_{[a,b]}(\omega)$ is finite a.e., for all $\omega \notin S_{[a,b]}$ where $P(S_{[a,b]}) = 0$. Define

$$S \equiv \cup \{S_{[a,b]} : a, b \in \mathbb{Q}, a < b\}.$$

Then $P(S) = 0$ and if $\omega \notin S$, $\{X_k\}_{k=1}^{\infty}$ has only finitely many upcrossings of every interval having rational endpoints. Thus, for $\omega \notin S$,

$$\begin{aligned} \limsup_{k \rightarrow \infty} X_k(\omega) &= \liminf_{k \rightarrow \infty} X_k(\omega) \\ &= \lim_{k \rightarrow \infty} X_k(\omega) \equiv X_{\infty}(\omega). \end{aligned}$$

Letting $X_{\infty}(\omega) = 0$ for $\omega \in S$, Fatou's lemma implies

$$\int_{\Omega} |X_{\infty}| dP = \int_{\Omega} \liminf_{n \rightarrow \infty} |X_n| dP \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |X_n| dP \leq K$$

and so this proves the theorem.

Another very interesting result about submartingales is the Doob submartingale estimate.

Theorem 48.2.8 *Let $\{(X_i, \mathcal{S}_i)\}_{i=1}^\infty$ be a submartingale. Then*

$$P\left(\max_{1 \leq k \leq n} X_k \geq \lambda\right) \leq \frac{1}{\lambda} \int_{\Omega} X_n^+ dP$$

Proof: Let

$$\begin{aligned} A_1 &\equiv [X_1 \geq \lambda], A_2 \equiv [X_2 \geq \lambda] \setminus A_1, \\ \dots, A_k &\equiv [X_k \geq \lambda] \setminus (\cup_{i=1}^{k-1} A_i) \dots \end{aligned}$$

Thus each A_k is $\mathcal{S}_k$ measurable, the A_k are disjoint, and their union equals $[\max_{1 \leq k \leq n} X_k \geq \lambda]$. Therefore from the definition of a submartingale and Jensen's inequality,

$$\begin{aligned} P\left(\max_{1 \leq k \leq n} X_k \geq \lambda\right) &= \sum_{k=1}^n P(A_k) \leq \frac{1}{\lambda} \sum_{k=1}^n \int_{A_k} X_k dP \\ &\leq \frac{1}{\lambda} \sum_{k=1}^n \int_{A_k} E(X_n | \mathcal{S}_k) dP \\ &\leq \frac{1}{\lambda} \sum_{k=1}^n \int_{A_k} E(X_n | \mathcal{S}_k)^+ dP \\ &\leq \frac{1}{\lambda} \sum_{k=1}^n \int_{A_k} E(X_n^+ | \mathcal{S}_k) dP \\ &= \frac{1}{\lambda} \sum_{k=1}^n \int_{A_k} X_n^+ dP \leq \frac{1}{\lambda} \int_{\Omega} X_n^+ dP. \end{aligned}$$

This proves the theorem.

48.3 Optional Sampling And Stopping Times

48.3.1 Stopping Times And Their Properties

First it is necessary to define the notion of a stopping time.

Definition 48.3.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n=1}^\infty$ be an increasing sequence of σ algebras each contained in $\mathcal{F}$. A stopping time is a measurable function, T which maps Ω to $\mathbb{N}$,*

$$T^{-1}(A) \in \mathcal{F} \text{ for all } A \in \mathcal{P}(\mathbb{N}),$$

such that for all $n \in \mathbb{N}$,

$$[T \leq n] \in \mathcal{F}_n.$$

Note this is equivalent to saying

$$[T = n] \in \mathcal{F}_n$$

because

$$[T = n] = [T \leq n] \setminus [T \leq n - 1].$$

For T a stopping time define $\mathcal{F}_T$ as follows.

$$\mathcal{F}_T \equiv \{A \in \mathcal{F} : A \cap [T \leq n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

These sets in $\mathcal{F}_T$ are referred to as “prior” to T .

The following lemma is fundamental to understand.

Lemma 48.3.2 *In the situation of Definition 48.3.1, if $S \leq T$ for two stopping times, S and T , then $\mathcal{F}_S \subseteq \mathcal{F}_T$. Also $\mathcal{F}_T$ is a σ algebra.*

Proof: Let $A \in \mathcal{F}_S$. Then this means

$$A \cap [S \leq n] \in \mathcal{F}_n \text{ for all } n.$$

Then

$$A \cap [T \leq n] = \cup_{i=1}^n (A \cap [S \leq i]) \cap [T \leq n] \quad (48.3.10)$$

Suppose ω is in the set on the left. Then if $T(\omega) < n$, it is clearly in the set on the right. If $T(\omega) = n$, then $\omega \in [S \leq i]$ for some $i \leq n$ and it is also in $[T \leq n]$. Thus the set on the left is contained in the set on the right. Next suppose ω is in the set on the right. Then $\omega \in [T \leq n]$ and it only remains to verify $\omega \in A$. However, $\omega \in A \cap [S \leq i]$ for some i and so $\omega \in A$ also.

Now from 48.3.10 it follows $A \cap [T \leq n] \in \mathcal{F}_n$ because

$$A \cap [S \leq i] \in \mathcal{F}_i \subseteq \mathcal{F}_n$$

and $[T \leq n] \in \mathcal{F}_n$ because T is a stopping time. Since n is arbitrary, this shows $\mathcal{F}_S \subseteq \mathcal{F}_T$.

It remains to verify $\mathcal{F}_T$ is a σ algebra. Suppose $\{A_i\}$ is a sequence of sets in $\mathcal{F}_T$. Then I need to show that $(\cup_{i=1}^{\infty} A_i) \cap [T \leq j] \in \mathcal{F}_j$ for all j .

$$\cup_{i=1}^{\infty} A_i \cap [T \leq j] = \cup_{i=1}^{\infty} (A_i \cap [T \leq j])$$

Now each $(A_i \cap [T \leq j])$ is in $\mathcal{F}_j$ and so the countable union of these sets is also in $\mathcal{F}_j$. Next suppose $A \in \mathcal{F}_T$. I need to verify $A^C \cap [T \leq j] \in \mathcal{F}_j$ for all j . However, $[T \leq j] \in \mathcal{F}_j$ and $\Omega \in \mathcal{F}_j$ so $\Omega \in \mathcal{F}_T$. Thus

$$\Omega \cap [T \leq j] = (A \cap [T \leq j]) \cup (A^C \cap [T \leq j])$$

and so

$$(A^C \cap [T \leq j]) = \Omega \cap [T \leq j] \setminus (A \cap [T \leq j]) \in \mathcal{F}_j.$$

This proves the lemma.

Lemma 48.3.3 *Let T be a stopping time and let $\{X_n\}$ be a sequence of random variables such that X_n is $\mathcal{F}_n$ measurable. Then $X_T(\omega) \equiv X_{T(\omega)}(\omega)$ is also a random variable and it is measurable with respect to $\mathcal{F}_T$.*

Proof: I assume the X_n have values in some topological space and each is measurable because the inverse image of an open set is in $\mathcal{F}_n$. I need to show $X_T^{-1}(U) \cap [T \leq n] \in \mathcal{F}_n$ for all n whenever U is open.

$$X_T^{-1}(U) = \cup_{i=1}^{\infty} X_i^{-1}(U) \cap [T = i].$$

It follows $X_T^{-1}(U) \in \mathcal{F}$. Furthermore,

$$\begin{aligned} X_T^{-1}(U) \cap [T \leq n] &= \cup_{i=1}^{\infty} X_i^{-1}(U) \cap [T = i] \cap [T \leq n] \\ &= \cup_{i=1}^n X_i^{-1}(U) \cap [T = i] \cap [T \leq n] \\ &= \cup_{i=1}^n \overbrace{X_i^{-1}(U) \cap [T = i]}^{\text{in } \mathcal{F}_i} \in \mathcal{F}_n \end{aligned}$$

and so it is $\mathcal{F}_T$ measurable as claimed. This proves the lemma.

Lemma 48.3.4 *Let $S \leq T$ be two stopping times such that T is **bounded above** and let $\{X_n\}$ be a submartingale (martingale) adapted to the increasing sequence of σ algebras, $\{\mathcal{F}_n\}$. Then*

$$E(X_T | \mathcal{F}_S) \geq X_S$$

in the case where $\{X_n\}$ is a submartingale and

$$E(X_T | \mathcal{F}_S) = X_S$$

in the case where $\{X_n\}$ is a martingale.

Proof: I will prove the case where $\{X_n\}$ is a submartingale and note the other case will only involve replacing $\geq$ with $=$. First recall that from Lemma 48.3.2 $\mathcal{F}_S \subseteq \mathcal{F}_T$. Also let m be an upper bound for T . Then it follows from this that

$$E(|X_T|) = \sum_{i=1}^m \int_{[T=i]} |X_i| dP < \infty$$

with a similar formula holding for $E(|X_S|)$. Thus it makes sense to speak of $E(X_T | \mathcal{F}_S)$.

I need to show that if $B \in \mathcal{F}_S$, so that $B \cap [S \leq n] \in \mathcal{F}_n$ for all n , then

$$\int_B X_T dP \geq \int_B X_S dP.$$

It suffices to do this for B of the special form

$$B = A \cap [S = i]$$

because if this is done, then the result follows from summing over all possible values of S . Then let B be of this form.

$$\int_{A \cap [S=i]} X_T dP = \sum_{j=i}^m \int_{A \cap [S=i] \cap [T=j]} X_T dP$$

$$= \sum_{j=i}^{m-1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \geq m]} X_m dP$$

And so

$$\begin{aligned} \int_{A \cap [S=i]} X_T dP &= \sum_{j=i}^{m-1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \geq m]} X_m dP \quad (48.3.11) \\ &= \sum_{j=i}^{m-1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \leq m-1]^C} X_m dP \\ &\geq \sum_{j=i}^{m-1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \leq m-1]^C} X_{m-1} dP \\ &= \sum_{j=i}^{m-1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T > m-1]} X_{m-1} dP \end{aligned}$$

provided $m-1 \geq i$ because $\{X_n\}$ is a submartingale and

$$A \cap [S=i] \cap [T \leq m-1]^C \in \mathcal{F}_{m-1}$$

Now combine the top term of the sum with the term on the right to obtain

$$= \sum_{j=i}^{m-2} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \geq m-1]} X_{m-1} dP$$

which is exactly the same form as 48.3.11 except m is replaced with $m-1$. Now repeat this process till you get the following inequality

$$\int_{A \cap [S=i]} X_T dP \geq \sum_{j=i}^{i+1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \geq i+2]} X_{i+2} dP$$

The right hand side equals

$$\begin{aligned} &\sum_{j=i}^{i+1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \leq i+1]^C} X_{i+2} dP \\ &\geq \sum_{j=i}^{i+1} \int_{A \cap [S=i] \cap [T=j]} X_T dP + \int_{A \cap [S=i] \cap [T \leq i+1]^C} X_{i+1} dP \\ &= \int_{A \cap [S=i] \cap [T=i]} X_T dP + \int_{A \cap [S=i] \cap [T=i+1]} X_T dP + \int_{A \cap [S=i] \cap [T \leq i+1]^C} X_{i+1} dP \\ &= \int_{A \cap [S=i] \cap [T=i]} X_i dP + \int_{A \cap [S=i] \cap [T=i+1]} X_{i+1} dP + \int_{A \cap [S=i] \cap [T > i+1]} X_{i+1} dP \end{aligned}$$

$$\begin{aligned}
&= \int_{A \cap [S=i] \cap [T=i]} X_i dP + \int_{A \cap [S=i] \cap [T \geq i+1]} X_{i+1} dP \\
&= \int_{A \cap [S=i] \cap [T=i]} X_i dP + \int_{A \cap [S=i] \cap [T \leq i]^C} X_{i+1} dP \\
&\geq \int_{A \cap [S=i] \cap [T=i]} X_i dP + \int_{A \cap [S=i] \cap [T \leq i]^C} X_i dP \\
&= \int_{A \cap [S=i] \cap [T=i]} X_i dP + \int_{A \cap [S=i] \cap [T > i]} X_i dP \\
&= \int_{A \cap [S=i] \cap [T \geq i]} X_i dP = \int_{A \cap [S=i]} X_i dP = \int_{A \cap [S=i]} X_S dP
\end{aligned}$$

In the case where $\{X_n\}$ is a martingale, you replace every occurrence of $\geq$ in the above argument with $=$. This proves the lemma.

This lemma is called the optional sampling theorem. Another version of this theorem is the case where you have an increasing sequence of stopping times, $\{T_n\}_{n=1}^\infty$. Thus if $\{X_n\}$ is a sequence of random variables each $\mathcal{F}_n$ measurable, the sequence $\{X_{T_n}\}$ is also a sequence of random variables such that X_{T_n} is measurable with respect to $\mathcal{F}_{T_n}$ where $\mathcal{F}_{T_n}$ is an increasing sequence of σ fields. In the case where X_n is a submartingale (martingale) it is reasonable to ask whether $\{X_{T_n}\}$ is also a submartingale (martingale). The optional sampling theorem says this is often the case.

Theorem 48.3.5 *Let $\{T_n\}$ be an increasing **bounded** sequence of stopping times and let $\{X_n\}$ be a submartingale (martingale) adapted to the increasing sequence of σ algebras, $\{\mathcal{F}_n\}$. Then $\{X_{T_n}\}$ is a submartingale (martingale) adapted to the increasing sequence of σ algebras $\{\mathcal{F}_{T_n}\}$.*

Proof: This follows from Lemma 48.3.4

Example 48.3.6 *Let $\{X_n\}$ be a sequence of real random variables such that X_n is $\mathcal{F}_n$ measurable and let A be a Borel subset of $\mathbb{R}$. Let $T(\omega)$ denote the first time $X_n(\omega)$ is in A . Then T is a stopping time. It is called the first hitting time.*

To see this is a stopping time,

$$[T \leq l] = \cup_{i=1}^l X_i^{-1}(A) \in \mathcal{F}_l.$$

48.4 Optional Stopping Times And Martingales

48.4.1 Stopping Times And Their Properties

The purpose of this section is to consider a special optional sampling theorem for martingales which is superior to the one presented earlier. I have presented a different treatment of the fundamental properties of stopping times also. See Kallenberg [40] for more.

Definition 48.4.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n=1}^\infty$ be an increasing sequence of σ algebras each contained in $\mathcal{F}$. A stopping time is a measurable function, τ which maps Ω to $\mathbb{N}$,

$$\tau^{-1}(A) \in \mathcal{F} \text{ for all } A \in \mathcal{P}(\mathbb{N}),$$

such that for all $n \in \mathbb{N}$,

$$[\tau \leq n] \in \mathcal{F}_n.$$

Note this is equivalent to saying

$$[\tau = n] \in \mathcal{F}_n$$

because

$$[\tau = n] = [\tau \leq n] \setminus [\tau \leq n-1].$$

For τ a stopping time define $\mathcal{F}_\tau$ as follows.

$$\mathcal{F}_\tau \equiv \{A \in \mathcal{F} : A \cap [\tau \leq n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

These sets in $\mathcal{F}_\tau$ are referred to as “prior” to τ .

First note that for τ a stopping time, $\mathcal{F}_\tau$ is a σ algebra. This is in the next proposition.

Proposition 48.4.2 For τ a stopping time, $\mathcal{F}_\tau$ is a σ algebra and if $Y(k)$ is $\mathcal{F}_k$ measurable for all k , then

$$\omega \rightarrow Y(\tau(\omega))$$

is $\mathcal{F}_\tau$ measurable.

Proof: Let $A_n \in \mathcal{F}_\tau$. I need to show $\cup_n A_n \in \mathcal{F}_\tau$. In other words, I need to show that

$$\cup_n A_n \cap [\tau \leq k] \in \mathcal{F}_k$$

The left side equals

$$\cup_n (A_n \cap [\tau \leq k])$$

which is a countable union of sets of $\mathcal{F}_k$ and so $\mathcal{F}_\tau$ is closed with respect to countable unions. Next suppose $A \in \mathcal{F}_\tau$.

$$(A^C \cap [\tau \leq k]) \cup (A \cap [\tau \leq k]) = \Omega \cap [\tau \leq k]$$

and $\Omega \cap [\tau \leq k] \in \mathcal{F}_k$. Therefore, so is $A^C \cap [\tau \leq k]$. It remains to verify the last claim.

$$[Y(\tau) \leq a] = \cup_k [\tau = k] \cap [Y(k) \leq a]$$

Thus

$$[Y(\tau) \leq a] \cap [\tau \leq l] = \cup_k [\tau = k] \cap [Y(k) \leq a] \cap [\tau \leq l]$$

Consider a term in the union. If $l \geq k$ the term reduces to $[\tau = k] \cap [Y(k) \leq a] \in \mathcal{F}_k$ while if $l < k$, this term reduces to $\emptyset$, also a set of $\mathcal{F}_k$. Therefore, $Y(\tau)$ must be $\mathcal{F}_\tau$ measurable. This proves the proposition.

The following lemma contains the fundamental properties of stopping times.

Lemma 48.4.3 *In the situation of Definition 48.4.1, let σ, τ be two stopping times. Then*

1. τ is $\mathcal{F}_\tau$ measurable
2. $\mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$
3. $\mathcal{F}_\tau = \mathcal{F}_k$ on $[\tau = k]$ for all k . That is if $A \in \mathcal{F}_k$, then $A \cap [\tau = k] \in \mathcal{F}_\tau$ and if $A \in \mathcal{F}_\tau$, then $A \cap [\tau = k] \in \mathcal{F}_k$. Also if $A \in \mathcal{F}_\tau$,

$$\int_{A \cap [\tau = k]} E(Y | \mathcal{F}_\tau) dP = \int_{A \cap [\tau = k]} E(Y | \mathcal{F}_k) dP$$

and

$$E(Y | \mathcal{F}_\tau) = E(Y | \mathcal{F}_k) \text{ a.e.}$$

on $[\tau = k]$.

Proof: Consider the first claim. I need to show that $[\tau \leq a] \cap [\tau \leq k] \in \mathcal{F}_k$ for every k . However, this is easy if $a \geq k$ because the left side is then $[\tau \leq k]$ which is given to be in $\mathcal{F}_k$ since τ is a stopping time. If $a < k$, it is also easy because then the left side is $[\tau \leq a] \in \mathcal{F}_{[a]}$ where $[a]$ is the greatest integer less than or equal to a .

Next consider the second claim. Let $A \in \mathcal{F}_\sigma$. I want to show first that

$$A \cap [\sigma \leq \tau] \in \mathcal{F}_\tau \quad (48.4.12)$$

In other words, I want to show

$$A \cap [\sigma \leq \tau] \cap [\tau \leq k] \in \mathcal{F}_k$$

for all k . This will be done if I can show

$$A \cap [\sigma \leq j] \cap [\tau \leq k] \in \mathcal{F}_k$$

for each $j \leq k$. However, since $\sigma \in \mathcal{F}_\sigma$, it follows $A \cap [\sigma \leq j] \in \mathcal{F}_j \subseteq \mathcal{F}_k$ for each $j \leq k$ and so this has shown what I wanted to show, $A \cap [\sigma \leq \tau] \in \mathcal{F}_\tau$. Now replace the stopping time, τ with the stopping time $\tau \wedge \sigma$ in what was just shown. This yields

$$A \cap [\sigma \leq \tau \wedge \sigma] \in \mathcal{F}_{\tau \wedge \sigma}$$

However the left side equals $A \cap [\sigma \leq \tau]$. Thus

$$A \cap [\sigma \leq \tau] \in \mathcal{F}_{\tau \wedge \sigma}$$

This has shown the first part of 2.), $\mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\tau \wedge \sigma}$. Now 48.4.12 implies if $A \in \mathcal{F}_{\sigma \wedge \tau}$,

$$A = A \cap \overbrace{[\sigma \wedge \tau \leq \tau]}^{\text{all of } \Omega} \in \mathcal{F}_\tau$$

and so $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\tau$. Similarly, $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma$ which shows

$$\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\tau \cap \mathcal{F}_\sigma.$$

Next let $A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$. Then is it in $\mathcal{F}_{\sigma \wedge \tau}$? Is $A \cap [\sigma \wedge \tau \leq k] \in \mathcal{F}_k$? Of course this is so because

$$\begin{aligned} A \cap [\sigma \wedge \tau \leq k] &= A \cap ([\sigma \leq k] \cup [\tau \leq k]) \\ &= (A \cap [\sigma \leq k]) \cup (A \cap [\tau \leq k]) \in \mathcal{F}_k \end{aligned}$$

since both σ, τ are stopping times. This proves part 2.).

Now consider part 3.). Let $A \in \mathcal{F}_k$. I need to show $A \cap [\tau = k] \in \mathcal{F}_\tau$. Is

$$A \cap [\tau = k] \cap [\tau \leq l] \in \mathcal{F}_l$$

for all l ? There are two cases. First suppose $l \geq k$. Then the expression on the left reduces to

$$A \cap [\tau = k] = A \cap ([\tau \leq k] \setminus [\tau \leq (k-1)]) \in \mathcal{F}_k.$$

If $l < k$, the left side equals $\emptyset \in \mathcal{F}_l$ and so this shows

$$\mathcal{F}_k \cap [\tau = k] \subseteq \mathcal{F}_\tau.$$

Now consider the other inclusion. Let $A \in \mathcal{F}_\tau$. Then I need to show $A \cap [\tau = k] \in \mathcal{F}_k$. Since $A \in \mathcal{F}_\tau$,

$$A \cap [\tau \leq j] \in \mathcal{F}_j$$

for all j . Therefore,

$$A \cap [\tau = k] = A \cap [\tau \leq k] \cap [\tau \leq k-1]^C \in \mathcal{F}_k$$

It only remains to verify the claim about the conditional expectations. Let $A \in \mathcal{F}_\tau$ be arbitrary.

$$\int_{A \cap [\tau=k]} E(Y|\mathcal{F}_\tau) dP \equiv \int_{A \cap [\tau=k]} Y dP \equiv \int_{A \cap [\tau=k]} E(Y|\mathcal{F}_k) dP$$

This is because the set, $A \cap [\tau = k]$ is in both $\mathcal{F}_k$ and $\mathcal{F}_\tau$. Since A is arbitrary, it follows

$$E(Y|\mathcal{F}_\tau) = E(Y|\mathcal{F}_k) \text{ a.e.}$$

on $[\tau = k]$. This proves the third claim and the Lemma.

With this lemma, here is a major theorem, the optional sampling theorem of Doob. This one is special for martingales.

Theorem 48.4.4 *Let $\{M(k)\}$ be a real valued martingale with respect to the increasing sequence of σ algebras, $\{\mathcal{F}_k\}$ and let σ, τ be two stopping times such that τ is bounded. Then $M(\tau)$ defined as*

$$\omega \rightarrow M(\tau(\omega))$$

is integrable and

$$M(\sigma \wedge \tau) = E(M(\tau) | \mathcal{F}_\sigma).$$

Proof: Consider first the measurability of $\omega \rightarrow M(\tau(\omega))$. Say $\tau \leq l$.

$$[M(\tau) \leq a] = \cup_{i=1}^l [\tau = i] \cap [M(i) \leq a] \in \mathcal{F}_l$$

so there is at least no problem with measurability. By Proposition 48.4.2 $M(\tau)$ is also $\mathcal{F}_\tau$ measurable.

Next note that since τ is bounded,

$$\int_{\Omega} |M(\tau(\omega))| dP \leq \sum_{i=1}^l \int_{[\tau=i]} |M(i)| dP < \infty.$$

This proves the first assertion and makes possible the consideration of conditional expectation.

Let $l \geq \tau$ as described above. Then for $k \leq l$, by Lemma 48.4.3,

$$\mathcal{F}_k \cap [\tau = k] = \mathcal{F}_\tau \cap [\tau = k]$$

and so if $A \in \mathcal{F}_k \cap [\tau = k]$,

$$\begin{aligned} \int_A E(M(l) | \mathcal{F}_\tau) dP &\equiv \int_A M(l) dP \\ &= \int_A E(M(l) | \mathcal{F}_k) dP \\ &= \int_A M(k) dP = \int_A M(\tau) dP. \end{aligned}$$

Therefore, since A was arbitrary,

$$E(M(l) | \mathcal{F}_\tau) = M(\tau) \text{ a.e.}$$

on $[\tau = k]$ for every $k \leq l$. It follows

$$E(M(l) | \mathcal{F}_\tau) = M(\tau) \text{ a.e.} \quad (48.4.13)$$

since it is true on each $[\tau = k]$ for all $k \leq l$.

Next I claim that $[\sigma \leq \tau] \in \mathcal{F}_\tau \cap \mathcal{F}_l \cap \mathcal{F}_\sigma$. This is because this set is of the form

$$\cup_{i=1}^l [\tau = i] \cap [\sigma \leq i] \in \mathcal{F}_l$$

This shows it is in $\mathcal{F}_l$. Now

$$[\tau = i] \cap [\sigma \leq i] \cap [\tau \leq k] \in \mathcal{F}_k$$

if $k \geq i$ while if $k < i$, it reduces to $\emptyset \in \mathcal{F}_k$. Thus $[\sigma \leq \tau] \in \mathcal{F}_\tau$ also because it is a finite union of these sets, each in $\mathcal{F}_\tau$. Now consider the claim $[\sigma \leq \tau] \in \mathcal{F}_\sigma$. This is because

$$\begin{aligned} [\sigma \leq \tau] &= \cup_{i=1}^{\infty} [\sigma = i] \cap [\tau \geq i] \\ &= \cup_{i=1}^l [\sigma = i] \cap [\tau \geq i] \end{aligned}$$

An individual term is in $\mathcal{F}_\sigma$ because

$$[\sigma = i] \cap [\tau \geq i] \cap [\sigma \leq k] = [\sigma = i] \cap [\tau \geq i] \in \mathcal{F}_i \subseteq \mathcal{F}_k$$

when $k \geq i$ and for $k < i$, this set,

$$[\sigma = i] \cap [\tau \geq i] \cap [\sigma \leq k] = \emptyset \in \mathcal{F}_k.$$

Thus $[\sigma \leq \tau]$ is the finite sum of sets in $\mathcal{F}_\sigma$ so it is also in $\mathcal{F}_\sigma$. I also claim that if $A \in \mathcal{F}_\sigma$, then

$$A \cap [\sigma \leq \tau] \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau. \quad (48.4.14)$$

Here is why. It is only necessary to show it is in $\mathcal{F}_\tau$.

$$A \cap [\sigma \leq \tau] \cap [\tau \leq k] = \cup_{i=1}^k A \cap [\sigma \leq i] \cap [\tau = i]$$

and by assumption $A \cap [\sigma \leq i] \in \mathcal{F}_i \subseteq \mathcal{F}_k$ while $[\tau = i] \in \mathcal{F}_i \subseteq \mathcal{F}_k$. Thus 48.4.14 holds.

Now let $A \in \mathcal{F}_\sigma$. Then using 48.4.13 and Lemma 48.4.3,

$$\begin{aligned} \int_{A \cap [\sigma \leq \tau]} E(M(\tau) | \mathcal{F}_\sigma) dP &= \int_{A \cap [\sigma \leq \tau]} E(E(M(l) | \mathcal{F}_\tau) | \mathcal{F}_\sigma) dP \\ &= \sum_{i=1}^l \int_{A \cap [\sigma \leq i] \cap [\tau = i]} E(E(M(l) | \mathcal{F}_i) | \mathcal{F}_\sigma) dP = \sum_{i=1}^l \int_{A \cap [\sigma \leq i] \cap [\tau = i]} E(M(i) | \mathcal{F}_\sigma) dP \\ &= \sum_{i=1}^l \sum_{j=1}^i \int_{A \cap [\sigma = j] \cap [\tau = i]} E(M(i) | \mathcal{F}_j) dP = \sum_{i=1}^l \sum_{j=1}^i \int_{A \cap [\sigma = j] \cap [\tau = i]} M(j) dP \\ &= \sum_{i=1}^l \sum_{j=1}^i \int_{A \cap [\sigma = j] \cap [\tau = i]} M(\sigma) dP = \sum_{i=1}^l \int_{A \cap [\sigma \leq i] \cap [\tau = i]} M(\sigma) dP \\ &= \int_{A \cap [\sigma \leq \tau]} M(\sigma) dP = \int_{A \cap [\sigma \leq \tau]} M(\sigma \wedge \tau) dP \end{aligned}$$

It follows that on $[\sigma \leq \tau]$,

$$E(M(\tau) | \mathcal{F}_\sigma) = M(\sigma \wedge \tau) \text{ a.e.} \quad (48.4.15)$$

because $M(\sigma \wedge \tau)$ is $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma$ measurable. Next consider the complement of this set, $[\sigma > \tau]$. I want to show that the same formula as above holds for this set also. Letting $A \in \mathcal{F}_\sigma$,

$$\begin{aligned} \int_{A \cap [\sigma > \tau]} E(M(\tau) | \mathcal{F}_\sigma) dP &= \int_{A \cap [\sigma > \tau]} E(M(\tau \wedge \sigma) | \mathcal{F}_\sigma) dP \\ &= \int_{A \cap [\sigma > \tau]} M(\tau \wedge \sigma) dP \end{aligned}$$

because by Lemma 48.4.3, $M(\tau \wedge \sigma)$ is $\mathcal{F}_{\sigma \wedge \tau}$ measurable and this is a subset of $\mathcal{F}_\sigma$. Thus 48.4.15 holds on $[\sigma > \tau]$. It follows 48.4.15 holds a.e. This proves the theorem.

This is a really amazing theorem. Note it says $M(\sigma \wedge \tau) = E(M(\tau) | \mathcal{F}_\sigma)$. This would not be so surprising if it had said

$$M(\sigma \wedge \tau) = E(M(\tau) | \mathcal{F}_{\sigma \wedge \tau}).$$

48.4.2 Maximal Inequalities

Next I will show that stopping times and the optional stopping theorem can be used to establish maximal inequalities for submartingales very easily.

Theorem 48.4.5 *Let $\{X_k\}$ be a real valued submartingale with respect to the σ algebras $\{\mathcal{F}_k\}$. Then for $\lambda > 0$*

$$\lambda P\left(\left[\max_{1 \leq k \leq n} X_k \geq \lambda\right]\right) \leq E(X_n^+), \quad (48.4.16)$$

$$\lambda P\left(\left[\min_{1 \leq k \leq n} X_k \leq -\lambda\right]\right) \leq E(|X_n| + |X_1|), \quad (48.4.17)$$

$$\lambda P\left(\left[\max_{1 \leq k \leq n} |X_k| \geq \lambda\right]\right) \leq 2E(|X_n| + |X_1|). \quad (48.4.18)$$

Proof: Let $T(\omega)$ be the first time $X_k(\omega) \geq \lambda$ or if this does not happen for $k \leq n$, then $T(\omega) \equiv n$. Thus

$$T(\omega) \equiv \min(\min\{k : T_k(\omega) \geq \lambda\}, n)$$

Then $1 \leq T(\omega) \leq n$ are stopping times. Therefore, from the optional stopping theorem, X_1, X_T, X_n is a submartingale. It follows

$$\begin{aligned} E(X_n) &\geq E(X_T) = \int_{[\max_k X_k \geq \lambda]} X_T dP + \int_{[\max_k X_k < \lambda]} X_T dP \\ &= \int_{[\max_k X_k \geq \lambda]} X_T dP + \int_{[\max_k X_k < \lambda]} X_n dP \end{aligned}$$

and so, subtracting the last term on the right from both sides,

$$\begin{aligned} E(X_n^+) &\geq \int_{[\max_k X_k \geq \lambda]} X_n dP = \int_{[\max_k X_k \geq \lambda]} X_T dP \\ &\geq \lambda P\left(\left[\max_k X_k \geq \lambda\right]\right) \end{aligned}$$

because $X_T(\omega) \geq \lambda$ on $[\max_k X_k \geq \lambda]$ from the definition of T . This establishes 48.4.16.

Next let $T(\omega)$ be the first time $X_k(\omega)$ is $\leq -\lambda$ or if this does not happen for $k \leq n$, then $T(\omega) \equiv n$. Then this is a stopping time and $1 \leq T(\omega) \leq n$ are stopping times and so by the optional stopping theorem, X_1, X_T, X_n is a submartingale. Therefore, on

$$\left[\min_k X_k \leq -\lambda \right], \quad X_T(\omega) \leq -\lambda$$

and $E(X_T | \mathcal{F}_1) \geq X_1$ and so

$$E(X_1) \leq E(E(X_T | \mathcal{F}_T)) = E(X_T)$$

which implies

$$\begin{aligned} E(X_1) &\leq E(X_T) = \int_{[\min_k X_k \leq -\lambda]} X_T dP + \int_{[\min_k X_k > -\lambda]} X_T dP \\ &= \int_{[\min_k X_k \leq -\lambda]} X_T dP + \int_{[\min_k X_k > -\lambda]} X_n dP \end{aligned}$$

and so

$$\begin{aligned} E(X_1) - \int_{[\min_k X_k > -\lambda]} X_n dP &\leq \int_{[\min_k X_k \leq -\lambda]} X_T dP \\ &\leq -\lambda P\left(\left[\min_k X_k \leq -\lambda\right]\right) \end{aligned}$$

which implies

$$\begin{aligned} \lambda P\left(\left[\min_k X_k \leq -\lambda\right]\right) &\leq \int_{[\min_k X_k > -\lambda]} X_n dP - E(X_1) \\ &\leq \int_{\Omega} (|X_n| + |X_1|) dP \end{aligned}$$

and this proves 48.4.17.

The last estimate follows from these. Here is why.

$$\left[\max_{1 \leq k \leq n} |X_k| \geq \lambda \right] \subseteq \left[\max_{1 \leq k \leq n} X_k \geq \lambda \right] \cup \left[\min_{1 \leq k \leq n} X_k \leq -\lambda \right]$$

and so

$$\begin{aligned} \lambda P\left(\left[\max_{1 \leq k \leq n} |X_k| \geq \lambda \right]\right) &\leq \lambda P\left(\left[\max_{1 \leq k \leq n} X_k \geq \lambda \right] \cup \left[\min_{1 \leq k \leq n} X_k \leq -\lambda \right]\right) \\ &\leq \lambda P\left(\left[\max_{1 \leq k \leq n} X_k \geq \lambda \right]\right) + \lambda P\left(\left[\min_{1 \leq k \leq n} X_k \leq -\lambda \right]\right) \\ &\leq 2E(|X_1| + |X_n|) \end{aligned}$$

and this proves the last estimate.

48.4.3 The Upcrossing Estimate

A very interesting example of stopping times is next. It has to do with upcrossings.

Definition 48.4.6 Let $\{X_k\}$ be a sequence of random variables adapted to the increasing sequence of σ algebras, $\{\mathcal{F}_k\}$. Let $[a, b]$ be an interval. An upcrossing is a sequence $X_n(\omega), \dots, X_{n+p}(\omega)$ such that $X_n(\omega) \leq a, X_{n+i}(\omega) \in (a, b)$ for $i < p$, and $X_{n+p}(\omega) \geq b$.

Example 48.4.7 Let $\{\mathcal{F}_n\}$ be an increasing sequence of σ algebras contained in $\mathcal{F}$ where $(\Omega, \mathcal{F}, P)$ is a probability space and let $\{X_n\}$ be a sequence of real valued random variables such that X_n is $\mathcal{F}_n$ measurable. Also let $a < b$. Define

$$\begin{aligned} T_0(\omega) &\equiv \inf \{n : X_n(\omega) \leq a\} \\ T_1(\omega) &\equiv \inf \{n > T_0(\omega) : X_n(\omega) \geq b\} \\ T_2(\omega) &\equiv \inf \{n > T_1(\omega) : X_n \leq a\} \\ &\vdots \\ T_{2k-1}(\omega) &\equiv \inf \{n > T_{2k-2}(\omega) : X_n \geq b\} \\ T_{2k}(\omega) &\equiv \inf \{n > T_{2k-1}(\omega) : X_n \leq a\} \end{aligned}$$

If $X_n(\omega)$ is never in the desired interval for any $n > T_j(\omega)$, then define $T_{j+1}(\omega) \equiv \infty$. Then this is an increasing sequence of stopping times.

The reason for the claim is that

$$\begin{aligned} [T_0 \leq l] &= \cup_{k=1}^l X_k^{-1}((-\infty, a]) \in \mathcal{F}_l \\ [T_1 \leq l] &\equiv \cup_{k=T_0(\omega)+1}^l X_k^{-1}([b, \infty)) \in \mathcal{F}_l \end{aligned}$$

and so forth. Thus each T_j is a stopping time as claimed.

Now there is an interesting application of these stopping times to the concept of upcrossings. Let $\{X_n\}$ be a submartingale such that X_n is $\mathcal{F}_n$ measurable and let $a < b$. Assume $X_0(\omega) \leq a$. The function, $x \rightarrow (x - a)^+$ is increasing and convex so $\{(X_n - a)^+\}$ is also a submartingale. Furthermore, $\{X_n\}$ goes from $\leq a$ to $\geq b$ if and only if $\{(X_n - a)^+\}$ goes from 0 to $\geq b - a$. That is, a subsequence of the form $Y_n(\omega), Y_{n+1}(\omega), \dots, Y_{n+r}(\omega)$ for Y equal to either X or $(X - a)^+$ starts out below a (0) and ends up above b ($b - a$). Such a sequence is called an upcrossing of $[a, b]$. The idea is to estimate the expected number of upcrossings for $n \leq N$. For the stopping times defined in Example 48.4.7, let $T'_k \equiv \min(T_k, N)$. Thus T'_k , a continuous function of the stopping time, is also a stopping time which is bounded. Moreover, $T'_k \leq T'_{k+1}$. Now pick n such that $2n > N$. Then for each $\omega \in \Omega$

$$(X_N(\omega) - a)^+ - (X_0(\omega) - a)^+$$

must equal the sum of all successive terms of the form

$$\left(\left(X_{T'_{k+1}}(\omega) - a \right)^+ - \left(X_{T'_k}(\omega) - a \right)^+ \right)$$

for $k = 1, 2, \dots, 2n$. This is because $\{T'_k(\omega)\}$ is a strictly increasing sequence which starts with 1 due to the assumption $X_0(\omega) \leq a$ and ends with $N < 2n$. Therefore,

$$\begin{aligned} (X_N - a)^+ - (X_1 - a)^+ &= \sum_{k=1}^{2n} \left(X_{T'_k} - a \right)^+ - \left(X_{T'_{k-1}} - a \right)^+ \\ &= \overbrace{\sum_{k=0}^{n-1} \left(\left(X_{T'_{2k+1}} - a \right)^+ - \left(X_{T'_{2k}} - a \right)^+ \right)}^{\text{odds} - \text{evens}} + \overbrace{\sum_{k=1}^n \left(\left(X_{T'_{2k}} - a \right)^+ - \left(X_{T'_{2k-1}} - a \right)^+ \right)}^{\text{evens} - \text{odds}}. \end{aligned}$$

Now denote by $U_{[a,b]}^N$ the number of upcrossings. When T'_k is such that k is odd, $\left(X_{T'_k} - a \right)^+$ is above $b - a$ and when k is even, it equals 0. Therefore, in the first sum $X_{T'_{2k+1}} - X_{T'_{2k}} \geq b - a$ and there are $U_{[a,b]}^N$ terms which are nonzero in this sum. (Note this might not be n because many of the terms in the sum could be 0 due to the definition of T'_k .) Hence

$$\begin{aligned} (X_N - a)^+ - (X_0 - a)^+ &= (X_N - a)^+ \\ &\geq (b - a) U_{[a,b]}^N + \sum_{k=1}^n \left(\left(X_{T'_{2k}} - a \right)^+ - \left(X_{T'_{2k-1}} - a \right)^+ \right). \end{aligned} \quad (48.4.19)$$

Now $U_{[a,b]}^N$ is a random variable. To see this, let $Z_k(\omega) = 1$ if $T'_{2k+1} > T'_{2k}$ and 0 otherwise. Thus $U_{[a,b]}^N(\omega) = \sum_{k=0}^{n-1} Z_k(\omega)$. Therefore, it makes sense to take the expected value of both sides of 48.4.19. By the optional sampling theorem, $\left\{ \left(X_{T'_k} - a \right)^+ \right\}$ is a submartingale and so

$$\begin{aligned} &E \left(\left(X_{T'_{2k}} - a \right)^+ - \left(X_{T'_{2k-1}} - a \right)^+ \right) \\ &= \int_{\Omega} E \left(\left(X_{T'_{2k}} - a \right)^+ \mid \mathcal{F}_{T'_{2k-1}} \right) dP - \int_{\Omega} \left(X_{T'_{2k-1}} - a \right)^+ dP \geq 0. \end{aligned}$$

Therefore,

$$E \left((X_N - a)^+ \right) \geq (b - a) E \left(U_{[a,b]}^N \right). \quad (48.4.20)$$

This proves most of the following fundamental upcrossing estimate.

Theorem 48.4.8 *Let $\{X_n\}$ be a real valued submartingale such that X_n is $\mathcal{F}_n$ measurable. Then letting $U_{[a,b]}^N$ denote the upcrossings of $\{X_n\}$ from a to b for $n \leq N$,*

$$E \left(U_{[a,b]}^N \right) \leq \frac{1}{b - a} E \left((X_N - a)^+ \right).$$

Proof: The estimate 48.4.20 was based on the assumption that $X_0(\omega) \leq a$. If this is not so, modify X_0 . Change it to $\min(X_0, a)$. Then the upcrossings for the modified submartingale are at least as many as the upcrossings for the original submartingale. Therefore, the inequality remains.

Note this theorem holds if the submartingale starts at the index 1 rather than 0. Just adjust the argument.

48.5 The Submartingale Convergence Theorem

With this estimate it is now possible to prove the amazing submartingale convergence theorem.

Theorem 48.5.1 *Let $\{X_n\}$ be a real valued submartingale such that*

$$E(|X_n|) < M$$

for all n . Then there exists $X \in L^1(\Omega, \mathcal{F})$ such that $X_n(\omega)$ converges to $X(\omega)$ a.e. ω and $X \in L^1(\Omega)$.

Proof: Let $a < b$ be two rational numbers. From Theorem 48.4.8 it follows that for all N ,

$$\begin{aligned} \int_{\Omega} U_{[a,b]}^N dP &\leq \frac{1}{b-a} E((X_N - a)^+) \\ &\leq \frac{1}{b-a} (E(|X_N|) + |a|) \leq \frac{M + |a|}{b-a}. \end{aligned}$$

Therefore, letting $N \rightarrow \infty$, it follows that for a.e. ω , there are only finitely many upcrossings of $[a, b]$. Denote by $S_{[a,b]}$ the exceptional set. Then letting $S \equiv \cup_{a,b \in \mathbb{Q}} S_{[a,b]}$, it follows that $P(S) = 0$ and for $\omega \notin S$, $\{X_n(\omega)\}$ is a Cauchy sequence because if

$$\limsup_{n \rightarrow \infty} X_n(\omega) > \liminf_{n \rightarrow \infty} X_n(\omega)$$

then you can pick $\liminf_{n \rightarrow \infty} X_n(\omega) < a < b < \limsup_{n \rightarrow \infty} X_n(\omega)$ with a, b rational and conclude $\omega \in S_{[a,b]}$.

Let $X(\omega) = \lim_{n \rightarrow \infty} X_n(\omega)$ if $\omega \notin S$ and let $X(\omega) = 0$ if $\omega \in S$. Then it only remains to verify $X \in L^1(\Omega)$. Since X is the pointwise limit of measurable functions, it follows X is measurable. By Fatou's lemma,

$$\int_{\Omega} |X(\omega)| dP \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |X_n(\omega)| dP$$

Thus $X \in L^1(\Omega)$. This proves the theorem.

As a simple application, here is an easy proof of a nice theorem about convergence of sums of independent random variables.

Theorem 48.5.2 Let $\{X_k\}$ be a sequence of independent real valued random variables such that $E(|X_k|) < \infty$, $E(X_k) = 0$, and

$$\sum_{k=1}^{\infty} E(X_k^2) < \infty.$$

Then $\sum_{k=1}^{\infty} X_k$ converges a.e.

Proof: Let $\mathcal{F}_n \equiv \sigma(X_1, \dots, X_n)$. Consider $S_n \equiv \sum_{k=1}^n X_k$.

$$E(S_{n+1}|\mathcal{F}_n) = S_n + E(X_{n+1}|\mathcal{F}_n).$$

Letting $A \in \mathcal{F}_n$ it follows from independence that

$$\begin{aligned} \int_A E(X_{n+1}|\mathcal{F}_n) dP &\equiv \int_A X_{n+1} dP \\ &= \int_{\Omega} \chi_A X_{n+1} dP \\ &= P(A) \int_{\Omega} X_{n+1} dP = 0 \end{aligned}$$

and so $E(X_{n+1}|\mathcal{F}_n) = 0$. Therefore, $\{S_n\}$ is a martingale. Now using independence again,

$$E(|S_n|) \leq E(|S_n^2|) = \sum_{k=1}^n E(X_k^2) \leq \sum_{k=1}^{\infty} E(X_k^2) < \infty$$

and so $\{S_n\}$ is an L^1 bounded martingale. Therefore, it converges a.e. and this proves the theorem.

Corollary 48.5.3 Let $\{X_k\}$ be a sequence of independent real valued random variables such that $E(|X_k|) < \infty$, $E(X_k) = m_k$, and

$$\sum_{k=1}^{\infty} E(|X_k - m_k|^2) < \infty.$$

Then $\sum_{k=1}^{\infty} (X_k - m_k)$ converges a.e.

This can be extended to the case where the random variables have values in a separable Hilbert space.

Theorem 48.5.4 Let $\{X_k\}$ be a sequence of independent H valued random variables where H is a real separable Hilbert space such that $E(|X_k|_H) < \infty$, $E(X_k) = 0$, and

$$\sum_{k=1}^{\infty} E(|X_k|_H^2) < \infty.$$

Then $\sum_{k=1}^{\infty} X_k$ converges a.e.

Proof: Let $\{e_k\}$ be an orthonormal basis for H . Then $\{(X_n, e_k)_H\}_{n=1}^\infty$ are real valued, independent, and their mean equals 0. Also

$$\sum_{n=1}^{\infty} E \left(\left| (X_n, e_k)_H \right|^2 \right) \leq \sum_{n=1}^{\infty} E \left(|X_n|_H^2 \right) < \infty$$

and so from Theorem 48.5.2, the series,

$$\sum_{n=1}^{\infty} (X_n, e_k)_H$$

converges a.e. Therefore, there exists a set of measure zero such that for ω not in this set, $\sum_n (X_n(\omega), e_k)_H$ converges for each k . For ω not in this exceptional set, define

$$Y_k(\omega) \equiv \sum_{n=1}^{\infty} (X_n(\omega), e_k)_H$$

Next define

$$S(\omega) \equiv \sum_{k=1}^{\infty} Y_k(\omega) e_k. \quad (48.5.21)$$

Of course it is not clear this even makes sense. I need to show $\sum_{k=1}^{\infty} |Y_k(\omega)|^2 < \infty$. Using the independence of the X_n

$$\begin{aligned} E(|Y_k|^2) &= E \left(\left(\sum_{n=1}^{\infty} (X_n, e_k)_H \right)^2 \right) \\ &= E \left(\left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (X_n, e_k)_H (X_m, e_k)_H \right) \right) \\ &\leq \liminf_{N \rightarrow \infty} E \left(\left(\sum_{n=1}^N \sum_{m=1}^N (X_n, e_k)_H (X_m, e_k)_H \right) \right) \\ &= \liminf_{N \rightarrow \infty} E \left(\sum_{n=1}^N (X_n, e_k)_H^2 \right) \\ &= \sum_{n=1}^{\infty} E \left((X_n(\omega), e_k)_H^2 \right) \end{aligned}$$

Hence

$$E \left(\sum_k |Y_k|^2 \right) = \sum_k E \left(|Y_k|^2 \right) \leq \sum_k \sum_n E \left((X_n, e_k)_H^2 \right)$$

and by the monotone convergence theorem,

$$\begin{aligned} &= E \left(\sum_k \sum_n (X_n, e_k)_H^2 \right) = E \left(\sum_n \sum_k (X_n, e_k)_H^2 \right) \\ &= E \left(\sum_n |X_n|_H^2 \right) = \sum_n E \left(|X_n|_H^2 \right) < \infty \end{aligned}$$

Therefore, for ω off a set of measure zero,

$$\sum_k |Y_k(\omega)|^2 < \infty$$

and also for these ω ,

$$\sum_n \sum_k (X_n(\omega), e_k)_H^2 < \infty.$$

It follows that for ω not on a suitable set of measure zero, $S(\omega)$ defined by 48.5.21 makes sense. Thus for these ω

$$\begin{aligned} S(\omega) &= \sum_l (S(\omega), e_l) e_l = \sum_l Y_l(\omega) e_l = \sum_l \sum_n (X_n(\omega), e_l)_H e_l \\ &= \sum_n \sum_l (X_n(\omega), e_l) e_l = \sum_n X_n(\omega). \end{aligned}$$

This proves the theorem.

Now with this theorem, here is a strong law of large numbers.

Theorem 48.5.5 Suppose $\{\mathbf{X}_k\}$ are independent random variables and $E(|\mathbf{X}_k|) < \infty$ for each k and $E(\mathbf{X}_k) = \mathbf{m}_k$. Suppose also

$$\sum_{j=1}^{\infty} \frac{1}{j^2} E(|\mathbf{X}_j - \mathbf{m}_j|^2) < \infty. \quad (48.5.22)$$

Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (\mathbf{X}_j - \mathbf{m}_j) = \mathbf{0}$$

Proof: Consider the sum

$$\sum_{j=1}^{\infty} \frac{\mathbf{X}_j - \mathbf{m}_j}{j}.$$

This sum converges *a.e.* because of 48.5.22 and Theorem 48.5.4 applied to the random vectors $\left\{ \frac{\mathbf{X}_j - \mathbf{m}_j}{j} \right\}$. Therefore, from Lemma 47.1.27 it follows that for a.e. ω ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n (\mathbf{X}_j(\omega) - \mathbf{m}_j) = \mathbf{0}$$

This proves the theorem.

The next corollary is often called the strong law of large numbers. It follows immediately from the above theorem.

Corollary 48.5.6 Suppose $\{\mathbf{X}_j\}_{j=1}^{\infty}$ are independent and identically distributed random vectors, $\lambda_{\mathbf{X}_i} = \lambda_{\mathbf{X}_j}$ for all i, j having mean $\mathbf{m}$ and variance equal to

$$\sigma^2 \equiv \int_{\Omega} |\mathbf{X}_j - \mathbf{m}|^2 dP < \infty.$$

Then for a.e. $\omega \in \Omega$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n \mathbf{X}_j(\omega) = \mathbf{m}$$

48.6 A Reverse Submartingale Convergence Theorem

Definition 48.6.1 Let $\{X_n\}_{n=0}^{\infty}$ be a sequence of real random variables such that $E(|X_n|) < \infty$ for all n and let $\{\mathcal{F}_n\}$ be a sequence of σ algebras such that $\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$ for all n . Then $\{X_n\}$ is called a reverse submartingale if for all n ,

$$E(X_n | \mathcal{F}_{n+1}) \geq X_{n+1}.$$

Note it is just like a submartingale only the indices are going the other way. Here is an interesting lemma. This lemma gives uniform integrability for a reverse submartingale.

Lemma 48.6.2 Suppose $E(|X_n|) < \infty$ for all n , X_n is $\mathcal{F}_n$ measurable, $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$, and there exist X_{∞} $\mathcal{F}_{\infty}$ measurable such that $\mathcal{F}_{\infty} \subseteq \mathcal{F}_n$ for all n and X_0 $\mathcal{F}_0$ measurable such that $\mathcal{F}_0 \supseteq \mathcal{F}_n$ for all n such that for all $n \in \{0, 1, \dots\}$,

$$E(X_n | \mathcal{F}_{n+1}) \geq X_{n+1}, \quad E(X_n | \mathcal{F}_{\infty}) \geq X_{\infty},$$

where $E(|X_{\infty}|) < \infty$. Then $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable.

Proof:

$$E(X_{n+1}) \leq E(E(X_n | \mathcal{F}_{n+1})) = E(X_n)$$

Therefore, the sequence $\{E(X_n)\}$ is a decreasing sequence bounded below by $E(X_{\infty})$ so it has a limit. I am going to show the functions are equiintegrable. Let k be large enough that

$$\left| E(X_k) - \lim_{m \rightarrow \infty} E(X_m) \right| < \varepsilon \quad (48.6.23)$$

and suppose $n > k$. Then if $\lambda > 0$,

$$\int_{|X_n| \geq \lambda} |X_n| dP = \int_{X_n \geq \lambda} X_n dP + \int_{X_n \leq -\lambda} (-X_n) dP$$

$$\begin{aligned}
&= \int_{[X_n \geq \lambda]} X_n dP + \int_{\Omega} (-X_n) dP - \int_{[-X_n < \lambda]} (-X_n) dP \\
&= \int_{[X_n \geq \lambda]} X_n dP - \int_{\Omega} X_n dP + \int_{[-X_n < \lambda]} X_n dP
\end{aligned}$$

From 48.6.23,

$$\leq \int_{[X_n \geq \lambda]} X_n dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[-X_n < \lambda]} X_n dP$$

By assumption,

$$E(X_k | \mathcal{F}_n) \geq X_n$$

and so the above

$$\begin{aligned}
&\leq \int_{[X_n \geq \lambda]} E(X_k | \mathcal{F}_n) dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[-X_n < \lambda]} E(X_k | \mathcal{F}_n) dP \\
&= \int_{[X_n \geq \lambda]} X_k dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[-X_n < \lambda]} X_k dP \\
&= \int_{[X_n \geq \lambda]} X_k dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[X_n > -\lambda]} X_k dP \\
&= \int_{[X_n \geq \lambda]} X_k dP + \left(\int_{\Omega} (-X_k) dP - \int_{[X_n > -\lambda]} (-X_k) dP \right) + \varepsilon \\
&= \int_{[X_n \geq \lambda]} X_k dP + \int_{[X_n \leq -\lambda]} (-X_k) dP + \varepsilon = \int_{[|X_n| \geq \lambda]} |X_k| dP + \varepsilon
\end{aligned}$$

Applying the maximal inequality for submartingales, Theorem 48.4.5,

$$P(\max\{|X_j| : j = n, \dots, 1\} \geq \lambda) \leq \frac{1}{\lambda} (E(|X_0|) + E(|X_{\infty}|)) \leq \frac{C}{\lambda}$$

and taking sup for all n ,

$$P(\sup\{|X_j|\} \geq \lambda) \leq \frac{C}{\lambda}$$

It follows since the single function, X_k is equiintegrable that for all λ large enough,

$$\int_{[|X_n| \geq \lambda]} |X_n| dP \leq 2\varepsilon$$

and since ε is arbitrary, this shows $\{X_n\}$ for $n > k$ is equiintegrable. Since there are only finitely many X_j for $j \leq k$, this shows $\{X_n\}$ is equiintegrable. Hence $\{X_n\}$ is uniformly integrable. This proves the lemma.

Now with this lemma and the upcrossing lemma it is easy to prove an important convergence theorem.

Theorem 48.6.3 *Let $\{X_n, \mathcal{F}_n\}_{n=0}^\infty$ be a backwards submartingale as described above and suppose $\sup_{n \geq 0} E(|X_n|) < \infty$. Then $\{X_n\}$ converges a.e. and in $L^1(\Omega)$ to a function, X_∞ .*

Proof: By the upcrossing lemma applied to the submartingale $\{X_k\}_{k=0}^N$, the number of upcrossings (Downcrossings is probably a better term. They are upcrossings as n gets smaller.) of the interval $[a, b]$ satisfies the inequality

$$E\left(U_{[a,b]}^N\right) \leq \frac{1}{b-a} E\left((X_0 - a)^+\right)$$

Letting $N \rightarrow \infty$, it follows the expected number of upcrossings, $E(U_{[a,b]})$ is bounded. Therefore, there exists a set of measure 0 N_{ab} such that if $\omega \notin N_{ab}$, $U_{[a,b]}(\omega) < \infty$. Let $N = \cup \{N_{ab} : a, b \in \mathbb{Q}\}$. Then for $\omega \notin N$,

$$\limsup_{n \rightarrow \infty} X_n(\omega) = \liminf_{n \rightarrow \infty} X_n(\omega)$$

because if inequality holds, then letting

$$\liminf_{n \rightarrow \infty} X_n(\omega) < a < b < \limsup_{n \rightarrow \infty} X_n(\omega)$$

it would follow $U_{[a,b]}(\omega) = \infty$, contrary to $\omega \notin N_{ab}$.

Let $X_\infty(\omega) \equiv \lim_{n \rightarrow \infty} X_n(\omega)$. Then by Fatou's lemma,

$$\int_{\Omega} |X_\infty(\omega)| dP \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |X_n| dP < \infty.$$

and so X_∞ is in $L^1(\Omega)$. By the Vitali convergence theorem and Lemma 50.5.10 which shows $\{|X_n|\}$ is uniformly integrable, it follows

$$\lim_{n \rightarrow \infty} \int_{\Omega} |X_\infty(\omega) - X_n(\omega)| dP = 0.$$

This proves the theorem.

48.7 Strong Law Of Large Numbers

There is a version of the strong law of large numbers which does not depend on the random variables having finite variance. First are some preparatory lemmas. The approach followed here is from Ash [6].

Lemma 48.7.1 *Let $\{X_n\}$ be a sequence of independent random variables such that $E(|X_k|) < \infty$ for all k and let $S_n \equiv \sum_{k=1}^n X_k$. Then for $k \leq n$,*

$$E(X_k | \sigma(S_n)) = E(X_k | \sigma(S_n, \mathbf{Y})) \quad \text{a.e.}$$

where $\mathbf{Y} = (X_{n+1}, X_{n+2}, \dots) \in \mathbb{R}^\mathbb{N}$. Also for $k \leq n$ as above,

$$\sigma(S_n, \mathbf{Y}) = \sigma(S_n, S_{n+1}, \dots).$$

Proof: Note that $\mathbb{R}^{\mathbb{N}}$ with the usual product topology has a countable basis. Here it is. Let $\mathcal{B}_N$ denote sets of the form $\prod_{i=1}^{\infty} D_i$ where for $i \leq N, D_i \in \mathcal{B}$, a countable basis for $\mathbb{R}$ and for $i > N, D_i = \mathbb{R}$. Then $\mathcal{B}_N$ is countable and so is $\mathcal{D} \equiv \cup_{N=1}^{\infty} \mathcal{B}_N$. From the definition of the product topology, this is a countable basis for the product topology.

Let $V \in \mathcal{D}$ and U be an open set of $\mathbb{R}$. Then if $A \in (S_n, \mathbf{Y})^{-1}(U \times V)$, by independence of the $\{X_n\}$,

$$\begin{aligned} & \int_{(S_n, \mathbf{Y})^{-1}(U \times V)} E(X_k | \sigma(S_n, \mathbf{Y})) dP \equiv \int_{(S_n, \mathbf{Y})^{-1}(U \times V)} X_k dP \\ &= \int_{\Omega} \mathcal{X}_{S_n^{-1}(U)}(\omega) \mathcal{X}_{\mathbf{Y}^{-1}(V)}(\omega) X_k dP = P(\mathbf{Y}^{-1}(V)) \int_{\Omega} \mathcal{X}_{S_n^{-1}(U)}(\omega) X_k dP \\ &= P(\mathbf{Y}^{-1}(V)) \int_{S_n^{-1}(U)} E(X_k | \sigma(S_n)) dP. \end{aligned}$$

Now by independence again, $\{S_n, X_{n+1}, X_{n+2}, \dots\}$ are independent and so the above equals

$$\int_{S_n^{-1}(U)} \mathcal{X}_{\mathbf{Y}^{-1}(V)} E(X_k | \sigma(S_n)) dP = \int_{(S_n, \mathbf{Y})^{-1}(U \times V)} E(X_k | \sigma(S_n)) dP.$$

Letting

$$\begin{aligned} \mathcal{S} &\equiv \left\{ A \in \mathcal{B}(\mathbb{R} \times \mathbb{R}^{\mathbb{N}}) : \int_{(S_n, \mathbf{Y})^{-1}(A)} E(X_k | \sigma(S_n)) dP \right. \\ &\quad \left. = \int_{(S_n, \mathbf{Y})^{-1}(A)} E(X_k | \sigma(S_n, \mathbf{Y})) dP \right\} \end{aligned}$$

the above has shown this is true for all A in a countable basis. Therefore, it is true for all A open in $\mathbb{R} \times \mathbb{R}^{\mathbb{N}}$. Finally, it is clear that $\mathcal{S}$ is a σ algebra which shows the above holds for all A Borel in $\mathbb{R} \times \mathbb{R}^{\mathbb{N}}$. Thus, for all $B \in \sigma(S_n, \mathbf{Y})$,

$$\int_B E(X_k | \sigma(S_n)) dP = \int_B E(X_k | \sigma(S_n, \mathbf{Y})) dP$$

and thus $E(X_k | \sigma(S_n)) = E(X_k | \sigma(S_n, \mathbf{Y}))$ a.e.

It only remains to prove the last assertion. For $k > 0$,

$$X_{n+k} = S_{n+k} - S_{n+k-1}$$

Thus

$$\begin{aligned} \sigma(S_n, \mathbf{Y}) &= \sigma(S_n, X_{n+1}, \dots) \\ &= \sigma(S_n, (S_{n+1} - S_n), (S_{n+2} - S_{n+1}), \dots) \\ &\subseteq \sigma(S_n, S_{n+1}, \dots) \end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma(S_n, S_{n+1}, \dots) &= \sigma(S_n, X_{n+1} + S_n, X_{n+2} + X_{n+1} + S_n, \dots) \\ &\subseteq \sigma(S_n, X_{n+1}, X_{n+2}, \dots)\end{aligned}$$

To see this, note that for an open set, and hence for a Borel set, B ,

$$\left(S_n + \sum_{k=n+1}^m X_k\right)^{-1}(B) = (S_n, X_{n+1}, \dots, X_m)^{-1}(B')$$

for some $B' \in \mathbb{R}^{m+1}$. Thus $(S_n + \sum_{k=n+1}^m X_k)^{-1}(B)$ for B a Borel set is contained in $\sigma(S_n, X_{n+1}, X_{n+2}, \dots)$. Similar considerations apply to the other inclusion stated earlier. This proves the lemma.

Lemma 48.7.2 *Let $\{X_k\}$ be a sequence of independent random variables such that $E(|X_k|) < \infty$. Then letting $S_n = \sum_{k=1}^n X_k$, it follows that for $k \leq n$*

$$E(X_k | \sigma(S_n, S_{n+1}, \dots)) = E(X_k | \sigma(S_n)) = \frac{S_n}{n}.$$

Proof: It was shown in Lemma 48.7.1 the first equality holds. It remains to show the second. Letting $A = S_n^{-1}(B)$ where B is Borel, it follows there exists $B' \subseteq \mathbb{R}^n$ a Borel set such that

$$S_n^{-1}(B) = (X_1, \dots, X_n)^{-1}(B').$$

Then

$$\begin{aligned}\int_A E(X_k | \sigma(S_n)) dP &= \int_{S_n^{-1}(B)} X_k dP \\ &= \int_{(X_1, \dots, X_n)^{-1}(B')} X_k dP = \int_{(X_1, \dots, X_n)^{-1}(B')} x_k d\lambda_{(X_1, \dots, X_n)} \\ &= \int \cdots \int \mathcal{X}_{(X_1, \dots, X_n)^{-1}(B')}(\mathbf{x}) x_k d\lambda_{X_1} d\lambda_{X_2} \cdots d\lambda_{X_n} \\ &= \int \cdots \int \mathcal{X}_{(X_1, \dots, X_n)^{-1}(B')}(\mathbf{x}) x_l d\lambda_{X_1} d\lambda_{X_2} \cdots d\lambda_{X_n} \\ &= \int_A E(X_l | \sigma(S_n)) dP\end{aligned}$$

and so since $A \in \sigma(S_n)$ is arbitrary,

$$E(X_l | \sigma(S_n)) = E(X_k | \sigma(S_n))$$

for each $k, l \leq n$. Therefore,

$$S_n = E(S_n | \sigma(S_n)) = \sum_{j=1}^n E(X_j | \sigma(S_n)) = nE(X_k | \sigma(S_n)) \text{ a.e.}$$

and so

$$E(X_k | \sigma(S_n)) = \frac{S_n}{n} \text{ a.e.}$$

as claimed. This proves the lemma.

With this preparation, here is the strong law of large numbers for identically distributed random variables.

Theorem 48.7.3 *Let $\{X_k\}$ be a sequence of independent identically distributed random variables such that $E(|X_k|) < \infty$ for all k . Letting $m = E(X_k)$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n X_k(\omega) = m \text{ a.e.}$$

and convergence also takes place in $L^1(\Omega)$.

Proof: Consider the reverse submartingale $\{E(X_1 | \sigma(S_n, S_{n+1}, \dots))\}$. By Theorem 48.6.3, this converges a.e. and in $L^1(\Omega)$ to a random variable, X_∞ . However, from Lemma 48.7.2, $E(X_1 | \sigma(S_n, S_{n+1}, \dots)) = S_n/n$. Therefore, S_n/n converges a.e. and in $L^1(\Omega)$ to X_∞ . I need to argue that X_∞ is constant and also that it equals m . For $a \in \mathbb{R}$ let

$$E_a \equiv [X_\infty \geq a]$$

For a small enough, $P(E_a) \neq 0$. Then since E_a is a tail event for the independent random variables, $\{X_k\}$ it follows from the Kolmogorov zero one law, Theorem 47.1.21, that $P(E_a) = 1$. Let $b \equiv \sup \{a : P(E_a) = 1\}$. The sets, E_a are decreasing as a increases. Let $\{a_n\}$ be a strictly increasing sequence converging to b . Then

$$[X_\infty \geq b] = \cap_n [X_\infty \geq a_n]$$

and so

$$1 = P(E_b) = \lim_{n \rightarrow \infty} P(E_{a_n}).$$

On the other hand, if $c > b$, then $P(E_c) < 1$ and so $P(E_c) = 0$. Hence $P([X = b]) = 1$. It remains to show $b = m$. This is easy because by the L^1 convergence,

$$b = \int_{\Omega} X_\infty dP = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{S_n}{n} dP = \lim_{n \rightarrow \infty} m = m.$$

This proves the theorem.

Probability In Infinite Dimensions

49.1 Conditional Expectation In Banach Spaces

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X \in L^1(\Omega; \mathbb{R})$. Also let $\mathcal{G} \subseteq \mathcal{F}$ where $\mathcal{G}$ is also a σ algebra. Then the usual conditional expectation is defined by

$$\int_A X dP = \int_A E(X|\mathcal{G}) dP$$

where $E(X|\mathcal{G})$ is $\mathcal{G}$ measurable and $A \in \mathcal{G}$ is arbitrary. Recall this is an application of the Radon Nikodym theorem. Also recall $E(X|\mathcal{G})$ is unique up to a set of measure zero.

I want to do something like this here. Denote by $L^1(\Omega; E, \mathcal{G})$ those functions in $L^1(\Omega; E)$ which are measurable with respect to $\mathcal{G}$.

Theorem 49.1.1 *Let E be a separable Banach space and let $X \in L^1(\Omega; E, \mathcal{F})$ where X is measurable with respect to $\mathcal{F}$ and let $\mathcal{G}$ be a σ algebra which is contained in $\mathcal{F}$. Then there exists a unique $Z \in L^1(\Omega; E, \mathcal{G})$ such that for all $A \in \mathcal{G}$,*

$$\int_A X dP = \int_A Z dP$$

Denoting this Z as $E(X|\mathcal{G})$, it follows

$$\|E(X|\mathcal{G})\| \leq E(\|X\| | \mathcal{G}).$$

Proof: First consider uniqueness. Suppose Z' is another in $L^1(\Omega; E, \mathcal{G})$ which works. Then let $A \equiv \{\omega : \|Z(\omega) - Z'(\omega)\| > \delta\}$ where $\delta > 0$. Now let $D = \{a_k\}$ be a countable dense subset of E and consider the balls $B(a_k, \frac{1}{4}\|a_k\|) \equiv B_k$ for which $\|a_k\| > \delta$. Since each of these balls has radius larger than $\delta/4$ and the $\{a_k\}$ are dense, it follows the union of these balls includes $\{x \in E : \|x\| > \delta\}$.

Now let

$$A_k \equiv \{\omega : \|Z(\omega) - Z'(\omega)\| > \delta\} \\ \cap \left\{ \omega : \|Z(\omega) - Z'(\omega) - a_k\| < \frac{1}{2} \|a_k\| \right\}.$$

These are the ω for which $Z(\omega) - Z'(\omega)$ is “close” to a_k . If $P(A) > 0$, then for some k , $P(A_k) > 0$ because the balls B_k cover the set $\{x : \|x\| > \delta\}$. Then

$$\begin{aligned} \frac{1}{2} \|a_k\| P(A_k) &\geq \int_{A_k} \|Z' - Z + a_k\| dP \\ &\geq \left\| \int_{A_k} (Z' - Z + a_k) dP \right\| \\ &= \left\| \int_{A_k} a_k dP \right\| = \|a_k\| P(A_k) \end{aligned}$$

which is a contradiction. Note that since both Z, Z' work,

$$\int_{A_k} (Z' - Z) dP = \int_{A_k} X dP - \int_{A_k} X dP = 0.$$

Hence $P(A) = 0$ after all. It follows $Z' = Z$ a.e. because $\delta > 0$ is arbitrary. This establishes the uniqueness part of the theorem.

Next I will show Z exists. To do this recall Theorem 23.2.5 on Page 708 which is stated below for convenience.

Theorem 49.1.2 *An E valued function, X , is Bochner integrable if and only if X is strongly measurable and*

$$\int_{\Omega} \|X(\omega)\| dP < \infty. \quad (49.1.1)$$

In this case there exists a sequence of simple functions $\{X_n\}$ satisfying

$$\int_{\Omega} \|X_n(\omega) - X_m(\omega)\| dP \rightarrow 0 \text{ as } m, n \rightarrow \infty. \quad (49.1.2)$$

$X_n(\omega)$ converging pointwise to $X(\omega)$,

$$\|X_n(\omega)\| \leq 2 \|X(\omega)\| \quad (49.1.3)$$

and

$$\lim_{n \rightarrow \infty} \int_{\Omega} \|X(\omega) - X_n(\omega)\| dP = 0. \quad (49.1.4)$$

Now let $\{X_n\}$ be the simple functions just defined and let

$$X_n(\omega) = \sum_{k=1}^m x_k \chi_{F_k}(\omega)$$

where $F_k \in \mathcal{F}$, the F_k being disjoint. Then define

$$Z_n \equiv \sum_{k=1}^m x_k E(\mathcal{X}_{F_k} | \mathcal{G}).$$

Thus, if $A \in \mathcal{G}$,

$$\begin{aligned} \int_A Z_n dP &= \sum_{k=1}^m x_k \int_A E(\mathcal{X}_{F_k} | \mathcal{G}) dP \\ &= \sum_{k=1}^m x_k \int_A \mathcal{X}_{F_k} dP \\ &= \sum_{k=1}^m x_k P(F_k \cap A) = \int_A X_n dP \end{aligned} \quad (49.1.5)$$

Then since $E(\mathcal{X}_{F_k} | \mathcal{G}) \geq 0$,

$$\begin{aligned} E(\|Z_n\|) &= E\left(\sum_{k=1}^m \|x_k\| E(\mathcal{X}_{F_k} | \mathcal{G})\right) \\ &\leq \sum_{k=1}^m \|x_k\| \int E(\mathcal{X}_{F_k} | \mathcal{G}) dP \\ &= \sum_{k=1}^m \|x_k\| \int \mathcal{X}_{F_k} dP = E(\|X_n\|). \end{aligned}$$

Note the use of $\leq$ in the first step in the above. Although the F_k are disjoint, all that is known about $E(\mathcal{X}_{F_k} | \mathcal{G})$ is that it is nonnegative. Similarly,

$$E(\|Z_n - Z_m\|) \leq E(\|X_n - X_m\|)$$

and this last term converges to 0 as $n, m \rightarrow \infty$ by the properties of the X_n . Therefore, $\{Z_n\}$ is a Cauchy sequence in $L^1(\Omega; E; \mathcal{G})$. It follows it converges to Z in $L^1(\Omega; E; \mathcal{G})$. Then letting $A \in \mathcal{G}$, and using 49.1.5,

$$\begin{aligned} \int_A Z dP &= \int \mathcal{X}_A Z dP \\ &= \lim_{n \rightarrow \infty} \int \mathcal{X}_A Z_n dP \\ &= \lim_{n \rightarrow \infty} \int_A Z_n dP \\ &= \lim_{n \rightarrow \infty} \int_A X_n dP \\ &= \int_A X dP. \end{aligned}$$

It remains to verify $\|E(X|\mathcal{G})\| \equiv \|Z\| \leq E(\|X\| |\mathcal{G})$. This follows similar to the above. Letting Z_n and Z have the same meaning as above and $A \in \mathcal{G}$, since $E(\mathcal{X}_{F_k}|\mathcal{G}) \geq 0$,

$$\begin{aligned} E(\mathcal{X}_A \|Z_n\|) &= \int_A \left\| \sum_{k=1}^m x_k E(\mathcal{X}_{F_k}|\mathcal{G}) \right\| dP \\ &\leq \sum_{k=1}^m \|x_k\| \int_A E(\mathcal{X}_{F_k}|\mathcal{G}) dP \\ &= \sum_{k=1}^m \|x_k\| \int_A \mathcal{X}_{F_k} dP \\ &= \int_A \sum_{k=1}^m \|x_k\| \mathcal{X}_{F_k} dP \\ &= E(\mathcal{X}_A \|X_n\|). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_A \|Z\| dP &= \int \mathcal{X}_A \|Z\| dP \\ &= \lim_{n \rightarrow \infty} \int \mathcal{X}_A \|Z_n\| dP \\ &\leq \lim_{n \rightarrow \infty} E(\mathcal{X}_A \|X_n\|) \\ &= \int_A \|X\| dP \end{aligned}$$

Thus for all $A \in \mathcal{G}$,

$$\begin{aligned} \int_A \|E(X|\mathcal{G})\| dP &\equiv \int_A \|Z\| dP \\ &\leq \int_A \|X\| dP = \int_A E(\|X\| |\mathcal{G}) dP \end{aligned}$$

which shows

$$\|E(X|\mathcal{G})\| \leq E(\|X\| |\mathcal{G})$$

as claimed. This proves the theorem.

In the case where E is reflexive, one could also use Corollary 23.7.5 on Page 739 to get the above result. You would define a vector measure on $\mathcal{G}$,

$$\nu(F) \equiv \int_F X dP$$

and then you would use the fact that reflexive separable Banach spaces have the Radon Nikodym property to obtain $Z \in L^1(\Omega; E, \mathcal{G})$ such that

$$\nu(F) = \int_F X dP = \int_F Z dP.$$

The function, Z whose existence and uniqueness is guaranteed by Theorem 49.1.2 is called $E(X|\mathcal{G})$.

49.2 Probability Measures And Tightness

Here and in what remains, $\mathcal{B}(E)$ will denote the Borel sets of E where E is a topological space, usually at least a Banach space. Because of the fact that probability measures are finite, you can use a simpler definition of what it means for a measure to be regular. Recall that there were two ingredients, inner regularity which said that the measure of a set is the supremum of the measures of compact subsets and outer regularity which says that the measure of a set is the infimum of the measures of the open sets which contain the given set. Here the definition will be similar but instead of using compact sets, closed sets are substituted. Thus the following definition is a little different than the earlier one. I will show, however, that in many interesting cases, this definition of regularity is actually the same as the earlier one.

Definition 49.2.1 *A measure, μ defined on $\mathcal{B}(E)$ will be called inner regular if for all $F \in \mathcal{B}(E)$,*

$$\mu(F) = \sup \{ \mu(K) : K \subseteq F \text{ and } K \text{ is closed} \}$$

A measure, μ defined on $\mathcal{B}(E)$ will be called outer regular if for all $F \in \mathcal{B}(E)$,

$$\mu(F) = \inf \{ \mu(V) : V \supseteq F \text{ and } V \text{ is open} \}$$

When a measure is both inner and outer regular, it is called regular.

For probability measures, regularity tends to come free.

Lemma 49.2.2 *Let μ be a finite measure defined on $\mathcal{B}(E)$ where E is a metric space. Then μ is regular.*

Proof: First note every open set is the countable union of closed sets and every closed set is the countable intersection of open sets. Here is why. Let V be an open set and let

$$K_k \equiv \{x \in V : \text{dist}(x, V^C) \geq 1/k\}.$$

Then clearly the union of the K_k equals V . Next, for K closed let

$$V_k \equiv \{x \in E : \text{dist}(x, K) < 1/k\}.$$

Clearly the intersection of the V_k equals K . Therefore, letting V denote an open set and K a closed set,

$$\begin{aligned} \mu(V) &= \sup \{ \mu(K) : K \subseteq V \text{ and } K \text{ is closed} \} \\ \mu(K) &= \inf \{ \mu(V) : V \supseteq K \text{ and } V \text{ is open} \}. \end{aligned}$$

Also since V is open and K is closed,

$$\begin{aligned}\mu(V) &= \inf \{ \mu(U) : U \supseteq V \text{ and } U \text{ is open} \} \\ \mu(K) &= \sup \{ \mu(L) : L \subseteq K \text{ and } L \text{ is closed} \}\end{aligned}$$

In words, μ is regular on open and closed sets. Let

$$\mathcal{F} \equiv \{F \in \mathcal{B}(E) \text{ such that } \mu \text{ is regular on } F\}.$$

Then $\mathcal{F}$ contains the open sets. I want to show $\mathcal{F}$ is a σ algebra and then it will follow $\mathcal{F} = \mathcal{B}(E)$.

First I will show $\mathcal{F}$ is closed with respect to complements. Let $F \in \mathcal{F}$. Then since μ is finite and F is inner regular, there exists $K \subseteq F$ such that $\mu(F \setminus K) < \varepsilon$. But $K^C \setminus F^C = F \setminus K$ and so $\mu(K^C \setminus F^C) < \varepsilon$ showing that F^C is outer regular. I have just approximated the measure of F^C with the measure of K^C , an open set containing F^C . A similar argument works to show F^C is inner regular. You start with $V \supseteq F$ such that $\mu(V \setminus F) < \varepsilon$, note $F^C \setminus V^C = V \setminus F$, and then conclude $\mu(F^C \setminus V^C) < \varepsilon$, thus approximating F^C with the closed subset, V^C .

Next I will show $\mathcal{F}$ is closed with respect to taking countable unions. Let $\{F_k\}$ be a sequence of sets in $\mathcal{F}$. Then μ is inner regular on each of these so there exist $\{K_k\}$ such that $K_k \subseteq F_k$ and $\mu(F_k \setminus K_k) < \varepsilon/2^{k+1}$. First choose m large enough that

$$\mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m F_k)) < \frac{\varepsilon}{2}.$$

Then

$$\mu((\cup_{k=1}^m F_k) \setminus (\cup_{k=1}^m K_k)) \leq \sum_{k=1}^m \frac{\varepsilon}{2^{k+1}} < \frac{\varepsilon}{2}$$

and so

$$\begin{aligned}\mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m K_k)) &\leq \mu((\cup_{k=1}^{\infty} F_k) \setminus (\cup_{k=1}^m F_k)) \\ &\quad + \mu((\cup_{k=1}^m F_k) \setminus (\cup_{k=1}^m K_k)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

showing μ is inner regular on $\cup_{k=1}^{\infty} F_k$. Since μ is outer regular on F_k , there exists V_k such that $\mu(V_k \setminus F_k) < \varepsilon/2^k$. Then

$$\begin{aligned}\mu((\cup_{k=1}^{\infty} V_k) \setminus (\cup_{k=1}^{\infty} F_k)) &\leq \sum_{k=1}^{\infty} \mu(V_k \setminus F_k) \\ &< \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon\end{aligned}$$

and this shows μ is outer regular on $\cup_{k=1}^{\infty} F_k$ and this proves the lemma.

Lemma 49.2.3 *Let μ be a finite measure on $\mathcal{B}(E)$, the Borel sets of E , a separable complete metric space. Then if C is a closed set,*

$$\mu(C) = \sup \{ \mu(K) : K \subseteq C \text{ and } K \text{ is compact} \}$$

Proof: Let $\{a_k\}$ be a countable dense subset of C . Thus $\cup_{k=1}^{\infty} B(a_k, \frac{1}{n}) \supseteq C$. Therefore, there exists m_n such that

$$\mu\left(C \setminus \overline{\cup_{k=1}^{m_n} B\left(a_k, \frac{1}{n}\right)}\right) \equiv \mu(C \setminus C_n) < \frac{\varepsilon}{2^n}.$$

Now let $K = C \cap (\cap_{n=1}^{\infty} C_n)$. Then K is a subset of C_n for each n and so for each $\varepsilon > 0$ there exists an ε net for K since C_n has a $1/n$ net, namely $a_1, \dots, a_{m_n}$. Since K is closed, it is complete and so it is also compact. Now

$$\mu(C \setminus K) = \mu(\cup_{n=1}^{\infty} (C \setminus C_n)) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

Thus $\mu(C)$ can be approximated by $\mu(K)$ for K a compact subset of C . This proves the lemma.

This shows that for a finite measure on the Borel sets of a separable metric space, the above definition of regular coincides with the earlier one.

Now here is a definition of what it means for a set of measures to be tight.

Definition 49.2.4 Let Λ be a set of probability measures defined on the Borel sets of a topological space. Then Λ is “tight” if for all $\varepsilon > 0$ there exists a compact set, K_ε such that

$$\mu([x \notin K_\varepsilon]) < \varepsilon$$

for all $\mu \in \Lambda$.

Lemma 49.2.3 implies a single probability measure on the Borel sets of a separable metric space is tight. The proof of that lemma generalizes slightly to give a simple criterion for a set of measures to be tight.

Lemma 49.2.5 Let E be a separable complete metric space and let Λ be a set of Borel probability measures. Then Λ is tight if and only if for every $\varepsilon > 0$ and $r > 0$ there exists a finite collection of balls, $\{B(a_i, r)\}_{i=1}^m$ such that

$$\mu\left(\overline{\cup_{i=1}^m B(a_i, r)}\right) > 1 - \varepsilon$$

for every $\mu \in \Lambda$.

Proof: If Λ is tight, then there exists a compact set, K_ε such that

$$\mu(K_\varepsilon) > 1 - \varepsilon$$

for all $\mu \in \Lambda$. Then consider the open cover, $\{B(x, r) : x \in K_\varepsilon\}$. Finitely many of these cover K_ε and this yields the above condition.

Now suppose the above condition and let

$$C_n \equiv \overline{\cup_{i=1}^{m_n} B(a_i^n, 1/n)}$$

satisfy $\mu(C_n) > 1 - \varepsilon/2^n$ for all $\mu \in \Lambda$. Then let $K_\varepsilon \equiv \cap_{n=1}^\infty C_n$. Then as in Lemma 49.2.3 $\mu(K_\varepsilon) > 1 - \varepsilon$ for all $\mu \in \Lambda$.

Prokhorov's theorem is an important result which also involves tightness. In order to give a proof of this important theorem, it is necessary to consider some simple results from topology which are interesting for their own sake.

Theorem 49.2.6 *Let H be a compact metric space. Then there exists a compact subset of $[0, 1]$, K and a continuous function, θ which maps K onto H .*

Proof: Without loss of generality, it can be assumed H is an infinite set since otherwise the conclusion is trivial. You could pick finitely many points of $[0, 1]$ for K .

Since H is compact, it is totally bounded. Therefore, there exists a 1 net for H $\{h_i\}_{i=1}^{m_1}$. Letting $H_i^1 \equiv \overline{B(h_i, 1)}$, it follows H_i^1 is also a compact metric space and so there exists a $1/2$ net for each H_i^1 , $\{h_j^i\}_{j=1}^{m_i}$. Then taking the intersection of $\overline{B(h_j^i, \frac{1}{2})}$ with H_i^1 to obtain sets denoted by H_j^2 and continuing this way, one can obtain compact subsets of H , $\{H_j^i\}$ which satisfies: each H_j^i is contained in some H_k^{i-1} , each H_j^i is compact with diameter less than i^{-1} , each H_j^i is the union of sets of the form H_k^{i+1} which are contained in it. Denoting by $\{H_j^i\}_{j=1}^{m_i}$ those sets corresponding to a superscript of i , it can also be assumed $m_i < m_{i+1}$. If this is not so, simply add in another point to the i^{-1} net. Now let $\{I_j^i\}_{j=1}^{m_i}$ be disjoint closed intervals in $[0, 1]$ each of length no longer than 2^{-m_i} which have the property that I_j^i is contained in I_k^{i-1} for some k . Letting $K_i \equiv \cup_{j=1}^{m_i} I_j^i$, it follows K_i is a sequence of nested compact sets. Let $K = \cap_{i=1}^\infty K_i$. Then each $x \in K$ is the intersection of a unique sequence of these closed intervals, $\{I_{j_k}^k\}_{k=1}^\infty$. Define $\theta x \equiv \cap_{k=1}^\infty H_{j_k}^k$. Since the diameters of the H_j^i converge to 0 as $i \rightarrow \infty$, this function is well defined. It is continuous because if $x_n \rightarrow x$, then ultimately x_n and x are both in $I_{j_k}^k$, the k^{th} closed interval in the sequence whose intersection is x . Hence, $d(\theta x_n, \theta x) \leq \text{diameter}(H_{j_k}^k) \leq 1/k$. To see the map is onto, let $h \in H$. Then from the construction, there exists a sequence $\{H_{j_k}^k\}_{k=1}^\infty$ of the above sets whose intersection equals h . Then $\theta(\cap_{i=1}^\infty I_{j_k}^k) = h$. This proves the theorem.

Note θ is maybe not one to one.

As an important corollary, it follows that the continuous functions defined on any compact metric space is separable.

Corollary 49.2.7 *Let H be a compact metric space and let $C(H)$ denote the continuous functions defined on H with the usual norm,*

$$\|f\|_\infty \equiv \max \{|f(x)| : x \in H\}$$

Then $C(H)$ is separable.

Proof: The proof is by contradiction. Suppose $C(H)$ is not separable. Let $\mathcal{H}_k$ denote a maximal collection of functions of $C(H)$ with the property that if

$f, g \in \mathcal{H}_k$, then $\|f - g\|_\infty \geq 1/k$. The existence of such a maximal collection of functions is a consequence of a simple use of the Hausdorff maximality theorem. Then $\cup_{k=1}^\infty \mathcal{H}_k$ is dense. Therefore, it cannot be countable by the assumption that $C(H)$ is not separable. It follows that for some k , $\mathcal{H}_k$ is uncountable. Now by Theorem 49.2.6 there exists a continuous function, θ defined on a compact subset, K of $[0, 1]$ which maps K onto H . Now consider the functions defined on K

$$\mathcal{G}_k \equiv \{f \circ \theta : f \in \mathcal{H}_k\}.$$

Then $\mathcal{G}_k$ is an uncountable set of continuous functions defined on K with the property that the distance between any two of them is at least as large as $1/k$. This contradicts separability of $C(K)$ which follows from the Weierstrass approximation theorem in which the separable countable set of functions is the restrictions of polynomials that involve only rational coefficients. This proves the corollary. Now here is Prokhorov's theorem.

Theorem 49.2.8 *Let $\Lambda = \{\mu_n\}_{n=1}^\infty$ be a sequence of probability measures defined on $\mathcal{B}(E)$ where E is a separable Banach space. If Λ is tight then there exists a probability measure, λ and a subsequence of $\{\mu_n\}_{n=1}^\infty$, still denoted by $\{\mu_n\}_{n=1}^\infty$ such that whenever ϕ is a continuous bounded complex valued function defined on E ,*

$$\lim_{n \rightarrow \infty} \int \phi d\mu_n = \int \phi d\lambda.$$

Proof: By tightness, there exists an increasing sequence of compact sets, $\{K_n\}$ such that

$$\mu(K_n) > 1 - \frac{1}{n}$$

for all $\mu \in \Lambda$. Now letting $\mu \in \Lambda$ and $\phi \in C(K_n)$ such that $\|\phi\|_\infty \leq 1$, it follows

$$\left| \int_{K_n} \phi d\mu \right| \leq \mu(K_n) \leq 1$$

and so the restrictions of the measures of Λ to K_n are contained in the unit ball of $C(K_n)'$. Recall from the Riesz representation theorem, the dual space of $C(K_n)$ is a space of complex Borel measures. Theorem 13.4.5 on Page 384 implies the unit ball of $C(K_n)'$ is weak * sequentially compact. This follows from the observation that $C(K_n)$ is separable which is proved in Corollary 49.2.7 and leads to the fact that the unit ball in $C(K_n)'$ is actually metrizable by Theorem 13.4.5 on Page 384. Therefore, there exists a subsequence of Λ , $\{\mu_{1k}\}$ such that their restrictions to K_1 converge weak * to a measure, $\lambda_1 \in C(K_1)'$. That is, for every $\phi \in C(K_1)$,

$$\lim_{k \rightarrow \infty} \int_{K_1} \phi d\mu_{1k} = \int_{K_1} \phi d\lambda_1$$

By the same reasoning, there exists a further subsequence $\{\mu_{2k}\}$ such that the restrictions of these measures to K_2 converge weak * to a measure $\lambda_2 \in C(K_2)'$

etc. Continuing this way,

$$\begin{aligned}\mu_{11}, \mu_{12}, \mu_{13}, \cdots &\rightarrow \text{Weak } * \text{ in } C(K_1)' \\ \mu_{21}, \mu_{22}, \mu_{23}, \cdots &\rightarrow \text{Weak } * \text{ in } C(K_2)' \\ \mu_{31}, \mu_{32}, \mu_{33}, \cdots &\rightarrow \text{Weak } * \text{ in } C(K_3)' \\ &\vdots\end{aligned}$$

Here the j^{th} sequence is a subsequence of the $(j-1)^{th}$. Let λ_n denote the measure in $C(K_n)'$ to which the sequence $\{\mu_{nk}\}_{k=1}^\infty$ converges weak*. Let $\{\mu_n\} \equiv \{\mu_{nn}\}$, the diagonal sequence. Thus this sequence is ultimately a subsequence of every one of the above sequences and so μ_n converges weak* in $C(K_m)'$ to λ_m for each m .

Claim: For $p > n$, the restriction of λ_p to the Borel sets of K_n equals λ_n .

Proof of claim: Let H be a compact subset of K_n . Then there are sets, V_l open in K_n which are decreasing and whose intersection equals H . This follows because this is a metric space. Then let $H \prec \phi_l \prec V_l$. It follows

$$\begin{aligned}\lambda_n(V_l) &\geq \int_{K_n} \phi_l d\lambda_n = \lim_{k \rightarrow \infty} \int_{K_n} \phi_l d\mu_k \\ &= \lim_{k \rightarrow \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \geq \lambda_p(H).\end{aligned}$$

Now considering the ends of this inequality, let $l \rightarrow \infty$ and pass to the limit to conclude

$$\lambda_n(H) \geq \lambda_p(H).$$

Similarly,

$$\begin{aligned}\lambda_n(H) &\leq \int_{K_n} \phi_l d\lambda_n = \lim_{k \rightarrow \infty} \int_{K_n} \phi_l d\mu_k \\ &= \lim_{k \rightarrow \infty} \int_{K_p} \phi_l d\mu_k = \int_{K_p} \phi_l d\lambda_p \leq \lambda_p(V_l).\end{aligned}$$

Then passing to the limit as $l \rightarrow \infty$, it follows

$$\lambda_n(H) \leq \lambda_p(H).$$

Thus the restriction of $\lambda_p, \lambda_p|_{K_n}$ to the compact sets of K_n equals λ_n . Then by inner regularity it follows the two measures, $\lambda_p|_{K_n}$, and λ_n are equal on all Borel sets of K_n . Recall that for finite measures on separable metric spaces, regularity is obtained for free.

It is fairly routine to exploit regularity of the measures to verify that $\lambda_m(F) \geq 0$ for all F a Borel subset of K_m . (Whenever $\phi \geq 0$, $\int_{K_m} \phi d\lambda_m \geq 0$ because $\int_{K_m} \phi d\mu_k \geq 0$. Now you can approximate χ_F with a suitable nonnegative ϕ using regularity of the measure.) Also, letting $\phi \equiv 1$,

$$1 \geq \lambda_m(K_m) \geq 1 - \frac{1}{m}. \tag{49.2.6}$$

Define for F a Borel set,

$$\lambda(F) \equiv \lim_{n \rightarrow \infty} \lambda_n(F \cap K_n).$$

The limit exists because the sequence on the right is increasing due to the above observation that $\lambda_n = \lambda_m$ on the Borel subsets of K_m whenever $n > m$. Thus for $n > m$

$$\lambda_n(F \cap K_n) \geq \lambda_n(F \cap K_m) = \lambda_m(F \cap K_m).$$

Now let $\{F_k\}$ be a sequence of disjoint Borel sets. Then

$$\begin{aligned} \lambda(\cup_{k=1}^{\infty} F_k) &\equiv \lim_{n \rightarrow \infty} \lambda_n(\cup_{k=1}^{\infty} F_k \cap K_n) = \lim_{n \rightarrow \infty} \lambda_n(\cup_{k=1}^{\infty} (F_k \cap K_n)) \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \lambda_n(F_k \cap K_n) = \sum_{k=1}^{\infty} \lambda(F_k) \end{aligned}$$

the last equation holding by the monotone convergence theorem.

It remains to verify

$$\lim_{k \rightarrow \infty} \int \phi d\mu_k = \int \phi d\lambda$$

for every ϕ bounded and continuous. This is where tightness is used again. Suppose $\|\phi\|_{\infty} < M$. Then as noted above,

$$\lambda_n(K_n) = \lambda(K_n)$$

because for $p > n$, $\lambda_p(K_n) = \lambda_n(K_n)$ and so letting $p \rightarrow \infty$, the above is obtained. Also, from 49.2.6,

$$\begin{aligned} \lambda(K_n^C) &= \lim_{p \rightarrow \infty} \lambda_p(K_n^C \cap K_p) \\ &\leq \lim_{p \rightarrow \infty} \sup (\lambda_p(K_p) - \lambda_p(K_n)) \\ &\leq \lim_{p \rightarrow \infty} \sup (\lambda_p(K_p) - \lambda_n(K_n)) \\ &\leq \lim_{p \rightarrow \infty} \sup \left(1 - \left(1 - \frac{1}{n} \right) \right) = \frac{1}{n} \end{aligned}$$

Consequently,

$$\begin{aligned} \left| \int \phi d\mu_k - \int \phi d\lambda \right| &\leq \left| \int_{K_n^C} \phi d\mu_k + \int_{K_n} \phi d\mu_k - \left(\int_{K_n} \phi d\lambda + \int_{K_n^C} \phi d\lambda \right) \right| \\ &\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n \right| + \left| \int_{K_n^C} \phi d\mu_k - \int_{K_n^C} \phi d\lambda \right| \\ &\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n \right| + \left| \int_{K_n^C} \phi d\mu_k \right| + \left| \int_{K_n^C} \phi d\lambda \right| \\ &\leq \left| \int_{K_n} \phi d\mu_k - \int_{K_n} \phi d\lambda_n \right| + \frac{M}{n} + \frac{M}{n} \end{aligned}$$

First let n be so large that $2M/n < \varepsilon/2$ and then pick k large enough that the above expression is less than ε . This proves the theorem.

Definition 49.2.9 Let E be a Banach space and let μ and the sequence of probability measures, $\{\mu_n\}$ defined on $\mathcal{B}(E)$ satisfy

$$\lim_{n \rightarrow \infty} \int \phi d\mu_n = \int \phi d\mu.$$

for every ϕ a bounded continuous function. Then μ_n is said to converge weakly to μ .

49.3 A Major Existence And Convergence Theorem

Here is an interesting lemma about weak convergence.

Lemma 49.3.1 Let μ_n converge weakly to μ and let U be an open set with $\mu(\partial U) = 0$. Then

$$\lim_{n \rightarrow \infty} \mu_n(U) = \mu(U).$$

Proof: Let $\{\psi_k\}$ be a sequence of bounded continuous functions which decrease to $\chi_{\bar{U}}$. Also let $\{\phi_k\}$ be a sequence of bounded continuous functions which increase to χ_U . For example, you could let

$$\begin{aligned} \psi_k(x) &\equiv (1 - k \operatorname{dist}(x, U))^+, \\ \phi_k(x) &\equiv 1 - (1 - k \operatorname{dist}(x, U^C))^+. \end{aligned}$$

Let $\varepsilon > 0$ be given. Then since $\mu(\partial U) = 0$, the dominated convergence theorem implies there exists $\psi = \psi_k$ and $\phi = \phi_k$ such that

$$\varepsilon > \int \psi d\mu - \int \phi d\mu$$

Next use the weak convergence to pick N large enough that if $n \geq N$,

$$\int \psi d\mu_n \leq \int \psi d\mu + \varepsilon, \quad \int \phi d\mu_n \geq \int \phi d\mu - \varepsilon.$$

Therefore, for n this large,

$$\mu(U), \mu_n(U) \in \left[\int \phi d\mu - \varepsilon, \int \psi d\mu + \varepsilon \right]$$

and so

$$|\mu(U) - \mu_n(U)| < 3\varepsilon.$$

since ε is arbitrary, this proves the lemma.

Definition 49.3.2 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \rightarrow E$ be a random variable where here E is some topological space. Then one can define a probability measure, λ_X on $\mathcal{B}(E)$ as follows:

$$\lambda_X(F) \equiv P([X \in F])$$

More generally, if μ is a probability measure on $\mathcal{B}(E)$, and X is a random variable defined on a probability space, $\mathcal{L}(X) = \mu$ means

$$\mu(F) \equiv P([X \in F]).$$

The following amazing theorem is due to Skorokhod. It starts with a measure, μ on $\mathcal{B}(E)$ and produces a random variable, X for which $\mathcal{L}(X) = \mu$. It also has something to say about the convergence of a sequence of such random variables.

Theorem 49.3.3 Let E be a separable Banach space and let $\{\mu_n\}$ be a sequence of Borel probability measures defined on $\mathcal{B}(E)$ such that μ_n converges weakly to μ another probability measure on $\mathcal{B}(E)$. Then there exist random variables, X_n, X defined on the probability space, $([0, 1], \mathcal{B}([0, 1]), m)$ where m is one dimensional Lebesgue measure such that

$$\mathcal{L}(X) = \mu, \mathcal{L}(X_n) = \mu_n, \quad (49.3.7)$$

each random variable, X, X_n is continuous off a set of measure zero, and

$$X_n(\omega) \rightarrow X(\omega) \text{ } m \text{ a.e.}$$

Proof: Let $\{a_k\}$ be a countable dense subset of E .

Construction of sets in E

First I will describe a construction. Letting $C \in \mathcal{B}(E)$ and $r > 0$,

$$\begin{aligned} C_1^r &\equiv C \cap B(a_1, r), C_2^r \equiv B(a_2, r) \cap C \setminus C_1^r, \dots, \\ C_n^r &\equiv B(a_n, r) \cap C \setminus (\cup_{k=1}^{n-1} C_k^r). \end{aligned}$$

Thus the sets, C_k^r for $k = 1, 2, \dots$ are disjoint Borel sets whose union is all of C . Now let $C = E$, the whole Banach space. Also let $\{r_k\}$ be a decreasing sequence of positive numbers which converges to 0. Let

$$A_k \equiv E_k^{r_1}, \quad k = 1, 2, \dots$$

Thus $\{A_k\}$ is a sequence of Borel sets, $A_k \subseteq B(a_k, r_1)$, and the union of the A_k equals E . For $(i_1, \dots, i_m) \in \mathbb{N}^m$, suppose $A_{i_1, \dots, i_m}$ has been defined. Then for $k \in \mathbb{N}$,

$$A_{i_1, \dots, i_m k} \equiv (A_{i_1, \dots, i_m})_k^{r_{m+1}}$$

Thus $A_{i_1, \dots, i_m k} \subseteq B(a_k, r_{m+1})$, is a Borel set, and

$$\cup_{k=1}^{\infty} A_{i_1, \dots, i_m k} = A_{i_1, \dots, i_m}. \quad (49.3.8)$$

Also note that $A_{i_1, \dots, i_m}$ could be empty. This is because $A_{i_1, \dots, i_m k} \subseteq B(a_k, r_{m+1})$ but $A_{i_1, \dots, i_m} \subseteq B(a_{i_m}, r_m)$ which might have empty intersection with $B(a_k, r_{m+1})$. However, applying 49.3.8 repeatedly,

$$E = \cup_{i_1} \cdots \cup_{i_m} A_{i_1, \dots, i_m}$$

and also, the construction shows the Borel sets, $A_{i_1, \dots, i_m}$ are disjoint.

Construction of intervals depending on the measure

Next I will construct intervals, $I_{i_1, \dots, i_n}^\nu$ in $[0, 1)$ corresponding to these $A_{i_1, \dots, i_n}$. In what follows, $\nu = \mu_n$ or μ . These intervals will depend on the measure chosen as indicated in the notation.

$$I_1^\nu \equiv [0, \nu(A_1)), \dots, I_j^\nu \equiv \left[\sum_{k=1}^{j-1} \nu(A_k), \sum_{k=1}^j \nu(A_k) \right)$$

for $j = 1, 2, \dots$. Note these are disjoint intervals whose union is $[0, 1)$. Also note

$$m(I_j^\nu) = \nu(A_j).$$

The endpoints of these intervals as well as their lengths depend on the measures of the sets A_k . Now supposing $I_{i_1, \dots, i_m}^\nu = [\alpha, \beta)$ where $\beta - \alpha = \nu(A_{i_1, \dots, i_m})$, define

$$I_{i_1, \dots, i_m, j}^\nu \equiv \left[\alpha + \sum_{k=1}^{j-1} \nu(A_{i_1, \dots, i_m, k}), \alpha + \sum_{k=1}^j \nu(A_{i_1, \dots, i_m, k}) \right)$$

Thus $m(I_{i_1, \dots, i_m, j}^\nu) = \nu(A_{i_1, \dots, i_m, j})$ and

$$\nu(A_{i_1, \dots, i_m}) = \sum_{k=1}^{\infty} \nu(A_{i_1, \dots, i_m, k}) = \sum_{k=1}^{\infty} m(I_{i_1, \dots, i_m, k}^\nu) = \beta - \alpha,$$

the intervals, $I_{i_1, \dots, i_m, j}^\nu$ being disjoint and

$$I_{i_1, \dots, i_m}^\nu = \cup_{j=1}^{\infty} I_{i_1, \dots, i_m, j}^\nu.$$

Choosing the sequence $\{r_k\}$ in an auspicious manner

There are at most countably many positive numbers, r such that for $\nu = \mu_n$ or μ , $\nu(\partial B(a_i, r)) > 0$. This is because ν is a finite measure. Taking the countable union of these countable sets, there are only countably many r such that $\nu(\partial B(a_i, r)) > 0$ for some a_i . Let the sequence avoid all these bad values of r . Thus for

$$F \equiv \cup_{m=1}^{\infty} \cup_{k=1}^{\infty} \partial B(a_k, r_m)$$

and $\nu = \mu$ or μ_n , $\nu(F) = 0$.

Claim 1: $\partial A_{i_1, \dots, i_k} \subseteq F$.

Proof of claim: Suppose C is a Borel set for which $\partial C \subseteq F$. I need to show $\partial C_k^{r_i} \in F$. First consider $k = 1$. Then $C_1^{r_i} \equiv B(a_1, r_i) \cap C$. If $x \in \partial C_1^{r_i}$, then $B(x, \delta)$ contains points of $B(a_1, r_i) \cap C$ and points of $B(a_1, r_i)^C \cup C^C$ for every $\delta > 0$. First suppose $x \in B(a_1, r_i)$. Then a small enough neighborhood of x has no points of $B(a_1, r_i)^C$ and so every $B(x, \delta)$ has points of C and points of C^C so that $x \in \partial C \subseteq F$ by assumption. If $x \in \partial C_1^{r_i}$, then it can't happen that $\|x - a_1\| > r_i$ because then there would be a neighborhood of x having no points of $C_1^{r_i}$. The only other case to consider is that $\|x - a_i\| = r_i$ but this says $x \in F$. Now assume $\partial C_j^{r_i} \subseteq F$ for $j \leq k - 1$ and consider $\partial C_k^{r_i}$.

$$\begin{aligned} C_k^{r_i} &\equiv B(a_k, r_i) \cap C \setminus \bigcup_{j=1}^{k-1} C_j^{r_i} \\ &= B(a_k, r_i) \cap C \cap \left(\bigcap_{j=1}^{k-1} (C_j^{r_i})^C \right) \end{aligned} \quad (49.3.9)$$

Consider $x \in \partial C_k^{r_i}$. If $x \in \text{int}(B(a_k, r_i) \cap C)$ ($\text{int} \equiv \text{interior}$) then a small enough ball about x contains no points of $(B(a_k, r_i) \cap C)^C$ and so every ball about x must contain points of

$$\left(\bigcap_{j=1}^{k-1} (C_j^{r_i})^C \right)^C = \bigcup_{j=1}^{k-1} C_j^{r_i}$$

Since there are only finitely many sets in the union, there exists $s \leq k - 1$ such that every ball about x contains points of $C_s^{r_i}$ but from 49.3.9, every ball about x contains points of $(C_s^{r_i})^C$ which implies $x \in \partial C_s^{r_i} \subseteq F$ by induction. It is not possible that $\|x - a_k\| > r_i$ and yet have x in $\partial C_k^{r_i}$. This follows from the description in 49.3.9. If $\|x - a_k\| = r_i$ then by definition, $x \in F$. The only other case to consider is that $x \notin \text{int}(B(a_k, r_i) \cap C)$ but $x \in B(a_k, r_i)$. From 49.3.9, every ball about x contains points of C . However, since $x \in B(a_k, r_i)$, a small enough ball is contained in $B(a_k, r_i)$. Therefore, every ball about x must also contain points of C^C since otherwise, $x \in \text{int}(B(a_k, r_i) \cap C)$. Thus $x \in \partial C \subseteq F$ by assumption. Now apply what was just shown to the case where $C = E$, the whole space. In this case, $\partial E \subseteq F$ because $\partial E = \emptyset$. Then keep applying what was just shown to the $A_{i_1, \dots, i_n}$. This proves the claim.

From the claim, $\nu(\text{int}(A_{i_1, \dots, i_n})) = \nu(A_{i_1, \dots, i_n})$ whenever $\nu = \mu$ or μ_n .

Some functions on $[0, 1)$

By the axiom of choice, there exists $x_{i_1, \dots, i_m} \in \text{int}(A_{i_1, \dots, i_m})$ whenever $\text{int}(A_{i_1, \dots, i_m}) \neq \emptyset$. For $\nu = \mu_n$ or μ , define the following functions. For $\omega \in I_{i_1, \dots, i_m}^\nu$

$$Z_m^\nu(\omega) \equiv x_{i_1, \dots, i_m}.$$

This defines the functions, $Z_m^{\mu_n}$ and Z_m^μ . Note these functions have the same values but on slightly different intervals. Here is an important claim.

Claim 2: For a.e. $\omega \in [0, 1)$, $\lim_{n \rightarrow \infty} Z_m^{\mu_n}(\omega) = Z_m^\mu(\omega)$.

Proof of the claim: This follows from the weak convergence of μ_n to μ and Lemma 49.3.1. This lemma implies $\mu_n(\text{int}(A_{i_1, \dots, i_m})) \rightarrow \mu(\text{int}(A_{i_1, \dots, i_m}))$. Thus by the construction described above, $\mu_n(A_{i_1, \dots, i_m}) \rightarrow \mu(A_{i_1, \dots, i_m})$ because of claim

1 and the construction of F in which it is always a set of measure zero. It follows that if $\omega \in \text{int}(I_{i_1, \dots, i_m}^\mu)$, then for all n large enough, $\omega \in \text{int}(I_{i_1, \dots, i_m}^{\mu_n})$ and so $Z_m^{\mu_n}(\omega) = Z_m^\mu(\omega)$. Note this convergence is very far from being uniform.

Claim 3: For $\nu = \mu_n$ or μ , $\{Z_m^\nu\}_{m=1}^\infty$ is uniformly Cauchy independent of n .

Proof of the claim: For $\omega \in I_{i_1, \dots, i_m}^\nu$, then by the construction, $\omega \in I_{i_1, \dots, i_m, i_{m+1}, \dots, i_n}^\nu$ for some $i_{m+1}, \dots, i_n$. Therefore, $Z_m^\nu(\omega)$ and $Z_n^\nu(\omega)$ are both contained in $A_{i_1, \dots, i_m}$ which is contained in $B(a_{i_m}, r_m)$. Since $\omega \in [0, 1)$ was arbitrary, and $r_m \rightarrow 0$, it follows these functions are uniformly Cauchy as claimed.

Let $X^\nu(\omega) = \lim_{m \rightarrow \infty} Z_m^\nu(\omega)$. Since each Z_m^ν is continuous off a set of measure zero, it follows from the uniform convergence that X^ν is also continuous off a set of measure zero.

Claim 4: For a.e. ω ,

$$\lim_{n \rightarrow \infty} X^{\mu_n}(\omega) = X^\mu(\omega).$$

Proof of the claim: From Claim 3 and letting $\varepsilon > 0$ be given, there exists m large enough that for all n ,

$$\|Z_m^{\mu_n} - X^{\mu_n}\|_\infty < \varepsilon/3, \|Z_m^\mu - X^\mu\|_\infty < \varepsilon/3.$$

Now pick $\omega \in [0, 1)$ such that ω is not equal to any of the end points of any of the intervals, $\{I_{i_1, \dots, i_m}^\nu\}$, a set of measure zero. Then by Claim 2, there exists N such that if $n \geq N$, then $\|Z_m^{\mu_n}(\omega) - Z_m^\mu(\omega)\|_E < \varepsilon/3$. Therefore, for such n and this ω ,

$$\begin{aligned} \|X^{\mu_n}(\omega) - X^\mu(\omega)\|_E &\leq \|X^{\mu_n}(\omega) - Z_m^{\mu_n}(\omega)\|_E + \|Z_m^{\mu_n}(\omega) - Z_m^\mu(\omega)\|_E \\ &\quad + \|Z_m^\mu(\omega) - X^\mu(\omega)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This proves the claim.

Showing $\mathcal{L}(X^\nu) = \nu$.

This has mostly proved the theorem except for the claim that $\mathcal{L}(X^\nu) = \nu$ for $\nu = \mu_n$ and μ . To do this, I will first show $m\left((X^\nu)^{-1}(\partial A_{i_1, \dots, i_m})\right) = 0$. By the construction, $\nu(\partial A_{i_1, \dots, i_m}) = 0$. Let $\varepsilon > 0$ be given and let $\delta > 0$ be small enough that

$$H_\delta \equiv \{x \in E : \text{dist}(x, \partial A_{i_1, \dots, i_m}) \leq \delta\}$$

is a set of measure less than $\varepsilon/2$. Denote by $\mathcal{G}_k$ the sets of the form $A_{i_1, \dots, i_k}$ where $(i_1, \dots, i_k) \in \mathbb{N}^k$. Recall also that corresponding to $A_{i_1, \dots, i_k}$ is an interval, $I_{i_1, \dots, i_k}^\nu$ having length equal to $\nu(A_{i_1, \dots, i_k})$. Denote by $\mathcal{B}_k$ those sets of $\mathcal{G}_k$ which have nonempty intersection with H_δ and let the corresponding intervals be denoted by $\mathcal{I}_k^\nu$. If $\omega \notin \cup \mathcal{I}_k^\nu$, then from the construction, $Z_p^\nu(\omega)$ is at a distance of at least δ from $\partial A_{i_1, \dots, i_m}$ for all $p \geq k$ and so, passing to the limit as $p \rightarrow \infty$, it follows $X^\nu(\omega) \notin \partial A_{i_1, \dots, i_m}$. Therefore,

$$(X^\nu)^{-1}(\partial A_{i_1, \dots, i_m}) \subseteq \cup \mathcal{I}_k^\nu$$

Recall that $A_{i_1, \dots, i_k} \subseteq B(a_{i_k}, r_k)$ and the $r_k \rightarrow 0$. Therefore, if k is large enough,

$$\nu(\cup \mathcal{B}_k) < \varepsilon$$

because $\cup \mathcal{B}_k$ approximates H_δ closely (In fact, $\cap_{k=1}^\infty (\cup \mathcal{B}_k) = H_\delta$). Therefore,

$$\begin{aligned} m\left((X^\nu)^{-1}(\partial A_{i_1, \dots, i_m})\right) &\leq m(\cup \mathcal{I}_k^\nu) \\ &= \sum_{I_{i_1, \dots, i_k}^\nu \in \mathcal{I}_k^\nu} m(I_{i_1, \dots, i_k}^\nu) \\ &= \sum_{A_{i_1, \dots, i_k} \in \mathcal{B}_k} \nu(A_{i_1, \dots, i_k}) \\ &= \nu(\cup \mathcal{B}_k) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows $m\left((X^\nu)^{-1}(\partial A_{i_1, \dots, i_m})\right) = 0$.

If $\omega \in I_{i_1, \dots, i_m}^\nu$, then from the construction, $Z_p^\nu(\omega) \in \text{int}(A_{i_1, \dots, i_m})$ for all $p \geq k$. Therefore, taking a limit, as $p \rightarrow \infty$,

$$X^\nu(\omega) \in \text{int}(A_{i_1, \dots, i_m}) \cup \partial A_{i_1, \dots, i_m}$$

and so

$$I_{i_1, \dots, i_m}^\nu \subseteq (X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}) \cup \partial A_{i_1, \dots, i_m})$$

but also, if $X^\nu(\omega) \in \text{int}(A_{i_1, \dots, i_m})$, then $Z_p^\nu(\omega) \in \text{int}(A_{i_1, \dots, i_m})$ for all p large enough and so

$$\begin{aligned} &(X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m})) \\ &\subseteq I_{i_1, \dots, i_m}^\nu \\ &\subseteq (X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}) \cup \partial A_{i_1, \dots, i_m}) \end{aligned}$$

Therefore,

$$\begin{aligned} &m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right) \\ &\leq m(I_{i_1, \dots, i_m}^\nu) \\ &\leq m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right) + m\left((X^\nu)^{-1}(\partial A_{i_1, \dots, i_m})\right) \\ &= m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right) \end{aligned}$$

which shows

$$m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right) = m(I_{i_1, \dots, i_m}^\nu) = \nu(A_{i_1, \dots, i_m}). \quad (49.3.10)$$

Also

$$\begin{aligned}
 & m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right) \\
 & \leq m\left((X^\nu)^{-1}(A_{i_1, \dots, i_m})\right) \\
 & \leq m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}) \cup \partial A_{i_1, \dots, i_m})\right) \\
 & = m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right)
 \end{aligned}$$

Hence from 49.3.10,

$$\begin{aligned}
 \nu(A_{i_1, \dots, i_m}) &= m\left((X^\nu)^{-1}(\text{int}(A_{i_1, \dots, i_m}))\right) \\
 &= m\left((X^\nu)^{-1}(A_{i_1, \dots, i_m})\right)
 \end{aligned} \tag{49.3.11}$$

Now let U be an open set in E . Then letting

$$H_k = \{x \in U : \text{dist}(x, U^C) \geq r_k\}$$

it follows

$$\cup_k H_k = U.$$

Next consider the sets of $\mathcal{G}_k$ which have nonempty intersection with $H_k, \mathcal{H}_k$. Then H_k is covered by $\mathcal{H}_k$ and every set of $\mathcal{H}_k$ is contained in U , the sets of $\mathcal{H}_k$ also being disjoint. Then from 49.3.11,

$$\begin{aligned}
 m\left((X^\nu)^{-1}(\cup \mathcal{H}_k)\right) &= \sum_{A \in \mathcal{H}_k} m\left((X^\nu)^{-1}(A)\right) \\
 &= \sum_{A \in \mathcal{H}_k} \nu(A) = \nu(\cup \mathcal{H}_k).
 \end{aligned}$$

Therefore, letting $k \rightarrow \infty$ and passing to the limit in the above,

$$m\left((X^\nu)^{-1}(U)\right) = \nu(U).$$

Since this holds for every open set, it is routine to verify using regularity that it holds for every Borel set and so $\mathcal{L}(X^\nu) = \nu$ as claimed. This proves the theorem.

49.4 Bochner's Theorem In Infinite Dimensions

Let X be a real vector space and let X^* denote the space of real valued linear mappings defined on X . Then you can consider each $x \in X$ as a linear transformation defined on X^* by the convention $x^* \rightarrow x^*(x)$. Now let Λ be a Hamel basis. For a description of what one of these is, see Page 1811. It is just the usual notion of a basis. Thus every vector of X is a finite linear combination of vectors of Λ in a unique way.

Now consider $\mathbb{R}^\Lambda$ the space of all mappings from Λ to $\mathbb{R}$. In different notation, this is of the form

$$\mathbb{R}^\Lambda \equiv \prod_{y \in \Lambda} \mathbb{R}$$

Since Λ is a Hamel basis, there exists a one to one and onto mapping, $\theta : X^* \rightarrow \mathbb{R}^\Lambda$ defined as

$$\theta(x^*) \equiv \prod_{y \in \Lambda} x^*(y).$$

Now denote by $\sigma(X)$ the smallest σ algebra of sets of X^* such that each x is measurable with respect to this σ algebra. Thus

$$\{x^* : x^*(x) \in B\} \in \sigma(X)$$

whenever B is a Borel set in $\mathbb{R}$.

Let $\mathcal{E}$ denote the algebra of disjoint unions of sets of $\mathbb{R}^\Lambda$ of the form

$$\prod_{y \in \Lambda} A_y$$

where $A_y = \mathbb{R}$ except for finitely many y .

Lemma 49.4.1 *Let $\mathcal{A}$ denote sets of the form*

$$\{x^* : \theta(x^*) \in U\}$$

where $U \in \mathcal{E}$. Then $\mathcal{A}$ is an algebra and $\sigma(\mathcal{A}) = \sigma(X)$. Also

$$\{\theta^{-1}(U) : U \in \sigma(\mathcal{E})\} = \sigma(X)$$

Proof: Since $\mathcal{E}$ is an algebra it is clear $\mathcal{A}$ is also an algebra. Also, $\mathcal{A} \subseteq \sigma(X)$ because you could let U have only one A_y not equal to $\mathbb{R}$ and all the others equal to $\mathbb{R}$ and then

$$\{x^* : \theta(x^*) \in U\} = \{x^* : y(x^*) \equiv x^*(y) \in A_y\} \in \sigma(X).$$

Therefore, $\sigma(\mathcal{A}) \subseteq \sigma(X)$. I need to verify that for an arbitrary x , it is measurable with respect to $\sigma(\mathcal{A})$. However, this is true because if x is arbitrary, it is a linear combination of $\{y_1, \dots, y_n\}$, some finite set of functions in Λ and so, x being a linear combination of measurable functions implies it is itself measurable.

By definition, $\theta^{-1}(U)$ is in $\mathcal{A}$ whenever $U \in \mathcal{E}$. Now let $\mathcal{G}$ denote those sets, U in $\sigma(\mathcal{E})$ such that $\theta^{-1}(U) \in \sigma(\mathcal{A})$. Then $\mathcal{G}$ is a σ algebra which contains $\mathcal{E}$ and so $\mathcal{G} \supseteq \sigma(\mathcal{E}) \supseteq \mathcal{G}$. This proves the last claim. This proves the lemma.

Definition 49.4.2 *Let $\psi : X \rightarrow \mathbb{C}$. Then ψ is said to be pseudo continuous if whenever $\{x_1, \dots, x_n\}$ is a finite subset of X and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$,*

$$\mathbf{a} \rightarrow \psi \left(\sum_{k=1}^n a_k x_k \right)$$

is continuous. ψ is said to be positive definite if

$$\sum_{j,k} \psi(x_k - x_j) \alpha_k \overline{\alpha_j} \geq 0$$

ψ is said to be a characteristic function if there exists a probability measure, μ defined on $\sigma(X)$ such that

$$\psi(x) = \int_{X^*} e^{ix^*(x)} d\mu(x^*)$$

Note that $x^* \rightarrow e^{ix^*(x)}$ is $\sigma(X)$ measurable.

Using Kolmogorov's extension theorem on Page 11.3.5, there exists a generalization of Bochner's theorem found in [65]. For convenience, here is Kolmogorov's theorem.

Theorem 49.4.3 (Kolmogorov extension theorem) For each finite set

$$J = (t_1, \dots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \dots t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ for $M_t = \mathbb{R}^{n_t}$ for n_t an integer, such that the following consistency condition holds. If

$$(t_1, \dots, t_n) \subseteq (s_1, \dots, s_p),$$

then

$$\nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) = \nu_{s_1 \dots s_p}(G_{s_1} \times \dots \times G_{s_p}) \quad (49.4.12)$$

where if $s_i = t_j$, then $G_{s_i} = F_{t_j}$ and if s_i is not equal to any of the indices, t_k , then $G_{s_i} = M_{s_i}$. Then for $\mathcal{E}$ defined as in Definition 11.3.1, adjusted so that $\pm\infty$ never appears as any endpoint of any interval, there exists a probability measure, P and a σ algebra $\mathcal{F} = \sigma(\mathcal{E})$ such that

$$\left(\prod_{t \in I} M_t, P, \mathcal{F} \right)$$

is a probability space. Also there exist measurable functions, $X_s : \prod_{t \in I} M_t \rightarrow M_s$ defined as

$$X_s \mathbf{x} \equiv x_s$$

for each $s \in I$ such that for each $(t_1 \dots t_n) \subseteq I$,

$$\begin{aligned} \nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) &= P([X_{t_1} \in F_{t_1}] \cap \dots \cap [X_{t_n} \in F_{t_n}]) \\ &= P\left((X_{t_1}, \dots, X_{t_n}) \in \prod_{j=1}^n F_{t_j}\right) = P\left(\prod_{t \in I} F_t\right) \end{aligned} \quad (49.4.13)$$

where $F_t = M_t$ for every $t \notin \{t_1 \cdots t_n\}$ and F_{t_i} is a Borel set. Also if f is a non-negative function of finitely many variables, $x_{t_1}, \dots, x_{t_n}$, measurable with respect to $\mathcal{B}\left(\prod_{j=1}^n M_{t_j}\right)$, then f is also measurable with respect to $\mathcal{F}$ and

$$\begin{aligned} & \int_{M_{t_1} \times \cdots \times M_{t_n}} f(x_{t_1}, \dots, x_{t_n}) d\nu_{t_1 \cdots t_n} \\ &= \int_{\prod_{t \in I} M_t} f(x_{t_1}, \dots, x_{t_n}) dP \end{aligned} \quad (49.4.14)$$

Theorem 49.4.4 Let X be a real vector space and let X^* be the space of linear functionals defined on X . Also let $\psi : X \rightarrow \mathbb{C}$. Then ψ is a characteristic function if and only if $\psi(0) = 1$ and ψ is pseudo continuous at 0.

Proof: Suppose first ψ is a characteristic function as just described. I need to show it is positive definite and pseudo continuous. It is obvious $\psi(0) = 1$ in this case. Also

$$\psi\left(\sum_k a_k x_k\right) = \int_{X^*} \exp\left(ix^*\left(\sum_k a_k x_k\right)\right) d\mu(x^*)$$

and this is obviously a continuous function of $\mathbf{a}$ by the dominated convergence theorem. It only remains to verify the function is positive definite. However,

$$\sum_{k,j} \exp(ix^*(x_k - x_j)) \alpha_k \overline{\alpha_j} = \sum_{k,j} e^{ix^*(x_k)} \alpha_k \overline{e^{ix^*(x_j)} \alpha_j} \geq 0$$

as in the earlier discussion of what it means to be positive definite given on Page 1365.

Next suppose the conditions hold. Define for $\mathbf{t} \in \mathbb{R}^n$ and $\{y_1, \dots, y_n\} \subseteq \Lambda$

$$\psi_{\{y_1, \dots, y_n\}}(\mathbf{t}) \equiv \psi\left(\sum_{j=1}^n t_j y_j\right).$$

Then $\psi_{\{y_1, \dots, y_n\}}$ is continuous at $\mathbf{0}$, equals 1 there, and is positive definite. It follows from Bochner's theorem, Theorem 47.13.7 on Page 1369 there exists a measure $\mu_{\{y_1, \dots, y_n\}}$ defined on the Borel sets of $\mathbb{R}^n$ such that

$$\psi\left(\sum_{j=1}^n t_j y_j\right) = \psi_{\{y_1, \dots, y_n\}}(\mathbf{t}) = \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} d\mu_{\{y_1, \dots, y_n\}}(x). \quad (49.4.15)$$

Thus if $\{y_1, \dots, y_n, y_{n+1}, \dots, y_p\} \subseteq \Lambda$,

$$\begin{aligned} \psi\left(\sum_{j=1}^n t_j y_j + \sum_{j=1}^{p-n} s_j y_{j+n}\right) &= \psi_{\{y_1, \dots, y_p\}}(\mathbf{t}, \mathbf{s}) \\ &= \int_{\mathbb{R}^{p-n}} e^{i\mathbf{s} \cdot \mathbf{x}} \int_{\mathbb{R}^n} e^{i\mathbf{t} \cdot \mathbf{x}} d\mu_{\{y_1, \dots, y_p\}}(x) \end{aligned}$$

I need to verify the measures are consistent to use Kolmogorov's theorem. Specifically, I need to show

$$\mu_{\{y_1, \dots, y_p\}}(A \times \mathbb{R}^{p-n}) = \mu_{\{y_1, \dots, y_n\}}(A).$$

Letting

$$\lambda(A) = \mu_{\{y_1, \dots, y_p\}}(A \times \mathbb{R}^{p-n})$$

it follows

$$\begin{aligned} \int_{\mathbb{R}^n} e^{it \cdot x} d\lambda &= \int_{\mathbb{R}^{p-n}} \int_{\mathbb{R}^n} e^{it \cdot x} d\mu_{\{y_1, \dots, y_p\}}(x) \\ &= \int_{\mathbb{R}^{p-n}} e^{i0 \cdot x} \int_{\mathbb{R}^n} e^{it \cdot x} d\mu_{\{y_1, \dots, y_p\}}(x) \\ &= \psi \left(\sum_{j=1}^n t_j y_j + \sum_{j=1}^{p-n} 0 y_{j+n} \right) \\ &= \psi \left(\sum_{j=1}^n t_j y_j \right) \\ &= \int_{\mathbb{R}^n} e^{it \cdot x} d\mu_{\{y_1, \dots, y_n\}}(x) \end{aligned}$$

and so, by uniqueness of characteristic functions,

$$\lambda = \mu_{\{y_1, \dots, y_n\}}$$

which verifies the necessary consistency condition for Kolmogorov's theorem.

It follows there exists a probability measure μ defined on $\sigma(\mathcal{E})$ and random variables, X_y for each $y \in \Lambda$ such that whenever $\{y_1, \dots, y_p\} \subseteq \Lambda$,

$$\mu_{\{y_1, \dots, y_p\}}(A_{y_1} \times \dots \times A_{y_n}) = \mu \left(\prod_{y \in \Lambda} A_y \right)$$

where $A_y = \mathbb{R}$ whenever $y \notin \{y_1, \dots, y_p\}$. This defines a measure on $\sigma(\mathcal{E})$ which consists of sets of $\mathbb{R}^\Lambda$.

By Lemma 49.4.1 it follows $\{\theta^{-1}(U) : U \in \sigma(\mathcal{E})\} = \sigma(\mathcal{A})$ which equals $\sigma(X)$. Define ν on $\sigma(X)$ by

$$\nu(F) \equiv \mu(\theta F).$$

Thus ν is a measure because μ is and θ is one to one.

I need to check whether ν works. Let $x = \sum_{k=1}^m t_k y_k$ and let a typical element of $\mathbb{R}^\Lambda$ be denoted by $\mathbf{z}$. Then by Kolmogorov's theorem above,

$$\int_{X^*} \exp \left(ix^* \left(\sum_{k=1}^m t_k y_k \right) \right) d\nu = \int_{X^*} \exp \left(i \left(\sum_{k=1}^m t_k x^*(y_k) \right) \right) d\nu$$

$$\begin{aligned}
&= \int_{X^*} \exp \left(i \left(\sum_{k=1}^m t_k \pi_{y_k} (\theta x^*) \right) \right) d\nu = \int_{\mathbb{R}^\Lambda} \exp \left(i \left(\sum_{k=1}^m t_k \pi_{y_k} \mathbf{z} \right) \right) d\mu \\
&= \int_{\mathbb{R}^m} \exp (i (\mathbf{t} \cdot \mathbf{x})) d\mu_{\{y_1, \dots, y_m\}} (x) = \psi \left(\sum_{k=1}^m t_k y_k \right)
\end{aligned}$$

where the last equality comes from 49.4.15. Since Λ is a Hamel basis, it follows that for every $x \in X$,

$$\psi(x) = \int_{X^*} \exp(ix^*(x)) d\nu$$

This proves the theorem.

49.5 The Multivariate Normal Distribution

Here I give a review of the main theorems and definitions about multivariate normal random variables. Recall that for a random vector (variable), $\mathbf{X}$ having values in $\mathbb{R}^p$, $\lambda_{\mathbf{X}}$ is the law of $\mathbf{X}$ defined by

$$P([\mathbf{X} \in E]) = \lambda_{\mathbf{X}}(E)$$

for all E a Borel set in $\mathbb{R}^p$. In different notation, $\mathcal{L}(\mathbf{X}) = \lambda_{\mathbf{X}}$. Then the following definitions and theorems are proved and presented starting on Page 1343

Definition 49.5.1 A random vector, $\mathbf{X}$, with values in $\mathbb{R}^p$ has a multivariate normal distribution written as $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ if for all Borel $E \subseteq \mathbb{R}^p$,

$$\lambda_{\mathbf{X}}(E) = \int_{\mathbb{R}^p} \mathcal{X}_E(\mathbf{x}) \frac{1}{(2\pi)^{p/2} \det(\Sigma)^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^* \Sigma^{-1}(\mathbf{x}-\mathbf{m})} d\mathbf{x}$$

for μ a given vector and Σ a given positive definite symmetric matrix.

Theorem 49.5.2 For $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$, $\mathbf{m} = E(\mathbf{X})$ and

$$\Sigma = E((\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^*).$$

Theorem 49.5.3 Suppose $\mathbf{X}_1 \sim N_p(\mathbf{m}_1, \Sigma_1)$, $\mathbf{X}_2 \sim N_p(\mathbf{m}_2, \Sigma_2)$ and the two random vectors are independent. Then

$$\mathbf{X}_1 + \mathbf{X}_2 \sim N_p(\mathbf{m}_1 + \mathbf{m}_2, \Sigma_1 + \Sigma_2). \quad (49.5.16)$$

Also, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then $-\mathbf{X} \sim N_p(-\mathbf{m}, \Sigma)$. Furthermore, if $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then

$$E(e^{i\mathbf{t} \cdot \mathbf{X}}) = e^{i\mathbf{t} \cdot \mathbf{m}} e^{-\frac{1}{2}\mathbf{t}^* \Sigma \mathbf{t}} \quad (49.5.17)$$

Also if a is a constant and $\mathbf{X} \sim N_p(\mathbf{m}, \Sigma)$ then $a\mathbf{X} \sim N_p(a\mathbf{m}, a^2\Sigma)$.

Following [54] a random vector has a generalized normal distribution if its characteristic function is given as

$$e^{it \cdot \mathbf{m}} e^{-\frac{1}{2} t^* \Sigma t} \quad (49.5.18)$$

where Σ is symmetric and has nonnegative eigenvalues. For a random real valued variable, $\mathbf{m}$ is scalar and so is Σ so the characteristic function of such a generalized normally distributed random variable is

$$e^{itm} e^{-\frac{1}{2} t^2 \sigma^2} \quad (49.5.19)$$

These generalized normal distributions do not require Σ to be invertible, only that the eigenvalues be nonnegative. In one dimension this would correspond to the characteristic function of a Dirac measure having point mass 1 at m . In higher dimensions, it could be a mixture of such things with more familiar things. I will often not bother to distinguish between generalized normal and normal distributions.

Here are some other interesting results about normal distributions found in [54]. The next theorem has to do with the question whether a random vector is normally distributed in the above generalized sense. It is proved on Page 1346.

Theorem 49.5.4 *Let $\mathbf{X} = (X_1, \dots, X_p)$ where each X_i is a real valued random variable. Then $\mathbf{X}$ is normally distributed in the above generalized sense if and only if every linear combination, $\sum_{j=1}^p a_j X_j$ is normally distributed. In this case the mean of $\mathbf{X}$ is*

$$\mathbf{m} = (E(X_1), \dots, E(X_p))$$

and the covariance matrix for $\mathbf{X}$ is

$$\Sigma_{jk} = E((X_j - m_j)(X_k - m_k))$$

where $m_j = E(X_j)$.

Also proved there is the interesting corollary listed next.

Corollary 49.5.5 *Let $\mathbf{X} = (X_1, \dots, X_p)$, $\mathbf{Y} = (Y_1, \dots, Y_p)$ where each X_i, Y_i is a real valued random variable. Suppose also that for every $\mathbf{a} \in \mathbb{R}^p$, $\mathbf{a} \cdot \mathbf{X}$ and $\mathbf{a} \cdot \mathbf{Y}$ are both normally distributed with the same mean and variance. Then $\mathbf{X}$ and $\mathbf{Y}$ are both multivariate normal random vectors with the same mean and variance.*

Theorem 49.5.6 *Suppose $\mathbf{X} = (X_1, \dots, X_p)$ is normally distributed with mean $\mathbf{m}$ and covariance Σ . Then if X_1 is uncorrelated with any of the X_i ,*

$$E((X_1 - m_1)(X_j - m_j)) = 0 \text{ for } j > 1$$

then X_1 and $(X_2, \dots, X_p)$ are both normally distributed and the two random vectors are independent. Here $m_j \equiv E(X_j)$.

Next I will consider the question of existence of independent random variables having a given law.

Lemma 49.5.7 *Let μ be a probability measure on $\mathcal{B}(E)$, the Borel subsets of a separable real Banach space. Then there exists a probability space $(\Omega, \mathcal{F}, P)$ and two independent random variables, X, Y mapping Ω to E such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$.*

Proof: First note that if A, B are Borel sets of E then $A \times B$ is a Borel set in $E \times E$ where the norm on $E \times E$ is given by

$$\|(x, y)\| \equiv \max(\|x\|, \|y\|).$$

This can be proved by letting A be open and considering

$$\mathcal{G} \equiv \{B \in \mathcal{B}(E) : A \times B \in \mathcal{B}(A \times B)\}.$$

Show $\mathcal{G}$ is a σ algebra and it contains the open sets. Therefore, this will show $A \times B$ is in $\mathcal{B}(A \times B)$ whenever A is open and B is Borel. Next repeat a similar argument to show that this is true whenever either set is Borel. Since E is separable, it is completely separable and so is $E \times E$. Thus every open set in $E \times E$ is the union of balls from a countable set. However, these balls are of the form $B_1 \times B_2$ where B_i is a ball in E . Now let

$$\mathcal{K} \equiv \{A \times B : A, B \text{ are Borel}\}$$

Then $\mathcal{K} \subseteq \mathcal{B}(E \times E)$ as was just shown and also every open set from $E \times E$ is in $\sigma(\mathcal{K})$. It follows $\sigma(\mathcal{K})$ equals the σ algebra of product measurable sets, $\mathcal{B}(E) \times \mathcal{B}(E)$ and you can consider the product measure, $\mu \times \mu$. By Skorokhod's theorem, Theorem 49.3.3, there exists (X, Y) a random variable with values in $E \times E$ and a probability space, $(\Omega, \mathcal{F}, P)$ such that $\mathcal{L}((X, Y)) = \mu \times \mu$. Then for A, B Borel sets in E

$$P(X \in A, Y \in B) = (\mu \times \mu)(A \times B) = \mu(A) \mu(B).$$

Also, $P(X \in A) = P(X \in A, Y \in E) = \mu(A)$ and similarly, $P(Y \in B) = \mu(B)$ showing $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$ and X, Y are independent.

Now here is an interesting theorem in [18].

Theorem 49.5.8 *Suppose ν is a probability measure on the Borel sets of $\mathbb{R}$ and suppose that ξ and ζ are independent random variables such that $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = \nu$ and whenever $\alpha^2 + \beta^2 = 1$ it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = \nu$. Then*

$$\mathcal{L}(\xi) = N(0, \sigma^2)$$

for some $\sigma \geq 0$. Also if $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = N(0, \sigma^2)$ where ξ, ζ are independent, then if $\alpha^2 + \beta^2 = 1$, it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = N(0, \sigma^2)$.

Proof: Let ξ, ζ be independent random variables with $\mathcal{L}(\xi) = \mathcal{L}(\zeta) = \nu$ and whenever $\alpha^2 + \beta^2 = 1$ it follows $\mathcal{L}(\alpha\xi + \beta\zeta) = \nu$.

By independence of ξ and ζ ,

$$\begin{aligned}\phi_\nu(t) &\equiv \phi_{\alpha\xi+\alpha\zeta}(t) \\ &= E\left(e^{it(\alpha\xi+\beta\zeta)}\right) \\ &= E\left(e^{it\alpha\xi}\right) E\left(e^{it\beta\zeta}\right) \\ &= \phi_\xi(\alpha t) \phi_\zeta(\beta t) \\ &\equiv \phi_\nu(\alpha t) \phi_\nu(\beta t)\end{aligned}$$

In simpler terms and suppressing the subscript,

$$\phi(t) = \phi(\cos(\theta)t) \phi(\sin(\theta)t). \quad (49.5.20)$$

Since ν is a probability measure, $\phi(0) = 1$. Also, letting $\theta = \pi/4$, this yields

$$\phi(t) = \phi\left(\frac{\sqrt{2}}{2}t\right)^2 \quad (49.5.21)$$

and so if ϕ has real values, then $\phi(t) \geq 0$.

Next I will show ϕ is real. To do this, it follows from the definition of ϕ_ν ,

$$\phi_\nu(-t) \equiv \int_{\mathbb{R}} e^{-itx} d\nu = \overline{\int_{\mathbb{R}} e^{itx} d\nu} = \overline{\phi_\nu(t)}.$$

Then letting $\theta = \pi$,

$$\phi(t) = \phi(-t) \cdot \phi(0) = \phi(-t) = \overline{\phi(t)}$$

showing ϕ has real values. It is positive near 0 because $\phi(0) = 1$ and ϕ is a continuous function of t thanks to the dominated convergence theorem. However, this and 49.5.21 implies it is positive everywhere. Here is why. If not, let t_m be the smallest positive value of t where $\phi(t) = 0$. Then $t_m > 0$ by continuity. Now from 49.5.21, an immediate contradiction results. Therefore, $\phi(t) > 0$ for all $t > 0$. Similar reasoning yields the same conclusion for $t < 0$.

Next note that $\phi(t) = \phi(-t)$ also implies ϕ depends only on $|t|$ because it takes the same value for t as for $-t$. More simply, ϕ depends only on t^2 . Thus one can define a new function of the form $\phi(t) = f(t^2)$ and 49.5.20 implies the following for $\alpha \in [0, 1]$.

$$f(t^2) = f(\alpha^2 t^2) f((1 - \alpha^2) t^2).$$

Taking $\ln$ of both sides, one obtains the following.

$$\ln f(t^2) = \ln f(\alpha^2 t^2) + \ln f((1 - \alpha^2) t^2).$$

Now let $x, y \geq 0$. Then choose t such that $t^2 = x + y$. Then for some $\alpha \in [0, 1]$, $x = \alpha^2 t^2$ and so $y = t^2(1 - \alpha^2)$. Thus for $x, y \geq 0$,

$$\ln f(x + y) = \ln f(x) + \ln f(y).$$

Hence $\ln f(x) = kx$ and so $\ln f(t^2) = kt^2$ and so $\phi(t) = f(t^2) = e^{kt^2}$ for all t . The constant, k must be nonpositive because $\phi(t)$ is bounded due to its definition. Therefore, the characteristic function of ν is

$$\phi_\nu(t) = e^{-\frac{1}{2}t^2\sigma^2}$$

for some $\sigma \geq 0$. That is, ν is the law of a generalized normal random variable.

Note the other direction of the implication is obvious. If $\xi, \zeta \sim N(0, \sigma)$ and they are independent, then if $\alpha^2 + \beta^2 = 1$, it follows

$$\alpha\xi + \beta\zeta \sim N(0, \sigma^2)$$

because

$$\begin{aligned} E\left(e^{it(\alpha\xi + \beta\zeta)}\right) &= E\left(e^{it\alpha\xi}\right) E\left(e^{it\beta\zeta}\right) \\ &= e^{-\frac{1}{2}(\alpha t)^2\sigma^2} e^{-\frac{1}{2}(\beta t)^2\sigma^2} \\ &= e^{-\frac{1}{2}t^2\sigma^2}, \end{aligned}$$

the characteristic function for a random variable which is $N(0, \sigma)$. This proves the theorem.

The next theorem is a useful gimmick for showing certain random variables are independent in the context of normal distributions.

Theorem 49.5.9 *Let $\mathbf{X}$ and $\mathbf{Y}$ be random vectors having values in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Suppose also that $(\mathbf{X}, \mathbf{Y})$ is multivariate normally distributed and*

$$E\left((\mathbf{X} - E(\mathbf{X}))(\mathbf{Y} - E(\mathbf{Y}))^*\right) = \mathbf{0}.$$

Then $\mathbf{X}$ and $\mathbf{Y}$ are independent random vectors.

Proof: Let $\mathbf{Z} = (\mathbf{X}, \mathbf{Y})$, $m = p + q$. Then by hypothesis, the characteristic function of $\mathbf{Z}$ is of the form

$$E\left(e^{it \cdot \mathbf{Z}}\right) = e^{it \cdot \mathbf{m}} e^{-\frac{1}{2}it^* \Sigma t}$$

where $\mathbf{m} = (\mathbf{m}_\mathbf{X}, \mathbf{m}_\mathbf{Y}) = E(\mathbf{Z}) = E(\mathbf{X}, \mathbf{Y})$ and

$$\begin{aligned} \Sigma &= \begin{pmatrix} E\left((\mathbf{X} - E(\mathbf{X}))(\mathbf{X} - E(\mathbf{X}))^*\right) & \mathbf{0} \\ \mathbf{0} & E\left((\mathbf{Y} - E(\mathbf{Y}))(\mathbf{Y} - E(\mathbf{Y}))^*\right) \end{pmatrix} \\ &\equiv \begin{pmatrix} \Sigma_\mathbf{X} & \mathbf{0} \\ \mathbf{0} & \Sigma_\mathbf{Y} \end{pmatrix}. \end{aligned}$$

Therefore, letting $\mathbf{t} = (\mathbf{u}, \mathbf{v})$ where $\mathbf{u} \in \mathbb{R}^p$ and $\mathbf{v} \in \mathbb{R}^q$

$$\begin{aligned} E\left(e^{it \cdot \mathbf{Z}}\right) &= E\left(e^{i(\mathbf{u}, \mathbf{v}) \cdot (\mathbf{X}, \mathbf{Y})}\right) = E\left(e^{i(\mathbf{u} \cdot \mathbf{X} + \mathbf{v} \cdot \mathbf{Y})}\right) \\ &= e^{i\mathbf{u} \cdot \mathbf{m}_\mathbf{X}} e^{-\frac{1}{2}\mathbf{u}^* \Sigma_\mathbf{X} \mathbf{u}} e^{i\mathbf{v} \cdot \mathbf{m}_\mathbf{Y}} e^{-\frac{1}{2}\mathbf{v}^* \Sigma_\mathbf{Y} \mathbf{v}} \\ &= E\left(e^{i\mathbf{u} \cdot \mathbf{X}}\right) E\left(e^{i\mathbf{v} \cdot \mathbf{Y}}\right). \end{aligned} \tag{49.5.22}$$

Where the last equality needs to be justified. When this is done it will follow from Proposition 47.4.1 on Page 1323 which is proved on Page 1323 that $\mathbf{X}$ and $\mathbf{Y}$ are independent. Thus all that remains is to verify

$$E(e^{i\mathbf{u}\cdot\mathbf{X}}) = e^{i\mathbf{u}\cdot\mathbf{m}_\mathbf{X}} e^{-\frac{1}{2}\mathbf{u}^*\Sigma_\mathbf{X}\mathbf{u}}, \quad E(e^{i\mathbf{v}\cdot\mathbf{Y}}) = e^{i\mathbf{v}\cdot\mathbf{m}_\mathbf{Y}} e^{-\frac{1}{2}\mathbf{v}^*\Sigma_\mathbf{Y}\mathbf{v}}.$$

However, this follows from 49.5.22. To get the first formula, let $\mathbf{v} = \mathbf{0}$. To get the second, let $\mathbf{u} = \mathbf{0}$. This proves the Theorem.

Note that to verify the conclusion of this theorem, it suffices to show

$$E(X_i - E(X_i)(Y_j - E(Y_j))) = 0.$$

49.6 Gaussian Measures

49.6.1 Definitions And Basic Properties

First suppose $\mathbf{X}$ is a random vector having values in $\mathbb{R}^n$ and its distribution function is $N(\mathbf{m}, \Sigma)$ where $\mathbf{m}$ is the mean and Σ is the covariance. Then the characteristic function of $\mathbf{X}$ or equivalently, the characteristic function of its distribution is

$$e^{it\cdot\mathbf{m}} e^{-\frac{1}{2}t^*\Sigma t}$$

What is the distribution of $\mathbf{a}\cdot\mathbf{X}$ where $\mathbf{a} \in \mathbb{R}^n$? In other words, if you take a linear functional and do it to $\mathbf{X}$ to get a scalar valued random variable, what is the distribution of this scalar valued random variable? Let $Y = \mathbf{a}\cdot\mathbf{X}$. Then

$$E(e^{itY}) = E(e^{it\mathbf{a}\cdot\mathbf{X}})$$

which from the above formula is

$$e^{i\mathbf{a}\cdot\mathbf{m}t} e^{-\frac{1}{2}\mathbf{a}^*\Sigma\mathbf{a}t^2}$$

which is the characteristic function of a random variable whose distribution is $N(\mathbf{a}\cdot\mathbf{m}, \mathbf{a}^*\Sigma\mathbf{a})$. In other words, it is normally distributed having mean equal to $\mathbf{a}\cdot\mathbf{m}$ and variance equal to $\mathbf{a}^*\Sigma\mathbf{a}$. Obviously such a concept generalizes to a Banach space in place of $\mathbb{R}^n$ and this motivates the following definition.

Definition 49.6.1 *Let E be a real separable Banach space. A probability measure, μ defined on $\mathcal{B}(E)$ is called a Gaussian measure if for every $h \in E'$, the law of h considered as a random variable defined on the probability space, $(E, \mathcal{B}(E), \mu)$ is normal. That is, for $A \subseteq \mathbb{R}$ a Borel set,*

$$\lambda_h(A) \equiv \mu(h^{-1}(A))$$

is given by

$$\int_A \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-m)^2} dx$$

for some σ and m . A Gaussian measure is called symmetric if m is always equal to 0.

There is another definition of symmetric. First here are a few simple conventions. For $f \in E'$, $x \rightarrow f(x)$ is normally distributed. In particular,

$$\int_E |f(x)| d\mu < \infty$$

and so it makes sense to define

$$m_\mu(f) \equiv \int_E f(x) d\mu.$$

Thus $m_\mu(f)$ is the mean of the random variable $x \rightarrow f(x)$. It is obvious that $f \rightarrow m_\mu(f)$ is linear. Also define the variance $\sigma^2(f)$ by

$$\sigma^2(f) \equiv \int_E (f(x) - m_\mu(f))^2 d\mu$$

This is finite because $x \rightarrow f(x)$ is normally distributed. The following lemma gives such an equivalent condition for μ to be symmetric.

Lemma 49.6.2 *Let μ be a Gaussian measure defined on $\mathcal{B}(E)$. Then $\mu(F) = \mu(-F)$ for all F Borel if and only if $m_\mu(f) = 0$ for all $f \in E'$. Such a Gaussian measure is called symmetric.*

Proof: Suppose first $m_\mu(f) = 0$ for all $f \in E'$. Let

$$G \equiv f_1^{-1}(F_1) \cap f_2^{-1}(F_2) \cap \cdots \cap f_m^{-1}(F_m)$$

where F_i is a Borel set of $\mathbb{R}$ and each $f_i \in E'$. Since every linear combination of the f_i is in E' , every such linear combination is normally distributed and so $\mathbf{f} \equiv (f_1, \dots, f_m)$ is multivariate normal. That is, $\lambda_{\mathbf{f}}$ the distribution measure, is multivariate normal. Since each $m_\mu(f) = 0$, it follows

$$\mu(G) = \lambda_{\mathbf{f}}\left(\prod_{i=1}^m F_i\right) = \lambda_{\mathbf{f}}\left(\prod_{i=1}^m -F_i\right) = \mu(-G) \quad (49.6.23)$$

By Lemma 23.1.4 on Page 699 there exists a countable subset, $D \equiv \{f_k\}_{k=1}^\infty$ of the closed unit ball such that for every $x \in E$,

$$\|x\| = \sup_{f \in D} |f(x)|.$$

Therefore, letting $D(a, r)$ denote the closed ball centered at a having radius r , it follows

$$D(a, r) = \cap_{k=1}^\infty f_k^{-1}(D(f_k(a), r))$$

Let

$$D_n(a, r) = \cap_{k=1}^n f_k^{-1}(D(f_k(a), r))$$

Then by 49.6.23

$$\mu(D_n(a, r)) = \mu(-D_n(a, r))$$

and letting $n \rightarrow \infty$, it follows

$$\mu(D(a, r)) = \mu(-D(a, r))$$

Therefore the same is true with $D(a, r)$ replaced with an open ball. Now consider

$$D(a, r_1) \cap D(b, r_2) = \cap_{k=1}^{\infty} f_k^{-1}(D(f_k(a), r_1)) \cap \cap_{k=1}^{\infty} f_k^{-1}(D(f_k(b), r_2))$$

The intersection of these two closed balls is the intersection of sets of the form

$$\cap_{k=1}^n f_k^{-1}(D(f_k(a), r_1)) \cap \cap_{k=1}^n f_k^{-1}(D(f_k(b), r_2))$$

to which 49.6.23 applies. Therefore, by continuing this way it follows that if G is any finite intersection of closed balls,

$$\mu(G) = \mu(-G).$$

Let $\mathcal{K}$ denote the set of finite intersections of closed balls, a π system. Thus for $G \in \mathcal{K}$ the above holds. Now let

$$\mathcal{G} \equiv \{F \in \sigma(\mathcal{K}) : \mu(F) = \mu(-F)\}$$

Thus $\mathcal{G}$ contains $\mathcal{K}$ and it is clearly closed with respect to complements and countable disjoint unions. By the π system lemma, $\mathcal{G} \supseteq \sigma(\mathcal{K})$ but $\sigma(\mathcal{K})$ clearly contains the open sets since every open ball is the countable union of closed disks and every open set is the countable union of open balls. Therefore, $\mu(G) = \mu(-G)$ for all Borel G .

Conversely suppose $\mu(G) = \mu(-G)$ for all G Borel. If for some $f \in E'$, $m_\mu(f) \neq 0$, then

$$\begin{aligned} \mu(f^{-1}(0, \infty)) &\equiv \lambda_f(0, \infty) \neq \lambda_f(-\infty, 0) \\ &\equiv \mu(f^{-1}(-\infty, 0)) = \mu(-f^{-1}(0, \infty)) \end{aligned}$$

a contradiction. This proves the lemma.

Lemma 49.6.3 *Let $\mu = \mathcal{L}(X)$ where X is a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ which has values in E , a Banach space. Suppose also that for all $\phi \in E'$, $\phi \circ X$ is normally distributed. Then μ is a Gaussian measure. Conversely, suppose μ is a Gaussian measure on $\mathcal{B}(E)$ and X is a random variable having values in E such that $\mathcal{L}(X) = \mu$. Then for every $h \in E'$, $h \circ X$ is normally distributed.*

Proof: First suppose μ is a Gaussian measure and X is a random variable such that $\mathcal{L}(X) = \mu$. Then if F is a Borel set in $\mathbb{R}$, and $h \in E'$

$$\begin{aligned} P((h \circ X)^{-1}(F)) &= P(X^{-1}(h^{-1}(F))) \\ &= \mu(h^{-1}(F)) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx \end{aligned}$$

for some m and σ^2 showing that $h \circ X$ is normally distributed.

Next suppose $h \circ X$ is normally distributed whenever $h \in E'$ and $\mathcal{L}(X) = \mu$. Then letting F be a Borel set in $\mathbb{R}$, I need to verify

$$\mu(h^{-1}(F)) = \frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx.$$

However, this is easy because

$$\begin{aligned} \mu(h^{-1}(F)) &= P(X^{-1}(h^{-1}(F))) \\ &= P((h \circ X)^{-1}(F)) \end{aligned}$$

which is given to equal

$$\frac{1}{\sqrt{2\pi}\sigma} \int_F e^{-\frac{|x-m|^2}{2\sigma^2}} dx$$

for some m and σ^2 . This proves the lemma.

Here is another important observation. Suppose X is as just described, a random variable having values in E such that $\mathcal{L}(X) = \mu$ and suppose $h_1, \dots, h_n$ are each in E' . Then for scalars, $t_1, \dots, t_n$,

$$\begin{aligned} &t_1 h_1 \circ X + \dots + t_n h_n \circ X \\ &= (t_1 h_1 + \dots + t_n h_n) \circ X \end{aligned}$$

and this last is assumed to be normally distributed because $(t_1 h_1 + \dots + t_n h_n) \in E'$. Therefore, by Theorem 49.5.4

$$(h_1 \circ X, \dots, h_n \circ X)$$

is distributed as a multivariate normal.

Obviously there exist examples of Gaussian measures defined on E , a Banach space. Here is why. Let ξ be a random variable defined on a probability space, $(\Omega, \mathcal{F}, P)$ which is normally distributed with mean 0 and variance σ^2 . Then let $X(\omega) \equiv \xi(\omega)e$ where $e \in E$. Then let $\mu \equiv \mathcal{L}(X)$. For A a Borel set of $\mathbb{R}$ and $h \in E'$,

$$\begin{aligned} \mu([h(x) \in A]) &\equiv P([X(\omega) \in [x : h(x) \in A]]) \\ &= P([h \circ X \in A]) = P([\xi(\omega)h(e) \in A]) \\ &= \frac{1}{|h(e)|\sigma\sqrt{2\pi}} \int_A e^{-\frac{1}{2|h(e)|^2\sigma^2}x^2} dx \end{aligned}$$

because $h(e)\xi$ is a random variable which has variance $|h(e)|^2\sigma^2$ and mean 0. Thus μ is indeed a Gaussian measure. Similarly, one can consider finite sums of the form

$$\sum_{i=1}^n \xi_i(\omega) e_i$$

where the ξ_i are independent normal random variables having mean 0 for convenience. However, this is a rather trivial case.

49.6.2 Fernique's Theorem

The following is an interesting lemma.

Lemma 49.6.4 *Suppose μ is a symmetric Gaussian measure on the real separable Banach space, E . Then there exists a probability space, $(\Omega, \mathcal{F}, P)$ and independent random variables, X and Y mapping Ω to E such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. Also, the two random variables,*

$$\frac{1}{\sqrt{2}}(X - Y), \quad \frac{1}{\sqrt{2}}(X + Y)$$

are independent and

$$\mathcal{L}\left(\frac{1}{\sqrt{2}}(X - Y)\right) = \mathcal{L}\left(\frac{1}{\sqrt{2}}(X + Y)\right) = \mu.$$

Proof: Letting $X' \equiv \frac{1}{\sqrt{2}}(X + Y)$ and $Y' \equiv \frac{1}{\sqrt{2}}(X - Y)$, it follows from Theorem 47.6.2 on Page 1329, that X' and Y' are independent if whenever $h_1, \dots, h_m \in E'$ and $g_1, \dots, g_k \in E'$, the two random vectors,

$$(h_1 \circ X', \dots, h_m \circ X') \quad \text{and} \quad (g_1 \circ Y', \dots, g_k \circ Y')$$

are linearly independent. Now consider linear combinations

$$\sum_{j=1}^m t_j h_j \circ X' + \sum_{i=1}^k s_i g_i \circ Y'.$$

This equals

$$\begin{aligned} & \frac{1}{\sqrt{2}} \sum_{j=1}^m t_j h_j(X) + \frac{1}{\sqrt{2}} \sum_{j=1}^m t_j h_j(Y) \\ & + \frac{1}{\sqrt{2}} \sum_{i=1}^k s_i g_i(X) - \frac{1}{\sqrt{2}} \sum_{i=1}^k s_i g_i(Y) \\ & = \frac{1}{\sqrt{2}} \left(\sum_{j=1}^m t_j h_j + \sum_{i=1}^k s_i g_i \right)(X) \\ & \quad + \frac{1}{\sqrt{2}} \left(\sum_{j=1}^m t_j h_j - \sum_{i=1}^k s_i g_i \right)(Y) \end{aligned}$$

and this is the sum of two independent normally distributed random variables so it is also normally distributed. Therefore, by Theorem 49.5.4

$$(h_1 \circ X', \dots, h_m \circ X', g_1 \circ Y', \dots, g_k \circ Y')$$

is a random variable with multivariate normal distribution and by Theorem 49.5.9 the two random vectors

$$(h_1 \circ X', \dots, h_m \circ X') \text{ and } (g_1 \circ Y', \dots, g_k \circ Y')$$

are linearly independent if

$$E((h_i \circ X')(g_j \circ Y')) = 0$$

for all i, j . This is what I will show next.

$$\begin{aligned} & E((h_i \circ X')(g_j \circ Y')) \\ &= \frac{1}{4} E((h_i(X) + h_i(Y))(g_j(X) - g_j(Y))) \\ &= \frac{1}{4} E(h_i(X)g_j(X)) - \frac{1}{4} E(h_i(X)g_j(Y)) \\ &\quad + \frac{1}{4} E(h_i(Y)g_j(X)) - \frac{1}{4} E(h_i(Y)g_j(Y)) \end{aligned} \quad (49.6.24)$$

Now from the above observation after the definition of Gaussian measure $h_i(X)g_j(X)$ and $h_i(Y)g_j(Y)$ are both in L^1 because each term in each product is normally distributed. Therefore, by Lemma 47.8.2,

$$\begin{aligned} E(h_i(X)g_j(X)) &= \int_{\Omega} h_i(Y)g_j(Y) dP \\ &= \int_E h_i(y)g_j(y) d\mu \\ &= \int_{\Omega} h_i(X)g_j(X) dP \\ &= E(h_i(Y)g_j(Y)) \end{aligned}$$

and so 49.6.24 reduces to

$$\frac{1}{4} (E(h_i(Y)g_j(X)) - E(h_i(X)g_j(Y))) = 0$$

because $h_i(X)$ and $g_j(Y)$ are independent due to the assumption that X and Y are independent. Thus

$$E(h_i(X)g_j(Y)) = E(h_i(X))E(g_j(Y)) = 0$$

due to the assumption that μ is symmetric which implies the mean of these random variables equals 0. The other term works out similarly. This has proved the independence of the random variables, X' and Y' .

Next consider the claim they have the same law and it equals μ . To do this, I will use Theorem 47.5.9 on Page 1328. Thus I need to show

$$E(e^{ih(X')}) = E(e^{ih(Y')}) = E(e^{ih(X)}) \quad (49.6.25)$$

for all $h \in E'$. Pick such an h . Then $h \circ X$ is normally distributed and has mean 0. Therefore, for some σ ,

$$E(e^{ith \circ X}) = e^{-\frac{1}{2}t^2\sigma^2}.$$

Now since X and Y are independent,

$$\begin{aligned} E(e^{ith \circ X'}) &= E\left(e^{ith\left(\frac{1}{\sqrt{2}}\right)(X+Y)}\right) \\ &= E\left(e^{ith\left(\frac{1}{\sqrt{2}}\right)X}\right) E\left(e^{ith\left(\frac{1}{\sqrt{2}}\right)Y}\right) \end{aligned}$$

the product of two characteristic functions of two random variables, $\frac{1}{\sqrt{2}}X$ and $\frac{1}{\sqrt{2}}Y$. The variance of these two random variables which are normally distributed with zero mean is $\frac{1}{2}\sigma^2$ and so

$$E(e^{ith \circ X'}) = e^{-\frac{1}{2}\left(\frac{1}{2}\sigma^2\right)} e^{-\frac{1}{2}\left(\frac{1}{2}\sigma^2\right)} = e^{-\frac{1}{2}\sigma^2} = E(e^{ith \circ X}).$$

Similar reasoning shows $E(e^{ith \circ Y'}) = E(e^{ith \circ Y}) = E(e^{ith \circ X})$. Letting $t = 1$, this yields 49.6.25. This proves the lemma.

With this preparation, here is an incredible theorem due to Fernique.

Theorem 49.6.5 *Let μ be a symmetric Gaussian measure on $\mathcal{B}(E)$ where E is a real separable Banach space. Then for λ sufficiently small and positive,*

$$\int_E e^{\lambda\|x\|^2} d\mu < \infty.$$

More specifically, if λ and r are chosen such that

$$\ln \left(\frac{\mu([x : \|x\| > r])}{\mu(\overline{B}(0, r))} \right) + 25\lambda r^2 < -1,$$

then

$$\int_E e^{\lambda\|x\|^2} d\mu \leq \exp(\lambda r^2) + \frac{e^2}{e^2 - 1}.$$

Proof: Let X, Y be independent random variables having values in E such that $\mathcal{L}(X) = \mathcal{L}(Y) = \mu$. Then by Lemma 49.6.4

$$\frac{1}{\sqrt{2}}(X - Y), \frac{1}{\sqrt{2}}(X + Y)$$

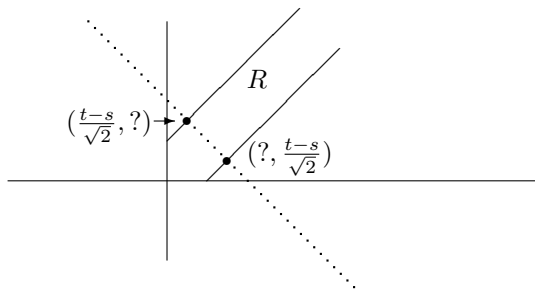
are also independent and have the same law. Now let $0 \leq s \leq t$ and use independence of the above random variables along with the fact they have the same law as X and Y to obtain

$$P(\|X\| \leq s, \|Y\| > t) = P(\|X\| \leq s) P(\|Y\| > t)$$

$$\begin{aligned}
&= P\left(\left\|\frac{1}{\sqrt{2}}(X-Y)\right\| \leq s\right) P\left(\left\|\frac{1}{\sqrt{2}}(X+Y)\right\| > t\right) \\
&= P\left(\left\|\frac{1}{\sqrt{2}}(X-Y)\right\| \leq s, \left\|\frac{1}{\sqrt{2}}(X+Y)\right\| > t\right) \\
&\leq P\left(\frac{1}{\sqrt{2}}\|X\| - \|Y\| \leq s, \frac{1}{\sqrt{2}}(\|X\| + \|Y\|) > t\right).
\end{aligned}$$

Now consider the following picture in which the region, R represents the points, $(\|X\|, \|Y\|)$ such that

$$\frac{1}{\sqrt{2}}\|X\| - \|Y\| \leq s \text{ and } \frac{1}{\sqrt{2}}(\|X\| + \|Y\|) > t.$$



Therefore, continuing with the chain of inequalities above,

$$\begin{aligned}
&P(\|X\| \leq s) P(\|Y\| > t) \\
&\leq P\left(\|X\| > \frac{t-s}{\sqrt{2}}, \|Y\| > \frac{t-s}{\sqrt{2}}\right) \\
&= P\left(\|X\| > \frac{t-s}{\sqrt{2}}\right)^2.
\end{aligned}$$

Since X, Y have the same law, this can be written as

$$P(\|X\| > t) \leq \frac{P\left(\|X\| > \frac{t-s}{\sqrt{2}}\right)^2}{P(\|X\| \leq s)}.$$

Now define a sequence as follows. $t_0 \equiv r > 0$ and $t_{n+1} \equiv r + \sqrt{2}t_n$. Also, in the above inequality, let $s \equiv r$ and then it follows

$$\begin{aligned}
P(\|X\| > t_{n+1}) &\leq \frac{P\left(\|X\| > \frac{t_{n+1}-r}{\sqrt{2}}\right)^2}{P(\|X\| \leq r)} \\
&= \frac{P(\|X\| > t_n)^2}{P(\|X\| \leq r)}.
\end{aligned}$$

Let

$$\alpha_n(r) \equiv \frac{P(\|X\| > t_n)}{P(\|X\| \leq r)}.$$

Then it follows

$$\alpha_{n+1}(r) \leq \alpha_n(r)^2, \quad \alpha_0(r) = \frac{P(\|X\| > r)}{P(\|X\| \leq r)}.$$

Consequently, $\alpha_n(r) \leq \alpha_0(r)^{2^n}$ and also

$$\begin{aligned} P(\|X\| > t_n) &= \alpha_n(r) P(\|X\| \leq r) \\ &\leq P(\|X\| \leq r) \alpha_0(r)^{2^n} \\ &= P(\|X\| \leq r) e^{\ln(\alpha_0(r))2^n}. \end{aligned} \quad (49.6.26)$$

Now using the distribution function technique and letting $\lambda > 0$,

$$\begin{aligned} \int_E e^{\lambda\|x\|^2} d\mu &= \int_0^\infty \mu\left([e^{\lambda\|x\|^2} > t]\right) dt \\ &= 1 + \int_1^\infty \mu\left([e^{\lambda\|x\|^2} > t]\right) dt \\ &= 1 + \int_1^\infty P\left([e^{\lambda\|X\|^2} > t]\right) dt. \end{aligned} \quad (49.6.27)$$

From 49.6.26,

$$P\left([\exp(\lambda\|X\|^2) > \exp(\lambda t_n^2)]\right) \leq P(\|X\| \leq r) e^{\ln(\alpha_0(r))2^n}.$$

Now split the above improper integral into intervals, $(\exp(\lambda t_n^2), \exp(\lambda t_{n+1}^2))$ for $n = 0, 1, \dots$ and note that $P([e^{\lambda\|X\|^2} > t])$ is decreasing in t . Then from 49.6.27,

$$\begin{aligned} \int_E e^{\lambda\|x\|^2} d\mu &\leq \exp(\lambda r^2) + \sum_{n=0}^\infty \int_{\exp(\lambda t_n^2)}^{\exp(\lambda t_{n+1}^2)} P([e^{\lambda\|X\|^2} > t]) dt \\ &\leq \exp(\lambda r^2) + \sum_{n=0}^\infty P([e^{\lambda\|X\|^2} > \exp(\lambda t_n^2)]) (\exp(\lambda t_{n+1}^2) - \exp(\lambda t_n^2)) \\ &\leq \exp(\lambda r^2) + \sum_{n=0}^\infty P(\|X\| \leq r) e^{\ln(\alpha_0(r))2^n} \exp(\lambda t_{n+1}^2) \\ &\leq \exp(\lambda r^2) + \sum_{n=0}^\infty e^{\ln(\alpha_0(r))2^n} \exp(\lambda t_{n+1}^2). \end{aligned}$$

It remains to estimate t_{n+1} . From the description of the t_n ,

$$t_n = \left(\sum_{k=0}^n (\sqrt{2})^k \right) r = r \frac{(\sqrt{2})^{n+1} - 1}{\sqrt{2} - 1} \leq \frac{\sqrt{2}}{\sqrt{2} - 1} r (\sqrt{2})^n$$

and so

$$t_{n+1} \leq 5r \left(\sqrt{2} \right)^n$$

Therefore,

$$\int_E e^{\lambda \|x\|^2} d\mu \leq \exp(\lambda r^2) + \sum_{n=0}^{\infty} e^{\ln(\alpha_0(r))2^n + \lambda 25r^2 2^n}.$$

Now first pick r large enough that $\ln(\alpha_0(r)) < -2$ and then let λ be small enough that $25\lambda r^2 < 1$ or some such scheme and you obtain $\ln(\alpha_0(r)) + \lambda 25r^2 < -1$. Then for this choice of r and λ , or for any other choice which makes $\ln(\alpha_0(r)) + \lambda 25r^2 < -1$,

$$\begin{aligned} \int_E e^{\lambda \|x\|^2} d\mu &\leq \exp(\lambda r^2) + \sum_{n=0}^{\infty} e^{-2^n} \\ &\leq \exp(\lambda r^2) + \sum_{n=0}^{\infty} e^{-2^n} \\ &= \exp(\lambda r^2) + \frac{e^2}{e^2 - 1}. \end{aligned}$$

This proves the theorem.

49.7 Gaussian Measures For A Separable Hilbert Space

First recall the Kolmogorov extension theorem, Theorem 11.3.5 on Page 332 which is stated here for convenience. In this theorem, I is an ordered index set, possibly infinite, even uncountable.

Theorem 49.7.1 (*Kolmogorov extension theorem*) For each finite set

$$J = (t_1, \dots, t_n) \subseteq I,$$

suppose there exists a Borel probability measure, $\nu_J = \nu_{t_1 \dots t_n}$ defined on the Borel sets of $\prod_{t \in J} M_t$ where $M_t = \mathbb{R}^{n_t}$ such that if

$$(t_1, \dots, t_n) \subseteq (s_1, \dots, s_p),$$

then

$$\nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) = \nu_{s_1 \dots s_p}(G_{s_1} \times \dots \times G_{s_p}) \quad (49.7.28)$$

where if $s_i = t_j$, then $G_{s_i} = F_{t_j}$ and if s_i is not equal to any of the indices, t_k , then $G_{s_i} = M_{s_i}$. Then there exists a probability space, $(\Omega, P, \mathcal{F})$ and measurable functions, $X_t : \Omega \rightarrow M_t$ for each $t \in I$ such that for each $(t_1 \dots t_n) \subseteq I$,

$$\nu_{t_1 \dots t_n}(F_{t_1} \times \dots \times F_{t_n}) = P([X_{t_1} \in F_{t_1}] \cap \dots \cap [X_{t_n} \in F_{t_n}]). \quad (49.7.29)$$

Lemma 49.7.2 *There exists a sequence, $\{\xi_k\}_{k=1}^\infty$ of random variables such that*

$$\mathcal{L}(\xi_k) = N(0, 1)$$

and $\{\xi_k\}_{k=1}^\infty$ is independent.

Proof: Let $i_1 < i_2 < \dots < i_n$ be positive integers and define

$$\mu_{i_1 \dots i_n}(F_1 \times \dots \times F_n) \equiv \frac{1}{(\sqrt{2\pi})^n} \int_{F_1 \times \dots \times F_n} e^{-|\mathbf{x}|^2/2} dx.$$

Then for the index set equal to $\mathbb{N}$ the measures satisfy the necessary consistency condition for the Kolmogorov theorem above. Therefore, there exists a probability space, $(\Omega, P, \mathcal{F})$ and measurable functions, $\xi_k : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} & P([\xi_{i_1} \in F_{i_1}] \cap [\xi_{i_2} \in F_{i_2}] \dots \cap [\xi_{i_n} \in F_{i_n}]) \\ &= \mu_{i_1 \dots i_n}(F_1 \times \dots \times F_n) \\ &= P([\xi_{i_1} \in F_{i_1}]) \dots P([\xi_{i_n} \in F_{i_n}]) \end{aligned}$$

which shows the random variables are independent as well as normal with mean 0 and variance 1. This proves the Lemma.

A random variable X defined on a probability space $(\Omega, \mathcal{F}, P)$ is called Gaussian if

$$P([X \in A]) = \frac{1}{\sqrt{2\pi\sigma(v)^2}} \int_A e^{-\frac{1}{2\sigma(v)^2}(x-m(v))^2} dx$$

for all A a Borel set in $\mathbb{R}$. Therefore, for the probability space $(X, \mathcal{B}(X), \mu)$ it is natural to say μ is a Gaussian measure if every x^* in the dual space X' is a Gaussian random variable. That is, normally distributed.

Definition 49.7.3 *Let μ be a measure defined on $\mathcal{B}(X)$, the Borel sets of X , a separable Banach space. It is called a Gaussian measure if each of the functions in the dual space X' is normally distributed. As a special case, when $X = U$ a separable real Hilbert space, μ is called a Gaussian measure if for each $v \in U$, the function $u \rightarrow (u, v)_U$ is normally distributed. That is, denoting this random variable as v' , it follows for A a Borel set in $\mathbb{R}$*

$$\lambda_{v'}(A) \equiv \mu([u : v'(u) \in A]) = \frac{1}{\sqrt{2\pi\sigma(v)^2}} \int_A e^{-\frac{1}{2\sigma(v)^2}(x-m(v))^2} dx$$

in case $\sigma(v) > 0$. In case $\sigma(v) = 0$

$$\lambda_{v'} \equiv \delta_{m(v)}$$

In other words, the random variables v' for $v \in U$ are all normally distributed on the probability space $(U, \mathcal{B}(U), \mu)$.

Also recall the definition of the characteristic function of a measure.

Definition 49.7.4 *The Borel sets in a topological space X will be denoted by $\mathcal{B}(X)$. For a Borel probability measure μ defined on $\mathcal{B}(U)$ for U a real separable Hilbert space, define its characteristic function as follows.*

$$\phi_\mu(u) \equiv \widehat{\mu}(u) \equiv \int_U e^{i(u,v)} d\mu(v) \quad (49.7.30)$$

More generally, if μ is a probability measure defined on $\mathcal{B}(X)$ where X is a separable Banach space, then the characteristic function is defined as

$$\phi_\mu(x^*) = \widehat{\mu}(x^*) \equiv \int_U e^{ix^*(x)} d\mu(x)$$

One can tell whether μ is a Gaussian measure by looking at its characteristic function. In fact you can show the following theorem. One part of this theorem is that if μ is Gaussian, then m and σ^2 have a certain form.

Theorem 49.7.5 *A measure μ on $\mathcal{B}(U)$ is Gaussian if and only if there exists $m \in U$ and $Q \in \mathcal{L}(U)$ such that Q is nonnegative symmetric with finite trace,*

$$\sum_k (Qe_k, e_k) < \infty$$

for a complete orthonormal basis for U , and

$$\phi_\mu(u) = \widehat{\mu}(u) = e^{i(m,u) - \frac{1}{2}(Qu,u)} \quad (49.7.31)$$

In this case μ is called $N(m, Q)$ where m is the mean and Q is called the covariance. The measure μ is uniquely determined by m and Q . Also for all $h, g \in U$

$$\int (x, h)_U d\mu(x) = (m, h)_U \quad (49.7.32)$$

$$\int ((x, h) - (m, h))((x, g) - (m, g)) d\mu(x) = (Qh, g) \quad (49.7.33)$$

$$\int \|x - m\|_U^2 d\mu(x) = \text{trace}(Q). \quad (49.7.34)$$

Proof: First of all suppose 49.7.31 holds. Why is μ Gaussian? Consider the random variable u' defined by $u'(v) \equiv (v, u)$. Why is $\lambda_{u'}$ a Gaussian measure on $\mathbb{R}$? By the definition in 49.7.30,

$$\begin{aligned} \int_U e^{itu'(v)} d\mu(v) &\equiv \int_U e^{it(v,u)} d\mu(v) = \int_{\mathbb{R}} e^{ix} d\lambda_{u'}(x) \\ &= \int_U e^{i(v,tu)} d\mu(v) = e^{it(m,u) - \frac{1}{2}t^2(Qu,u)} \end{aligned}$$

and this is the characteristic equation for a random variable having mean (m, u) and variance (Qu, u) . In case $(Qu, u) = 0$, you get $e^{it(m, u)}$ which is the characteristic function for a random variable having distribution $\delta_{(m, u)}$. Thus if 49.7.31 holds, then u' is normally distributed as desired. Thus μ is Gaussian by definition.

The next task is to suppose μ is Gaussian and show the existence of m, Q which have the desired properties. This involves the following lemma.

Lemma 49.7.6 *Let U be a real separable Hilbert space and let μ be a probability measure defined on $\mathcal{B}(U)$. Suppose for some positive integer, k*

$$\int_U |(x, z)|^k d\mu(x) < \infty$$

for all $z \in U$. Then the transformation,

$$(h_1, \dots, h_k) \rightarrow \int_U (h_1, x) \cdots (h_k, x) d\mu(x) \quad (49.7.35)$$

is a continuous k -linear form.

Proof: I need to show that for each $\mathbf{h} \in U^k$, the integral in 49.7.35 exists. From this it is obvious it is k -linear, meaning linear in each argument. Then it is shown it is continuous.

First note

$$|(h_1, x) \cdots (h_k, x)| \leq |(h_1, x)|^k + \cdots + |(h_k, x)|^k$$

This follows from observing that one of $|(h_j, x)|$ is largest. Then the left side is smaller than $|(h_j, x)|^k$. Therefore, the above inequality is valid. This inequality shows the integral in 49.7.35 makes sense.

I need to establish an estimate of the form

$$\int_U |(x, h)|^k d\mu(x) < C < \infty$$

for every $h \in U$ such that $\|h\|$ is small enough.

Let

$$U_n \equiv \left\{ z \in U : \int_U |(x, z)|^k d\mu(x) \leq n \right\}$$

Then by assumption $U = \bigcup_{n=1}^{\infty} U_n$ and it is also clear from Fatou's lemma that each U_n is closed. Therefore, by the Baire category theorem, at least one of these U_{n_0} contains an open ball, $B(z_0, r)$. Then letting $|y| < r$,

$$\int_U |(x, z_0 + y)|^k d\mu(x), \int_U |(x, z_0)|^k d\mu(x) \leq n_0,$$

and so for such y ,

$$\begin{aligned} \int_U |(x, y)|^k d\mu &= \int_U |(x, z_0 + y) - (x, z_0)|^k d\mu \\ &\leq \int_U 2^k |(x, z_0 + y)|^k + 2^k |(x, z_0)|^k d\mu(x) \\ &\leq 2^k (n_0 + n_0) = 2^{k+1} n_0. \end{aligned}$$

It follows that for arbitrary nonzero $y \in U$

$$\int_U \left| \left(x, \frac{(r/2)y}{\|y\|} \right) \right|^k d\mu \leq 2^{k+1} n_0$$

and so

$$\int_U |(x, y)|^k d\mu \leq (2^{k+2}/r) n_0 \|y\|^k \equiv C \|y\|^k.$$

Thus by Holder's inequality,

$$\begin{aligned} \int_U |(h_1, x) \cdots (h_k, x)| d\mu(x) &\leq \prod_{j=1}^k \left(\int_U |(h_j, x)|^k d\mu(x) \right)^{1/k} \\ &\leq C \prod_{j=1}^k \|h_j\| \end{aligned}$$

This proves the lemma.

Now continue with the proof of the theorem. I need to identify m and Q . It is assumed μ is Gaussian. Recall this means h' is normally distributed for each $h \in U$. Then using

$$\begin{aligned} |x| &\leq |x - m(h)| + |m(h)| \\ \int_U |(x, h)_U| d\mu(x) &= \int_{\mathbb{R}} |x| d\lambda_{h'}(x) \\ &= \frac{1}{\sqrt{2\pi\sigma^2(h)}} \int_{\mathbb{R}} |x| e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2(h)}} \int_{\mathbb{R}} |x - m(h)| e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx \\ &\quad + |m(h)| \end{aligned}$$

Then using the Cauchy Schwarz inequality, with respect to the probability measure

$$\begin{aligned} &\frac{1}{\sqrt{2\pi\sigma^2(h)}} e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx, \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2(h)}} \left(\int_{\mathbb{R}} |x - m(h)|^2 e^{-\frac{1}{2\sigma^2}(x-m(h))^2} dx \right)^{1/2} + |m(h)| < \infty \end{aligned}$$

Thus by Lemma 49.7.6

$$h \rightarrow \int_U (x, h) d\mu(x)$$

is a continuous linear transformation and so by the Riesz representation theorem, there exists a unique $m \in U$ such that

$$(h, m)_U = \int_U (h, x) d\mu(x)$$

Also the above says (h, m) is the mean of the random variable $x \rightarrow (x, h)$ so in the above,

$$m(h) = (h, m)_U.$$

Next it is necessary to find Q . To do this let Q be given by 49.7.33. Thus

$$\begin{aligned}(Qh, g) &\equiv \int_U ((x, h) - (m, h)) ((x, g) - (m, g)) d\mu(x) \\ &= \int_U (x - m, h) (x - m, g) d\mu(x)\end{aligned}$$

It is clear Q is linear and the above is a bilinear form (The integral makes sense because of the assumption that h', g' are normally distributed.) but is it continuous? Does $(Qh, h) = \sigma^2(h)$?

First, the above equals

$$\begin{aligned}&\int_U (x, h) (x - m, g) d\mu - \int_U (m, h) (x - m, g) d\mu(x) \\ &= \int_U (x, h) (x - m, g) d\mu\end{aligned}\tag{49.7.36}$$

because from the first part,

$$\int_U (x - m, g) d\mu(x) = \int_U (x, g) d\mu(x) - (m, g)_U = 0.$$

Now by the first part, the term in 49.7.36 is

$$\begin{aligned}&\int_U (x, h) (x, g) d\mu(x) - (m, g) \int_U (x, h) d\mu(x) \\ &= \int_U (x, h) (x, g) d\mu(x) - (m, g) (m, h).\end{aligned}$$

Thus

$$|(Qh, g)| \leq \int_U |(x, h) (x, g)| d\mu(x) + \|m\|^2 \|h\| \|g\|$$

and since the random variables h' and g' given by $x \rightarrow (x, h)$ and $x \rightarrow (x, g)$ respectively are given to be normally distributed with variance $\sigma^2(h)$ and $\sigma^2(g)$ respectively, the above integral is finite. Also for all h ,

$$\int_U |(x, h)|^2 d\mu(x) < \infty$$

because the random variable h' is given to be normally distributed. Therefore from Lemma 49.7.6, there exists some constant C such that

$$|(Qh, g)| \leq C \|h\| \|g\|$$

which shows Q is continuous as desired.

Why is $\sigma^2(h) = (Qh, h)$? This follows because from the above

$$\begin{aligned} (Qh, h) &\equiv \int_U (h, x - m)^2 d\mu(x) \\ &= \int_U ((x, h) - (h, m))^2 d\mu(x) = \int_{\mathbb{R}} (t - (h, m))^2 d\lambda_{h'}(t) \\ &= \frac{1}{\sqrt{2\pi\sigma^2(h)}} \int_{\mathbb{R}} (t - (h, m))^2 e^{-\frac{1}{2\sigma^2(h)}(t - (h, m))^2} dt = \sigma^2(h) \end{aligned}$$

from a standard result for the normal distribution function which follows from an easy change of variables argument.

Why must Q have finite trace? For $h \in U$, it follows from the above that h' is normally distributed with mean (h, m) and variance (Qh, h) . Therefore, the characteristic function of h' is known. In fact

$$\int_U e^{it(x, h)} d\mu(x) = e^{it(h, m)} e^{-\frac{1}{2}t^2(Qh, h)}$$

Thus also

$$\int_U e^{it(x-m, h)} d\mu(x) = e^{-\frac{1}{2}t^2(Qh, h)}$$

and letting $t = 1$ this yields

$$\int_U e^{i(x-m, h)} d\mu(x) = e^{-\frac{1}{2}(Qh, h)}$$

From this it follows

$$\int_U (1 - e^{i(x-m, h)}) d\mu = 1 - e^{-\frac{1}{2}(Qh, h)}$$

and since the right side is real, this implies

$$\int_U (1 - \cos(x - m, h)) d\mu(x) = 1 - e^{-\frac{1}{2}(Qh, h)}$$

Thus

$$\begin{aligned} 1 - e^{-\frac{1}{2}(Qh, h)} &\leq \int_{[||x-m|| \leq c]} (1 - \cos(x - m, h)) d\mu(x) \\ &\quad + 2 \int_{[||x-m|| > c]} d\mu(x) \end{aligned}$$

Now it is routine to show

$$1 - \cos t \leq \frac{1}{2}t^2$$

and so

$$\begin{aligned} 1 - e^{-\frac{1}{2}(Qh, h)} &\leq \frac{1}{2} \int_{[||x-m|| \leq c]} |(x - m, h)|^2 d\mu(x) \\ &\quad + 2\mu([||x - m|| > c]) \end{aligned}$$

Pick c large enough that the last term is smaller than $1/8$. This can be done because the sets decrease to $\emptyset$ as $c \rightarrow \infty$ and μ is given to be a finite measure. Then with this choice of c ,

$$\frac{7}{8} - \frac{1}{2} \int_{[\|x-m\| \leq c]} |(x-m, h)|^2 d\mu(x) \leq e^{-\frac{1}{2}(Qh, h)} \quad (49.7.37)$$

For each h the integral in the above is finite. In fact

$$\int_{[\|x-m\| \leq c]} |(x-m, h)|^2 d\mu(x) \leq c^2 \|h\|^2$$

Let

$$(Q_ch, h_1) \equiv \int_{[\|x-m\| \leq c]} (x-m, h)(x-m, h_1) d\mu(x)$$

and let A denote those $h \in U$ such that

$$(Q_ch, h) < 1.$$

Then from 49.7.37 it follows that for $h \in A$,

$$\frac{3}{8} = \frac{7}{8} - \frac{1}{2} \leq \frac{7}{8} - \frac{1}{2} (Q_ch, h) \leq e^{-\frac{1}{2}(Qh, h)}$$

Therefore, for such h ,

$$\frac{8}{3} \geq e^{\frac{1}{2}(Qh, h)} \geq 1 + \frac{1}{2} (Qh, h)$$

and so for $h \in A$,

$$(Qh, h) \leq \left(\frac{8}{3} - 1\right) 2 = \frac{10}{3}$$

Now let h be arbitrary. Then for each $\varepsilon > 0$

$$\frac{h}{\varepsilon + \sqrt{(Q_ch, h)}} \in A$$

and so

$$\left(Q \left(\frac{h}{\varepsilon + \sqrt{(Q_ch, h)}} \right), \frac{h}{\varepsilon + \sqrt{(Q_ch, h)}} \right) \leq \frac{10}{3}$$

which implies

$$(Qh, h) \leq \frac{10}{3} \left(\varepsilon + \sqrt{(Q_ch, h)} \right)^2$$

Since ε is arbitrary,

$$(Qh, h) \leq \frac{10}{3} (Q_ch, h). \quad (49.7.38)$$

However, Q_c has finite trace. To see this, let $\{e_k\}$ be an orthonormal basis in U . Then

$$\sum_k (Q_c e_k, e_k) = \sum_k \int_{[\|x-m\| \leq c]} |(x-m, e_k)|^2 d\mu(x)$$

$$= \int_{\|x-m\| \leq c} \sum_k |(x-m, e_k)|^2 d\mu(x) = \int_{\|x-m\| \leq c} \|x-m\|^2 d\mu(x) \leq c^2$$

It follows from 49.7.38 Q that Q must also have finite trace.

That μ is uniquely determined by m and Q follows from Theorem 47.5.9. This proves the theorem.

Suppose you have a given Q having finite trace and $m \in U$. Does there exist a Gaussian measure on $\mathcal{B}(U)$ having these as the covariance and mean respectively?

Proposition 49.7.7 *Let U be a real separable Hilbert space and let $m \in U$ and Q be a positive, symmetric operator defined on U which has finite trace. Then there exists a Gaussian measure with mean m and covariance Q .*

Proof: By Lemma 49.7.2 which comes from Kolmogorov's extension theorem, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a sequence $\{\xi_j\}$ of independent random variables which are normally distributed with mean 0 and variance 1. Then let

$$X(\omega) \equiv m + \sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j(\omega) e_j$$

where the $\{e_j\}$ are the eigenvectors of Q such that $Qe_j = \lambda_j e_j$. The series in the above converges in $L^2(\Omega; U)$ because

$$\left\| \sum_{j=m}^n \sqrt{\lambda_j} \xi_j e_j \right\|_{L^2(\Omega; U)}^2 = \int_{\Omega} \sum_{j=m}^n \lambda_j \xi_j^2(\omega) dP = \sum_{j=m}^n \lambda_j$$

and so the partial sums form a Cauchy sequence in $L^2(\Omega; U)$.

Now if $h \in U$, I need to show that $\omega \rightarrow (X(\omega), h)$ is normally distributed. From this it will follow that $\mathcal{L}(X)$ is Gaussian. A subsequence

$$\left\{ m + \sum_{j=1}^{n_k} \sqrt{\lambda_j} \xi_j(\omega) e_j \right\} \equiv \{S_{n_k}(\omega)\}$$

of the above sequence converges pointwise a.e. to X .

$$\begin{aligned} & E(\exp(it(X, h))) \\ &= \lim_{k \rightarrow \infty} E(\exp(it(S_{n_k}, h))) \\ &= \exp(it(m, h)) \lim_{k \rightarrow \infty} E\left(\exp\left(it \sum_{j=1}^{n_k} \sqrt{\lambda_j} \xi_j(\omega) (e_j, h)\right)\right) \end{aligned}$$

Since the ξ_j are independent,

$$= \exp(it(m, h)) \lim_{k \rightarrow \infty} \prod_{j=1}^{n_k} E\left(\exp(it \sqrt{\lambda_j} (e_j, h) \xi_j)\right)$$

$$\begin{aligned}
&= \exp(it(m, h)) \lim_{k \rightarrow \infty} \prod_{j=1}^{n_k} e^{-\frac{1}{2}t^2 \lambda_j(e_j, h)^2} \\
&= \exp(it(m, h)) \lim_{k \rightarrow \infty} \exp\left(-\frac{1}{2}t^2 \sum_{j=1}^{n_k} \lambda_j(e_j, h)^2\right). \tag{49.7.39}
\end{aligned}$$

Now

$$\begin{aligned}
(Qh, h) &= \left(Q \sum_{k=1}^{\infty} (e_k, h) e_k, \sum_{j=1}^{\infty} (e_j, h) e_j \right) \\
&= \left(\sum_{k=1}^{\infty} (e_k, h) \lambda_k e_k, \sum_{j=1}^{\infty} (e_j, h) e_j \right) \\
&= \sum_{j=1}^{\infty} \lambda_j (e_j, h)^2
\end{aligned}$$

and so, passing to the limit in 49.7.39 yields

$$\exp(it(m, h)) \exp\left(-\frac{1}{2}t^2 (Qh, h)\right) \tag{49.7.40}$$

which implies that $\omega \rightarrow (X(\omega), h)$ is normally distributed with mean (m, h) and variance (Qh, h) .

Now let $\mu = \mathcal{L}(X)$. That is, for all $B \in \mathcal{B}(U)$,

$$\mu(B) \equiv P([X \in B])$$

In particular, B could be the cylindrical set

$$B \equiv [x : (x, h) \in A]$$

for A a Borel set in $\mathbb{R}$. Then by definition, if $h \in U$, and A is a Borel set in $\mathbb{R}$,

$$\begin{aligned}
\mu(B) &= \mu([x : (x, h) \in A]) \equiv P(\{\omega : (X(\omega), h) \in A\}) \\
&= \int_A \frac{1}{\sqrt{2\pi(Qh, h)}} e^{-\frac{(t - (m, h))^2}{2(Qh, h)}} dt
\end{aligned}$$

and so $x \rightarrow (x, h)$ is normally distributed. Therefore by definition, μ is a Gaussian measure.

Letting $t = 1$ in 49.7.40 it follows

$$\int_U e^{i(x, h)} d\mu(x) = \int_{\Omega} e^{i(X(\omega), h)} dP = \exp(i(m, h)) \exp\left(-\frac{1}{2}(Qh, h)\right)$$

which is the characteristic function of a Gaussian measure on U having covariance Q and mean m . This proves the proposition.

49.8 Abstract Wiener Spaces

This material follows [13], [36] and [29]. More can be found on this subject in these references. Here H will be a separable real Hilbert space.

Definition 49.8.1 *Cylinder sets in H are of the form*

$$\{x \in H : ((x, e_1), \dots, (x, e_n)) \in F\}$$

where $F \in \mathcal{B}(\mathbb{R}^n)$, the Borel sets of $\mathbb{R}^n$ and $\{e_k\}$ are given vectors in H . Denote this collection of cylinder sets as $\mathcal{C}$.

Proposition 49.8.2 *The cylinder sets form an algebra of sets.*

Proof: First note the complement of a cylinder set is a cylinder set.

$$\begin{aligned} & \{x \in H : ((x, e_1), \dots, (x, e_n)) \in F\}^C \\ &= \{x \in H : ((x, e_1), \dots, (x, e_n)) \in F^C\}. \end{aligned}$$

Now consider the intersection of two of these cylinder sets. Let the cylinder sets be

$$\begin{aligned} & \{x \in H : ((x, e_1), \dots, (x, e_n)) \in E\}, \\ & \{x \in H : ((x, f_1), \dots, (x, f_m)) \in F\} \end{aligned}$$

The first of these equals

$$\{x \in H : ((x, e_1), \dots, (x, e_n), (x, f_1), \dots, (x, f_m)) \in E \times \mathbb{R}^m\}$$

and the second equals

$$\{x \in H : ((x, e_1), \dots, (x, e_n), (x, f_1), \dots, (x, f_m)) \in \mathbb{R}^n \times F\}$$

Therefore, their intersection equals

$$\begin{aligned} & \{x \in H : ((x, e_1), \dots, (x, e_n), (x, f_1), \dots, (x, f_m)) \\ & \in E \times \mathbb{R}^m \cap \mathbb{R}^n \times F\}, \end{aligned}$$

a cylinder set.

Now it is clear the whole of H and $\emptyset$ are cylinder sets given by

$$\{x \in H : (e, x) \in \mathbb{R}\}, \quad \{x \in H : (e, x) \in \emptyset\}$$

respectively and so if C_1, C_2 are two cylinder sets,

$$C_1 \setminus C_2 \equiv C_1 \cap C_2^C,$$

which was just shown to be a cylinder set. Hence

$$C_1 \cup C_2 = (C_1^C \cap C_2^C)^C,$$

a cylinder set. This proves the proposition.

It is good to have a more geometrical description of cylinder sets. Letting A be a cylinder set as just described, let P denote the orthogonal projection onto $\text{span}(e_1, \dots, e_n)$. Also let $\alpha : PH \rightarrow \mathbb{R}^n$ be given by

$$\alpha(x) \equiv ((x, e_1), \dots, (x, e_n)).$$

This is continuous but might not be one to one if the e_i are not a basis for example. Then consider $\alpha^{-1}(F)$, those $x \in PH$ such that

$$((x, e_1), \dots, (x, e_n)) \in F.$$

For any $x \in H$,

$$((I - P)x, e_k) = 0$$

for each k and so

$$((x, e_1), \dots, (x, e_n)) = ((Px, e_1), \dots, (Px, e_n)) \in F$$

Thus $Px \in \alpha^{-1}(F)$, which is a Borel set of PH and

$$x = Px + (I - P)x$$

so the cylinder set is contained in

$$\alpha^{-1}(F) + (PH)^\perp$$

which is of the form

$$(\text{Borel set of } PH) + (PH)^\perp$$

On the other hand, consider a set of the form

$$G + (PH)^\perp$$

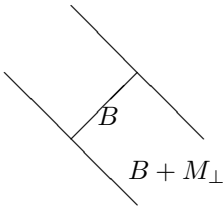
where G is a Borel set in PH . There is a basis for PH consisting of a subset of $\{e_1, \dots, e_n\}$. For simplicity, suppose it is $\{e_1, \dots, e_k\}$. Then let $\alpha_1 : PH \rightarrow \mathbb{R}^k$ be given by

$$\alpha_1(x) \equiv ((x, e_1), \dots, (x, e_k))$$

Thus α is a homeomorphism of PH and $\mathbb{R}^k$ so $\alpha_1(G)$ is a Borel set of $\mathbb{R}^k$. Now

$$\alpha^{-1}(\alpha_1(G) \times \mathbb{R}^{n-k}) = G$$

and $\alpha_1(G) \times \mathbb{R}^{n-k}$ is a Borel set of $\mathbb{R}^n$. This has proved the following important Proposition illustrated by the following picture.



Proposition 49.8.3 *The cylinder sets are sets of the form*

$$B + M^\perp$$

where M is a finite dimensional subspace and B is a Borel subset of M . Furthermore, the collection of cylinder sets is an algebra.

Lemma 49.8.4 $\sigma(\mathcal{C})$, the smallest σ algebra containing $\mathcal{C}$, contains the Borel sets of $H, \mathcal{B}(H)$.

Proof: It follows from the definition of these cylinder sets that if $f_i(x) \equiv (x, e_i)$, so that $f_i \in H'$, then with respect to $\sigma(\mathcal{C})$, each f_i is measurable. It follows that every linear combination of the f_i is also measurable with respect to $\sigma(\mathcal{C})$. However, this set of linear combinations is dense in H' and so the conclusion of the lemma follows from Lemma 47.1.16 on Page 1307. This proves the lemma.

Also note that the mapping

$$x \rightarrow ((x, e_1), \dots, (x, e_n))$$

is a $\sigma(\mathcal{C})$ measurable map. Restricting it to $\text{span}(e_1, \dots, e_n)$, it is Borel measurable. Next is a definition of a Gaussian measure defined on $\mathcal{C}$. While this is what it is called, it is a fake measure in general because it cannot be extended to a countably additive measure on $\sigma(\mathcal{C})$. This will be shown below.

Definition 49.8.5 Let $Q \in \mathcal{L}(H, H)$ be self adjoint and satisfy

$$(Qx, x) > 0$$

for all $x \in H, x \neq 0$. Define ν on the cylinder sets, $\mathcal{C}$ by the following rule. For $\{e_k\}_{k=1}^n$ an orthonormal set in H ,

$$\begin{aligned} & \nu(\{x \in H : ((x, e_1), \dots, (x, e_n)) \in F\}) \\ & \equiv \frac{1}{(2\pi)^{n/2} (\det(\theta^* Q \theta))^{1/2}} \int_F e^{-\frac{1}{2} \mathbf{t}^* \theta^* Q^{-1} \theta \mathbf{t}} d\mathbf{t}. \end{aligned}$$

where here

$$\theta \mathbf{t} \equiv \sum_{i=1}^n t_i e_i.$$

Note that the cylinder set is of the form

$$\theta F + \text{span}(e_1, \dots, e_n)^\perp.$$

Thus if $B + M^\perp$ is a typical cylinder set, choose an orthonormal basis for $M, \{e_k\}_{k=1}^n$ and do the above definition with $F = \theta^{-1}B$.

To see this last claim which is like what was done earlier, let

$$((x, e_1), \dots, (x, e_n)) \in F.$$

Then $\theta((x, e_1), \dots, (x, e_n)) = \sum_i (x, e_i) e_i = Px$ and so

$$\begin{aligned} x &= x - Px + Px = x - Px + \theta((x, e_1), \dots, (x, e_n)) \\ &\in \theta F + \text{span}(e_1, \dots, e_n)^\perp \end{aligned}$$

Thus

$$\{x \in H : ((x, e_1), \dots, (x, e_n)) \in F\} \subseteq \theta F + \text{span}(e_1, \dots, e_n)^\perp$$

To see the other inclusion, if $\mathbf{t} \in F$ and $y \in \text{span}(e_1, \dots, e_n)^\perp$, then if $x = \theta \mathbf{t}$, it follows

$$t_i = (x, e_i)$$

and so $((x, e_1), \dots, (x, e_n)) \in F$. But $(y, e_k) = 0$ for all k and so $x + y$ is in

$$\{x \in H : ((x, e_1), \dots, (x, e_n)) \in F\}.$$

Lemma 49.8.6 *The above definition is well defined.*

Proof: Let $\{f_k\}$ be another orthonormal set such that for F, G Borel sets in $\mathbb{R}^n$,

$$\begin{aligned} A &= \{x \in H : ((x, e_1), \dots, (x, e_n)) \in F\} \\ &= \{x \in H : ((x, f_1), \dots, (x, f_n)) \in G\} \end{aligned}$$

I need to verify $\nu(A)$ is the same using either $\{f_k\}$ or $\{e_k\}$. Let $\mathbf{a} \in G$. Then

$$x \equiv \sum_{i=1}^n a_i f_i \in A$$

because $(x, f_k) = a_k$. Therefore, for this x it is also true that $((x, e_1), \dots, (x, e_n)) \in F$. In other words for $\mathbf{a} \in G$,

$$\left(\sum_{i=1}^n (e_1, f_i) a_i, \dots, \sum_{i=1}^n (e_n, f_i) a_i \right) \in F$$

Let $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ be defined by

$$L\mathbf{a} \equiv \sum_i L_{ji} a_i, \quad L_{ji} \equiv (e_j, f_i).$$

Since the $\{e_j\}$ and $\{f_k\}$ are orthonormal, this mapping is unitary. Also this has shown that

$$LG \subseteq F.$$

Similarly

$$L^*F \subseteq G$$

where L^* has the ij^{th} entry $L_{ij}^* = (f_i, e_j)$ as above and L^* is the inverse of L because L is unitary. Thus

$$F = L(L^*(F)) \subseteq L(G) \subseteq F$$

showing that $LG = F$ and $L^*F = G$.

Now let $\theta_e \mathbf{t} \equiv \sum_i t_i e_i$ with θ_f defined similarly. Then the definition of $\nu(A)$ corresponding to $\{e_i\}$ is

$$\nu(A) \equiv \frac{1}{(2\pi)^{n/2} (\det(\theta_e^* Q \theta_e))^{1/2}} \int_F e^{-\frac{1}{2} \mathbf{t}^* \theta_e^* Q^{-1} \theta_e \mathbf{t}} d\mathbf{t}$$

Now change the variables letting $\mathbf{t} = L\mathbf{s}$ where $\mathbf{s} \in G$.

From the definition,

$$\begin{aligned} \theta_e L\mathbf{s} &= \sum_j \sum_i (e_j, f_i) s_i e_j \\ &= \sum_j \left(e_j, \sum_i f_i s_i \right) e_j = \sum_j (e_j, \theta_f \mathbf{s}) e_j \end{aligned}$$

and so

$$L\mathbf{s} = ((e_1, \theta_f \mathbf{s}), \dots, (e_n, \theta_f \mathbf{s})) = \theta_e^* \theta_f \mathbf{s}$$

where from the definition,

$$\begin{aligned} (\theta_e^* \theta_e \mathbf{s}, \mathbf{t}) &= \sum_i t_i \left(\sum_j s_j e_j, e_i \right) \\ &= \sum_i t_i s_i = (\mathbf{s}, \mathbf{t}) \end{aligned}$$

and so $\theta_e^* \theta_e$ is the identity on $\mathbb{R}^n$ and similar reasoning yields $\theta_e \theta_e^*$ is the identity on $\theta_e(\mathbb{R}^n)$. Then using the change of variables formula and the fact $|\det(L)| = 1$,

$$\begin{aligned} & \frac{1}{(2\pi)^{n/2} (\det(\theta_e^* Q \theta_e))^{1/2}} \int_F e^{-\frac{1}{2} \mathbf{t}^* \theta_e^* Q^{-1} \theta_e \mathbf{t}} d\mathbf{t} \\ &= \frac{1}{(2\pi)^{n/2} (\det(\theta_e^* Q \theta_e))^{1/2}} \int_G e^{-\frac{1}{2} \mathbf{s}^* L^* \theta_e^* Q^{-1} \theta_e L \mathbf{s}} d\mathbf{s} \\ &= \frac{1}{(2\pi)^{n/2} (\det(\theta_f^* Q \theta_f))^{1/2}} \int_G e^{-\frac{1}{2} \mathbf{s}^* \theta_f^* \theta_e \theta_e^* Q^{-1} \theta_e \theta_e^* \theta_f \mathbf{s}} d\mathbf{s} \\ &= \frac{1}{(2\pi)^{n/2} (\det(\theta_f^* Q \theta_f))^{1/2}} \int_G e^{-\frac{1}{2} \mathbf{s}^* \theta_f^* Q^{-1} \theta_f \mathbf{s}} d\mathbf{s} \end{aligned}$$

where part of the justification is as follows.

$$\det(\theta_f^* Q \theta_f) = \det(\theta_f^* \theta_e \theta_e^* Q \theta_e \theta_e^* \theta_f)$$

$$\begin{aligned} &= \det(\theta_f^* \theta_e) \det(\theta_e^* Q \theta_e) \det(\theta_e^* \theta_f) \\ &= \det(\theta_e^* Q \theta_e) \end{aligned}$$

because

$$\det(\theta_f^* \theta_e) \det(\theta_e^* \theta_f) = \det(\theta_f^* \theta_e \theta_e^* \theta_f) = \det(\theta_f^* \theta_f) = 1.$$

This proves the lemma.

It would be natural to try to extend ν to the σ algebra determined by $\mathcal{C}$ and obtain a measure defined on this σ algebra. However, this is always impossible in the case where $Q = I$.

Proposition 49.8.7 *For $Q = I$, ν cannot be extended to a measure defined on $\sigma(\mathcal{C})$ whenever H is infinite dimensional.*

Proof: Let $\{e_n\}$ be a complete orthonormal set of vectors in H . Then first note that H is a cylinder set.

$$H = \{x \in H : (x, e_1) \in \mathbb{R}\}$$

and so

$$\nu(H) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}t^2} dt = 1.$$

However, H is also equal to the countable union of the sets,

$$A_n \equiv \{x \in H : ((x, e_1)_H, \dots, (x, e_{a_n})_H) \in B(\mathbf{0}, n)\}$$

where $a_n \rightarrow \infty$.

$$\begin{aligned} \nu(A_n) &\equiv \frac{1}{(\sqrt{2\pi})^{a_n}} \int_{B(\mathbf{0}, n)} e^{-\frac{1}{2}|\mathbf{t}|^2} d\mathbf{t} \\ &\leq \frac{1}{(\sqrt{2\pi})^{a_n}} \int_{-n}^n \dots \int_{-n}^n e^{-|\mathbf{t}|^2/2} dt_1 \dots dt_{a_n} \\ &= \left(\frac{\int_{-n}^n e^{-x^2/2} dx}{\sqrt{2\pi}} \right)^{a_n} \end{aligned}$$

Now pick a_n so large that the above is smaller than $1/2^{n+1}$. This can be done because for no matter what choice of n ,

$$\frac{\int_{-n}^n e^{-x^2/2} dx}{\sqrt{2\pi}} < 1.$$

Then

$$\sum_{n=1}^{\infty} \nu(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

This proves the proposition and shows something else must be done to get a countably additive measure from ν .

However, let $\mu(C) \equiv \nu_M(C)$ where C is a cylinder set of the form $C = B + M^\perp$ for M a finite dimensional subspace.

Proposition 49.8.8 μ is finitely additive on $\mathcal{C}$ the algebra of cylinder sets.

Proof: Let

$$A \equiv \{x \in H : ((x, e_1), \dots, (x, e_n)) \in E\},$$

$$B \equiv \{x \in H : ((x, f_1), \dots, (x, f_m)) \in F\}$$

be two disjoint cylinder sets. Then writing them differently as was done earlier they are

$$\{x \in H : ((x, e_1), \dots, (x, e_n), (x, f_1), \dots, (x, f_m)) \in E \times \mathbb{R}^m\}$$

and

$$\{x \in H : ((x, e_1), \dots, (x, e_n), (x, f_1), \dots, (x, f_m)) \in \mathbb{R}^n \times F\}$$

respectively. Hence the two sets $E \times \mathbb{R}^m, \mathbb{R}^n \times F$ must be disjoint. Then the definition yields $\mu(A \cup B) = \mu(A) + \mu(B)$. This proves the proposition.

Definition 49.8.9 Let H be a separable Hilbert space and let $\|\cdot\|$ be a norm defined on H which has the following property. Whenever $\{e_n\}$ is an orthonormal sequence of vectors in H and $\mathcal{F}(\{e_n\})$ consists of the set of all orthogonal projections onto the span of finitely many of the e_k the following condition holds. For every $\varepsilon > 0$ there exists $P_\varepsilon \in \mathcal{F}(\{e_n\})$ such that if $P \in \mathcal{F}(\{e_n\})$ and $PP_\varepsilon = 0$, then

$$\nu(\{x \in H : \|Px\| > \varepsilon\}) < \varepsilon.$$

Then $\|\cdot\|$ is called Gross measurable.

The following lemma is a fundamental result about Gross measurable norms. It is about the continuity of $\|\cdot\|$. It is obvious that with respect to the topology determined by $\|\cdot\|$ that $x \rightarrow \|x\|$ is continuous. However, it would be interesting if this were the case with respect to the topology determined by the norm on $H, |\cdot|$. This lemma shows this is the case and so the funny condition above implies $x \rightarrow \|x\|$ is a continuous, hence Borel measurable function.

Lemma 49.8.10 Let $\|\cdot\|$ be Gross measurable. Then there exists $c > 0$ such that

$$\|x\| \leq c|x|$$

for all $x \in H$. Furthermore, the above definition is well defined.

Proof: First it is important to consider the question whether the above definition is well defined. To do this note that on PH , the two norms are equivalent because PH is a finite dimensional space. Let $G = \{y \in PH : \|y\| > \varepsilon\}$ so G is an open set in PH . Then

$$\{x \in H : \|Px\| > \varepsilon\}$$

equals

$$\{x \in H : Px \in G\}$$

which equals a set of the form

$$\{x \in H : ((x, e_{i_1})_H, \dots, (x, e_{i_m})_H) \in G'\}$$

for G' an open set in $\mathbb{R}^m$ and so everything makes sense in the above definition.

Now it is necessary to verify $\|\cdot\| \leq c|\cdot|$. If it is not so, there exists e_1 such that

$$\|e_1\| \geq 1, |e_1| = 1.$$

Suppose $\{e_k\}_{k=1}^n$ have been chosen such that each is a unit vector in H and $\|e_k\| \geq k$. Then considering $\text{span}(e_1, \dots, e_n)^\perp$ if for every $x \in \text{span}(e_1, \dots, e_n)^\perp$, $\|x\| \leq c|x|$, then if $z \in H$ is arbitrary, $z = x + y$ where $y \in \text{span}(e_1, \dots, e_n)$ and so since the two norms are equivalent on a finite dimensional subspace, there exists c' corresponding to $\text{span}(e_1, \dots, e_n)$ such that

$$\begin{aligned} \|z\|^2 &\leq (\|x\| + \|y\|)^2 \leq 2\|x\|^2 + 2\|y\|^2 \\ &\leq 2c^2|x|^2 + 2c'|y|^2 \\ &\leq (2c^2 + 2c'^2)(|x|^2 + |y|^2) \\ &= (2c^2 + 2c'^2)|z|^2 \end{aligned}$$

and the lemma is proved. Therefore it can be assumed, there exists

$$e_{n+1} \in \text{span}(e_1, \dots, e_n)^\perp$$

such that $|e_{n+1}| = 1$ and $\|e_{n+1}\| \geq n + 1$.

This constructs an orthonormal set of vectors, $\{e_k\}$. Letting $0 < \varepsilon < \frac{1}{2}$, it follows since $\|\cdot\|$ is measurable, there exists $P_\varepsilon \in \mathcal{F}(\{e_n\})$ such that if $PP_\varepsilon = 0$ where $P \in \mathcal{F}(\{e_n\})$, then

$$\nu(\{x \in H : \|Px\| > \varepsilon\}) < \varepsilon.$$

Say P_ε is the projection onto the span of finitely many of the e_k , the last one being e_N . Then for $n > N$ and P_n the projection onto e_n , it follows $P_\varepsilon P_n = 0$ and from the definition of ν ,

$$\begin{aligned} \varepsilon &> \nu(\{x \in H : \|P_n x\| > \varepsilon\}) \\ &= \nu(\{x \in H : |(x, e_n)| \|e_{n+1}\| > \varepsilon\}) \\ &= \nu(\{x \in H : |(x, e_n)| > \varepsilon / \|e_{n+1}\|\}) \\ &\geq \nu(\{x \in H : |(x, e_n)| > \varepsilon / (n+1)\}) \\ &> \frac{1}{\sqrt{2\pi}} \int_{\varepsilon/(n+1)}^\infty e^{-x^2/2} dx \end{aligned}$$

which yields a contradiction for all n large enough. This proves the lemma.

What are examples of Gross measurable norms defined on a separable Hilbert space, H ? The following lemma gives an important example.

Lemma 49.8.11 *Let H be a separable Hilbert space and let $A \in \mathcal{L}_2(H, H)$, a Hilbert Schmidt operator. Thus A is a continuous linear operator with the property that for any orthonormal set, $\{e_k\}$,*

$$\sum_{k=1}^{\infty} |Ae_k|^2 < \infty.$$

Then define $||\cdot||$ by

$$||x|| \equiv |Ax|_H.$$

Then if $||\cdot||$ is a norm, it is measurable¹.

Proof: Let $\{e_k\}$ be an orthonormal sequence. Let P_n denote the orthogonal projection onto $\text{span}(e_1, \dots, e_n)$. Let $\varepsilon > 0$ be given. Since A is a Hilbert Schmidt operator, there exists N such that

$$\sum_{k=N}^{\infty} |Ae_k|^2 < \alpha$$

where α is chosen very small. In fact, α is chosen such that $\alpha < \varepsilon^2/r^2$ where r is sufficiently large that

$$\frac{2}{\sqrt{2\pi}} \int_r^{\infty} e^{-t^2/2} dt < \varepsilon. \quad (49.8.41)$$

Let P denote an orthogonal projection in $\mathcal{F}(\{e_k\})$ such that $PP_N = 0$. Thus P is the projection on to $\text{span}(e_{i_1}, \dots, e_{i_m})$ where each $i_k > N$. Then

$$\begin{aligned} & \nu(\{x \in H : ||Px|| > \varepsilon\}) \\ = & \nu(\{x \in H : |APx| > \varepsilon\}) \end{aligned}$$

Now $Px = \sum_{j=1}^m (x, e_{i_j}) e_{i_j}$ and the above reduces to

$$\nu \left(\left\{ x \in H : \left| \sum_{j=1}^m (x, e_{i_j}) Ae_{i_j} \right| > \varepsilon \right\} \right) \leq$$

¹If it is only a seminorm, it satisfies the same conditions.

$$\begin{aligned}
& \nu \left(\left\{ x \in H : \left(\sum_{j=1}^m |(x, e_{i_j})|^2 \right)^{1/2} \left(\sum_{j=1}^m |Ae_{i_j}|^2 \right)^{1/2} > \varepsilon \right\} \right) \\
& \leq \nu \left(\left\{ x \in H : \left(\sum_{j=1}^m |(x, e_{i_j})|^2 \right)^{1/2} \alpha^{1/2} > \varepsilon \right\} \right) \\
& = \nu \left(\left\{ x \in H : \left(\sum_{j=1}^m |(x, e_{i_j})|^2 \right)^{1/2} > \frac{\varepsilon}{\alpha^{1/2}} \right\} \right) \\
& = \nu \left(\left\{ x \in H : ((x, e_{i_1}), \dots, (x, e_{i_m})) \in B \left(\mathbf{0}, \frac{\varepsilon}{\alpha^{1/2}} \right)^C \right\} \right) \\
& \leq \nu \left(\left\{ x \in H : \max \{ |(x, e_{i_j})| \} > \frac{\varepsilon}{\sqrt{m}\alpha^{1/2}} \right\} \right)
\end{aligned}$$

This is no larger than

$$\begin{aligned}
& \frac{1}{(\sqrt{2\pi})^m} \int_{|t_1| > \frac{\varepsilon}{\sqrt{m}\sqrt{\alpha}}} \int_{|t_2| > \frac{\varepsilon}{\sqrt{m}\sqrt{\alpha}}} \cdots \int_{|t_m| > \frac{\varepsilon}{\sqrt{m}\sqrt{\alpha}}} e^{-|t|^2/2} dt_m \cdots dt_1 \\
& = \left(\frac{2 \int_{\varepsilon/(\sqrt{m}\alpha^{1/2})}^{\infty} e^{-t^2/2} dt}{\sqrt{2\pi}} \right)^m
\end{aligned}$$

which by Jensen's inequality is no larger than

$$\begin{aligned}
\frac{2 \int_{\varepsilon/(\sqrt{m}\alpha^{1/2})}^{\infty} e^{-mt^2/2} dt}{\sqrt{2\pi}} &= \frac{2 \frac{1}{\sqrt{m}} \int_{\varepsilon/(\alpha^{1/2})}^{\infty} e^{-t^2/2} dt}{\sqrt{2\pi}} \\
&\leq \frac{2 \int_{\varepsilon/(\varepsilon/r)}^{\infty} e^{-t^2/2} dt}{\sqrt{2\pi}} \\
&= \frac{2 \int_r^{\infty} e^{-t^2/2} dt}{\sqrt{2\pi}} < \varepsilon
\end{aligned}$$

By 49.8.41. This proves the lemma.

Definition 49.8.12 A triple, (i, H, B) is called an abstract Wiener space if B is a separable Banach space and H is a separable Hilbert space such that H is dense and continuously embedded in B and the norm $\|\cdot\|$ on B is Gross measurable.

Next consider a weaker norm for H which comes from the inner product

$$(x, y)_E \equiv \sum_{k=1}^{\infty} \frac{1}{k^2} (x, e_k)_H (y, e_k)_H.$$

Then let E be the completion of H with respect to this new norm. Thus $\{ke_k\}$ is a complete orthonormal basis for E . This follows from the density of H in E along with the obvious observation that in the above inner product, $\{ke_k\}$ is an orthonormal set of vectors.

Lemma 49.8.13 *There exists a countably additive Gaussian measure, λ defined on $\mathcal{B}(E)$. This measure is the law of the random variable,*

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega) e_k,$$

where $\{\xi_k\}$ denotes a sequence of independent normally distributed random variables having mean 0 and variance 1, the series converging pointwise a.e. in E . Also

$$k^2 (X(\omega), e_k)_E = \xi_k(\omega) \text{ a.e.}$$

Proof: Observe that $\sum_{k=1}^{\infty} \frac{1}{k^2} (ke_k) \otimes (ke_k)$ is a nuclear operator on the Hilbert space, E . Letting $\{\xi_k\}$ be a sequence of independent random variables each normally distributed with mean 0 and variance 1, that

$$X(\omega) \equiv \sum_{k=1}^{\infty} \frac{1}{k} \xi_k(\omega) ke_k = \sum_{k=1}^{\infty} \xi_k(\omega) e_k \quad (49.8.42)$$

is a random variable with values in E and $\mathcal{L}(X)$ is a Gaussian measure on $\mathcal{B}(E)$, the series converging pointwise a.e. in E . Let λ be the name of this Gaussian measure and denote the probability space on which the ξ_k are defined as $(\Omega, \mathcal{F}, P)$. Thus for $F \in \mathcal{B}(E)$,

$$\lambda(F) \equiv P(\{\omega \in \Omega : X(\omega) \in F\})$$

Finally, denoting by X_N , the partial sum,

$$X_N(\omega) \equiv \sum_{k=1}^N \xi_k(\omega) e_k,$$

the definition of $(\cdot, \cdot)_E$ on H and a simple computation yields

$$\begin{aligned} \xi_k(\omega) &= \lim_{N \rightarrow \infty} k^2 (X_N(\omega), e_k)_E \\ &= k^2 (X(\omega), e_k)_E. \end{aligned} \quad (49.8.43)$$

One can pass to the limit because $X_N(\omega)$ converges to $X(\omega)$ in E . This proves the lemma.

Theorem 49.8.14 *Let (i, H, B) be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of B .*

Proof: Let E be defined above as the completion of H with respect to that weaker norm. Then from Lemma 49.8.13 and $X(\omega)$ given above in 49.8.42,

$$k^2 (X(\omega), e_k)_E = \xi_k(\omega) \text{ a.e. } \omega.$$

Let $\{e_n\}$ be a complete orthonormal set for H . There exists an increasing sequence of projections, $\{Q_n\} \subseteq \mathcal{F}(\{e_n\})$ such that $Q_n x \rightarrow x$ in H for each $x \in H$. Say Q_n is the orthogonal projection onto $\text{span}(e_1, \dots, e_{p_n})$. Then since $\|\cdot\|$ is measurable, these can be chosen such that if Q is the orthogonal projection onto $\text{span}(e_1, \dots, e_k)$ for some $k > p_n$ then

$$\nu(\{x : \|Qx - Q_n x\| > 2^{-n}\}) < 2^{-n}.$$

In particular,

$$\nu(\{x : \|Q_n x - Q_m x\| > 2^{-m}\}) < 2^{-m}$$

whenever $n \geq m$.

I would like to consider the infinite series,

$$S(\omega) \equiv \sum_{k=1}^{\infty} k^2 (X(\omega), e_k)_E e_k \in B.$$

converging in B but of course this might make no sense because the series might not converge. It was shown above that the series converges in E but it has not been shown to converge in B .

Suppose the series did converge a.e. Then let $f \in B'$ and consider the random variable $f \circ S$ which maps Ω to $\mathbb{R}$. I would like to verify this is normally distributed. First note that the following finite sum is weakly measurable and separably valued so it is strongly measurable with values in B .

$$S_{p_n}(\omega) \equiv \sum_{k=1}^{p_n} k^2 (X(\omega), e_k)_E e_k,$$

Since $f \in B'$ which is a subset of H' due to the assumption that H is dense in B , there exists a unique $v \in H$ such that $f(x) = (x, v)$ for all $x \in H$. Then from the above sum,

$$f(S_{p_n}(\omega)) = (S_{p_n}(\omega), v) = \sum_{k=1}^{p_n} k^2 (X(\omega), e_k)_E (e_k, v)$$

which by Lemma 49.8.13 equals

$$\sum_{k=1}^{p_n} (e_k, v)_H \xi_k(\omega)$$

a finite linear combination of the independent $N(0, 1)$ random variables, $\xi_k(\omega)$. Then it follows

$$\omega \rightarrow f(S_{p_n}(\omega))$$

is also normally distributed and has mean 0 and variance equal to

$$\sum_{k=1}^{p_n} (e_k, v)_H^2.$$

Then it seems reasonable to suppose

$$\begin{aligned} E(e^{itf \circ S}) &= \lim_{n \rightarrow \infty} E(e^{itf \circ S_{p_n}}) \\ &= \lim_{n \rightarrow \infty} e^{-t^2 \sum_{k=1}^{p_n} (e_k, v)_H^2} \\ &= e^{-t^2 \sum_{k=1}^{\infty} (e_k, v)_H^2} \\ &= e^{-t^2 |v|_H^2} \end{aligned} \quad (49.8.44)$$

the characteristic function of a random variable which is $N(0, |v|_H^2)$. Thus at least formally, this would imply for all $f \in B'$, $f \circ S$ is normally distributed and so if $\mu = \mathcal{L}(S)$, then by Lemma 49.6.3 it follows μ is a Gaussian measure.

What is missing to make the above a proof? First of all, there is the issue of the sum. Next there is the problem of passing to the limit in the little argument above in which the characteristic function is used.

First consider the sum. Note that $Q_n X(\omega) \in H$. Then for any $n > p_m$,

$$\begin{aligned} &P(\{\omega \in \Omega : \|S_n(\omega) - S_{p_m}(\omega)\| > 2^{-m}\}) \\ &= P\left(\left\{\omega \in \Omega : \left\| \sum_{k=p_m+1}^n k^2 (X(\omega), e_k)_E e_k \right\| > 2^{-m} \right\}\right) \\ &= P\left(\left\{\omega \in \Omega : \left\| \sum_{k=p_m+1}^n \xi_k(\omega) e_k \right\| > 2^{-m} \right\}\right) \end{aligned} \quad (49.8.45)$$

Let Q be the orthogonal projection onto $\text{span}(e_1, \dots, e_n)$. Define

$$F \equiv \{x \in (Q - Q_m)H : \|x\| > 2^{-m}\}$$

Then continuing the chain of equalities ending with 49.8.45,

$$\begin{aligned} &= P\left(\left\{\omega \in \Omega : \sum_{k=p_m+1}^n \xi_k(\omega) e_k \in F\right\}\right) \\ &= P(\{\omega \in \Omega : (\xi_n(\omega), \dots, \xi_{p_m+1}(\omega)) \in F'\}) \\ &= \nu(\{x \in H : ((x, e_n)_H, \dots, (x, e_{p_m+1})_H) \in F'\}) \\ &= \nu(\{x \in H : Q(x) - Q_m(x) \in F\}) \\ &= \nu(\{x \in H : \|Q(x) - Q_m(x)\| > 2^{-m}\}) < 2^{-m}. \end{aligned}$$

This has shown that

$$P(\{\omega \in \Omega : \|S_n(\omega) - S_{p_m}(\omega)\| > 2^{-m}\}) < 2^{-m} \quad (49.8.46)$$

for all $n > p_m$. In particular, the above is true if $n = p_n$ for $n > m$.

If $\{S_{p_n}(\omega)\}$ fails to converge, then ω must be contained in the set,

$$A \equiv \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega \in \Omega : \|S_{p_n}(\omega) - S_{p_m}(\omega)\| > 2^{-m}\} \quad (49.8.47)$$

because if ω is in the complement of this set,

$$\bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{\omega \in \Omega : \|S_{p_n}(\omega) - S_{p_m}(\omega)\| \leq 2^{-m}\},$$

it follows $\{S_{p_n}(\omega)\}_{n=1}^{\infty}$ is a Cauchy sequence and so it must converge. However, the set in 49.8.47 is a set of measure 0 because of 49.8.46 and the observation that for all m ,

$$\begin{aligned} P(A) &\leq \sum_{n=m}^{\infty} P(\{\omega \in \Omega : \|S_{p_n}(\omega) - S_{p_m}(\omega)\| > 2^{-m}\}) \\ &\leq \sum_{n=m}^{\infty} \frac{1}{2^m} \end{aligned}$$

Thus the subsequence $\{S_{p_n}\}$ of the sequence of partial sums of the above series does converge pointwise in B and so the dominated convergence theorem also verifies that the computations involving the characteristic function in 49.8.44 are correct.

The random variable obtained as the limit of the partial sums, $\{S_{p_n}(\omega)\}$ described above is strongly measurable because each $S_{p_n}(\omega)$ is strongly measurable due to each of these being weakly measurable and separably valued. Thus the measure given as the law of S defined as

$$S(\omega) \equiv \lim_{n \rightarrow \infty} S_{p_n}(\omega)$$

is defined on the Borel sets of B . This proves the theorem.

Also, there is an important observation from the proof which I will state as the following corollary.

Corollary 49.8.15 *Let (i, H, B) be an abstract Wiener space. Then there exists a Gaussian measure on the Borel sets of B . This Gaussian measure equals $\mathcal{L}(S)$ where $S(\omega)$ is the a.e. limit of a subsequence of the sequence of partial sums,*

$$S_{p_n}(\omega) \equiv \sum_{k=1}^{p_n} \xi_k(\omega) e_k$$

for $\{\xi_k\}$ a sequence of independent random variables which are normal with mean 0 and variance 1 which are defined on a probability space, $(\Omega, \mathcal{F}, P)$. Furthermore, for any $k > p_n$,

$$P(\{\omega \in \Omega : \|S_k(\omega) - S_{p_n}(\omega)\| > 2^{-n}\}) < 2^{-n}.$$

49.9 White Noise

In an abstract Wiener space as discussed above there is a Gaussian measure, μ defined on the Borel sets of B . This measure is the law of a random variable having values in B which is the limit of a subsequence of a sequence of partial sums. I will show here that the sequence of partial sums also converges pointwise a.e.

Now with this preparation, here is the theorem about white noise.

Theorem 49.9.1 *Let (i, H, B) be an abstract Wiener space and $\{e_k\}$ is a complete orthonormal sequence in H . Then there exists a Gaussian measure on the Borel sets of B . This Gaussian measure equals $\mathcal{L}(S)$ where $S(\omega)$ is the a.e. limit of the sequence of partial sums,*

$$S_n(\omega) \equiv \sum_{k=1}^n \xi_k(\omega) e_k$$

for $\{\xi_k\}$ a sequence of independent random variables which are normal with mean 0 and variance 1, defined on a probability space, $(\Omega, \mathcal{F}, P)$

Proof: By Corollary 49.8.15 there is a subsequence, $\{S_{p_n}\}$ of these partial sums which converge pointwise a.e. to $S(\omega)$. However, this corollary also states that

$$P(\{\omega \in \Omega : \|S_k(\omega) - S_{p_n}(\omega)\| > 2^{-n}\}) < 2^{-n}$$

whenever $k > p_n$ and so by Lemma 47.8.6 the original sequence of partial sums also converges uniformly of a set of measure zero. The reason this lemma applies is that $\xi_k(\omega) e_k$ has symmetric distribution. This proves the theorem.

49.10 Existence Of Abstract Wiener Spaces

It turns out that if E is a separable Banach space, then it is the top third of an abstract Wiener space. This is what will be shown in this section. Therefore, it follows from the above that there exists a Gaussian measure on E which is the law of an a.e. convergent series as discussed above. First recall Lemma 13.3.2 on Page 379.

Lemma 49.10.1 *Let E be a separable Banach space. Then there exists an increasing sequence of subspaces, $\{F_n\}$ such that $\dim(F_{n+1}) - \dim(F_n) \leq 1$ and equals 1 for all n if the dimension of E is infinite. Also $\cup_{n=1}^{\infty} F_n$ is dense in E .*

Lemma 49.10.2 *Let E be a separable Banach space. Then there exists a sequence $\{e_n\}$ of points of E such that whenever $|\beta| \leq 1$ for $\beta \in \mathbb{F}^n$,*

$$\sum_{k=1}^n \beta_k e_k \in B(0, 1)$$

the unit ball in E .

Proof: By Lemma 49.10.1, let $\{z_1, \dots, z_n\}$ be a basis for F_n where $\cup_{n=1}^{\infty} F_n$ is dense in E . Then let α_1 be such that $e_1 \equiv \alpha_1 z_1 \in B(0, 1)$. Thus $\beta_1 e_1 \in B(0, 1)$ whenever $|\beta_1| \leq 1$. Suppose α_i has been chosen for $i = 1, 2, \dots, n$ such that for all $\beta \in D_n \equiv \{\alpha \in \mathbb{F}^n : |\alpha| \leq 1\}$, it follows

$$\sum_{k=1}^n \beta_k \alpha_k z_k \in B(0, 1).$$

Then

$$C_n \equiv \left\{ \sum_{k=1}^n \beta_k \alpha_k z_k : \beta \in D_n \right\}$$

is a compact subset of $B(0, 1)$ and so it is at a positive distance from the complement of $B(0, 1)$, δ . Now let $0 < \alpha_{n+1} < \delta / \|z_{n+1}\|$. Then for $\beta \in D_{n+1}$,

$$\sum_{k=1}^n \beta_k \alpha_k z_k \in C_n$$

and so

$$\begin{aligned} \left\| \sum_{k=1}^{n+1} \beta_k \alpha_k z_k - \sum_{k=1}^n \beta_k \alpha_k z_k \right\| &= \|\beta_{n+1} \alpha_{n+1} z_{n+1}\| \\ &< \|\alpha_{n+1} z_{n+1}\| < \delta \end{aligned}$$

which shows

$$\sum_{k=1}^{n+1} \beta_k \alpha_k z_k \in B(0, 1).$$

This proves the lemma. Let $e_k \equiv \alpha_k z_k$.

Now the main result is the following. It says that any separable Banach space is the upper third of some abstract Wiener space.

Theorem 49.10.3 *Let E be a real separable Banach space with norm $\|\cdot\|$. Then there exists a separable Hilbert space, H such that H is dense in E and the inclusion map is continuous. Furthermore, if ν is the Gaussian measure defined earlier on the cylinder sets of H , $\|\cdot\|$ is Gross measurable.*

Proof: Let $\{e_k\}$ be the points of E described in Lemma 49.10.2. Then let H_0 denote the subspace of all finite linear combinations of the $\{e_k\}$. It follows H_0 is dense in E . Next decree that $\{e_k\}$ is an orthonormal basis for H_0 . Thus for

$$\begin{aligned} \sum_{k=1}^n c_k e_k, \sum_{j=1}^n d_j e_j &\in H_0, \\ \left(\sum_{k=1}^n c_k e_k, \sum_{j=1}^n d_j e_j \right)_{H_0} &\equiv \sum_{k=1}^n c_k d_k \end{aligned}$$

this being well defined because the $\{e_k\}$ are linearly independent. Let the norm on H_0 be denoted by $|\cdot|_{H_0}$. Let H_1 be the completion of H_0 with respect to this norm.

I want to show that $|\cdot|_{H_0}$ is stronger than $\|\cdot\|$. Suppose then that

$$\left| \sum_{k=1}^n \beta_k e_k \right|_{H_0} \leq 1.$$

It follows then from the definition of $|\cdot|_{H_0}$ that

$$\left| \sum_{k=1}^n \beta_k e_k \right|_{H_0}^2 = \sum_{k=1}^n \beta_k^2 \leq 1$$

and so from the construction of the e_k , it follows that

$$\left\| \sum_{k=1}^n \beta_k e_k \right\| < 1$$

Stated more simply, this has just shown that if $h \in H_0$ then since $|h|/|h|_{H_0}|_{H_0} \leq 1$, it follows that

$$\|h\|/|h|_{H_0} < 1$$

and so

$$\|h\| < |h|_{H_0}.$$

It follows that the completion of H_0 must lie in E because this shows that every Cauchy sequence in H_0 is a Cauchy sequence in E . Thus H_1 embeds continuously into E and is dense in E . Denote its norm by $|\cdot|_{H_1}$.

Now consider the nuclear operator,

$$A = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$$

where each $\lambda_k > 0$ and $\sum_k \lambda_k < \infty$. This operator is clearly one to one. Also it is clear the operator is Hilbert Schmidt because $\sum_k \lambda_k^2 < \infty$. Let

$$H \equiv AH_1.$$

and for $x \in H$, define

$$|x|_H \equiv |A^{-1}x|_{H_1}.$$

Since each e_k is in H it follows that H is dense in E . Note also that $H \subseteq H_1$ because A maps H_1 to H_1 .

$$Ax \equiv \sum_{k=1}^{\infty} \lambda_k (x, e_k) e_k$$

and the series converges in H_1 because

$$\sum_{k=1}^{\infty} \lambda_k |(x, e_k)| \leq \left(\sum_{k=1}^{\infty} \lambda_k^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |(x, e_k)|^2 \right)^{1/2} < \infty.$$

Also H is a Hilbert space with inner product given by

$$(x, y)_H \equiv (A^{-1}x, A^{-1}y)_{H_1}.$$

H is complete because if $\{x_n\}$ is a Cauchy sequence in H , this is the same as $\{A^{-1}x_n\}$ being a Cauchy sequence in H_1 which implies $A^{-1}x_n \rightarrow y$ for some $y \in H_1$. Then it follows $x_n = A(A^{-1}x_n) \rightarrow Ay$ in H .

For $x \in H \subseteq H_1$,

$$\|x\| \leq |x|_{H_1} = |AA^{-1}x|_{H_1} \leq \|A\| |A^{-1}x|_{H_1} \equiv \|A\| |x|_H$$

and so the embedding of H into E is continuous. Why is $\|\cdot\|$ a measurable norm on H ? Note first that for $x \in H \subseteq H_1$,

$$|Ax|_H \equiv |A^{-1}Ax|_{H_1} = |x|_{H_1} \geq \|x\|_E. \quad (49.10.48)$$

Therefore, if it can be shown A is a Hilbert Schmidt operator on H , the desired measurability will follow from Lemma 49.8.11 on Page 1457.

Claim: A is a Hilbert Schmidt operator on H .

Proof of the claim: From the definition of the inner product in H , it follows an orthonormal basis for H is $\{\lambda_k e_k\}$. This is because

$$(\lambda_k e_k, \lambda_j e_j)_H \equiv (\lambda_k A^{-1} e_k, \lambda_j A^{-1} e_j)_{H_1} = (e_k, e_j)_{H_1} = \delta_{jk}.$$

To show that A is Hilbert Schmidt, it suffices to show that

$$\sum_k |A(\lambda_k e_k)|_H^2 < \infty$$

because this is the definition of an operator being Hilbert Schmidt. However, the above equals

$$\sum_k |A^{-1}A(\lambda_k e_k)|_{H_1}^2 = \sum_k \lambda_k^2 < \infty.$$

This proves the claim.

Now consider 49.10.48. By Lemma 49.8.11, it follows the norm $\|x\|' \equiv |Ax|_H$ is Gross measurable on H . Therefore, $\|\cdot\|_E$ is also Gross measurable because it is smaller. This proves the theorem.

Using Theorem 49.10.3 and Theorem 49.9.1 this proves most of the following important corollary.

Corollary 49.10.4 *Let E be any real separable Banach space. Then there exists a sequence, $\{e_k\} \subseteq E$ such that for any $\{\xi_k\}$ a sequence of independent random variables such that $\mathcal{L}(\xi_k) = N(0, 1)$, it follows*

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega) e_k$$

converges a.e. and its law is a Gaussian measure defined on $\mathcal{B}(E)$. Furthermore, $\|e_k\|_E \leq \lambda_k$ where $\sum_k \lambda_k < \infty$.

Proof: From the proof of Theorem 49.10.3 a basis for H is $\{\lambda_k e_k\}$. Therefore, by Theorem 49.9.1, if $\{\xi_k\}$ is a sequence of independent $N(0, 1)$ random variables, then $\sum_{k=1}^{\infty} \xi_k(\omega) \lambda_k e_k$ converges a.e. to a random variable whose law is Gaussian. Also from the proof of Theorem 49.9.1, each e_k in that proof has the property that $\|e_k\| \leq 1$ because if $\|e_k\| > 1$, then you could consider $\beta \equiv (0, 0, \dots, 1)$ and from the construction of the e_k , you would need $1e_k \in B(0, 1)$ which is a contradiction. Thus $\|\lambda_k e_k\| \leq \lambda_k$ and changing the notation, replacing $\lambda_k e_k$ with e_k , this proves the corollary.

Stochastic Processes

50.1 Fundamental Definitions And Properties

Here E will be a separable Banach space and $\mathcal{B}(E)$ will be the Borel sets of E . Let $(\Omega, \mathcal{F}, P)$ be a probability space and I will be an interval of $\mathbb{R}$. A set of E valued random variables, one for each $t \in I$, $\{X(t) : t \in I\}$ is called a stochastic process. Thus for each t , $X(t)$ is a measurable function of $\omega \in \Omega$. Set $X(t, \omega) \equiv X(t)(\omega)$. Functions $t \rightarrow X(t, \omega)$ are called trajectories. Thus there is a trajectory for each $\omega \in \Omega$. A stochastic process, Y is called a version or a modification of a stochastic process, X if for all $t \in I$,

$$X(t, \omega) = Y(t, \omega) \text{ a.e. } \omega$$

There are several descriptions of stochastic processes.

1. X is measurable if $X(\cdot, \cdot) : I \times \Omega \rightarrow E$ is $B(I) \times \mathcal{F}$ measurable. Note that a stochastic process, X is not necessarily measurable.
2. X is stochastically continuous at $t_0 \in I$ means: for all $\varepsilon > 0$ and $\delta > 0$ there exists $\rho > 0$ such that

$$P([||X(t) - X(t_0)|| \geq \varepsilon]) \leq \delta \text{ whenever } |t - t_0| < \rho, t \in I.$$

Note the above condition says that for each $\varepsilon > 0$,

$$\lim_{t \rightarrow t_0} P([||X(t) - X(t_0)|| \geq \varepsilon]) = 0.$$

3. X is stochastically continuous if it is stochastically continuous at every $t \in I$.
4. X is stochastically uniformly continuous if for every $\varepsilon, \delta > 0$ there exists $\rho > 0$ such that whenever $s, t \in I$ with $|s - t| < \rho$, it follows

$$P([||X(t) - X(s)|| \geq \varepsilon]) \leq \delta.$$

5. X is mean square continuous at $t_0 \in I$ if

$$\lim_{t \rightarrow t_0} E \left(\|X(t) - X(t_0)\|^2 \right) \equiv \lim_{t \rightarrow t_0} \int_{\Omega} \|X(t)(\omega) - X(t_0)(\omega)\|^2 dP = 0.$$

6. X is mean square continuous in I if it is mean square continuous at every point of I .

7. X is continuous with probability 1 or continuous if $t \rightarrow X(t, \omega)$ is continuous for all ω outside some set of measure 0.

8. X is Hölder continuous if $t \rightarrow X(t, \omega)$ is Hölder continuous for a.e. ω .

Lemma 50.1.1 *A stochastically continuous process on $[a, b] \equiv I$ is uniformly stochastically continuous on $[a, b] \equiv I$.*

Proof: If this is not so, there exists $\varepsilon, \delta > 0$ and points of I, s_n, t_n such that even though

$$|t_n - s_n| < \frac{1}{n},$$

$$P(\|X(s_n) - X(t_n)\| \geq \varepsilon) > \delta. \quad (50.1.1)$$

Taking a subsequence, still denoted by s_n and t_n there exists $t \in I$ such that the above hold and

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} t_n = t.$$

Then

$$\begin{aligned} & P(\|X(s_n) - X(t_n)\| \geq \varepsilon) \\ & \leq P(\|X(s_n) - X(t)\| \geq \varepsilon/2) + P(\|X(t) - X(t_n)\| \geq \varepsilon/2). \end{aligned}$$

But the sum of the last two terms converges to 0 as $n \rightarrow \infty$ by stochastic continuity of X at t , violating 50.1.1 for all n large enough. This proves the lemma.

For a stochastically continuous process defined on a closed and bounded interval, there always exists a measurable version. This is significant because then you can do things with product measure and iterated integrals.

Proposition 50.1.2 *Let X be a stochastically continuous process defined on a closed interval, $I \equiv [a, b]$. Then there exists a measurable version of X .*

Proof: By Lemma 50.1.1 X is uniformly stochastically continuous and so there exists a sequence of positive numbers, $\{\rho_n\}$ such that if $|s - t| < \rho_n$, then

$$P\left(\|X(t) - X(s)\| \geq \frac{1}{2^n}\right) \leq \frac{1}{2^n}. \quad (50.1.2)$$

Then let $\{t_0^n, t_1^n, \dots, t_{m_n}^n\}$ be a partition of $[a, b]$ in which $|t_i^n - t_{i-1}^n| < \rho_n$. Now define X_n as follows:

$$\begin{aligned} X_n(t) &\equiv \sum_{i=1}^{m_n} X(t_{i-1}^n) \mathcal{X}_{[t_{i-1}^n, t_i^n)}(t) \\ X_n(b) &\equiv X(b). \end{aligned}$$

Then X_n is obviously $B(I) \times \mathcal{F}$ measurable because it is the sum of functions which are. Consider the set, A on which $\{X_n(t, \omega)\}$ is a Cauchy sequence. This set is of the form

$$A = (I \times \Omega) \setminus \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \bigcup_{p=k}^{\infty} \left[\|X_m - X_p\| > \frac{1}{n} \right]$$

and so it is a $B(I) \times \mathcal{F}$ measurable set. Now define

$$Y(t, \omega) \equiv \begin{cases} \lim_{n \rightarrow \infty} X_n(t, \omega) & \text{if } (t, \omega) \in A \\ 0 & \text{if } (t, \omega) \notin A \end{cases}$$

I claim $Y(t, \omega) = X(t, \omega)$ for a.e. ω . To see this, consider 50.1.2. From the construction of X_n , it follows that for each t ,

$$P\left(\left[\|X_n(t) - X(t)\| \geq \frac{1}{2^n}\right]\right) \leq \frac{1}{2^n}$$

Also, for a fixed t , if $X_n(t, \omega)$ fails to converge to $X(t, \omega)$, then ω must be in infinitely many of the sets,

$$B_n \equiv \left[\|X_n(t) - X(t)\| \geq \frac{1}{2^n}\right]$$

which is a set of measure zero by the Borel Cantelli lemma. Recall why this is so.

$$P(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B_n) \leq \sum_{n=k}^{\infty} P(B_n) < \frac{1}{2^{k-1}}$$

Therefore, for each t , $(t, \omega) \in A$ for a.e. ω . Hence $X(t) = Y(t)$ a.e. and so Y is a measurable version of X .

Lemma 50.1.3 *Let D be a dense subset of an interval, $I = [0, T]$ and suppose $X : D \rightarrow E$ satisfies*

$$\|X(d) - X(d')\| \leq C |d - d'|^\gamma$$

for all $d', d \in D$. Then X extends uniquely to a continuous Y defined on $[0, T]$ such that

$$\|Y(t) - Y(t')\| \leq C |t - t'|^\gamma.$$

Proof: Let $t \in I$ and let $d_k \rightarrow t$ where $d_k \in D$. Then $\{X(d_k)\}$ is a Cauchy sequence because $\|X(d_k) - X(d_m)\| \leq C|d_k - d_m|^\gamma$. Therefore, $X(d_k)$ converges. The thing it converges to will be called $Y(t)$. Note this is well defined, giving $X(t)$ if $t \in D$. Also, if $d_k \rightarrow t$ and $d'_k \rightarrow t$, then $\|X(d_k) - X(d'_k)\| \leq C|d_k - d'_k|^\gamma$ and so $X(d_k)$ and $X(d'_k)$ converge to the same thing. Therefore, it makes sense to define $Y(t) \equiv \lim_{d \rightarrow t} X(d)$. It only remains to verify the estimate. But letting $|d - t|$ and $|d' - t'|$ be small enough,

$$\begin{aligned} \|Y(t) - Y(t')\| &= \|X(d) - X(d')\| + \varepsilon \\ &\leq C|d' - d| + \varepsilon \leq C|t - t'| + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary, this proves the existence part of the lemma. Uniqueness follows from observing that $Y(t)$ must equal $\lim_{d \rightarrow t} X(d)$. This proves the lemma.

The following is a very interesting theorem called the Kolmogorov Čentsov continuity theorem[41].

Theorem 50.1.4 Suppose X is a stochastic process for each $t \in [0, T]$. Suppose also that there exists a constant, C and positive numbers, α, β such that

$$E(\|X(t) - X(s)\|^\alpha) \leq C|t - s|^{1+\beta} \quad (50.1.3)$$

Then there exists a stochastic process Y such that for a.e. ω , $t \rightarrow Y(t)(\omega)$ is Hölder continuous with exponent $\gamma < \frac{\beta}{\alpha}$ and for each t , $P(\|X(t) - Y(t)\| > 0) = 0$.

Proof: Let r_j^m denote $j(\frac{T}{2^m})$ where $j \in \{0, 1, \dots, 2^m\}$. Also let $D_m = \{r_j^m\}_{j=1}^{2^m}$ and $D = \cup_{m=1}^\infty D_m$. Consider the set,

$$\{ \|X(t) - X(s)\| > \delta \}$$

By 50.1.3,

$$\begin{aligned} P(\|X(t) - X(s)\| > \delta) \delta^\alpha &\leq \int_{\|X(t) - X(s)\| > \delta} \|X(t) - X(s)\|^\alpha dP \\ &\leq C|t - s|^{1+\beta}. \end{aligned} \quad (50.1.4)$$

Letting $t = r_{j+1}^k$, $s = r_j^k$, and $\delta = 2^{-\gamma k}$ where

$$\gamma \in \left(0, \frac{\beta}{\alpha}\right),$$

this yields

$$P(\|X(r_{j+1}^k) - X(r_j^k)\| > 2^{-\gamma k}) \leq C2^{\alpha\gamma k} (T2^{-k})^{1+\beta}.$$

There are 2^k of these differences and so letting

$$N_k = \cup_{j=1}^{2^k} [\|X(r_{j+1}^k) - X(r_j^k)\| > 2^{-\gamma k}]$$

it follows

$$P(N_k) \leq C 2^{\alpha \gamma k} (T 2^{-k})^{1+\beta} 2^k = C 2^{k(\alpha \gamma - \beta)} T^{1+\beta}.$$

Since $\gamma < \beta/\alpha$,

$$\sum_{k=1}^{\infty} P(N_k) \leq C T^{1+\beta} \sum_{k=1}^{\infty} 2^{k(\alpha \gamma - \beta)} < \infty$$

and so by the Borel Cantelli lemma, Lemma 47.1.2, there exists a set of measure zero, N , such that if $\omega \notin N$, then ω is in only finitely many N_k . In other words, for $\omega \notin N$, there exists $M(\omega)$ such that if $k > M(\omega)$, then for each j ,

$$||X(r_{j+1}^k)(\omega) - X(r_j^k)(\omega)|| \leq 2^{-\gamma k}. \quad (50.1.5)$$

Claim: If $n \geq M(\omega)$ for $\omega \notin N$ and if $d, d' \in D_m$ for $m > n$ such that $|d - d'| < T 2^{-n}$, then

$$||X(d')(\omega) - X(d)(\omega)|| \leq 2 \sum_{j=n+1}^m 2^{-\gamma j}.$$

Proof of the claim: Suppose $d' < d$. Suppose first $m = n + 1$. Then $d = (k + 1) T 2^{-(n+1)}$ and $d' = k T 2^{-(n+1)}$. Then from 50.1.5

$$||X(d')(\omega) - X(d)(\omega)|| \leq 2^{-\gamma(n+1)} \leq 2 \sum_{j=n+1}^{n+1} 2^{-\gamma j}.$$

Suppose the claim is true for some $m > n$ and let $d, d' \in D_{m+1}$ with $|d - d'| < T 2^{-n}$. If there is no point of D_m between these, then d', d are either adjacent points in D_m or in D_{m+1} . Consequently,

$$||X(d') - X(d)|| \leq 2^{-\gamma m} < 2 \sum_{j=n+1}^{m+1} 2^{-\gamma j}.$$

Assume therefore, there exist points of D_m between d' and d . Let $d' \leq d'_1 \leq d_1 \leq d$ where d_1, d'_1 are in D_m and d'_1 is the smallest element of D_m which is at least as large as d' and d_1 is the largest element of D_m which is no larger than d . Then $|d' - d'_1| \leq T 2^{-(m+1)}$ and $|d_1 - d| \leq T 2^{-(m+1)}$ while all of these are still in D_{m+1} which contains D_m . Therefore, from 50.1.5 and induction,

$$\begin{aligned} & ||X(d')(\omega) - X(d)(\omega)|| \\ & \leq ||X(d')(\omega) - X(d'_1)(\omega)|| + ||X(d'_1)(\omega) - X(d_1)(\omega)|| + ||X(d_1)(\omega) - X(d)(\omega)|| \\ & \leq 2 \times 2^{-\gamma(m+1)} + 2 \sum_{j=n+1}^m 2^{-\gamma j} = 2 \sum_{j=n+1}^{m+1} 2^{-\gamma j} \end{aligned}$$

which proves the claim.

From this estimate, it follows that if $d, d' \in D$ and $|d - d'| < T2^{-n}$ where $n \geq M(\omega)$, then d, d' are both in some D_m where $m > n$ and so

$$\begin{aligned} \|X(d')(\omega) - X(d)(\omega)\| &\leq 2 \sum_{j=n+1}^m 2^{-\gamma j} \leq 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \\ &= \frac{2}{1 - 2^{-\gamma}} (2^{-\gamma})^{n+1}. \end{aligned} \quad (50.1.6)$$

Now let $d, d' \in D$ and suppose $|d - d'| < T2^{-M(\omega)}$. Then let $n \geq M(\omega)$ such that

$$T2^{-(n+1)} \leq |d - d'| < T2^{-n}$$

Then from 50.1.6,

$$\begin{aligned} \|X(d')(\omega) - X(d)(\omega)\| &\leq \frac{2}{T^\gamma (1 - 2^{-\gamma})} \left(T2^{-(n+1)}\right)^\gamma \\ &\leq \frac{2}{T^\gamma (1 - 2^{-\gamma})} (|d - d'|)^\gamma \end{aligned}$$

which shows $t \rightarrow X(t)(\omega)$ is Holder continuous on D .

By Lemma 50.1.3, one can define $Y(t)(\omega)$ to be the unique function which extends $d \rightarrow X(d)(\omega)$ off D for $\omega \notin N$ and let $Y(t)(\omega) = 0$ if $\omega \in N$. Thus by Lemma 50.1.3 $t \rightarrow Y(t)(\omega)$ is Holder continuous. Also, $\omega \rightarrow Y(t)(\omega)$ is measurable because it is the pointwise limit of measurable functions

$$Y(t)(\omega) = \lim_{d \rightarrow t} X(d)(\omega) \mathcal{X}_{N^c}(\omega).$$

It remains to verify the claim that $Y(t)(\omega) = X(t)(\omega)$ a.e.

$$\mathcal{X}_{\{|Y(t)-X(t)|>\varepsilon\} \cap N^c}(\omega) \leq \liminf_{d \rightarrow t} \mathcal{X}_{\{|X(d)-X(t)|>\varepsilon\} \cap N^c}(\omega)$$

because if $\omega \in N^c$ then the above limit holds and so if $\|Y(t)(\omega) - X(t)(\omega)\| > \varepsilon$, the same is true of $\|X(d)(\omega) - X(t)(\omega)\|$ whenever d is close enough to t and so by Fatou's theorem

$$\begin{aligned} P(\{|Y(t) - X(t)| > \varepsilon\}) &= \int \mathcal{X}_{\{|Y(t)-X(t)|>\varepsilon\} \cap N^c}(\omega) dP \\ &\leq \int \liminf_{d \rightarrow t} \mathcal{X}_{\{|X(d)-X(t)|>\varepsilon\}}(\omega) dP \\ &\leq \liminf_{d \rightarrow t} \int \mathcal{X}_{\{|X(d)-X(t)|>\varepsilon\}}(\omega) dP \\ &\leq \liminf_{d \rightarrow t} P(\{|X(d) - X(t)|^\alpha > \varepsilon^\alpha\}) \\ &\leq \liminf_{d \rightarrow t} \varepsilon^{-\alpha} \int_{\{|X(d)-X(t)|^\alpha > \varepsilon^\alpha\}} \|X(d) - X(t)\|^\alpha dP \\ &\leq \liminf_{d \rightarrow t} \frac{C}{\varepsilon^\alpha} |d - t|^{1+\beta} = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} P(\|Y(t) - X(t)\| > 0) &= P\left(\bigcup_{k=1}^{\infty} \left[\|Y(t) - X(t)\| > \frac{1}{k}\right]\right) \\ &\leq \sum_{k=1}^{\infty} P\left(\left[\|Y(t) - X(t)\| > \frac{1}{k}\right]\right) = 0. \end{aligned}$$

This proves the theorem.

50.2 Filtrations

Definition 50.2.1 Let X be a stochastic process defined on an interval, $I = [0, T]$ or $[0, \infty)$. Suppose the probability space, $(\Omega, \mathcal{F}, P)$ has an increasing family of σ algebras, $\{\mathcal{F}_t\}$. This is called a filtration. Denote by $\mathcal{F}_{t+}$ the intersection of all $\mathcal{F}_s$ for $s > t$. The filtration is normal if

1. $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $P(A) = 0$
2. $\mathcal{F}_t = \mathcal{F}_{t+}$ for all $t \in I$.

If for arbitrary $t \in I$ the random variable $X(t)$ is $\mathcal{F}_t$ measurable, then X is said to be adapted to the family, $\{\mathcal{F}_t\}$. X is called progressively measurable if for every $t \in I$, the mapping

$$(s, \omega) \in [0, t] \times \Omega, \quad (s, \omega) \rightarrow X(s, \omega)$$

is $B([0, t]) \times \mathcal{F}_t$ measurable. I shall generally assume the filtration is normal.

Example 50.2.2 For example, you could have a stochastic process, $X(t)$ and you could define

$$\mathcal{G}_t \equiv \overline{\sigma(X(s) : s \leq t)},$$

the completion of the smallest σ algebra such that each $X(s)$ is measurable for all $s \leq t$. This gives an example of a filtration to which $X(t)$ is adapted which satisfies

1. More generally, suppose $X(t)$ is adapted to a filtration, $\mathcal{G}_t$. Define

$$\mathcal{F}_t \equiv \bigcap_{s>t} \mathcal{G}_s$$

Then

$$\mathcal{F}_{t+} \equiv \bigcap_{s>t} \mathcal{F}_s = \bigcap_{s>t} \bigcap_{r>s} \mathcal{G}_r = \bigcap_{s>t} \mathcal{G}_s \equiv \mathcal{F}_t.$$

and each $X(t)$ is measurable with respect to $\mathcal{F}_t$. Thus there is no harm in assuming a stochastic process adapted to a filtration can be modified so the filtration is normal.

The following is similar to Proposition 50.1.2.

Proposition 50.2.3 Let X be a stochastically continuous adapted process for a normal filtration defined on a closed interval, $I \equiv [0, T]$. Then X has a progressively measurable adapted version.

Proof: By Lemma 50.1.1 X is uniformly stochastically continuous and so there exists a sequence of positive numbers, $\{\rho_n\}$ such that if $|s - t| < \rho_n$, then

$$P\left(\left[\|X(t) - X(s)\| \geq \frac{1}{2^n}\right]\right) \leq \frac{1}{2^n}. \quad (50.2.7)$$

Then let $\{t_0^n, t_1^n, \dots, t_{m_n}^n\}$ be a partition of $[0, T]$ in which $|t_i^n - t_{i-1}^n| < \rho_n$. Now define X_n as follows:

$$\begin{aligned} X_n(t)(\omega) &\equiv \sum_{i=1}^{m_n} X(t_{i-1}^n)(\omega) \mathcal{X}_{[t_{i-1}^n, t_i^n)}(t) \\ X_n(T) &\equiv X(T). \end{aligned}$$

Then $(s, \omega) \rightarrow X_n(s, \omega)$ for $(s, \omega) \in [0, t] \times \Omega$ is obviously $B([0, t]) \times \mathcal{F}$ measurable. Consider the set, A on which $\{X_n(t, \omega)\}$ is a Cauchy sequence. This set is of the form

$$A = (I \times \Omega) \setminus \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcup_{m=k}^{\infty} \bigcup_{p=k}^{\infty} \left[\|X_m - X_p\| > \frac{1}{n} \right]$$

and so it is a $B(I) \times \mathcal{F}$ measurable set. Now define

$$Y(t, \omega) \equiv \begin{cases} \lim_{n \rightarrow \infty} X_n(t, \omega) & \text{if } (t, \omega) \in A \\ 0 & \text{if } (t, \omega) \notin A \end{cases}$$

Thus for each t , $(s, \omega) \rightarrow X_n(s, \omega)$ for $(s, \omega) \in [0, t] \times \Omega$ is $B([0, t]) \times \mathcal{F}$ measurable and $A \cap [0, t] \times \Omega$ is $B([0, t]) \times \mathcal{F}$ measurable.

I claim that for each t , $Y(t, \omega) = X(t, \omega)$ for a.e. ω . To see this, consider 50.2.7. From the construction of X_n , it follows that for each t ,

$$P\left(\left[\|X_n(t) - X(t)\| \geq \frac{1}{2^n}\right]\right) \leq \frac{1}{2^n}$$

Also, for a fixed t , if $X_n(t, \omega)$ fails to converge to $X(t, \omega)$, then ω must be in infinitely many of the sets,

$$B_n \equiv \left[\|X_n(t) - X(t)\| \geq \frac{1}{2^n} \right]$$

which is a set of measure zero by the Borel Cantelli lemma. Recall why this is so.

$$P(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} B_n) \leq \sum_{n=k}^{\infty} P(B_n) < \frac{1}{2^{k-1}}$$

Therefore, for each t , $(t, \omega) \in A$ for a.e. ω . Hence $X(t) = Y(t)$ a.e. and so Y is a measurable version of X . Y is adapted because the filtration is normal and hence $\mathcal{F}_t$ contains all sets of measure zero. Therefore, $Y(t)$ differs from $X(t)$ on a set which is $\mathcal{F}_t$ measurable. This proves the proposition.

There is a situation in which the measurability of a stochastic process automatically implies it is adapted.

Definition 50.2.4 Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, P)$ and denote by $\mathcal{P}_\infty$ the smallest σ algebra of sets of $[0, \infty) \times \Omega$ containing the sets

$$(s, t] \times F, F \in \mathcal{F}_s \quad \{0\} \times F, F \in \mathcal{F}_0.$$

This is called the predictable σ algebra. and the sets in this σ algebra are called the predictable sets. Denote by $\mathcal{P}_T$ the intersection of $\mathcal{P}_\infty$ to $[0, T] \times \Omega$. A stochastic process X which maps either $[0, T] \times \Omega$ or $[0, \infty) \times \Omega$ to E , a separable real Banach space is called predictable if for every Borel set $A \in \mathcal{B}(E)$, it follows $X^{-1}(A) \in \mathcal{P}_T$ or $\mathcal{P}_\infty$.

This is a lot like product measure except one of the σ algebras is changing.

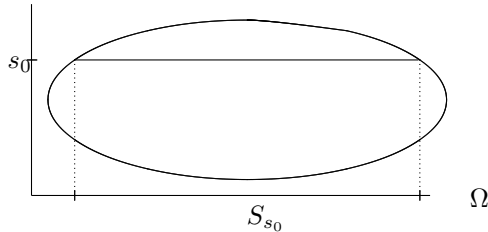
Proposition 50.2.5 Let $\mathcal{F}_t$ be a filtration as above and let X be a predictable stochastic process. Then X is $\mathcal{F}_t$ adapted.

Proof: Let $s_0 > 0$ and define

$$\mathcal{G}_{s_0} \equiv \{S \in \mathcal{P}_\infty : S_{s_0} \in \mathcal{F}_{s_0}\}$$

where

$$S_{s_0} \equiv \{\omega \in \Omega : (s_0, \omega) \in S\}.$$



It is clear $\mathcal{G}_{s_0}$ is a σ algebra. The next step is to show $\mathcal{G}_{s_0}$ contains the sets

$$(s, t] \times F, F \in \mathcal{F}_s \tag{50.2.8}$$

and

$$\{0\} \times F, F \in \mathcal{F}_0. \tag{50.2.9}$$

It is clear $\{0\} \times F$ is contained in $\mathcal{G}_{s_0}$ because $(\{0\} \times F)_{s_0} = \emptyset \in \mathcal{F}_{s_0}$. Similarly, if $s \geq s_0$ or if $s, t < s_0$ then $((s, t] \times F)_{s_0} = \emptyset \in \mathcal{F}_{s_0}$. The only case left is for $s < s_0$ and $t \geq s_0$. In this case, letting $A_s \in \mathcal{F}_s$, $((s, t] \times A_s)_{s_0} = A_s \in \mathcal{F}_s \subseteq \mathcal{F}_{s_0}$. Therefore, $\mathcal{G}_{s_0}$ contains all the sets of the form given in 50.2.8 and 50.2.9 and so since $\mathcal{P}_\infty$ is the smallest σ algebra containing these sets, it follows $\mathcal{P}_\infty = \mathcal{G}_{s_0}$. The case where $s_0 = 0$ is entirely similar but shorter.

Therefore, if X is predictable, letting $A \in \mathcal{B}(E)$, $X^{-1}(A) \in \mathcal{P}_\infty$ or $\mathcal{P}_T$ and so

$$(X^{-1}(A))_s \equiv \{\omega \in \Omega : X(s, \omega) \in A\} = X(s)^{-1}(A) \in \mathcal{F}_s$$

showing $X(t)$ is $\mathcal{F}_t$ adapted. This proves the proposition.

Proposition 50.2.6 *Let $X(t)$ be a stochastic process having values in E a complete metric space and let it be $\mathcal{F}_t$ adapted and left continuous. Then it is predictable. Also, if $X(t)$ is stochastically continuous and adapted on $[0, T]$, then it has a predictable version.*

Proof: Define $I_{m,k} \equiv ((k-1)2^{-m}T, k2^{-m}T]$ if $k \geq 1$ and $I_{m,0} = \{0\}$ if $k = 1$. Then define

$$\begin{aligned} X_m(t) &\equiv \sum_{k=1}^{2^m} X((k-1)2^{-m}T) \mathcal{X}_{((k-1)2^{-m}T, k2^{-m}T]}(t) \\ &\quad + X(0) \mathcal{X}_{[0,0]}(t) \end{aligned}$$

Here the sum means that $X_m(t)$ has value $X((k-1)2^{-m}T)$ on the interval

$$((k-1)2^{-m}T, k2^{-m}T].$$

Thus X_m is predictable because each term in the formal sum is. Thus

$$\begin{aligned} X_m^{-1}(U) &= \cup_{k=1}^{2^m} (X((k-1)2^{-m}T) \mathcal{X}_{((k-1)2^{-m}T, k2^{-m}T]}^{-1}(U)) \\ &= \cup_{k=1}^{2^m} ((k-1)2^{-m}T, k2^{-m}T] \times (X((k-1)2^{-m}T))^{-1}(U), \end{aligned}$$

a finite union of predictable sets. Since X is left continuous,

$$X(t, \omega) = \lim_{m \rightarrow \infty} X_m(t, \omega)$$

and so X is predictable.

Next consider the other claim. Since X is stochastically continuous on $[0, T]$ it is uniformly stochastically continuous on this interval by Lemma 50.1.1. Therefore, there exists a sequence of partitions of $[0, T]$, the m^{th} being

$$0 = t_{m,0} < t_{m,1} < \cdots < t_{m,n(m)} = T$$

such that for X_m defined as above, then for each t

$$P([d(X_m(t), X(t)) \geq 2^{-m}]) \leq 2^{-m} \quad (50.2.10)$$

Then as above, X_m is predictable. Let A denote those points of $\mathcal{P}_T$ at which $X_m(t, \omega)$ converges. Thus A is a predictable set because it is just the set where $X_m(t, \omega)$ is a Cauchy sequence. Now define the predictable function Y

$$Y(t, \omega) \equiv \begin{cases} \lim_{m \rightarrow \infty} X_m(t, \omega) & \text{if } (t, \omega) \in A \\ 0 & \text{if } (t, \omega) \notin A \end{cases}$$

From 50.2.10 it follows from the Borel Cantelli lemma that for fixed t , the set of ω which are in infinitely many of the sets,

$$[d(X_m(t), X(t)) \geq 2^{-m}]$$

has measure zero. Therefore, for each t , there exists a set of measure zero, $N(t)$ such that for $\omega \notin N(t)$ and all m large enough

$$[d(X_m(t, \omega), X(t, \omega)) < 2^{-m}]$$

Hence for $\omega \notin N(t)$, $(t, \omega) \in A$ and so $X_m(t, \omega) \rightarrow Y(t, \omega)$ which shows

$$d(Y(t, \omega), X(t, \omega)) = 0 \text{ if } \omega \notin N(t).$$

The predictable version of $X(t)$ is $Y(t)$. This proves the proposition.

50.3 Martingales

Definition 50.3.1 Let X be a stochastic process defined on an interval, I with values in a separable Banach space, E . It is called integrable if $E(\|X(t)\|) < \infty$ for each $t \in I$. Also let $\mathcal{F}_t$ be a filtration. An integrable and adapted stochastic process X is called a martingale if for $s \leq t$

$$E(X(t) | \mathcal{F}_s) = X(s) \text{ P a.e. } \omega.$$

Recalling the definition of conditional expectation this says that for $F \in \mathcal{F}_s$

$$\int_F X(t) dP = \int_F E(X(t) | \mathcal{F}_s) dP = \int_F X(s) dP$$

for all $F \in \mathcal{F}_s$. A real valued stochastic process is called a submartingale if whenever $s \leq t$,

$$E(X(t) | \mathcal{F}_s) \geq X(s) \text{ a.e.}$$

and a supermartingale if

$$E(X(t) | \mathcal{F}_s) \leq X(s) \text{ a.e.}$$

Proposition 50.3.2 The following statements hold for a stochastic process defined on $[0, T] \times \Omega$ having values in a real separable Banach space, E .

1. If $X(t)$ is a martingale then $\|X(t)\|, t \in [0, T]$ is a submartingale.
2. If g is an increasing convex function from $[0, \infty)$ to $[0, \infty)$ and $E(g(\|X(t)\|)) < \infty$ for all $t \in [0, T]$ then $g(\|X(t)\|), t \in [0, T]$ is a submartingale.

Proof: Let $s \leq t$

$$\begin{aligned} \|X(s)\| &= \|E(X(s) - X(t) | \mathcal{F}_s) + E(X(t) | \mathcal{F}_s)\| \\ &\leq \overbrace{\|E(X(s) - X(t) | \mathcal{F}_s)\|}^{=0 \text{ a.e.}} + \|E(X(t) | \mathcal{F}_s)\| \\ &\leq \|E(X(t) | \mathcal{F}_s)\|. \end{aligned}$$

Now by Theorem 49.1.1 on Page 1403

$$||E(X(t) | \mathcal{F}_s)|| \leq E(||X(t)|| | \mathcal{F}_s).$$

Thus $||X(s)|| \leq E(||X(t)|| | \mathcal{F}_s)$ which shows $||X||$ is a submartingale as claimed.

Consider the second claim. Recall Jensen's inequality for submartingales, Theorem 48.1.5 on Page 1374. From the first part

$$||X(s)|| \leq E(||X(t)|| | \mathcal{F}_s) \text{ a.e.}$$

and so from Jensen's inequality,

$$g(||X(s)||) \leq g(E(||X(t)|| | \mathcal{F}_s)) \leq E(g(||X(t)||) | \mathcal{F}_s) \text{ a.e.,}$$

showing that $g(||X(t)||)$ is also a submartingale. This proves the proposition.

50.3.1 Some Maximal Estimates

Martingales and submartingales have some very interesting maximal estimates. I will present some of these here. The proofs are fairly general and do not require the filtration to be normal.

Lemma 50.3.3 *Let $\{\mathcal{M}_t\}$ be a filtration and let $\{M(t)\}$ be a nonnegative valued submartingale for $t \in [S, T]$. Then for $\lambda > 0$ and any $p \geq 1$, if A_t is a $\mathcal{M}_t$ measurable subset of $[M(t) \geq \lambda]$, then*

$$P(A_t) \leq \frac{1}{\lambda^p} \int_{A_t} M(T)^p dP.$$

Proof: From Jensen's inequality,

$$\begin{aligned} \lambda^p P(A_t) &\leq \int_{A_t} M(t)^p dP \leq \int_{A_t} |E(M(T) | \mathcal{M}_t)|^p dP \\ &\leq \int_{A_t} E(M(T)^p | \mathcal{M}_t) dP = \int_{A_t} M(T)^p dP \end{aligned}$$

and this proves the lemma.

The following theorem is the main result.

Theorem 50.3.4 *Let $\{\mathcal{M}_t\}$ be a filtration and let $\{M(t)\}$ be a nonnegative valued right continuous¹ submartingale for $t \in [S, T]$. Then for all $\lambda > 0$ and $p \geq 1$,*

$$P\left(\left[\sup_{t \in [S, T]} M(t) \geq \lambda\right]\right) \leq \frac{1}{\lambda^p} \int_{\mathcal{X}_{[\sup_{t \in [S, T]} M(t) \geq \lambda]}} M(T)^p dP$$

In the case that $p > 1$, it is also true that

$$E\left(\left(\sup_{t \in [S, T]} M(t)\right)^p\right)^{1/p} \leq \left(\frac{p}{p-1}\right) E(M(T)^p)^{1/p}$$

¹ $t \rightarrow M(t)(\omega)$ is continuous from the right for a.e. ω .

Proof: Let $S \leq t_0^m < t_1^m < \cdots < t_{N_m}^m = T$ where $t_{j+1}^m - t_j^m = (T - S) 2^{-m}$. First consider $m = 1$.

$$A_{t_0^1} \equiv \{\omega \in \Omega : M(t_0^1)(\omega) \geq \lambda\}, \quad A_{t_1^1} \equiv \{\omega \in \Omega : M(t_1^1)(\omega) \geq \lambda\} \setminus A_{t_0^1}$$

$$A_{t_2^1} \equiv \{\omega \in \Omega : M(t_2^1)(\omega) \geq \lambda\} \setminus (A_{t_0^1} \cup A_{t_1^1}).$$

Do this type of construction for $m = 2, 3, 4, \dots$ yielding disjoint sets, $\{A_{t_j^m}\}_{j=0}^{2^m}$ whose union equals

$$\cup_{t \in D_m} [M(t) \geq \lambda]$$

where $D_m = \{t_j^m\}_{j=0}^{2^m}$. Thus $D_m \subseteq D_{m+1}$. Then also, $D \equiv \cup_{m=1}^{\infty} D_m$ is dense and countable. From Lemma 50.3.3,

$$\begin{aligned} P(\cup_{t \in D_m} [M(t) \geq \lambda]) &= \sum_{j=0}^{2^m} P(A_{t_j^m}) \\ &\leq \frac{1}{\lambda^p} \sum_{j=0}^{2^m} \int_{A_{t_j^m}} \mathcal{X}_{[\sup_{t \in D_m} M(t) \geq \lambda]} M(T)^p dP \\ &\leq \frac{1}{\lambda^p} \int_{\Omega} \mathcal{X}_{[\sup_{t \in D} M(t) \geq \lambda]} M(T)^p dP. \end{aligned}$$

Let $m \rightarrow \infty$ in the above to obtain

$$P(\cup_{t \in D} [M(t) \geq \lambda]) \leq \frac{1}{\lambda^p} \int_{\Omega} \mathcal{X}_{[\sup_{t \in D} M(t) \geq \lambda]} M(T)^p dP. \quad (50.3.11)$$

From now on, assume that for a.e. $\omega \in \Omega$, $t \rightarrow M(t)(\omega)$ is right continuous. Then with this assumption, the following claim holds.

Claim: For $\lambda > \varepsilon > 0$,

$$\cup_{t \in D} [M(t) \geq \lambda - \varepsilon] \supseteq \left[\sup_{t \in [S, T]} M(t) \geq \lambda \right]$$

Proof of the claim: Suppose $\omega \in \left[\sup_{t \in [S, T]} M(t) \geq \lambda \right]$. Then there exists s such that $|M(s)(\omega)| > \lambda - \varepsilon$. If $s < T$, then by right continuity, this situation persists for all $t > s$ for t close enough to s . In particular, it is true for some $t \in D$. If $s = T$, then $s \in D$. This proves the claim.

Letting P' denote the outer measure determined by P it follows from the claim and 50.3.11 that

$$\begin{aligned} P' \left(\left[\sup_{t \in [S, T]} M(t) \geq \lambda \right] \right) &\leq P(\cup_{t \in D} [M(t) \geq \lambda - \varepsilon]) \\ &\leq \frac{1}{(\lambda - \varepsilon)^p} \int_{\Omega} \mathcal{X}_{[\sup_{t \in D} M(t) \geq \lambda - \varepsilon]} M(T)^p dP. \end{aligned}$$

Since ε is arbitrary, this shows

$$P' \left(\left[\sup_{t \in [S, T]} M(t) \geq \lambda \right] \right) \leq \frac{1}{\lambda^p} \int_{\Omega} \mathcal{X}_{[\sup_{t \in D} M(t) \geq \lambda]} M(T)^p dP. \quad (50.3.12)$$

It would be interesting to consider whether $\left[\sup_{t \in [S, T]} M(t) \geq \lambda \right]$ is measurable. However, for every $\varepsilon \in (0, \lambda)$ it follows from the right continuity of M that

$$\left[\sup_{t \in D} M(t) \geq \lambda - \varepsilon \right] \supseteq \left[\sup_{t \in [S, T]} M(t) \geq \lambda \right] \supseteq \left[\sup_{t \in D} M(t) \geq \lambda \right]$$

Now let ε_n be a sequence decreasing to 0 and then

$$\begin{aligned} \cap_{n=1}^{\infty} \left[\sup_{t \in D} M(t) \geq \lambda - \varepsilon_n \right] &= \left[\sup_{t \in D} M(t) \geq \lambda \right] \\ &\supseteq \left[\sup_{t \in [S, T]} M(t) \geq \lambda \right] \\ &\supseteq \left[\sup_{t \in D} M(t) \geq \lambda \right] \end{aligned}$$

Thus $\left[\sup_{t \in [S, T]} M(t) \geq \lambda \right]$ is a measurable set due to countability of D . Also from 50.3.12 this shows

$$P \left(\left[\sup_{t \in [S, T]} M(t) \geq \lambda \right] \right) \leq \frac{1}{\lambda^p} \int_{\Omega} \mathcal{X}_{[\sup_{t \in [S, T]} M(t) \geq \lambda]} M(T)^p dP$$

Now consider the other inequality. Using the distribution function technique and the above estimate obtained in the first part,

$$\begin{aligned} E \left(\left(\sup_{t \in [S, T]} M(t) \right)^p \right) &= \int_0^{\infty} p\alpha^{p-1} P \left(\left[\sup_{t \in [S, T]} M(t) > \alpha \right] \right) d\alpha \\ &\leq \int_0^{\infty} p\alpha^{p-1} P \left(\left[\sup_{t \in [S, T]} M(t) \geq \alpha \right] \right) d\alpha \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\infty p\alpha^{p-1} \frac{1}{\alpha} \int_\Omega \mathcal{X}_{[\sup_{t \in [S, T]} M(t) \geq \alpha]} M(T) dP d\alpha \\
&= p \int_\Omega \int_0^{\sup_{t \in [S, T]} M(t)} \alpha^{p-2} d\alpha M(T) dP \\
&= \frac{p}{p-1} \int_\Omega \left(\sup_{t \in [S, T]} M(t) \right)^{p-1} M(T) dP \\
&\leq \frac{p}{p-1} \left(\int_\Omega \left(\sup_{t \in [S, T]} M(t) \right)^p \right)^{1/p'} \left(\int_\Omega M(T)^p \right)^{1/p} \\
&= \frac{p}{p-1} E \left(\left(\sup_{t \in [S, T]} M(t) \right)^p \right)^{1/p'} E(M(T)^p)^{1/p}.
\end{aligned}$$

Therefore, dividing both sides by $E \left(\left(\sup_{t \in [S, T]} M(t) \right)^p \right)^{1/p'}$ yields

$$E \left(\left(\sup_{t \in [S, T]} M(t) \right)^p \right)^{1/p} \leq \left(\frac{p}{p-1} \right) E(M(T)^p)^{1/p}$$

This proves the theorem.

With Theorem 50.3.4, here is an important maximal estimate for martingales having values in E , a real separable Banach space.

Theorem 50.3.5 *Let $X(t)$ for $t \in I = [0, T]$ be an E valued right continuous martingale with respect to a filtration, $\mathcal{F}_t$. Then for $p \geq 1$,*

$$P \left(\left[\sup_{t \in I} \|X(t)\| \geq \lambda \right] \right) \leq \frac{1}{\lambda^p} E(\|X(T)\|^p). \quad (50.3.13)$$

If $p > 1$,

$$E \left(\left(\sup_{t \in [S, T]} \|X(t)\| \right)^p \right)^{1/p} \leq \left(\frac{p}{p-1} \right) E(\|X(T)\|^p)^{1/p} \quad (50.3.14)$$

Proof: By Proposition 50.3.2 $\|X(t)\|, t \in I$ is a submartingale and so from Theorem 50.3.4, it follows 50.3.13 and 50.3.14 hold. This proves the theorem.

Definition 50.3.6 *Let K be a set of functions of $L^1(\Omega, \mathcal{F}, P)$. Then K is called equi integrable if*

$$\lim_{\lambda \rightarrow \infty} \sup_{f \in K} \int_{\|f\| \geq \lambda} |f| dP = 0.$$

Recall that from Corollary 16.7.6 on Page 520 such an equi integrable set of functions is weakly sequentially precompact in $L^1(\Omega, \mathcal{F}, P)$ in the sense that if $\{f_n\} \subseteq K$, there exists a subsequence, $\{f_{n_k}\}$ and a function, $f \in L^1(\Omega, \mathcal{F}, P)$ such that for all $g \in L^1(\Omega, \mathcal{F}, P)$,

$$g(f_{n_k}) \rightarrow g(f).$$

50.4 Optional Sampling Theorems

50.4.1 Stopping Times And Their Properties

The optional sampling theorem is very useful in the study of martingales and submartingales as will be shown.

First it is necessary to define the notion of a stopping time.

Definition 50.4.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be an increasing sequence of σ algebras each contained in $\mathcal{F}$, called a discrete filtration. A stopping time is a measurable function, τ which maps Ω to $\mathbb{N}$,

$$\tau^{-1}(A) \in \mathcal{F} \text{ for all } A \in \mathcal{P}(\mathbb{N}),$$

such that for all $n \in \mathbb{N}$,

$$[\tau \leq n] \in \mathcal{F}_n.$$

Note this is equivalent to saying

$$[\tau = n] \in \mathcal{F}_n$$

because

$$[\tau = n] = [\tau \leq n] \setminus [\tau \leq n-1].$$

For τ a stopping time define $\mathcal{F}_\tau$ as follows.

$$\mathcal{F}_\tau \equiv \{A \in \mathcal{F} : A \cap [\tau \leq n] \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

These sets in $\mathcal{F}_\tau$ are referred to as “prior” to τ .

Proposition 50.4.2 For τ a stopping time, $\mathcal{F}_\tau$ is a σ algebra and if $Y(k)$ is $\mathcal{F}_k$ measurable for all k , $Y(k)$ having values in a separable Banach space E , then

$$\omega \rightarrow Y(\tau(\omega))$$

is $\mathcal{F}_\tau$ measurable.

Proof: Let $A_n \in \mathcal{F}_\tau$. I need to show $\cup_n A_n \in \mathcal{F}_\tau$. In other words, I need to show that

$$\cup_n A_n \cap [\tau \leq k] \in \mathcal{F}_k$$

The left side equals

$$\cup_n (A_n \cap [\tau \leq k])$$

which is a countable union of sets of $\mathcal{F}_k$ and so $\mathcal{F}_\tau$ is closed with respect to countable unions. Next suppose $A \in \mathcal{F}_\tau$.

$$(A^C \cap [\tau \leq k]) \cup (A \cap [\tau \leq k]) = \Omega \cap [\tau \leq k]$$

and $\Omega \cap [\tau \leq k] \in \mathcal{F}_k$. Therefore, so is $A^C \cap [\tau \leq k]$. It remains to verify the last claim. Let B be an open set in E

$$[Y(\tau) \in B] = \cup_k [\tau = k] \cap [Y(k) \in B]$$

Thus

$$[Y(\tau) \in B] \cap [\tau \leq l] = \cup_k [\tau = k] \cap [Y(k) \in B] \cap [\tau \leq l]$$

Consider a term in the union. If $l \geq k$ the term reduces to $[\tau = k] \cap [Y(k) \in B] \in \mathcal{F}_k$ while if $l < k$, this term reduces to $\emptyset$, also a set of $\mathcal{F}_k$. Therefore, $Y(\tau)$ must be $\mathcal{F}_\tau$ measurable. This proves the proposition.

The following lemma contains the fundamental properties of stopping times for discrete filtrations.

Lemma 50.4.3 *In the situation of Definition 50.4.1, let σ, τ be two stopping times. Then*

1. τ is $\mathcal{F}_\tau$ measurable

2. $\mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\sigma \wedge \tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$

3. $\mathcal{F}_\tau = \mathcal{F}_k$ on $[\tau = k]$ for all k . That is if $A \in \mathcal{F}_k$, then $A \cap [\tau = k] \in \mathcal{F}_\tau$ and if $A \in \mathcal{F}_\tau$, then $A \cap [\tau = k] \in \mathcal{F}_k$. Also if $A \in \mathcal{F}_\tau$, and $Y \in L^1(\Omega; E)$,

$$\int_{A \cap [\tau = k]} E(Y | \mathcal{F}_\tau) dP = \int_{A \cap [\tau = k]} E(Y | \mathcal{F}_k) dP$$

and

$$E(Y | \mathcal{F}_\tau) = E(Y | \mathcal{F}_k) \text{ a.e.}$$

on $[\tau = k]$.

Proof: Consider the first claim. I need to show that $[\tau \leq a] \cap [\tau \leq k] \in \mathcal{F}_k$ for every k . However, this is easy if $a \geq k$ because the left side is then $[\tau \leq k]$ which is given to be in $\mathcal{F}_k$ since τ is a stopping time. If $a < k$, it is also easy because then the left side is $[\tau \leq a] \in \mathcal{F}_{[a]}$ where $[a]$ is the greatest integer less than or equal to a .

Next consider the second claim. Let $A \in \mathcal{F}_\sigma$. I want to show first that

$$A \cap [\sigma \leq \tau] \in \mathcal{F}_\tau \quad (50.4.15)$$

In other words, I want to show

$$A \cap [\sigma \leq \tau] \cap [\tau \leq k] \in \mathcal{F}_k$$

for all k . This will be done if I can show

$$A \cap [\sigma \leq j] \cap [\tau \leq k] \in \mathcal{F}_k$$

for each $j \leq k$ because the two sets on the left are equal. However, since σ is $\mathcal{F}_\sigma$ measurable, it follows $A \cap [\sigma \leq j] \in \mathcal{F}_j \subseteq \mathcal{F}_k$ for each $j \leq k$ and so this has shown what I wanted to show, $A \cap [\sigma \leq \tau] \in \mathcal{F}_\tau$. Now replace the stopping time, τ with the stopping time $\tau \wedge \sigma$ in what was just shown. First note $\tau \wedge \sigma$ is a stopping time because

$$[\tau \wedge \sigma \leq k] = [\tau \leq k] \cup [\sigma \leq k] \in \mathcal{F}_k.$$

Thus from 50.4.15

$$A \cap [\sigma \leq \tau \wedge \sigma] \in \mathcal{F}_{\tau \wedge \sigma}$$

However the left side equals $A \cap [\sigma \leq \tau]$. Thus

$$A \cap [\sigma \leq \tau] \in \mathcal{F}_{\tau \wedge \sigma}$$

This has shown the first part of 2.), $\mathcal{F}_\sigma \cap [\sigma \leq \tau] \subseteq \mathcal{F}_{\tau \wedge \sigma}$.

Now 50.4.15 implies if $A \in \mathcal{F}_{\sigma \wedge \tau}$,

$$A = A \cap \overbrace{[\sigma \wedge \tau \leq \tau]}^{\text{all of } \Omega} \in \mathcal{F}_\tau$$

and so $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\tau$. Similarly, $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma$ which shows

$$\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\tau \cap \mathcal{F}_\sigma.$$

Next let $A \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$. Then is it in $\mathcal{F}_{\sigma \wedge \tau}$? Is $A \cap [\sigma \wedge \tau \leq k] \in \mathcal{F}_k$? Of course this is so because

$$\begin{aligned} A \cap [\sigma \wedge \tau \leq k] &= A \cap ([\sigma \leq k] \cup [\tau \leq k]) \\ &= (A \cap [\sigma \leq k]) \cup (A \cap [\tau \leq k]) \in \mathcal{F}_k \end{aligned}$$

since both σ, τ are stopping times. This proves part 2.).

Now consider part 3.). Let $A \in \mathcal{F}_k$. I need to show $A \cap [\tau = k] \in \mathcal{F}_\tau$. Is

$$A \cap [\tau = k] \cap [\tau \leq l] \in \mathcal{F}_l$$

for all l ? There are two cases. First suppose $l \geq k$. Then the expression on the left reduces to

$$A \cap [\tau = k] = A \cap ([\tau \leq k] \setminus [\tau \leq (k-1)]) \in \mathcal{F}_k.$$

If $l < k$, the left side equals $\emptyset \in \mathcal{F}_l$ and so this shows

$$\mathcal{F}_k \cap [\tau = k] \subseteq \mathcal{F}_\tau.$$

Now consider the other inclusion. Let $A \in \mathcal{F}_\tau$. Then I need to show $A \cap [\tau = k] \in \mathcal{F}_k$. Since $A \in \mathcal{F}_\tau$,

$$A \cap [\tau \leq j] \in \mathcal{F}_j$$

for all j . Therefore,

$$A \cap [\tau = k] = A \cap [\tau \leq k] \cap (A \cap [\tau \leq k-1])^C \in \mathcal{F}_k$$

It only remains to verify the claim about the conditional expectations. Let $A \in \mathcal{F}_\tau$ be arbitrary.

$$\int_{A \cap [\tau=k]} E(Y|\mathcal{F}_\tau) dP \equiv \int_{A \cap [\tau=k]} Y dP \equiv \int_{A \cap [\tau=k]} E(Y|\mathcal{F}_k) dP$$

This is because the set, $A \cap [\tau = k]$ is in both $\mathcal{F}_k$ and $\mathcal{F}_\tau$. Since A is arbitrary, it follows

$$E(Y|\mathcal{F}_\tau) = E(Y|\mathcal{F}_k) \text{ a.e.}$$

on $[\tau = k]$. This proves the third claim and the Lemma.

50.4.2 Doob Optional Sampling Theorem

With this lemma, here is a major theorem, the optional sampling theorem of Doob. This one is for martingales having values in a Banach space. To begin with, consider the case of a martingale defined on a countable set.

Theorem 50.4.4 *Let $\{M(k)\}$ be a martingale having values in E a separable real Banach space with respect to the increasing sequence of σ algebras, $\{\mathcal{F}_k\}$ and let σ, τ be two stopping times such that τ is bounded. Then $M(\tau)$ defined as*

$$\omega \rightarrow M(\tau(\omega))$$

is integrable and

$$M(\sigma \wedge \tau) = E(M(\tau) | \mathcal{F}_\sigma).$$

Proof: By Proposition 50.4.2 $M(\tau)$ is $\mathcal{F}_\tau$ measurable.

Next note that since τ is bounded,

$$\int_{\Omega} \|M(\tau(\omega))\| dP \leq \sum_{i=1}^l \int_{[\tau=i]} \|M(i)\| dP < \infty.$$

This proves the first assertion and makes possible the consideration of conditional expectation.

Let $l \geq \tau$ as described above. Then for $k \leq l$, by Lemma 50.4.3,

$$\mathcal{F}_k \cap [\tau = k] = \mathcal{F}_\tau \cap [\tau = k]$$

and so if $A \in \mathcal{F}_k \cap [\tau = k] = \mathcal{F}_\tau \cap [\tau = k]$,

$$\begin{aligned} \int_A E(M(l) | \mathcal{F}_\tau) dP &\equiv \int_A M(l) dP \\ &= \int_A E(M(l) | \mathcal{F}_k) dP \\ &= \int_A M(k) dP = \int_A M(\tau) dP. \end{aligned}$$

Therefore, since A was arbitrary,

$$E(M(l) | \mathcal{F}_\tau) = M(\tau) \text{ a.e.}$$

on $[\tau = k]$ for every $k \leq l$. It follows

$$E(M(l) | \mathcal{F}_\tau) = M(\tau) \text{ a.e.} \quad (50.4.16)$$

since it is true on each $[\tau = k]$ for all $k \leq l$.

Claim: $[\sigma \leq \tau] \in \mathcal{F}_\tau \cap \mathcal{F}_l \cap \mathcal{F}_\sigma$.

Proof of claim: First I show $[\sigma \leq \tau] \in \mathcal{F}_l$.

$$[\sigma \leq \tau] = \cup_{i=1}^l [\tau = i] \cap [\sigma \leq i] \in \mathcal{F}_l$$

This shows it is in $\mathcal{F}_l$. Next I show $[\sigma \leq \tau] \in \mathcal{F}_\tau$. To do this I have to show

$$[\sigma \leq \tau] \cap [\tau \leq k] \in \mathcal{F}_k$$

for all k . But

$$[\sigma \leq \tau] \cap [\tau \leq k] = \cup_{i=1}^\infty [\tau = i] \cap [\sigma \leq i] \cap [\tau \leq k]$$

and for $i > k$ the terms of the union equal $\emptyset$ while for $i \leq k$, the terms reduce to

$$[\tau = i] \cap [\sigma \leq i] \in \mathcal{F}_i \subseteq \mathcal{F}_k.$$

Thus $[\sigma \leq \tau] \in \mathcal{F}_\tau$. Now consider the claim $[\sigma \leq \tau] \in \mathcal{F}_\sigma$. This occurs because

$$\begin{aligned} [\sigma \leq \tau] &= \cup_{i=1}^\infty [\sigma = i] \cap [i \leq \tau] \\ &= \cup_{i=1}^l [\sigma = i] \cap [i \leq \tau] \end{aligned}$$

An individual term is in $\mathcal{F}_\sigma$ because

$$[\sigma = i] \cap [i \leq \tau] \cap [\sigma \leq k] = [\sigma = i] \cap [i \leq \tau] \in \mathcal{F}_i \subseteq \mathcal{F}_k$$

when $k \geq i$ and for $k < i$, this set,

$$[\sigma = i] \cap [i \leq \tau] \cap [\sigma \leq k] = \emptyset \in \mathcal{F}_k.$$

Thus $[\sigma \leq \tau]$ is the finite union of sets in $\mathcal{F}_\sigma$ so it is also in $\mathcal{F}_\sigma$.

I also claim that if $A \in \mathcal{F}_\sigma$, then

$$A \cap [\sigma \leq \tau] \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau. \quad (50.4.17)$$

Here is why. It is only necessary to show $A \cap [\sigma \leq \tau]$ is in $\mathcal{F}_\tau$ because the above claim verifies $[\sigma \leq \tau] \in \mathcal{F}_\sigma$.

$$A \cap [\sigma \leq \tau] \cap [\tau \leq k] = \cup_{i=1}^k A \cap [\sigma \leq i] \cap [\tau = i]$$

and by assumption $A \cap [\sigma \leq i] \in \mathcal{F}_i \subseteq \mathcal{F}_k$ while $[\tau = i] \in \mathcal{F}_i \subseteq \mathcal{F}_k$. Thus 50.4.17 holds.

Now let $A \in \mathcal{F}_\sigma$. Then using 50.4.16 and Lemma 50.4.3,

$$\begin{aligned}
 \int_{A \cap [\sigma \leq \tau]} E(M(\tau) | \mathcal{F}_\sigma) dP &= \int_{A \cap [\sigma \leq \tau]} E(E(M(l) | \mathcal{F}_\tau) | \mathcal{F}_\sigma) dP \\
 &= \sum_{i=1}^l \int_{A \cap [\sigma \leq i] \cap [\tau = i]} E(E(M(l) | \mathcal{F}_i) | \mathcal{F}_\sigma) dP = \sum_{i=1}^l \int_{A \cap [\sigma \leq i] \cap [\tau = i]} E(M(i) | \mathcal{F}_\sigma) dP \\
 &= \sum_{i=1}^l \sum_{j=1}^i \int_{A \cap [\sigma = j] \cap [\tau = i]} E(M(i) | \mathcal{F}_j) dP = \sum_{i=1}^l \sum_{j=1}^i \int_{A \cap [\sigma = j] \cap [\tau = i]} M(j) dP \\
 &= \sum_{i=1}^l \sum_{j=1}^i \int_{A \cap [\sigma = j] \cap [\tau = i]} M(\sigma) dP = \sum_{i=1}^l \int_{A \cap [\sigma \leq i] \cap [\tau = i]} M(\sigma) dP \\
 &= \int_{A \cap [\sigma \leq \tau]} M(\sigma) dP = \int_{A \cap [\sigma \leq \tau]} M(\sigma \wedge \tau) dP \quad (50.4.18)
 \end{aligned}$$

It follows that on $[\sigma \leq \tau]$,

$$E(M(\tau) | \mathcal{F}_\sigma) = M(\sigma \wedge \tau) \text{ a.e.} \quad (50.4.19)$$

because $M(\sigma \wedge \tau)$ is $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma$ measurable thanks to Proposition 50.4.2.

Next consider the complement of this set, $[\sigma > \tau]$. I want to show that 50.4.18 for $[\sigma > \tau]$ in place of $[\sigma \leq \tau]$. Letting $A \in \mathcal{F}_\sigma$,

$$\begin{aligned}
 \int_{A \cap [\sigma > \tau]} E(M(\tau) | \mathcal{F}_\sigma) dP &= \int_{A \cap [\sigma > \tau]} E(M(\tau \wedge \sigma) | \mathcal{F}_\sigma) dP \\
 &= \int_{A \cap [\sigma > \tau]} M(\tau \wedge \sigma) dP
 \end{aligned}$$

because by Lemma 50.4.3, $M(\tau \wedge \sigma)$ is $\mathcal{F}_{\sigma \wedge \tau}$ measurable and this is a subset of $\mathcal{F}_\sigma$. Thus 50.4.19 holds on $[\sigma > \tau]$ also. It follows 50.4.19 holds a.e. This proves the theorem.

What about submartingales? Recall $\{X(k)\}_{k=1}^\infty$ is a submartingale if

$$E(X(k+1) | \mathcal{F}_k) \geq X(k)$$

where the $\mathcal{F}_k$ are an increasing sequence of σ algebras in the usual way. The following is a very interesting result.

Lemma 50.4.5 *Let $\{X(k)\}_{k=0}^\infty$ be a submartingale adapted to the increasing sequence of σ algebras, $\{\mathcal{F}_k\}$. Then there exists a unique increasing process $\{A(k)\}_{k=0}^\infty$ such that $A(0) = 0$ and $A(k+1)$ is $\mathcal{F}_k$ measurable for all k and a martingale, $\{M(k)\}_{k=0}^\infty$ such that*

$$X(k) = A(k) + M(k).$$

Proof: First consider the uniqueness assertion. Suppose A is a process which does what is supposed to do.

$$\begin{aligned} & \sum_{k=0}^{n-1} E(X(k+1) - X(k) | \mathcal{F}_k) \\ &= \sum_{k=0}^{n-1} E(A(k+1) - A(k) | \mathcal{F}_k) \\ & \quad + \sum_{k=0}^{n-1} E(M(k+1) - M(k) | \mathcal{F}_k) \end{aligned}$$

Then since $\{M(k)\}$ is a martingale,

$$\sum_{k=0}^{n-1} E(X(k+1) - X(k) | \mathcal{F}_k) = \sum_{k=0}^{n-1} A(k+1) - A(k) = A(n)$$

This shows uniqueness and gives a formula for $A(n)$ assuming it exists. It is only a matter of verifying this does work. Define

$$A(n) \equiv \sum_{k=0}^{n-1} E(X(k+1) - X(k) | \mathcal{F}_k).$$

Then A is increasing because from the definition,

$$A(n+1) - A(n) = E(X(n+1) - X(n) | \mathcal{F}_n) \geq 0.$$

Also from the definition above, $A(n)$ is $\mathcal{F}_{n-1}$ measurable. Consider

$$\{X(k) - A(k)\}.$$

Why is this a martingale?

$$\begin{aligned} & E(X(k+1) - A(k+1) | \mathcal{F}_k) \\ &= E(X(k+1) | \mathcal{F}_k) - A(k+1) \\ &= E(X(k+1) | \mathcal{F}_k) - \sum_{j=0}^k E(X(j+1) - X(j) | \mathcal{F}_j) \\ &= E(X(k+1) | \mathcal{F}_k) - E(X(k+1) - X(k) | \mathcal{F}_k) \\ & \quad - \sum_{j=0}^{k-1} E(X(j+1) - X(j) | \mathcal{F}_j) \\ &= X(k) - \sum_{j=0}^{k-1} E(X(j+1) - X(j) | \mathcal{F}_j) = X(k) - A(k) \end{aligned}$$

Letting $M(k) \equiv X(k) - A(k)$, this proves the lemma.

Note the nonnegative integers could be replaced with any finite set or ordered countable set of numbers with no change in the conclusions of this lemma or the above optional sampling theorem.

Next consider the case of a submartingale.

Theorem 50.4.6 *Let $\{X(k)\}$ be a submartingale with respect to the increasing sequence of σ algebras, $\{\mathcal{F}_k\}$ and let σ, τ be two stopping times such that τ is bounded. Then $X(\tau)$ defined as*

$$\omega \rightarrow X(\tau(\omega))$$

is integrable and

$$X(\sigma \wedge \tau) \leq E(X(\tau) | \mathcal{F}_\sigma).$$

Proof: The claim about $X(\tau)$ being integrable is the same as in Theorem 50.4.4. By Lemma 50.4.5 there is a martingale, $\{M(k)\}$ and an increasing process $\{A(k)\}$ such that $A(k+1)$ is $\mathcal{F}_k$ measurable such that

$$X(k) = M(k) + A(k).$$

Then using Theorem 50.4.4 on the martingale and the fact A is increasing

$$\begin{aligned} E(X(\tau) | \mathcal{F}_\sigma) &= E(M(\tau) + A(\tau) | \mathcal{F}_\sigma) = M(\tau \wedge \sigma) + E(A(\tau) | \mathcal{F}_\sigma) \\ &\geq M(\tau \wedge \sigma) + E(A(\tau \wedge \sigma) | \mathcal{F}_\sigma) \\ &= M(\tau \wedge \sigma) + A(\tau \wedge \sigma) = X(\tau \wedge \sigma). \end{aligned}$$

This is so provided that for τ a stopping time, $A(\tau)$ is $\mathcal{F}_\tau$ measurable because in the above, it will follow

$$E(A(\tau \wedge \sigma) | \mathcal{F}_\sigma) = A(\tau \wedge \sigma).$$

I need to verify that for each l ,

$$[A(\tau) \leq l] \cap [\tau \leq k] \in \mathcal{F}_k$$

for each k . However, this is easy to show.

$$[A(\tau) \leq l] = \cup_{i \geq 0} [A(i) \leq l] \cap [\tau = i].$$

Therefore,

$$[A(\tau) \leq l] \cap [\tau \leq k] = \cup_{i \geq 0} [A(i) \leq l] \cap [\tau = i] \cap [\tau \leq k]$$

Consider a term. If $k \geq i$

$$\begin{aligned} &[A(i) \leq l] \cap [\tau = i] \cap [\tau \leq k] \\ &= [A(i) \leq l] \cap [\tau = i] \in \mathcal{F}_i \subseteq \mathcal{F}_k \end{aligned}$$

while if $k < i$ the term equals $\emptyset$ also in $\mathcal{F}_k$. Therefore, $A(\tau)$ is $\mathcal{F}_\tau$ measurable as hoped. This proves the theorem.

50.5 Doob Optional Sampling Continuous Case

50.5.1 Stopping Times

As in the case of discrete martingales, there is something called a stopping time.

Definition 50.5.1 Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_t$ be a filtration. A measurable function, $\tau : \Omega \rightarrow [0, \infty]$ is called a stopping time if

$$[\tau \leq t] \in \mathcal{F}_t$$

for all $t \geq 0$. Associated with a stopping time is the σ algebra, $\mathcal{F}_\tau$ defined by

$$\mathcal{F}_\tau \equiv \{A \in \mathcal{F} : A \cap [\tau \leq t] \in \mathcal{F}_t \text{ for all } t\}.$$

These sets are also called those “prior” to τ .

Proposition 50.5.2 Let B be an open subset of a Banach space, E and let $X(t)$ be a right continuous $\mathcal{F}_t$ adapted stochastic process such that $\mathcal{F}_t$ is normal. Then define

$$\tau(\omega) \equiv \inf \{t > 0 : X(t)(\omega) \in B\}.$$

This is called the first hitting time. Then τ is a stopping time. If $X(t)$ is continuous and adapted to $\mathcal{F}_t$, an arbitrary filtration, then if H is a nonempty closed set,

$$\tau(\omega) \equiv \inf \{t > 0 : X(t)(\omega) \in H\}$$

is also a stopping time.

Proof: Since $X(t)$ is right continuous and B is open,

$$\tau(\omega) = \inf \{t > 0, t \in \mathbb{Q} \cap (0, \infty) : X(t)(\omega) \in B\}$$

I need to verify that $[\tau \leq a] \in \mathcal{F}_a$. First consider $[\tau < a]$. Suppose that $\tau(\omega) < a$. It follows then that for each $\varepsilon > 0$, there exists

$$t \in \mathbb{Q} \cap (0, \infty) \cap [0, a) \equiv \mathbb{Q}^+ \cap [0, a)$$

such that

$$X(t)(\omega) \in B$$

That is,

$$[\tau < a] \equiv \tau^{-1}([0, a)) \subseteq \cup_{t \in \mathbb{Q}^+ \cap [0, a)} X(t)^{-1}(B) \in \mathcal{F}_a.$$

Also if $\omega \in \cup_{t \in \mathbb{Q}^+ \cap [0, a)} X(t)^{-1}(B)$ then for some $t \in \mathbb{Q}^+ \cap [0, a)$, $X(t)(\omega) \in B$. Thus $\tau(\omega) \leq t < a$ and so

$$[\tau < a] = \cup_{t \in \mathbb{Q}^+ \cap [0, a)} X(t)^{-1}(B)$$

which shows τ is measurable. Also this shows $[\tau < a] \in \mathcal{F}_a$.

It remains to verify $\tau^{-1}([0, a]) \equiv [\tau \leq a] \in \mathcal{F}_a$. But $\tau^{-1}([0, a]) = \cap_{k=1}^{\infty} \tau^{-1}([0, a + \frac{1}{k}])$ and it was just shown $\tau^{-1}([0, a + \frac{1}{k}]) \in \mathcal{F}_{a+1/k}$ and so $\tau^{-1}([0, a]) \in \mathcal{F}_a$ because $\tau^{-1}([0, a]) \in \mathcal{F}_{a+1/k}$ for all $k \in \mathbb{N}$ and the intersection of these equals $\cap_{s>t} \mathcal{F}_s \equiv \mathcal{F}_{t+} = \mathcal{F}_t$.

Now consider the case of a closed set where X is continuous and you are hitting a closed set, H . Let

$$H = \cap_{n=1}^{\infty} U_n$$

where $\{U_n\}$ is a decreasing sequence of open sets,

$$\text{dist}(H, U_n^C) = \frac{1}{n}.$$

Then $\tau(\omega) \leq t < \infty$ means for all n there exists $r_n \in [0, t]$ such that $X(r_n)(\omega) \in U_n$. This is because if this condition holds then taking a subsequence still denoted by r_n it can be assumed to converge to $s \leq t$ and $X(s)(\omega) \in H$. Conversely, if $X(s)(\omega) \in H$ for some $s \leq t$, then $X(s)(\omega) \in U_n$ for all n and so you could let $r_n \equiv s$. By continuity of X you can replace $[0, t]$ with $[0, t] \cap \mathbb{Q}$. Thus

$$[\tau \leq t] = \cap_{n=1}^{\infty} \cup_{r \in \mathbb{Q} \cap [0, t]} [X(r) \in U_n] \in \mathcal{F}_t$$

Now consider the case where $\tau(\omega) = \infty$. This is the case where $X(t)(\omega)$ is never in H for any t . In this case, there are infinitely many of the U_n such that for all $r \in \mathbb{Q} \cap [0, t]$, $X(r)(\omega)$ fails to be in any of these U_n . Since they are decreasing, this set is of the form

$$[\tau = \infty] = \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} \cap_{r \in \mathbb{Q} \cap [0, t]} [X(r) \in U_n^C] \in \mathcal{F}_t$$

Now note that if $t \in [0, T]$ where $T \leq \infty$,

$$[\tau < T] = \cup_{n=1}^{\infty} [\tau < n]$$

and this completes showing this type of hitting time is a stopping time. This proves the proposition.

Thus there do exist stopping times, the first hitting time being an example. The following is a fundamental observation.

Proposition 50.5.3 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\sigma \leq \tau$ be two stopping times with respect to a filtration, $\mathcal{F}_t$. Then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$. If $X(t)$ is a right continuous stochastic process adapted to $\mathcal{F}_t$, a normal filtration, and τ is a stopping time, $\omega \rightarrow X(\tau(\omega))$ is $\mathcal{F}_\tau$ measurable.*

Proof: Let $A \in \mathcal{F}_\sigma$. Then $A \cap [\sigma \leq t] \in \mathcal{F}_t$ for all $t \geq 0$. I claim

$$A \cap [\tau \leq t] = (\cap_{s \leq t, s \in \mathbb{Q} \cup \{t\}} A \cap [\sigma \leq s]) \cap [\tau \leq t]$$

Suppose $\omega \in A \cap [\tau \leq t]$. First suppose $\tau(\omega) = t$. Then $\omega \in [\tau \leq t]$ and $\omega \in A \cap [\sigma \leq t]$ and so ω is in the right side of the above expression. If $\tau(\omega) < t$, then

for some $s \in \mathbb{Q} \cap \{t\}$, $\tau(\omega) < s$ and so $\omega \in [\sigma \leq s] \cap A$. Thus the left side is contained in the right side. Next suppose ω is in the right side. Then it is obviously in the left. This proves the first part because $A \cap [\tau \leq t] \in \mathcal{F}_t$ for all $t \geq 0$ and so $A \in \mathcal{F}_\tau$.

Next consider $X(\tau)$. Let U be an open set. Since X is right continuous,

$$X(\tau)^{-1}(U) \equiv \cup_{[\tau=s, s \geq 0]} X(s)^{-1}(U) = \cup_{[\tau=s, s \in \mathbb{Q}^+]} X(s)^{-1}(U) \in \mathcal{F} \quad (50.5.20)$$

It remains to verify $X(\tau)$ is $\mathcal{F}_\tau$ measurable. To do this, I need to show that $X(\tau)^{-1}(U) \in \mathcal{F}_\tau$ for all U open. That is,

$$X(\tau)^{-1}(U) \cap [\tau \leq t] \in \mathcal{F}_t$$

for all U open. However,

$$X(\tau)^{-1}(U) \equiv \cup_{[\tau=s, s \geq 0]} X(s)^{-1}(U) = \cup_{[\tau=s, s \in \mathbb{Q}^+ \cup \{t\}]} X(s)^{-1}(U)$$

and so letting $[\tau = s, s \in \mathbb{Q}^+ \cup \{t\}, s \leq t]$ equal

$$\{\omega : \tau(\omega) = s \text{ for some } s \in \mathbb{Q}^+ \cap [0, t] \cup \{t\}\},$$

$$X(\tau)^{-1}(U) \cap [\tau \leq t] = \cup_{[\tau=s, s \in \mathbb{Q}^+ \cup \{t\}, s \leq t]} X(s)^{-1}(U) \in \mathcal{F}_t.$$

This proves the proposition.

Now consider an increasing family of stopping times, $\tau(t)$ ($\omega \rightarrow \tau(t)(\omega)$). It turns out this is a submartingale.

Example 50.5.4 Let $\{\tau(t)\}$ be an increasing family of stopping times. Then $\tau(t)$ is adapted to the σ algebras $\mathcal{F}_{\tau(t)}$ and $\{\tau(t)\}$ is a submartingale adapted to these σ algebras.

First I need to show that a stopping time, τ is $\mathcal{F}_\tau$ measurable. Consider $[\tau \leq s]$. Is this in $\mathcal{F}_\tau$? Is $[\tau \leq s] \cap [\tau \leq r] \in \mathcal{F}_r$ for each r ? This is obviously so if $s \leq r$ because the intersection reduces to $[\tau \leq s] \in \mathcal{F}_s \subseteq \mathcal{F}_r$. On the other hand, if $s > r$ then the intersection reduces to $[\tau \leq r] \in \mathcal{F}_r$ and so it is clear that τ is $\mathcal{F}_\tau$ measurable. It remains to verify it is a submartingale.

Let $s < t$ and let $A \in \mathcal{F}_{\tau(s)}$

$$\int_A E(\tau(t) | \mathcal{F}_{\tau(s)}) dP \equiv \int_A \tau(t) dP \geq \int_A \tau(s) dP$$

and this shows $E(\tau(t) | \mathcal{F}_{\tau(s)}) \geq \tau(s)$.

Now here is an important example.

Example 50.5.5 Let τ be a stopping time and let X be continuous and adapted to the filtration $\mathcal{F}_t$. Then for $a > 0$, define σ as

$$\sigma(\omega) \equiv \inf \{t > \tau(\omega) : \|X(t)(\omega) - X(\tau(\omega))\| = a\}$$

Then σ is also a stopping time.

To see this is so, let

$$Y(t)(\omega) = \begin{cases} \|X(t)(\omega) - X(\tau(\omega))\| & \text{if } t \geq \tau(\omega) \\ 0 & \text{if } t < \tau(\omega) \end{cases}$$

Then Y is continuous and adapted and

$$\begin{aligned} \sigma(\omega) &= \inf \{t > \tau(\omega) : \|X(t)(\omega) - X(\tau(\omega))\| = a\} \\ &= \inf \{t > 0 : Y(t)(\omega) = a\} \end{aligned}$$

By Proposition 50.5.2 σ is a stopping time.

50.5.2 The Optional Sampling Theorem Continuous Case

Next I want a version of the Doob optional sampling theorem which applies to martingales defined on $[0, L]$, $L \leq \infty$. First here is a little lemma.

Lemma 50.5.6 *Let $f \in L^1(\Omega; E, \mathcal{F})$ where E is a separable Banach space. Then if $\mathcal{G}$ is a σ algebra $\mathcal{G} \subseteq \mathcal{F}$,*

$$\|E(f|\mathcal{G})\| \leq E(\|f\||\mathcal{G}).$$

Proof: First suppose f is a simple function,

$$f(\omega) = \sum_{k=1}^n a_k \mathcal{X}_{F_k}(\omega)$$

Letting $A \in \mathcal{G}$,

$$\begin{aligned} \int_A \|E(f|\mathcal{G})\| dP &= \int_A \left\| E\left(\sum_{k=1}^n a_k \mathcal{X}_{F_k}(\omega) | \mathcal{G}\right) \right\| dP \\ &= \int_A \left\| \sum_k a_k E(\mathcal{X}_{F_k} | \mathcal{G}) \right\| dP \leq \int_A \sum_k \|a_k\| E(\mathcal{X}_{F_k} | \mathcal{G}) dP \\ &= \int_A E\left(\sum_k \|a_k\| \mathcal{X}_{F_k} | \mathcal{G}\right) dP = \int_A E(\|f\| | \mathcal{G}) dP \end{aligned}$$

and since A is arbitrary, the lemma is proved when f is simple. The general case follows from this and an approximating sequence of simple functions. This proves the lemma.

Here is a technical lemma.

Lemma 50.5.7 *Let $X(t)$ be a right continuous adapted process having values in a separable Banach space E such that the filtration $\{\mathcal{F}_t\}$ is normal. Recall this includes*

$$\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s.$$

Let $\{\sigma_n\}$ be a sequence of stopping times whose values are $\{kT2^{-n}\}_{k=0}^\infty$ instead of the positive integers and suppose σ is a measurable function for which

$$T2^{-n} \geq \sigma_n - \sigma \geq 0 \quad (50.5.21)$$

and $X(\sigma_n)$ is $\mathcal{F}_{\sigma_n}$ measurable. Then σ is a stopping time and so $X(\sigma)$ given by $\omega \rightarrow X(\sigma(\omega))$ is $\mathcal{F}_\sigma$ measurable.

Proof: I need to verify

$$[\sigma \leq t] \in \mathcal{F}_t.$$

Note that for each p

$$[\sigma \leq t] = \cap_{n=p}^\infty [\sigma_n \leq k_t^n T2^{-n}] \quad (50.5.22)$$

where $k_t^n T2^{-n}$ is the first value of σ_n at least as large as t . Here is why. If $\omega \in [\sigma \leq t]$, then for each $n \geq p$, the assumed inequality in 50.5.21 implies $\sigma_n(\omega)$ cannot be any larger than $k_t^n T2^{-n}$ since if it were, it would have to be at least as large as $(k_t^n + 1)T2^{-n}$ and the inequality would be violated. Hence the left side of 50.5.22 is contained in the right. If ω is in the right side, then for all n ,

$$T2^{-n} \geq k_t^n T2^{-n} - \sigma(\omega) \geq 0$$

Since $\{k_t^n T2^{-n}\}$ is a decreasing sequence converging to t , passing to a limit implies

$$t - \sigma(\omega) \geq 0$$

and this shows the two sides are equal.

The sequence $\{\mathcal{F}_{k_t^n T2^{-n}}\}$ is decreasing in n because $\{k_t^n T2^{-n}\}$ is a decreasing sequence. Therefore, for each p

$$[\sigma \leq t] = \cap_{n=p}^\infty [\sigma_n \leq k_t^n T2^{-n}] \in \mathcal{F}_{k_t^p T2^{-p}}$$

Consequently

$$[\sigma \leq t] \in \cap_p \mathcal{F}_{k_t^p T2^{-p}} = \mathcal{F}_t$$

because the filtration is normal. It follows from Proposition 50.5.3 that $X(\sigma)$ is $\mathcal{F}_\sigma$ measurable. This proves the lemma.

Theorem 50.5.8 Let $\{M(t)\}$ be a right continuous martingale having values in E a separable real Banach space with respect to the increasing sequence of σ algebras, $\{\mathcal{F}_t\}$ which is assumed to be a normal filtration satisfying,

$$\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s,$$

for $t \in [0, L]$, $L \leq \infty$ and let σ, τ be two stopping times with τ bounded. Then $M(\tau)$ defined as

$$\omega \rightarrow M(\tau(\omega))$$

is integrable and

$$M(\sigma \wedge \tau) = E(M(\tau) | \mathcal{F}_\sigma).$$

Proof: To begin with I need to show the random variable $M(\tau)$ is integrable given that τ is a stopping time having values in $[0, T]$, T being the bound for τ . Otherwise nothing makes sense. To do this, let

$$\tau_n(\omega) \equiv \sum_{k \geq 0} 2^{-n} (k+1) T \mathcal{X}_{\tau^{-1}((k2^{-n}T, (k+1)T2^{-n})]}(\omega).$$

Then τ_n is a stopping time for the filtration $\mathcal{F}_t$ because

$$[\tau_n \leq t] = \tau^{-1}((0, (k+1)T2^{-n}]) \in \mathcal{F}_{(k+1)T2^{-n}}$$

where $(k+1)T2^{-n} \leq t$. Also τ_n takes values at equally spaced points from 0 up to T and for each ω ,

$$|\tau_n(\omega) - \tau(\omega)| < 2^{-n}T$$

for all $\omega \in \Omega$ while $\tau_n(\omega) \geq \tau(\omega)$. Consider $\{M(kT2^{-n})\}_{k=0}^{2^n}$. It follows easily that this is a martingale on $\{kT2^{-n}\}_{k=0}^{2^n}$, the σ algebras being $\mathcal{F}_{kT2^{-n}}$. Also τ_n is a stopping time on $\{kT2^{-n}\}_{k=0}^{2^n}$ with respect to these σ algebras since

$$[\tau_n \leq (k+1)T2^{-n}] = [\tau \leq (k+1)T2^{-n}] \in \mathcal{F}_{(k+1)T2^{-n}}$$

This is because from the definition of τ_n , $\tau_n = (k+1)T2^{-n}$ corresponds to

$$\tau \in (k2^{-n}T, (k+1)T2^{-n}]$$

and so forth. The function $M(\tau_n)$ is measurable because for B an open set,

$$\begin{aligned} M(\tau_n)^{-1}(B) &= \cup_{j=0}^{2^n} M(jT2^{-n})^{-1}(B) \cap [\tau_n = jT2^{-n}] \\ &= \cup_{j=0}^{2^n} M(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \in \mathcal{F}_{jT2^{-n}} \end{aligned}$$

In addition,

$$\begin{aligned} &M(\tau_n)^{-1}(B) \cap [\tau_n \leq kT2^{-n}] \\ &= \cup_{j=0}^{2^n} M(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \cap [\tau_n \leq kT2^{-n}] \\ &= \cup_{j=0}^{2^n} M(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \cap [\tau \leq kT2^{-n}] \end{aligned}$$

which is in $\mathcal{F}_{kT2^{-n}}$. This follows from looking at the individual terms. If $k \geq j$ the term reduces to

$$M(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \in \mathcal{F}_{jT2^{-n}} \subseteq \mathcal{F}_{kT2^{-n}}$$

while if $k < j$, the term reduces to $\emptyset$ which is still in $\mathcal{F}_{kT2^{-n}}$. Thus $M(\tau_n)$ is $\mathcal{F}_{\tau_n}$ measurable.

Now consider the random variables $M(\tau_n)$. Also let $A \in \mathcal{F}_T$. From Lemma 50.5.6,

$$\int_A \|M(\tau_n)\| dP = \sum_{k=0}^{2^n} \int_{[\tau_n = kT2^{-n}] \cap A} \|M(kT2^{-n})\| dP$$

$$\begin{aligned}
&= \sum_{k=0}^{2^n} \int_{[\tau_n=kT2^{-n}] \cap A} \|E(M(T) | \mathcal{F}_{kT2^{-n}})\| dP \\
&\leq \sum_{k=0}^{2^n} \int_{[\tau_n=kT2^{-n}] \cap A} E(\|M(T)\| | \mathcal{F}_{kT2^{-n}}) dP \\
&= \sum_{k=0}^{2^n} \int_{[\tau_n=kT2^{-n}] \cap A} \|M(T)\| dP = \int_A \|M(T)\| dP \quad (50.5.23)
\end{aligned}$$

Letting $A = \Omega$, this shows the random variables $M(\tau_n)$ are at least in $L^1(\Omega; E)$. Since M is right continuous,

$$\|M(\tau)\| = \lim_{n \rightarrow \infty} \|M(\tau_n)\|$$

and so by Fatou's lemma,

$$\int_{\Omega} \|M(\tau)\| dP \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \|M(\tau_n)\| dP \leq E(\|M(T)\|) < \infty.$$

This shows the random variable, $M(\tau)$ is at least in $L^1(\Omega; E)$. However, more importantly, 50.5.23 also shows the random variables $\{M(\tau_n)\}_{n=1}^{\infty}$ are uniformly integrable.

Consider the main claim now. Letting σ, τ be stopping times with τ bounded, it follows that for σ_n and τ_n as above, it follows from Theorem 50.4.4

$$M(\sigma_n \wedge \tau_n) = E(M(\tau_n) | \mathcal{F}_{\sigma_n})$$

Thus, taking $A \in \mathcal{F}_{\sigma}$ and recalling $\sigma \leq \sigma_n$ so that by Proposition 50.5.3, $\mathcal{F}_{\sigma} \subseteq \mathcal{F}_{\sigma_n}$,

$$\int_A M(\sigma_n \wedge \tau_n) dP = \int_A E(M(\tau_n) | \mathcal{F}_{\sigma_n}) dP = \int_A M(\tau_n) dP.$$

Now passing to a limit as $n \rightarrow \infty$, the Vitali convergence theorem, Theorem 8.5.3 on Page 213 and the right continuity of M implies one can pass to the limit in the above and conclude

$$\int_A M(\sigma \wedge \tau) dP = \int_A M(\tau) dP.$$

By Proposition 50.5.3, $M(\sigma \wedge \tau)$ is $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_{\sigma}$ measurable showing

$$E(M(\tau) | \mathcal{F}_{\sigma}) = M(\sigma \wedge \tau).$$

This proves the theorem.

A similar theorem is available for submartingales defined on $[0, L]$, $L \leq \infty$. The hard part is showing the uniform integrability of the approximate processes.

Theorem 50.5.9 *Let $\{X(t)\}$ be a right continuous submartingale with respect to the increasing sequence of σ algebras, $\{\mathcal{F}_t\}$ which is assumed to be a normal filtration,*

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s,$$

for $t \in [0, L]$, $L \leq \infty$ and let σ, τ be two stopping times with τ bounded. Then $X(\tau)$ defined as

$$\omega \rightarrow X(\tau(\omega))$$

is integrable and

$$X(\sigma \wedge \tau) \leq E(X(\tau) | \mathcal{F}_\sigma).$$

Proof: To begin with I need to show the random variable $X(\tau)$ is integrable given that τ is a stopping time having values in $[0, T]$, T being the bound for τ . Otherwise nothing makes sense. To do this, let

$$\tau_n(\omega) \equiv \sum_{k \geq 0} 2^{-n} (k+1) T \mathcal{X}_{\tau^{-1}((k2^{-n}T, (k+1)T2^{-n}))}(\omega).$$

Then τ_n is a stopping time for the filtration $\mathcal{F}_t$ because

$$[\tau_n \leq t] = \tau^{-1}((0, (k+1)T2^{-n}]) \in \mathcal{F}_{(k+1)T2^{-n}}$$

where $(k+1)T2^{-n} \leq t$. Also τ_n takes values at equally spaced points from 0 up to T and for each ω ,

$$|\tau_n(\omega) - \tau(\omega)| < 2^{-n}T$$

for all $\omega \in \Omega$ while $\tau_n(\omega) \geq \tau(\omega)$. Consider $\{X(kT2^{-n})\}_{k=0}^{2^n}$. It follows easily that this is a submartingale on $\{kT2^{-n}\}_{k=0}^{2^n}$, the σ algebras being $\mathcal{F}_{kT2^{-n}}$. Also τ_n is a stopping time on $\{kT2^{-n}\}_{k=0}^{2^n}$ with respect to these σ algebras since

$$[\tau_n \leq (k+1)T2^{-n}] = [\tau \leq (k+1)T2^{-n}] \in \mathcal{F}_{(k+1)T2^{-n}}$$

This is because from the definition of τ_n , $\tau_n = (k+1)T2^{-n}$ corresponds to

$$\tau \in (k2^{-n}T, (k+1)T2^{-n})$$

and so forth. The function $X(\tau_n)$ is measurable because for B an open set,

$$\begin{aligned} X(\tau_n)^{-1}(B) &= \cup_{j=0}^{2^n} X(jT2^{-n})^{-1}(B) \cap [\tau_n = jT2^{-n}] \\ &= \cup_{j=0}^{2^n} X(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \in \mathcal{F}_{jT2^{-n}} \end{aligned}$$

In addition,

$$\begin{aligned} &X(\tau_n)^{-1}(B) \cap [\tau_n \leq kT2^{-n}] \\ &= \cup_{j=0}^{2^n} X(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \cap [\tau_n \leq kT2^{-n}] \\ &= \cup_{j=0}^{2^n} X(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \cap [\tau \leq kT2^{-n}] \end{aligned}$$

which is in $\mathcal{F}_{kT2^{-n}}$. This follows from looking at the individual terms. If $k \geq j$ the term reduces to

$$X(jT2^{-n})^{-1}(B) \cap [\tau \in ((j-1)2^{-n}T, jT2^{-n}]] \in \mathcal{F}_{jT2^{-n}} \subseteq \mathcal{F}_{kT2^{-n}}$$

while if $k \leq j-1$, the term reduces to $\emptyset$ which is still in $\mathcal{F}_{kT2^{-n}}$. Thus $X(\tau_n)$ is $\mathcal{F}_{\tau_n}$ measurable.

Now consider the random variables $X(\tau_n)$. Each is in $L^1(\Omega)$ thanks to the assumption that τ is bounded.

$$\int_{\Omega} |X(\tau_n)| dP = \sum_{k=0}^{2^n} \int_{[\tau_n=kT2^{-n}]} |X(kT2^{-n})| dP < \infty. \quad (50.5.24)$$

The challenge is to show $\{X(\tau_n)\}$ are uniformly integrable. The idea is that $X(\tau)$ is the limit of these due to the right continuity of X . Then the uniform integrability can be used to transfer integrability of these random variables to $X(\tau)$.

By the discrete optional sampling theorem, Theorem 50.4.6,

$$E(X(\tau_n) | \mathcal{F}_{\tau_{n+1}}) \geq X(\tau_{n+1})$$

and so

$$E(X(\tau_n)) = E(E(X(\tau_n) | \mathcal{F}_{\tau_{n+1}})) \geq E(X(\tau_{n+1}))$$

Also since $\tau_n \geq 0$, it also follows from the same reasoning and the discrete optional sampling theorem,

$$E(X(\tau_n)) \geq E(X(0)).$$

Now let $\varepsilon > 0$ be given. Since $\{E(X(\tau_n))\}$ is decreasing and bounded below, there exists fixed k large enough, depending on ε such that

$$\left| E(X(\tau_k)) - \lim_{m \rightarrow \infty} E(X(\tau_m)) \right| < \varepsilon. \quad (50.5.25)$$

In what follows $n > k$. Let $\lambda > 0$. Then

$$\begin{aligned} \int_{[|X(\tau_n)| \geq \lambda]} |X(\tau_n)| dP &= \int_{[X(\tau_n) \geq \lambda]} X(\tau_n) dP + \int_{[X(\tau_n) \leq -\lambda]} (-X(\tau_n)) dP \\ &= \int_{[X(\tau_n) \geq \lambda]} X(\tau_n) dP + \int_{[-X(\tau_n) \geq \lambda]} (-X(\tau_n)) dP \\ &= \int_{[X(\tau_n) \geq \lambda]} X(\tau_n) dP + \int_{\Omega} (-X(\tau_n)) dP + \int_{[-X(\tau_n) < \lambda]} X(\tau_n) dP \end{aligned}$$

Since the τ_n are decreasing, it follows from the discrete optional sampling theorem

$$\begin{aligned} &\leq \int_{[X(\tau_n) \geq \lambda]} E(X(\tau_k) | \mathcal{F}_{\tau_n}) dP + \int_{\Omega} (-X(\tau_n)) dP \\ &\quad + \int_{[-X(\tau_n) < \lambda]} E(X(\tau_k) | \mathcal{F}_{\tau_n}) dP \end{aligned}$$

Here is where 50.5.25 is useful. The middle term can be replaced and the inequality can be strengthened as follows.

$$\begin{aligned} &\leq \int_{[X(\tau_n) \geq \lambda]} E(X(\tau_k) | \mathcal{F}_{\tau_n}) dP + \int_{\Omega} (-X(\tau_k)) dP \\ &\quad + \varepsilon + \int_{[-X(\tau_n) < \lambda]} E(X(\tau_k) | \mathcal{F}_{\tau_n}) dP \end{aligned}$$

$$\begin{aligned}
&\leq \int_{[X(\tau_n) \geq \lambda]} X(\tau_k) dP + \int_{\Omega} (-X(\tau_k)) dP \\
&\quad + \varepsilon - \int_{[-X(\tau_n) < \lambda]} (-X(\tau_k)) dP \\
&= \int_{[X(\tau_n) \geq \lambda]} X(\tau_k) dP + \int_{[X(\tau_n) \leq -\lambda]} (-X(\tau_k)) dP + \varepsilon \\
&= \int_{[|X(\tau_n)| \geq \lambda]} |X(\tau_k)| dP + \varepsilon.
\end{aligned} \tag{50.5.26}$$

Now apply the maximal inequality, Theorem 48.4.5,

$$P(|\sup X(\tau_n)| \geq \lambda) \leq \frac{1}{\lambda} (E(|X(0)|) + E(|X(T)|)) \leq \frac{C}{\lambda}$$

In this theorem and by the discrete optional sampling theorem,

$$X(0), X(\tau_n), X(T)$$

is a submartingale for the σ algebras $\mathcal{F}_0, \mathcal{F}_{\tau_n}, \mathcal{F}_T$. Note that the right side does not depend on $n > k$. Therefore,

$$P\left(\left[\sup_{n > k} |X(\tau_n)| \geq \lambda\right]\right) \leq \frac{C}{\lambda}$$

The single random variable $X(\tau_k)$ is in $L^1(\Omega)$ by 50.5.24 and so 50.5.26 is less than 2ε provided λ is large enough for all $n > k$. Also there are only finitely many $X(\tau_m)$ for $m \leq k$, each of them in $L^1(\Omega)$. Therefore, if $\varepsilon > 0$ is given, there exists λ large enough that for all $n > 0$,

$$\int_{[|X(\tau_n)| \geq \lambda]} |X(\tau_n)| dP < \varepsilon.$$

Thus the functions $\{X(\tau_n)\}_{n=1}^{\infty}$ are equiintegrable and so they are uniformly integrable. It follows from this that it is also the case that for any σ a stopping time, bounded or not, $\{X(\sigma_n \wedge \tau_n)\}_{n=1}^{\infty}$ are uniformly integrable. You just repeat the argument with $\sigma \wedge \tau$ in place of τ .

Consider the main claim now. Letting σ, τ be stopping times with τ bounded, it follows that for σ_n and τ_n as above and the optional sampling theorem for discrete submartingales,

$$X(\sigma_n \wedge \tau_n) \leq E(X(\tau_n) | \mathcal{F}_{\sigma_n})$$

Thus, taking $A \in \mathcal{F}_{\sigma}$ and recalling $\sigma \leq \sigma_n$ so that $\mathcal{F}_{\sigma} \subseteq \mathcal{F}_{\sigma_n}$, both σ and σ_n being considered as stopping times for $\mathcal{F}_t$,

$$\int_A X(\sigma_n \wedge \tau_n) dP \leq \int_A E(X(\tau_n) | \mathcal{F}_{\sigma_n}) dP \equiv \int_A X(\tau_n) dP.$$

Now passing to a limit as $n \rightarrow \infty$, the Vitali convergence theorem, Theorem 8.5.3 on Page 213 and the right continuity of X implies one can pass to the limit in the above and conclude

$$\int_A X(\sigma \wedge \tau) dP \leq \int_A X(\tau) dP = \int_A E(X(\tau) | \mathcal{F}_\sigma) dP$$

By Proposition 50.5.3, $X(\sigma \wedge \tau)$ is $\mathcal{F}_{\sigma \wedge \tau} \subseteq \mathcal{F}_\sigma$ measurable which shows that since $A \in \mathcal{F}_\sigma$ was arbitrary,

$$E(X(\tau) | \mathcal{F}_\sigma) \geq X(\sigma \wedge \tau).$$

This proves the theorem.

Note that a function defined on a countable ordered set such as the integers or equally spaced points is right continuous.

Here is a useful lemma which was essentially proved in the above. However, it is interesting and so I will list it again and also give a proof.

Lemma 50.5.10 *Suppose $E(|X_n|) < \infty$ for all n , X_n is $\mathcal{F}_n$ measurable, $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$ for all $n \in \mathbb{N}$, and there exist X_∞ $\mathcal{F}_\infty$ measurable such that $\mathcal{F}_\infty \subseteq \mathcal{F}_n$ for all n and X_0 $\mathcal{F}_0$ measurable such that $\mathcal{F}_0 \supseteq \mathcal{F}_n$ for all n such that for all $n \in \{0, 1, \dots\}$,*

$$E(X_n | \mathcal{F}_{n+1}) \geq X_{n+1}, \quad E(X_n | \mathcal{F}_\infty) \geq X_\infty.$$

Then $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable.

Proof:

$$E(X_{n+1}) \leq E(E(X_n | \mathcal{F}_{n+1})) = E(X_n)$$

Therefore, the sequence $E(X_n)$ is a decreasing sequence bounded below by $E(X_\infty)$ so it has a limit. Let k be large enough that

$$\left| E(X_k) - \lim_{m \rightarrow \infty} E(X_m) \right| < \varepsilon \quad (50.5.27)$$

and suppose $n > k$. Then if $\lambda > 0$,

$$\begin{aligned} \int_{[|X_n| \geq \lambda]} |X_n| dP &= \int_{[X_n \geq \lambda]} X_n dP + \int_{[X_n \leq -\lambda]} (-X_n) dP \\ &= \int_{[X_n \geq \lambda]} X_n dP + \int_{\Omega} (-X_n) dP - \int_{[-X_n < \lambda]} (-X_n) dP \\ &= \int_{[X_n \geq \lambda]} X_n dP - \int_{\Omega} X_n dP + \int_{[-X_n < \lambda]} X_n dP \end{aligned}$$

From 50.5.27,

$$\leq \int_{[X_n \geq \lambda]} X_n dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[-X_n < \lambda]} X_n dP$$

By assumption,

$$E(X_k | \mathcal{F}_n) \geq X_n$$

and so

$$\begin{aligned} &\leq \int_{[X_n \geq \lambda]} E(X_k | \mathcal{F}_n) dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[-X_n < \lambda]} E(X_k | \mathcal{F}_n) dP \\ &= \int_{[X_n \geq \lambda]} X_k dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[-X_n < \lambda]} X_k dP \\ &= \int_{[X_n \geq \lambda]} X_k dP - \int_{\Omega} X_k dP + \varepsilon + \int_{[X_n > -\lambda]} X_k dP \\ &= \int_{[X_n \geq \lambda]} X_k dP + \left(\int_{\Omega} (-X_k) dP - \int_{[X_n > -\lambda]} (-X_k) dP \right) + \varepsilon \\ &= \int_{[X_n \geq \lambda]} X_k dP + \int_{[X_n \leq -\lambda]} (-X_k) dP + \varepsilon = \int_{[|X_n| \geq \lambda]} |X_k| dP + \varepsilon \end{aligned}$$

Applying the maximal inequality for submartingales, Theorem 48.4.5,

$$P(\max\{|X_j| : j = n, \dots, 1\} \geq \lambda) \leq \frac{1}{\lambda} (E(|X_0|) + E(|X_{\infty}|)) \leq \frac{C}{\lambda}$$

and taking sup for all n ,

$$P(\sup\{|X_j|\} \geq \lambda) \leq \frac{C}{\lambda}$$

It follows that for all λ large enough,

$$\int_{[|X_n| \geq \lambda]} |X_n| dP \leq 2\varepsilon$$

and since ε is arbitrary, this shows $\{X_n\}$ for $n > k$ is equiintegrable. Since there are only finitely many X_j for $j \leq k$, this shows $\{X_n\}$ is equiintegrable. Hence $\{X_n\}$ is uniformly integrable. This proves the lemma.

50.6 Right Continuity Of Submartingales

The following theorem is an attempt to consider the question of right continuity. It turns out that you can always assume right continuity of a submartingale by going to a suitable version and this theorem is a first step in this direction.

Theorem 50.6.1 *Let $\{X(t)\}$ be a real valued submartingale adapted to the filtration $\mathcal{F}_t$. Then there exists a set of measure zero, N , $P(N) = 0$, such that if $\omega \notin N$ then,*

$$\lim_{r \rightarrow t+, r \in \mathbb{Q}} X(r, \omega), \quad \lim_{r \rightarrow t-, r \in \mathbb{Q}} X(r, \omega)$$

both exist. There also exists a set of measure zero N such that for $\mathbb{Q}^+$ the nonnegative rationals and $\omega \notin N$,

$$\sup_{t \in \mathbb{Q}^+ \cap [0, M]} |X(t, \omega)| < \infty$$

is bounded for each $M \in \mathbb{N}$. $\mathbb{Q}$ can be replaced with any countable dense subset of $\mathbb{R}$.

Proof: Let $\{r_k\}_{k=1}^\infty$ be an enumeration of the nonnegative rationals. Let $t > 0$ be given. Then let $\{s_1, s_2, \dots, s_n\}$ be such that these are in order and $\{s_2, \dots, s_{n-1}\}$ are the first $n - 2$ rationals less than t listed in order and $s_1 = 0$ while $s_n = t$. Then let $Y_k \equiv X(s_k)$. It follows $\{Y_k\}$ is a submartingale and so from the maximal inequality in Theorem 48.4.5,

$$\begin{aligned} P\left(\left[\max_{1 \leq k \leq n} |Y_k| \geq 2^m\right]\right) &\leq \frac{1}{2^m} (2E(|Y_n| + |Y_1|)) \\ &= 2^{-m} (2E(|X(t)| + |X(0)|)) \end{aligned}$$

Then letting $n \rightarrow \infty$, it follows upon letting $C_t = 2E(|X(t)| + |X(0)|)$,

$$P\left(\left[\sup_{r \in \mathbb{Q}^+ \cap [0, t]} |X(r)| \geq 2^m\right]\right) \leq 2^{-m} C_t.$$

By the Borel Cantelli lemma, there exists a set of measure 0, N_t such that for $\omega \notin N_t$, ω is contained in only finitely many of the sets

$$\left[\sup_{r \in \mathbb{Q}^+ \cap [0, t]} |X(r)| \geq 2^m\right]$$

which shows that for $\omega \notin N_t$, $\sup_{r \in \mathbb{Q}^+ \cap [0, t]} |X(r)|$ is bounded. Now let $N = \cup_{j=1}^\infty N_j$. This proves the second claim.

Next consider the first claim. By the upcrossing estimate, Theorem 48.4.8 or Lemma 48.2.6, and letting $a < b$ and $U_{[a, b]}^n[c, d]$ the upcrossings of Y_k from a to b on $[c, d]$ for $d \leq t$ and $c \geq 0$,

$$\begin{aligned} E\left(U_{[a, b]}^n[0, t]\right) &\leq \frac{1}{b-a} E\left((Y_n - a)^+\right) \\ &= \frac{1}{b-a} E\left((X(t) - a)^+\right). \end{aligned}$$

Hence

$$P\left(\left[U_{[a, b]}^n[0, t] \geq M\right]\right) \leq \frac{1}{M} \left(\frac{1}{b-a} E\left((X(t) - a)^+\right)\right). \quad (50.6.28)$$

Suppose for some $s < t$,

$$\lim_{r \rightarrow s+, r \in \mathbb{Q}} \sup_{r \in \mathbb{Q}} X(r, \omega) > b > a > \lim_{r \rightarrow s+, r \in \mathbb{Q}} \inf_{r \in \mathbb{Q}} X(r, \omega). \quad (50.6.29)$$

If this is so, then in $(s, t) \cap \mathbb{Q}$ there must be infinitely many values of $r \in \mathbb{Q}$ such that $X(r, \omega) \geq b$ as well as infinitely many values of $r \in \mathbb{Q}$ such that $X(r, \omega) \leq a$. Note this involves the consideration of a limit from one side. Thus, since it is a limit from one side only, there are an arbitrarily large number of upcrossings between s and t . Therefore, letting M be a large positive number, it follows that for all n sufficiently large,

$$U_{[a,b]}^n [0, t] (\omega) \geq M$$

which implies

$$\omega \in \left[U_{[a,b]}^n [0, t] \geq M \right]$$

which from 50.6.28 is a set of measure no more than

$$\frac{1}{M} \left(\frac{1}{b-a} E \left((X(t) - a)^+ \right) \right).$$

This has shown that the set of ω such that for some $s \in [0, t]$ 50.6.29 holds is contained in the set

$$N_{[a,b]} \equiv \cap_{M=1}^{\infty} \cup_{n=1}^{\infty} \left[U_{[a,b]}^n [0, t] \geq M \right]$$

Now the sets,

$$\left[U_{[a,b]}^n [0, t] \geq M \right]$$

are increasing in n and each has measure less than

$$\frac{1}{M} \left(\frac{1}{b-a} E \left((X(t) - a)^+ \right) \right)$$

and so

$$P \left(\cup_{n=1}^{\infty} \left[U_{[a,b]}^n [0, t] \geq M \right] \right) \leq \frac{1}{M} \left(\frac{1}{b-a} E \left((X(t) - a)^+ \right) \right).$$

which shows that

$$P(N_{[a,b]}) \leq \frac{1}{M} \left(\frac{1}{b-a} E \left((X(t) - a)^+ \right) \right)$$

for every M and therefore, $P(N_{[a,b]}) = 0$.

Therefore, corresponding to $a < b$, there exists a set of measure 0, $N_{[a,b]}$ such that for $\omega \notin N_{[a,b]}$ 50.6.29 is not true for any $s \in [0, t]$. Let $N \equiv \cup_{a,b \in \mathbb{Q}} N_{[a,b]}$, a set of measure 0 with the property that if $\omega \notin N$, then 50.6.29 fails to hold for any pair of rational numbers, $a < b$ for any $s \in [0, t]$. Thus for $\omega \notin N$,

$$\lim_{r \rightarrow s+, r \in \mathbb{Q}} X(r, \omega)$$

exists for all $s \in [0, t]$. Similar reasoning applies to show the existence of the limit

$$\lim_{r \rightarrow s-, r \in \mathbb{Q}} X(r, \omega).$$

for all $s \in (0, t]$ whenever ω is outside of a set of measure zero. Of course, this exceptional set depends on t . However, if this exceptional set is denoted as N_t , one could consider $N \equiv \bigcup_{n=1}^{\infty} N_n$. It is obvious there is no change if $\mathbb{Q}$ is replaced with any countable dense subset. This proves the theorem.

Of course the above theorem does not say the left and right limits are equal, just that they exist in some way for ω not in some set of measure zero. Also it has not been shown that $\lim_{r \rightarrow s+, r \in \mathbb{Q}} X(r, \omega) = X(r, \omega)$ for a.e. ω .

Corollary 50.6.2 *In the situation of Theorem 50.6.1, let $s > 0$ and let D_1 and D_2 be two countable dense subsets of $\mathbb{R}$. Then*

$$\lim_{r \rightarrow s-, r \in D_1} X(r, \omega) = \lim_{r \rightarrow s-, r \in D_2} X(r, \omega) \quad \text{a.e. } \omega$$

$$\lim_{r \rightarrow s+, r \in D_1} X(r, \omega) = \lim_{r \rightarrow s+, r \in D_2} X(r, \omega) \quad \text{a.e. } \omega$$

Proof: Let $\{r_n^i\}$ be an increasing sequence from D_i converging to s and let N be the exceptional set corresponding to the countable dense set $D_1 \cup D_2$. Then for $\omega \notin N$, and $i = 1, 2$,

$$\lim_{r \rightarrow s-, r \in D_1 \cup D_2} X(r, \omega) = \lim_{n \rightarrow \infty} X(r_n^i, \omega) = \lim_{r \rightarrow s-, r \in D_i} X(r, \omega)$$

The other claim is similar. This proves the corollary.

Now here is an impressive lemma about submartingales and uniform integrability.

Lemma 50.6.3 *Let $X(t)$ be a submartingale adapted to a filtration $\mathcal{F}_t$. Let $\{r_k\} \subseteq [s, t]$ be a decreasing sequence converging to s . Then $\{X(r_j)\}_{j=1}^{\infty}$ is uniformly integrable.*

Proof: First I will show the sequence is equiintegrable. I need to show that for all $\varepsilon > 0$ there exists λ large enough that for all n

$$\int_{|X(r_n)| \geq \lambda} |X(r_n)| dP < \varepsilon.$$

Let $\varepsilon > 0$ be given. Since $\{X(r)\}_{r \geq 0}$ is a submartingale, $E(X(r_n))$ is a decreasing sequence bounded below by $E(X(s))$. This is because for $r_n < r_k$,

$$E(X(r_n)) \leq E(E(X(r_k) | \mathcal{F}_n)) = E(X(r_k))$$

Pick k such that

$$\begin{aligned} & E(X(r_k)) - \lim_{n \rightarrow \infty} E(X(r_n)) \\ &= \left| E(X(r_k)) - \lim_{n \rightarrow \infty} E(X(r_n)) \right| < \varepsilon/2. \end{aligned}$$

Then for $n > k$,

$$\begin{aligned}
 \int_{[|X(r_n)| \geq \lambda]} |X(r_n)| dP &= \int_{[X(r_n) \geq \lambda]} X(r_n) dP + \int_{[X(r_n) \leq -\lambda]} -X(r_n) dP \\
 &= \int_{[X(r_n) \geq \lambda]} X(r_n) dP + \int_{[X(r_n) > -\lambda]} X(r_n) dP - \int_{\Omega} X(r_n) dP \\
 &\leq \int_{[X(r_n) \geq \lambda]} X(r_n) dP + \int_{[X(r_n) > -\lambda]} E(X(r_k) | \mathcal{F}_n) dP - \int_{\Omega} X(r_n) dP \\
 &\leq \int_{[X(r_n) \geq \lambda]} X(r_k) dP + \int_{[X(r_n) > -\lambda]} X(r_k) dP - \int_{\Omega} X(r_k) dP + \varepsilon/2 \\
 &= \int_{[X(r_n) \geq \lambda]} X(r_k) dP + \int_{[X(r_n) \leq -\lambda]} (-X(r_k)) dP + \varepsilon/2 \\
 &= \int_{[|X(r_n)| \geq \lambda]} |X(r_k)| dP + \varepsilon/2 \\
 &\leq \int_{\left[\sup\{|X(r)| \geq \lambda : r \in \{r_j\}_{j=1}^{\infty}\}\right]} |X(r_k)| dP + \varepsilon/2 \tag{50.6.30}
 \end{aligned}$$

From maximal inequalities of Theorem 48.4.5

$$P\left(\left[\sup_{r \in \{r_n, r_{n-1}, \dots, r_1\}} |X(r)| \geq \lambda\right]\right) \leq \frac{2E(|X(t)| + |X(0)|)}{\lambda} \equiv \frac{C}{\lambda}$$

and so, letting $n \rightarrow \infty$,

$$P\left(\left[\sup_{r \in \{r_n\}_{n=1}^{\infty}} |X(r)| \geq \lambda\right]\right) \leq \frac{C}{\lambda}.$$

It follows that for λ sufficiently large the first term in 50.6.30 is smaller than $\varepsilon/2$ because k is fixed. Now this shows there is a choice of λ such that for all $n > k$,

$$\int_{[|X(r_n)| \geq \lambda]} |X(r_n)| dP < \varepsilon$$

There are only finitely many r_n for $n \leq k$ and by choosing λ sufficiently large the above formula can be made to hold for these also, thus showing $\{X(r_n)\}$ is equiintegrable.

Now this implies the sequence of random variables is uniformly integrable as well. Let $\varepsilon > 0$ be given and choose λ large enough that for all n ,

$$\int_{[|X(r_n)| \geq \lambda]} |X(r_n)| dP < \varepsilon/2$$

Then let A be a measurable set.

$$\begin{aligned} \int_A |X(r_n)| dP &= \int_{A \cap [|X(r_n)| \geq \lambda]} |X(r_n)| dP + \int_{A \cap [|X(r_n)| < \lambda]} |X(r_n)| dP \\ &< \varepsilon/2 + \int_{A \cap [|X(r_n)| < \lambda]} |X(r_n)| dP \leq \frac{\varepsilon}{2} + \lambda P(A) \end{aligned}$$

and now you see that if $P(A)$ is sufficiently small then for all n ,

$$\int_A |X(r_n)| dP < \varepsilon$$

which shows the set of functions is uniformly integrable as claimed. This proves the lemma.

You can often consider a submartingale to be right continuous. This is the importance of the following theorem.

Theorem 50.6.4 *Let $\{X(t)\}$ be a submartingale adapted to a normal filtration $\mathcal{F}_t$. There exists a right continuous submartingale having left limits, $\{Y(t)\}$ such that $Y(t) = X(t)$ a.e. for every $t \in \mathbb{Q}^+$. Furthermore $\{X(t)\}$ has a right continuous left limits version if and only if*

$$t \rightarrow E(X(t))$$

is right continuous. More generally, $Y(t) = X(t)$ a.e. at every point where the above function is right continuous.

Proof: From Theorem 50.6.1, there exists a set of measure zero, N such that for $\omega \notin N$, left and right limits of the following form exist.

$$\lim_{r \rightarrow t+, r \in \mathbb{Q}} X(r, \omega), \quad \lim_{r \rightarrow t-, r \in \mathbb{Q}} X(r, \omega).$$

Then define for each t and $\omega \notin N$,

$$Y(t, \omega) \equiv \lim_{r \rightarrow t+, r \in \mathbb{Q}} X(r, \omega).$$

and for $\omega \in N$,

$$Y(t, \omega) \equiv 0$$

Thus $Y(t)(\omega) = X(t)(\omega)$ a.e. for $t \in \mathbb{Q}^+$. For each $\omega \notin N$, there exists $\delta > 0$ such that if $r \in \mathbb{Q}$, $t < r < t + 2\delta$, then

$$|Y(t, \omega) - X(r, \omega)| < \varepsilon/2.$$

Now suppose $s \in (t, t + \delta)$. Then there exists $\delta_1 < \delta$ such that if $s < r < s + \delta_1$ then

$$|Y(s, \omega) - X(r, \omega)| < \varepsilon/2.$$

pick $r \in \mathbb{Q} \cap (s, t + \delta)$. Then both of the above two inequalities hold and so it follows

$$\begin{aligned} |Y(t, \omega) - Y(s, \omega)| &< |Y(t, \omega) - X(r, \omega)| + |X(r, \omega) - Y(s, \omega)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, $t \rightarrow Y(t, \omega)$ is right continuous as claimed.

From the definition of $Y(t, \omega)$, it follows $\omega \rightarrow Y(t, \omega)$ is measurable in $\mathcal{F}_{t+}$ because it is the limit of a sequence, $X(r_n, \omega)$ where $r_n \rightarrow t +$. Now each $X(r_n, \cdot)$ is $\mathcal{F}_{r_n}$ measurable and so $Y(t, \cdot)$ is $\mathcal{F}_{r_n}$ measurable also for each r_n . Thus $Y(t, \cdot)$ is $\mathcal{F}_{t+}$ measurable. Since the filtration is normal, $\mathcal{F}_t = \mathcal{F}_{t+}$ and it follows $Y(t, \cdot)$ is $\mathcal{F}_t$ measurable. Why is $Y(t, \cdot) \in L^1(\Omega)$?

From Lemma 50.6.3, the collection $\{X(r_n)\}$ is uniformly integrable. Therefore, from the Vitali convergence theorem, Theorem 8.5.3 on Page 213,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |Y(s) - X(r_n)| dP = 0 \quad (50.6.31)$$

and $Y(s) \in L^1(\Omega)$.

It remains to verify $\{Y(s)\}$ is a submartingale. For $s < t$, is it true that

$$E(Y(t) | \mathcal{F}_s) \geq Y(s)?$$

Fix $A \in \mathcal{F}_s$. From the above construction, there exists $w \in \mathbb{Q}$ and $w \geq t$ such that

$$\int_A Y(t) dP \geq \int_A X(w) dP - \varepsilon.$$

Then also, there exists $r \in \mathbb{Q} \cap (s, t)$ such that

$$\int_A X(r) dP \geq \int_A Y(s) dP - \varepsilon.$$

Now

$$\begin{aligned} \int_A E(Y(t) | \mathcal{F}_s) dP &= \int_A Y(t) dP \geq \int_A X(w) dP - \varepsilon \\ &= \int_A E(X(w) | \mathcal{F}_r) dP - \varepsilon \\ &\geq \int_A X(r) dP - \varepsilon \geq \int_A Y(s) dP - 2\varepsilon. \end{aligned}$$

Since ε was arbitrary, this shows

$$\int_A E(Y(t) | \mathcal{F}_s) dP \geq \int_A Y(s) dP$$

for any $A \in \mathcal{F}_s$ and so this verifies since A is arbitrary that

$$E(Y(t) | \mathcal{F}_s) \geq Y(s)$$

so Y is a submartingale.

By Theorem 50.6.1 there exists a set of measure 0, N such that the left limits of $Y(r, \omega)$ exist through rational numbers if $\omega \notin N$.

Is $\{Y(t)\}$ a version of $\{X(t)\}$? This is where the assumption $t \rightarrow E(X(t))$ is continuous is used. We know $E(X(r_n)|\mathcal{F}_s) \geq X(s)$ and also for $A \in \mathcal{F}_s$

$$\int_A X(r_n) dP = \int_A E(X(r_n)|\mathcal{F}_s) dP \geq \int_A X(s) dP.$$

Hence taking a limit yields

$$\int_A Y(s) dP \geq \int_A X(s) dP$$

and since A is arbitrary, $Y(s) \geq X(s)$. Now since $t \rightarrow E(X(t))$ is continuous,

$$\begin{aligned} \int_{\Omega} |Y(s) - X(s)| dP &= E(Y(s)) - E(X(s)) \\ &= \lim_{n \rightarrow \infty} (E(X(r_n)) - E(X(r_n))) = 0. \end{aligned}$$

It only remains to verify the only way $X(t)$ has a right continuous version is for $t \rightarrow E(X(t))$ to be continuous. Suppose then that $\{X(t)\}$ has a right continuous version, $\{Y(t)\}$. Letting $r_n \downarrow s$, Lemma 50.6.3 implies $\{Y(r_n)\}$ is uniformly integrable. Also $Y(s)(\omega) = \lim_{n \rightarrow \infty} Y(r_n)(\omega)$ a.e. and so by the Vitali convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\Omega} |X(r_n) - X(s)| dP = \lim_{n \rightarrow \infty} \int_{\Omega} |Y(r_n) - Y(s)| dP = 0.$$

This proves the theorem.

Note that the condition $t \rightarrow E(X(t))$ being continuous holds for any martingale. Therefore, every martingale has a right continuous version. The condition that $t \rightarrow E(X(t))$ is right continuous is not a very stringent assumption. For $\{X(t)\}$ a submartingale, this is an increasing function. Therefore, the only points where the condition might not hold comprise a countable set.

50.7 Some Maximal Inequalities

As in the case of discrete martingales and submartingales, there are maximal inequalities available.

Theorem 50.7.1 *Let $\{X(t)\}$ be a right continuous nonnegative submartingale adapted to the normal filtration $\mathcal{F}_t$ for $t \in [0, M]$. Let $p \geq 1$. Define*

$$X^*(t) \equiv \sup \{X(s) : 0 < s < t\}, \quad X^*(0) \equiv 0.$$

Then for $\lambda > 0$

$$P([X^*(M) > \lambda]) \leq \frac{1}{\lambda^p} \int \mathcal{X}_{[X^*(M) > \lambda]} X(M)^p dP \quad (50.7.32)$$

In case $p > 1$ then for each $t \leq M$,

$$\left(\int_{\Omega} |X^*(t)|^p dP \right)^{1/p} \leq \frac{p}{p-1} \left(\int_{\Omega} X(M)^p dP \right)^{1/p} \quad (50.7.33)$$

Proof: First note that from the right continuity of X , $X^*(t)$ is a random variable. This is because this right continuity makes it possible to replace $(0, t)$ with $(0, t) \cap \mathbb{Q}$ in the definition of $X^*(t)$. Define the stopping time

$$T \equiv \min(\inf\{t > 0 : X(t) > \lambda\}, M).$$

(The infimum over an empty set will equal ∞ .) This is a stopping time because it is a continuous function of the stopping time of Proposition 50.5.2. Since $f(x) = x^p$ is convex and increasing $\{X(t)^p\}$ is also a submartingale. Then by the optional sampling theorem, $X(0)^p, X(T)^p, X(M)^p$ is a submartingale and so

$$E(X(T)^p) \leq E(X(M)^p)$$

which implies

$$\int_{[T < M]} X(T)^p dP + \int_{[T = M]} X(T)^p dP \leq \int_{\Omega} X(M)^p dP.$$

On $[T < M]$, then since $X(t)$ is right continuous, $X(T) \geq \lambda$. On $[T = M]$,

$$X(T) = X(M)$$

and so

$$\lambda^p P([T < M]) \leq \int \mathcal{X}_{[T < M]} X(M)^p dP$$

Also $[T < M] = [X^*(M) > \lambda]$. Therefore, this yields 50.7.32.

Next if $p > 1$, then from the first part,

$$\begin{aligned} \int_{\Omega} |X^*(t)|^p dP &\leq \int_{\Omega} |X(M)|^p dP = \int_0^{\infty} p\lambda^{p-1} P([X^*(M) > \lambda]) d\lambda \\ &\leq p \int_0^{\infty} \lambda^{p-1} \frac{1}{\lambda} \int \mathcal{X}_{[X^*(M) > \lambda]} X(M) dP d\lambda \\ &= p \int_{\Omega} X(M) \int_0^{X^*(M)} \lambda^{p-2} d\lambda dP \\ &= p \int_{\Omega} X(M) \frac{X^*(M)^{p-1}}{p-1} dP \\ &\leq \frac{p}{p-1} \left(\int_{\Omega} X^*(M)^p dP \right)^{1/p'} \left(\int_{\Omega} X(M)^p dP \right)^{1/p} \end{aligned}$$

This establishes 50.7.33.

Here is another sort of maximal inequality in which $X(t)$ is not assumed non-negative.

Theorem 50.7.2 *Let $\{X(t)\}$ be a submartingale adapted to the normal filtration $\mathcal{F}_t$ for $t \in [0, M]$ and $X^*(t)$ defined as in Theorem 50.7.1*

$$P([X^*(M) > \lambda]) \leq \frac{1}{\lambda} E(|X(M)|) \quad (50.7.34)$$

For $t > 0$, let

$$X_*(t) = \inf \{X(s) : s < t\}.$$

Then

$$P([X_*(M) < -\lambda]) \leq \frac{1}{\lambda} E(|X(M)| + |X(0)|) \quad (50.7.35)$$

Also

$$\begin{aligned} P([\sup \{|X(s)| : s < M\} > \lambda]) \\ \leq \frac{2}{\lambda} E(|X(M)| + |X(0)|) \end{aligned} \quad (50.7.36)$$

Proof: First let the stopping time T be given by

$$T = \min(\inf \{t : X(t) > \lambda\}, M)$$

Then by optional sampling theorem $X(0), X(T), X(M)$ is a submartingale and so

$$E(X(T)) \leq E(X(M))$$

which implies

$$\int_{[T < M]} X(T) dP + \int_{[T=M]} X(T) dP \leq \int_{\Omega} X(M) dP.$$

On $[T < M]$, $X(T) \geq \lambda$ due to right continuity. Hence

$$\lambda P([T < M]) \leq \int_{[T < M]} X(M) dP \leq E(X(M)^+).$$

It follows

$$P([X^*(M) > \lambda]) \leq \frac{1}{\lambda} E(|X(M)|).$$

which verifies 50.7.34.

Next let

$$T = \min(\inf \{t : X(t) < -\lambda\}, M)$$

then as before,

$$\int_{[T < M]} X(T) dP + \int_{[T=M]} X(T) dP \geq \int_{\Omega} X(0) dP$$

and so

$$-\lambda \int_{[T < M]} dP \geq - \int_{[T=M]} X(M) dP + \int_{\Omega} X(0) dP$$

and so since $[T < M] = [X_*(M) < -\lambda]$

$$\begin{aligned} P([X_*(M) < -\lambda]) &= P([T < M]) \\ &\leq \frac{1}{\lambda} \int_{[T=M]} X(M) dP - \frac{1}{\lambda} \int_{\Omega} X(0) dP \\ &\leq \frac{1}{\lambda} E(|X(M)| + |X(0)|) \end{aligned}$$

and this proves 50.7.35.

Finally, combining the above two inequalities,

$$\begin{aligned} &P([\sup\{|X(s)| : s < M\} > \lambda]) \\ &= P([X_*(M) < -\lambda]) + P([X^*(M) > \lambda]) \\ &\leq \frac{2}{\lambda} E(|X(M)| + |X(0)|) \end{aligned}$$

This verifies 50.7.36 and proves the theorem.

50.8 Continuous Submartingale Convergence Theorem

In this section, $\{X(t)\}$ will be a right continuous submartingale.

Lemma 50.8.1 *Let $\{X(t)\}$ be a right continuous submartingale adapted to a normal filtration, $\mathcal{F}_t$ and let S be a stopping time. Let*

$$T \equiv \inf\{t > S : X(t) \in B\}$$

where B is an open set. Then T is also a stopping time.

Proof: By the right continuity of X ,

$$T(\omega) = \inf\{t > S, t \in \mathbb{Q} : X(t) \in B\}$$

Therefore for $p \in \mathbb{N}$,

$$\left[T < t + \frac{1}{p}\right] = \cup_{s \in \mathbb{Q} \cap (S, t+1/p)} [X(s) \in B],$$

a measurable set contained in $\mathcal{F}_{t+1/p}$. It follows

$$[T \leq t] = \cap_{p=1}^{\infty} \left[T < t + \frac{1}{p}\right]$$

Now

$$\cap_{p=1}^N [T < t + 1/p] = [T < t + 1/N] \in \mathcal{F}_{t+1/N}$$

and so $[T \leq t] \in \cap_{n=1}^{\infty} \mathcal{F}_{t+1/n} = \mathcal{F}_t$ since the filtration is normal. Thus T is a stopping time as claimed. This proves the lemma.

Now the next task is to consider an upcrossing estimate as was done before for discrete submartingales. Let $a < b$ and define

$$\begin{aligned} T_0 &\equiv \min(\inf\{t > 0 : X(t) < a\}, M), T_1 \equiv \min(\inf\{t > T_0 : X(t) > b\}, M), \\ T_2 &\equiv \min(\inf\{t > T_1 : X(t) < a\}, M), T_3 \equiv \min(\inf\{t > T_2 : X(t) > b\}, M), \end{aligned}$$

etc. where the infimum of an empty set is ∞ . Then by Lemma 50.8.1 this is an increasing sequence of stopping times. Let

$$\begin{aligned} U_{[a,b]}^{nM} &\equiv \lim_{\varepsilon \rightarrow 0} \sum_{k=0}^n \frac{X(T_{2k+1}) - X(T_{2k})}{\varepsilon + |X(T_{2k+1}) - X(T_{2k})|} \\ &\leq \frac{1}{b-a} \sum_{k=0}^n X(T_{2k+1}) - X(T_{2k}) \\ &\leq \frac{1}{b-a} \sum_{k=0}^n X(T_{2k+1}) - X(T_{2k}) \end{aligned}$$

Note that an upcrossing occurs when $T_{2k+1} > T_{2k}$. Then $U_{[a,b]}^{nM}$ is clearly a random variable and it records the number of upcrossings occurring for $t \leq M$ using only n of the stopping times. Then by taking the expected value of both sides and using the optional sampling theorem which implies $E(X(T_{2k})) - E(X(T_{2k-1})) \geq 0$ due to the fact $\{X(T_k)\}$ is a submartingale,

$$\begin{aligned} E(U_{[a,b]}^{nM}) &\leq \frac{1}{b-a} \sum_{k=0}^n E(X(T_{2k+1})) - E(X(T_{2k})) \\ &\leq \frac{1}{b-a} \sum_{k=0}^n E(X(T_{2k+1})) - E(X(T_{2k})) + \frac{1}{b-a} \sum_{k=1}^n E(X(T_{2k})) - E(X(T_{2k-1})) \\ &\leq \frac{1}{b-a} (E(X(T_{2n+1})) - E(X(T_0))) \\ &\leq \frac{1}{b-a} (E(X(M)) - E(X(0))) \end{aligned}$$

which does not depend on n . Let $n \rightarrow \infty$ to obtain

$$E(U_{[a,b]}^M) \leq \frac{1}{b-a} (E(X(M)) - E(X(0)))$$

where $U_{[a,b]}^M$ is the number of upcrossings of $\{X(t)\}$ on $[0, M]$. This proves the following interesting upcrossing estimate.

Lemma 50.8.2 *Let $\{X(t)\}$ be a right continuous submartingale adapted to a normal filtration $\mathcal{F}_t$ for $t \in [0, M]$. Then if $U_{[a,b]}^M$ is defined to be the number of upcrossings of $\{X(t)\}$ for $t \in [0, M]$, then this is a random variable and*

$$E\left(U_{[a,b]}^M\right) \leq \frac{1}{b-a} (E(X(M)) - E(X(0))).$$

With this it is easy to prove a continuous submartingale convergence theorem.

Theorem 50.8.3 *Let $\{X(t)\}$ be a right continuous submartingale such that*

$$\sup \{E(|X(t)|)\} = C < \infty.$$

Then there exists $X_\infty \in L^1(\Omega)$ such that

$$\lim_{t \rightarrow \infty} X(t)(\omega) = X_\infty(\omega) \text{ a.e. } \omega.$$

Proof: Let $U_{[a,b]}$ denote the upcrossings for $t \geq 0$. Then

$$U_{[a,b]} = \lim_{M \rightarrow \infty} U_{[a,b]}^M.$$

From Lemma 50.8.2 and the assumption of this theorem, there exists a constant, C independent of M such that

$$E\left(U_{[a,b]}^M\right) \leq \frac{2C}{b-a}.$$

Letting $M \rightarrow \infty$, it follows from monotone convergence theorem that

$$E(U_{[a,b]}) \leq \frac{2C}{b-a}$$

also. Therefore, there exists a set of measure 0, N_{ab} such that if $\omega \notin N_{ab}$, then $U_{[a,b]}(\omega) < \infty$. That is, there are only finitely many upcrossings. Now let

$$N = \cup \{N_{ab} : a, b \in \mathbb{Q}\}.$$

It follows that for $\omega \notin N$, it cannot happen that

$$\limsup_{t \rightarrow \infty} X(t)(\omega) - \liminf_{t \rightarrow \infty} X(t)(\omega) > 0$$

because if this expression is positive, there would be arbitrarily large values of t where $X(t)(\omega) > b$ and arbitrarily large values of t where $X(t)(\omega) < a$ where a, b are rational numbers chosen such that

$$\limsup_{t \rightarrow \infty} X(t)(\omega) > b > a > \liminf_{t \rightarrow \infty} X(t)(\omega)$$

Thus there would be infinitely many upcrossings which is not allowed for $\omega \notin N$. Therefore, the limit $\lim_{t \rightarrow \infty} X(t)(\omega)$ exists for a.e. ω . Let $X_\infty(\omega)$ equal this limit

for $\omega \notin N$ and let $X_\infty(\omega) = 0$ for $\omega \in N$. Then X_∞ is measurable and by Fatou's lemma,

$$\int_{\Omega} |X_\infty(\omega)| dP \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |X(n)(\omega)| dP < C$$

This proves the theorem.

Now here is an interesting result due to Doob.

Theorem 50.8.4 *Let $\{M(t)\}$ be a right continuous real martingale adapted to the normal filtration $\mathcal{F}_t$. Then the following are equivalent.*

1. *The random variables $M(t)$ are equiintegrable.*
2. *There exists $M(\infty) \in L^1(\Omega)$ such that $\lim_{t \rightarrow \infty} \|M(\infty) - M(t)\|_{L^1(\Omega)} = 0$.*

In this case, $M(t) = E(M(\infty) | \mathcal{F}_t)$ and convergence also takes place pointwise.

Proof: Suppose the equiintegrable condition. Then there exists λ large enough that for all t ,

$$\int_{[|M(t)| \geq \lambda]} |M(t)| dt < 1.$$

It follows that for all t ,

$$\begin{aligned} \int_{\Omega} |M(t)| dP &= \int_{[|M(t)| \geq \lambda]} |M(t)| dt + \int_{[|M(t)| < \lambda]} |M(t)| dt \\ &\leq 1 + \lambda. \end{aligned}$$

Since the martingale is bounded in L^1 , Theorem 50.8.3 there exists $M(\infty) \in L^1(\Omega)$ such that $\lim_{t \rightarrow \infty} M(t)(\omega) = M(\infty)(\omega)$ pointwise a.e. By the assumption $\{M(t)\}$ are equiintegrable, it follows from the Vitali convergence theorem, Theorem 8.5.3 this convergence takes place in $L^1(\Omega)$ also.

Next suppose there is a function $M(\infty)$ to which $M(t)$ converges in $L^1(\Omega)$. Then in $L^1(\Omega)$,

$$E(M(\infty) | \mathcal{F}_t) = \lim_{s \rightarrow \infty} E(M(s) | \mathcal{F}_t) = \lim_{s \rightarrow \infty} M(t) = M(t)$$

which shows $E(M(\infty) | \mathcal{F}_t) = M(t)$ a.e. Now by Lemma 50.5.6,

$$\begin{aligned} \int_{[|M(t)| \geq \lambda]} |M(t)| dP &= \int_{[|M(t)| \geq \lambda]} |E(M(\infty) | \mathcal{F}_t)| dP \\ &\leq \int_{[|M(t)| \geq \lambda]} E(|M(\infty)| | \mathcal{F}_t) dP \\ &= \int_{[|M(t)| \geq \lambda]} |M(\infty)| dP \end{aligned} \tag{50.8.37}$$

Now from this,

$$\lambda P([|M(t)| \geq \lambda]) \leq \int_{\Omega} |M(t)| dP \leq \int_{\Omega} |M(\infty)| dP$$

and so

$$P(|M(t)| \geq \lambda) \leq \frac{C}{\lambda}$$

From 50.8.37, this shows $\{M(t)\}$ is uniformly integrable. This proves the theorem.

50.9 Hitting This Before That

Let $\{M(t)\}$ be a real valued martingale for $t \in [0, T]$ where $T \leq \infty$ and $M(0) = 0$. In case $T = \infty$, assume the conditions of Theorem 50.8.4 are satisfied. Thus there exists $M(\infty)$ and the $M(t)$ are equiintegrable. With the Doob optional sampling theorem it is possible to estimate the probability that $M(t)$ hits a before it hits b where $a < 0 < b$. There is no loss of generality in assuming $T = \infty$ since if it is less than ∞ , you could just let $M(t) \equiv M(T)$ for all $t > T$. In this case, the equiintegrability of the $M(t)$ follows because for $t < T$,

$$\begin{aligned} \int_{|M(t)| > \lambda} |M(t)| dP &= \int_{|M(t)| > \lambda} |E(M(T) | \mathcal{F}_t)| dP \\ &\leq \int_{|M(t)| > \lambda} |M(T)| dP \end{aligned}$$

and from Theorem 50.7.2,

$$P(|M(t)| > \lambda) \leq P([M^*(t) > \lambda]) \leq \frac{1}{\lambda} \int_{\Omega} |M(T)| dP.$$

Definition 50.9.1 Let M be a process adapted to the filtration $\mathcal{F}_t$ and let τ be a stopping time. Then M^τ , called the stopped process is defined by

$$M^\tau(t) \equiv M(\tau \wedge t).$$

With this definition, here is a simple lemma.

Lemma 50.9.2 Let M be a martingale adapted to the normal filtration $\mathcal{F}_t$ and let τ be a stopping time. Then M^τ is also a martingale adapted to the filtration $\mathcal{F}_t$.

Proof: Let $s < t$. By the Doob optional sampling theorem,

$$E(M^\tau(t) | \mathcal{F}_s) \equiv E(M(\tau \wedge t) | \mathcal{F}_s) = M(\tau \wedge t \wedge s) = M^\tau(s).$$

Theorem 50.9.3 Let $\{M(t)\}$ be a continuous real valued martingale adapted to the normal filtration $\mathcal{F}_t$ and let

$$M^* \equiv \sup \{|M(t)| : t \geq 0\}$$

and $M(0) = 0$. Letting

$$\tau_x \equiv \inf \{t > 0 : M(t) = x\}$$

Then if $a < 0 < b$ the following inequalities hold.

$$(b-a)P([\tau_b \leq \tau_a]) \geq -aP([M^* > 0]) \geq (b-a)P([\tau_b < \tau_a])$$

and

$$(b-a)P([\tau_a < \tau_b]) \leq bP([M^* > 0]) \leq (b-a)P([\tau_a \leq \tau_b]).$$

Proof: For $x \in \mathbb{R}$, define

$$\tau_x \equiv \inf \{t \in \mathbb{R} \text{ such that } M(t) = x\}$$

with the usual convention that $\inf(\emptyset) = \infty$. Let $a < 0 < b$ and let

$$\tau = \tau_a \wedge \tau_b$$

Then the following claim will be important.

Claim: $E(M(\tau)) = 0$.

Proof of the claim: Let $t > 0$. Then by the Doob optional sampling theorem,

$$E(M(\tau \wedge t)) = E(E(M(t) | \mathcal{F}_\tau)) = E(M(t)) \quad (50.9.38)$$

$$= E(E(M(t) | \mathcal{F}_0)) = E(M(0)) = 0. \quad (50.9.39)$$

Observe the martingale M^τ must be bounded. There are two cases according to whether $\tau = \infty$. If $\tau = \infty$, then $M(t)$ never hits a or b so $M(t)$ has values between a and b . In this case $M^\tau(t) = M(t) \in [a, b]$. On the other hand, you could have $\tau < \infty$. Then in this case $M^\tau(t)$ is eventually equal to either a or b depending on which it hits first. In either case, the martingale M^τ is bounded and

$$\lim_{t \rightarrow \infty} M^\tau(t)(\omega) = M^\tau(\infty)(\omega) = M(\tau)(\omega)$$

and since the $M^\tau(t)$ are bounded, the dominated convergence theorem implies

$$E(M(\tau)) = \lim_{t \rightarrow \infty} E(M(\tau \wedge t)) = 0.$$

This proves the claim.

Recall

$$M^*(\omega) \equiv \sup \{|M(t)(\omega)| : t \in [0, \infty]\}.$$

Also note that $[\tau_a = \tau_b] = [\tau = \infty]$. Now from the claim,

$$\begin{aligned} 0 &= E(M(\tau)) = \int_{[\tau_a < \tau_b]} M(\tau) dP \\ &\quad + \int_{[\tau_b < \tau_a]} M(\tau) dP + \int_{[\tau_a = \tau_b] \cap [M^* > 0]} M(\infty) dP \\ &\quad + \int_{[\tau_a = \tau_b] \cap [M^* = 0]} M(\infty) dP \end{aligned} \quad (50.9.40)$$

The last term equals 0. Thus

$$0 = E(M(\tau)) = aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a]) + \int_{[\tau_a = \tau_b] \cap [M^* > 0]} M(\infty) dP \quad (50.9.41)$$

Consider this last term. By the definition, $[\tau_a = \tau_b]$ corresponds to $M(t)$ never hitting either a or b . Since $M(0) = 0$, this can only happen if $M(t)$ has values in $[a, b]$. Therefore, this last term satisfies

$$\begin{aligned} & aP([\tau_a = \tau_b] \cap [M^* > 0]) \\ & \leq \int_{[\tau_a = \tau_b] \cap [M^* > 0]} M(\infty) dP \\ & \leq bP([\tau_a = \tau_b] \cap [M^* > 0]) \end{aligned} \quad (50.9.42)$$

Thus from 50.9.41

$$0 \leq aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a]) + bP([\tau_a = \tau_b] \cap [M^* > 0])$$

Note that $[\tau_b < \tau_a] \subseteq [M^* > 0]$. Thus the above reduces to

$$0 \leq aP([\tau_a < \tau_b]) + bP([\tau_b \leq \tau_a] \cap [M^* > 0])$$

and so

$$0 \leq aP([\tau_a < \tau_b]) + b(P([M^* > 0]) - P([\tau_a < \tau_b]))$$

which implies

$$(b - a)P([\tau_a < \tau_b]) \leq bP([M^* > 0]) \quad (50.9.43)$$

This gives an estimate for $P([\tau_a < \tau_b])$.

Next I will use the bottom half of 50.9.42 to get an estimate for $P([\tau_a \leq \tau_b])$. Thus from 50.9.41,

$$\begin{aligned} 0 & \geq aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a]) + aP([\tau_a = \tau_b] \cap [M^* > 0]) \\ & = aP([\tau_a \leq \tau_b] \cap [M^* > 0]) + bP([\tau_b < \tau_a]) \\ & = aP([\tau_a \leq \tau_b] \cap [M^* > 0]) + b(P([M^* > 0]) - P([\tau_a \leq \tau_b] \cap [M^* > 0])) \end{aligned}$$

which shows

$$\begin{aligned} (b - a)P([\tau_a \leq \tau_b]) & \geq (b - a)P([\tau_a \leq \tau_b] \cap [M^* > 0]) \\ & \geq bP([M^* > 0]) \end{aligned} \quad (50.9.44)$$

Combining these inequalities gives

$$(b - a)P([\tau_a < \tau_b]) \leq$$

$$bP([M^* > 0]) \leq (b - a)P([\tau_a \leq \tau_b]). \quad (50.9.45)$$

Note that if you knew that it is not possible that τ_a could ever equal τ_b this would yield an equality

$$bP([M^* > 0]) = (b - a)P([\tau_a < \tau_b]).$$

Using the same reasoning, it is possible to estimate

$$P([\tau_b < \tau_a]), \quad P([\tau_b \leq \tau_a]).$$

Begin with 50.9.41 again and use the inequality 50.9.42 again. Thus

$$\begin{aligned} 0 &\leq aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a]) + bP([\tau_a = \tau_b] \cap [M^* > 0]) \\ &= a(P([M^* > 0]) - P([\tau_b \leq \tau_a] \cap [M^* > 0])) + bP([\tau_b \leq \tau_a] \cap [M^* > 0]) \end{aligned}$$

and so

$$\begin{aligned} (b - a)P([\tau_b \leq \tau_a]) &\geq \\ (b - a)P([\tau_b \leq \tau_a] \cap [M^* > 0]) &\geq -aP([M^* > 0]) \end{aligned} \quad (50.9.46)$$

Using the other half of the inequality 50.9.42 this yields

$$\begin{aligned} 0 &\geq aP([\tau_a < \tau_b]) + bP([\tau_b < \tau_a]) + aP([\tau_a = \tau_b] \cap [M^* > 0]) \\ &= aP([\tau_a \leq \tau_b] \cap [M^* > 0]) + bP([\tau_b < \tau_a]) \\ &= a(P([M^* > 0]) - P([\tau_b < \tau_a])) + bP([\tau_b < \tau_a]) \end{aligned}$$

and so

$$aP([M^* > 0]) \leq -(b - a)P([\tau_b < \tau_a])$$

which implies

$$-aP([M^* > 0]) \geq (b - a)P([\tau_b < \tau_a])$$

combining these yields

$$(b - a)P([\tau_b \leq \tau_a]) \geq -aP([M^* > 0]) \geq (b - a)P([\tau_b < \tau_a])$$

This proves Theorem 50.9.3.

Note $P([\tau_a < \tau_b])$ means $M(t)$ hits a before it hits b with other occurrences of similar expressions being defined similarly.

50.10 The Space $\mathcal{M}_T^p(E)$

Here $p > 1$.

Definition 50.10.1 Let M be an E valued martingale. Then $M \in \mathcal{M}_T^p(E)$ if $t \rightarrow M(t)(\omega)$ is continuous for a.e. ω and

$$E \left(\sup_{t \in [0, T]} \|M(t)\|^p \right) < \infty$$

Here E is a separable Banach space.

Proposition 50.10.2 Define a norm on $\mathcal{M}_T^p(E)$ by

$$\|M\|_{\mathcal{M}_T^p(E)} \equiv E \left(\sup_{t \in [0, T]} \|M(t)\|^p \right)^{1/p}.$$

Then with this norm, $\mathcal{M}_T^p(E)$ is a Banach space.

Proof: First it is good to observe that $\sup_{t \in [0, T]} \|M(t)\|^p$ is measurable. This follows because of the continuity of $t \rightarrow M(t)$. Let D be a dense countable set in $[0, T]$. Then by continuity,

$$\sup_{t \in [0, T]} \|M(t)\|^p = \sup_{t \in D} \|M(t)\|^p$$

and the expression on the right is measurable because D is countable.

Next it is necessary to show this is a norm. It is clear that

$$\|M\|_{\mathcal{M}_T^p(E)} \geq 0$$

and equals 0 only if $E(\|M(t)\|^p) = 0$ for all t which requires $M(t) = 0$ for each t so that $M = 0$. It is also clear that

$$\|\alpha M\|_{\mathcal{M}_T^p(E)} = |\alpha| \|M\|_{\mathcal{M}_T^p(E)}.$$

It remains to check the triangle inequality. Let $M, N \in \mathcal{M}_T^p(E)$.

$$\begin{aligned} \|M + N\|_{\mathcal{M}_T^p(E)} &\equiv E \left(\sup_{t \in [0, T]} \|M(t) + N(t)\|^p \right)^{1/p} \\ &\leq E \left(\sup_{t \in [0, T]} (\|M(t)\| + \|N(t)\|)^p \right)^{1/p} \\ &\leq E \left(\left(\sup_{t \in [0, T]} \|M(t)\| + \sup_{t \in [0, T]} \|N(t)\| \right)^p \right)^{1/p} \\ &\equiv \left(\int_{\Omega} \left(\sup_{t \in [0, T]} \|M(t)\| + \sup_{t \in [0, T]} \|N(t)\| \right)^p dP \right)^{1/p} \\ &\leq \left(\int_{\Omega} \left(\sup_{t \in [0, T]} \|M(t)\| \right)^p dP \right)^{1/p} + \left(\int_{\Omega} \left(\sup_{t \in [0, T]} \|N(t)\| \right)^p dP \right)^{1/p} \\ &\equiv \|M\|_{\mathcal{M}_T^p(E)} + \|N\|_{\mathcal{M}_T^p(E)} \end{aligned}$$

Next consider the claim that $\mathcal{M}_T^p(E)$ is a Banach space. Let $\{M_n\}$ be a Cauchy sequence. Then

$$E \left(\sup_{t \in [0, T]} \|M_n(t) - M_m(t)\|^p \right) \rightarrow 0 \quad (50.10.47)$$

as $m, n \rightarrow \infty$. From Theorem 50.3.5

$$P \left(\sup_{t \in [0, T]} \|M_n(t) - M_m(t)\| \geq \lambda \right) \leq \frac{1}{\lambda^p} E (\|M_n(T) - M_m(T)\|^p)$$

Therefore, one can extract a subsequence $\{M_{n_k}\}$ such that

$$P \left(\sup_{t \in [0, T]} \|M_{n_k}(t) - M_{n_{k+1}}(t)\| \geq 2^{-k} \right) \leq 2^{-k}.$$

By the Borel Cantelli lemma, it follows $\{M_{n_k}(t)(\omega)\}$ converges uniformly on $[0, T]$ for a.e. ω . Denote by $M(t)(\omega)$ the thing to which it converges, a continuous process because of the uniform convergence. Also, from 50.10.47 and Fatou's lemma

$$\begin{aligned} & \int_{\Omega} \sup_{t \in [0, T]} \|M_n(t) - M(t)\|^p dP \\ & \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \sup_{t \in [0, T]} \|M_n(t) - M_{n_k}(t)\|^p dP \leq \varepsilon \end{aligned}$$

whenever n is large enough. Thus

$$\lim_{n \rightarrow \infty} E \left(\sup_{t \in [0, T]} \|M_n(t) - M(t)\|^p \right) = 0$$

and so for each t , $M_n(t) \rightarrow M(t)$ in $L^p(\Omega)$. This also shows

$$E \left(\sup_{t \in [0, T]} \|M(t)\|^p \right) < \infty.$$

It only remains to verify M is a martingale. Let $s \leq t$ and let $B \in \mathcal{F}_s$. Then from what was just observed that $M_n(t)$ converges to $M(t)$ in $L^p(\Omega)$ for each t ,

$$\begin{aligned} \int_B M(s) dP &= \lim_{n \rightarrow \infty} \int_B M_n(s) dP \\ &= \lim_{n \rightarrow \infty} \int_B E(M_n(t) | \mathcal{F}_s) dP \\ &= \lim_{n \rightarrow \infty} \int_B M_n(t) dP \\ &= \int_B M(t) dP \end{aligned}$$

and so by definition, $E(M(t) | \mathcal{F}_s) = M(s)$ which shows M is a martingale. This proves the proposition.

Proposition 50.10.3 $\mathcal{M}_T^p(E)$ is equi integrable.

Proof: This follows because

$$\int_{[\|M(t)\| \geq \lambda]} \|M(t)\|^p dP \leq \int_{[\sup_{t \in [0, T]} \|M(t)\| \geq \lambda]} \left(\sup_{t \in [0, T]} \|M(t)\|^p \right) dP \quad (50.10.48)$$

which converges to 0 due to the definition of $\mathcal{M}_T^p(E)$ which requires that

$$\sup_{t \in [0, T]} \|M(t)\|^p \in L^1(\Omega, \mathcal{F}, P).$$

Since the sets $[\sup_{t \in [0, T]} \|M(t)\| \geq \lambda]$ decrease to $\emptyset$ as $\lambda \rightarrow \infty$, the dominated convergence theorem implies the integral on the right in 50.10.48 converges to 0. This proves the proposition.

The Quadratic Variation Of A Martingale

51.1 How To Recognize A Martingale

The main ideas are most easily understood in the special case where it is assumed the martingale is bounded. Then one can extend to more general situations.

Let $\{M(t)\}$ be a continuous martingale having values in a separable Hilbert space. The idea is to consider the submartingale, $\{\|M(t)\|^2\}$ and write it as the sum of a martingale and a submartingale. An important part of the argument is the following lemma which gives a checkable criterion for a stochastic process to be a martingale.

Lemma 51.1.1 *Let $\{X(t)\}$ be a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$ for $t \geq 0$. Then it is a martingale for the given filtration if for every stopping time, σ it follows*

$$E(X(t)) = E(X(\sigma)).$$

In fact, it suffices to check this on stopping times which have two values.

Proof: Let $s < t$ and $A \in \mathcal{F}_s$. Define a stopping time

$$\sigma(\omega) \equiv s\mathcal{X}_A(\omega) + t\mathcal{X}_{A^c}(\omega)$$

Then

$$\int X(t) dP = \int X(\sigma) dP = \int_A X(s) dP + \int_{A^c} X(t) dP$$

Therefore,

$$\int_A X(t) dP = \int_A X(s) dP$$

and since $X(s)$ is $\mathcal{F}_s$ measurable, it follows $E(X(t) | \mathcal{F}_s) = X(s)$ a.e. and this shows $\{X(t)\}$ is a martingale.

The following lemma will be useful.

Lemma 51.1.2 Suppose $X_n \rightarrow X$ in $L^1(\Omega, \mathcal{F}, P; E)$ where E is a separable Banach space. Then letting $\mathcal{G}$ be a σ algebra contained in $\mathcal{F}$,

$$E(X_n | \mathcal{G}) \rightarrow E(X | \mathcal{G})$$

in $L^1(\Omega)$.

Proof: This follows from the definitions and Theorem 49.1.1 on Page 1403.

$$\begin{aligned} \int_{\Omega} \|E(X | \mathcal{G}) - E(X_n | \mathcal{G})\| dP &= \int_{\Omega} \|E(X_n - X | \mathcal{G})\| dP \\ &\leq \int_{\Omega} E(\|X_n - X\| | \mathcal{G}) dP \\ &= \int_{\Omega} \|X_n - X\| dP \end{aligned}$$

which converges to 0.

Corollary 51.1.3 Let X, Y be in $L^2(\Omega, \mathcal{F}, P; H)$ where H is a separable Hilbert space and let X be $\mathcal{G}$ measurable where $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E((X, Y) | \mathcal{G}) = (X, E(Y | \mathcal{G})) \text{ a.e.}$$

Proof: Let X_n be a sequence of $\mathcal{G}$ measurable simple functions converging pointwise to X such that $\|X_n\| \leq \|X\|$. Say $X_n = \sum_{k=1}^m c_k \mathcal{X}_{F_k}$. Then

$$\begin{aligned} E((X_n, Y) | \mathcal{G}) &= E\left(\left(\sum_{k=1}^m c_k \mathcal{X}_{F_k}, Y\right) | \mathcal{G}\right) \\ &= \sum_{k=1}^m \mathcal{X}_{F_k} E((c_k, Y) | \mathcal{G}) \end{aligned} \tag{51.1.1}$$

Now I claim $E((c_k, Y) | \mathcal{G}) = (c_k, E(Y | \mathcal{G}))$ a.e. Let $A \in \mathcal{G}$. Then

$$\int_A E((c_k, Y) | \mathcal{G}) dP = \int_A (c_k, Y) dP$$

while

$$\begin{aligned} \int_A (c_k, E(Y | \mathcal{G})) &= \left(c_k, \int_A E(Y | \mathcal{G})\right) dP \\ &= \left(c_k, \int_A Y dP\right) = \int_A (c_k, Y) dP \end{aligned}$$

Since A is arbitrary these must be the same a.e. Thus 51.1.1 equals

$$\left(\sum_{k=1}^m \mathcal{X}_{F_k} c_k, E(Y | \mathcal{G})\right) = (X_n, E(Y | \mathcal{G}))$$

and so the desired result follows for each of these simple functions approximating X . Now let $A \in \mathcal{G}$. Then by Lemma 51.1.2 above,

$$\begin{aligned} \int_A (X, E(Y|\mathcal{G})) dP &= \lim_{n \rightarrow \infty} \int_A (X_n, E(Y|\mathcal{G})) dP \\ &= \lim_{n \rightarrow \infty} \int_A E((X_n, Y) | \mathcal{G}) dP \\ &= \int_A E((X, Y) | \mathcal{G}) dP. \end{aligned}$$

This proves the corollary.

The following is related to something called a martingale transform.

Proposition 51.1.4 *Let $\{\tau_k\}$ be an increasing sequence of stopping times for the filtration $\{\mathcal{F}_t\}$ such that*

$$\lim_{k \rightarrow \infty} \tau_k = \infty.$$

in the case where $t \in [0, \infty)$ and

$$\tau_m \geq T$$

for all m large enough in the case where $t \in [0, T]$. Also let ξ_k be $\mathcal{F}_{\tau_k}$ measurable with values in H , a separable Hilbert space and let $M(t)$ be a continuous martingale adapted to the normal filtration $\mathcal{F}_t$ which has the property that $M(t) \in L^2(\Omega; H)$ for all t and $M(0) = 0$. Then if $|\xi_k| \leq C$,

$$\begin{aligned} E \left(\left(\sum_{k \geq 0} (\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t))) \right)^2 \right) \\ \leq C^2 E (||M(t)||^2) \end{aligned} \quad (51.1.2)$$

Proof: First of all, the sum converges because eventually $\tau_k \wedge t = t$. Therefore, for large enough k , $M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t) = 0$. When the sum on the left is multiplied out, you get mixed terms. Consider one of these mixed terms

$$E((\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t))) (\xi_j, (M(\tau_{j+1} \wedge t) - M(\tau_j \wedge t))))$$

where $j < k$. Using Corollary 51.1.3 this equals

$$\begin{aligned} E(E((\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t))) (\xi_j, (M(\tau_{j+1} \wedge t) - M(\tau_j \wedge t))) | \mathcal{F}_{\tau_k})) \\ = E((\xi_j, (M(\tau_{j+1} \wedge t) - M(\tau_j \wedge t))) (\xi_k, E((M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t)) | \mathcal{F}_{\tau_k}))) \end{aligned}$$

and by Doob's optional sampling theorem, Theorem 48.4.4, the above equals

$$E((\xi_j, (M(\tau_{j+1} \wedge t) - M(\tau_j \wedge t))) (\xi_k, (M(\tau_{k+1} \wedge t \wedge \tau_k) - M(\tau_k \wedge t))))$$

which equals 0. Note that in using the optional sampling theorem, the stopping time $\tau_{k+1} \wedge t$ is bounded. Therefore, the only terms which survive in 51.1.2 are the non mixed terms and the expression on the left in 51.1.2 reduces to

$$\begin{aligned}
 & \sum_{k \geq 0} E \left(((\xi_k, (M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t))))^2 \right) \\
 & \leq C^2 \sum_{k \geq 0} E \left(\|M(\tau_{k+1} \wedge t) - M(\tau_k \wedge t)\|^2 \right) \\
 & = C^2 \sum_{k \geq 0} E \left(\|M(\tau_{k+1} \wedge t)\|^2 \right) + E \left(\|M(\tau_k \wedge t)\|^2 \right) \\
 & \quad - 2E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t))) \tag{51.1.3}
 \end{aligned}$$

Consider the term $E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t)))$. By Doob's optional sampling theorem for martingales again, this equals

$$\begin{aligned}
 & E(E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t)) | \mathcal{F}_{\tau_k})) \\
 & = E((M(\tau_k \wedge t), E(M(\tau_{k+1} \wedge t) | \mathcal{F}_{\tau_k}))) \\
 & = E((M(\tau_k \wedge t), M(\tau_{k+1} \wedge t \wedge \tau_k))) \\
 & = E(\|M(\tau_k \wedge t)\|^2)
 \end{aligned}$$

It follows 51.1.3 equals

$$C^2 \sum_{k \geq 0} E \left(\|M(\tau_{k+1} \wedge t)\|^2 \right) - E \left(\|M(\tau_k \wedge t)\|^2 \right) = C^2 E \left(\|M(t)\|^2 \right).$$

This proves the proposition.

Now here is an interesting lemma which will be used to prove uniqueness in the main result.

Lemma 51.1.5 *Let $\mathcal{F}_t$ be a normal filtration and let $A(t), B(t)$ be adapted to $\mathcal{F}_t$, continuous, and increasing with $A(0) = B(0) = 0$ and suppose $A(t) - B(t)$ is a martingale. Then $A(t) - B(t) = 0$ for all t .*

Proof: I shall show $A(l) = B(l)$ where l is arbitrary. Let $M(t)$ be the name of the martingale. Define a stopping time

$$\tau \equiv \inf \{t > 0 : |M(t)| > C\} \wedge l$$

where $\inf(\emptyset) \equiv \infty$ and denote the stopped martingale

$$M^\tau(t) \equiv M(t \wedge \tau).$$

Then I claim this is also a martingale with respect to the filtration $\mathcal{F}_t$ because by Doob's optional sampling theorem for martingales, if $s < t$,

$$E(M^\tau(t) | \mathcal{F}_s) \equiv E(M(\tau \wedge t) | \mathcal{F}_s) = M(\tau \wedge t \wedge s) = M(\tau \wedge s) = M^\tau(s)$$

Note the bounded stopping time is $\tau \wedge t$ and the other one is $\sigma = s$ in this theorem. Then M^τ is a continuous martingale which is also uniformly bounded. It equals $A^\tau - B^\tau$. Since τ is also bounded being no larger than l , it must be the case that since A and B are both increasing, A^τ, B^τ are bounded uniformly also. Let D be a large enough constant that it bounds all of $|M^\tau(t)|, B^\tau(t), A^\tau(t)$ on $[0, l]$. Now let $\mathcal{P}_n \equiv \{t_k\}_{k=1}^n$ be a uniform partition of $[0, l]$ and let $M^\tau(\mathcal{P}_n)$ denote

$$\max \{|M^\tau(t_{i+1}) - M^\tau(t_i)|\}_{i=1}^n.$$

Then

$$E\left(M^\tau(l)^2\right) = E\left(\left(\sum_{k=0}^{n-1} M^\tau(t_{k+1}) - M^\tau(t_k)\right)^2\right)$$

Now consider a mixed term in the sum where $j < k$.

$$\begin{aligned} & E((M^\tau(t_{k+1}) - M^\tau(t_k))(M^\tau(t_{j+1}) - M^\tau(t_j))) \\ &= E(E((M^\tau(t_{k+1}) - M^\tau(t_k))(M^\tau(t_{j+1}) - M^\tau(t_j)) | \mathcal{F}_{t_k})) \\ &= E((M^\tau(t_{j+1}) - M^\tau(t_j)) E((M^\tau(t_{k+1}) - M^\tau(t_k)) | \mathcal{F}_{t_k})) \\ &= E((M^\tau(t_{j+1}) - M^\tau(t_j))(M^\tau(t_k) - M^\tau(t_k))) = 0 \end{aligned}$$

It follows

$$\begin{aligned} E\left(M^\tau(l)^2\right) &= E\left(\sum_{k=0}^{n-1} (M^\tau(t_{k+1}) - M^\tau(t_k))^2\right) \\ &\leq E\left(\sum_{k=0}^{n-1} M^\tau(\mathcal{P}_n) |M^\tau(t_{k+1}) - M^\tau(t_k)|\right) \\ &\leq E\left(\sum_{k=0}^{n-1} M^\tau(\mathcal{P}_n) (|A^\tau(t_{k+1}) - A^\tau(t_k)| + |B^\tau(t_{k+1}) - B^\tau(t_k)|)\right) \\ &\leq E\left(M^\tau(\mathcal{P}_n) \sum_{k=0}^{n-1} (|A^\tau(t_{k+1}) - A^\tau(t_k)| + |B^\tau(t_{k+1}) - B^\tau(t_k)|)\right) \\ &\leq E(M^\tau(\mathcal{P}_n) 2D) \end{aligned}$$

the last step holding because A and B are increasing. Now letting $n \rightarrow \infty$, the right side converges to 0 by the dominated convergence theorem and the observation that for a.e. ω ,

$$\lim_{n \rightarrow \infty} M^\tau(\mathcal{P}_n)(\omega) = 0$$

and $M^\tau(\mathcal{P}_n) \leq 2D$. Thus for $\tau = \tau_C$ given above,

$$M(l \wedge \tau_C) = 0 \text{ a.e.}$$

Since M is a continuous martingale, it follows that for a.e. ω , $|M(t)(\omega)| \leq C$ for all $t \in [0, l]$ provided C is large enough and so, $M(l \wedge \tau_C)$ reduces to $M(l)$. Thus $M(l) = 0$ a.e. This proves the lemma.

The following proposition is the main result. One can generalize it in many ways.

Proposition 51.1.6 *Let $\{M(t)\}$ be a continuous martingale having values in H a separable Hilbert space adapted to the normal filtration $\{\mathcal{F}_t\}$ such that $\|M(t)(\omega)\|$ is bounded and $M(0) = 0$. Then there exists a unique continuous, increasing, nonnegative, submartingale $\{[M](t)\}$ called the quadratic variation such that*

$$\|M(t)\|^2 - [M](t)$$

is a real martingale and $[M](0) = 0$. Here $t \in [0, T]$. Also letting

$$N(t) \equiv \|M(t)\|^2 - [M](t),$$

it follows

$$\|N(t)\|_{L^2(\Omega)} \leq 2 \|M(t)\|_{L^2(\Omega)}. \quad (51.1.4)$$

Proof: Define stopping times $\tau_0^n \equiv \eta_0^n \equiv 0$. Then

$$\eta_{k+1}^n \equiv \inf \{T \geq s \geq \eta_k^n : \|M(s) - M(\eta_k^n)\| = 2^{-n}\}, \quad \tau_k^n \equiv \eta_k^n \wedge T$$

where $\inf \emptyset \equiv \infty$. These are stopping times by Example 50.5.5 on Page 1494. Then for $t > 0$ and fixed ω , eventually for large enough k , $t \in (\tau_k^n(\omega), \tau_{k+1}^n(\omega)]$. Here is why. The sequence $\{\eta_k^n(\omega)\}_{k=1}^\infty$ eventually equals ∞ . This is because if it did not, it would converge, being bounded above and then by continuity of M , $\{M(\eta_k^n(\omega))\}_{k=1}^\infty$ would be a Cauchy sequence contrary to the requirement that

$$\|M(\eta_{k+1}^n(\omega)) - M(\eta_k^n(\omega))\| = 2^{-n}.$$

Thus

$$M(t) \equiv \sum_{k \geq 0} M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n) \quad (51.1.5)$$

and the terms of the series are eventually 0, as soon as $\eta_k^n = \infty$. Therefore,

$$\|M(t)\|^2 = \left\| \sum_{k \geq 0} M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n) \right\|^2$$

Then this equals

$$\begin{aligned} &= \sum_{k \geq 0} \|M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)\|^2 \\ &\quad + \sum_{j \neq k} ((M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)), (M(t \wedge \tau_{j+1}^n) - M(t \wedge \tau_j^n))) \end{aligned}$$

Consider the second sum. It equals

$$\begin{aligned}
 & 2 \sum_{k \geq 0} \sum_{j=0}^{k-1} \left((M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)), (M(t \wedge \tau_{j+1}^n) - M(t \wedge \tau_j^n)) \right) \\
 &= 2 \sum_{k \geq 0} \left((M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)), \sum_{j=0}^{k-1} (M(t \wedge \tau_{j+1}^n) - M(t \wedge \tau_j^n)) \right) \\
 &= 2 \sum_{k \geq 0} \left((M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)), M(t \wedge \tau_k^n) \right)
 \end{aligned}$$

Now for $t \in (\tau_m^n, \tau_{m+1}^n]$, $M(t \wedge \tau_k^n) = M(\tau_k^n)$ whenever $k \leq m$. However, for $k \geq m+1$,

$$M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n) = M(t) - M(t) = 0$$

and so this last sum equals

$$2 \sum_{k \geq 0} (M(\tau_k^n), (M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n))) \equiv P_n(t)$$

By adding in the values of $\{\tau_k^{n+1}\}$ $P_n(t)$ can be written in the form

$$2 \sum_{k \geq 0} (M(\tau_k^{n+1'}), (M(t \wedge \tau_{k+1}^{n+1}) - M(t \wedge \tau_k^{n+1})))$$

where $\tau_k^{n+1'}$ has some repeats. From the definition,

$$\|M(\tau_k^{n+1'}) - M(\tau_k^{n+1})\| \leq 2^{-n}$$

and so from Proposition 51.1.4

$$E\left(\|P_n(t) - P_{n+1}(t)\|^2\right) \leq 2^{-2n} E\left(\|M(t)\|^2\right).$$

Note this shows that since $P_0(t) = 0$,

$$\begin{aligned}
 \|P_n(t)\|_{L^2(\Omega)} &\leq \sum_{k=0}^{n-1} \|P_{k+1}(t) - P_k(t)\|_{L^2(\Omega)} \\
 &\leq \sum_{k=0}^{n-1} 2^{-k} \|M(t)\|_{L^2(\Omega)} \leq 2 \|M(t)\|_{L^2(\Omega)} \quad (51.1.6)
 \end{aligned}$$

Now $t \rightarrow P_n(t)$ is continuous because it is a finite sum of continuous functions. It is also the case that $\{P_n(t)\}$ is a martingale. To see this use Lemma 51.1.1. Let

σ be a stopping time having two values. Then using Corollary 51.1.3 and the Doob optional sampling theorem, Theorem 50.5.8

$$\begin{aligned}
 & E \left(\sum_{k \geq 0} (M(\tau_k^n), (M(\sigma \wedge \tau_{k+1}^n) - M(\sigma \wedge \tau_k^n))) \right) \\
 &= \sum_{k \geq 0} E((M(\tau_k^n), (M(\sigma \wedge \tau_{k+1}^n) - M(\sigma \wedge \tau_k^n)))) \\
 &= \sum_{k \geq 0} E((E(M(\tau_k^n), (M(\sigma \wedge \tau_{k+1}^n) - M(\sigma \wedge \tau_k^n))) | \mathcal{F}_{\tau_k^n})) \\
 &= \sum_{k \geq 0} E((M(\tau_k^n), E(M(\sigma \wedge \tau_{k+1}^n) - M(\sigma \wedge \tau_k^n)) | \mathcal{F}_{\tau_k^n})) \\
 &= \sum_{k \geq 0} E((M(\tau_k^n), E(M(\sigma \wedge \tau_{k+1}^n \wedge \tau_k^n) - M(\sigma \wedge \tau_k^n)))) = 0
 \end{aligned}$$

Note the Doob theorem applies because $\sigma \wedge \tau_{k+1}^n$ is a bounded stopping time due to the fact σ has only two values. You don't need to use the fact that τ_k^n is bounded. Similarly

$$\begin{aligned}
 & E \left(\sum_{k \geq 0} (M(\tau_k^n), (M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n))) \right) \\
 &= \sum_{k \geq 0} E((M(\tau_k^n), (M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)))) \\
 &= \sum_{k \geq 0} E((E(M(\tau_k^n), (M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n))) | \mathcal{F}_{\tau_k^n})) \\
 &= \sum_{k \geq 0} E((M(\tau_k^n), E(M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)) | \mathcal{F}_{\tau_k^n})) \\
 &= \sum_{k \geq 0} E((M(\tau_k^n), E(M(t \wedge \tau_{k+1}^n \wedge \tau_k^n) - M(t \wedge \tau_k^n)))) = 0
 \end{aligned}$$

Note the Doob theorem applies because $t \wedge \tau_{k+1}^n$ is a bounded stopping time.

Thus $\{P_n(t)\}$ is a Cauchy sequence in $\mathcal{M}_T^2(\mathbb{R})$. Therefore, by Proposition 50.10.2 there exists $\{N(t)\} \in \mathcal{M}_T^2(\mathbb{R})$ such that $P_n \rightarrow N$ in $\mathcal{M}_T^2(H)$. That is

$$\lim_{n \rightarrow \infty} E \left(\sup_{t \in [0, T]} |P_n(t) - N(t)|^2 \right)^{1/2} = 0.$$

Also from 51.1.6

$$\|N(t)\|_{L^2(\Omega)} \leq 2 \|M(t)\|_{L^2(\Omega)}.$$

Since $\{N(t)\} \in \mathcal{M}_T^2(\mathbb{R})$ it is a continuous martingale and $N(t) \in L^2(\Omega)$, and $N(0) = 0$ because this is true of each $P_n(0)$. From the above,

$$\|M(t)\|^2 = Q_n(t) + P_n(t) \quad (51.1.7)$$

where

$$Q_n(t) = \sum_{k \geq 0} \|M(t \wedge \tau_{k+1}^n) - M(t \wedge \tau_k^n)\|^2$$

and $P_n(t)$ is a martingale. Then from 51.1.7, $Q_n(t)$ is a submartingale and converges for each t to something, denoted as $[M](t)$ in $L^1(\Omega)$ uniformly in $t \in [0, T]$. This is because $P_n(t)$ converges in $L^2(\Omega)$ to $N(t)$ and $\|M(t)\|^2$ converges to $\|M(t)\|^2$ in $L^1(\Omega)$. Then also $[M]$ is a submartingale which equals 0 at 0 because this is true of Q_n and because if $A \in \mathcal{F}_s$ where $s < t$,

$$\begin{aligned} \int_A E([M](t) | \mathcal{F}_s) dP &\equiv \int_A [M](t) dP = \lim_{n \rightarrow \infty} \int_A (\|M(t)\|^2 - P_n(t)) dP \\ &= \lim_{n \rightarrow \infty} \int_A E(\|M(t)\|^2 - P_n(t) | \mathcal{F}_s) dP \geq \lim_{n \rightarrow \infty} \inf \int_A \|M(s)\|^2 - P_n(s) dP \\ &= \lim_{n \rightarrow \infty} \inf \int_A Q_n(s) dP = \int_A [M](s) dP. \end{aligned}$$

Note that $Q_n(t)$ is increasing because as t increases, the definition allows for the possibility of more nonzero terms in the sum. Therefore, $[M](t)$ is also increasing in t . The function $t \rightarrow [M](t)$ is continuous because $\|M(t)\|^2 = [M](t) + N(t)$ and $t \rightarrow N(t)$ is continuous as is $t \rightarrow \|M(t)\|^2$. That is, off a set of measure zero, these are both continuous functions of t and so the same is true of $[M]$.

It remains to consider the claim about the uniqueness. Suppose then there are two which work, $[M]$, and $[M]_1$. Then $[M] - [M]_1$ equals a martingale, G which is 0 when $t = 0$. Thus the uniqueness assertion follows from Lemma 51.1.5. This proves the proposition.

Here is a corollary which tells how to manipulate stopping times.

Corollary 51.1.7 *In the situation of Proposition 51.1.6 let τ be a stopping time. Then*

$$[M^\tau] = [M]^\tau.$$

Proof:

$$[M]^\tau(t) + N_1(t) = \left(\|M\|^2\right)^\tau(t) = \|M^\tau\|^2(t) = [M^\tau](t) + N_2(t)$$

where N_i is a martingale. Therefore,

$$[M]^\tau(t) - [M^\tau](t) = N_2(t) - N_1(t),$$

a martingale. Therefore, by Lemma 51.1.5, this shows $[M]^\tau(t) - [M^\tau](t) = 0$. This proves the corollary.

Definition 51.1.8 *The covariation of two continuous H valued martingales for H a separable Hilbert space M, N is defined as follows.*

$$[M, N] \equiv \frac{1}{4} ([M + N] - [M - N])$$

Thus $[M] = [M, M]$. In writing this formula, it is only assumed M is such that $[M]$ has been defined.

Corollary 51.1.9 *Let M, N be two continuous bounded martingales, $M(0) = N(0) = 0$, as in Proposition 51.1.6. Then $[M, N]$ is of bounded variation and*

$$(M, N)_H - [M, N]$$

is a martingale. Also for τ a stopping time,

$$[M, N]^\tau = [M^\tau, N^\tau] = [M^\tau, N] = [M, N^\tau].$$

In addition to this,

$$[M - M^\tau] = [M] - [M^\tau] \leq [M]$$

and also

$$(M, N) \rightarrow [M, N]$$

is bilinear and symmetric.

Proof: Since $[M, N]$ is the difference of increasing functions, it is of bounded variation.

$$\begin{aligned} (M, N)_H - [M, N] &= \frac{1}{4} \left(\|M + N\|^2 - \|M - N\|^2 \right) \\ &\quad - \frac{1}{4} ([M + N] - [M - N]) \end{aligned}$$

which equals a martingale from the definition of $[M + N]$ and $[M - N]$. It remains to verify the claim about the stopping time. Using Corollary 51.1.7

$$\begin{aligned} [M, N]^\tau &= \frac{1}{4} ([M + N] - [M - N])^\tau \\ &= \frac{1}{4} ([M + N]^\tau - [M - N]^\tau) \\ &= \frac{1}{4} ([M^\tau + N^\tau] - [M^\tau - N^\tau]) \equiv [M^\tau, N^\tau]. \end{aligned}$$

The really interesting part is the next equality. This will involve Lemma 51.1.5.

$$[M, N]^\tau - [M^\tau, N] = [M^\tau, N^\tau] - [M^\tau, N]$$

$$\begin{aligned}
&= \frac{1}{4} ([M^\tau + N^\tau] - [M^\tau - N^\tau]) - \frac{1}{4} ([M^\tau + N] - [M^\tau - N]) \\
&= \frac{1}{4} ([M^\tau + N^\tau] + [M^\tau - N]) - \frac{1}{4} ([M^\tau + N] + [M^\tau - N^\tau]), \quad (51.1.8)
\end{aligned}$$

the difference of two increasing adapted processes. Also, this equals

$$\text{martingale} + (M^\tau, N) - (M^\tau, N^\tau)$$

I claim $(M^\tau, N) - (M^\tau, N^\tau) = (M^\tau, N - N^\tau)$ is a martingale. This follows from Lemma 51.1.1. Let σ be a stopping time with two values. By the Doob optional sampling theorem,

$$\begin{aligned}
E((M^\tau(\sigma), N(\sigma) - N^\tau(\sigma))) &= E(E((M^\tau(\sigma), N(\sigma) - N^\tau(\sigma)) | \mathcal{F}_\tau)) \\
&= E(M^\tau(\sigma), E(N(\sigma) - N^\tau(\sigma) | \mathcal{F}_\tau)) \\
&= E(M^\tau(\sigma), N(\sigma \wedge \tau) - N(\tau \wedge \sigma)) = 0
\end{aligned}$$

while

$$\begin{aligned}
E((M^\tau(t), N(t) - N^\tau(t))) &= E(E((M^\tau(t), N(t) - N^\tau(t)) | \mathcal{F}_\tau)) \\
&= E((M^\tau(t), E(N(t) - N^\tau(t) | \mathcal{F}_\tau))) \\
&= E((M^\tau(t), E(N(t \wedge \tau) - N(\tau \wedge t)))) = 0
\end{aligned}$$

This shows the claim is true. Now from 51.1.8 and Lemma 51.1.5,

$$[M, N]^\tau - [M^\tau, N] = 0.$$

Similarly

$$[M, N]^\tau - [M, N^\tau]$$

Now consider the last claim. From the definition, it follows

$$[M - M^\tau] - ([M] + [M^\tau] - 2[M, M^\tau]) = \text{a martingale}.$$

By the first part of the corollary, the left side is the difference of two increasing adapted processes and so the left side equals 0. Thus from what was just shown,

$$\begin{aligned}
[M - M^\tau] &= [M] + [M^\tau] - 2[M, M^\tau] \\
&= [M] + [M^\tau] - 2[M^\tau, M^\tau] \\
&= [M] + [M^\tau] - 2[M^\tau] \\
&= [M] - [M^\tau] \leq [M]
\end{aligned}$$

Finally consider the claim that $[M, N]$ is bilinear. From the definition, letting M_1, M_2, N be bounded H valued martingales,

$$\begin{aligned}
(aM_1 + bM_2, N)_H &= [aM_1 + bM_2, N] + \text{martingale} \\
a(M_1, N) + b(M_2, N)_H &= a[M_1, N] + b[M_2, N] + \text{martingale}
\end{aligned}$$

Hence

$$[aM_1 + bM_2, N] - (a[M_1, N] + b[M_2, N]) = \text{martingale}.$$

The left side can be written as the difference of two increasing functions and so by Lemma 51.1.5 it equals 0. $[M, N]$ is obviously symmetric. This proves the corollary.

51.2 The Burkholder Davis Gundy Inequality

The next topic is the Burkholder Davis Gundy inequality. Before presenting this, here is the good lambda inequality, Theorem 9.7.7 on Page 250 listed here for convenience.

Theorem 51.2.1 *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let F be a continuous increasing function defined on $[0, \infty)$ such that F is continuous at 0 and $F(0) = 0$. Suppose also that for all $\alpha > 1$, there exists a constant C_α such that for all $x \in [0, \infty)$,*

$$F(\alpha x) \leq C_\alpha F(x).$$

Also suppose f, g are nonnegative measurable functions and there exists $\beta > 1$ such that for all $\lambda > 0$ and $1 > \delta > 0$,

$$\mu([f > \beta\lambda] \cap [g \leq \delta\lambda]) \leq \phi(\delta) \mu([f > \lambda]) \quad (51.2.9)$$

where $\lim_{\delta \rightarrow 0+} \phi(\delta) = 0$ and ϕ is increasing. Under these conditions, there exists a constant C depending only on β, ϕ such that

$$\int_{\Omega} F(f(\omega)) d\mu(\omega) \leq C \int_{\Omega} F(g(\omega)) d\mu(\omega).$$

Then the Burkholder Davis Gundy inequality is as follows. Generalizations will be presented later.

Theorem 51.2.2 *Let $\{M(t)\}$ be a continuous H valued martingale which is uniformly bounded, $M(0) = 0$, where H is a separable Hilbert space and $t \in [0, T]$. Then if F is a function of the sort described in the good lambda inequality above, there are constants, C and c independent of such martingales M such that*

$$c \int_{\Omega} F([M]^{1/2}(T)) dP \leq \int_{\Omega} F(M^*) dP \leq C \int_{\Omega} F([M]^{1/2}(T)) dP$$

where

$$M^*(\omega) \equiv \sup \{ \|M(t)(\omega)\| : t \in [0, T] \}.$$

Proof: Using Corollary 51.1.9, let

$$\begin{aligned} N(t) &\equiv \|M(t) - M^\tau(t)\|^2 - [M - M^\tau](t) \\ &= \|M(t) - M^\tau(t)\|^2 - [M](t) + [M]^\tau(t) \end{aligned}$$

where

$$\tau \equiv \inf \{t \in [0, T] : \|M(t)\| > \lambda\}$$

Thus N is a martingale and $N(0) = 0$. In fact $N(t) = 0$ as long as $t \leq \tau$. As usual $\inf(\emptyset) \equiv \infty$. Note

$$[\tau < \infty] = [M^* > \lambda] \supseteq [N^* > 0].$$

This is because to say $\tau < \infty$ is to say there exists $t < T$ such that $\|M(t)\| > \lambda$ which is the same as saying $M^* > \lambda$. Thus the first two sets are equal. If $\tau = \infty$, then from the formula for $N(t)$ above, $N(t) = 0$ for all $t \in [0, T]$ and so it can't happen that $N^* > 0$. Thus the third set is contained in $[\tau < \infty]$ as claimed.

Let $\beta > 2$ and let $\delta \in (0, 1)$. Then

$$\beta - 1 > 1 > \delta > 0$$

Consider

$$S_r \equiv [M^* > \beta\lambda] \cap \left[[M]^{1/2}(T) \leq r\delta\lambda \right]$$

where $0 < r < 1$.

Claim: For $\omega \in S_r$, $N(t)$ hits $(\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2$.

Proof of claim: For $\omega \in S_r$, there exists a $t < T$ such that $\|M(t)\| > \beta\lambda$ and so using Corollary 51.1.9,

$$\begin{aligned} N(t) &\geq \|M(t)\| - \|M^\tau(t)\|^2 - [M - M^\tau](t) \geq |\beta\lambda - \lambda|^2 - [M](t) \\ &\geq (\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2 \end{aligned}$$

which shows that $N(t)$ hits $(\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2$ for $\omega \in S_r$. This proves the claim.

Claim: $N(t)(\omega)$ never hits $-\delta^2 \lambda^2$ for $\omega \in S_r$.

Proof of claim: Suppose t is the first time $N(t)$ reaches $-\delta^2 \lambda^2$. Then

$$0 \leq \|M(t) - M^\tau(t)\|^2 = [M - M^\tau](t) - \delta^2 \lambda^2$$

and by Corollary 51.1.9,

$$\leq [M](t) - \delta^2 \lambda^2 \leq r^2 \delta^2 \lambda^2 - \delta^2 \lambda^2 < 0,$$

a contradiction. This proves the claim.

Therefore, for all $\omega \in S_r$, $N(t)(\omega)$ reaches $(\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2$ before it reaches $-\delta^2 \lambda^2$. It follows

$$P(S_r) \leq P\left(N(t) \text{ reaches } (\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2 \text{ first.}\right)$$

and because of Theorem 50.9.3 this is no larger than

$$P([N^* > 0]) \frac{\delta^2 \lambda^2}{\left((\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2\right) - (-\delta^2 \lambda^2)} = P([N^* > 0]) \frac{\delta^2}{(\beta - 1)^2}.$$

Since this is true for each $r < 1$, it follows

$$P\left([M^* > \beta\lambda] \cap \left[[M]^{1/2}(T) \leq \delta\lambda\right]\right) \leq P([N^* > 0]) \frac{\delta^2}{(\beta-1)^2}$$

and as noted above, this implies

$$P\left([M^* > \beta\lambda] \cap \left[[M]^{1/2}(T) \leq \delta\lambda\right]\right) \leq P([M^* > \lambda]) \frac{\delta^2}{(\beta-1)^2}$$

By the good lambda inequality,

$$\int_{\Omega} F(M^*) dP \leq C \int_{\Omega} F\left([M]^{1/2}(T)\right) dP$$

which is one half the inequality.

Now consider the other half. This time define the stopping time τ by

$$\tau \equiv \inf \left\{ t \in [0, T] : [M]^{1/2}(t) > \lambda \right\}$$

and let

$$S_r \equiv \left[[M]^{1/2}(T) > \beta\lambda\right] \cap [2M^* \leq r\delta\lambda]$$

Then there exists $t < T$ such that $[M](t) > \beta^2\lambda^2$. This time, let

$$N(t) = [M](t) - [M^\tau](t) - \|M(t) - M^\tau(t)\|^2$$

This is still a martingale since by Corollary 51.1.9

$$[M](t) - [M^\tau](t) = [M - M^\tau](t)$$

Claim: $N(t)(\omega)$ hits $(\beta-1)^2\lambda^2 - \delta^2\lambda^2$ for some $t < T$ for $\omega \in S_r$.

Proof of claim: Fix such a $\omega \in S_r$. Let $t < T$ be such that $[M](t) > \beta^2\lambda^2$. Then $t > \tau$ and so for that ω ,

$$\begin{aligned} N(t) &> \beta^2\lambda^2 - \lambda^2 - \|M(t) - M(\tau)\|^2 \\ &\geq (\beta-1)^2\lambda^2 - (\|M(t)\| + \|M(\tau)\|)^2 \\ &\geq (\beta-1)^2\lambda^2 - r^2\delta^2\lambda^2 \geq (\beta-1)^2\lambda^2 - \delta^2\lambda^2 \end{aligned}$$

This proves the claim.

Claim: $N(t)(\omega)$ never hits $-\delta^2\lambda^2$ for $\omega \in S_r$.

Proof of claim: By Corollary 51.1.9, if it did at t , then $t > \tau$ and

$$\begin{aligned} 0 &\leq [M](t) - [M^\tau](t) = \|M(t) - M(\tau)\|^2 - \delta^2\lambda^2 \\ &\leq (\|M(t)\| + \|M(\tau)\|)^2 - \delta^2\lambda^2 \leq r^2\delta^2\lambda^2 - \delta^2\lambda^2 < 0, \end{aligned}$$

a contradiction. This proves the claim.

It follows that for each $r \in (0, 1)$,

$$P(S_r) \leq P\left(N(t) \text{ hits } (\beta - 1)^2 \lambda^2 - \delta^2 \lambda^2 \text{ before } -\delta^2 \lambda^2\right)$$

By Theorem 50.9.3 this is no larger than

$$\begin{aligned} P([N^* > 0]) \frac{\delta^2 \lambda^2}{(\beta - 1)^2 \lambda^2} &= P([N^* > 0]) \frac{\delta^2}{(\beta - 1)^2} \\ &\leq P([\tau < \infty]) \frac{\delta^2}{(\beta - 1)^2} = P\left([M]^{1/2}(T) > \lambda\right) \frac{\delta^2}{(\beta - 1)^2} \end{aligned}$$

Since the inequality holds for all $r < 1$, it follows that the same inequality holds letting $r \rightarrow 1$. Thus

$$P\left([M]^{1/2}(T) > \beta \lambda\right) \cap [2M^* \leq \delta \lambda] \leq P\left([M]^{1/2}(T) > \lambda\right) \frac{\delta^2}{(\beta - 1)^2}$$

Now by the good lambda inequality, there is a constant k independent of M such that

$$\int_{\Omega} F\left([M]^{1/2}(T)\right) dP \leq k \int_{\Omega} F(2M^*) dP \leq kC_2 \int_{\Omega} F(M^*) dP$$

by the assumptions about F . Therefore, combining this result with the first part,

$$\begin{aligned} (kC_2)^{-1} \int_{\Omega} F\left([M]^{1/2}(T)\right) dP &\leq \int_{\Omega} F(M^*) dP \\ &\leq C \int_{\Omega} F\left([M]^{1/2}(T)\right) dP \end{aligned}$$

and this proves the inequality.

51.3 The Case Of $\mathcal{M}_T^2(H)$

Now of course it would be more interesting if the assumption that $\{M(t)\}$ is uniformly bounded were dropped. In particular, it would be better to have the above result for $M \in \mathcal{M}_T^2(H)$.

Theorem 51.3.1 *Let $\{M(t)\}$ be a continuous martingale in $\mathcal{M}_T^2(H)$ where H is a separable Hilbert space adapted to the filtration $\{\mathcal{F}_t\}$ such that $M(0) = 0$. Then there exists a unique continuous, increasing, nonnegative, submartingale*

$$\{[M](t)\}, [M](0) = 0$$

called the quadratic variation such that

$$\|M(t)\|^2 - [M](t) \equiv N(t)$$

is in $\mathcal{M}_T^2(\mathbb{R})$. Here $t \in [0, T]$. Also

$$\|N(t)\|_{L^2(\Omega)} \leq 2 \|M(t)\|_{L^2(\Omega)}.$$

Proof: Since $t \rightarrow M(t)(\omega)$ is continuous, there exists an increasing sequence of stopping times of the form

$$\sigma_m \equiv \inf \{t : \|M(t)\| > m\} \wedge T$$

such that $\sigma_m = T$ for all m large enough. Here $\inf \emptyset \equiv \infty$. Now from Proposition 51.1.6 there exists a martingale N_m and a submartingale $[M^{\sigma_m}]$ satisfying the conditions of that proposition such that

$$\|M^{\sigma_m}(t)\|^2 = [M^{\sigma_m}](t) + N_m(t)$$

where $M^{\sigma_m}(t)$ denotes the stopped martingale given by

$$M^{\sigma_m}(t) \equiv M(t \wedge \sigma_m)$$

Recall this is a martingale with respect to the same filtration because if $s < t$, it follows from the Doob optional sampling theorem for martingales, Theorem 50.5.8

$$\begin{aligned} E(M^{\sigma_m}(t) | \mathcal{F}_s) &= E(M(t \wedge \sigma_m) | \mathcal{F}_s) \\ &= M(t \wedge \sigma_m \wedge s) = M(\sigma_m \wedge s) = M^{\sigma_m}(s). \end{aligned}$$

Also from Proposition 51.1.6 and the Doob optional sampling theorem and Jensen's inequality,

$$\begin{aligned} \|N_m(t)\|_{L^2(\Omega)}^2 &\leq 2E(\|M^{\sigma_m}(t)\|^2) \\ &= 2 \int_{\Omega} \|M(\sigma_m \wedge t)\|^2 dP = 2 \int_{\Omega} \|E(M(t) | \mathcal{F}_{\sigma_m})\|^2 dP \\ &\leq 2 \int_{\Omega} E(\|M(t)\|^2 | \mathcal{F}_{\sigma_m \wedge t}) dP = 2 \int_{\Omega} \|M(t)\|^2 dP \end{aligned} \quad (51.3.10)$$

and so

$$E(|N_m(t)|^2) \leq 2E(\|M(t)\|^2)$$

Also from the uniqueness part of Proposition 51.1.6, pointwise convergence takes place as $m \rightarrow \infty$ uniformly in t for each fixed ω . This is because for each ω , eventually $\sigma_m(\omega)$ is so large that $\|M(t)(\omega)\| < m$ for all $t \in [0, T]$ and so no further change can take place as a function of m . Thus $N_m(t)$ converges pointwise to a function, $N(t)$. From Fatou's lemma

$$E(|N(t)|^2) \leq 2E(\|M(t)\|^2) \quad (51.3.11)$$

also. From 51.3.10 the set $\left\{|N_m(t)|^{3/2}, |N(t)|^{3/2}\right\}_{m=1}^{\infty}$ is uniformly integrable and so the pointwise convergence uniformly in $t \in [0, T]$ implies

$$N_m(t) \rightarrow N(t) \quad (51.3.12)$$

uniformly on $[0, T]$ in $L^{6/5}(\Omega)$. Here is why. First note that $C([0, T])$ is separable from the Weierstrass approximation theorem. By Egoroff's theorem, Theorem

8.3.11 on Page 198 applied to $X = C([0, T])$ there is a subsequence, still denoted by m and a set of measure ε called B_ε such that for $\omega \notin B_\varepsilon$ then for each $\eta > 0$ there exists N_η such that if $m > N_\eta$, then

$$\sup_{t \in [0, T]} |N_m(t) - N(t)| < \eta$$

Then for all $t \in [0, T]$,

$$\begin{aligned} \int_{\Omega} |N_m(t) - N(t)|^{6/5} dP &\leq 2^{1/5} \int_{B_\varepsilon} (|N_m(t)|^{6/5} + |N(t)|^{6/5}) dP \\ &\quad + \int_{B_\varepsilon^c} |N_m(t) - N(t)|^{6/5} dP \\ &\leq 2^{1/5} \left[\left(\int_{\Omega} \chi_{B_\varepsilon} dP \right)^{2/5} \left(\int_{\Omega} |N_m(t)|^2 dP \right)^{3/5} \right. \\ &\quad \left. + \left(\int_{\Omega} \chi_{B_\varepsilon^c} dP \right)^{2/5} \left(\int_{\Omega} |N(t)|^2 dP \right)^{3/5} \right] \\ &\quad + \int_{B_\varepsilon^c} |N_m(t) - N(t)|^{6/5} dP + \int_{B_\varepsilon} |N_m(t) - N(t)|^{6/5} dP \\ &\leq 4 \cdot 2^{1/5} \varepsilon^{2/5} \left(E \left(\|M(t)\|^2 \right) \right) + \int_{B_\varepsilon^c} |N_m(t) - N(t)|^{6/5} dP \\ &\leq 4 \cdot 2^{1/5} \varepsilon^{2/5} \|M\|_{\mathcal{M}_T^2(H)}^2 + \int_{B_\varepsilon^c} \varepsilon^{6/5} dP \end{aligned}$$

and so since ε and $t \in [0, T]$ are arbitrary, this shows

$$\lim_{m \rightarrow \infty} \left(\sup_{t \in [0, T]} \int_{\Omega} |N_m(t) - N(t)|^{6/5} dP \right) = 0$$

It follows from Proposition 50.10.2 that $N \in \mathcal{M}_T^{6/5}(\mathbb{R})$ and so, in particular it is a continuous martingale. In fact it is in $\mathcal{M}_T^2(\mathbb{R})$ because of the above estimate 51.3.11.

Letting $[M](t)$ be the pointwise limit of $[M^{\sigma_m}](t)$

$$\|M(t)\|^2 = [M](t) + N(t)$$

and N is a martingale. It follows $[M]$ must be a submartingale since $\|M\|^2$ is a submartingale. It is increasing because it is a limit of increasing functions of t for each ω . This proves the theorem.

51.4 The Case Of Local Martingales

There is also the notion of a local martingale which can be used to generalize the above results.

Definition 51.4.1 *Let $\{M(t)\}$ be adapted to the normal filtration $\mathcal{F}_t$ for $t > 0$. Then $\{M(t)\}$ is a local martingale if there exist stopping times τ_n increasing to infinity such that for each n , the process*

$$M^{\tau_n}(t) - M(0)$$

is a martingale with respect to the given filtration. The sequence of stopping times is called a localizing sequence.

As in the proof of Lemma 51.1.5 if $\{M(t)\}$ is a martingale, then $M^{\tau_n}(t) - M(0)$ is a martingale with respect to the σ algebras, $\mathcal{F}_t$ as shown by the following computation;

$$\begin{aligned} E(M^{\tau_n}(t) - M(0) | \mathcal{F}_s) &\equiv E(M(\tau \wedge t) - M(0) | \mathcal{F}_s) \\ &= M(\tau \wedge t \wedge s) - M(0) \\ &= M(\tau \wedge s) - M(0) \\ &= M^{\tau_n}(s) - M(0); \end{aligned}$$

so the idea of a local martingale is just an adapted process which preserves this property having to do with stopping times coming from a localizing sequence. The above reasoning is used in the following simple proposition which states you can assume the localizing sequence can be assumed such that $\{M^{\tau_n}(t) - M(0)\}$ is a bounded martingale.

Proposition 51.4.2 *If M is a local martingale, there exists a localizing sequence $\{\sigma_m\}$ such that $M^{\sigma_m} - M(0)$ is a bounded martingale.*

Proof: Let

$$\delta_n \equiv \inf \{t : \|M(t) - M(0)\| > n\}$$

Thus $M^{\delta_n} - M(0)$ is a bounded adapted process, bounded by n . Also $\delta_n \rightarrow \infty$ because of continuity of M . (If ω is fixed and $b > 0$, eventually $\|M(t)(\omega) - M(0)\| < n$ for all $t \in [0, b]$ and so $\delta_n(\omega) > b$.) Then let $\sigma_n = \delta_n \wedge \tau_n$. Thus $M^{\delta_n \wedge \tau_n} - M(0) = (M^{\tau_n} - M(0))^{\delta_n}$, a martingale since $M^{\tau_n} - M(0)$ is given to be one. This proves the proposition.

One can also give a generalization of Lemma 51.1.5 to conclude a local martingale must be constant or else they must fail to be of bounded variation.

Corollary 51.4.3 *Let $\mathcal{F}_t$ be a normal filtration and let $A(t), B(t)$ be adapted to $\mathcal{F}_t$, continuous, and increasing with $A(0) = B(0) = 0$ and suppose $A(t) - B(t) \equiv M(t)$ is a local martingale. Then $A(t) - B(t) = 0$ a.e. for all t .*

Proof: Let $\{\tau_n\}$ be a localizing sequence for M . For given n consider the martingale,

$$M^{\tau_n}(t) = A^{\tau_n}(t) - B^{\tau_n}(t)$$

Also let

$$\sigma \equiv \sigma_C \equiv \inf \{t \in [0, l] : |A^{\tau_n}(t)| + |B^{\tau_n}(t)| > C\}$$

As usual $\inf(\emptyset) \equiv \infty$. Letting $N = M^{\tau_n}$,

$$N^\sigma(l) = \sum_{k=0}^{n-1} (N^\sigma(t_{k+1}) - N^\sigma(t_k))$$

where $\{t_k\}_{k=0}^m$ is a uniform partition of $[0, l]$. Then since the mixed terms have 0 expectation due to the fact N^σ is a martingale,

$$\begin{aligned} E(N^\sigma(l)^2) &= E\left(\left(\sum_{k=0}^{m-1} (N^\sigma(t_{k+1}) - N^\sigma(t_k))\right)^2\right) \\ &= \sum_{k=0}^{m-1} E\left((N^\sigma(t_{k+1}) - N^\sigma(t_k))^2\right) \\ &\leq \sum_{k=0}^{m-1} \int_{\Omega} \Delta_m^\sigma(\omega) |N^\sigma(t_{k+1}) - N^\sigma(t_k)| dP \\ &= \int_{\Omega} \Delta_m^\sigma(\omega) \sum_{k=0}^{m-1} |N^\sigma(t_{k+1}) - N^\sigma(t_k)| dP \end{aligned}$$

where

$$\Delta_m^\sigma(\omega) \equiv \max_k \{|N^\sigma(t_{k+1})(\omega) - N^\sigma(t_k)(\omega)|\}$$

Since

$$\begin{aligned} |N^\sigma(t_{k+1})(\omega) - N^\sigma(t_k)(\omega)| &= |(A^{\tau_n})^\sigma(t_{k+1}) - (B^{\tau_n})^\sigma(t_{k+1}) \\ &\quad - ((A^{\tau_n})^\sigma(t_k) - (B^{\tau_n})^\sigma(t_k))| \\ &\leq (A^{\tau_n})^\sigma(t_{k+1}) - (A^{\tau_n})^\sigma(t_k) + (B^{\tau_n})^\sigma(t_{k+1}) - (B^{\tau_n})^\sigma(t_k), \end{aligned}$$

it follows

$$\sum_{k=0}^{m-1} |N^\sigma(t_{k+1}) - N^\sigma(t_k)| \leq (A^{\tau_n})^\sigma(l) + (B^{\tau_n})^\sigma(l)$$

and so

$$\begin{aligned} E(N^\sigma(l)^2) &\leq \int_{\Omega} \Delta_m^\sigma(\omega) ((A^{\tau_n})^\sigma(l) + (B^{\tau_n})^\sigma(l)) dP \\ &\leq \int_{\Omega} \Delta_m^\sigma(\omega) (A^\sigma(l) + B^\sigma(l)) dP. \end{aligned}$$

Since N^σ is uniformly bounded, the dominated convergence theorem yields this converges to 0 as $m \rightarrow \infty$ due to the continuity of N^σ . Thus

$$N^{\sigma_C}(l) = 0$$

for all $C > 0$ and letting $C \rightarrow \infty$, it follows from continuity of M that for each ω , eventually $\sigma_C(\omega) > l$ and so the above reduces to

$$N(l) \equiv M^{\tau_n}(l) = 0.$$

Since n is arbitrary and $\tau_n \rightarrow \infty$, it follows for large enough $n, \tau_n > l$ and so

$$M(l) = M^{\tau_n}(l) = 0.$$

This proves the corollary.

The next task is to generalize everything to the case of local martingales.

Theorem 51.4.4 *Let $\{M(t)\}$ be a continuous local martingale adapted to the normal filtration $\mathcal{F}_t$ for $t \in [0, T]$, $M(t)$ having values in a separable Hilbert space H , $M(0) = 0$. Then there exists a unique adapted process, $[M](t)$ which is increasing, equal to 0 at $t = 0$ such that*

$$\left\{ \|M\|^2(t) - [M](t) \right\}$$

is a local martingale adapted to $\mathcal{F}_t$. Furthermore, if σ is any stopping time

$$[M]^\sigma(t) = [M^\sigma](t).$$

Also there exists an increasing localizing sequence $\{\sigma_m\}$ for the local martingale

$$\left\{ \|M\|^2(t) - [M](t) \right\}$$

such that $\{M^{\sigma_m}(t)\}$ is a uniformly bounded martingale and

$$[M](t) = \lim_{m \rightarrow \infty} [M^{\sigma_m}](t)$$

Proof: Let $\{\tau_n\}_{n=1}^\infty$ be a localizing sequence for $\{M(t)\}$ so that $\{M^{\tau_n}(t)\}$ is a martingale. Also let σ be a stopping time. Thus $\{M^{\sigma \wedge \tau_n}\}$ is a martingale and $\{\tau_n\}_{n=1}^\infty$ is a localizing sequence for M^σ .

Lemma 51.4.5 *Let*

$$\sigma_m \equiv \inf \{t : \|M(t)\| > m\}$$

Then if σ is any stopping time, $\{M^\sigma(t)\}$ a local martingale and $\{\sigma_m\}$ is a localizing sequence for it. Also

$$\left\{ \left(\|M\|^2 \right)^{\sigma \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}](t) \right\}$$

is a martingale for each σ_m .

Proof: It follows from $\{\tau_n\}$ is a localizing sequence that $\{M^{\sigma \wedge \tau_n \wedge \sigma_m}(t)\}$ is a bounded martingale and so by Proposition 51.1.6 there exists a unique continuous increasing submartingale $[M^{\sigma \wedge \tau_n \wedge \sigma_m}]$ such that $[M^{\sigma \wedge \tau_n \wedge \sigma_m}](0) = 0$ and

$$\left\{ \left(\|M\|^2 \right)^{\sigma \wedge \tau_n \wedge \sigma_m}(t) - [M^{\sigma \wedge \tau_n \wedge \sigma_m}](t) \right\}$$

is a martingale. By Corollary 51.1.7, for each t it equals

$$\left(\|M\|^2 \right)^{\sigma \wedge \tau_n \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}]^{\tau_n}(t)$$

Since $\tau_n \rightarrow \infty$, it follows that for each ω , it converges to

$$\left(\|M\|^2 \right)^{\sigma \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}](t)$$

because eventually $\tau_n > t$. Why is this a martingale? This is easy to see because convergence is pointwise and the functions are bounded so the dominated convergence theorem applies. Thus letting $s < t$ and $A \in \mathcal{F}_s$,

$$\begin{aligned} & \int_A E \left(\left(\|M\|^2 \right)^{\sigma \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}](t) \mid \mathcal{F}_s \right) dP \\ &= \int_A \left(\|M\|^2 \right)^{\sigma \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}](t) dP \\ &= \lim_{n \rightarrow \infty} \int_A \left(\|M\|^2 \right)^{\sigma \wedge \tau_n \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}]^{\tau_n}(t) dP \\ &= \lim_{n \rightarrow \infty} \int_A E \left(\left(\|M\|^2 \right)^{\sigma \wedge \tau_n \wedge \sigma_m}(t) - [M^{\sigma \wedge \sigma_m}]^{\tau_n}(t) \mid \mathcal{F}_s \right) dP \\ &= \lim_{n \rightarrow \infty} \int_A \left(\|M\|^2 \right)^{\sigma \wedge \tau_n \wedge \sigma_m}(s) - [M^{\sigma \wedge \sigma_m}]^{\tau_n}(s) dP \\ &= \int_A \left(\|M\|^2 \right)^{\sigma \wedge \sigma_m}(s) - [M^{\sigma \wedge \sigma_m}](s) dP \end{aligned} \tag{51.4.13}$$

Also

$$\begin{aligned} \int_A E(M^{\sigma \wedge \sigma_m}(t) \mid \mathcal{F}_s) dP &= \int_A M^{\sigma \wedge \sigma_m}(t) dP = \lim_{n \rightarrow \infty} \int_A (M^{\sigma \wedge \sigma_m})^{\tau_n}(t) dP \\ &= \lim_{n \rightarrow \infty} \int_A E((M^{\sigma \wedge \sigma_m})^{\tau_n}(t) \mid \mathcal{F}_s) dP \\ &= \lim_{n \rightarrow \infty} \int_A (M^{\sigma \wedge \sigma_m})^{\tau_n}(s) dP = \int_A M^{\sigma \wedge \sigma_m}(s) dP \end{aligned}$$

Thus $\{\sigma_m\}$ is also a localizing sequence for $\{M^\sigma(t)\}$. This proves the lemma.

Note that if you take $\sigma = \infty$, the argument has shown M^{σ_m} is a bounded martingale.

Now it was just shown

$$\left(\|M\|^2 \right)^{\sigma \wedge \sigma_m} (t) = [M^{\sigma \wedge \sigma_m}] (t) + N_{1m\sigma} (t)$$

where $N_{1m\sigma}$ is a martingale. For each ω the left side is eventually constant in m because of continuity of M . Therefore, the same is true of the right side. Define

$$[M^\sigma] (t) + N_{1\sigma} (t) \equiv \lim_{m \rightarrow \infty} ([M^{\sigma \wedge \sigma_m}] (t) + N_{1m\sigma} (t)) \quad (51.4.14)$$

as a pointwise limit. Note the limit of the $[M^{\sigma \wedge \sigma_m}] (t)$ must exist because for each ω these are increasing in m . Therefore, the limit of the $N_{1m\sigma} (t)$ also exists and equals something referred to here as $N_{1\sigma} (t)$. If you let $\sigma \equiv \infty$, this has defined

$$[M] (t) \equiv \lim_{m \rightarrow \infty} [M^{\sigma_m}] (t). \quad (51.4.15)$$

However, from Corollary 51.1.7, the right side of the 51.4.14 equals

$$\lim_{m \rightarrow \infty} ([M^{\sigma_m}]^\sigma (t) + N_{1m\sigma} (t)). \quad (51.4.16)$$

Using 51.4.15,

$$\begin{aligned} \lim_{m \rightarrow \infty} [M^{\sigma_m}]^\sigma (t) &\equiv \lim_{m \rightarrow \infty} [M^{\sigma_m}] (\sigma \wedge t) \\ &= [M] (\sigma \wedge t) \equiv [M]^\sigma (t) \end{aligned}$$

and so 51.4.16 equals

$$[M]^\sigma (t) + N_{1\sigma} (t)$$

and so comparing this with 51.4.14,

$$[M]^\sigma (t) = [M^\sigma] (t)$$

as claimed.

Letting $\sigma = \infty$ in the above lemma and using the definition of $[M]$ given above, this has shown $\|M\|^2 (t) - [M] (t)$ is a local martingale with localizing sequence $\{\sigma_m\}$. Uniqueness follows from Corollary 51.4.3 as before. If $[M]_1$ works then

$$[M] (t) - [M]_1 (t) = \text{a local martingale}$$

and so is equal to 0. This proves the theorem.

With this theorem, a similar one can be established for the covariation.

Theorem 51.4.6 *Let M, N be local martingales having values in a separable Hilbert space, H for $t \in [0, T]$ with respect to the normal filtration $\mathcal{F}_t$, and let $M(0) = N(0) = 0$. Then there exists a unique bounded variation process adapted to the filtration $[M, N]$ such that $[M, N](0) = 0$ and*

$$(M, N)_H - [M, N]$$

is a local martingale. Also if σ is a stopping time

$$[M, N]^\sigma = [M^\sigma, N^\sigma] = [M^\sigma, N]$$

Also

$$0 \leq [M - M^\sigma] = [M] - [M^\sigma].$$

For $\{\sigma_m\}$ a sequence of stopping times converging to ∞ such that M^{σ_m} and N^{σ_m} are uniformly bounded martingales,

$$[M, N](t) = [M^{\sigma_m}, N^{\sigma_m}](t)$$

The map $(M, N) \rightarrow [M, N]$ is also bilinear and symmetric.

Proof: Define

$$[M, N](t) \equiv \frac{1}{4} ([M + N] - [M - N]) \quad (51.4.17)$$

where $[M + N]$ and $[M - N]$ come from Theorem 51.4.4. Then it is routine to verify this satisfies $(M, N)_H - [M, N]$ is a local martingale and $[M, N]$ is of bounded variation. Uniqueness is also easy to establish. If $[M, N]_1$ is another one,

$$[M, N]_1 - [M, N]$$

is a local martingale and since this expression can be written as the difference of two increasing functions which are equal to 0 at $t = 0$, it follows from Corollary 51.4.3 this expression equals 0.

It only remains to verify the claims about the stopping time.

$$\begin{aligned} [M, N]^\sigma + \text{local martingale} &= (M, N)_H^\sigma = (M^\sigma, N^\sigma)_H \\ &= [M^\sigma, N^\sigma] + \text{local martingale} \end{aligned}$$

and so by Corollary 51.4.3 again

$$[M, N]^\sigma = [M^\sigma, N^\sigma]$$

The other claim is more surprising. This will involve a use of Lemma 51.1.1 which tells how to recognize a martingale when you see one. First note there exists a single localizing sequence $\{\tau_n\}$ which works for M and N simultaneously. You can get it by simply taking $\tau_n = \sigma_n \wedge \delta_n$ where the σ_n and the δ_n are localizing sequences for M and N respectively.

Claim: $(M^{\tau_n \wedge \sigma}, N^{\tau_n \wedge \sigma} - N^{\tau_n})_H$ is a martingale and so $(M^\sigma, N^\sigma - N)_H$ is a local martingale.

Proof of the claim: Let α be a stopping time with two values. Then using the Doob optional sampling theorem,

$$\begin{aligned} &E((M^{\tau_n \wedge \sigma}(\alpha), N^{\tau_n \wedge \sigma}(\alpha) - N^{\tau_n}(\alpha))_H) \\ &= E(E((M^{\tau_n \wedge \sigma}(\alpha), N^{\tau_n \wedge \sigma}(\alpha) - N^{\tau_n}(\alpha))_H | \mathcal{F}_\sigma)) \end{aligned}$$

$$\begin{aligned}
&= E((M^{\tau_n \wedge \sigma}(\alpha), E(N^{\tau_n \wedge \sigma}(\alpha) - N^{\tau_n}(\alpha) | \mathcal{F}_\sigma))) \\
&= E((M^{\tau_n \wedge \sigma}(\alpha), N^{\tau_n \wedge \sigma}(\alpha \wedge \sigma) - N^{\tau_n}(\alpha \wedge \sigma))) = 0
\end{aligned}$$

Also

$$\begin{aligned}
&E((M^{\tau_n \wedge \sigma}(t), N^{\tau_n \wedge \sigma}(t) - N^{\tau_n}(t))_H) \\
&= E(E((M^{\tau_n \wedge \sigma}(t), N^{\tau_n \wedge \sigma}(t) - N^{\tau_n}(t))_H | \mathcal{F}_\sigma)) \\
&= E((M^{\tau_n \wedge \sigma}(t), E(N^{\tau_n \wedge \sigma}(t) - N^{\tau_n}(t) | \mathcal{F}_\sigma))) \\
&= E((M^{\tau_n \wedge \sigma}(t), N^{\tau_n \wedge \sigma}(t \wedge \sigma) - N^{\tau_n}(t \wedge \sigma))) = 0
\end{aligned}$$

Therefore by Lemma 51.1.1 this proves the claim.

Now with the claim,

$$\begin{aligned}
[M^\sigma, N^\sigma] - [M^\sigma, N] &= (M^\sigma, N^\sigma) - (M^\sigma, N) + \text{local martingale} \\
&= (M^\sigma, N^\sigma - N) + \text{local martingale}
\end{aligned}$$

which by the claim is a local martingale. Thus the result follows from Corollary 51.4.3.

Finally, consider the last claim. By Theorem 51.4.4

$$[M^\sigma](t) = \lim_{m \rightarrow \infty} [M^{\sigma \wedge \sigma_m}](t)$$

where $\{\sigma_m\}$ is an increasing localizing sequence of stopping times for which M^{σ_m} is bounded. Similarly

$$[M - M^\sigma](t) = \lim_{m \rightarrow \infty} [M^{\sigma_m} - M^{\sigma \wedge \sigma_m}](t)$$

By Corollary 51.1.9

$$= \lim_{m \rightarrow \infty} [M^{\sigma_m}](t) - [M^{\sigma \wedge \sigma_m}](t) = [M](t) - [M^\sigma](t).$$

The formula follows from Theorem 51.4.4 and the definition, 51.4.17.

It remains to prove the claim about $[M, N]$ being bilinear. From the first part, letting M_1, M_2, N be H valued continuous local martingales equal to 0 at $t = 0$,

$$\begin{aligned}
(aM_1 + bM_2, N)_H &= [aM_1 + bM_2, N] + \text{local martingale} \\
a(M_1, N) + b(M_2, N)_H &= a[M_1, N] + b[M_2, N] + \text{local martingale}
\end{aligned}$$

Hence

$$[aM_1 + bM_2, N] - (a[M_1, N] + b[M_2, N]) = \text{local martingale}.$$

It follows from Corollary 51.4.3, the left side, being the difference of two increasing functions equals 0. $[M, N]$ is obviously symmetric thanks to the definition. This proves the theorem.

Now here is a generalization of the Burkholder Gundy inequality to local martingales.

Theorem 51.4.7 *Let $\{M(t)\}$ be a continuous H valued local martingale, $M(0) = 0$, where H is a separable Hilbert space and $t \in [0, T]$. Then if F is a function of the sort described in the good lambda inequality, that is,*

$$F(0) = 0, \quad F \text{ continuous, } F \text{ increasing,}$$

$$F(\alpha x) \leq c_\alpha F(x),$$

there are constants, C and c independent of such local martingales M such that

$$c \int_{\Omega} F([M]^{1/2}(T)) dP \leq \int_{\Omega} F(M^*) dP \leq C \int_{\Omega} F([M]^{1/2}(T)) dP$$

where

$$M^*(\omega) \equiv \sup \{ \|M(t)(\omega)\| : t \in [0, T] \}.$$

Proof: Let $\{\tau_n\}$ be a localizing sequence for M such that M^{τ_n} is uniformly bounded. Such a localizing sequence exists from Theorem 51.4.4. Then from Theorem 51.2.2 there exist constants c, C independent of τ_n such that

$$\begin{aligned} c \int_{\Omega} F([M^{\tau_n}]^{1/2}(T)) dP &\leq \int_{\Omega} F((M^{\tau_n})^*) dP \\ &\leq C \int_{\Omega} F([M^{\tau_n}]^{1/2}(T)) dP \end{aligned}$$

By Theorem 51.4.4, this implies

$$\begin{aligned} c \int_{\Omega} F([M]^{\tau_n})^{1/2}(T) dP &\leq \int_{\Omega} F((M^{\tau_n})^*) dP \\ &\leq C \int_{\Omega} F([M]^{\tau_n})^{1/2}(T) dP \end{aligned}$$

and now note that $([M]^{\tau_n})^{1/2}(T)$ and $(M^{\tau_n})^*$ increase in n to $[M]^{1/2}(T)$ and M^* respectively. Then the result follows from the monotone convergence theorem.

Here is a corollary [55].

Corollary 51.4.8 *Let $\{M(t)\}$ be a continuous H valued local martingale and let $\varepsilon, \delta \in (0, \infty)$. Then there is a constant independent of ε, δ such that*

$$P \left(\left[\sup_{t \in [0, T]} \|M(t)\| \geq \varepsilon \right] \right) \leq \frac{C}{\varepsilon} E \left([M]^{1/2}(T) \wedge \delta \right) + P \left([M]^{1/2}(T) > \delta \right)$$

Proof: Let the stopping time τ be defined by

$$\tau \equiv \inf \left\{ t \geq 0 : [M]^{1/2}(t) > \delta \right\}$$

and if the set is empty, $\tau \equiv T$.

$$P([M^* \geq \varepsilon]) = P([M^* \geq \varepsilon] \cap [\tau = T]) + P([M^* \geq \varepsilon] \cap [\tau < T])$$

On the set where $[\tau = T]$, $M^\tau = M$ and so

$$\leq \frac{1}{\varepsilon} \int_{\Omega} (M^\tau)^* dP + P \left([M^* \geq \varepsilon] \cap \left[[M]^{1/2}(T) > \delta \right] \right)$$

By Theorem 51.4.7 and Theorem 51.4.4

$$\begin{aligned} &\leq \frac{C}{\varepsilon} \int_{\Omega} [M^\tau]^{1/2}(T) dP + P \left([M^* \geq \varepsilon] \cap \left[[M]^{1/2}(T) > \delta \right] \right) \\ &\leq \frac{C}{\varepsilon} \int_{\Omega} ([M]^\tau)^{1/2}(T) dP + P \left([M^* \geq \varepsilon] \cap \left[[M]^{1/2}(T) > \delta \right] \right) \\ &\leq \frac{C}{\varepsilon} \int_{\Omega} [M]^{1/2}(T) \wedge \delta dP + P \left([M^* \geq \varepsilon] \cap \left[[M]^{1/2}(T) > \delta \right] \right) \\ &\leq \frac{C}{\varepsilon} \int_{\Omega} [M]^{1/2}(T) \wedge \delta dP + P \left(\left[[M]^{1/2}(T) > \delta \right] \right) \end{aligned}$$

This proves the corollary.

51.5 The Quadratic Variation And Stochastic Integration

Let $\mathcal{F}_t$ be a normal filtration and let $\{M(t)\}$ be a local martingale adapted to $\mathcal{F}_t$ having values in U a separable real Hilbert space.

Definition 51.5.1 *Let $\mathcal{F}_t$ be a normal filtration and let*

$$f(t) \equiv \sum_{k=0}^{n-1} f_k \mathcal{X}_{(t_k, t_{k+1}]}(t)$$

where $\{t_k\}_{k=0}^n$ is a partition of $[0, T]$ and each f_k is $\mathcal{F}_{t_k}$ measurable, $f_k M^* \in L^2(\Omega)$ where

$$M^*(\omega) \equiv \sup_{t \in [0, T]} \|M(t)(\omega)\|$$

Such a function is called an elementary function. Also let $\{M(t)\}$ be a local martingale adapted to $\mathcal{F}_t$ which has values in a separable real Hilbert space, U such that $M(0) = 0$. For such an elementary real valued function define

$$\int_0^t f dM \equiv \sum_{k=0}^{n-1} f_k (M(t \wedge t_{k+1}) - M(t \wedge t_k)).$$

Then with this definition, here is a wonderful lemma.

Lemma 51.5.2 *For f an elementary function as above, $\left\{\int_0^t f dM\right\}$ is a continuous local martingale and*

$$E\left(\left\|\int_0^t f dM\right\|_U^2\right) = \int_{\Omega} \int_0^t f(s)^2 d[M](s) dP. \quad (51.5.18)$$

If N is another continuous local martingale adapted to $\mathcal{F}_t$ and both f, g are elementary functions such that for each k ,

$$f_k M^*, f_k N^*, g_k M^*, g_k N^* \in L^2(\Omega),$$

then

$$E\left(\left(\int_0^t f dM, \int_0^t g dN\right)_U\right) = \int_{\Omega} \int_0^t f g d[M, N] \quad (51.5.19)$$

and both sides make sense.

Proof: Let $\{\tau_l\}$ be a localizing sequence for M such that M^{τ_l} is a bounded martingale. Then from the definition, for each ω

$$\int_0^t f dM = \lim_{l \rightarrow \infty} \int_0^t f dM^{\tau_l} = \lim_{l \rightarrow \infty} \left(\int_0^t f dM\right)^{\tau_l}$$

and it is clear that $\left\{\int_0^t f dM^{\tau_l}\right\}$ is a martingale because it is just the sum of some martingales. Thus $\{\tau_l\}$ is a localizing sequence for $\int_0^t f dM$. It is also clear $\int_0^t f dM$ is continuous because it is a finite sum of continuous random variables.

Next consider the formula which is really a version of the Ito isometry. There is no loss of generality in assuming the mesh points are the same for the two elementary functions because if not, one can simply add in points to make this happen. It suffices to consider 51.5.19 because the other formula is a special case. To begin with, let $\{\tau_l\}$ be a localizing sequence which makes both M^{τ_l} and N^{τ_l} into bounded martingales. Consider the stopped process.

$$\begin{aligned} & E\left(\left(\int_0^t f dM^{\tau_l}, \int_0^t g dN^{\tau_l}\right)_U\right) \\ &= E\left(\left(\sum_{k=0}^{n-1} f_k (M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)), \sum_{k=0}^{n-1} g_k (N^{\tau_l}(t \wedge t_{k+1}) - N^{\tau_l}(t \wedge t_k))\right)\right) \\ &= E\left(\sum_{k,j} f_k g_j ((M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)) (N^{\tau_l}(t \wedge t_{j+1}) - N^{\tau_l}(t \wedge t_j)))\right) \end{aligned}$$

Now consider one of the mixed terms with $j < k$.

$$\begin{aligned}
 & E(f_k(M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)), g_j(N^{\tau_l}(t \wedge t_{j+1}) - N^{\tau_l}(t \wedge t_j))) \\
 &= E(E(f_k(M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)), g_j(N^{\tau_l}(t \wedge t_{j+1}) - N^{\tau_l}(t \wedge t_j))) | \mathcal{F}_{t_k}) \\
 &= E((g_j(N^{\tau_l}(t \wedge t_{j+1}) - N^{\tau_l}(t \wedge t_j)), f_k E((M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)) | \mathcal{F}_{t_k}))) \\
 &\text{but } E((M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)) | \mathcal{F}_{t_k}) = 0 \text{ by the Doob optional sampling theorem. Thus}
 \end{aligned}$$

$$E\left(\left(\int_0^t f dM^{\tau_l}, \int_0^t g dN^{\tau_l}\right)_U\right) = \quad (51.5.20)$$

$$\begin{aligned}
 &= \sum_{k=0}^{n-1} E(f_k g_k ((M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)), (N^{\tau_l}(t \wedge t_{k+1}) - N^{\tau_l}(t \wedge t_k)))) \\
 &= \sum_{k=0}^{n-1} E\left(f_k g_k \left((M^{\tau_l})^{t_{k+1}} - (M^{\tau_l})^{t_k}, (N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k}\right)(t)\right) \\
 &= \sum_{k=0}^{n-1} E\left(f_k g_k \left(\left[(M^{\tau_l})^{t_{k+1}} - (M^{\tau_l})^{t_k}, (N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k}\right](t) + \mathcal{N}_k(t)\right)\right) \quad (51.5.21)
 \end{aligned}$$

where $\mathcal{N}_k(t)$ is a martingale such that $\mathcal{N}_k(t) = 0$ for $t \leq t_k$ since the martingale $(N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k}$ equals 0 for such t ; and so $E(\mathcal{N}_k(t)) = 0$. Now use Theorems 51.4.4 and 51.4.6 as needed to do the following computations on the right side of the above which equals

$$= \sum_{k=0}^{n-1} E\left(f_k g_k \left(\left[(M^{\tau_l})^{t_{k+1}} - (M^{\tau_l})^{t_k}, (N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k}\right](t) + \mathcal{N}_k(t)\right)\right) \quad (51.5.22)$$

Now

$$E(f_k g_k \mathcal{N}_k(t)) = E(f_k g_k E(\mathcal{N}_k(t) | \mathcal{F}_{t_k})) = 0$$

because $\mathcal{N}_k$ is martingale which equals 0 when $t = t_k$. Thus 51.5.22 equals

$$= \sum_{k=0}^{n-1} E\left(f_k g_k \left(\left[(M^{\tau_l})^{t_{k+1}} - (M^{\tau_l})^{t_k}, (N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k}\right](t)\right)\right)$$

which equals

$$\begin{aligned}
 &= \frac{1}{4} \sum_{k=0}^{n-1} E\left(f_k g_k \left(\left[(M^{\tau_l})^{t_{k+1}} - (M^{\tau_l})^{t_k} + (N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k}\right] \right. \right. \\
 &\quad \left. \left. - \left[(M^{\tau_l})^{t_{k+1}} - (M^{\tau_l})^{t_k} - ((N^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_k})\right]\right)\right)
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \sum_{k=0}^{n-1} E \left(f_k g_k \left(\left[(M^{\tau_l})^{t_{k+1}} + (N^{\tau_l})^{t_{k+1}} - \left((M^{\tau_l})^{t_k} + (N^{\tau_l})^{t_k} \right) \right] \right. \right. \\
&\quad \left. \left. - \left[(M^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_{k+1}} - \left((M^{\tau_l})^{t_k} - (N^{\tau_l})^{t_k} \right) \right] \right) \right) \\
&= \frac{1}{4} \sum_{k=0}^{n-1} E \left(f_k g_k \left(\left[(M^{\tau_l})^{t_{k+1}} + (N^{\tau_l})^{t_{k+1}} \right] - \left[(M^{\tau_l})^{t_k} + (N^{\tau_l})^{t_k} \right] \right. \right. \\
&\quad \left. \left. - \left[(M^{\tau_l})^{t_{k+1}} - (N^{\tau_l})^{t_{k+1}} \right] - \left[(M^{\tau_l})^{t_k} - (N^{\tau_l})^{t_k} \right] \right) \right) \\
&= \frac{1}{4} \sum_{k=0}^{n-1} E \left(f_k g_k \left(\left[(M^{\tau_l}) + (N^{\tau_l}) \right]^{t_{k+1}} - \left[(M^{\tau_l}) + (N^{\tau_l}) \right]^{t_k} \right. \right. \\
&\quad \left. \left. - \left[(M^{\tau_l}) - (N^{\tau_l}) \right]^{t_{k+1}} - \left[(M^{\tau_l}) - (N^{\tau_l}) \right]^{t_k} \right) \right) \\
&= \frac{1}{4} \sum_{k=0}^{n-1} E \left(f_k g_k \left(\left[(M^{t_{k+1}}) + (N^{t_{k+1}}) \right]^{\tau_l} - \left[(M^{t_k}) + (N^{t_k}) \right]^{\tau_l} \right. \right. \\
&\quad \left. \left. - \left[(M^{t_{k+1}}) - (N^{t_{k+1}}) \right]^{\tau_l} - \left[(M^{t_k}) - (N^{t_k}) \right]^{\tau_l} \right) \right)
\end{aligned}$$

Letting $l \rightarrow \infty$ this reduces to

$$\begin{aligned}
&= \frac{1}{4} \sum_{k=0}^{n-1} E \left(f_k g_k \left(\left[(M^{t_{k+1}}) + (N^{t_{k+1}}) \right] - \left[(M^{t_k}) + (N^{t_k}) \right] \right. \right. \\
&\quad \left. \left. - \left[(M^{t_{k+1}}) - (N^{t_{k+1}}) \right] - \left[(M^{t_k}) - (N^{t_k}) \right] \right) \right) \\
&= \frac{1}{4} \left(\int_{\Omega} \int_0^t f g (d[M + N] - d[M - N]) \right) \\
&= \int_{\Omega} \int_0^t f g d[M, N]
\end{aligned}$$

Now consider the left side of 51.5.21.

$$\begin{aligned}
&E \left(\left(\int_0^t f dM^{\tau_l}, \int_0^t g dN^{\tau_l} \right)_U \right) \\
&\equiv \int_{\Omega} \sum_{k,j} f_k g_j \left((M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)) \right. \\
&\quad \left. (N^{\tau_l}(t \wedge t_{j+1}) - N^{\tau_l}(t \wedge t_j)) \right) dP
\end{aligned}$$

Then for each ω , the integrand converges as $l \rightarrow \infty$ to

$$\sum_{k,j} f_k g_j ((M(t \wedge t_{k+1}) - M(t \wedge t_k)), (N(t \wedge t_{j+1}) - N(t \wedge t_j)))$$

But also you can do a sloppy estimate which will allow the use of the dominated convergence theorem.

$$\begin{aligned} & \left\| \sum_{k,j} f_k g_j ((M^{\tau_l}(t \wedge t_{k+1}) - M^{\tau_l}(t \wedge t_k)), (N^{\tau_l}(t \wedge t_{j+1}) - N^{\tau_l}(t \wedge t_j))) \right\| \\ & \leq \sum_{k,j} |f_k| |g_j| 4M^* N^* \in L^1(\Omega) \end{aligned}$$

by assumption. Thus the left side of 51.5.21 converges as $l \rightarrow \infty$ to

$$\begin{aligned} & \int_{\Omega} \sum_{k,j} f_k g_j ((M(t \wedge t_{k+1}) - M(t \wedge t_k)), (N(t \wedge t_{j+1}) - N(t \wedge t_j))) dP \\ & = \int_{\Omega} \left(\int_0^t f dM, \int_0^t g dN \right)_U dP \end{aligned}$$

This proves the lemma.

Note for each ω , the inside integral in 51.5.18 is just a Stieltjes integral taken with respect to the increasing integrating function $[M]$.

Of course, with this estimate it is obvious how to extend the integral to a larger class of functions.

Definition 51.5.3 Let $\nu(\omega)$ denote the Radon measure representing the functional

$$\Lambda(\omega)(g) \equiv \int_0^T g d[M](t)(\omega)$$

($t \rightarrow [M](t)(\omega)$ is a continuous increasing function and $\nu(\omega)$ is the measure representing the Stieltjes integral, one for each ω .) Then let $\mathcal{G}_M$ denote functions $f(s, \omega)$ which are the limit of such elementary functions in the space $L^2(\Omega; L^2([0, T], \nu(\cdot)))$, the norm of such functions being

$$\|f\|_{\mathcal{G}}^2 \equiv \int_{\Omega} \int_0^t f(s)^2 d[M](s) dP$$

For $f \in \mathcal{G}$ just defined,

$$\int_0^t f dM \equiv \lim_{n \rightarrow \infty} \int_0^t f_n dM$$

where $\{f_n\}$ is a sequence of elementary functions converging to f in

$$L^2(\Omega; L^2([0, T], \nu(\cdot))).$$

Now here is an interesting lemma.

Lemma 51.5.4 *Let M, N be continuous local martingales, $M(0) = N(0) = 0$ having values in a separable Hilbert space, U . Then*

$$[M + N]^{1/2} \leq ([M]^{1/2} + [N]^{1/2}) \quad (51.5.23)$$

$$[M + N] \leq 2([M] + [N]) \quad (51.5.24)$$

Also, letting ν_{M+N} denote the measure obtained from the increasing function $[M + N]$ and ν_N, ν_M defined similarly,

$$\nu_{M+N} \leq 2(\nu_M + \nu_N) \quad (51.5.25)$$

on all Borel sets.

Proof: Since $(M, N) \rightarrow [M, N]$ is bilinear and satisfies

$$\begin{aligned} [M, N] &= [N, M] \\ [aM + bM_1, N] &= a[M, N] + b[M_1, N] \\ [M, M] &\geq 0 \end{aligned}$$

which follows from Theorem 51.4.6, the usual Cauchy Schwarz inequality holds and so

$$|[M, N]| \leq [M]^{1/2} [N]^{1/2}$$

Thus

$$\begin{aligned} [M + N] &\equiv [M + N, M + N] = [M, M] + [N, N] + 2[M, N] \\ &\leq [M] + [N] + 2[M]^{1/2} [N]^{1/2} = ([M]^{1/2} + [N]^{1/2})^2 \end{aligned}$$

This proves 51.5.23. Now the right side is no larger than

$$2([M] + [N])$$

and this shows 51.5.24.

Now consider the claim about the measures. It was just shown that

$$[(M + N) - (M + N)^s] \leq 2([M - M^s] + [N - N^s])$$

and from Theorem 51.4.6 this implies that for $t > s$

$$\begin{aligned} &[M + N](t) - [M + N](s \wedge t) \\ &= [M + N](t) - [M + N](s) \\ &\leq 2([M - M^s] + [N - N^s])(t) \\ &= 2[M - M^s](t) + 2[N - N^s](t) \\ &\leq 2([M](t) - [M](s)) + 2([N](t) - [N](s)) \end{aligned}$$

Thus

$$\nu_{M+N}([s, t]) \leq 2(\nu_M([s, t]) + \nu_N([s, t]))$$

By regularity of the measures, this continues to hold with any Borel set, F in place of $[s, t]$. This proves the lemma.

Theorem 51.5.5 *The integral is well defined and has a continuous version which is a local martingale. Furthermore it satisfies the Ito isometry,*

$$E \left(\left\| \int_0^t f dM \right\|_U^2 \right) = \int_{\Omega} \int_0^t f(s)^2 d[M](s) dP$$

Let the norm on $\mathcal{G}_N \cap \mathcal{G}_M$ be the maximum of the norms on $\mathcal{G}_N$ and $\mathcal{G}_M$ and denote by $\mathcal{E}_N$ and $\mathcal{E}_M$ the elementary functions corresponding to the martingales N and M respectively. Define $\mathcal{G}_{NM}$ as the closure in $\mathcal{G}_N \cap \mathcal{G}_M$ of $\mathcal{E}_N \cap \mathcal{E}_M$. Then for $f, g \in \mathcal{G}_{NM}$,

$$E \left(\left(\int_0^t f dM, \int_0^t g dN \right) \right) = \int_{\Omega} \int_0^t f g d[M, N] \quad (51.5.26)$$

Proof: It is clear the definition is well defined because if $\{f_n\}$ and $\{g_n\}$ are two sequences of elementary functions converging to f in $L^2(\Omega; L^2([0, T], \nu(\cdot)))$ and if $\int_0^{1t} f dM$ is the integral which comes from $\{g_n\}$,

$$\begin{aligned} & \int_{\Omega} \left\| \int_0^{1t} f dM - \int_0^t f dM \right\|^2 dP \\ &= \lim_{n \rightarrow \infty} \int_{\Omega} \left\| \int_0^t g_n dM - \int_0^t f_n dM \right\|^2 dP \\ &\leq \lim_{n \rightarrow \infty} \int_{\Omega} \int_0^T \|g_n - f_n\|^2 d\nu dP = 0 \end{aligned}$$

Consider the claim the integral has a continuous version. Recall Theorem 50.7.1, part of which is listed here for convenience.

Theorem 51.5.6 *Let $\{X(t)\}$ be a right continuous nonnegative submartingale adapted to the normal filtration $\mathcal{F}_t$ for $t \in [0, T]$. Let $p \geq 1$. Define*

$$X^*(t) \equiv \sup \{X(s) : 0 < s < t\}, \quad X^*(0) \equiv 0.$$

Then for $\lambda > 0$

$$P([X^*(T) > \lambda]) \leq \frac{1}{\lambda^p} \int_{\Omega} X(T)^p dP \quad (51.5.27)$$

Let $\{f_n\}$ be a sequence of elementary functions converging to f in

$$L^2(\Omega; L^2([0, T], \nu(\cdot))).$$

Then letting

$$\begin{aligned} X_{n,m}^{\tau_l}(t) &= \left\| \int_0^t (f_n - f_m) dM^{\tau_l} \right\|_U, \\ X_{n,m}(t) &= \left\| \int_0^t (f_n - f_m) dM \right\|_U \\ &= \left\| \int_0^t f_n dM - \int_0^t f_m dM \right\|_U \end{aligned}$$

It follows $X_{n,m}^{\tau_l}$ is a continuous nonnegative submartingale and from Theorem 50.7.1 just listed,

$$\begin{aligned} P([X_{n,m}^{\tau_l^*}(T) > \lambda]) &\leq \frac{1}{\lambda^2} \int_{\Omega} X_{n,m}^{\tau_l}(T)^2 dP \\ &\leq \frac{1}{\lambda^2} \int_{\Omega} \int_0^T |f_n - f_m|^2 d[M^{\tau_l}] dP \\ &\leq \frac{1}{\lambda^2} \int_{\Omega} \int_0^T |f_n - f_m|^2 d[M] dP \end{aligned}$$

Letting $l \rightarrow \infty$,

$$P([X_{n,m}^*(T) > \lambda]) \leq \frac{1}{\lambda^2} \int_{\Omega} \int_0^T |f_n - f_m|^2 d[M] dP$$

Therefore, there exists a subsequence, still denoted by $\{f_n\}$ such that

$$P([X_{n,n+1}^*(T) > 2^{-n}]) < 2^{-n}$$

Then by the Borel Cantelli lemma, the ω in infinitely many of the sets

$$[X_{n,n+1}^*(T) > 2^{-n}]$$

has measure 0. Denoting this exceptional set as N , it follows that for $\omega \notin N$, there exists $n(\omega)$ such that for $n > n(\omega)$,

$$\sup_{t \in [0, T]} \left\| \int_0^t f_n dM - \int_0^t f_{n+1} dM \right\| \leq 2^{-n}$$

and this implies uniform convergence of $\left\{ \int_0^t f_n dM \right\}$. Letting

$$G(t) = \lim_{n \rightarrow \infty} \int_0^t f_n dM,$$

for $\omega \notin N$ and $G(t) = 0$ for $\omega \in N$, it follows that for each t , the continuous adapted process $G(t)$ equals $\int_0^t f dM$ a.e. Thus $\left\{ \int_0^t f dM \right\}$ has a continuous version.

It suffices to verify 51.5.26. Let $\{f_n\}$ and $\{g_n\}$ be sequences of elementary functions converging to f and g in $\mathcal{G}_M \cap \mathcal{G}_N$. By Lemma 51.5.2,

$$E \left(\left(\int_0^t f_n dM, \int_0^t g_n dN \right)_U \right) = \int_{\Omega} \int_0^t f_n g_n d[M, N]$$

Then by the Holder inequality and the above definition,

$$\lim_{n \rightarrow \infty} E \left(\left(\int_0^t f_n dM, \int_0^t g_n dN \right)_U \right) = E \left(\left(\int_0^t f dM, \int_0^t g dN \right)_U \right)$$

Consider the right side which equals

$$\frac{1}{4} \int_{\Omega} \int_0^t f_n g_n d[M + N] dP - \frac{1}{4} \int_{\Omega} \int_0^t f_n g_n d[M - N] dP$$

Now from Lemma 51.5.4,

$$\begin{aligned} & \left| \int_{\Omega} \int_0^t f_n g_n d[M + N] dP - \int_{\Omega} \int_0^t f g d[M + N] dP \right| \\ &= \left| \int_{\Omega} \int_0^t f_n g_n d\nu_{M+N} dP - \int_{\Omega} \int_0^t f g d\nu_{M+N} dP \right| \\ &\leq \int_{\Omega} \int_0^t |f_n g_n - f g| d\nu_M dP + \int_{\Omega} \int_0^t |f_n g_n - f g| d\nu_N dP \end{aligned}$$

and by the choice of the f_n and g_n , these both converge to 0. Similar considerations apply to

$$\left| \int_{\Omega} \int_0^t f_n g_n d[M - N] dP - \int_{\Omega} \int_0^t f g d[M - N] dP \right|$$

and show

$$\lim_{n \rightarrow \infty} \int_{\Omega} \int_0^t f_n g_n d[M, N] = \int_{\Omega} \int_0^t f g d[M, N]$$

This proves the theorem.

51.6 Doob Meyer Decomposition

This section is on the Doob Meyer decomposition which is a way of starting with a submartingale and writing it as the sum of a martingale and an increasing adapted stochastic process of a certain form. This is more general than what was done above with the submartingales $\|M(t)\|^2$ for $M(t) \in \mathcal{M}_T^2(H)$ where M is a continuous martingale. There are two forms for this theorem, one for discrete martingales and one for martingales defined on an interval of the real line which is much harder. According to [37], this material is found in [41] however, I am following [37] for the continuous version of this theorem.

Theorem 51.6.1 *Let $\{X_n\}$ be a submartingale. Then there exists a unique stochastic process, $\{A_n\}$ and martingale, $\{M_n\}$ such that*

1. $A_n(\omega) \leq A_{n+1}(\omega)$, $A_1(\omega) = 0$,
 2. A_n is $\mathcal{F}_{n-1}$ adapted for all $n \geq 1$ where $\mathcal{F}_0 \equiv \mathcal{F}_1$.
- and also $X_n = M_n + A_n$.

Proof: Let $A_1 \equiv 0$ and define

$$A_n \equiv \sum_{k=2}^n E(X_k - X_{k-1} | \mathcal{F}_{k-1}).$$

It follows A_n is $\mathcal{F}_{n-1}$ measurable. Since $\{X_k\}$ is a submartingale, A_n is increasing because

$$A_{n+1} - A_n = E(X_{n+1} - X_n | \mathcal{F}_n) \geq 0 \quad (51.6.28)$$

It is a submartingale because

$$E(A_n | \mathcal{F}_{n-1}) = A_n \geq A_{n-1}.$$

Now let M_n be defined by

$$X_n = M_n + A_n.$$

Then from 51.6.28,

$$\begin{aligned} E(M_{n+1} | \mathcal{F}_n) &= E(X_{n+1} | \mathcal{F}_n) - E(A_{n+1} | \mathcal{F}_n) \\ &= E(X_{n+1} | \mathcal{F}_n) - E(A_{n+1} - A_n | \mathcal{F}_n) - A_n \\ &= E(X_{n+1} | \mathcal{F}_n) - E(E(X_{n+1} - X_n | \mathcal{F}_n) | \mathcal{F}_n) - A_n \\ &= E(X_{n+1} | \mathcal{F}_n) - E(X_{n+1} - X_n | \mathcal{F}_n) - A_n \\ &= E(X_n | \mathcal{F}_n) - A_n \\ &= X_n - A_n \equiv M_n \end{aligned}$$

This proves the existence part.

It remains to verify uniqueness. Suppose then that

$$X_n = M_n + A_n = M'_n + A'_n$$

where $\{A_n\}$ and $\{A'_n\}$ both satisfy the conditions of the theorem and $\{M_n\}$ and $\{M'_n\}$ are both martingales. Then

$$M_n - M'_n = A'_n - A_n$$

and so, since $A'_n - A_n$ is $\mathcal{F}_{n-1}$ measurable and $\{M_n - M'_n\}$ is a martingale,

$$\begin{aligned} M_{n-1} - M'_{n-1} &= E(M_n - M'_n | \mathcal{F}_{n-1}) \\ &= E(A'_n - A_n | \mathcal{F}_{n-1}) \\ &= A'_n - A_n = M_n - M'_n. \end{aligned}$$

Continuing this way shows $M_n - M'_n$ is a constant. However, since $A'_1 - A_1 = 0 = M_1 - M'_1$, it follows $M_n = M'_n$ and this proves uniqueness. This proves the theorem.

Definition 51.6.2 A stochastic process, $\{A_n\}$ which satisfies the conditions of Theorem 51.6.1,

$$A_n(\omega) \leq A_{n+1}(\omega)$$

and

$$A_n \text{ is } \mathcal{F}_{n-1} \text{ adapted for all } n \geq 1$$

where $\mathcal{F}_0 \equiv \mathcal{F}_1$ is said to be natural.

The Doob Meyer theorem needs to be extended to continuous submartingales and this will require another description of what it means for a stochastic process to be natural. To get an idea of what this condition should be, here is a lemma.

Lemma 51.6.3 Let a stochastic process, $\{A_n\}$ be natural. Then for every martingale, $\{M_n\}$,

$$E(M_n A_n) = E\left(\sum_{j=1}^{n-1} M_j (A_{j+1} - A_j)\right)$$

Proof: Start with the right side.

$$\begin{aligned} E\left(\sum_{j=1}^{n-1} M_j (A_{j+1} - A_j)\right) &= E\left(\sum_{j=2}^n M_{j-1} A_j - \sum_{j=1}^{n-1} M_j A_j\right) \\ &= E\left(\sum_{j=2}^{n-1} A_j (M_{j-1} - M_j)\right) + E(M_{n-1} A_n) \end{aligned}$$

Then the first term equals zero because since A_j is $\mathcal{F}_{j-1}$ measurable,

$$\begin{aligned} \int_{\Omega} A_j M_{j-1} dP - \int_{\Omega} A_j M_j &= \int_{\Omega} A_j E(M_j | \mathcal{F}_{j-1}) dP - \int_{\Omega} A_j M_j dP \\ &= \int_{\Omega} E(A_j M_j | \mathcal{F}_{j-1}) dP - \int_{\Omega} A_j M_j dP \\ &= \int_{\Omega} A_j M_j dP - \int_{\Omega} A_j M_j dP = 0. \end{aligned}$$

The last term equals

$$\begin{aligned} \int_{\Omega} M_{n-1} A_n dP &= \int_{\Omega} E(M_n | \mathcal{F}_{n-1}) A_n dP \\ &= \int_{\Omega} E(M_n A_n | \mathcal{F}_{n-1}) dP = E(M_n A_n). \end{aligned}$$

This proves the lemma.

Definition 51.6.4 Let A be an increasing function defined on $\mathbb{R}$. By Theorem 3.3.4 on Page 42 there exists a positive linear functional, L defined on $C_c(\mathbb{R})$ given by

$$Lf \equiv \int_a^b f dA \text{ where } \text{spt}(f) \subseteq [a, b]$$

where the integral is just the Riemann Stieltjes integral. Then by the Riesz representation theorem, Theorem 9.4.2 on Page 234, there exists a unique Radon measure, μ which extends this functional, as described in the Riesz representation theorem. Then for B a measurable set, I will write either

$$\int_B f d\mu \text{ or } \int_B f dA$$

to denote the Lebesgue integral,

$$\int \chi_B f d\mu.$$

Lemma 51.6.5 Let f be right continuous. Then f is Borel measurable. Also, if the limit from the left exists, then $f_-(x) \equiv f(x)_- \equiv \lim_{y \rightarrow x-} f(y)$ is also Borel measurable. If A is an increasing right continuous function and f is right continuous and f_- , the left limit function exists, then if f is bounded, on $[a, b]$, and if

$$\left\{ x_0^p, \dots, x_{n_p}^p \right\}_{p=1}^{\infty}$$

is a sequence of partitions of $[a, b]$ such that

$$\lim_{p \rightarrow \infty} \max \left\{ |x_k^p - x_{k-1}^p| : k = 1, 2, \dots, n_p \right\} = 0 \quad (51.6.29)$$

then

$$\int_{(a,b]} f_- dA = \lim_{p \rightarrow \infty} \sum_{k=1}^{n_p} f(x_{k-1}^p) (A(x_k^p) - A(x_{k-1}^p)) \quad (51.6.30)$$

More generally, let

$$D \equiv \cup_{p=1}^{\infty} \left\{ x_0^p, \dots, x_{n_p}^p \right\}$$

and

$$f_-(t) = \lim_{s \rightarrow t-, s \in D} f(s).$$

Then 51.6.30 holds.

Proof: For $x \in f^{-1}((a, \infty))$, denote by I_x the union of all intervals containing x such that $f(y)$ is larger than a for all y in the interval. Since f is right continuous, each I_x has positive length. Now if I_x and I_y are two of these intervals, then either they must have empty intersection or they are the same interval. Thus $f^{-1}((a, \infty))$

is of the form $\cup_{x \in f^{-1}((a, \infty))} I_x$ and there can only be countably many distinct intervals because each has positive length and $\mathbb{R}$ is separable. Hence $f^{-1}((a, \infty))$ equals the countable union of intervals and is therefore, Borel measurable. Now

$$f_{-}(x) = \lim_{n \rightarrow \infty} f(x - r_n) \equiv \lim_{n \rightarrow \infty} f_{r_n}(x)$$

where r_n is a decreasing sequence converging to 0. Now each f_{r_n} is Borel measurable by the first part of the proof because it is right continuous and so it follows the same is true of f_{-} .

Finally consider the claim about the integral. Since A is right continuous, a simple argument involving the dominated convergence theorem and approximating $(c, d]$ with a piecewise linear continuous function nonzero only on $(c, d + h)$ which approximates $\mathcal{X}_{(c, d]}$ will show that for μ the measure of Definition 51.6.4

$$\mu((c, d]) = A(d) - A(c).$$

Therefore, the sum in 51.6.30 is of the form

$$\sum_{k=1}^{n_p} f(x_{k-1}^p) \mu((x_{k-1}, x_k]) = \int_{(a, b]} \sum_{k=1}^{n_p} f(x_{k-1}^p) \mathcal{X}_{(x_{k-1}, x_k]} d\mu$$

and by 51.6.29

$$\lim_{p \rightarrow \infty} \sum_{k=1}^{n_p} f(x_{k-1}^p) \mathcal{X}_{(x_{k-1}, x_k]}(x) = f_{-}(x)$$

for each $x \in (a, b]$. Therefore, since f is bounded, 51.6.30 follows from the dominated convergence theorem. The last claim follows the same way. This proves the lemma.

Definition 51.6.6 *An increasing stochastic process, $\{A(t)\}$ which is right continuous is said to be natural if $A(0) = 0$ and whenever $\{\xi(t)\}$ is a bounded right continuous martingale,*

$$E(A(t)\xi(t)) = E\left(\int_{(0, t]} \xi_{-}(s) dA(s)\right). \quad (51.6.31)$$

Here

$$\xi_{-}(s, \omega) \equiv \lim_{r \rightarrow s-, r \in D} \xi(r, \omega)$$

a.e. where D is a countable dense subset of $[0, t]$. By Corollary 50.6.2 the right side of 51.6.31 is not dependent on the choice of D since if ξ_{-} is computed using two different dense subsets, the two random variables are equal a.e.

Some discussion is in order for this definition. Pick $\omega \in \Omega$. Then since A is right continuous, the function $t \rightarrow A(t, \omega)$ is increasing and right continuous. Therefore, one can do the Lebesgue Stieltjes integral defined in Definition 51.6.4 for each ω whenever f is Borel measurable and bounded. Now it is assumed $\{\xi(t)\}$ is bounded

and right continuous. By Lemma 51.6.5 $\xi_-(t) \equiv \lim_{r \rightarrow t-, r \in D} \xi(r)$ is measurable and by this lemma,

$$\int_{(0,t]} \xi_-(s) dA(s) = \lim_{p \rightarrow \infty} \sum_{k=1}^{n_p} \xi(t_{k-1}^p) (A(t_k^p) - A(t_{k-1}^p))$$

where $\{t_k^p\}_{k=1}^{n_p}$ is a sequence of partitions of $[0, t]$ such that

$$\lim_{p \rightarrow \infty} \max \{ |t_k^p - t_{k-1}^p| : k = 1, 2, \dots, n_p \} = 0. \quad (51.6.32)$$

and $D \equiv \cup_{p=1}^{\infty} \cup_{k=1}^{n_p} \{t_k^p\}_{k=1}^{n_p}$.

Also, if $t \rightarrow A(t, \omega)$ is right continuous, hence Borel measurable, then for $\xi(t)$ the above bounded right continuous martingale, it follows it makes sense to write

$$\int_{(0,t]} \xi(s) dA(s).$$

Consider the right sum,

$$\sum_{k=1}^{n_p} \xi(t_k^p) (A(t_k^p) - A(t_{k-1}^p))$$

This equals

$$\int_{(0,t]} \sum_{k=1}^{n_p} \xi(t_k^p) \mathcal{X}_{(t_{k-1}^p, t_k^p]}(s) dA(s)$$

and by right continuity, it follows

$$\lim_{p \rightarrow \infty} \sum_{k=1}^{n_p} \xi(t_k^p) \mathcal{X}_{(t_{k-1}^p, t_k^p]}(s) = \xi(s)$$

and so the dominated convergence theorem applies and it follows

$$\lim_{p \rightarrow \infty} \sum_{k=1}^{n_p} \xi(t_k^p) (A(t_k^p) - A(t_{k-1}^p)) = \int_{(0,t]} \xi(s) dA(s)$$

where this is a random variable. Thus

$$E \left(\int_{(0,t]} \xi(s) dA(s) \right) = \int_{\Omega} \left(\lim_{p \rightarrow \infty} \int_{(0,t]} \sum_{k=1}^{n_p} \xi(t_k^p) \mathcal{X}_{(t_{k-1}^p, t_k^p]}(s) dA(s) \right) dP \quad (51.6.33)$$

Now as mentioned above,

$$\int_{(0,t]} \sum_{k=1}^{n_p} \xi(t_k^p) \mathcal{X}_{(t_{k-1}^p, t_k^p]}(s) dA(s) = \sum_{k=1}^{n_p} \xi(t_k^p) (A(t_k^p) - A(t_{k-1}^p))$$

and since A is increasing, this is bounded above by an expression of the form $CA(t)$, a function in L^1 . Therefore, by the dominated convergence theorem, 51.6.33 reduces to

$$\begin{aligned}
 & \lim_{p \rightarrow \infty} \int_{\Omega} \int_{(0,t]} \sum_{k=1}^{n_p} \xi(t_k^p) \mathcal{X}_{(t_{k-1}^p, t_k^p]}(s) dA(s) dP \\
 &= \lim_{p \rightarrow \infty} \int_{\Omega} \sum_{k=1}^{n_p} \xi(t_k^p) (A(t_k^p) - A(t_{k-1}^p)) dP \\
 &= \lim_{p \rightarrow \infty} \int_{\Omega} \left(\sum_{k=1}^{n_p} \xi(t_k^p) A(t_k^p) - \sum_{k=0}^{n_p-1} \xi(t_{k+1}^p) A(t_k^p) \right) dP \\
 &= \lim_{p \rightarrow \infty} \sum_{k=1}^{n_p-1} \int_{\Omega} (\xi(t_k^p) - \xi(t_{k+1}^p)) A(t_k^p) dP + \int_{\Omega} \xi(t) A(t) dP. \quad (51.6.34)
 \end{aligned}$$

Since ξ is a martingale,

$$\begin{aligned}
 \int_{\Omega} \xi(t_{k+1}^p) A(t_k^p) dP &= \int_{\Omega} E\left(\xi(t_{k+1}^p) A(t_k^p) \mid \mathcal{F}_{t_k^p}\right) dP \\
 &= \int_{\Omega} A(t_k^p) E\left(\xi(t_{k+1}^p) \mid \mathcal{F}_{t_k^p}\right) dP \\
 &= \int_{\Omega} A(t_k^p) \xi(t_k^p) dP
 \end{aligned}$$

and so in 51.6.34 the term with the sum equals 0 and it reduces to

$$E(\xi(t) A(t)).$$

This is sufficiently interesting to state as a lemma.

Lemma 51.6.7 *Let A be an increasing adapted stochastic process which is right continuous. Also let $\xi(t)$ be a bounded right continuous martingale. Then*

$$E(\xi(t) A(t)) = E\left(\int_{(0,t]} \xi(s) dA(s)\right)$$

and A is natural, if and only if for all such bounded right continuous martingales,

$$E(\xi(t) A(t)) = E\left(\int_{(0,t]} \xi(s) dA(s)\right) = E\left(\int_{(0,t]} \xi_{-}(s) dA(s)\right)$$

Lemma 51.6.8 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{G}$ be a σ algebra contained in $\mathcal{F}$. Suppose also that $\{f_n\}$ is a sequence in $L^1(\Omega)$ which converges weakly to f in $L^1(\Omega)$. That is, for every $h \in L^\infty(\Omega)$,*

$$\int_{\Omega} f_n h dP \rightarrow \int_{\Omega} f h dP.$$

Then $E(f_n | \mathcal{G})$ converges weakly in $L^1(\Omega)$ to $E(f | \mathcal{G})$.

Proof: First note that if $h \in L^\infty(\Omega, \mathcal{F})$, then $E(h|\mathcal{G}) \in L^\infty(\Omega, \mathcal{G})$ because if $A \in \mathcal{G}$,

$$\int_A |E(h|\mathcal{G})| dP \leq \int_A E(|h||\mathcal{G}) dP = \int_A |h| dP$$

and so if $A = [|E(h|\mathcal{G})| > \|h\|_\infty]$, then if $P(A) > 0$,

$$\|h\|_\infty P(A) < \int_A |E(h|\mathcal{G})| dP \leq \int_A |h| dP \leq \|h\|_\infty P(A),$$

a contradiction. Hence $P(A) = 0$ and so $E(h|\mathcal{G}) \in L^\infty(\Omega, \mathcal{G})$ as claimed. Let $h \in L^\infty(\Omega, \mathcal{G})$.

$$\begin{aligned} \int_\Omega E(f_n|\mathcal{G}) h dP &= \int_\Omega E(E(f_n|\mathcal{G}) h|\mathcal{G}) dP \\ &= \int_\Omega E(f_n|\mathcal{G}) E(h|\mathcal{G}) dP \\ &= \int_\Omega E(f_n E(h|\mathcal{G})|\mathcal{G}) dP \\ &= \int_\Omega f_n E(h|\mathcal{G}) dP \end{aligned}$$

and so

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_\Omega E(f_n|\mathcal{G}) h dP &= \lim_{n \rightarrow \infty} \int_\Omega f_n E(h|\mathcal{G}) dP \\ &= \int_\Omega f E(h|\mathcal{G}) dP \\ &= \int_\Omega E(f E(h|\mathcal{G})|\mathcal{G}) dP \\ &= \int_\Omega E(h|\mathcal{G}) E(f|\mathcal{G}) dP \\ &= \int_\Omega E(E(f|\mathcal{G}) h|\mathcal{G}) dP \\ &= \int_\Omega E(f|\mathcal{G}) h dP \end{aligned}$$

and this proves the lemma.

Next suppose $\{X(t)\}$ is a real submartingale and suppose $X(t) = M(t) + A(t)$ where $A(t)$ is an increasing stochastic process adapted to $\mathcal{F}_t$ such that $A(0) = 0$ and $\{M(t)\}$ is a martingale adapted to $\mathcal{F}_t$. Also let T be a stopping time bounded above by a . Then by the optional sampling theorem, and the observation that $\{|M(t)|\}$

is a submartingale

$$\begin{aligned}
 & \int_{[|X(T)| \geq \lambda]} |X(T)| dP \\
 & \leq \int_{[|X(T)| \geq \lambda]} |M(T)| dP + \int_{[|X(T)| \geq \lambda]} A(T) dP \\
 & \leq \int_{[|X(T)| \geq \lambda]} E(|M(a)| | \mathcal{F}_T) dP + \int_{[|X(T)| \geq \lambda]} E(A(a) | \mathcal{F}_T) dP \\
 & \leq \int_{[|X(T)| \geq \lambda]} |M(a)| dP + \int_{[|X(T)| \geq \lambda]} A(a) dP
 \end{aligned}$$

Now by Theorem 48.4.5,

$$P(|X_T| \geq \lambda) \leq \frac{2}{\lambda} E(|X(0)| + |X(a)|)$$

and so $P(|X(T)| \geq \lambda) \rightarrow 0$ uniformly for T a stopping time bounded by a as $\lambda \rightarrow \infty$ and so this shows equi integrability of $\{X(T)\}$ because $A(t, \omega) \geq 0$.

This motivates the following definition.

Definition 51.6.9 A stochastic process, $\{X(t)\}$ is called *DL* if for all $a > 0$, the set of random variables, $\{X(T)\}$ for T a stopping time bounded by a is equi integrable.

Example 51.6.10 Let $\{M(t)\}$ be a continuous martingale. Then $\{M(t)\}$ is of class *DL*.

To show this, let $a > 0$ be given and let T be a stopping time bounded by a . Then by the optional sampling theorem, $M(0), M(T), M(a)$ is a martingale and so

$$E(M(a) | \mathcal{F}_T) = M(T)$$

and so by Jensen's inequality, $|M(T)| \leq E(|M(a)| | \mathcal{F}_T)$. Therefore,

$$\begin{aligned}
 \int_{[|M(T)| \geq \lambda]} |M(T)| dP & \leq \int_{[|M(T)| \geq \lambda]} E(|M(a)| | \mathcal{F}_T) dP \\
 & = \int_{[|M(T)| \geq \lambda]} |M(a)| dP.
 \end{aligned} \tag{51.6.35}$$

Now by Theorem 50.3.5,

$$P(|M(T)| \geq \lambda) \leq \frac{1}{\lambda} E(|M(a)|)$$

and so since a given L^1 function is uniformly integrable, there exists δ such that if $P(A) < \delta$ then

$$\int_A |M(a)| dP < \varepsilon.$$

Now choose λ large enough that

$$\frac{1}{\lambda} E(|M(a)|) < \delta.$$

Then for such λ , it follows from 51.6.35 that for any stopping time bounded by a ,

$$\int_{[|M(T)| \geq \lambda]} |M(T)| dP < \varepsilon.$$

This shows M is DL .

Example 51.6.11 Let $\{X(t)\}$ be a nonnegative submartingale with $t \rightarrow E(X(t))$ right continuous so $\{X(t)\}$ can be considered right continuous. Then $\{X(t)\}$ is DL .

To show this, let T be a stopping time bounded by $a > 0$. Then by the optional sampling theorem,

$$\int_{[X(T) \geq \lambda]} X(T) dP \leq \int_{[X(T) \geq \lambda]} X(a) dP$$

and now by Theorem 48.4.5 on Page 1389

$$P([X(T) \geq \lambda]) \leq \frac{1}{\lambda} E(X_a^+).$$

Thus if $\varepsilon > 0$ is given, there exists λ large enough that for any stopping time, $T \leq a$,

$$\int_{[X(T) \geq \lambda]} X(T) dP \leq \varepsilon$$

Thus the submartingale is DL .

Now with this preparation, here is the Doob Meyer decomposition.

Theorem 51.6.12 Let $\{X(t)\}$ be a submartingale of class DL . Then there exists a martingale, $\{M(t)\}$ and an increasing submartingale, $\{A(t)\}$ such that for each t ,

$$X(t) = M(t) + A(t).$$

If $\{A(t)\}$ is chosen to be natural and $A(0) = 0$, then with this condition, $\{M(t)\}$ and $\{A(t)\}$ are unique.

Proof: First I will show uniqueness. Suppose then that

$$X(t) = M(t) + A(t) = M'(t) + A'(t)$$

where M, M' and A, A' satisfy the given conditions. Let $t > 0$ and consider $s \in [0, t]$. Then

$$A(s) - A'(s) = M'(s) - M(s)$$

Since A, A' are natural, it follows that for $\xi(t)$ a right continuous bounded martingale,

$$\begin{aligned} E(\xi(t)(A(t) - A'(t))) &= E\left(\int_{(0,t]} \xi_{-}(s) dA(s)\right) - E\left(\int_{(0,t]} \xi_{-}(s) dA'(s)\right) \\ &= E\left(\lim_{n \rightarrow \infty} \sum_{k=1}^{m_n} \xi(t_{k-1}^n) (A(t_k^n) - A(t_{k-1}^n)) - \sum_{k=1}^{m_n} \xi(t_{k-1}^n) (A'(t_k^n) - A'(t_{k-1}^n))\right) \end{aligned}$$

where $\{t_k^n\}_{k=0}^{m_n}$ is a sequence of partitions of $[0, t]$ such that these are equally spaced points, $\lim_{n \rightarrow \infty} t_{k+1}^n - t_k^n = 0$, and $\{t_k^n\}_{k=0}^{m_n} \subseteq \{t_k^{n+1}\}_{k=0}^{m_{n+1}}$. Then since $A(t)$ and $A'(t)$ are increasing, the absolute value of each sum is bounded above by an expression of the form

$$CA(t) \text{ or } CA'(t)$$

and so the dominated convergence theorem can be applied to get the above expression to equal

$$\lim_{n \rightarrow \infty} E\left(\sum_{k=1}^{m_n} \xi(t_{k-1}^n) (A(t_k^n) - A(t_{k-1}^n)) - \sum_{k=1}^{m_n} \xi(t_{k-1}^n) (A'(t_k^n) - A'(t_{k-1}^n))\right)$$

Now using $X = A + M$ and $X = A' + M'$

$$= \lim_{n \rightarrow \infty} E\left(\sum_{k=1}^{m_n} \xi(t_{k-1}^n) (M(t_k^n) - M(t_{k-1}^n)) - \sum_{k=1}^{m_n} \xi(t_{k-1}^n) (M'(t_k^n) - M'(t_{k-1}^n))\right).$$

Both terms in the above equal 0. Here is why.

$$\begin{aligned} E(\xi(t_{k-1}^n) M(t_k^n)) &= E\left(E(\xi(t_{k-1}^n) M(t_k^n) | \mathcal{F}_{t_{k-1}^n})\right) \\ &= E\left(\xi(t_{k-1}^n) E(M(t_k^n) | \mathcal{F}_{t_{k-1}^n})\right) \\ &= E(\xi(t_{k-1}^n) M(t_{k-1}^n)). \end{aligned}$$

Thus the expected value of the first sum equals 0. Similarly, the expected value of the second sum equals 0. Hence this has shown that for any bounded right continuous martingale, $\{\xi(s)\}$ and $t > 0$,

$$E(\xi(t)(A(t) - A'(t))) = 0.$$

Now let ξ be a bounded random variable and let $\xi(t)$ be a right continuous version of the martingale $E(\xi | \mathcal{F}_t)$. Then

$$\begin{aligned} 0 &= E(E(\xi | \mathcal{F}_t)(A(t) - A'(t))) = E(E(\xi(A(t) - A'(t)) | \mathcal{F}_t)) \\ &= E(\xi(A(t) - A'(t))) \end{aligned}$$

and since ξ is arbitrary, it follows that $A(t) = A'(t)$ a.e. which proves uniqueness.

Because of the uniqueness assertion, it suffices to prove the theorem on an arbitrary interval, $[0, a]$.

Without loss of generality, it can be assumed $X(0) = 0$ since otherwise, you could simply consider $X(t) - X(0)$ in its place and then at the end, add $X(0)$ to $M(t)$. Let $\{t_k^n\}_{k=0}^{m_n}$ be a sequence of partitions of $[0, a]$ such that these are equally spaced points, $\lim_{n \rightarrow \infty} t_{k+1}^n - t_k^n = 0$, and $\{t_k^n\}_{k=0}^{m_n} \subseteq \{t_k^{n+1}\}_{k=0}^{m_{n+1}}$. Then consider the submartingale, $\{X(t_k^n)\}_{k=0}^{m_n}$. Theorem 51.6.1 implies there exists a unique martingale, and increasing submartingale, $\{M(t_k^n)\}_{k=0}^{m_n}$ and $\{A(t_k^n)\}_{k=0}^{m_n}$ respectively such that $M(0) = 0 = A(0)$,

$$X(t_k^n) = M(t_k^n) + A(t_k^n).$$

and $A^n(t_k^n)$ is $\mathcal{F}_{t_{k-1}^n}$ measurable. Recall how these were defined.

$$A^n(t_k^n) = \sum_{j=1}^k E\left(X(t_j^n) - X(t_{j-1}^n) \mid \mathcal{F}_{t_{j-1}^n}\right), \quad A^n(0) = 0$$

$$M^n(t_k^n) = X(t_k^n) - A^n(t_k^n).$$

I want to show that $\{A^n(a)\}$ is equi integrable. From this there will be a weakly convergent subsequence and nice things will happen. Define $T^n(\omega)$ to equal t_{j-1}^n where t_j^n is the first time where $A^n(t_j^n, \omega) \geq \lambda$ or $T^n(\omega) = a$ if this never happens. I want to say that T^n is a stopping time and so I need to verify that $[T^n \leq t_j^n] \in \mathcal{F}_{t_j^n}$ for each j . If $\omega \in [T^n \leq t_j^n]$, then this means the first time, t_k^n , where $A^n(t_k^n, \omega) \geq \lambda$ is such that $t_k^n \leq t_{j+1}^n$. Since A_k^n is increasing in k ,

$$\begin{aligned} [T^n \leq t_j^n] &= \cup_{k=0}^{j+1} [A^n(t_k^n) \geq \lambda] \\ &= [A^n(t_{j+1}^n) \geq \lambda] \in \mathcal{F}_{t_j^n}. \end{aligned}$$

Note T^n only has the values t_k^n . Thus for $t \in [t_{j-1}^n, t_j^n)$,

$$[T^n \leq t] = [T^n \leq t_{j-1}^n] \in \mathcal{F}_{t_{j-1}^n} \subseteq \mathcal{F}_t.$$

Thus T^n is one of those stopping times bounded by a . Since $\{X(t)\}$ is DL , this shows $\{X(T^n)\}$ is equi integrable. Now from the definition of T^n , it follows

$$A^n(T^n) \leq \lambda$$

Recall $T^n(\omega) = t_{j-1}^n$ where t_j^n is the first time where $A^n(t_j^n, \omega) \geq \lambda$ or $T^n(\omega) = a$ if this never happens. Thus T^n is such that it is before A^n gets larger than λ . Thus,

$$\begin{aligned} \int_{[A^n(a) \geq 2\lambda]} \frac{1}{2} A^n(a) dP &\leq \int_{[A^n(a) \geq 2\lambda]} (A^n(a) - \lambda) dP \\ &\leq \int_{[A^n(a) \geq 2\lambda]} (A^n(a) - A^n(T^n)) dP \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\Omega} (A^n(a) - A^n(T^n)) dP \\
&= \int_{\Omega} (X(a) - M^n(a) - (X(T^n) - M^n(T^n))) dP \\
&= \int_{\Omega} (X(a) - X(T^n)) dP
\end{aligned}$$

Because by the discrete optional sampling theorem,

$$\int_{\Omega} (M^n(a) - M^n(T^n)) dP = 0.$$

Remember $\{M^n(t_k^n)\}_{k=0}^{m_n}$ was a martingale.

$$\begin{aligned}
\int_{\Omega} (X(a) - X(T^n)) dP &= \int_{[A^n(a) \geq \lambda]} (X(a) - X(T^n)) dP \\
&\quad + \int_{[A^n(a) < \lambda]} (X(a) - X(T^n)) dP.
\end{aligned}$$

The second of the integrals on the right is such that for ω in this set, $T^n(\omega) = a$ and so the second integral equals 0. Hence from the above,

$$\int_{[A^n(a) \geq 2\lambda]} \frac{1}{2} A^n(a) dP \leq \int_{[A^n(a) \geq \lambda]} (X(a) - X(T^n)) dP$$

and since $\{X(t)\}$ is DL , this shows $\{A^n(a)\}_{n=1}^{\infty}$ is equi integrable.

By Corollary 16.7.6 on Page 520 there exists a subsequence $\{A^{n_k}(a)\}_{k=1}^{\infty}$ which converges weakly in $L^1(\Omega)$ to $A(a)$. By Lemma 51.6.8 it also follows that $E(A^{n_k}(a) | \mathcal{F}_t)$ converges weakly to $E(A(a) | \mathcal{F}_t)$ in $L^1(\Omega)$. Now define

$$M(t) \equiv E(X(a) - A(a) | \mathcal{F}_t).$$

Thus it is obvious from properties of conditional expectation that $\{M(t)\}$ is a martingale adapted to $\mathcal{F}_t$ and without loss of generality, it is a right continuous version. Let

$$A(t) \equiv X(t) - M(t).$$

Then since $\{X(t)\}$ is a submartingale, it follows $\{A(t)\}$ is also a submartingale.

It remains to show several things. First, it is necessary to show $A(t)$ is increasing in t and $A(0) = 0$. To see this, let $s < t$, $s, t \in \cup_{n=1}^{\infty} \cup_{k=0}^{m_n} t_k^n$. Then letting n large enough both s, t are in $\cup_{k=0}^{m_n} t_k^n$. Only consider such n . Let $t = t_{k(t)}^n$, $s = t_{k(s)}^n$ and let $h \in L^{\infty}(\Omega)$, $h \geq 0$. Then

$$\begin{aligned}
\int_{\Omega} (A(t) - A(s)) h dP &= \int_{\Omega} (X(t) - M(t) - (X(s) - M(s))) h dP \\
\int_{\Omega} (X(t) - E(X(a) - A(a) | \mathcal{F}_t) - (X(s) - E(X(a) - A(a) | \mathcal{F}_s))) h dP.
\end{aligned} \tag{51.6.36}$$

Now by Lemma 51.6.8, the following weak limit holds.

$$\begin{aligned} E(X(a) - A(a) | \mathcal{F}_t) &= \lim_{k \rightarrow \infty} E \left(\overbrace{X(a) - A^{n_k}(a)}^{M^{n_k}(a)} | \mathcal{F}_t \right) \\ &= \lim_{k \rightarrow \infty} M^{n_k}(t) \end{aligned}$$

A similar formula holds for s in place of t . Then the expression in 51.6.36 equals

$$\begin{aligned} &= \lim_{k \rightarrow \infty} \int_{\Omega} (X(t) - M^{n_k}(t) - (X(s) - M^{n_k}(s))) h dP \\ &= \lim_{k \rightarrow \infty} \int_{\Omega} (A^{n_k}(t) - A^{n_k}(s)) h dP \geq 0 \end{aligned}$$

Since $h \geq 0$ is arbitrary, this shows $A(t) - A(s) \geq 0$ a.e. Not requiring $h \geq 0$, the above argument also shows that for $s, t \in \cup_{n=1}^{\infty} \cup_{k=0}^{m_n} t_k^n$,

$$A(t) - A(s) = \text{weak} \lim_{p \rightarrow \infty} A^{n_p}(t) - A^{n_p}(s). \quad (51.6.37)$$

Now consider the claim that $A(0) = 0$. Recall

$$A(0) \equiv X(0) - E(X(a) - A(a) | \mathcal{F}_0) = -E(X(a) - A(a) | \mathcal{F}_0)$$

and so

$$\begin{aligned} A(0) &= \lim_{k \rightarrow \infty} -E(X(a) - A^{n_k}(a) | \mathcal{F}_0) \\ &= \lim_{k \rightarrow \infty} -E(M^{n_k}(a) | \mathcal{F}_0) = \lim_{k \rightarrow \infty} -M^{n_k}(0) = 0. \end{aligned}$$

This proves the theorem except for the claim that $A(t)$ is natural. Let $\xi(t)$ be a bounded right continuous martingale. I need to consider

$$E \left(\int_{(0,t]} \xi_-(s) dA(s) \right)$$

and show it equals $\xi(t) A(t)$. First consider the case $t = a$. By Lemma 51.6.5,

$$E \left(\int_{(0,a]} \xi_-(s) dA(s) \right) \equiv E \left(\lim_{k \rightarrow \infty} \sum_{j=1}^{m_{n_k}} \xi(t_{j-1}^{n_k}) (A(t_j^{n_k}) - A(t_{j-1}^{n_k})) \right) \quad (51.6.38)$$

Since ξ is bounded, you can take the limit outside. This follows from the dominated convergence theorem and the fact, shown above that A is increasing and nonnegative. Here is why.

$$0 \leq |\xi(t_{j-1}^{n_k})| A(t_j^{n_k}) \leq A(a) C$$

where C is a constant larger than the values of ξ . Thus the above equals

$$\begin{aligned}
 & \lim_{k \rightarrow \infty} E \left(\sum_{j=1}^{m_{n_k}} \xi(t_{j-1}^{n_k}) (A(t_j^{n_k}) - A(t_{j-1}^{n_k})) \right) \\
 &= \lim_{k \rightarrow \infty} E \left(\sum_{j=1}^{m_{n_k}} \xi(t_{j-1}^{n_k}) (X(t_j^{n_k}) - M(t_j^{n_k}) - (X(t_{j-1}^{n_k}) - M(t_{j-1}^{n_k}))) \right) \\
 &= \lim_{k \rightarrow \infty} E \left(\sum_{j=1}^{m_{n_k}} \xi(t_{j-1}^{n_k}) (X(t_j^{n_k}) - X(t_{j-1}^{n_k})) \right) \tag{51.6.39}
 \end{aligned}$$

because

$$\begin{aligned}
 E(\xi(t_{j-1}^{n_k}) M(t_j^{n_k})) &= E(E(\xi(t_{j-1}^{n_k}) M(t_j^{n_k}) | \mathcal{F}_{t_{j-1}^{n_k}})) \\
 &= E(\xi(t_{j-1}^{n_k}) E(M(t_j^{n_k}) | \mathcal{F}_{t_{j-1}^{n_k}})) \\
 &= E(\xi(t_{j-1}^{n_k}) M(t_{j-1}^{n_k}))
 \end{aligned}$$

since M is a martingale. Now by a similar trick, this time using that $\{M^{n_k}(t_j^{n_k})\}_{j=0}^{m_{n_k}}$ is a martingale, 51.6.39 equals

$$\lim_{k \rightarrow \infty} E \left(\sum_{j=1}^{m_{n_k}} \xi(t_{j-1}^{n_k}) (A^{n_k}(t_j^{n_k}) - A^{n_k}(t_{j-1}^{n_k})) \right) \tag{51.6.40}$$

and now recall that $A^{n_k}(t_j^{n_k})$ is $\mathcal{F}_{t_{j-1}^{n_k}}$ measurable. This will now be used to change the subscript of $t_{j-1}^{n_k}$ in $\xi(t_{j-1}^{n_k})$ to a j . 51.6.40 equals

$$\begin{aligned}
 &= \lim_{k \rightarrow \infty} \sum_{j=1}^{m_{n_k}} E \left(E(\xi(t_j^{n_k}) | \mathcal{F}_{t_{j-1}^{n_k}}) (A^{n_k}(t_j^{n_k}) - A^{n_k}(t_{j-1}^{n_k})) \right) \\
 &= \lim_{k \rightarrow \infty} \sum_{j=1}^{m_{n_k}} E \left(E(\xi(t_j^{n_k}) (A^{n_k}(t_j^{n_k}) - A^{n_k}(t_{j-1}^{n_k})) | \mathcal{F}_{t_{j-1}^{n_k}}) \right) \\
 &= \lim_{k \rightarrow \infty} \sum_{j=1}^{m_{n_k}} E(\xi(t_j^{n_k}) (A^{n_k}(t_j^{n_k}) - A^{n_k}(t_{j-1}^{n_k}))) \\
 &= \lim_{k \rightarrow \infty} E \left(\sum_{j=1}^{m_{n_k}} \xi(t_j^{n_k}) (A^{n_k}(t_j^{n_k}) - A^{n_k}(t_{j-1}^{n_k})) \right)
 \end{aligned}$$

From this all that remains is to write the above as

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(\sum_{j=1}^{m_{n_k}} \xi(t_j^{n_k}) A^{n_k}(t_j^{n_k}) - \sum_{j=0}^{m_{n_k}-1} \xi(t_{j+1}^{n_k}) A^{n_k}(t_j^{n_k}) \right) \\ &= \lim_{k \rightarrow \infty} \left(E(\xi(a) A^{n_k}(a)) + E \left(\sum_{j=1}^{m_{n_k}-1} (\xi(t_j^{n_k}) - \xi(t_{j+1}^{n_k})) A^{n_k}(t_j^{n_k}) \right) \right) \end{aligned}$$

Now using the fact ξ is a martingale, this last term equals 0. Here is why.

$$\begin{aligned} E(\xi(t_{j+1}^{n_k}) A^{n_k}(t_j^{n_k})) &= E \left(E(\xi(t_{j+1}^{n_k}) A^{n_k}(t_j^{n_k}) | \mathcal{F}_{t_j^{n_k}}) \right) \\ &= E(A^{n_k}(t_j^{n_k}) E(\xi(t_{j+1}^{n_k}) | \mathcal{F}_{t_j^{n_k}}) | \mathcal{F}_{t_j^{n_k}}) \\ &= E(A^{n_k}(t_j^{n_k}) \xi(t_j^{n_k}) | \mathcal{F}_{t_j^{n_k}}) \end{aligned}$$

The first term converges to $E(\xi(a) A(a))$ because this was how $A(a)$ was obtained, as a weak limit in $L^1(\Omega)$ of $A^{n_k}(a)$. Also by Lemma 51.6.7,

$$E(\xi(a) A(a)) = E \left(\int_{(0,a]} \xi(s) dA(s) \right)$$

From 51.6.38 this has now shown that

$$E(\xi(a) A(a)) = E \left(\int_{(0,a]} \xi_-(s) dA(s) \right)$$

To get the desired result on $(0, t]$, apply what was just shown to a “stopped martingale”,

$$\xi^t(s) \equiv \begin{cases} \xi(s) & \text{if } s \leq t \\ \xi(t) & \text{if } s > t \end{cases}$$

$$\begin{aligned} & E \left(\int_{(0,t]} \xi(s) dA(s) \right) + (A(a) - A(t)) E(\xi(t)) \\ &= E \left(\int_{(0,a]} \xi^t(s) dA(s) \right) \end{aligned}$$

From what was shown above,

$$\begin{aligned} &= E \left(\int_{(0,a]} \xi_-^t(s) dA(s) \right) \\ &= E \left(\int_{(0,t]} \xi_-(s) dA(s) + \int_{(t,a]} \xi(t) sA(s) \right) \\ &= E \left(\int_{(0,t]} \xi_-(s) dA(s) \right) + (A(a) - A(t)) E(\xi(t)) \end{aligned}$$

and so

$$E \left(\int_{(0,t]} \xi(s) dA(s) \right) = E \left(\int_{(0,t]} \xi_{-}(s) dA(s) \right)$$

which shows A is natural by Lemma 51.6.7. This proves the theorem.

There is another interesting variation of the above theorem. It involves the following definition.

Definition 51.6.13 *A submartingale, $\{X(t)\}$ is said to be D if*

$$\{X_T : T < \infty \text{ is a stopping time}\}$$

is equi integrable.

In this case, you can consider partitions of the entire positive real line and the martingales, $\{M(t_k^n)\}_{k=0}^{\infty}$ and $\{A(t_k^n)\}_{k=0}^{\infty}$ as before. This time you don't stop at m_n . By the submartingale convergence theorem, you can argue there exists $A_{\infty}^n = \lim_{k \rightarrow \infty} A(t_k^n)$. Then repeat the above argument using A_{∞}^n in place of $A^n(a)$. This time you get $\{A(t)\}$ equi integrable. Thus the following corollary is obtained.

Corollary 51.6.14 *Let $\{X(t)\}$ be a right continuous submartingale of class D . Then there exists a right continuous martingale, $\{M(t)\}$ and a right continuous increasing submartingale, $\{A(t)\}$ such that for each t ,*

$$X(t) = M(t) + A(t).$$

If $\{A(t)\}$ is chosen to be natural and $A(0) = 0$, then with this condition, $\{M(t)\}$ and $\{A(t)\}$ are unique. Furthermore $\{M(t)\}$ and $\{A(t)\}$ are equi integrable on $[0, \infty)$.

In the above theorem, $\{X(t)\}$ was a submartingale and so it has a right continuous version. What if $\{X(t)\}$ is actually continuous? Can one conclude that $A(t)$ and $M(t)$ are also continuous? The answer is yes.

Theorem 51.6.15 *Let $\{X(t)\}$ be a right continuous submartingale of class DL . Then there exists a right continuous martingale, $\{M(t)\}$ and a right continuous increasing submartingale, $\{A(t)\}$ such that for each t ,*

$$X(t) = M(t) + A(t).$$

If $\{A(t)\}$ is chosen to be natural and $A(0) = 0$, then with this condition, $\{M(t)\}$ and $\{A(t)\}$ are unique. Also, if $\{X(t)\}$ is continuous, $(t \rightarrow X(t, \omega))$ is continuous for a.e. ω then the same is true of $\{A(t)\}$ and $\{M(t)\}$.

Proof: The first part is done above. Let $\{X(t)\}$ be continuous. As before, let $\{t_k^n\}_{k=0}^{m_n}$ be a sequence of partitions of $[0, a]$ such that these are equally spaced points, $\lim_{n \rightarrow \infty} t_{k+1}^n - t_k^n = 0$, and $\{t_k^n\}_{k=0}^{m_n} \subseteq \{t_k^{n+1}\}_{k=0}^{m_{n+1}}$ where here $a > 0$ is an arbitrary positive number and let $\lambda > 0$ be an arbitrary positive number. Define

$$\xi^n(t) \equiv E(\min(\lambda, A(t_j^n)) | \mathcal{F}_t) \text{ for } t_{j-1}^n < t \leq t_j^n$$

Thus on $(t_{j-1}^n, t_j^n]$ $\xi^n(t)$ is a bounded martingale. Assuming we are dealing with a right continuous version of this martingale so there are no measurability questions, it follows since A is natural,

$$E \left(\int_{(t_{j-1}^n, t_j^n]} \xi^n(s) dA(s) \right) = E \left(\int_{(t_{j-1}^n, t_j^n]} \xi_-^n(s) dA(s) \right)$$

where here

$$\xi_-^n(s, \omega) \equiv \lim_{r \rightarrow s-, r \in D} \xi^n(r, \omega) \text{ a.e.}$$

for $D \equiv \cup_{n=1}^{\infty} \cup_{k=1}^{m_n} \{t_k^n\}_{k=0}^{m_n}$. Thus, adding these up for all the intervals, $(t_{j-1}^n, t_j^n]$ yields

$$E \left(\int_{(0, a]} \xi^n(s) dA(s) \right) = E \left(\int_{(0, a]} \xi_-^n(s) dA(s) \right)$$

I want to show that for a.e. ω , $\xi^{n_k}(t, \omega)$ converges uniformly to

$$\min(\lambda, A(t, \omega)) \equiv \lambda \wedge A(t, \omega)$$

on $(0, a]$. From this it will follow

$$E \left(\int_{(0, a]} \lambda \wedge A(s, \omega) dA(s) \right) = E \left(\int_{(0, a]} \lambda \wedge A_-(s, \omega) dA(s) \right)$$

Now since $s \rightarrow A(s, \omega)$ is increasing, there is no problem in writing $A_-(s, \omega)$ and the above equation will suffice to show with simple considerations that for a.e. ω , $s \rightarrow A(s, \omega)$ is left continuous. Since $\{A(s)\}$ is a submartingale already, it has a right continuous version which we are using in the above. Thus for a.e. ω it must be the case that $s \rightarrow A(s, \omega)$ is continuous. Let $t \in (t_{j-1}^n, t_j^n]$. Then since $\lambda \wedge A(t)$ is $\mathcal{F}_t$ measurable,

$$\xi^n(t) - \lambda \wedge A(t) \equiv E(\lambda \wedge A(t_j^n) - \lambda \wedge A(t) | \mathcal{F}_t) \geq 0$$

because $A(t)$ is increasing.

Now define a stopping time, $T^n(\varepsilon)$ for $\varepsilon > 0$ by letting $T^n(\varepsilon)$ be the infimum of all $t \in [0, a]$ with the property that

$$\xi^n(t) - \lambda \wedge A(t) > \varepsilon$$

or if this does not happen, then $T^n(\varepsilon) = a$. Thus

$$T^n(\varepsilon)(\omega) = a \wedge \inf \{t \in [0, a] : \xi^n(t, \omega) - \lambda \wedge A(t, \omega) > \varepsilon\}$$

I need to verify $T^n(\varepsilon)$ really is a stopping time. Letting $s < a$, it follows that if $\omega \in [T^n(\varepsilon) \leq s]$, then for each N , there exists $t \in [s, s + \frac{1}{N})$ such that $\xi^n(t, \omega) - \lambda \wedge A(t, \omega) > \varepsilon$. Then by right continuity it follows there exists $r \in D \cap [s, s + \frac{1}{N})$ such that

$$\xi^n(r, \omega) - \lambda \wedge A(r, \omega) > \varepsilon$$

Thus

$$[T^n(\varepsilon) \leq s] = \cap_{N=1}^{\infty} \cup_{r \in D \cap [s, s + \frac{1}{N})} [\xi^n(r, \omega) - \lambda \wedge A(r, \omega) > \varepsilon]$$

and each $\cup_{r \in D \cap [s, s + \frac{1}{N})} [\xi^n(r, \omega) - \lambda \wedge A(r, \omega) > \varepsilon] \in \mathcal{F}_{s+1/N}$ and so $[T^n(\varepsilon) \leq s] \in \cap_{r \in D, r \geq s} \mathcal{F}_r = \mathcal{F}_{s+} = \mathcal{F}_s$ due to the assumption that the filtration is normal. What if $s \geq a$? Then from the definition, $[T^n(\varepsilon) \leq a] = \Omega \in \mathcal{F}_a$. Thus this really is a stopping time.

Now let $B_j \equiv [t_{j-1}^n < T^n(\varepsilon) \leq t_j^n]$. Note that $T^n(\varepsilon) \wedge t_j^n$ is also a stopping time.

$$\begin{aligned} \int_{\Omega} \xi_{T^n(\varepsilon)}^n dP &= \sum_{j=1}^{m_n} \int_{B_j} \xi_{T^n(\varepsilon)}^n dP = \sum_{j=1}^{m_n} \int_{B_j} \xi_{T^n(\varepsilon) \wedge t_j^n}^n dP \\ &= \sum_{j=1}^{m_n} \int_{B_j} E \left(\xi_{T^n(\varepsilon) \wedge t_j^n}^n | \mathcal{F}_{T^n(\varepsilon) \wedge t_j^n} \right) dP. \end{aligned}$$

This is because $B_j \in \mathcal{F}_{T^n(\varepsilon) \wedge t_j^n}$. Thus from the definition, the above equals

$$\begin{aligned} &= \sum_{j=1}^{m_n} \int_{B_j} E \left(E \left(\lambda \wedge A(t_j^n) | \mathcal{F}_{T^n(\varepsilon) \wedge t_j^n} \right) | \mathcal{F}_{T^n(\varepsilon) \wedge t_j^n} \right) dP \\ &= \sum_{j=1}^{m_n} \int_{B_j} E \left(\lambda \wedge A(t_j^n) | \mathcal{F}_{T^n(\varepsilon) \wedge t_j^n} \right) dP \\ &= \sum_{j=1}^{m_n} \int_{B_j} \lambda \wedge A(t_j^n) dP = \int_{\Omega} \lambda \wedge A([T^n(\varepsilon)]) dP \end{aligned} \quad (51.6.41)$$

where on $(t_{j-1}^n, t_j^n]$, $[T^n(\varepsilon)] \equiv t_j^n$. Now $[T^n(\varepsilon)]$ is also a bounded stopping time. Here is why. Suppose $s \in (t_{j-1}^n, t_j^n]$. Then

$$[[T^n(\varepsilon)] \leq s] = [T^n(\varepsilon) \leq t_{j-1}^n] \in \mathcal{F}_{t_{j-1}^n} \subseteq \mathcal{F}_s.$$

Now let

$$Q_n \equiv \sup_{t \in [0, a]} |\xi^n(t) - \lambda \wedge A(t)|.$$

Then first note that

$$[Q_n > \varepsilon] = \left[\sup_{t \in [0, a]} |\xi^n(t) - \lambda \wedge A(t)| > \varepsilon \right]$$

because $Q_n(a) = 0$ follows from the definition of $\xi^n(t)$ as

$$E(\lambda \wedge A(t_j^n) | \mathcal{F}_t) \text{ for } t_{j-1}^n < t \leq t_j^n$$

and so

$$\xi^n(a) = E(\lambda \wedge A(a) | \mathcal{F}_a) = \lambda \wedge A(a).$$

Thus it suffices to take the supremum over the half open interval, $[0, a)$. It follows

$$[Q_n > \varepsilon] = [T^n(\varepsilon) < a]$$

By right continuity,

$$\xi^n(T^n(\varepsilon)) - \lambda \wedge A(T^n(\varepsilon)) \geq \varepsilon$$

on $[Q_n > \varepsilon]$.

$$\begin{aligned} \varepsilon P([Q_n > \varepsilon]) &= \varepsilon P([T^n(\varepsilon) < a]) \\ &\leq \int_{[Q_n > \varepsilon]} (\xi^n(T^n(\varepsilon)) - \lambda \wedge A(T^n(\varepsilon))) dP \\ &\leq \int_{\Omega} (\xi^n(T^n(\varepsilon)) - \lambda \wedge A(T^n(\varepsilon))) dP \end{aligned}$$

Therefore, from 51.6.41,

$$\begin{aligned} P([Q_n > \varepsilon]) &\leq \frac{1}{\varepsilon} \int_{\Omega} (\lambda \wedge A(\lceil T^n(\varepsilon) \rceil) - \lambda \wedge A(T^n(\varepsilon))) dP \\ &\leq \frac{1}{\varepsilon} \int_{\Omega} (A(\lceil T^n(\varepsilon) \rceil) - A(T^n(\varepsilon))) dP \end{aligned} \quad (51.6.42)$$

By optional sampling theorem,

$$E(M(T^n(\varepsilon))) = E(M(0)) = 0$$

and also

$$E(M(\lceil T^n(\varepsilon) \rceil)) = E(M(0)) = 0.$$

Therefore, 51.6.42 reduces to

$$P([Q_n > \varepsilon]) \leq \frac{1}{\varepsilon} \int_{\Omega} (X(\lceil T^n(\varepsilon) \rceil) - X(T^n(\varepsilon))) dP$$

By the assumption that $\{X(t)\}$ is *DL*, it follows the functions in the above integrand are equi integrable and so since $\lim_{n \rightarrow \infty} X(\lceil T^n(\varepsilon) \rceil) - X(T^n(\varepsilon)) = 0$, the above integral converges to 0 as $n \rightarrow \infty$ by Vitali's convergence theorem, Theorem 8.5.3 on Page 213. It follows that there is a subsequence, n_k such that

$$P([Q_{n_k} > 2^{-k}]) \leq 2^{-k}$$

and so from the definition of Q_n ,

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, a]} |\xi^{n_k}(t) - \lambda \wedge A(t)|$$

giving uniform convergence. Now recall that

$$E\left(\int_{(0, a]} \xi^{n_k}(s) dA(s)\right) = E\left(\int_{(0, a]} \xi_-^{n_k}(s) dA(s)\right)$$

and so passing to the limit as $k \rightarrow \infty$ with the uniform convergence yields

$$E \left(\int_{(0,a]} \lambda \wedge A(s) dA(s) \right) = E \left(\int_{(0,a]} \lambda \wedge A_-(s) dA(s) \right)$$

Now let $\lambda \rightarrow \infty$. Then from the monotone convergence theorem,

$$E \left(\int_{(0,a]} A(s) dA(s) \right) = E \left(\int_{(0,a]} A_-(s) dA(s) \right)$$

and so for a.e. ω ,

$$\int_{(0,a]} (A(s) - A_-(s)) dA(s) = 0.$$

Thus letting the measure associated with this Lebesgue integral be denoted by μ ,

$$A(s) - A_-(s) = 0 \quad \mu \text{ a.e.}$$

Suppose then that $A(s) - A_-(s) > 0$. Then $\mu(\{s\}) = 0 = A(s) - A(s-)$, a contradiction. Hence $A(s) - A_-(s) = 0$ for all s . It is already the case that $s \rightarrow A(s)$ is right continuous. Therefore, this proves the theorem.

Example 51.6.16 Suppose $\{M(t)\}$ is a continuous martingale. Assume

$$\sup_{t \in [0,a]} \|M(t)\|_{L^2(\Omega)} < \infty$$

Then $\{\|M(t)\|\}$ is a submartingale and so is $\{\|M(t)\|^2\}$. By Example 51.6.11, this is DL. Then there exists a unique Doob Meyer decomposition,

$$\|M(t)\|^2 = Y(t) + \langle \|M(t)\| \rangle$$

where $Y(t)$ is a martingale and $\{\langle \|M(t)\| \rangle\}$ is a submartingale which is continuous, natural, increasing and equal to 0 when $t = 0$. This submartingale is called the quadratic variation.

51.7 Levy's Theorem

This remarkable theorem has to do with when a martingale is a Wiener process. The proof I am giving here follows [21].

Definition 51.7.1 Let $W(t)$ be a stochastic process which has the properties that whenever $t_1 < t_2 < \dots < t_m$, the increments $\{W(t_i) - W(t_{i-1})\}$ are independent and whenever $s < t$, it follows $W(t) - W(s)$ is normally distributed with variance $t - s$ and mean 0. Also $W(0) = 0$. This is called a Wiener process.

First here is a lemma.

Lemma 51.7.2 *Let $\{X(t)\}$ be a real martingale adapted to the filtration $\mathcal{F}_t$ for $t \in [a, b]$ some interval such that for all $t \in [a, b]$, $E(X(t)^2) < \infty$. Then $\{X(t)^2 - t\}$ is also a martingale if and only if whenever $s < t$,*

$$E\left((X(t) - X(s))^2 | \mathcal{F}_s\right) = t - s.$$

Proof: Suppose first $\{X(t)^2 - t\}$ is a real martingale. Then since $\{X(t)\}$ is a martingale,

$$\begin{aligned} E\left((X(t) - X(s))^2 | \mathcal{F}_s\right) &= E\left(X(t)^2 - 2X(t)X(s) + X(s)^2 | \mathcal{F}_s\right) \\ &= E\left(X(t)^2 | \mathcal{F}_s\right) - 2E(X(t)X(s) | \mathcal{F}_s) + X(s)^2 \\ &= E\left(X(t)^2 | \mathcal{F}_s\right) - 2X(s)E(X(t) | \mathcal{F}_s) + X(s)^2 \\ &= E\left(X(t)^2 | \mathcal{F}_s\right) - 2X(s)^2 + X(s)^2 \\ &= E\left(X(t)^2 - t | \mathcal{F}_s\right) + t - X(s)^2 \\ &= X(s)^2 - s + t - X(s)^2 = t - s \end{aligned}$$

Next suppose $E\left((X(t) - X(s))^2 | \mathcal{F}_s\right) = t - s$. Then since $\{X(t)\}$ is a martingale,

$$\begin{aligned} t - s &= E\left(X(t)^2 - X(s)^2 | \mathcal{F}_s\right) \\ &= E\left(X(t)^2 - t | \mathcal{F}_s\right) + t - X(s)^2 \end{aligned}$$

and so

$$0 = E\left(X(t)^2 - t | \mathcal{F}_s\right) - \left(X(s)^2 - s\right)$$

which proves the converse.

Theorem 51.7.3 *Suppose $\{X(t)\}$ is a real stochastic process which satisfies all the conditions of a real Wiener process except the requirement that it be continuous. Then both $\{X(t)\}$ and $\{X(t)^2 - t\}$ are martingales.*

Proof: First define the filtration to be

$$\mathcal{F}_t \equiv \sigma(X(s) - X(r) : r \leq s \leq t).$$

Claim: If $A \in \mathcal{F}_s$, then

$$\int_{\Omega} \mathcal{X}_A(X(t) - X(s)) dP = P(A) \int_{\Omega} (X(t) - X(s)) dP.$$

Proof of claim: Let $\mathcal{G}$ denote those sets of $\mathcal{F}_s$ for which the above formula holds. Then it is clear that $\mathcal{G}$ is closed with respect to countable unions of disjoint sets and complements. Let $\mathcal{K}$ denote those sets which are finite intersections of sets of the form $(X(u) - X(r))^{-1}(B)$ where B is a Borel set and $r \leq u \leq s$. Say a set, A of $\mathcal{K}$ is of the form

$$\cap_{i=1}^m (X(u_i) - X(r_i))^{-1}(B_i)$$

Then since disjoint increments are independent, linear combinations of the random variables, $X(u_i) - X(r_i)$ are normally distributed. Consequently,

$$(X(u_1) - X(r_1), \dots, X(u_m) - X(r_m), X(t) - X(s))$$

is multivariate normal. The covariance matrix is of the form

$$\begin{pmatrix} A & 0 \\ 0 & t-s \end{pmatrix}$$

and so the random vector, $(X(u_1) - X(r_1), \dots, X(u_m) - X(r_m))$ and the random variable $X(t) - X(s)$ are independent. Consequently, $\mathcal{X}_A$ is independent of $X(t) - X(s)$ for any $A \in \mathcal{K}$. Then by the lemma on π systems, Lemma 9.11.3 on Page 275, $\mathcal{F}_s \supseteq \mathcal{G} \supseteq \sigma(\mathcal{K}) = \mathcal{F}_s$. This proves the claim.

Thus

$$\begin{aligned} \int_A (X(t) - X(s)) dP &= \int_{\Omega} (X(t) - X(s)) \mathcal{X}_A dP \\ &= P(A) \int_{\Omega} (X(t) - X(s)) dP = 0 \end{aligned}$$

which shows that since $A \in \mathcal{F}_s$ was arbitrary,

$$E(X(t) | \mathcal{F}_s) = X(s)$$

and $\{X(t)\}$ is a martingale.

Now consider whether $\{X(t)^2 - t\}$ is a martingale. By assumption,

$$\mathcal{L}(X(t) - X(s)) = \mathcal{L}(X(t-s)) = N(0, t-s).$$

Then for $A \in \mathcal{F}_s$, the independence of $\mathcal{X}_A$ and $X(t) - X(s)$ shows

$$\begin{aligned} \int_A E((X(t) - X(s))^2 | \mathcal{F}_s) dP &= \int_A (X(t) - X(s))^2 dP \\ &= P(A)(t-s) = \int_A (t-s) dP \end{aligned}$$

and since $A \in \mathcal{F}_s$ is arbitrary,

$$E((X(t) - X(s))^2 | \mathcal{F}_s) = t-s$$

and so the result follows from Lemma 51.7.2. This proves the theorem.

The next lemma is the main result from which Levy's theorem will be established.

Lemma 51.7.4 *Let $\{X(t)\}$ be a real continuous martingale adapted to the filtration $\mathcal{F}_t$ for $t \in [a, b]$ some interval such that for all $t \in [a, b]$, $E(X(t)^2) < \infty$. Suppose also that $\{X(t)^2 - t\}$ is a martingale. Then for λ real,*

$$E\left(e^{i\lambda X(b)}\right) = E\left(e^{i\lambda X(a)}\right) e^{-(b-a)\frac{\lambda^2}{2}}$$

Proof: Let $\lambda \in [-p, p]$ where for most of the proof, p is fixed but arbitrary. Let $\{t_k^n\}_{k=0}^{2^n}$ be uniform partitions such that $t_k^n - t_{k-1}^n = \delta_n \equiv (b-a)/2^n$. Now for $\varepsilon > 0$ define a stopping time $\tau_{\varepsilon, n}$ to be the first time, t such that there exist $s_1, s_2 \in [a, t]$ with $|s_1 - s_2| < \delta_n$ but

$$|X(s_1) - X(s_2)| = \varepsilon.$$

If no such time exists, then $\tau_{\varepsilon, n} \equiv b$.

Then $\tau_{\varepsilon, n}$ really is a stopping time because from continuity of $X(t)$ and denoting by r, r_1 elements of $\mathbb{Q}$, then

$$[\tau_{\varepsilon, n} > t] = \bigcup_{m=1}^{\infty} \bigcap_{0 \leq r_1, r_2 \leq t, |r_1 - r_2| \leq \delta_n} \left[|X(r_1) - X(r_2)| \leq \varepsilon - \frac{1}{m} \right] \in \mathcal{F}_t$$

because to be in $[\tau_{\varepsilon, n} > t]$ it means that by t the absolute value of the differences must always be less than ε . Hence $[\tau_{\varepsilon, n} \leq t] = \Omega \setminus [\tau_{\varepsilon, n} > t] \in \mathcal{F}_t$.

Now consider $[\tau_{\varepsilon, n} \leq b]$ for various n . By continuity, it follows that for each $\omega \in \Omega$,

$$\tau_{\varepsilon, n}(\omega) = b$$

for all n large enough. Thus

$$\emptyset = \bigcap_{n=1}^{\infty} [\tau_{\varepsilon, n} < b],$$

the sets in the intersection decreasing. Thus there exists $n(\varepsilon)$ such that

$$P([\tau_{\varepsilon, n(\varepsilon)} < b]) < \varepsilon. \quad (51.7.43)$$

Denote $\tau_{\varepsilon, n(\varepsilon)}$ as τ_{ε} for short and it will always be assumed that $n(\varepsilon)$ is at least this large and that $\lim_{\varepsilon \rightarrow 0+} n(\varepsilon) = \infty$. In addition to this, $n(\varepsilon)$ will also be large enough that

$$1 - \frac{\lambda^2}{2} \delta_{n(\varepsilon)} > 0$$

for all $\lambda \in [-p, p]$. To save on notation, t_j will take the place of t_j^n . Then consider the stopping times $\tau_{\varepsilon} \wedge t_j$ for $j = 0, 1, \dots, 2^{n(\varepsilon)}$.

Let $y_j \equiv X(\tau_{\varepsilon} \wedge t_j) - X(\tau_{\varepsilon} \wedge t_{j-1})$, it follows from the definition of the stopping time that

$$|y_j| \leq \varepsilon \quad (51.7.44)$$

because both $\tau_\varepsilon \wedge t_j$ and $\tau_\varepsilon \wedge t_{j-1}$ are less than τ_ε and closer together than $\delta_{n(\varepsilon)}$ and so if $|y_j| > \varepsilon$, then $\tau_\varepsilon \leq t_j, t_{j-1}$ and so y_j would need to equal 0.

By the optional stopping theorem, $\{X(\tau_\varepsilon \wedge t_j)\}_j$ is a martingale as is also

$$\{X(\tau_\varepsilon \wedge t_j) - \tau_\varepsilon \wedge t_j\}_j.$$

Thus for $A \in \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}$,

$$\begin{aligned} \int_A E(y_j^2 | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) dP &= \int_A E\left((X(\tau_\varepsilon \wedge t_j) - X(\tau_\varepsilon \wedge t_{j-1}))^2 | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}\right) dP \\ &= \int_A E\left(X(\tau_\varepsilon \wedge t_j)^2 | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}\right) + X(\tau_\varepsilon \wedge t_{j-1})^2 \\ &\quad - 2X(\tau_\varepsilon \wedge t_{j-1}) E(X(\tau_\varepsilon \wedge t_j) | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) dP \\ &= \int_A E\left(X(\tau_\varepsilon \wedge t_j)^2 - \tau_\varepsilon \wedge t_j | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}\right) dP + \int_A E(\tau_\varepsilon \wedge t_j | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) dP \\ &\quad + \int_A X(\tau_\varepsilon \wedge t_{j-1})^2 dP - 2 \int_A X(\tau_\varepsilon \wedge t_{j-1})^2 dP \\ &= \int_A X(\tau_\varepsilon \wedge t_{j-1})^2 dP - \int_A \tau_\varepsilon \wedge t_{j-1} dP + \int_A E(\tau_\varepsilon \wedge t_j | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) dP \\ &\quad + \int_A X(\tau_\varepsilon \wedge t_{j-1})^2 dP - 2 \int_A X(\tau_\varepsilon \wedge t_{j-1})^2 dP \\ &= \int_A E(\tau_\varepsilon \wedge t_j | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) dP - \int_A \tau_\varepsilon \wedge t_{j-1} dP \\ &= \int_A (\tau_\varepsilon \wedge t_j - \tau_\varepsilon \wedge t_{j-1}) dP \leq \int_A t_j - t_{j-1} dP. \end{aligned}$$

Thus, since A is arbitrary,

$$\sigma_j^2 \equiv \int_A E(y_j^2 | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) dP =$$

$$E\left((X(\tau_\varepsilon \wedge t_j) - X(\tau_\varepsilon \wedge t_{j-1}))^2 | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}\right) \leq t_j - t_{j-1} = \delta_{n(\varepsilon)} \quad (51.7.45)$$

Also,

$$E(y_j | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) = E(X(\tau_\varepsilon \wedge t_j) - X(\tau_\varepsilon \wedge t_{j-1}) | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}}) = 0. \quad (51.7.46)$$

Now it is time to find $E(e^{i\lambda X(\tau_\varepsilon \wedge t_j)})$.

$$E(e^{i\lambda X(\tau_\varepsilon \wedge t_j)}) = E(e^{i\lambda(X(\tau_\varepsilon \wedge t_{j-1}) + y_j)})$$

$$= E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} E \left(e^{i\lambda y_j} | \mathcal{F}_{\tau_\varepsilon \wedge t_{j-1}} \right) \right). \quad (51.7.47)$$

Now let $o(1)$ denote any quantity which converges to 0 as $\varepsilon \rightarrow 0$ for all $\lambda \in [-p, p]$ and $O(1)$ is a quantity which is bounded as $\varepsilon \rightarrow 0$. Then from 51.7.46 and 51.7.47 you can consider the power series for $e^{i\lambda y_j}$ which converges uniformly due to 51.7.44 and write 51.7.47 as

$$E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} \left(1 - \frac{\lambda^2}{2} \sigma_j^2 (1 + o(1)) \right) \right).$$

then noting that from 51.7.45 which shows σ_j^2 is $o(1)$, it is routine to verify

$$1 - \frac{\lambda^2}{2} \sigma_j^2 (1 + o(1)) = e^{-\frac{\lambda^2}{2} \sigma_j^2 (1 + o(1))}.$$

Now this shows

$$E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) = E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \sigma_j^2 (1 + o(1))} \right)$$

Recall that $\sigma_j^2 \leq \delta_n = t_j - t_{j-1}$. Consider

$$\begin{aligned} & \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) - E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \right) \right| \\ &= \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \sigma_j^2 (1 + o(1))} \right) - E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \right) \right| \\ &= \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} \left(e^{-\frac{\lambda^2}{2} \sigma_j^2 (1 + o(1))} - e^{-\frac{\lambda^2}{2} \delta_n} \right) \right) \right| \\ &= \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \left(e^{-\frac{\lambda^2}{2} \sigma_j^2 (1 + o(1)) + \frac{\lambda^2}{2} \delta_n} - 1 \right) \right) \right| \\ &\leq E \left(\left| e^{\frac{\lambda^2}{2} (\delta_n - \sigma_j^2) + \sigma_j^2 o(1)} - 1 \right| \right) \end{aligned}$$

Everything in the exponent is $o(1)$ and so the above expression is bounded by

$$\begin{aligned} & O(1) E \left(\left| \frac{\lambda^2}{2} (\delta_n - \sigma_j^2) + \sigma_j^2 o(1) \right| \right) \\ &\leq O(1) E \left((\delta_n - \sigma_j^2) + \delta_n |o(1)| \right) \\ &= O(1) [\delta_n - E(y_j^2) + \delta_n o(1)]. \end{aligned} \quad (51.7.48)$$

Therefore,

$$\begin{aligned} & \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) - E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{-\frac{\lambda^2}{2} \delta_n} \right) \right| \\ &\leq O(1) [\delta_n - E(y_j^2) + \delta_n o(1)] \end{aligned}$$

and so it also follows

$$\begin{aligned} & \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) e^{\frac{\lambda^2}{2} t_j} - E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{\frac{\lambda^2}{2} t_{j-1}} \right) \right| \\ & \leq O(1) [\delta_n - E(y_j^2) + \delta_n o(1)] \end{aligned}$$

Now also remember

$$y_j = X(\tau_\varepsilon \wedge t_j) - X(\tau_\varepsilon \wedge t_{j-1})$$

and that $\{X(\tau_\varepsilon \wedge t_j)\}_j$ is a martingale. Therefore it is routine to show,

$$E(y_j^2) = E(X(\tau_\varepsilon \wedge t_j)^2) - E(X(\tau_\varepsilon \wedge t_{j-1})^2).$$

and so

$$\begin{aligned} & \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_j)} \right) e^{\frac{\lambda^2}{2} t_j} - E \left(e^{i\lambda X(\tau_\varepsilon \wedge t_{j-1})} e^{\frac{\lambda^2}{2} t_{j-1}} \right) \right| \\ & \leq O(1) [\delta_n - (E(X(\tau_\varepsilon \wedge t_j)^2) - E(X(\tau_\varepsilon \wedge t_{j-1})^2))] + \delta_n o(1) \end{aligned}$$

and so, summing over all $j = 1, \dots, 2^{n(\varepsilon)}$,

$$\begin{aligned} & \left| E \left(e^{i\lambda X(\tau_\varepsilon \wedge b)} \right) e^{\frac{\lambda^2}{2} b} - E \left(e^{i\lambda X(a)} e^{\frac{\lambda^2}{2} a} \right) \right| \\ & \leq O(1) \left((1 + o(1))(b - a) - (E(X(\tau_\varepsilon \wedge b)^2) - E(X(a)^2)) \right) \end{aligned} \quad (51.7.49)$$

Now recall 51.7.43 which said

$$P([\tau_\varepsilon < b]) < \varepsilon.$$

Let $\varepsilon_k \equiv 2^{-k}$ and then by the Borel Cantelli lemma,

$$X(\tau_\varepsilon \wedge b) \rightarrow X(b)$$

a.e. since if ω is such that convergence does not take place, ω must be in infinitely many of the sets, $[\tau_{\varepsilon_k} < b]$, a set of measure 0. Also since $\{X(\tau_\varepsilon \wedge t_j)\}_j$ is a martingale, it follows from optional sampling theorem that $\{X(a)^2, X(\tau_\varepsilon \wedge b)^2, X(b)^2\}$ is a submartingale and so

$$\int_{[X(\tau_\varepsilon \wedge b)^2 \geq \alpha]} X(\tau_\varepsilon \wedge b)^2 dP \leq \int_{[X(\tau_\varepsilon \wedge b)^2 \geq \alpha]} X(b)^2 dP$$

and also from the maximal inequalities, Theorem 48.4.5 on Page 1389 it follows

$$P\left([X(\tau_\varepsilon \wedge b)^2 \geq \alpha]\right) \leq \frac{1}{\alpha} E(X(b)^2)$$

and so the functions, $\left\{X(\tau_{\varepsilon_k} \wedge b)^2\right\}_{\varepsilon_k}$ are uniformly integrable which implies by the Vitali convergence theorem, Theorem 8.5.3 on Page 213, that you can pass to the limit as $\varepsilon_k \rightarrow 0$ in the inequality, 51.7.49 and conclude

$$\begin{aligned} & \left| E\left(e^{i\lambda X(b)}\right) e^{\frac{\lambda^2}{2}b} - E\left(e^{i\lambda X(a)} e^{\frac{\lambda^2}{2}a}\right) \right| \\ & \leq O(1) \left((b-a) - \left(E\left(X(b)^2\right) - E\left(X(a)^2\right) \right) \right) = 0. \end{aligned}$$

Therefore,

$$E\left(e^{i\lambda X(b)}\right) = E\left(e^{i\lambda X(a)}\right) e^{-\frac{\lambda^2}{2}(b-a)}$$

This proves the lemma because p was arbitrary.

Now from this lemma, it is not hard to establish Levy's theorem.

Theorem 51.7.5 *Let $\{X(t)\}$ be a real continuous martingale adapted to the filtration $\mathcal{F}_t$ for $t \in [0, a]$ some interval such that for all $t \in [0, a]$, $E\left(X(t)^2\right) < \infty$. Suppose also that $\{X(t)^2 - t\}$ is a martingale. Then for $s < t$, $X(t) - X(s)$ is normally distributed with mean 0 and variance $t - s$. Also if $0 \leq t_0 < t_1 < \dots < t_m \leq b$, then the increments $\{X(t_j) - X(t_{j-1})\}$ are independent.*

Proof: Let the t_j be as described above and consider the interval $[t_{m-1}, t_m]$ in place of $[a, b]$ in Lemma 51.7.4. Also let λ_k for $k = 1, 2, \dots, m$ be given. For $t \in [t_{m-1}, t_m]$, and $\lambda_m \neq 0$,

$$Z_{\lambda_m}(t) = \frac{1}{\lambda_m} \sum_{j=1}^{m-1} \lambda_j (X(t_j) - X(t_{j-1})) + (X(t) - X(t_{m-1}))$$

Then it is clear that $\{Z_{\lambda_m}(t)\}$ is a martingale on $[t_{m-1}, t_m]$. What is possibly less clear is that $\{Z_{\lambda_m}(t)^2 - t\}$ is also a martingale. Note that $Z_{\lambda_m}(t) = X(t) + Y$ where Y is measurable in $\mathcal{F}_{t_{m-1}}$. Therefore, for $s < t$, $s \in [t_{m-1}, t_m]$,

$$\begin{aligned} E\left(Z_{\lambda_m}(t)^2 - t | \mathcal{F}_s\right) &= E\left(X(t)^2 + 2X(t)Y + Y^2 - t | \mathcal{F}_s\right) \\ &= X(s)^2 - s + 2E(X(t)Y | \mathcal{F}_s) + Y^2 \\ &= X(s)^2 - s + 2YX(s) + Y^2 = Z_{\lambda_m}(s)^2 - s \end{aligned}$$

and so Lemma 51.7.4 can be applied to conclude

$$E\left(e^{i\lambda Z_{\lambda_m}(t_m)}\right) = E\left(e^{i\lambda Z_{\lambda_m}(t_{m-1})}\right) e^{-\frac{\lambda^2}{2}(t_m - t_{m-1})}.$$

Now letting $\lambda = \lambda_m$,

$$E\left(e^{i\sum_{j=1}^m \lambda_j (X(t_j) - X(t_{j-1}))}\right) = E\left(e^{i\sum_{j=1}^{m-1} \lambda_j (X(t_j) - X(t_{j-1}))}\right) e^{-\frac{\lambda_m^2}{2}(t_m - t_{m-1})}.$$

By continuity, this equation continues to hold for $\lambda_m = 0$. Then iterate this, using a similar argument on the first factor of the right side to eventually obtain

$$E \left(e^{i \sum_{j=1}^m \lambda_j (X(t_j) - X(t_{j-1}))} \right) = \prod_{j=1}^m e^{-\frac{\lambda_j^2}{2} (t_j - t_{j-1})}.$$

Then letting all but one λ_j equal zero, this shows the increment, $X(t_j) - X(t_{j-1})$ is a random variable which is normally distributed having variance $t_j - t_{j-1}$ and mean 0. The above formula also shows from Proposition 47.4.1 on Page 1323 that the increments are independent. This proves the theorem.

Wiener Processes

52.1 Real Wiener Processes

Do Wiener processes exist? Yes, they do. First here is a simple lemma which has really been done before. It depends on the Kolmogorov extension theorem, Theorem 11.3.5 on Page 332.

Lemma 52.1.1 *There exists a sequence, $\{\xi_k\}_{k=1}^\infty$ of random variables such that*

$$\mathcal{L}(\xi_k) = N(0, 1)$$

and $\{\xi_k\}_{k=1}^\infty$ is independent.

Proof: Let $i_1 < i_2 < \dots < i_n$ be positive integers and define

$$\mu_{i_1 \dots i_n}(F_1 \times \dots \times F_n) \equiv \frac{1}{(\sqrt{2\pi})^n} \int_{F_1 \times \dots \times F_n} e^{-|\mathbf{x}|^2/2} dx.$$

Then for the index set equal to $\mathbb{N}$ the measures satisfy the necessary consistency condition for the Kolmogorov theorem. Therefore, there exists a probability space, $(\Omega, P, \mathcal{F})$ and measurable functions, $\xi_k : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} & P([\xi_{i_1} \in F_{i_1}] \cap [\xi_{i_2} \in F_{i_2}] \dots \cap [\xi_{i_n} \in F_{i_n}]) \\ &= \mu_{i_1 \dots i_n}(F_1 \times \dots \times F_n) \\ &= P([\xi_{i_1} \in F_{i_1}]) \dots P([\xi_{i_n} \in F_{i_n}]) \end{aligned}$$

which shows the random variables are independent as well as normal with mean 0 and variance 1. This proves the Lemma.

Recall that the sum of independent normal random variables is normal. The Wiener process is just an infinite weighted sum of the above independent normal random variables, the weights depending on t . Therefore, if the sum converges, it is not too surprising that the result will be normally distributed and the variance will depend on t . This is the idea behind the following theorem.

Theorem 52.1.2 *There exists a real Wiener process as defined in Definition 51.7.1. Furthermore, the distribution of $W(t) - W(s)$ is the same as the distribution of $W(t - s)$ and W is Holder continuous with exponent γ for any $\gamma < 1/2$.*

Proof: Let $\{g_m\}_{m=1}^\infty$ be a complete orthonormal set in $L^2(0, \infty)$. Thus, if $f \in L^2(0, \infty)$,

$$f = \sum_{i=1}^\infty (f, g_i)_{L^2} g_i.$$

The Wiener process is defined as

$$W(t, \omega) \equiv \sum_{i=1}^\infty (\mathcal{X}_{(0,t)}, g_i)_{L^2} \xi_i(\omega)$$

where the random variables, $\{\xi_i\}$ are as described in Lemma 52.1.1. The series converges in $L^2(\Omega)$ where $(\Omega, \mathcal{F}, P)$ is the probability space on which the random variables, ξ_i are defined. This will first be shown. Note first that from the independence of the ξ_i ,

$$\int_\Omega \xi_i \xi_j dP = 0$$

Therefore,

$$\begin{aligned} \int_\Omega \left| \sum_{i=m}^n (\mathcal{X}_{(0,t)}, g_i)_{L^2} \xi_i(\omega) \right|^2 dP &= \sum_{i=m}^n (\mathcal{X}_{(0,t)}, g_i)_{L^2}^2 \int_\Omega |\xi_i|^2 dP \\ &= \sum_{i=m}^n (\mathcal{X}_{(0,t)}, g_i)_{L^2}^2 \end{aligned}$$

which converges to 0 as $m, n \rightarrow \infty$. Thus the partial sums are a Cauchy sequence in $L^2(\Omega, P)$.

It just remains to verify this definition satisfies the desired conditions. First I will show that $\omega \rightarrow W(t, \omega)$ is normally distributed with mean 0 and variance t . Selecting a suitable subsequence, $\{n_k\}$ it can be assumed

$$W(t, \omega) = \lim_{k \rightarrow \infty} \sum_{i=1}^{n_k} (\mathcal{X}_{(0,t)}, g_i)_{L^2} \xi_i(\omega) \text{ a.e.}$$

and so from the dominated convergence theorem and the independence of the ξ_i ,

$$\begin{aligned}
 E(\exp(i\lambda W(t))) &= \lim_{k \rightarrow \infty} E\left(\exp\left(i\lambda \sum_{j=1}^{n_k} (\mathcal{X}_{(0,t)}, g_j)_{L^2} \xi_j(\omega)\right)\right) \\
 &= \lim_{k \rightarrow \infty} E\left(\prod_{j=1}^{n_k} \exp(i\lambda (\mathcal{X}_{(0,t)}, g_j)_{L^2} \xi_j(\omega))\right) \\
 &= \lim_{k \rightarrow \infty} \prod_{j=1}^{n_k} E(\exp(i\lambda (\mathcal{X}_{(0,t)}, g_j)_{L^2} \xi_j(\omega))) \\
 &= \lim_{k \rightarrow \infty} \prod_{j=1}^{n_k} e^{-\frac{1}{2}\lambda^2 (\mathcal{X}_{(0,t)}, g_j)_{L^2}^2} \\
 &= \lim_{k \rightarrow \infty} \exp\left(\sum_{j=1}^{n_k} -\frac{1}{2}\lambda^2 (\mathcal{X}_{(0,t)}, g_j)_{L^2}^2\right) \\
 &= \exp\left(-\frac{1}{2}\lambda^2 \|\mathcal{X}_{(0,t)}\|_{L^2}^2\right) = \exp\left(-\frac{1}{2}\lambda^2 t\right),
 \end{aligned}$$

the characteristic function of a normally distributed random variable having variance t and mean 0..

It is clear $W(0) = 0$. It remains to verify the increments are independent. First note that for a, b scalars and $r < s < t$, the above definition shows

$$\begin{aligned}
 &a(W(t) - W(s)) + b(W(s) - W(r)) \\
 &= \sum_j (a\mathcal{X}_{(s,t)} + b\mathcal{X}_{(r,s)}, g_j)_{L^2} \xi_j
 \end{aligned}$$

and by reasoning identical to the above this is normal. Since this holds for every linear combination, it follows the vector $(W(s) - W(r), W(t) - W(s))$ is multivariate normal. The increments, $W(s) - W(r)$, $W(t) - W(s)$ will be independent if the covariance matrix of this multivariate normal distribution is diagonal. However, the off diagonal terms are of the form

$$\begin{aligned}
 &E((W(s) - W(r))(W(t) - W(s))) \\
 &= E\left(\left(\sum_{k=1}^{\infty} (\mathcal{X}_{(r,s)}, g_k)_{L^2} \xi_k(\omega)\right)\left(\sum_{i=1}^{\infty} (\mathcal{X}_{(s,t)}, g_i)_{L^2} \xi_i(\omega)\right)\right)
 \end{aligned}$$

$$\begin{aligned}
 &= E \left(\sum_{i,k} (\mathcal{X}_{(r,s)}, g_k)_{L^2} (\mathcal{X}_{(s,t)}, g_i)_{L^2} \xi_k \xi_i \right) \\
 &= \sum_{i,k} (\mathcal{X}_{(r,s)}, g_k)_{L^2} (\mathcal{X}_{(s,t)}, g_i)_{L^2} E (\xi_k \xi_i) \\
 &= \sum_i (\mathcal{X}_{(r,s)}, g_i)_{L^2} (\mathcal{X}_{(s,t)}, g_i)_{L^2} = (\mathcal{X}_{(r,s)}, \mathcal{X}_{(s,t)})_{L^2} = 0
 \end{aligned}$$

and so this shows the increments are independent. Similarly for $0 \leq t_0 < t_1 < \dots < t_m$, the random variables $\{W(t_i) - W(t_{i-1})\}_{i=1}^m$ are independent. You observe in the same way as above that $(W(t_1) - W(t_0), \dots, W(t_m) - W(t_{m-1}))$ is multivariate normal and by the same reasoning as above the covariance matrix is diagonal.

From the definition, if $t > s$

$$W(t-s) = \sum_{k=1}^{\infty} (\mathcal{X}_{(0,t-s)}, g_k)_{L^2} \xi_k$$

while

$$W(t) - W(s) = \sum_{k=1}^{\infty} (\mathcal{X}_{(s,t)}, g_k)_{L^2} \xi_k.$$

Then the same argument given above involving the characteristic function to show $W(t)$ is normally distributed shows both of these random variables are normally distributed with mean 0 and variance $t-s$.

Finally note the distribution of $W(t-s)$ is the same as the distribution of

$$W(1)(t-s)^{1/2} = \sum_{k=1}^{\infty} (\mathcal{X}_{(0,1)}, g_k)_{L^2} \xi_k (t-s)^{1/2}$$

because the characteristic function of this last random variable is the same as the characteristic function of $W(t-s)$ which is $e^{-\frac{1}{2}\lambda^2(t-s)}$ which follows from a simple computation. Hence for any positive α ,

$$\begin{aligned}
 E(|W(t) - W(s)|^\alpha) &= E(|W(t-s)|^\alpha) \\
 &= E\left(\left|(t-s)^{1/2} W(1)\right|^\alpha\right) \\
 &= |t-s|^{\alpha/2} E(|W(1)|^\alpha)
 \end{aligned} \tag{52.1.1}$$

It follows from Theorem 50.1.4 that $W(t)$ is Hölder continuous with exponent γ where γ is any positive number less than β/α where $\alpha/2 = 1 + \beta$. Thus γ is any constant less than

$$\frac{\frac{\alpha}{2} - 1}{\alpha}.$$

Thus γ is any constant less than $\frac{1}{2}$. This proves the theorem.

The proof of the theorem also implies the following corollary.

Corollary 52.1.3 *Let $\{\xi_i\}_{i=1}^\infty$ be independent random variables each normal with mean 0 and variance 1. Then*

$$W(t, \omega) \equiv \sum_{i=1}^{\infty} (\mathcal{X}_{[0,t]}, g_i)_{L^2} \xi_i(\omega)$$

is a real Wiener process.

52.2 Nowhere Differentiability Of Wiener Processes

If $W(t)$ is a Wiener process, it turns out that $t \rightarrow W(t, \omega)$ is nowhere differentiable for a.e. ω . This fact is based on the independence of the increments and the fact that these increments are normally distributed.

First note that $W(t) - W(s)$ has the same distribution as $(t - s)^{1/2} W(1)$. This is because they have the same characteristic function. Next it follows that because of the independence of the increments and what was just noted that,

$$\begin{aligned} & P(\cap_{r=1}^5 [|W(t + r\delta) - W(t + (r-1)\delta)| \leq K\delta]) \\ &= \prod_{r=1}^5 P(|W(t + r\delta) - W(t + (r-1)\delta)| \leq K\delta) \\ &= \prod_{r=1}^5 P(|\delta^{1/2} W(1)| \leq K\delta) = \left(\frac{1}{\sqrt{2\pi}} \int_{-K\sqrt{\delta}}^{K\sqrt{\delta}} e^{-\frac{1}{2}t^2} dt \right)^5 \\ &\leq C\delta^{5/2}. \end{aligned} \tag{52.2.2}$$

With this observation, here is the proof which follows [62] and according to this reference is due to Payley, Wiener and Zygmund and the proof is like one given by Dvoretzky, Erdős and Kakutani.

Theorem 52.2.1 *Let $W(t)$ be a Wiener process. Then there exists a set of measure 0, N such that for all $\omega \notin N$,*

$$t \rightarrow W(t, \omega)$$

is nowhere differentiable.

Proof: Let $[0, a]$ be an interval. If for some $\omega, t \rightarrow W(t, \omega)$ is differentiable at some s , then for some $n, p > 0$,

$$\left| \frac{W(t, \omega) - W(s, \omega)}{t - s} \right| \leq p$$

whenever $|t - s| < 5a2^{-n} \equiv 5\delta_n$. Define C_{np} by

$$\left\{ \omega : \text{for some } s \in [0, a), \left| \frac{W(t, \omega) - W(s, \omega)}{t - s} \right| \leq p \text{ if } |t - s| \leq 5\delta_n \right\}. \tag{52.2.3}$$

Thus $\cup_{n,p \in \mathbb{N}} C_{np}$ contains the set of ω such that $t \rightarrow W(t, \omega)$ is differentiable for some $s \in [0, a)$.

Now define uniform partitions of $[0, a)$, $\{t_k^n\}_{k=0}^{2^n}$ such that

$$|t_k^n - t_{k-1}^n| = a2^{-n} \equiv \delta_n$$

Let

$$D_{np} \equiv \cup_{i=0}^{2^n-1} (\cap_{r=1}^5 [|W(t_i^n + r\delta_n) - W(t_i^n + (r-1)\delta_n)| \leq 10p\delta_n])$$

If $\omega \in C_{np}$, then for some $s \in [0, a)$, the condition of 52.2.3 holds. Suppose k is the number such that $s \in [t_{k-1}^n, t_k^n)$. Then for $r \in \{1, 2, 3, 4, 5\}$,

$$\begin{aligned} & |W(t_{k-1}^n + r\delta_n, \omega) - W(t_{k-1}^n + (r-1)\delta_n, \omega)| \\ & \leq |W(t_{k-1}^n + r\delta_n, \omega) - W(s, \omega)| + |W(s, \omega) - W(t_{k-1}^n + (r-1)\delta_n, \omega)| \\ & \leq 5p\delta_n + 5p\delta_n = 10p\delta_n \end{aligned}$$

Thus $C_{np} \subseteq D_{np}$. Now from 52.2.2,

$$P(D_{np}) \leq 2^n C \delta_n^{5/2} = C a^{5/2} 2^n (2^{-n})^{5/2} = C (\sqrt{a})^5 2^{-\frac{3}{2}n} \quad (52.2.4)$$

Let

$$C_p = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} C_{kp} \subseteq \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} D_{kp}.$$

It was just shown in 52.2.4 that $P(\cap_{k=n}^{\infty} D_{kp}) = 0$ and so C_p has measure 0. Thus $\cup_{p=1}^{\infty} C_p$, the set of points, ω where $t \rightarrow W(t, \omega)$ could have a derivative has measure 0. Taking the union of the exceptional sets corresponding to intervals $[0, n)$ for $n \in \mathbb{N}$, this proves the theorem.

This theorem on nowhere differentiability is very important because it shows it is doubtful one can define an integral $\int f(s) dW(s)$ by simply fixing ω and then doing some sort of Stieltjes integral in time. The reason for this is that the nowhere differentiability of W implies it is also not of bounded variation on any interval since if it were, it would equal the difference of two increasing functions and would therefore have a derivative at a.e. point.

I have presented the theorem on nowhere differentiability for one dimensional Wiener processes but the same proof holds with minor modifications if you have defined the Wiener process in $\mathbb{R}^n$ or you could simply consider the components and apply the above result.

52.3 Wiener Processes In Separable Banach Space

Here is an important lemma on which the existence of Wiener processes will be based.

Lemma 52.3.1 *There exists a sequence of real Wiener processes, $\{\psi_k(t)\}_{k=1}^\infty$ which have the following properties. Let $t_0 < t_1 < \dots < t_n$ be an arbitrary sequence. Then the random variables*

$$\{\psi_k(t_q) - \psi_k(t_{q-1}) : (q, k) \in (1, 2, \dots, n) \times (k_1, \dots, k_m)\}$$

are independent. Also each ψ_k is Holder continuous with exponent γ for any $\gamma < 1/2$ and for each $m \in \mathbb{N}$ there exists a constant C_m independent of k such that

$$\int_{\Omega} |\psi_k(t) - \psi_k(s)|^{2m} dP \leq C_m |t - s|^m$$

Proof: First, there exists a sequence $\{\xi_{ij}\}_{(i,j) \in \mathbb{N} \times \mathbb{N}}$ such that the $\{\xi_{ij}\}$ are independent and each normally distributed with mean 0 and variance 1. This follows from Lemma 52.1.1. (Let θ be a one to one and onto map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Then define $\xi_{ij} \equiv \xi_{\theta^{-1}(i,j)}.$)

Let

$$\psi_k(t) = \sum_{j=1}^{\infty} (\mathcal{X}_{[0,t]}, g_j)_{L^2} \xi_{kj}$$

where $\{g_j\}$ is a orthonormal basis for $L^2(0, \infty)$. By Corollary 52.1.3 this defines a real Wiener process. It remains to show the sequence of these Wiener processes has the desired properties.

From the definition of $\psi_k(t)$,

$$\psi_k(t) - \psi_k(s) = \sum_{j=1}^{\infty} (\mathcal{X}_{[s,t]}, g_j)_{L^2} \xi_{kj} \quad (52.3.5)$$

and this series converges in $L^2(\Omega)$ to a normal random variable having variance $t - s$ and mean 0. The convergence follows from

$$\begin{aligned} & \int_{\Omega} \left| \sum_{j=m}^n (\mathcal{X}_{[s,t]}, g_j)_{L^2} \xi_{kj} \right|^2 dP \\ &= \int_{\Omega} \sum_{j,l} (\mathcal{X}_{[s,t]}, g_j)_{L^2} \xi_{kj} (\mathcal{X}_{[s,t]}, g_l)_{L^2} \xi_{kl} dP \\ &= \sum_{j=m}^n (\mathcal{X}_{[s,t]}, g_j)_{L^2}^2 \end{aligned}$$

which converges to 0 as $m, n \rightarrow \infty$ due to $\mathcal{X}_{[s,t]} \in L^2(0, \infty)$. Thus

$$\begin{aligned} \int_{\Omega} |\psi_k(t) - \psi_k(s)|^2 dP &= \sum_{j=1}^{\infty} (\mathcal{X}_{[s,t]}, g_j)_{L^2}^2 \int_{\Omega} \xi_{kj}^2 dP \\ &= \sum_{j=1}^{\infty} (\mathcal{X}_{[s,t]}, g_j)_{L^2}^2 = (t - s) \end{aligned}$$

The rest of the claim that $\psi_k(t) - \psi_k(s)$ is normally distributed with mean 0 and variance $(t - s)$ follows from routine considerations involving partial sums and passing to a limit in the characteristic function of a partial sum. It will also follow from the next part of the argument. Given this is so it follows from considering the characteristic function as in 47.10.29 that an inequality of the following sort holds.

$$\int_{\Omega} |\psi_k(t) - \psi_k(s)|^{2m} dP \leq C_m |t - s|^m$$

By Theorem 50.1.4, the Kolmogorov Čentsov theorem, it follows $\psi_k(t)$ is Hölder continuous with exponent γ for $\gamma = (m - 1)/2m$.

Why are the random variables

$$\{\psi_k(t_q) - \psi_k(t_{q-1}) : (q, k) \in (1, 2, \dots, n) \times (k_1, \dots, k_m)\} \quad (52.3.6)$$

linearly independent? Let

$$P = \sum_{q=1}^n \sum_{r=1}^m s_{qr} (\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1}))$$

and consider $E(e^{iP})$. I want to use Proposition 47.4.1 on Page 1323. To do this I need to show $E(e^{iP})$ equals

$$\prod_{q=1}^n \prod_{r=1}^m E(\exp(is_{qr}(\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1})))) .$$

Using 52.3.5 $E(e^{iP})$ equals

$$\begin{aligned} & E \left(\exp \left(i \sum_{q=1}^n \sum_{r=1}^m s_{qr} \sum_{j=1}^{\infty} (\mathcal{X}_{[t_{q-1}, t_q]}, g_j)_{L^2} \xi_{k_r j} \right) \right) \\ &= \lim_{N \rightarrow \infty} E \left(\exp \left(i \sum_{q=1}^n \sum_{r=1}^m s_{qr} \sum_{j=1}^N (\mathcal{X}_{[t_{q-1}, t_q]}, g_j)_{L^2} \xi_{k_r j} \right) \right) \end{aligned}$$

Now the $\xi_{k_r j}$ are independent by construction. Therefore, the above equals

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \prod_{q=1}^n \prod_{r=1}^m \prod_{j=1}^N E \left(\exp \left(is_{qr} (\mathcal{X}_{[t_{q-1}, t_q]}, g_j)_{L^2} \xi_{k_r j} \right) \right) \\ &= \lim_{N \rightarrow \infty} \prod_{q=1}^n \prod_{r=1}^m \prod_{j=1}^N \exp \left(-\frac{1}{2} s_{qr}^2 (\mathcal{X}_{[t_{q-1}, t_q]}, g_j)_{L^2}^2 \right) \end{aligned}$$

$$\begin{aligned}
&= \prod_{q=1}^n \prod_{r=1}^m \lim_{N \rightarrow \infty} \exp \left(-\frac{1}{2} \sum_{j=1}^N s_{qr}^2 (\mathcal{X}_{[t_{q-1}, t_q]}, g_j)_{L^2}^2 \right) \\
&= \prod_{q=1}^n \prod_{r=1}^m \exp \left(-\frac{1}{2} s_{qr}^2 (t_q - t_{q-1}) \right)
\end{aligned}$$

Letting $m = n = 1$ this shows the earlier claim about $\psi_k(t) - \psi_k(s)$ being normally distributed with variance $(t - s)$. Now the above equals

$$= \prod_{q=1}^n \prod_{r=1}^m E \left(\exp \left(i s_{qr} (\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1})) \right) \right)$$

and so this verifies

$$E(e^{iP}) = \prod_{q=1}^n \prod_{r=1}^m E \left(\exp \left(i s_{qr} (\psi_{k_r}(t_q) - \psi_{k_r}(t_{q-1})) \right) \right)$$

and by Proposition 47.4.1 on Page 1323 it follows the random variables in 52.3.6 are independent. Note that as a special case, this also shows the random variables, $\{\psi_k(t)\}_{k=1}^\infty$ are independent due to the fact $\psi_k(0) = 0$. This proves the lemma.

Recall Corollary 49.10.4 which is stated here for convenience.

Corollary 52.3.2 *Let E be any real separable Banach space. Then there exists a sequence, $\{e_k\} \subseteq E$ such that for any $\{\xi_k\}$ a sequence of independent random variables such that $\mathcal{L}(\xi_k) = N(0, 1)$, it follows*

$$X(\omega) \equiv \sum_{k=1}^{\infty} \xi_k(\omega) e_k$$

converges a.e. and its law is a Gaussian measure defined on $\mathcal{B}(E)$. Furthermore, $\|e_k\|_E \leq \lambda_k$ where $\sum_k \lambda_k < \infty$.

Now let $\{\psi_k(t)\}$ be the sequence of Wiener processes described in Lemma 52.3.1. Then define a process with values in E by

$$W(t) \equiv \sum_{k=1}^{\infty} \psi_k(t) e_k \quad (52.3.7)$$

Then $\psi_k(t) / \sqrt{t}$ is $N(0, 1)$ and so by Corollary 49.10.4 the law of

$$W(t) / \sqrt{t} = \sum_{k=1}^{\infty} \left(\psi_k(t) / \sqrt{t} \right) e_k$$

is a Gaussian measure. Therefore, the same is true of $W(t)$. Similar reasoning applies to the increments, $W(t) - W(s)$ to conclude the law of each of these is

Gaussian. Consider the question whether the increments are independent. Let $0 \leq t_0 < t_1 < \cdots < t_m$ and let $\phi_j \in E'$. Then by the dominated convergence theorem and the properties of the $\{\psi_k\}$,

$$\begin{aligned}
 & E \left(\exp \left(i \sum_{j=1}^m \phi_j (W(t_j) - W(t_{j-1})) \right) \right) \\
 &= E \left(\exp \left(i \sum_{j=1}^m \left(\sum_{k=1}^{\infty} (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \right) \\
 &= E \left(\prod_{j=1}^m \exp \left(i \sum_{k=1}^{\infty} (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \lim_{n \rightarrow \infty} E \left(\prod_{j=1}^m \exp \left(i \sum_{k=1}^n (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \lim_{n \rightarrow \infty} \prod_{j=1}^m \prod_{k=1}^n E \left(\exp \left(i (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \lim_{n \rightarrow \infty} \prod_{j=1}^m E \left(\exp \left(i \sum_{k=1}^n (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \lim_{n \rightarrow \infty} E \left(\prod_{j=1}^m \exp \left(i \sum_{k=1}^n (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \lim_{n \rightarrow \infty} \prod_{j=1}^m \prod_{k=1}^n E \left(\exp \left(i (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \lim_{n \rightarrow \infty} \prod_{j=1}^m E \left(\exp \left(i \sum_{k=1}^n (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \prod_{j=1}^m E \left(\exp \left(i \sum_{k=1}^{\infty} (\psi_k(t_j) - \psi_k(t_{j-1})) \phi_j(e_k) \right) \right) \\
 &= \prod_{j=1}^m E \left(\exp \left(i \phi_j \left(\sum_{k=1}^{\infty} (\psi_k(t_j) - \psi_k(t_{j-1})) e_k \right) \right) \right) \\
 &= \prod_{j=1}^m E \left(\exp \left(i \phi_j (W(t_j) - W(t_{j-1})) \right) \right)
 \end{aligned}$$

which shows by Theorem 47.6.3 on Page 1329 that the random vectors, $\{W(t_j) - W(t_{j-1})\}_{j=1}^m$ are independent.

It is also routine to verify using properties of the ψ_k and characteristic functions that $\mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t-s))$. To see this, let $\phi \in E'$

$$\begin{aligned}
 & E(\exp(i\phi(W(t) - W(s)))) \\
 &= E\left(\left(\exp\left(i\phi\sum_{k=1}^{\infty}(\psi_k(t) - \psi_k(s))e_k\right)\right)\right) \\
 &= \lim_{n \rightarrow \infty} E\left(\left(\exp\left(i\phi\sum_{k=1}^n(\psi_k(t) - \psi_k(s))e_k\right)\right)\right) \\
 &= \lim_{n \rightarrow \infty} \prod_{k=1}^n E(\exp(i\phi(e_k)(\psi_k(t) - \psi_k(s)))) \\
 &= \lim_{n \rightarrow \infty} \prod_{k=1}^n E\left(\exp\left(-\frac{1}{2}\phi(e_k)^2(t-s)\right)\right) \\
 &= \lim_{n \rightarrow \infty} E\left(\exp\sum_{k=1}^n\left(-\frac{1}{2}\phi(e_k)^2(t-s)\right)\right)
 \end{aligned}$$

which is the same as the result for

$$E(\exp(i\phi(W(t-s))))$$

and

$$E(\exp(i\phi(\sqrt{t-s}W(1)))).$$

This has proved the following lemma.

Lemma 52.3.3 *Let E be a real separable Banach space. Then there exists an E valued stochastic process, $W(t)$ such that $\mathcal{L}(W(t))$ and $\mathcal{L}(W(t) - W(s))$ are Gaussian measures and the increments, $\{W(t) - W(s)\}$ are independent. Furthermore, the increment $W(t) - W(s)$ has the same distribution as $W(t-s)$ and $W(t)$ has the same distribution as $\sqrt{t}W(1)$.*

Now I want to consider the question of Holder continuity of the functions, $t \rightarrow W(t, \omega)$.

$$\begin{aligned}
 \int_{\Omega} \|W(t) - W(s)\|^\alpha dP &= \int_E \|x\|^\alpha d\mu_{W(t)-W(s)} \\
 &= \int_E \|x\|^\alpha d\mu_{W(t-s)} = \int_E \|x\|^\alpha d\mu_{\sqrt{t-s}W(1)} \\
 &= \int_{\Omega} \|\sqrt{t-s}W(1)\|^\alpha dP \\
 &= |t-s|^{\alpha/2} \int_{\Omega} \|W(1)\|^\alpha dP = C_\alpha |t-s|^{\alpha/2}
 \end{aligned}$$

by Fernique's theorem, Theorem 49.6.5. From the Kolmogorov Čentsov theorem, Theorem 50.1.4, it follows $\{W(t)\}$ is Hölder continuous with exponent $\gamma < (\frac{\alpha}{2} - 1)/\alpha$.

This completes the proof of the following theorem.

Theorem 52.3.4 *Let E be a separable real Banach space. Then there exists a stochastic process, $\{W(t)\}$ such that the distribution of $W(t)$ and every increment, $W(t) - W(s)$ is Gaussian. Furthermore, the increments corresponding to disjoint intervals are independent, $\mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t-s)) = \mathcal{L}(\sqrt{t-s}W(1))$. Also for a.e. ω , $t \rightarrow W(t, \omega)$ is Hölder continuous with exponent $\gamma < 1/2$.*

52.4 An Example Of Martingales, Independent Increments

Here is an interesting lemma.

Lemma 52.4.1 *Let $W(t)$ be a stochastic process having values in E a real separable Banach space which has independent increments. Let*

$$A \in \mathcal{H}_s \equiv \sigma(W(u) - W(r) : 0 \leq r < u \leq s)$$

Suppose $g(W(t) - W(s)) \in L^1(\Omega; E)$. Then $g(W(t) - W(s))$ and $\mathcal{X}_A$ are independent and

$$\int_{\Omega} \mathcal{X}_A g(W(t) - W(s)) dP = P(A) \int_{\Omega} g(W(t) - W(s)) dP \quad (52.4.8)$$

Proof: Let $\mathcal{G}$ denote the set, of all $A \in \mathcal{H}_s$ such that 52.4.8 holds. Then it is obvious $\mathcal{G}$ is closed with respect to complements and countable disjoint unions. Let $\mathcal{K}$ denote those sets which are finite intersections of the form

$$A = \cap_{i=1}^m A_i$$

where each A_i is in a set of $\sigma(W(u_i) - W(r_i))$ for some $0 \leq r_i < u_i \leq s$. For such A , it follows $A \in \sigma(W(u_i) - W(r_i), i = 1, \dots, m)$. Now consider the random vector having values in E^{m+1} ,

$$(W(u_1) - W(r_1), \dots, W(u_m) - W(r_m), g(W(t) - W(s)))$$

Let $\mathbf{t}^* \in (E')^m$ and $s^* \in E'$.

$$\mathbf{t}^* \cdot (W(u_1) - W(r_1), \dots, W(u_m) - W(r_m))$$

can be written in the form $\mathbf{g}^* \cdot (W(\tau_1) - W(\eta_1), \dots, W(\tau_l) - W(\eta_l))$ where the intervals, (τ_j, η_j) are disjoint and each $\eta_j \leq s$. Therefore, by independence of the increments,

$$E(\exp i(\mathbf{t}^* \cdot (W(u_1) - W(r_1), \dots, W(u_m) - W(r_m)) + s^*(g(W(t) - W(s))))))$$

$$\begin{aligned}
&= E(\exp i(\mathbf{g}^* \cdot (W(\tau_1) - W(\eta_1), \dots, W(\tau_l) - W(\eta_l)) + s^*(g(W(t) - W(s)))))) \\
&= \prod_{j=1}^l E(\exp(ig_j(W(\tau_j) - W(\eta_j)))) E(\exp(is^*(g(W(t) - W(s)))))) \\
&= E(\exp(i(\mathbf{t}^* \cdot (W(u_1) - W(r_1), \dots, W(u_m) - W(r_m)))))) \cdot \\
&\quad E(\exp(is^*(g(W(t) - W(s))))).
\end{aligned}$$

By Theorem 47.6.3, it follows the vector $(W(u_1) - W(r_1), \dots, W(u_m) - W(r_m))$ is independent of the random variable $g(W(t) - W(s))$ which shows that for $A \in \mathcal{K}$, $\mathcal{X}_A$, measurable in $\sigma(W(u_1) - W(r_1), \dots, W(u_m) - W(r_m))$ is independent of $g(W(t) - W(s))$. Therefore,

$$\begin{aligned}
\int_{\Omega} \mathcal{X}_A g(W(t) - W(s)) dP &= \int_{\Omega} \mathcal{X}_A dP \int_{\Omega} g(W(t) - W(s)) dP \\
&= P(A) \int_{\Omega} g(W(t) - W(s)) dP
\end{aligned}$$

Thus $\mathcal{K} \subseteq \mathcal{G}$ and so by the lemma on π systems, Lemma 9.11.3 on Page 275, it follows $\mathcal{G} \supseteq \sigma(\mathcal{K}) \supseteq \mathcal{F}_s \supseteq \mathcal{G}$. This proves the lemma.

Lemma 52.4.2 *Let $\{W(t)\}$ be a stochastic process having values in a separable Banach space which has the property that if $t_1 < t_2 < \dots < t_n$, then the increments,*

$$\{W(t_k) - W(t_{k-1})\}$$

are independent and integrable and $E(W(t) - W(s)) = 0$. Suppose also that $W(t)$ is right continuous meaning that for ω off a set of measure zero, $t \rightarrow W(t)(\omega)$ is right continuous. Also suppose that for some $q > 1$

$$\|W(t) - W(s)\|_{L^q(\Omega)}$$

is bounded independent of $s \leq t$. Then it is also a martingale with respect to the normal filtration defined by

$$\mathcal{F}_s \equiv \cap_{t>s} \overline{\sigma(W(u) - W(r) : 0 \leq r < u \leq t)}$$

where this denotes the intersection of the completions of the σ algebras

$$\sigma(W(u) - W(r) : 0 \leq r < u \leq t)$$

Also, in the same situation but without the assumption that $E(W(t) - W(s)) = 0$, if $t > s$ and $A \in \mathcal{F}_s$ it follows that if g is a continuous function such that

$$g(W(t) - W(s)) \in L^1(\Omega; E)$$

$$\|g(W(t) - W(s))\|_{L^q(\Omega)} \quad (52.4.9)$$

is bounded independent of $s \leq t$ for some $q > 1$ then

$$\int_{\Omega} \mathcal{X}_A g(W(t) - W(s)) dP = P(A) \int_{\Omega} g(W(t) - W(s)) dP. \quad (52.4.10)$$

Proof: Consider first the claim, 52.4.10. To begin with I show that if $A \in \mathcal{F}_s$ then for all ε small enough that $t > s + \varepsilon$,

$$\int_{\Omega} \mathcal{X}_A g(W(t) - W(s + \varepsilon)) dP = P(A) \int_{\Omega} g(W(t) - W(s + \varepsilon)) dP \quad (52.4.11)$$

This will happen if $\mathcal{X}_A$ and $g(W(t) - W(s + \varepsilon))$ are independent. First note that from the definition

$$A \in \overline{\sigma(W(u) - W(r) : 0 \leq r < u \leq s + \varepsilon)}$$

and so from the process of completion of a measure space, there exists

$$B \in \sigma(W(u) - W(r) : 0 \leq r < u \leq s + \varepsilon)$$

such that $B \supseteq A$ and $P(B \setminus A) = 0$. Therefore, letting $\phi \in E'$,

$$\begin{aligned} & E(\exp(it\mathcal{X}_A + i\phi(g(W(t) - W(s + \varepsilon)))) \\ &= E(\exp(it\mathcal{X}_B + i\phi(g(W(t) - W(s + \varepsilon)))) \\ &= E(\exp(it\mathcal{X}_B)) E(\exp(i\phi(g(W(t) - W(s + \varepsilon)))) \end{aligned}$$

because $\mathcal{X}_B$ is independent of $g(W(t) - W(s + \varepsilon))$ by Lemma 52.4.1 above. Then the above equals

$$= E(\exp(it\mathcal{X}_A)) E(\exp(i\phi(g(W(t) - W(s + \varepsilon))))$$

Now by Theorem 47.6.3, 52.4.11 follows. Next pass to the limit in both sides of 52.4.11 as $\varepsilon \rightarrow 0$. One can do this because of 52.4.9 which implies the functions in the integrands are uniformly integrable and Vitali's convergence theorem, Theorem 23.5.7. This yields 52.4.10.

Now consider the part about the stochastic process being a martingale. If $A \in \mathcal{F}_s$, the assumptions imply

$$\begin{aligned} \int_A E(W(t) | \mathcal{F}_s) - W(s) dP &= \int_{\Omega} \mathcal{X}_A E((W(t) - W(s)) | \mathcal{F}_s) dP \\ &= \int_{\Omega} E(\mathcal{X}_A (W(t) - W(s)) | \mathcal{F}_s) dP \\ \int_{\Omega} \mathcal{X}_A (W(t) - W(s)) dP &= \int_{\Omega} \mathcal{X}_A dP \int_{\Omega} W(t) - W(s) dP = 0 \end{aligned}$$

and so since A is arbitrary, $E(W(t) | \mathcal{F}_s) = W(s)$. This proves the lemma.

Note this implies immediately from Lemma 51.1.5 that Wiener process is not of bounded variation on any interval. This is because this lemma implies if it were of bounded variation, then it would be constant which is not the case due to

$$\mathcal{L}(W(t) - W(s)) = \mathcal{L}(W(t - s)) = \mathcal{L}(\sqrt{t - s}W(1)).$$

Here is an interesting theorem about approximation.

Theorem 52.4.3 *Let $\{W(t)\}$ be a Wiener process having values in a separable Banach space as described in Theorem 52.3.4. There exists a set of measure 0, N such that for $\omega \notin N$, the sum in 52.3.7 converges uniformly to $W(t, \omega)$ on any interval, $[0, T]$. That is, for each ω not in a set of measure zero, the partial sums of the sum in that formula converge uniformly to $t \rightarrow W(t, \omega)$ on $[0, T]$.*

Proof: By Lemma 52.4.2 the independence of the increments imply

$$\sum_{k=m}^n \psi_k(t) e_k$$

is a martingale and so by Theorem 50.3.5,

$$P\left(\left[\sup_{t \in [0, T]} \left\| \sum_{k=m}^n \psi_k(t) e_k \right\| \geq \alpha\right]\right) \leq \frac{1}{\alpha} \int_{\Omega} \left\| \sum_{k=m}^n \psi_k(T) e_k \right\| dP$$

From Corollary 52.3.2

$$\begin{aligned} \int_{\Omega} \left\| \sum_{k=m}^n \psi_k(T) e_k \right\| dP &\leq \sum_{k=m}^n \int_{\Omega} |\psi_k(T)| dP \lambda_k \\ &\leq \sum_{k=m}^n \lambda_k \end{aligned}$$

which shows that there exists a subsequence, m_l such that whenever $n > m_l$,

$$P\left(\left[\sup_{t \in [0, T]} \left\| \sum_{k=m_l}^n \psi_k(t) e_k \right\| \geq 2^{-k}\right]\right) \leq 2^{-k}.$$

Recall Lemma 47.8.6 stated below for convenience.

Lemma 52.4.4 *Let $\{\zeta_k\}$ be a sequence of random variables having values in a separable real Banach space, E whose distributions are symmetric. Letting $S_k \equiv \sum_{i=1}^k \zeta_i$, suppose $\{S_{n_k}\}$ converges a.e. Also suppose that for every $m > n_k$,*

$$P\left(\left[\|S_m - S_{n_k}\|_E > 2^{-k}\right]\right) < 2^{-k}. \quad (52.4.12)$$

Then in fact,

$$S_k(\omega) \rightarrow S(\omega) \text{ a.e. } \omega \quad (52.4.13)$$

Apply this lemma to the situation in which the Banach space, E is $C([0, T]; E)$ and $\zeta_k = \psi_k e_k$. Then you can conclude uniform convergence of the partial sums,

$$\sum_{k=1}^m \psi_k(t) e_k.$$

This proves the theorem.

Why is $C([0, T]; E)$ separable? Consider uniform partitions $\{t_k^n\}_{k=0}^{2^n}$ such that $t_k^n - t_{k-1}^n = T2^{-n}$. Let S denote all the step functions which are constant on the subintervals of one of these partitions which also have values in a countable dense subset of E . If $f \in C([0, T]; E)$, there exists something in S which is uniformly close to f . Now let $\{\psi_n\}$ be a mollifier and let S_n denote the convolution of elements of S with ψ_n . Then $\cup_{n=1}^\infty S_n$ is a countable dense subset of $C([0, T]; E)$. You could also develop a proof based on the Bernstein polynomials as in Lemma 7.1.3 on Page 165.

52.5 Hilbert Space Valued Wiener Processes

Next I will consider the case of Hilbert space valued Wiener processes. This will include the case of $\mathbb{R}^n$ valued Wiener processes. I will present this material independent of the more general case of E valued Wiener processes.

Definition 52.5.1 *Let $W(t)$ be a stochastic process with values in H , a real separable Hilbert space which has the properties that $t \rightarrow W(t, \omega)$ is continuous, whenever $t_1 < t_2 < \dots < t_m$, the increments $\{W(t_i) - W(t_{i-1})\}$ are independent, $W(0) = 0$, and whenever $s < t$,*

$$\mathcal{L}(W(t) - W(s)) = N(0, (t - s)Q)$$

which means that whenever $h \in H$,

$$\mathcal{L}((h, W(t) - W(s))) = N(0, (t - s)(Qh, h))$$

Also

$$E((h_1, W(t) - W(s))(h_2, W(t) - W(s))) = (Qh_1, h_2)(t - s).$$

Here Q is a nonnegative trace class operator. Recall this means

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$$

where $\{e_i\}$ is a complete orthonormal basis, $\lambda_i \geq 0$, and

$$\sum_{i=1}^{\infty} \lambda_i < \infty$$

Such a stochastic process is called a Q Wiener process.

Note the characteristic function of a Q Wiener process is

$$E\left(e^{i(h, W(t))}\right) = e^{-\frac{1}{2}t^2(Qh, h)} \quad (52.5.14)$$

Note that by Theorem 49.7.5 it you simply say that the distribution measure of $W(t)$ is Gaussian, then it follows there exists a trace class operator Q_t and $m_t \in H$

such that this measure is $N(m_t, Q_t)$. Thus for $W(t)$ a Wiener process, $Q_t = tQ$ and $m_t = 0$. In addition, the increments are independent so this is much more specific than the earlier definition of a Gaussian measure.

What is a Q Wiener process if the Hilbert space is $\mathbb{R}^n$? In particular, what is Q ? It is given that

$$\mathcal{L}((h, W(t) - W(s))) = N(0, (t-s)(Qh, h))$$

In this case everything is a vector in $\mathbb{R}^n$ and so for $h \in \mathbb{R}^n$,

$$E\left(e^{i\lambda(h, W(t) - W(s))}\right) = e^{-\frac{1}{2}\lambda^2(t-s)(Qh, h)}$$

In particular, letting $\lambda = 1$ this shows $W(t) - W(s)$ is normally distributed with covariance $(t-s)Q$ because its characteristic function is $e^{-\frac{1}{2}h^*(t-s)Qh}$.

With this and definition, one can describe Hilbert space valued Wiener processes in a fairly general setting.

Theorem 52.5.2 *Let U be a real separable Hilbert space and let $J : U_0 \rightarrow U$ be a Hilbert Schmidt operator where U_0 is a real separable Hilbert space. Then let $\{g_k\}$ be a complete orthonormal basis for U_0 and define for $t \in [0, T]$*

$$W(t) \equiv \sum_{k=1}^{\infty} \psi_k(t) Jg_k$$

Then $W(t)$ is a Q Wiener process for $Q = JJ^*$ as in Definition 52.5.1. Furthermore, the distribution of $W(t) - W(s)$ is the same as the distribution of $W(t-s)$ and W is Holder continuous with exponent γ for any $\gamma < 1/2$. There also is a subsequence denoted by N such that the convergence of the series

$$\sum_{k=1}^N \psi_k(t) Jg_k$$

is uniform for all ω not in some set of measure zero.

Proof: First it is necessary to show the series converges in $L^2(\Omega; U)$ for each t . For convenience I will consider the series for $W(t) - W(s)$. Then since $\psi_k(t) - \psi_k(s)$ is normal with mean 0 and variance $(t-s)$ and $\psi_k(t) - \psi_k(s)$ and $\psi_l(t) - \psi_l(s)$ are independent,

$$\begin{aligned} & \int_{\Omega} \left| \sum_{k=m}^n (\psi_k(t) - \psi_k(s)) Jg_k \right|_U^2 dP \\ &= \int_{\Omega} \sum_{k,l=m}^n ((\psi_k(t) - \psi_k(s)) Jg_k, (\psi_l(t) - \psi_l(s)) Jg_l) \\ &= (t-s) \sum_{k=m}^n (Jg_k, Jg_k) = (t-s) \sum_{k=m}^n \|Jg_k\|_U^2 \end{aligned}$$

which converges to 0 as $m, n \rightarrow \infty$ thanks to the assumption that J is Hilbert Schmidt. It follows the above sum converges in $L^2(\Omega; U)$ for each t .

Now letting $m < n$, it follows by the maximal estimate, Theorem 50.3.5, and the above

$$\begin{aligned} & P \left(\left| \sum_{k=1}^m \psi_k(t) Jg_k - \sum_{k=1}^n \psi_k(t) Jg_k \right|_U \geq \lambda \right) \\ & \leq \frac{1}{\lambda} E \left(\left| \sum_{k=m+1}^n \psi_k(t) Jg_k \right|_U \right) \leq \frac{1}{\lambda} T \sum_{k=m}^n \|Jg_k\|_U^2 \end{aligned}$$

and so there exists a subsequence n_l such that for all $p \geq 0$,

$$P \left(\left| \sum_{k=1}^{n_l} \psi_k(t) Jg_k - \sum_{k=1}^{n_l+p} \psi_k(t) Jg_k \right|_U \geq 2^{-l} \right) < 2^{-l}$$

Therefore, by Borel Cantelli lemma, there is a set of measure zero such that for ω not in this set,

$$\lim_{l \rightarrow \infty} \sum_{k=1}^{n_l} \psi_k(t) Jg_k = \sum_{k=1}^{\infty} \psi_k(t) Jg_k$$

is uniform. From now on denote this subsequence by N to save on notation.

I need to consider the characteristic function of $(h, W(t) - W(s))_U$ for $h \in U$. Then since convergence is in $L^2(\Omega; U)$,

$$\begin{aligned} & E(\exp(ir(h, (W(t) - W(s))_U))) \\ &= \lim_{N \rightarrow \infty} E \left(\exp \left(ir \left(\sum_{j=1}^N (\psi_j(t) - \psi_j(s)) (h, Jg_j) \right) \right) \right) \\ &= \lim_{N \rightarrow \infty} E \left(\prod_{j=1}^N e^{ir(\psi_j(t) - \psi_j(s))(h, Jg_j)} \right) \end{aligned}$$

Since the random variables $\psi_j(t) - \psi_j(s)$ are independent,

$$= \lim_{N \rightarrow \infty} \prod_{j=1}^N E \left(e^{ir(h, Jg_j)(\psi_j(t) - \psi_j(s))} \right)$$

Since $\psi_j(t) - \psi_j(s)$ is a Gaussian random variable having mean 0 and variance $(t-s)$, the above equals

$$= \lim_{N \rightarrow \infty} \prod_{j=1}^N e^{-\frac{1}{2} r^2 (h, Jg_j)^2 (t-s)}$$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \exp \left(\sum_{j=1}^N -\frac{1}{2} r^2 (h, Jg_j)^2 (t-s) \right) \\
&= \exp \left(-\frac{1}{2} r^2 (t-s) \sum_{j=1}^{\infty} (h, Jg_j)_U^2 \right) \\
&= \exp \left(-\frac{1}{2} r^2 (t-s) \sum_{j=1}^{\infty} (J^* h, g_j)_{U_0}^2 \right) \\
&= \exp \left(-\frac{1}{2} r^2 (t-s) \|J^* h\|_U^2 \right) = \exp \left(-\frac{1}{2} r^2 (t-s) (JJ^* h, h)_U \right) \\
&= \exp \left(-\frac{1}{2} r^2 (t-s) (Qh, h)_U \right) \tag{52.5.15}
\end{aligned}$$

which shows $(h, W(t) - W(s))_U$ is normally distributed with mean 0 and variance $(t-s)(Qh, h)$ where $Q \equiv JJ^*$. It is obvious from the definition that $W(0) = 0$. Note that Q is of trace class because if $\{e_k\}$ is an orthonormal basis for U ,

$$\begin{aligned}
\sum_k (Qe_k, e_k)_U &= \sum_k \|J^* e_k\|_{U_0}^2 = \sum_k \sum_l (J^* e_k, g_l)_{U_0}^2 \\
&= \sum_l \sum_k (e_k, Jg_l)_U^2 = \sum_l \|Jg_l\|_U^2 < \infty
\end{aligned}$$

To find the covariance, consider

$$E((h_1, W(t) - W(s)) (h_2, W(t) - W(s))),$$

and use 52.5.19 to obtain this is equal to

$$E \left(\sum_{k=1}^{\infty} (\psi_k(t) - \psi_k(s)) (h_1, Jg_k) \sum_{j=1}^{\infty} (\psi_j(t) - \psi_j(s)) (h_2, Jg_j) \right).$$

Since the series converge in $L^2(\Omega; U)$, the above equals

$$= \lim_{n \rightarrow \infty} E \left(\sum_{k=1}^n (\psi_k(t) - \psi_k(s)) (h_1, Jg_k) \sum_{j=1}^n (\psi_j(t) - \psi_j(s)) (h_2, Jg_j) \right)$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} (t-s) \sum_{k=1}^n (h_1, Jg_k) (h_2, Jg_k) \\
&= \lim_{n \rightarrow \infty} (t-s) \sum_{k=1}^n (J^* h_1, g_k)_{U_0} (J^* h_2, g_k)_{U_0} \\
&= (t-s) \sum_{k=1}^{\infty} (J^* h_1, g_k)_{U_0} (J^* h_2, g_k)_{U_0} \\
&= (t-s) (J^* h_1, J^* h_2) = (t-s) (Qh_1, h_2).
\end{aligned}$$

Next consider the claim that the increments are independent. Let $W^N(t)$ be given by the appropriate partial sum and let $\{h_j\}_{j=1}^m$ be a finite list of vectors of U . Then from the independence properties of ψ_j explained above,

$$\begin{aligned}
&E \left(\exp \sum_{j=1}^m i (h_j, W^N(t_j) - W^N(t_{j-1}))_U \right) \\
&= E \left(\exp \sum_{j=1}^m i \left(h_j, \sum_{k=1}^N Jg_k (\psi_k(t_j) - \psi_k(t_{j-1})) \right)_U \right) \\
&= E \left(\exp \sum_{j=1}^m \sum_{k=1}^N i (h_j, Jg_k)_U (\psi_k(t_j) - \psi_k(t_{j-1})) \right) \\
&= E \left(\prod_{j,k} \exp (i (h_j, Jg_k)_U (\psi_k(t_j) - \psi_k(t_{j-1}))) \right) \\
&= \prod_{j,k} E (\exp (i (h_j, Jg_k)_U (\psi_k(t_j) - \psi_k(t_{j-1}))))
\end{aligned}$$

This can be done because of the independence of the random variables

$$\{\psi_k(t_j) - \psi_k(t_{j-1})\}_{j,k}.$$

Thus the above equals

$$\begin{aligned}
&\prod_{j,k} \exp \left(-\frac{1}{2} (h_j, Jg_k)_U^2 (t_j - t_{j-1}) \right) \\
&= \prod_{j=1}^m \exp \left(-\frac{1}{2} \sum_{k=1}^N (h_j, Jg_k)_U^2 (t_j - t_{j-1}) \right)
\end{aligned}$$

because $\psi_k(t_j) - \psi_k(t_{j-1})$ is normally distributed having variance $t_j - t_{j-1}$. Now letting $N \rightarrow \infty$, this implies

$$\begin{aligned}
 & E \left(\exp \sum_{j=1}^m i(h_j, W(t_j) - W(t_{j-1}))_U \right) \\
 &= \prod_{j=1}^m \exp \left(-\frac{1}{2} \sum_{k=1}^{\infty} (h_j, Jg_k)_U^2 (t_j - t_{j-1}) \right) \\
 &= \prod_{j=1}^m \exp \left(-\frac{1}{2} (t_j - t_{j-1}) \sum_{k=1}^{\infty} (J^* h_j, g_k)_{U_0}^2 \right) \\
 &= \prod_{j=1}^m \exp \left(-\frac{1}{2} (t_j - t_{j-1}) \|J^* h_j\|_{U_0}^2 \right) \\
 &= \prod_{j=1}^m \exp \left(-\frac{1}{2} (t_j - t_{j-1}) (Qh, h)_U \right) \\
 &= \prod_{j=1}^m \exp (i(h_j, W(t_j) - W(t_{j-1}))_U) \tag{52.5.16}
 \end{aligned}$$

Now from 52.5.15, letting $r = 1$, the above equals

$$\prod_{j=1}^m E (i(h_j, W(t_j) - W(t_{j-1}))_U)$$

By Theorem 47.6.3 on Page 1329, this shows the increments are independent.

It remains to verify the Holder continuity. Recall

$$W(t) = \sum_{k=1}^{\infty} Jg_k \psi_k(t)$$

where ψ_k is a real Wiener process.

Let $m > 2$ and let $W^N(t)$ denote the appropriate partial sum up to N in the definition of $W(t)$. Also let N be a subsequence such that in addition to convergence in $L^2(\Omega; U)$, pointwise convergence takes place for both $W^N(t)$ and $W^N(s)$. Then using 52.1.1 for ψ_k and Jensen's inequality along with Lemma 52.3.1,

$$\left(E \left(\|W^N(t) - W^N(s)\|_U^{2m} \right) \right)^{1/m} = \left(E \left(\left\| \sum_{k=1}^N Jg_k (\psi_k(t) - \psi_k(s)) \right\|_U^{2m} \right) \right)^{1/m}$$

$$\begin{aligned}
&= \left(E \left(\left\{ \sum_{k,l=1}^N (Jg_k Jg_l)_U (\psi_k(t) - \psi_k(s)) (\psi_l(t) - \psi_l(s)) \right\}^m \right) \right)^{1/m} \\
&\leq \sum_{k,l=1}^N (E (|(Jg_k Jg_l)_U|^m |\psi_k(t) - \psi_k(s)|^m |\psi_l(t) - \psi_l(s)|^m))^{1/m} \\
&= \sum_{k,l=1}^N |(Jg_k Jg_l)_U|^m (E (|\psi_k(t) - \psi_k(s)|^m |\psi_l(t) - \psi_l(s)|^m))^{1/m} \\
&= \sum_{k=1}^N \|Jg_k\|_U^{2m} \left(E \left(|\psi_k(t) - \psi_k(s)|^{2m} \right) \right)^{1/m} \\
&\quad + \sum_{k \neq l} \|Jg_k\|^m \|Jg_l\|^m \left(E \left(|\psi_k(t) - \psi_k(s)|^{2m} \right) \right)^{1/2m} \cdot \\
&\quad \left(E \left(|\psi_l(t) - \psi_l(s)|^{2m} \right) \right)^{1/2m} \\
&\leq C_m |t - s| \sum_{k=1}^N \|Jg_k\|_U^{2m} \\
&\quad + C_m \sum_k \sum_l \|Jg_k\|^m (|t - s|^m)^{1/2m} \|Jg_l\|^m (|t - s|^m)^{1/2m} \\
&\leq C_m |t - s| \sum_{k=1}^{\infty} \|Jg_k\|_U^{2m} + \\
&\quad C_m |t - s| \sum_{k=1}^{\infty} \|Jg_k\|^m \sum_{l=1}^{\infty} \|Jg_l\|^m \leq C'_m |t - s|
\end{aligned}$$

It follows an expression of the following form holds.

$$E \left(|W^N(t) - W^N(s)|_U^{2m} \right) \leq C_m |t - s|^m, m > 2$$

where C_m is some constant independent of N . Now an application of Fatou's lemma as $N \rightarrow \infty$ yields

$$E \left(|W(t) - W(s)|_U^{2m} \right) \leq C_m |t - s|^m.$$

By the Kolmogorov Čentsov Theorem, Theorem 50.1.4, it follows that off a set of measure 0, $t \rightarrow W(t, \omega)$ is Hölder continuous with exponent γ such that

$$\gamma < \frac{m-1}{2m}, m > 2.$$

Finally, from 52.5.15 with $r = 1$,

$$E(\exp i(h, W(t) - W(s))_U) = \exp\left(-\frac{1}{2}(t-s)(Qh, h)\right)$$

which is the same as $E(\exp i(h, W(t-s)))_U$ due to the fact $W(0) = 0$. This proves the theorem.

Now suppose you start with a nonnegative trace class operator Q . Then in this case also one can define a Q Wiener process. It is possible to get this theorem from Theorem 52.5.2 but this will not be done here.

Theorem 52.5.3 *Let U be a real separable Hilbert space and let Q be a nonnegative trace class operator defined on U . Then there exists a Q Wiener process as defined in Definition 52.5.1. Furthermore, the distribution of $W(t) - W(s)$ is the same as the distribution of $W(t-s)$ and W is Holder continuous with exponent γ for any $\gamma < 1/2$.*

Proof: One can obtain this theorem as a corollary of Theorem 52.5.2 but this will not be done here.

Let

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$$

where $\{e_i\}$ is a complete orthonormal set and $\lambda_i \geq 0$ and $\sum \lambda_i < \infty$. Now the definition of the Q Wiener process is

$$W(t) \equiv \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \psi_k(t) \quad (52.5.17)$$

where $\{\psi_k(t)\}$ are the real Wiener processes defined in Lemma 52.3.1.

Now consider 52.5.17. From this formula, if $s < t$

$$W(t) - W(s) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k (\psi_k(t) - \psi_k(s)) \quad (52.5.18)$$

First it is necessary to show this sum converges. Since $\psi_j(t)$ is a Wiener process,

$$\begin{aligned} & \int_{\Omega} \left| \sum_{j=m}^n \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s)) e_j \right|_U^2 dP \\ &= \int_{\Omega} \sum_{j=m}^n \lambda_j (\psi_j(t) - \psi_j(s))^2 dP \\ &= (t-s) \sum_{j=m}^n \lambda_j \end{aligned}$$

and this converges to 0 as $m, n \rightarrow \infty$ because it was given that

$$\sum_{j=1}^{\infty} \lambda_j < \infty$$

so the series in 52.5.18 converges in $L^2(\Omega; U)$.

Therefore, there exists a subsequence

$$\left\{ \sum_{k=1}^N \sqrt{\lambda_k} e_k (\psi_k(t) - \psi_k(s)) \right\}$$

which converges pointwise a.e. to $W(t) - W(s)$ as well as in $L^2(\Omega; U)$ as $N \rightarrow \infty$. Then letting $h \in U$,

$$(h, W(t) - W(s))_U = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (\psi_k(t) - \psi_k(s)) (h, e_k) \quad (52.5.19)$$

Then by the dominated convergence theorem,

$$\begin{aligned} & E(\exp(ir(h, (W(t) - W(s))_U))) \\ &= \lim_{N \rightarrow \infty} E \left(\exp \left(ir \left(\sum_{j=1}^N \sqrt{\lambda_j} (\psi_j(t) - \psi_j(s)) (h, e_j) \right) \right) \right) \\ &= \lim_{N \rightarrow \infty} E \left(\prod_{j=1}^N e^{ir\sqrt{\lambda_j}(\psi_j(t) - \psi_j(s))(h, e_j)} \right) \end{aligned}$$

Since the random variables $\psi_j(t) - \psi_j(s)$ are independent,

$$= \lim_{N \rightarrow \infty} \prod_{j=1}^N E \left(e^{ir\sqrt{\lambda_j}(\psi_j(t) - \psi_j(s))(h, e_j)} \right)$$

Since $\psi_j(t)$ is a real Wiener process,

$$\begin{aligned} &= \lim_{N \rightarrow \infty} \prod_{j=1}^N e^{-\frac{1}{2} r^2 \lambda_j (t-s) (h, e_j)^2} \\ &= \lim_{N \rightarrow \infty} \exp \left(\sum_{j=1}^N -\frac{1}{2} r^2 \lambda_j (t-s) (h, e_j)^2 \right) \\ &= \exp \left(-\frac{1}{2} r^2 (t-s) \sum_{j=1}^{\infty} \lambda_j (h, e_j)^2 \right) \\ &= \exp \left(-\frac{1}{2} r^2 (t-s) (Qh, h) \right) \end{aligned} \quad (52.5.20)$$

Thus $(h, W(t) - W(s))$ is normally distributed with mean 0 and variance $(t - s)(Qh, h)$. It is obvious from the definition that $W(0) = 0$. Also to find the covariance, consider

$$E((h_1, W(t) - W(s))(h_2, W(t) - W(s))),$$

and use 52.5.19 to obtain this is equal to

$$\begin{aligned} & E\left(\sum_{k=1}^{\infty} \sqrt{\lambda_k}(\psi_k(t) - \psi_k(s))(h_1, e_k) \sum_{j=1}^{\infty} \sqrt{\lambda_j}(\psi_j(t) - \psi_j(s))(h_2, e_j)\right) \\ &= \lim_{n \rightarrow \infty} E\left(\sum_{k=1}^n \sqrt{\lambda_k}(\psi_k(t) - \psi_k(s))(h_1, e_k) \sum_{j=1}^n \sqrt{\lambda_j}(\psi_j(t) - \psi_j(s))(h_2, e_j)\right) \\ &= \lim_{n \rightarrow \infty} (t - s) \sum_{k=1}^n \lambda_k (h_1, e_k)(h_2, e_k) = (t - s)(Qh_1, h_2) \end{aligned}$$

(Recall $Q \equiv \sum_k \lambda_k e_k \otimes e_k$.)

Next I show the increments are independent. Let N be the subsequence defined above and let $W^N(t)$ be given by the appropriate partial sum and let $\{h_j\}_{j=1}^m$ be a finite list of vectors of U . Then from the independence properties of ψ_j explained above,

$$\begin{aligned} & E\left(\exp \sum_{j=1}^m i(h_j, W^N(t_j) - W^N(t_{j-1}))_U\right) \\ &= E\left(\exp \sum_{j=1}^m i\left(h_j, \sum_{k=1}^N \sqrt{\lambda_k} e_k (\psi_k(t_j) - \psi_k(t_{j-1}))\right)_U\right) \\ &= E\left(\exp \sum_{j=1}^m \sum_{k=1}^N i \sqrt{\lambda_k} (h_j, e_k)_U (\psi_k(t_j) - \psi_k(t_{j-1}))\right) \\ &= E\left(\prod_{j,k} \exp\left(i \sqrt{\lambda_k} (h_j, e_k)_U (\psi_k(t_j) - \psi_k(t_{j-1}))\right)\right) \\ &= \prod_{j,k} E\left(\exp\left(i \sqrt{\lambda_k} (h_j, e_k)_U (\psi_k(t_j) - \psi_k(t_{j-1}))\right)\right) \end{aligned}$$

This can be done because of the independence of the random variables

$$\{\psi_k(t_j) - \psi_k(t_{j-1})\}_{j,k}.$$

Thus the above equals

$$= \prod_{j=1}^m \exp \left(-\frac{1}{2} \sum_{k=1}^N \lambda_k (h_j, e_k)_U^2 (t_j - t_{j-1}) \right)$$

because $\psi_k(t_j) - \psi_k(t_{j-1})$ is normally distributed having variance $t_j - t_{j-1}$ and mean 0. Now letting $N \rightarrow \infty$, this implies

$$\begin{aligned} & E \left(\exp \sum_{j=1}^m i(h_j, W(t_j) - W(t_{j-1}))_U \right) \\ &= \prod_{j=1}^m \exp \left(-\frac{1}{2} (t_j - t_{j-1}) \sum_{k=1}^{\infty} \lambda_k (h_j, e_k)_U^2 \right) \\ &= \prod_{j=1}^m \exp \left(-\frac{1}{2} (t_j - t_{j-1}) (Qh, h)_U \right) \\ &= \prod_{j=1}^m \exp (i(h_j, W(t_j) - W(t_{j-1}))_U) \end{aligned} \quad (52.5.21)$$

because of the fact shown above that $(h, W(t) - W(s))$ is normally distributed with mean 0 and variance $(t - s)(Qh, h)$. By Theorem 47.6.3 on Page 1329, this shows the increments are independent.

Next consider the continuity assertion. Recall

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k \psi_k(t)$$

where ψ_k is a real Wiener process. Therefore, letting $2m > 2, m \in \mathbb{N}$ and using 52.1.1 for ψ_k and Jensen's inequality along with Lemma 52.3.1,

$$\begin{aligned} E(|W(t) - W(s)|^{2m}) &= E \left(\left| \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k (\psi_k(t) - \psi_k(s)) \right|^{2m} \right) \\ &= E \left(\left(\sum_{k=1}^{\infty} \lambda_k |\psi_k(t) - \psi_k(s)|^2 \right)^m \right) \\ &\leq E \left(\left(\sum_{k=1}^{\infty} \lambda_k \right)^{m-1} \sum_{k=1}^{\infty} \lambda_k |\psi_k(t) - \psi_k(s)|^{2m} \right) \\ &\leq C_m \sum_{k=1}^{\infty} \lambda_k E(|\psi_k(t) - \psi_k(s)|^{2m}) \end{aligned} \quad (52.5.22)$$

$$\leq C_m |t - s|^m \quad (52.5.23)$$

By the Kolmogorov Čentsov Theorem, Theorem 50.1.4, it follows that off a set of measure 0, $t \rightarrow W(t, \omega)$ is Hölder continuous with exponent γ such that

$$\gamma < \frac{m-1}{2m}.$$

Finally, from 52.5.20 taking $r = 1$,

$$E(\exp i(h, W(t) - W(s))_U) = \exp\left(-\frac{1}{2}(t-s)(Qh, h)\right)$$

which is the same as $E(\exp i(h, W(t-s))_U)$ due to the fact $W(0) = 0$. This proves the theorem.

The above shows there exists Q Wiener processes in any separable Hilbert space. Next I will show the way described above is the only way it can happen.

Theorem 52.5.4 *Suppose $\{W(t)\}$ is a Q Wiener process in U , a real separable Hilbert space. Then letting*

$$Q = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$$

where the $\{e_k\}$ are orthonormal, $\lambda_k \geq 0$, and $\sum_{k=1}^{\infty} \lambda_k < \infty$, it follows

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \psi_k(t) e_k \quad (52.5.24)$$

where

$$\psi_k(t) \equiv \begin{cases} \frac{1}{\sqrt{\lambda_k}} (W(t), e_k)_U & \text{if } \lambda_k \neq 0 \\ 0 & \text{if } \lambda_k = 0 \end{cases}$$

then $\{\psi_k(t)\}$ is a Wiener process and for $t_0 < t_1 < \cdots < t_n$ the random variables

$$\{\psi_k(t_q) - \psi_k(t_{q-1}) : (q, k) \in (1, 2, \dots, n) \times (k_1, \dots, k_m)\}$$

are independent. Furthermore, the sum in 52.5.24 converges uniformly for a.e. ω on any closed interval, $[0, T]$.

Proof: First of all, the fact that $W(t)$ has values in U and that $\{e_k\}$ is an orthonormal basis implies the sum in 52.5.24 converges for each ω . Consider

$$E(\exp(ir\psi_k(t))) = E\left(\exp\left(ir\frac{1}{\sqrt{\lambda_k}}(W(t), e_k)_U\right)\right)$$

Since $W(t)$ is given to be a Q Wiener process, $(W(t), e_k)_U$ is normally distributed with mean 0 and variance $t(Qh, h)$. Therefore, the above equals

$$= e^{-\frac{1}{2}r^2 \frac{1}{\lambda_k} t(Qe_k, e_k)} = e^{-\frac{1}{2}r^2 \frac{1}{\lambda_k} t\lambda_k} = e^{-\frac{1}{2}r^2 t},$$

the characteristic function for a random variable which is $N(0, t)$. The independence of the increments for a given $\psi_k(t)$ follows right away from the independence of the increments of $W(t)$ and the distribution of the increments being $N(0, (t-s))$ follows similarly to the above.

For $t_1 < t_2 < \cdots < t_n$, why are the random variables,

$$\{(W(t_q), e_k)_U - (W(t_{q-1}), e_k)_U : (q, k) \in (1, 2, \dots, n) \times (k_1, \dots, k_m)\} \quad (52.5.25)$$

independent? Let

$$P = \sum_{q=1}^n \sum_{j=1}^m s_{qj} ((W(t_q), e_{k_j})_U - (W(t_{q-1}), e_{k_j})_U)$$

and consider $E(e^{iP})$. This equals

$$\begin{aligned} e^{iP} &= E \left(\exp \left(i \sum_{q=1}^n \sum_{j=1}^m s_{qj} ((W(t_q), e_{k_j})_U - (W(t_{q-1}), e_{k_j})_U) \right) \right) \quad (52.5.26) \\ &= E \left(\exp \left(i \sum_{q=1}^n \left(\left(W(t_q), \sum_{j=1}^m s_{qj} e_{k_j} \right)_U - \left(W(t_{q-1}), \sum_{j=1}^m s_{qj} e_{k_j} \right)_U \right) \right) \right) \\ &= E \left(\prod_{q=1}^n \exp \left(i \left(W(t_q) - W(t_{q-1}), \sum_{j=1}^m s_{qj} e_{k_j} \right)_U \right) \right) \end{aligned}$$

Now recall that by assumption the increments $W(t) - W(s)$ are independent. Therefore, the above equals

$$\prod_{q=1}^n E \left(\exp \left(i \left(W(t_q) - W(t_{q-1}), \sum_{j=1}^m s_{qj} e_{k_j} \right)_U \right) \right)$$

Recall that by assumption $(W(t) - W(s), h)_U$ is normally distributed with variance $(t-s)(Qh, h)$ and mean 0. Therefore, the above equals

$$\begin{aligned} &\prod_{q=1}^n \exp \left(-\frac{1}{2} (t_q - t_{q-1}) \left(Q \sum_{j=1}^m s_{qj} e_{k_j}, \sum_{j=1}^m s_{qj} e_{k_j} \right) \right) \\ &= \prod_{q=1}^n \exp \left(-\frac{1}{2} (t_q - t_{q-1}) \sum_{j=1}^m s_{qj}^2 \lambda_{k_j} \right) \\ &\quad \exp \left(-\frac{1}{2} \sum_{q=1}^n \sum_{j=1}^m (t_q - t_{q-1}) s_{qj}^2 \lambda_{k_j} \right) \quad (52.5.27) \end{aligned}$$

Also

$$\begin{aligned}
 & \prod_{q=1}^n \prod_{j=1}^n E \left(\exp \left(i s_{qj} \left((W(t_q), e_{k_j})_U - (W(t_{q-1}), e_{k_j})_U \right) \right) \right) \quad (52.5.28) \\
 &= \prod_{q=1}^n \prod_{j=1}^n E \left(\exp \left(i s_{qj} \left((W(t_q) - W(t_{q-1}), e_{k_j})_U \right) \right) \right) \\
 &= \prod_{q=1}^n \prod_{j=1}^n \exp \left(-\frac{1}{2} (t_q - t_{q-1}) s_{qj}^2 (Q e_{k_j}, e_{k_j}) \right) \\
 &= \prod_{q=1}^n \prod_{j=1}^n \exp \left(-\frac{1}{2} (t_q - t_{q-1}) s_{qj}^2 \lambda_{k_j} \right) \\
 &= \exp \left(-\frac{1}{2} \sum_{q=1}^n \sum_{j=1}^m (t_q - t_{q-1}) s_{qj}^2 \lambda_{k_j} \right)
 \end{aligned}$$

Therefore, e^{iP} equals the expression in 52.5.28 because both equal the expression in 52.5.27 and it follows from Proposition 47.4.1 on Page 1323 that the random variables of 52.5.25 are independent.

What about the claim of uniform convergence? By the independence of the increments, it follows from Lemma 52.4.2 that $\{W(t)\}$ is a martingale and each real valued function, $(W(t), e_k)_U$ is also a martingale. Therefore, Theorem 50.3.5 can be applied to conclude

$$\begin{aligned}
 & P \left(\left[\sup_{t \in [0, T]} \left| \sum_{k=m}^n (W(t), e_k)_U e_k \right| \geq \alpha \right] \right) \leq \frac{1}{\alpha} \int_{\Omega} \left| \sum_{k=m}^n (W(T), e_k)_U e_k \right| dP \\
 & \leq \frac{1}{\alpha} \int_{\Omega} \left| \sum_{k=m}^n (W(T), e_k)_U e_k \right|^2 dP = \frac{1}{\alpha} \sum_{k=m}^n \int_{\Omega} (W(T), e_k)_U^2 dP \\
 & = \frac{1}{\alpha} \sum_{k=m}^n (Q e_k, e_k) T = \frac{T}{\alpha} \sum_{k=m}^n \lambda_k \leq \frac{T}{\alpha} \sum_{k=m}^{\infty} \lambda_k
 \end{aligned}$$

Since $\sum_{k=1}^{\infty} \lambda_k < \infty$, there exists a sequence, $\{m_l\}$ such that if $n > m_l$

$$P \left(\left[\sup_{t \in [0, T]} \left| \sum_{k=m_l}^n (W(t), e_k)_U e_k \right| > 2^{-k} \right] \right) < 2^{-k}$$

and so by the Borel Cantelli lemma, off a set of measure 0 the partial sums

$$\left\{ \sum_{k=1}^{m_l} (W(t), e_k)_U e_k \right\}$$

converge uniformly on $[0, T]$. This is very interesting but more can be said. In fact the original partial sums converge.

Recall Lemma 47.8.6 stated below for convenience.

Lemma 52.5.5 *Let $\{\zeta_k\}$ be a sequence of random variables having values in a separable real Banach space, E whose distributions are symmetric. Letting $S_k \equiv \sum_{i=1}^k \zeta_i$, suppose $\{S_{n_k}\}$ converges a.e. Also suppose that for every $m > n_k$,*

$$P\left(\|S_m - S_{n_k}\|_E > 2^{-k}\right) < 2^{-k}. \quad (52.5.29)$$

Then in fact,

$$S_k(\omega) \rightarrow S(\omega) \text{ a.e.} \quad (52.5.30)$$

Apply this lemma to the situation in which the Banach space, E is $C([0, T]; U)$. Then you can conclude uniform convergence of the partial sums,

$$\sum_{k=1}^m (W(t), e_k)_U e_k.$$

This proves the theorem.

52.6 Levy's Theorem In Hilbert Space

Recall the concept of quadratic variation. Let $W(t)$ be a Q Wiener process. Does it follow $\{W(t)\} \in \mathcal{M}_T^2(H)$? The Wiener process is continuous. Furthermore,

$$E\left(|W(t)|_H^2\right) < \infty$$

for each $t \in [0, T]$. Since $\{W(t)\}$ is a martingale, Theorem 50.3.5 can be applied to conclude

$$E\left(|W(t)|_H^2\right)^{1/2} \leq E\left(\left(\sup_{t \in [0, T]} |W(t)|\right)^2\right)^{1/2} \leq 2E\left(|W(T)|_H^2\right)^{1/2}$$

and so $\{W(t)\} \in \mathcal{M}_T^2(H)$. Therefore, by the Doob Meyer decomposition, Theorem 51.6.15, there exists an increasing natural process, $A(t)$ and a martingale, $Y(t)$ such that

$$|W(t)|_H^2 = Y(t) + A(t).$$

What is $A(t)$? Consider the process

$$|W(t)|^2$$

From Theorem 52.5.4 this equals

$$\sum_{k=1}^{\infty} \lambda_k \psi_k(t)^2$$

where $\psi_k(t)$ is a one dimensional Wiener process and

$$Q = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k, \quad \sum_{k=1}^{\infty} \lambda_k < \infty.$$

By Lemma 52.4.2, $\{W(t)\}$ is a martingale. Therefore, for $s < t$ and $A \in \mathcal{F}_s$, it follows since $\mathcal{X}_A$ is independent of $W(t) - W(s)$ as in the proof of Lemma 52.4.2 that the following holds.

$$\begin{aligned} & \int_A E \left(|W(t)|^2 | \mathcal{F}_s \right) - |W(s)|^2 dP \\ &= \int_A E \left(|W(t)|^2 + |W(s)|^2 - 2W(t) \cdot W(s) | \mathcal{F}_s \right) dP \\ &= \int_A E \left(|W(t) - W(s)|^2 | \mathcal{F}_s \right) dP = \int_A |W(t) - W(s)|^2 dP \\ &= P(A) \int_{\Omega} |W(t) - W(s)|^2 dP \\ &= P(A) \sum_{k=1}^{\infty} \lambda_k E \left((\psi_k(t) - \psi_k(s))^2 \right) \\ &= P(A) (t - s) \sum_{k=1}^{\infty} \lambda_k = P(A) (t - s) \operatorname{tr}(Q). \end{aligned}$$

Therefore,

$$\int_A E \left(|W(t)|^2 - t \operatorname{tr}(Q) | \mathcal{F}_s \right) - \left(|W(s)|^2 - s \operatorname{tr}(Q) \right) dP = 0$$

and since $A \in \mathcal{F}_s$ is arbitrary, this shows $\left\{ |W(t)|^2 - t \operatorname{tr}(Q) \right\}$ is a martingale. Hence the Doob Meyer decomposition for $|W(t)|^2$ is

$$|W(t)|^2 = Y(t) + t \operatorname{tr}(Q)$$

where $Y(t)$ is a martingale.

There is a generalization of Levy's theorem to Hilbert space valued Wiener processes.

Theorem 52.6.1 *Let $\{W(t)\} \in \mathcal{M}_T^2(H)$, $E(W(t)) = 0$, where H is a real separable Hilbert space. Then for Q a nonnegative symmetric trace class operator, $\{W(t)\}$ is a Q Wiener process if and only if both $\{W(t)\}$ and $\left\{ (W(t), h)^2 - t(Qh, h) \right\}$ are martingales for every $h \in H$.*

Proof: First suppose $\{W(t)\}$ is a Q Wiener process. Then defining the filtration to be

$$\mathcal{F}_t \equiv \sigma(W(s) - W(u) : u \leq s \leq t),$$

it follows from Lemma 52.4.2 that $\{W(t)\}$ is a martingale. Consider

$$\left\{ (W(t), h)^2 - t(Qh, h) \right\}.$$

Let $A \in \mathcal{F}_s$ where $s \leq t$. Then using the fact $\{W(t)\}$ is a martingale,

$$\begin{aligned} & \int_A E\left((W(t) - W(s), h)^2 | \mathcal{F}_s\right) dP \\ &= \int_A E\left((W(t), h)^2 + (W(s), h)^2 - 2(W(t), h)(W(s), h) | \mathcal{F}_s\right) dP \\ &= \int_A E\left((W(t), h)^2 | \mathcal{F}_s\right) + (W(s), h)^2 - E(2(W(t), h)(W(s), h) | \mathcal{F}_s) dP \\ &= \int_A E\left((W(t), h)^2 | \mathcal{F}_s\right) dP + \int_A (W(s), h)^2 dP \\ &\quad - \int_A (W(s), h) E(2(W(t), h) | \mathcal{F}_s) dP \\ &= \int_A E\left((W(t), h)^2 | \mathcal{F}_s\right) dP - \int_A (W(s), h)^2 dP. \end{aligned}$$

Also since $\mathcal{X}_A$ is independent of $(W(t) - W(s), h)^2$ as in the proof of Lemma 52.4.2, and $\{W(t)\}$ is a Q Wiener process,

$$\begin{aligned} & \int_A E\left((W(t) - W(s), h)^2 | \mathcal{F}_s\right) dP \\ &= \int_A (W(t) - W(s), h)^2 dP \\ &= P(A) \int_{\Omega} (W(t) - W(s), h)^2 dP \\ &= P(A)(t - s)(Qh, h). \end{aligned}$$

Thus, this has shown that for all $A \in \mathcal{F}_s$,

$$\begin{aligned} & \int_A E\left((W(t), h)^2 | \mathcal{F}_s\right) dP - \int_A (W(s), h)^2 dP \\ &= P(A)(t - s)(Qh, h) = \int_A (t - s)(Qh, h) dP \end{aligned}$$

and since $A \in \mathcal{F}_s$ is arbitrary, this proves

$$E\left((W(t), h)^2 - t(Qh, h) | \mathcal{F}_s\right) = (W(s), h)^2 - s(Qh, h)$$

This proves one half of the theorem.

Next suppose both $\{W(t)\}$ and $\{(W(t), h)^2 - t(Qh, h)\}$ are martingales for any $h \in H$. It follows that both $\{(W(t), h)\}$ and $\{(W(t), h)^2 - t(Qh, h)\}$ are martingales also. Therefore, by Levy's theorem, Theorem 51.7.5, $\{(W(t), h)\}$ is a Wiener process with the property that its variance at t equals $(Qh, h)t$ instead of t . Thus the time increments are normal and independent. I need to verify that $\{W(t)\}$ is a Q Wiener process. One of the things which needs to be shown is that

$$E((W(t) - W(s), h_1)(W(t) - W(s), h_2)) = (Qh_1, h_2)(t - s). \quad (52.6.31)$$

I have just shown

$$E((W(t) - W(s), h)^2) = (t - s)(Qh, h) \quad (52.6.32)$$

which follows from Levy's theorem which concludes $\{(W(t), h)\}$ is a Wiener process. Therefore,

$$\begin{aligned} & E((W(t) - W(s), h_1 + h_2)(W(t) - W(s), h_2 + h_1)) \\ &= (Q(h_1 + h_2), (h_1 + h_2))(t - s) \end{aligned}$$

Now using 52.6.32, it follows from this that

$$E((W(t) - W(s), h_1)(W(t) - W(s), h_2)) = (Qh_1, h_2)(t - s)$$

which shows 52.6.31. This completes the proof.

Stochastic Integration

53.1 Integrals Of Elementary Processes

Stochastic integration starts with a Q Wiener process having values in a separable Hilbert space, U . Thus it satisfies the following definition.

Definition 53.1.1 *Let $W(t)$ be a stochastic process with values in U , a real separable Hilbert space which has the properties that $t \rightarrow W(t, \omega)$ is continuous. Whenever $t_1 < t_2 < \dots < t_m$, the increments $\{W(t_i) - W(t_{i-1})\}$ are independent, $W(0) = 0$, and whenever $s < t$,*

$$\mathcal{L}(W(t) - W(s)) = N(0, (t - s)Q)$$

which means that whenever $h \in H$,

$$\mathcal{L}((h, W(t) - W(s))) = N(0, (t - s)(Qh, h))$$

Also

$$E((h_1, W(t) - W(s))(h_2, W(t) - W(s))) = (Qh_1, h_2)(t - s).$$

Here Q is a nonnegative trace class operator. Recall this means

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$$

where $\{e_i\}$ is a complete orthonormal basis, $\lambda_i \geq 0$, and

$$\sum_{i=1}^{\infty} \lambda_i < \infty$$

Such a stochastic process is called a Q Wiener process.

Recall the definition of $\mathcal{L}_2(U, H) \equiv \mathcal{L}_2$ the space of Hilbert Schmidt operators. $\Psi \in \mathcal{L}_2(U, H)$ means Ψ has the property that for some (equivalently all) orthonormal basis of U $\{e_k\}$, it follows

$$\sum_{k=1}^{\infty} \|\Psi(e_k)\|^2 < \infty$$

and the inner product for two of these, Ψ, Φ is given by

$$(\Psi, \Phi)_{\mathcal{L}_2} \equiv \sum_k (\Psi(e_k), \Phi(e_k))$$

Then for such a Hilbert Schmidt operator, the norm in $\mathcal{L}_2$ is given by

$$\left(\sum_{k=1}^{\infty} \|\Psi(e_k)\|^2 \right)^{1/2} \equiv \|\Psi\|_{\mathcal{L}_2}.$$

Note this is the same as

$$\left(\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\Psi(e_k), f_j)^2 \right)^{1/2} \quad (53.1.1)$$

where $\{f_j\}$ is an orthonormal basis for H . This is the analog of the Frobenius norm for matrices obtained as

$$\text{trace}(MM^*)^{1/2} = \left(\sum_i (MM^*)_{ii} \right)^{1/2} = \left(\sum_{i,j} M_{ij}^2 \right)^{1/2}$$

Also 53.1.1 shows right away that if $\Psi \in \mathcal{L}_2(U, H)$, then

$$\begin{aligned} \|\Psi\|_{\mathcal{L}_2(U, H)}^2 &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\Psi e_k, f_j)_H^2 \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (e_k, \Psi^* f_j)_U^2 = \|\Psi^*\|_{\mathcal{L}_2(H, U)}^2 \end{aligned}$$

The filtration will continue to be denoted by $\mathcal{F}_t$. It will be defined as the following normal filtration in which

$$\overline{\sigma(W(s) - W(r) : 0 \leq r < s \leq r)}$$

is the completion of $\sigma(W(s) - W(r) : 0 \leq r < s \leq r)$.

$$\mathcal{F}_t \equiv \cap_{r>t} \overline{\sigma(W(s) - W(r) : 0 \leq r < s \leq r)}. \quad (53.1.2)$$

and $\sigma(W(s) - W(r) : 0 \leq r < s \leq r)$ denotes the σ algebra of all sets of the form

$$(W(s) - W(r))^{-1} \text{ (Borel)}$$

where $0 \leq r < s \leq t$.

Definition 53.1.2 Let $\Phi(t) \in \mathcal{L}(U, H)$ be constant on each interval, $(t_m, t_{m+1}]$ determined by a partition of $[0, T]$, $0 = t_0 < t_1 \cdots < t_n = T$. Then $\Phi(t)$ is said to be elementary if also $\Phi(t_m)$ is $\mathcal{F}_{t_m}$ measurable and $\Phi(t_m)$ takes on only finitely many values. What does this mean? It means that if O is an open (Borel) set in the topological space $\mathcal{L}(U, H)$, $\Phi(t_m)^{-1}(O) \in \mathcal{F}_{t_m}$. Thus an elementary function is of the form

$$\Phi(t) = \sum_{k=0}^{n-1} \Phi(t_k) \chi_{(t_k, t_{k+1}]}(t).$$

Then for Φ elementary, the stochastic integral is defined by

$$\int_0^t \Phi(s) dW(s) \equiv \sum_{k=0}^{n-1} \Phi(t_k) (W(t \wedge t_{k+1}) - W(t \wedge t_k)).$$

It is also denoted by $\Phi \cdot W(t)$.

The above definition is the same as saying that for $t \in (t_m, t_{m+1}]$,

$$\begin{aligned} \int_0^t \Phi(s) dW(s) &= \sum_{k=0}^{m-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) \\ &\quad + \Phi(t_m) (W(t) - W(t_m)). \end{aligned} \quad (53.1.3)$$

The following lemma will be useful.

Lemma 53.1.3 Let $f, g \in L^2(\Omega; H)$ and suppose g is $\mathcal{G}$ measurable and f is $\mathcal{F}$ measurable where $\mathcal{F} \supseteq \mathcal{G}$. Then

$$E((f, g)_H | \mathcal{G}) = (E(f | \mathcal{G}), g)_H \text{ a.e.}$$

Similarly if Φ is $\mathcal{G}$ measurable as a map into $\mathcal{L}(U, H)$ with

$$\int_{\Omega} \|\Phi\|^2 dP < \infty$$

and f is $\mathcal{F}$ measurable as a map into H such that $f \in L^2(\Omega; H)$

$$E(\Phi f | \mathcal{G}) = \Phi E(f | \mathcal{G}).$$

Proof: Let $A \in \mathcal{G}$. Let $\{g_n\}$ be a sequence of simple functions, measurable with respect to $\mathcal{G}$,

$$g_n(\omega) \equiv \sum_{k=1}^{m_n} a_k^n \chi_{E_k^n}(\omega)$$

which converges in $L^2(\Omega; H)$ and pointwise to g . Then

$$\int_A (E(f | \mathcal{G}), g)_H dP = \lim_{n \rightarrow \infty} \int_A (E(f | \mathcal{G}), g_n)_H dP$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \int_A \sum_{k=1}^{m_n} (E(f|\mathcal{G}), a_k^n \mathcal{X}_{E_k^n})_H dP = \lim_{n \rightarrow \infty} \int_A \sum_{k=1}^{m_n} E((f, a_k^n)_H | \mathcal{G}) \mathcal{X}_{E_k^n} dP \\
&= \lim_{n \rightarrow \infty} \int_A \sum_{k=1}^{m_n} E\left((f, a_k^n \mathcal{X}_{E_k^n})_H | \mathcal{G}\right) dP = \lim_{n \rightarrow \infty} \int_A E\left(\left(f, \sum_{k=1}^{m_n} a_k^n \mathcal{X}_{E_k^n}\right)_H | \mathcal{G}\right) dP \\
&= \lim_{n \rightarrow \infty} \int_A E((f, g_n)_H | \mathcal{G}) dP = \lim_{n \rightarrow \infty} \int_A (f, g_n)_H dP = \int_A (f, g)_H dP
\end{aligned}$$

which shows

$$(E(f|\mathcal{G}), g)_H = E((f, g)_H | \mathcal{G})$$

as claimed.

Consider the other claim. Let

$$\Phi_n(\omega) = \sum_{k=1}^{m_n} \Phi_k^n \mathcal{X}_{E_k^n}(\omega), \quad E_k^n \in \mathcal{G}$$

where $\Phi_k^n \in \mathcal{L}(U, H)$ be such that Φ_n converges to Φ pointwise in $\mathcal{L}(U, H)$ and also

$$\int_{\Omega} \|\Phi_n - \Phi\|^2 dP \rightarrow 0.$$

Then letting $A \in \mathcal{G}$ and using Corollary 23.2.6 as needed,

$$\begin{aligned}
\int_A \Phi E(f|\mathcal{G}) dP &= \lim_{n \rightarrow \infty} \int_A \Phi_n E(f|\mathcal{G}) dP = \lim_{n \rightarrow \infty} \int_A \sum_{k=1}^{m_n} \Phi_k^n E(f|\mathcal{G}) \mathcal{X}_{E_k^n} dP \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^{m_n} \Phi_k^n \int_A E(f|\mathcal{G}) \mathcal{X}_{E_k^n} dP = \lim_{n \rightarrow \infty} \sum_{k=1}^{m_n} \Phi_k^n \int_A E(\mathcal{X}_{E_k^n} f | \mathcal{G}) dP \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^{m_n} \Phi_k^n \int_A \mathcal{X}_{E_k^n} f dP = \lim_{n \rightarrow \infty} \int_A \sum_{k=1}^{m_n} \Phi_k^n \mathcal{X}_{E_k^n} f dP \\
&= \lim_{n \rightarrow \infty} \int_A \Phi_n f dP = \lim_{n \rightarrow \infty} \int_A \Phi f dP \equiv \int_A E(\Phi f | \mathcal{G}) dP
\end{aligned}$$

Since $A \in \mathcal{G}$ is arbitrary, this proves the lemma.

With this lemma, here is a major theorem.

Theorem 53.1.4 *Let $\Phi(t)$ be an elementary process as defined in Definition 53.1.2 and let $W(t)$ be a Q Wiener process. Then $\Phi \cdot W(t)$ is a continuous square integrable H valued martingale with respect to the σ algebras of 53.1.2 on $[0, T]$ and*

$$E\left(|\Phi \cdot W(t)|_H^2\right) = \int_0^t E\left(\sum_{i=1}^{\infty} \lambda_i |\Phi(s) e_i|_H^2\right) ds$$

where

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i.$$

Proof: Let $\Phi(t)$ be an elementary process as described in Definition 53.1.2 so the stochastic integral $\Phi \cdot W(t)$ is of the form

$$\sum_{k=0}^{n-1} \Phi(t_k) (W(t \wedge t_{k+1}) - W(t \wedge t_k))$$

Why is $\Phi \cdot W(t)$ a continuous martingale? It is clearly continuous because it is the sum of continuous functions. It is also adapted to $\mathcal{F}_t$ from the definition because $t \wedge t_k, t \wedge t_{k+1}$ are both no larger than t . It remains to verify it is a martingale.

Suppose then that $t \in (t_m, t_{m+1}]$ and consider $s < t$. By Lemma 52.4.2 $W(t)$ is a martingale with respect to the σ algebras of 53.1.2. This follows from the estimate

$$\begin{aligned} E \left(\|W(t) - W(s)\|_U^2 \right) &= \sum_{k=1}^{\infty} E \left((W(t) - W(s), e_i)^2 \right) \\ &= \sum_{k=1}^{\infty} (t - s) (Qe_i, e_i) \leq (t - s) \sum_{k=1}^{\infty} \lambda_k. \end{aligned}$$

By adding in both s and t into $\{t_k\}$ and keeping Φ the same at the possibly new points, $\Phi \cdot W(t)$ and $\Phi \cdot W(s)$ are both unchanged. Therefore, there is no loss of generality in assuming $s = t_l, t = t_m$ where $m > l$. Thus

$$\begin{aligned} E(\Phi \cdot W(t) | \mathcal{F}_s) &= E(\Phi \cdot W(t_m) | \mathcal{F}_{t_l}) \\ &= E \left(\sum_{k=0}^{m-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_l} \right) \\ &= E \left(\sum_{k=0}^{l-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) + \sum_{k=l}^{m-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_l} \right) \end{aligned}$$

Consider a typical term of the second sum. Since $W(t)$ is a martingale,

$$\begin{aligned} E(\Phi(t_k) (W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_l}) &= E(E(\Phi(t_k) (W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_k}) | \mathcal{F}_{t_l}) \\ &= E(\Phi(t_k) E((W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_k}) | \mathcal{F}_{t_l}) = E(\Phi(t_k) \cdot 0 | \mathcal{F}_{t_l}) = 0. \end{aligned}$$

Thus this reduces to

$$\begin{aligned} E \left(\sum_{k=0}^{l-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) | \mathcal{F}_{t_l} \right) &= \sum_{k=0}^{l-1} \Phi(t_k) (W(t_{k+1}) - W(t_k)) \\ &= \Phi \cdot W(t_l) = \Phi \cdot W(s) \end{aligned}$$

This proves the claim about $\Phi \cdot W(t)$ being a martingale.

Now consider the estimate. To save space, let

$$\Delta_k W(t) \equiv W(t \wedge t_{k+1}) - W(t \wedge t_k)$$

Then

$$E \left(|\Phi \cdot W(t)|_H^2 \right) = E \left(\sum_{k=0}^{n-1} \sum_{j=0}^{n-1} (\Phi(t_k) (\Delta_k W(t)), \Phi(t_j) (\Delta_j W(t))) \right)$$

Consider the expectation of a mixed term for which $j < k$.

$$E((\Phi(t_k) \Delta_k W(t), \Phi(t_j) \Delta_j W(t))) \quad (53.1.4)$$

Recall that from the definition of elementary functions, each of $\Phi(t_j)$ and $\Phi(t_k)$ take only finitely many values in $\mathcal{L}(U, H)$ so there is no question that

$$(\Phi(t_k) \Delta_k W(t), \Phi(t_j) \Delta_j W(t)) \in L^1(\Omega).$$

Thus 53.1.4 above equals

$$E(E((\Phi(t_k) (\Delta_k W(t)), \Phi(t_j) (\Delta_j W(t))) | \mathcal{F}_{t_{j+1}}))$$

By Lemma 53.1.3 this equals

$$E(E(\Phi(t_k) \Delta_k W(t) | \mathcal{F}_{t_{j+1}}), \Phi(t_j) \Delta_j W(t))$$

Now consider

$$E(\Phi(t_k) \Delta_k W(t) | \mathcal{F}_{t_{j+1}})$$

By properties of conditional expectation this is

$$\begin{aligned} & E(E(\Phi(t_k) \Delta_k W(t) | \mathcal{F}_{t_k}) | \mathcal{F}_{j+1}) \\ &= E(\Phi(t_k) E(\Delta_k W(t) | \mathcal{F}_{t_k}) | \mathcal{F}_{j+1}) \\ &= E(\Phi(t_k) 0 | \mathcal{F}_{j+1}) = 0 \end{aligned}$$

because $W(t)$ is a martingale by Lemma 52.4.2. Therefore the mixed term in 53.1.4 vanishes. It follows

$$E(|\Phi \cdot W(t)|_H^2) = E \left(\sum_{k=0}^{n-1} |\Phi(t_k) (\Delta_k W(t))|_H^2 \right) \quad (53.1.5)$$

Let $\{f_k\}$ be a complete orthonormal basis for H and let $\{e_k\}$ be the complete orthonormal basis for U such that

$$Q = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k \quad (53.1.6)$$

Then 53.1.5 reduces to

$$E \left(\sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (\Phi(t_k) (\Delta_k W(t)), f_j)^2 \right)$$

$$= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} E \left(((\Delta_k W(t)), \Phi(t_k)^* f_j)^2 \right) \quad (53.1.7)$$

Claim:

$$E \left(((\Delta_k W(t)), \Phi(t_k)^* f_j)^2 \right) = (t \wedge t_{k+1} - t \wedge t_k) E \left((Q\Phi(t_k)^* f_j, \Phi(t_k)^* f_j) \right)$$

Proof of claim: First note that there is nothing to prove if $t < t_k$ since both sides are 0 in this case. This would be a lot easier if $\Phi(t_k)$ were a constant, but unfortunately it is not. Consider the simpler formula in which $\Phi(t_k)$ is replaced with $\Phi\mathcal{X}_A$ where $A \in \mathcal{F}_{t \wedge t_k}$, $\Phi \in \mathcal{L}(U, H)$. Then

$$E \left((\Delta_k W(t), \Phi^* \mathcal{X}_A f_j)_U^2 \right) = E \left(\mathcal{X}_A (\Delta_k W(t), \Phi^* f_j)_U^2 \right) \quad (53.1.8)$$

Now

$$A \in \cap_{r > t \wedge t_k} \overline{\sigma(W(s) - W(r) : 0 \leq r < s \leq r)}$$

and so by Lemma 52.4.2, 53.1.8 equals

$$\begin{aligned} & P(A) \int_{\Omega} (\Delta_k W(t), \Phi^* f_j)_U^2 dP \\ &= P(A) (t_{k+1} \wedge t - t_k \wedge t) (Q\Phi^* f_j, \Phi^* f_j)_U \end{aligned}$$

By assumption $\Phi(t_k)$ is a finite sum of functions of this form. Say

$$\Phi(t_k)(\omega) = \sum_{j=1}^{m_n} \Phi_j \mathcal{X}_{A_j^n}(\omega)$$

Then the above computation implies

$$\begin{aligned} & \int_{\Omega} (\Phi(t_k) \Delta_k W(t), f_j)_H^2 dP \\ &= \sum_{j=1}^{m_n} \int_{\Omega} \mathcal{X}_{A_j^n}(\omega) (\Delta_k W(t), \Phi_j^* f_j)_U^2 dP \\ &= \sum_{j=1}^{m_n} P(A_j^n) \int_{\Omega} (\Delta_k W(t), \Phi_j^* f_j)_U^2 dP \\ &= (t_{k+1} \wedge t - t_k \wedge t) \sum_{j=1}^{m_n} P(A_j^n) (Q\Phi_j^* f_j, \Phi_j^* f_j)_U \\ &= (t_{k+1} \wedge t - t_k \wedge t) \int_{\Omega} (Q\Phi(t_k)^* f_j, \Phi(t_k)^* f_j)_U dP \end{aligned}$$

This proves the claim.

Now returning to 53.1.7 and 53.1.5 and using the claim,

$$\begin{aligned}
 & E \left(|\Phi \cdot W(t)|_H^2 \right) \\
 &= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} E \left(((\Delta_k W(t)), \Phi(t_k)^* f_j)^2 \right) \\
 &= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (t_{k+1} \wedge t - t_k \wedge t) E \left((Q\Phi(t_k)^* f_j, \Phi(t_k)^* f_j)_U \right) \\
 &= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (t_{k+1} \wedge t - t_k \wedge t) E \left(\left(Q^{1/2} \Phi(t_k)^* f_j, Q^{1/2} \Phi(t_k)^* f_j \right)_U \right) \\
 &= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (t_{k+1} \wedge t - t_k \wedge t) E \left(\left| Q^{1/2} \Phi(t_k)^* f_j \right|_U^2 \right) \\
 &= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (t_{k+1} \wedge t - t_k \wedge t) E \left(\sum_{i=1}^{\infty} \left(Q^{1/2} \Phi(t_k)^* f_j, e_i \right)_U^2 \right) \\
 &= \sum_{k=0}^{n-1} \sum_{j=1}^{\infty} (t_{k+1} \wedge t - t_k \wedge t) E \left(\sum_{i=1}^{\infty} \left(\Phi(t_k)^* f_j, Q^{1/2} e_i \right)_U^2 \right) \\
 &= E \left(\sum_{k=0}^{n-1} \sum_{i=1}^{\infty} \lambda_i \sum_{j=1}^{\infty} (f_j, \Phi(t_k) e_i)_H^2 (t_{k+1} \wedge t - t_k \wedge t) \right) \\
 &= \sum_{k=0}^{n-1} E \left(\sum_{i=1}^{\infty} \lambda_i |\Phi(t_k) e_i|_H^2 \right) (t_{k+1} \wedge t - t_k \wedge t) \\
 &= \int_0^t E \left(\sum_{i=1}^{\infty} \lambda_i |\Phi(s) e_i|_H^2 \right) ds
 \end{aligned}$$

This proves the theorem.

Consider the expression

$$\sum_{i=1}^{\infty} \lambda_i |\Phi(s) e_i|_H^2 = \sum_{i=1}^{\infty} \left| \Phi(s) \sqrt{\lambda_i} e_i \right|_H^2 \quad (53.1.9)$$

which occurs in the conclusion of the above theorem. It is convenient to define a new Hilbert space called $U_0 \subseteq U$ such that this expression equals

$$\|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2$$

From 53.1.9 this will be so if and only if

$$\left\{ \sqrt{\lambda_i} e_i : \lambda_i \neq 0 \right\}$$

is an orthonormal basis for U_0 . This is the motivation for the following definition.

Definition 53.1.5 *Let U be a separable Hilbert space and let Q be a positive nuclear operator. That is*

$$Q = \sum_{k=1}^{\infty} \lambda_k e_k \otimes e_k$$

where each $\lambda_k \geq 0$, $\sum_{k=1}^{\infty} \lambda_k < \infty$, and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for U . Assume without loss of generality the λ_k form a decreasing sequence. (See Theorem 23.3.9.) Let

$$L = \sup \{k : \lambda_k > 0\}$$

Thus $L \leq \infty$. Define

$$\rho(x)^2 \equiv \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2$$

and let W_0 consist of all $x \in U$ such that

$$\rho(x) < \infty$$

Then U_0 will be defined as

$$U_0 \equiv \left\{ x \in W_0 : x = \sum_{k=1}^L (x, e_k)_U e_k \right\}, \quad \|x\|_{U_0} \equiv \rho(x)$$

Lemma 53.1.6 *In the above definition, U_0 is a Hilbert space properly contained in U and*

$$C \|x\|_{U_0} \geq \|x\|_U$$

The inner product in U_0 is

$$(x, y)_{U_0} = \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U \left(y, \frac{e_k}{\sqrt{\lambda_k}} \right)_U \quad (53.1.10)$$

and an orthonormal basis for U_0 is $\{\sqrt{\lambda_i} e_i\}_{i=1}^L \equiv \{g_i\}_{i=1}^L$. Also U_0 is dense in U if and only if $L = \infty$.

Proof: It is clear U_0 is contained in U from the definition. Why is ρ a norm on U_0 ? It is clear it comes from the bilinear

$$(x, y) \equiv \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U \left(y, \frac{e_k}{\sqrt{\lambda_k}} \right)_U$$

Therefore, the triangle inequality holds as well as $\|\alpha x\|_{U_0} = |\alpha| \|x\|_{U_0}$. If $\rho(x) \equiv \|x\|_{U_0} = 0$, why does $x = 0$? Since $x \in U_0$,

$$x = \sum_{k=1}^L (x, e_k)_U e_k = \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U \sqrt{\lambda_k} e_k$$

and so

$$\begin{aligned}
 \|x\|_U &\leq \sum_{k=1}^L \left| \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U \right| \sqrt{\lambda_k} \\
 &\leq \left(\sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 \right)^{1/2} \left(\sum_{k=1}^L \lambda_k \right)^{1/2} \\
 &= \rho(x) \left(\sum_{k=1}^L \lambda_k \right)^{1/2} = 0 \left(\sum_{k=1}^L \lambda_k \right)^{1/2} = 0
 \end{aligned}$$

Thus U_0 is indeed an inner product space. Why is it complete?

Let $\{x_n\}$ be a Cauchy sequence in U_0 . From the definition of ρ ,

$$\begin{aligned}
 \rho(x_n - x_m) &\geq \frac{1}{\sqrt{\lambda_1}} \left(\sum_{k=1}^L (x_n - x_m, e_k)_U^2 \right)^{1/2} \\
 &= \frac{1}{\sqrt{\lambda_1}} \|x_n - x_m\|_U
 \end{aligned} \tag{53.1.11}$$

which shows $\{x_n\}$ is Cauchy in U also. Let $x_n \rightarrow x$ in U . Let $\varepsilon > 0$ be given. Then there exists n_ε such that if $m, n > n_\varepsilon$, then $\rho(x_n - x_m) < \varepsilon$. Now pick $m \geq n_\varepsilon$. By Fatou's lemma

$$\sum_{k=1}^L \left(x - x_m, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 \leq \liminf_{n \rightarrow \infty} \sum_{k=1}^L \left(x_n - x_m, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 \leq \varepsilon^2$$

Thus

$$\begin{aligned}
 \left(\sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 \right)^{1/2} &\leq \left(\sum_{k=1}^L \left(x - x_m, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 \right)^{1/2} \\
 + \left(\sum_{k=1}^L \left(x_m, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 \right)^{1/2} &< \infty
 \end{aligned}$$

and so $x \in W_0$ and

$$\rho(x - x_n) \rightarrow 0.$$

Why is $x \in U_0$? This is so because from 53.1.11, $\{x_n\}$ is a Cauchy sequence in the closed subspace of U , $\overline{\text{span} \left(\{e_k\}_{k=1}^L \right)}$ and so it converges in U to some $x' \in \overline{\text{span} \left(\{e_k\}_{k=1}^L \right)}$ which must equal x since x was defined as the point of U to which $\{x_n\}$ converges. Hence $x \in U_0$ as claimed. The formula for $\|\cdot\|_{U_0}$ shows that on U_0 ,

$$\sqrt{\lambda_1} \|x\|_{U_0} \geq \|x\|_U.$$

Finally

$$\left(\sqrt{\lambda_i}e_i, \sqrt{\lambda_j}e_j\right)_{U_0} \equiv \sum_{k=1}^L \left(\sqrt{\lambda_i}e_i, \frac{e_k}{\sqrt{\lambda_k}}\right)_U \left(\sqrt{\lambda_j}e_j, \frac{e_k}{\sqrt{\lambda_k}}\right)_U = \delta_{ij}$$

and so $\{\sqrt{\lambda_i}e_i\}_{i=1}^L$ is an orthonormal set of vectors in U_0 . Also, every $x \in U_0$ is by construction given by

$$x = \sum_{k=1}^L (x, e_k) e_k = \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}}\right)_U \sqrt{\lambda_k} e_k$$

showing $\{\sqrt{\lambda_i}e_i\}_{i=1}^L$ is a complete orthonormal set of vectors.

Finally, if $L < \infty$, then U_0 cannot be dense in U because you could pick $m > L$ and then for all $x \in U_0$,

$$\|e_m - x\|_U^2 \geq 1$$

If $L = \infty$, then if $\varepsilon > 0$ is given

$$\left\|x - \sum_{k=1}^m (x, e_k) e_k\right\|_U < \varepsilon$$

whenever m is large enough. However, each e_k is in U_0 and so U_0 is indeed dense in U . This proves the lemma.

Note this last part about density adapts easily if U is a finite dimensional Hilbert space. In this case, just replace ∞ with n where n is the dimension.

Now with the above lemma and the description of the new Hilbert space U_0 , Theorem 53.1.4 can be restated in the following form.

Corollary 53.1.7 *Let $\Phi(t)$ be an elementary process as defined in Definition 53.1.2. Then $\Phi \cdot W$ is a continuous square integrable H valued martingale with respect to the σ algebras of 53.1.2 on $[0, T]$ and*

$$E\left(|\Phi \cdot W(t)|_H^2\right) = \int_0^t E\left(\|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2\right) ds \quad (53.1.12)$$

where

$$Q = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$$

is the covariance of $W(t)$. Here U_0 is a Hilbert space contained in U for which the nonzero elements of $\{\sqrt{\lambda_i}e_i\}_{i=1}^{\infty}$ is an orthonormal basis for U_0 . This Hilbert space U_0 equals the Hilbert space $Q^{1/2}U$ where this Hilbert space is discussed in Theorem 15.2.2 for $L = Q^{1/2}$.

Proof: It only remains to verify the claim about $U_0 = Q^{1/2}U$. Suppose then that $x \in U_0$ so that

$$x = \sum_{k=1}^L (x, e_k)_U e_k = \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U \sqrt{\lambda_k} e_k$$

Is $x = Q^{1/2}y$ for some $y \in U$?

$$Q^{1/2}y = \sum_{k=1}^L \sqrt{\lambda_k} e_k \otimes e_k(y) = \sum_{k=1}^L (e_k, y) \sqrt{\lambda_k} e_k$$

and so if y is such that $(\sqrt{\lambda_k} e_k, y) = (x, e_k)$, then $Q^{1/2}y = x$ as desired. Let

$$y = \sum_{k=1}^L \frac{1}{\sqrt{\lambda_k}} (x, e_k)_U e_k$$

This is in U because by definition, since $x \in U_0$,

$$\sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 < \infty.$$

Next suppose $x \in Q^{1/2}U$ so for some $y \in U$,

$$x = \sum_{k=1}^L \sqrt{\lambda_k} e_k \otimes e_k(y) = \sum_{k=1}^L (e_k, y)_U \sqrt{\lambda_k} e_k$$

and so it follows x is of the form $\sum_{k=1}^L a_k e_k$. Since $\{e_k\}$ is an orthonormal basis,

$$x = \sum_{k=1}^L (x, e_k)_U e_k, \quad (x, e_k)_U = (e_k, y)_U \sqrt{\lambda_k}$$

Also

$$\rho(x)^2 \equiv \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 = \sum_{k=1}^L (e_k, y)_U^2 < \infty$$

so $x \in U_0$.

What about the norm? Is $\|x\|_{U_0} = \|x\|_{Q^{1/2}U}$? By definition and Theorem 15.2.2

$$\|x\|_{U_0}^2 \equiv \sum_{k=1}^L \left(x, \frac{e_k}{\sqrt{\lambda_k}} \right)_U^2 = \sum_{k=1}^L \lambda_k^{-1} (x, e_k)_U^2$$

and

$$\begin{aligned} \|x\|_{Q^{1/2}U}^2 &= (Q^{-1/2}x, Q^{-1/2}x)_U \\ &= \left(\sum_{k=1}^L \lambda_k^{-1/2} e_k (x, e_k), \sum_{k=1}^L \lambda_k^{-1/2} e_k (x, e_k) \right)_U \end{aligned}$$

$$= \sum_{k=1}^L \lambda_k^{-1} (x, e_k)_U^2$$

which is the same thing. This proves the corollary.

This shows you can think of U_0 as the space $Q^{1/2}U$ where this is a Hilbert space defined in Definition 15.2.1. Now this is an interesting observation because one can define such a space without assuming Q is trace class. See Theorem 15.10.4 on Page 462.

The formula 53.1.12 is called the Ito isometry.

Note

$$Q^{1/2} = \sum_{i=1}^{\infty} \sqrt{\lambda_i} e_i \otimes e_i, \quad Q^{1/2} e_k = \sqrt{\lambda_k} e_k.$$

For $\Psi \in \mathcal{L}_2(U_0, H)$, and using that $\{g_k\} = \{\sqrt{\lambda_k} e_k\}$ is an orthonormal basis for U_0 ,

$$\begin{aligned} \|\Psi\|_{\mathcal{L}_2(U_0, H)}^2 &\equiv \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\Psi g_k, f_j)_H^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (\Psi Q^{1/2} e_k, f_j)_H^2 \\ &= \left\| \Psi Q^{1/2} \right\|_{\mathcal{L}_2(U, H)}^2 = \left\| (Q^{1/2} \Psi)^* \right\|_{\mathcal{L}_2(H, U)}^2 \end{aligned} \quad (53.1.13)$$

Thus also,

$$\begin{aligned} &\left\| (Q^{1/2} \Psi)^* \right\|_{\mathcal{L}_2(H, U)}^2 \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (Q^{1/2} \Psi^* f_j, e_k)_U^2 = \sum_{j=1}^{\infty} (Q^{1/2} \Psi^* f_j, Q^{1/2} \Psi^* f_j)_U \\ &= \sum_{j=1}^{\infty} (\Psi Q \Psi^* f_j, f_j)_H = \sum_{j=1}^{\infty} (\Psi Q^{1/2} (\Psi Q^{1/2})^* f_j, f_j)_H \end{aligned}$$

and this last expression is defined to be $\text{trace}(\Psi Q \Psi^*)$ by analogy with the case of matrices.

Definition 53.1.8 Define $\mathcal{L}(U, H)_0$ to be the restrictions of $\Phi \in \mathcal{L}(U, H)$ to U_0 .

Lemma 53.1.9 $\mathcal{L}(U, H)_0$ is dense in $\mathcal{L}_2(U_0, H)$, the space of Hilbert Schmidt operators mapping U_0 to H . Also, letting

$$\rho_0(\Phi) \equiv \sup \{ \|\Phi(x)\|_H : x \in U_0, \|x\|_U \leq 1 \}$$

Then if $\rho_0(\Phi_n) \rightarrow 0$, it follows $\|\Phi_n\|_{\mathcal{L}_2^0} \rightarrow 0$.

Proof: First why is $\mathcal{L}(U, H)_0$ contained in $\mathcal{L}_2(U_0, H)$? Let $\Phi \in \mathcal{L}(U, H)_0$. From the above, an orthonormal basis for U_0 is $\{\sqrt{\lambda_k} e_k\}$ for $\lambda_k > 0$. Then

$$\sum_{k=1}^L \left| \Phi \sqrt{\lambda_k} e_k \right|_H^2 \leq \sum_{k=1}^L \lambda_k \|\Phi\|^2 < \infty$$

and so this shows $\Phi \in \mathcal{L}_2(U_0, H)$.

Why is $\mathcal{L}(U, H)_0$ dense in $\mathcal{L}_2(U_0, H)$? Consider $\Phi \in \mathcal{L}_2(U_0, H)$. Then by Theorem 15.7.12 $\Phi = \sum_{i,j} a_{ij} f_i \otimes \sqrt{\lambda_j} e_j$ where $\{f_j\}$ is an orthonormal basis in H and as explained above, $\{\sqrt{\lambda_i} e_i\}$ is an orthonormal basis in U_0 , the sum converging in $\mathcal{L}_2(U_0, H)$. Then replacing the infinite sum with a suitable finite sum, Φ is approximated in $\mathcal{L}_2(U_0, H)$ by something in $\mathcal{L}(U, H)_0$,

$$\sum_{i,j < N} a_{ij} f_i \otimes \sqrt{\lambda_j} e_j$$

Finally, suppose $\rho_0(\Phi_n) \rightarrow 0$. Then

$$\|\Phi_n\|_{\mathcal{L}_2^0}^2 \equiv \sum_{k=1}^L \left| \Phi_n \sqrt{\lambda_k} e_k \right|_H^2 \leq \rho_0(\Phi_n)^2 \sum_{k=1}^L \lambda_k$$

This proves the lemma.

Note that this last estimate doesn't work out unless Q is trace class. If you just have $U_0 = Q^{1/2}U$ and $\Phi \in \mathcal{L}(U, H)_0$, then letting an orthonormal basis for U be e_k so that an orthonormal basis for $Q^{1/2}U$ is $\{Q^{1/2}e_k\}$,

$$\begin{aligned} \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 &= \sum_{j=1}^{\infty} \left| \Phi Q^{1/2} e_k \right|_H^2 \leq \rho_0(\Phi)^2 \sum_{j=1}^{\infty} \left\| Q^{1/2} e_j \right\|_U^2 \\ &= \rho_0(\Phi)^2 \sum_{j=1}^{\infty} (Q e_j, e_j) \end{aligned}$$

and this last term is not finite unless Q is trace class.

Recall the definition of the strange symbol $\Phi \cdot W(t)$ given above in Definition 53.1.2 for elementary processes, those of the form

$$\Phi(t) = \sum_{k=0}^{n-1} \Phi(t_k) \mathcal{X}_{(t_k, t_{k+1}]}(t), \quad \Phi(t_k) \text{ } \mathcal{F}_{t_k} \text{ measurable}$$

It is also written as $\int_0^t \Phi(s) dW(s)$ and has only been defined for elementary Φ .

53.2 Integrals Of More General Processes

Recall the definition of what it means to be predictable in Definition 50.2.4 stated here for convenience.

Definition 53.2.1 Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, P)$ and denote by $\mathcal{P}_{\infty}$ the smallest σ algebra of sets of $[0, \infty) \times \Omega$ containing the sets

$$(s, t] \times F, F \in \mathcal{F}_s \quad \{0\} \times F, \quad F \in \mathcal{F}_0.$$

This is called the predictable σ algebra. and the sets in this σ algebra are called the predictable sets. Denote by $\mathcal{P}_T$ the intersection of $\mathcal{P}_\infty$ to $[0, T] \times \Omega$. A stochastic process X which maps either $[0, T] \times \Omega$ or $[0, \infty) \times \Omega$ to E , a separable real Banach space is called predictable if for every Borel set $A \in \mathcal{B}(E)$, it follows

$$X^{-1}(A) \equiv \{(t, \omega) : X(t, \omega) \in A\} \in \mathcal{P}_T \text{ or } \mathcal{P}_\infty.$$

This is a lot like product measure except the $\mathcal{F}$ is constantly changing. The main result is the next proposition which is like the theorem which says a product measurable set has measurable sections.

Then with this definition, here is an important proposition.

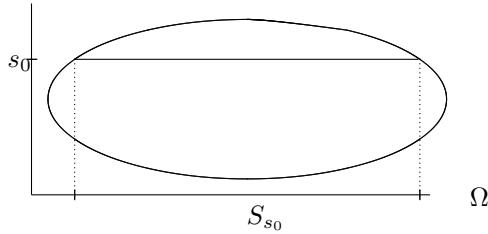
Proposition 53.2.2 *Let $\mathcal{F}_t$ be a filtration as above and let X be a predictable stochastic process. Then X is $\mathcal{F}_t$ adapted.*

Proof: Let $s_0 > 0$ and define

$$\mathcal{G}_{s_0} \equiv \{S \in \mathcal{P}_\infty : S_{s_0} \in \mathcal{F}_{s_0}\}$$

where

$$S_{s_0} \equiv \{\omega \in \Omega : (s_0, \omega) \in S\}.$$



It is clear $\mathcal{G}_{s_0}$ is a σ algebra. The next step is to show $\mathcal{G}_{s_0}$ contains the sets

$$(s, t] \times F, F \in \mathcal{F}_s \quad (53.2.14)$$

and

$$\{0\} \times F, F \in \mathcal{F}_0. \quad (53.2.15)$$

It is clear $\{0\} \times F$ is contained in $\mathcal{G}_{s_0}$ because $(\{0\} \times F)_{s_0} = \emptyset \in \mathcal{F}_{s_0}$. Similarly, if $s \geq s_0$ or if $s, t < s_0$ then $((s, t] \times F)_{s_0} = \emptyset \in \mathcal{F}_{s_0}$. The only case left is for $s < s_0$ and $t \geq s_0$. In this case, letting $A_s \in \mathcal{F}_s$, $((s, t] \times A_s)_{s_0} = A_s \in \mathcal{F}_s \subseteq \mathcal{F}_{s_0}$. Therefore, $\mathcal{G}_{s_0}$ contains all the sets of the form given in 53.2.14 and 53.2.15 and so since $\mathcal{P}_\infty$ is the smallest σ algebra containing these sets, it follows $\mathcal{P}_\infty = \mathcal{G}_{s_0}$. The case where $s_0 = 0$ is entirely similar but shorter.

Therefore, if X is predictable, letting $A \in \mathcal{B}(E)$, $X^{-1}(A) \in \mathcal{P}_\infty$ or $\mathcal{P}_T$ and so

$$(X^{-1}(A))_s \equiv \{\omega \in \Omega : X(s, \omega) \in A\} = X(s)^{-1}(A) \in \mathcal{F}_s$$

showing $X(t)$ is $\mathcal{F}_t$ adapted. This proves the proposition.

In the situation of the above definition, a predictable stochastic process will be called E predictable. Here is an interesting lemma, also like the sort of thing which occurs with product measure.

Lemma 53.2.3 *Let E be a separable Banach space and let $\Phi : [0, \infty) \times \Omega \rightarrow E$ be E predictable. Then the following iterated integrals make sense and are equal.*

$$\int_0^\infty \int_\Omega \phi(\Phi) dP dt = \int_\Omega \int_0^\infty \phi(\Phi) dt dP \quad (53.2.16)$$

where ϕ is any nonnegative continuous or Borel measurable function defined on E .

Proof: Let $B \in \mathcal{P}_\infty$ be such that

$$\int_0^\infty \int_\Omega \mathcal{X}_B dP dt = \int_\Omega \int_0^\infty \mathcal{X}_B dt dP \quad (53.2.17)$$

and both integrals make sense. Also let $\mathcal{K}$ denote sets of the form $(s, t] \times A_s$ where $A_s \in \mathcal{F}_s$. Thus $\mathcal{K}$ is a π system because $(s, t] \times A_s \cap (s_1, t_1] \times A_{s_1} = (s \vee s_1, t \wedge t_1] \times A_s \cap A_{s_1}$ and so $A_s \cap A_{s_1} \in \mathcal{F}_{s \vee s_1}$. The iterated integrals make sense for sets in $\mathcal{K}$ and are also equal. Also if $\mathcal{G}$ denotes those sets for which 53.2.17 holds, then $\mathcal{G} \supseteq \mathcal{K}$ and it is clear that $\mathcal{G}$ is closed under countable disjoint unions and also for complements. Therefore, by the Lemma on π systems, Lemma 9.11.3 on Page 275 it follows $\mathcal{G} \supseteq \sigma(\mathcal{K}) = \mathcal{P}_\infty$. If Φ is E predictable then $\phi(\Phi)$ is $\mathbb{R}$ predictable. Hence, approximating with simple functions, the iterated integrals in 53.2.16 make sense and are equal. This proves the lemma.

A similar lemma holds if you replace $[0, \infty)$ with $[0, T]$ in the above. Now here is an important observation.

Lemma 53.2.4 *Let $f : [0, T] \times \Omega \rightarrow [0, \infty)$ be real predictable. Then letting*

$$F(t, \omega) \equiv \int_0^t f(s, \omega) ds,$$

it follows F is also real predictable.

Proof: First suppose

$$f(t, \omega) = \mathcal{X}_{(p, q] \times A_p}(t, \omega) = \mathcal{X}_{(p, q]}(t) \mathcal{X}_{A_p}(\omega), \quad A_p \in \mathcal{F}_p$$

Is the assertion true in this case? Here $(p, q] \times A_p$ is one of the sets which generates $\mathcal{P}_T$. Doing the integral yields that in this case

$$F(t, \omega) = l(t) \mathcal{X}_{A_p}(\omega)$$

where l is piecewise linear and continuous, equal to 0 till p and then increasing to $q - p$ and remaining constant for $t > q$. Now let $\alpha \in [0, q - p]$ and consider

$$[F > \alpha] \quad (53.2.18)$$

This is of the form

$$(r, q] \times A_p$$

for some $r \in (p, q)$. This is a set which generates $\mathcal{P}_T$. If $\alpha \geq q - p$, 53.2.18 is $\emptyset$ also in $\mathcal{P}_T$. Let $\mathcal{K}$ consist of the sets of this sort which generate $\mathcal{P}_T$. These sets form a π system. Let $\mathcal{G}$ denote those sets E of $\mathcal{P}_T$ for which

$$(t, \omega) \rightarrow \int_0^t \mathcal{X}_E(s, \omega) ds \quad (53.2.19)$$

makes sense and is real predictable. Then it was shown above that $\mathcal{K} \subseteq \mathcal{G}$. It is also the case that $\mathcal{G}$ is closed under countable disjoint unions because if $\{E_i\}$ is a countable sequence of disjoint sets of $\mathcal{G}$ then letting $E = \cup_{i=1}^{\infty} E_i$

$$\mathcal{X}_E(s, \omega) = \sum_{i=1}^{\infty} \mathcal{X}_{E_i}(s, \omega)$$

and so by the monotone convergence theorem,

$$\int_0^t \mathcal{X}_E(s, \omega) ds = \sum_{i=1}^{\infty} \int_0^t \mathcal{X}_{E_i}(s, \omega) ds$$

and so the integral makes sense and the function of 53.2.19 is predictable. It is also easy to see that $\mathcal{G}$ is closed under complements. By the lemma on π systems, Lemma 9.11.3, it follows $\mathcal{G} = \mathcal{P}_T$.

In the general case, let f_n be an increasing sequence of $\mathbb{R}$ predictable simple functions converging pointwise to f . Thus from the above

$$(t, \omega) \rightarrow \int_0^t f_n(s, \omega) ds$$

makes sense and is predictable. By the monotone convergence theorem, the same must be true of

$$(t, \omega) \rightarrow \int_0^t f(s, \omega) ds$$

This proves the lemma.

Lemma 53.2.5 *The elementary functions are $\mathcal{L}_2(U_0, H)$ predictable.*

Proof: First of all, the elementary functions have values in $\mathcal{L}(U, H)$ and by Lemma 53.1.9 this means the restriction to U_0 is in $\mathcal{L}_2(U_0, H)$.

Consider a typical term in an elementary function $\mathcal{X}_{(s,t]} \Phi$ where Φ has values in $\mathcal{L}(U, H)$ and is measurable with respect to $\mathcal{F}_s$ and has finitely many values. Say

$$\mathcal{X}_{(s,t]} \Phi = \mathcal{X}_{(s,t]} \sum_{k=1}^m \Phi_k \mathcal{X}_{A_k}(\omega)$$

where each $A_k \in \mathcal{F}_s$. Then taking the restrictions of each Φ_k to U_0 , it follows from Lemma 53.1.9 the restricted simple function is of the form

$$\sum_{k=1}^m \Phi_k \mathcal{X}_{(s,t]} \mathcal{X}_{A_k}(\omega) = \sum_{k=1}^m \Phi_k \mathcal{X}_{(s,t] \times A_k}(t, \omega)$$

which is clearly predictable because the sets $(s, t] \times A_k$ are among those which generate $\mathcal{P}_T$. This proves the lemma.

The following proposition is like the theorem which says the simple functions are dense in L^p except that here it will be elementary functions and these are like step functions. It is this proposition which shows the need for Φ to be predictable in order to define the stochastic integral in terms of limits of integrals of elementary functions.

Proposition 53.2.6 *Let $\mathcal{F}_t$ be the filtration determined for a Q Wiener process as above. If Φ is a $\mathcal{L}_2(U_0, H) \equiv \mathcal{L}_2^0$ predictable process such that for some $p \geq 1$,*

$$\int_0^T \int_{\Omega} \|\Phi(t)\|_{\mathcal{L}_2^0}^p dP dt < \infty$$

then there exists a sequence $\{\Phi_n\}$ of elementary processes such that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \|\Phi(t) - \Phi_n(t)\|_{\mathcal{L}_2^0}^p dP dt = 0.$$

Proof: Let Φ be $\mathcal{L}_2(U_0, H)$ predictable. Then there exists a sequence of predictable simple functions, $\{\Phi_n\}$ converging to Φ pointwise such that

$$\|\Phi_n(t, \omega)\|_{\mathcal{L}_2(U_0, H)} < \|\Phi(t, \omega)\|_{\mathcal{L}_2(U_0, H)}$$

Then adjusting the finitely many values of Φ_n and using Lemma 53.1.9, the density of $\mathcal{L}(U, H)_0$ in $\mathcal{L}_2(U_0, H)$, it can also be assumed that the values of these simple functions lie in $\mathcal{L}(U, H)_0$.

Unfortunately, the simple functions are not elementary. The sets on which they are constant are not products of an interval, $(s, t]$ and a set $A \in \mathcal{F}_s$. I will argue that if $B \in \mathcal{P}_T$, then it differs from a finite disjoint union of sets of the form $(s, t] \times A_s$ where $A_s \in \mathcal{F}_s$ on a set of arbitrarily small measure.

Let $\mathcal{K}$ denote the set of all finite disjoint unions of sets of the form $(s, t] \times A_s$ as just described. If $(s, t] \times A_s$ and $(s_1, t_1] \times A_{s_1}$ are two of these sets having nonempty intersection, the intersection being

$$(s \vee s_1, t \wedge t_1] \times A_s \cap A_{s_1},$$

then $A_s \cap A_{s_1} \in \mathcal{F}_{s \vee s_1}$ and so such an intersection is one of the sets whose finite disjoint unions constitute $\mathcal{K}$. Thus it is routine to show that $\mathcal{K}$ is a π system, but it is also an algebra. To see it is an algebra, let $s < s_1 < t < t_1$ so there is possibly a nonempty intersection of two sets of the sort whose finite disjoint unions yield $\mathcal{K}$,

$$(s, t] \times A_s, (s_1, t_1] \times A_{s_1}$$

then

$$\begin{aligned} & (s_1, t_1] \times A_{s_1} \setminus (s, t] \times A_s \\ &= ((s_1, t] \times (A_{s_1} \setminus A_s)) \cup ((t, t_1] \times A_{s_1}) \in \mathcal{K}. \end{aligned}$$

By Lemma 9.9.2 on Page 264, $\mathcal{K}$ is an algebra.

Now letting Γ denote such a set of $\mathcal{K}$, let $\mathcal{G}$ denote those sets, $B \in \mathcal{P}_T$ such that for any $\varepsilon > 0$ there exists some Γ , such that

$$(m_1 \times P)((B \setminus \Gamma) \cup (\Gamma \setminus B)) < \varepsilon$$

It is clear $\mathcal{K} \subseteq \mathcal{G}$ because if $B \in \mathcal{K}$ then you could simply pick $\Gamma = B$.

Now suppose $\{B_k\}$ is a disjoint union of sets of $\mathcal{G}$ and pick Γ_k such that

$$(m_1 \times P)(B_k \setminus \Gamma_k) + (m_1 \times P)(\Gamma_k \setminus B_k) < \varepsilon/2^{k+1}$$

Then

$$\begin{aligned} & (m_1 \times P)((\cup_k B_k \setminus \cup_k \Gamma_k) \cup (\cup_k \Gamma_k \setminus \cup_k B_k)) \\ & < (m_1 \times P)(\cup_k (B_k \setminus \Gamma_k)) + (m_1 \times P)(\cup_k (\Gamma_k \setminus B_k)) < \varepsilon \end{aligned}$$

and so by taking a large enough finite union of the Γ_k , it follows there exists $\Gamma \in \mathcal{K}$ such that

$$(m_1 \times P)((\cup_k B_k \setminus \Gamma) \cup (\Gamma \setminus \cup_k B_k)) < \varepsilon.$$

Next suppose $B \in \mathcal{G}$. I need to show B^C is also in $\mathcal{G}$. But the condition states that

$$(m_1 \times P)((B \cap \Gamma^C) \cup (\Gamma \cap B^C)) < \varepsilon$$

which is unchanged if B is replaced with B^C due to the observation that $\mathcal{K}$ is an algebra which implies that for $\Gamma \in \mathcal{K}$ then $\Gamma^C \in \mathcal{K}$. Thus you replace Γ with Γ^C in the above and obtain the necessary result for B^C . Therefore, by the lemma on π systems, Lemma 9.11.3 on Page 275, it follows $\mathcal{G} \supseteq \sigma(\mathcal{K})$ but from the definition of $\mathcal{P}_T$, $\sigma(\mathcal{K}) = \mathcal{P}_T$. This proves that every term in the simple function approximating Φ ,

$$\Phi \mathcal{X}_B(t, \omega)$$

where $B \in \mathcal{P}_T$ can be approximated by an expression of the form

$$\Phi \sum_{k=1}^m \mathcal{X}_{(s_k, t_k]}(t) \mathcal{X}_{A_{s_k}}(\omega)$$

and the set on which these functions differ can be taken to have arbitrarily small measure. Choosing the exceptional set to be small enough for each of the finitely many values of Φ_n , the simple function, Φ_n can be approximated in

$$L^p(0, T; L^p(\Omega; \mathcal{L}_2(U_0, H)))$$

by an elementary function. This proves the proposition.

With the proposition, the definition of the stochastic integral can be extended.

Definition 53.2.7 Let Φ have values in $\mathcal{L}_2(U_0, H) \equiv \mathcal{L}_2^0$ and be $\mathcal{L}_2(U_0, H)$ predictable where as usual, the filtration is the one described earlier which comes from the Q Wiener process. Suppose also that

$$\int_0^T \int_{\Omega} \|\Phi\|_{\mathcal{L}_2^0}^2 dP dt < \infty.$$

This collection of operators is denoted by $\mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$. Then letting $\{\Phi_n\}$ be a sequence of predictable elementary functions such that

$$\int_0^T \int_{\Omega} \|\Phi_n - \Phi\|_{\mathcal{L}_2^0}^2 dP dt \rightarrow 0 \quad (53.2.20)$$

Then

$$\Phi \cdot W(t) \equiv \lim_{n \rightarrow \infty} \Phi_n \cdot W(t) \quad (53.2.21)$$

the limit taking place in $L^2(\Omega; H)$. Note the existence of this sequence satisfying 53.2.20 is guaranteed by Proposition 53.2.6 which used the assumption that Φ is predictable.

The following is the Ito isometry.

Theorem 53.2.8 Let $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$. Then the above limit in 53.2.21 exists in $L^2(\Omega; H)$. Furthermore, $\Phi \cdot W(t)$ is a continuous martingale and is $\mathcal{F}_t$ adapted. Also

$$E\left(|\Phi \cdot W(t)|_H^2\right) = \int_0^t \int_{\Omega} \|\Phi\|_{\mathcal{L}_2^0}^2 dP ds \quad (53.2.22)$$

Proof: By Theorem 53.1.4 and the observation that the difference of two elementary functions is another elementary function

$$E\left(|(\Phi_n - \Phi_m) \cdot W(t)|_H^2\right) \leq \|\Phi_n - \Phi_m\|_{L^2(0, T; L^2(\Omega; \mathcal{L}_2(U_0, H)))}^2$$

which shows $\{\Phi_n \cdot W(t)\}$ is a uniformly Cauchy sequence in $C(0, T; L^2(\Omega; H))$. Therefore, there exists a unique $\Phi \cdot W(\cdot) \in C(0, T; L^2(\Omega; H))$ which is the limit of the sequence $\{\Phi_n \cdot W(t)\}$ in $C(0, T; L^2(\Omega; H))$. Similarly, the definition is independent of the choice of sequence converging to Φ .

Now by the maximal inequality of Theorem 50.3.4

$$\begin{aligned} & P\left(\left[\sup_{t \in [0, T]} |\Phi_n \cdot W(t) - \Phi_m \cdot W(t)| \geq \lambda\right]\right) \\ & \leq \frac{1}{\lambda} E(|\Phi_n \cdot W(t) - \Phi_m \cdot W(t)|_H) \\ & \leq \frac{1}{\lambda} E\left(|\Phi_n \cdot W(t) - \Phi_m \cdot W(t)|_H^2\right) \\ & \leq \frac{1}{\lambda} \|\Phi_n - \Phi_m\|_{L^2(0, T; L^2(\Omega; \mathcal{L}_2(U_0, H)))}^2 \end{aligned}$$

It follows there exists a subsequence n_k such that

$$P \left(\left[\sup_{t \in [0, T]} |\Phi_{n_{k+1}} \cdot W(t) - \Phi_{n_k} \cdot W(t)| \geq 2^{-k} \right] \right) = P(A_k) < 2^{-k}$$

Therefore, by the Borel Cantelli lemma, there is a set of measure 0 N such that for $\omega \notin N$, ω is in only finitely many of the sets A_k . Therefore, for such ω , $\Phi_{n_k} \cdot W(t)$ converges uniformly to a continuous function $t \rightarrow g(t, \omega)$. Then by completeness of the σ algebras in the filtration this function $g(t)$ is $\mathcal{F}_t$ adapted. (This is where it was important to have $\mathcal{F}_t$ be a normal filtration.) Furthermore, for each t , $g(t)$ must equal $\Phi \cdot W(t)$ a.e. Therefore, regard $\Phi \cdot W(t)$ as this function and it follows $\Phi \cdot W(t)$ is adapted. This also shows $t \rightarrow \Phi \cdot W(t)$ is continuous. That is, $\Phi \cdot W(t)$ has a version with the property that for $\omega \notin N$ a set of measure zero $t \rightarrow \Phi \cdot W(t)(\omega)$ is continuous.

The last claim follows from the definition and the fact the equation holds for the elementary functions. This proves the theorem.

53.3 Progressively Measurable

Progressively measurable processes are at least as general as predictable ones. It is the most general type of $\mathcal{B}([0, T]) \times \mathcal{F}_T$ measurable process which is adapted. Therefore, it is interesting to do things in terms of these more general processes. It turns out that when the filtration is normal, the above can be generalized to these processes.

Definition 53.3.1 Let $\Phi(t)$ be a stochastic process $\Phi : [0, T] \times \Omega \rightarrow E$ where E is some topological space. Also suppose $\mathcal{F}_t, t \in [0, T]$ is a filtration. Then $\Phi(t)$ is progressively measurable if

$$\Phi : [0, t] \times \Omega \rightarrow E \text{ is } \mathcal{B}([0, t]) \times \mathcal{F}_t \text{ measurable}$$

where this is the usual product measure.

Note that every predictable process is also progressively measurable. This follows from observing that the sets which generate the predictable σ algebra are each progressively measurable.

Proposition 53.3.2 Let Φ be progressively measurable as above where $\mathcal{F}_t$ is a filtration. Then $\Phi(t)$ is $\mathcal{F}_t$ adapted.

Proof: Is $\Phi(t)^{-1}(U) \in \mathcal{F}_t$ whenever $U \in \mathcal{B}(E)$? Consider the restriction of Φ to $[0, t] \times \Omega$

$$\Phi^{-1}(U) \equiv \{(s, \omega) \in [0, t] \times \Omega : \Phi(s, \omega) \in U\}$$

By assumption this is a $\mathcal{B}([0, t]) \times \mathcal{F}_t$ measurable set. Therefore, its s sections are $\mathcal{F}_t$ measurable. In particular, the t section

$$\{\omega : (t, \omega) \in \Phi^{-1}(U)\}$$

is $\mathcal{F}_t$ measurable. That is,

$$\{\omega : \Phi(t, \omega) \in U\} \equiv \Phi(t)^{-1}(U) \in \mathcal{F}_t.$$

This proves the proposition.

Here is a really surprising result about approximating with step functions which is due to Doob.

Lemma 53.3.3 *Let E be a Banach space and let $\Phi : [0, T] \times \Omega \rightarrow E$, be $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable (Note this includes the case of progressively measurable Φ .) and suppose*

$$\Phi \in L^p([0, T] \times \Omega; E).$$

Then there exists a sequence of nested partitions, $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$,

$$\mathcal{P}_k \equiv \{t_0^k, \dots, t_{m_k}^k\}$$

such that the step functions given by

$$\Phi_k^l(t) \equiv \sum_{j=1}^{m_k} \Phi(t_{j-1}^k) \mathcal{X}_{(t_{j-1}^k, t_j^k]}(t)$$

converges to Φ in $L^p([0, T] \times \Omega; E)$ as $k \rightarrow \infty$ and

$$\lim_{k \rightarrow \infty} \max \{ |t_j^k - t_{j+1}^k| : j \in \{0, \dots, m_k\} \} = 0$$

Here define

$$\Phi_k^l(0) \equiv \begin{cases} \Phi(0) & \text{if } \Phi(0) \in L^p(\Omega; E) \\ 0 & \text{if } \Phi(0) \notin L^p(\Omega; E) \end{cases}$$

Also for each $t \in \cup_k \mathcal{P}_k$, it can be assumed each $\Phi(t) \in L^p(\Omega, E)$.

Proof: For $t \in \mathbb{R}$ let $\gamma_n(t) \equiv k/2^n$ where $t \in (k/2^n, (k+1)/2^n]$. Also suppose Φ is defined to equal 0 on $[0, T]^C \times \Omega$. There exists a set of measure zero N such that for $\omega \notin N$, $t \rightarrow \Phi(t, \omega)$ is in $L^p(\mathbb{R})$. Therefore by continuity of translation, as $n \rightarrow \infty$ it follows that for $\omega \notin N$, and $t \in [0, T]$,

$$\int_{\mathbb{R}} \|\Phi(\gamma_n(t) + s) - \Phi(t + s)\|_E^p ds \rightarrow 0$$

The above is dominated by

$$\int_{\mathbb{R}} 2^{p-1} (\|\Phi(s)\|^p + \|\Phi(s)\|^p) \mathcal{X}_{[-2T, 2T]}(s) ds < \infty$$

Consider

$$\int_{\Omega} \int_0^T \left(\int_{\mathbb{R}} \|\Phi(\gamma_n(t) + s) - \Phi(t + s)\|_E^p ds \right) dt dP$$

By the dominated convergence theorem, this converges to 0 as $n \rightarrow \infty$. Now Fubini. This yields

$$\int_{\Omega} \int_0^T \left(\int_{\mathbb{R}} \|\Phi(\gamma_n(t) + s) - \Phi(t + s)\|_E^p ds \right) dt dP$$

Next change the variable in the integrand.

$$= \int_{\Omega} \int_0^T \left(\int_{\mathbb{R}} \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p ds \right) dt dP$$

which converges to 0 as $n \rightarrow \infty$. Therefore,

$$\int_{\Omega} \int_0^T \left(\int_0^T \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p ds \right) dt dP \rightarrow 0$$

also. Now interchange the order of integration again to obtain

$$\int_0^T \left(\int_{\Omega} \int_0^T \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p dt dP \right) ds \rightarrow 0$$

At this point note $t \rightarrow \Phi(\gamma_n(t - s) + s)$ is a step function of the form $\Phi_n^l(t)$ for each $s \in [0, T]$. This is because $\gamma_n(t - s) + s \leq t$ and for fixed s , $\gamma_n(t - s) + s$ is constant on small intervals of the form $[a, b]$ where $b - a = 2^{-n}$. Also $\gamma_n(t - s) + s \geq -2^{-n}$ for every $t \in [0, T]$. It follows the function

$$s \rightarrow \int_{\Omega} \int_0^T \|\Phi(\gamma_n(t - s) + s) - \Phi(t)\|_E^p dt dP \quad (53.3.23)$$

converges to 0 in $L^1([0, T])$.

The problem here is that $\gamma_n(t - s) + s$ could be negative. Therefore, replace it with $(\gamma_n(t - s) + s)^+$. This will not affect the convergence of the integral to 0 in 53.3.23 as $n \rightarrow \infty$ because the modification takes place at most on an interval of length 2^{-n} . Therefore, there exists a subsequence n_k such that

$$\int_{\Omega} \int_0^T \left\| \Phi \left((\gamma_{n_k}(t - s) + s)^+ \right) - \Phi(t) \right\|_E^p dt dP$$

converges to 0 for a.e. $s \in [0, T]$. Pick such an s . Let

$$\begin{aligned} \Phi_k^l(t) &\equiv \Phi \left((\gamma_{n_k}(t - s) + s)^+ \right) \\ \Phi_k^l(0) &\equiv \begin{cases} \Phi(0) & \text{if } \Phi(0) \in L^p(\Omega; E) \\ 0 & \text{if } \Phi(0) \notin L^p(\Omega; E) \end{cases} \end{aligned}$$

Note that s can be chosen so that all the mesh points miss any given set of measure zero which does not contain 0. This is because when you slide s it just moves the mesh points of the various $\mathcal{P}_k$ except for the first point. Thus for each k

there is an interval $[a_k, b_k]$ such that for s in this interval, none of the points of $\mathcal{P}_k$ are in the given set of measure zero. You can also assume these intervals are nested. Therefore, the claim follows. It will be assumed that all these mesh points miss the set of all t such that $\omega \rightarrow \Phi(t, \omega)$ is not in $L^p(\Omega; E)$. This proves the lemma.

Note that if Φ is progressively measurable, then it is adapted and so these step functions are predictable in this case.

Now here is an approximation result analogous to that of Proposition 53.2.6 in which an element of $L^2(\Omega \times [0, T]; E)$ is approximated with elementary functions. In this proposition, let E be a separable Banach space as above and let E_0 be a dense subspace. (Like $\mathcal{L}_2(U_0, H)$ and $\mathcal{L}(U, H)_0$ respectively.) Also let Ψ be an elementary function if it is of the form

$$\Psi(t, \omega) = \sum_{k=0}^{n-1} \Psi(t_j) \mathcal{X}_{(t_k, t_{k+1}]}(t)$$

where $\Psi(t_j)$ is $\mathcal{F}_{t_j}$ measurable, has values in E_0 , and has only finitely many values.

Proposition 53.3.4 *Let $\Phi \in L^p([0, T] \times \Omega, E)$ be progressively measurable. Then there exists a sequence of elementary functions which converges to Φ in $L^p([0, T] \times \Omega, E)$.*

Proof: By Lemma 53.3.3 there exists a sequence of step functions

$$\Phi_k^l(t) = \sum_{j=1}^{m_k} \Phi(t_{j-1}^k) \mathcal{X}_{(t_{j-1}^k, t_j^k]}(t)$$

which converges to Φ in $L^p([0, T] \times \Omega, E)$ where at the left endpoint $\Phi(0)$ can be modified as described above. Now each $\Phi(t_{j-1}^k)$ is in $L^p(\Omega, E)$ and is $\mathcal{F}(t_{j-1}^k)$ measurable and so it can be approximated as closely as desired with a simple function

$$s(t_{j-1}^k) \equiv \sum_{i=1}^{m_k} c_i^j \mathcal{X}_{F_i}(\omega), \quad F_i \in \mathcal{F}(t_{j-1}^k).$$

Furthermore, by density of E_0 in E , it can be assumed each $c_i^j \in E_0$. Replacing each $\Phi(t_{j-1}^k)$ with $s(t_{j-1}^k)$, the result is an elementary function which approximates Φ_k^l . This proves the proposition.

Here is another interesting proposition about the time integral being adapted.

Proposition 53.3.5 *Suppose $f \geq 0$ is progressively measurable and the filtration $\mathcal{F}_t$ is a filtration. Then*

$$\omega \rightarrow \int_0^t f(s, \omega) ds$$

is $\mathcal{F}_t$ adapted.

Proof: This follows right away from the fact f is $\mathcal{B}([0, t]) \times \mathcal{F}_t$ measurable. This is just product measure and so the integral from 0 to t is $\mathcal{F}_t$ measurable.

It follows from these considerations that everything which was done earlier assuming things are predictable can be done with the assumption the functions are progressively measurable.

It will also be important to consider the composition of functions. The following is the main result. With the explanation of progressively measurable given, it says the composition of progressively measurable functions is progressively measurable.

Proposition 53.3.6 *Let $A : [0, T] \times V \times \Omega \rightarrow U$ where V, U are topological spaces and suppose A satisfies its restriction to $[0, t] \times V \times \Omega$ is $\mathcal{B}([0, t]) \times \mathcal{B}(V) \times \mathcal{F}_t$ measurable. This will be referred to as A is progressively measurable. Then if $X : [0, T] \times \Omega \rightarrow V$ is progressively measurable, then so is the map*

$$(t, \omega) \rightarrow A(t, X(t, \omega), \omega)$$

Proof: Consider the restriction of this map to $[0, t_0] \times \Omega$. For such (t, ω) , to say

$$A(t, X(t, \omega), \omega) \in O$$

for O a Borel set in U is to say that

$$X(t, \omega) \in \{v : (t, v, \omega) \in A^{-1}(O)\}$$

and this (t, ω) section is in $\mathcal{B}(V)$ due to the assumption that A is progressively measurable. Therefore, the set of interest is X^{-1} (Borel set) which is given to be in $\mathcal{B}([0, t_0]) \times \mathcal{F}_{t_0}$ due to the assumption that X is progressively measurable. This proves the proposition.

53.4 Localization

The notion of localization will allow the definition of the integral for more general situations, in particular for the case that it is only assumed that Φ is $\mathcal{L}_2^0$ predictable or progressively measurable and

$$P\left(\left[\int_0^T \|\Phi(t, \omega)\|_{\mathcal{L}_2^0}^2 dt < \infty\right]\right) = 1$$

Thus for ω not in a set of measure zero, the above integral is finite. Such Φ are referred to as being in $N_W(0, T; \mathcal{L}_2^0)$.

Definition 53.4.1 *Let $\mathcal{F}_t$ be a filtration. τ is a stopping time if it is a measurable function which has the property that $[\tau \leq t] \in \mathcal{F}_t$ for all t . A simple stopping time is one which has only finitely many values.*

Here is a simple lemma about stopping times.

Lemma 53.4.2 *Let τ be a stopping time which is bounded by T . Then there exists a decreasing sequence of simple stopping times, $\{\tau_n\}$ converging uniformly on Ω to τ and*

$$\mathcal{X}_{[0, \tau_n(\omega)]}(t) \rightarrow \mathcal{X}_{[0, \tau(\omega)]}(t)$$

for each (t, ω) . Furthermore the function

$$(t, \omega) \rightarrow \mathcal{X}_{[0, \tau]}(t)$$

is real predictable, hence progressively measurable.

Proof: Let $\{t_k^n\}_{k=0}^{2^n}$ for $n = 1, 2, 3, \dots$ be a sequence of partitions of $[0, T]$ such that $t_{k+1}^n - t_k^n = T2^{-n}$. Then define

$$\tau_n(\omega) \equiv \sum_{k=0}^{2^n-1} t_{k+1}^n \mathcal{X}_{\tau^{-1}((t_k^n, t_{k+1}^n])}(\omega).$$

It is clear that $\tau_n(\omega) \geq \tau(\omega)$ and $\tau_n(\omega) \rightarrow \tau(\omega)$ for each ω and the convergence is uniform because $|\tau_n(\omega) - \tau(\omega)| \leq 2^{-n}T$. It remains to verify τ_n is a stopping time. But $\omega \in [\tau_n \leq t]$ if and only if for some k , $\tau_n(\omega) = t_{k+1}^n$ where $t_{k+1}^n \leq t$. That is, $\omega \in \tau^{-1}((t_k^n, t_{k+1}^n]) \in \mathcal{F}_{t_{k+1}^n} \subseteq \mathcal{F}_t$ and so τ_n is indeed a stopping time. Since it has only finitely many values, it is simple.

It remains to prove the claim about the given function being real predictable. First consider the function $\mathcal{X}_{[0, \tau_n]}(t)$. Let $\alpha \in (0, 1]$ and consider the set

$$\{(t, \omega) : \mathcal{X}_{[0, \tau_n]}(t) < \alpha\}$$

where the τ_n are defined above. Then by the choice of α , the set above is the set where the function equals 0. This is the set where $t > \tau_n$. Of course t is in exactly one of the intervals $(t_p^n, t_{p+1}^n]$. Therefore, the above set equals

$$\bigcup_{p=0}^{2^n-1} \left((t_p^n, t_{p+1}^n] \times \bigcup_{k=0}^{p-1} \overbrace{\{\omega : \tau(\omega) \in (t_k^n, t_{k+1}^n]\}}^{\in \mathcal{F}_{t_p}} \right)$$

Thus this is a finite union of sets which generate the predictable sets. Next suppose $\alpha > 1$. Then

$$\begin{aligned} \{(t, \omega) : \mathcal{X}_{[0, \tau_n]}(t) < \alpha\} &= [0, T] \times \Omega \\ &= \{0\} \times \Omega \cup (0, T] \times \Omega \end{aligned}$$

the union of two of the sets which generate the predictable sets $\mathcal{P}_T$. Therefore, the function

$$(t, \omega) \rightarrow \mathcal{X}_{[0, \tau_n]}(t) = \mathcal{X}_{[t, T]}(\tau_n(\omega))$$

is predictable.

Now what about $(t, \omega) \rightarrow \mathcal{X}_{[0, \tau]}(t) = \mathcal{X}_{[t, T]}(\tau(\omega))$? If (t, ω) is such that $\tau(\omega) < t$, so that $\mathcal{X}_{[t, T]}(\tau(\omega)) = 0$, then for all n large enough, it is also the case that

$\mathcal{X}_{[t,T]}(\tau_n(\omega)) = 0$. It follows that for such (t, ω) , the sequence $\mathcal{X}_{[t,T]}(\tau_n(\omega))$ converges to $\mathcal{X}_{[t,T]}(\tau(\omega))$. If (t, ω) is such that $t \leq \tau(\omega)$ so that $\mathcal{X}_{[0,\tau]}(t) = \mathcal{X}_{[t,T]}(\tau(\omega)) = 1$, then for all n $t \leq \tau_n(\omega)$ because $\tau_n(\omega) \geq \tau(\omega)$. Therefore, $\mathcal{X}_{[t,T]}(\tau_n(\omega)) = 1$ for all n . Either way $\mathcal{X}_{[t,T]}(\tau_n(\omega)) \rightarrow \mathcal{X}_{[t,T]}(\tau(\omega))$ and this shows $(t, \omega) \rightarrow \mathcal{X}_{[0,\tau]}(t) = \mathcal{X}_{[t,T]}(\tau(\omega))$ is real predictable because it is the pointwise limit of real predictable functions. This proves the lemma.

Corollary 53.4.3 *If $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$ and τ is a stopping time, then $\mathcal{X}_{[0,\tau]}\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$.*

Proof: The only thing which is not obvious is that $\mathcal{X}_{[0,\tau]}\Phi$ is $\mathcal{L}_2^0$ predictable or progressively measurable. Letting $\Psi \in \mathcal{L}_2^0$, it follows by Pettis theorem that it suffices to verify

$$(\mathcal{X}_{[0,\tau]}\Phi, \Psi)_{\mathcal{L}_2^0}$$

is real predictable or progressively measurable. However, this equals

$$\mathcal{X}_{[0,\tau]}(\Phi, \Psi)_{\mathcal{L}_2^0}$$

and by Lemma 53.4.2 and the assumed fact that Φ is predictable or progressively measurable, both functions in the above product are real predictable or progressively measurable. Since the product of measurable functions is measurable, the above is measurable with respect to $\mathcal{P}_T$, hence predictable or progressively measurable in case Φ is only progressively measurable. Therefore, this proves the corollary.

Since $\mathcal{X}_{[0,\tau]}\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$ for $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$, it follows it makes sense to consider the stochastic integral

$$\mathcal{X}_{[0,\tau]}\Phi \cdot W(t).$$

The next lemma says this equals $\Phi \cdot W(t \wedge \tau)$.

Lemma 53.4.4 *Let $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$ meaning Φ is $\mathcal{L}_2^0$ predictable or progressively measurable and*

$$\int_0^T \int_{\Omega} \|\Phi\|_{\mathcal{L}_2^0}^2 dP dt < \infty,$$

and let τ be a bounded stopping time such that $\tau \leq T$. Then $\mathcal{X}_{[0,\tau]}\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$ and

$$\mathcal{X}_{[0,\tau]}\Phi \cdot W(t) = \Phi \cdot W(t \wedge \tau).$$

Proof: First suppose τ is a simple stopping time so it has only finitely many values, $\{a_1, a_2, \dots, a_m\}$. Also suppose Φ is elementary. Recall this means Φ has values in $\mathcal{L}(U, H)$ is predictable, is constant on each of finitely many intervals which divide $[0, T]$, and has only finitely many values. Then without loss of generality, it can be assumed $\{a_1, a_2, \dots, a_m\}$ is contained in the set of mesh points, $\{t_k\}$ which

determine the intervals on which Φ is constant and that the a_i are given in order from smallest to largest. If not, just add these in. Say

$$\Phi(t) = \sum_{k=0}^{m-1} \Phi(t_k) \mathcal{X}_{(t_k, t_{k+1}]}(t)$$

and τ has one of the values t_k . Then $\mathcal{X}_{[0, \tau]}(s) = 1$ as long as $s \leq \tau$ and equals 0 if $s > \tau$. Also recall $\Phi(t_k)$ takes only finitely many values. Thus $\Phi(t)$ is of the form

$$\begin{aligned} \Phi(t)(\omega) &= \sum_{k=0}^{m-1} \sum_{j=1}^{m_k} \Phi_j^k \mathcal{X}_{A_j^k}(\omega) \mathcal{X}_{(t_k, t_{k+1}]}(t) \\ &= \sum_{k=0}^{m-1} \sum_{j=1}^{m_k} \Phi_j^k \mathcal{X}_{(t_k, t_{k+1}] \times A_j^k}(t, \omega) \end{aligned}$$

where each $A_j^k \in \mathcal{F}_{t_k}$.

Claim 53.4.5 *If Φ is elementary, then $\mathcal{X}_{[0, \tau]} \Phi$ is elementary and $\mathcal{L}(U, H)$ predictable.*

Proof of claim: By Lemma 53.4.2, $(t, \omega) \rightarrow \mathcal{X}_{[0, \tau(\omega)]}(t)$ is real predictable. Also from the above formula each term in the sum is $\mathcal{L}(U, H)$ predictable because the sets

$$(t_k, t_{k+1}] \times A_j^k, A_j^k \in \mathcal{F}_{t_k}$$

generate $\mathcal{P}_T$ and are therefore, predictable. Therefore, the claim follows.

Now consider $\mathcal{X}_{[0, \tau]} \Phi \cdot W(t)$. Say $\tau(\omega) = t_p$. It equals one of these. Then for such ω ,

$$\mathcal{X}_{[0, \tau]}(t) \Phi(t) = \sum_{k=0}^{m-1} \mathcal{X}_{[t_p > t_k]} \Phi(t_k) \mathcal{X}_{(t_k, t_{k+1}]}(t)$$

In other words $k \leq p-1$ and so this reduces to

$$\begin{aligned} \sum_{k=0}^{p-1} \Phi(t_k) \mathcal{X}_{(t_k, t_{k+1}]}(t) &= \sum_{k=0}^{p-1} \Phi(t_k) \mathcal{X}_{(t_k, t_{k+1}]}(t \wedge t_p) \\ &= \sum_{k=0}^{m-1} \Phi(t_k) \mathcal{X}_{(t_k, t_{k+1}]}(t \wedge t_p) \end{aligned}$$

and so from the definition, for these ω ,

$$\begin{aligned} \mathcal{X}_{[0, \tau]} \Phi \cdot W(t) &= \sum_{k=0}^{m-1} \Phi(t_k) (W(t \wedge t_p \wedge t_{k+1}) - W(t \wedge t_p \wedge t_k)) \\ &\equiv \Phi \cdot W(t \wedge t_p) \equiv \Phi \cdot W(t \wedge \tau) \end{aligned}$$

This takes care of the case where τ has finitely many values and Φ is elementary.

Next suppose τ is just a stopping time bounded by T . Then by Lemma 53.4.2 there exists a decreasing sequence, $\{\tau_n\}$ of simple stopping times converging to τ uniformly on Ω where $|\tau_n - \tau| \leq T2^{-n}$ such that $\mathcal{X}_{[0, \tau_n(\omega)]}(t) \rightarrow \mathcal{X}_{[0, \tau(\omega)]}(t)$. Also recall from Lemma 53.1.9, $\mathcal{L}(U, H)_0$ defined by the restriction to U_0 is dense in $\mathcal{L}_2^0$. Thus for Φ elementary, having values in $\mathcal{L}(U, H)$, I will consider it in $\mathcal{L}_2^0$ by restriction of each Φ_k to U_0 . Then letting $m < n$,

$$\begin{aligned} \int_0^T \int_{\Omega} \|\mathcal{X}_{[0, \tau_n]} \Phi - \mathcal{X}_{[0, \tau_m]} \Phi\|_{\mathcal{L}_2^0}^2 dP dt &= \int_{\Omega} \int_0^T \|\mathcal{X}_{(\tau_n, \tau_m]} \Phi\|_{\mathcal{L}_2^0}^2 dt dP \\ &\leq T2^{-m} \int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \end{aligned}$$

and so letting $n \rightarrow \infty$, Fatou's lemma and Lemma 53.4.2 implies

$$\int_0^T \int_{\Omega} \|\mathcal{X}_{[0, \tau]} \Phi - \mathcal{X}_{[0, \tau_m]} \Phi\|_{\mathcal{L}_2^0}^2 dP dt \leq T2^{-m} \int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP$$

The integral $\mathcal{X}_{[0, \tau]} \Phi \cdot W(t)$ makes sense by Corollary 53.4.3 even though $\mathcal{X}_{[0, \tau]} \Phi$ is no longer elementary and by the Ito isometry,

$$\begin{aligned} E \left(\|\mathcal{X}_{[0, \tau]} \Phi - \mathcal{X}_{[0, \tau_m]} \Phi\|_{\mathcal{L}_2^0}^2 \right) &\leq \int_{\Omega} \int_0^T \|\mathcal{X}_{[0, \tau]} \Phi - \mathcal{X}_{[0, \tau_m]} \Phi\|_{\mathcal{L}_2^0}^2 dP ds \\ &\leq T2^{-m} \int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \end{aligned}$$

Thus $\mathcal{X}_{[0, \tau_m]} \Phi \cdot W(t)$ converges uniformly on $[0, T]$ to $\mathcal{X}_{[0, \tau]} \Phi \cdot W(t)$ in $L^2(\Omega; H)$.

Since $t \rightarrow \Phi \cdot W(t)$ is continuous, it follows $\Phi \cdot W(t \wedge \tau_n)$ converges uniformly to $\Phi \cdot W(t \wedge \tau)$ and it follows from this uniform convergence that

$$\begin{aligned} \mathcal{X}_{[0, \tau]} \Phi \cdot W(t) &= \lim_{m \rightarrow \infty} \mathcal{X}_{[0, \tau_m]} \Phi \cdot W(t) \\ &= \lim_{m \rightarrow \infty} \Phi \cdot W(t \wedge \tau_m) = \Phi \cdot W(t \wedge \tau) \end{aligned}$$

whenever Φ is elementary and $\mathcal{L}(U, H)$ predictable.

Now let $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$ be arbitrary. Then by Proposition 53.2.6 or Proposition 53.3.4 there exists a sequence $\{\Phi_n\}$ of elementary $\mathcal{L}(U, H)$ valued processes such that

$$e_n \equiv \int_0^T \int_{\Omega} \|\Phi_n - \Phi\|_{\mathcal{L}_2^0}^2 dP dt \rightarrow 0 \quad (53.4.24)$$

which also implies

$$\int_0^T \int_{\Omega} \|\mathcal{X}_{[0, \tau]} \Phi_n - \mathcal{X}_{[0, \tau]} \Phi\|_{\mathcal{L}_2^0}^2 dP dt \rightarrow 0. \quad (53.4.25)$$

Therefore, from 53.4.24 and Theorem 53.2.8 along with the maximal estimate of Theorem 50.3.5,

$$\begin{aligned}
 & P \left(\left[\sup_{t \in [0, T]} |\mathcal{X}_{[0, \tau]} \Phi_n \cdot W(t) - \mathcal{X}_{[0, \tau]} \Phi \cdot W(t)|_H \geq \alpha \right] \right) \\
 & \leq P \left(\left[\sup_{t \in [0, T]} |\Phi_n \cdot W(t) - \Phi \cdot W(t)|_H \geq \alpha \right] \right) \\
 & \leq \frac{1}{\alpha} E(|\Phi_n \cdot W(T) - \Phi \cdot W(T)|_H) \leq \frac{1}{\alpha} E(|\Phi_n \cdot W(T) - \Phi \cdot W(T)|_H^2) \\
 & = \frac{1}{\alpha} \int_0^T \int_{\Omega} \|\Phi_n - \Phi\|_{\mathcal{L}_2^0}^2 dP dt = \frac{e_n}{\alpha}
 \end{aligned}$$

where $e_n \rightarrow 0$. Therefore, there exists a subsequence, n_k such that

$$P \left(\left[\sup_{t \in [0, T]} |\mathcal{X}_{[0, \tau]} \Phi_{n_k} \cdot W(t) - \mathcal{X}_{[0, \tau]} \Phi \cdot W(t)|_H \geq 2^{-k} \right] \right) \leq 2^{-k}$$

and so there exists a set of measure 0, N such that if $\omega \notin N$, then ω is in only finitely many of the sets,

$$\left[\sup_{t \in [0, T]} |\mathcal{X}_{[0, \tau]} \Phi_{n_k} \cdot W(t) - \mathcal{X}_{[0, \tau]} \Phi \cdot W(t)|_H \geq 2^{-k} \right]$$

and so for such ω , $\{\mathcal{X}_{[0, \tau]} \Phi_{n_k} \cdot W(t)\}$ converges uniformly to $\mathcal{X}_{[0, \tau]} \Phi \cdot W(t)$ for $t \in [0, T]$. Similarly, there exists a subsequence and a set of measure zero such that for ω not in this set, $\{\Phi_{n_k} \cdot W(t)\}$ converges uniformly to $\Phi \cdot W(t)$ for $t \in [0, T]$. Therefore, for ω not in this exceptional set,

$$\Phi_{n_k} \cdot W(t \wedge \tau) \rightarrow \Phi \cdot W(t \wedge \tau)$$

It follows that for a suitable exceptional set and subsequence, n_k then for all $t \in [0, T]$,

$$\begin{aligned}
 \mathcal{X}_{[0, \tau]} \Phi \cdot W(t) &= \lim_{k \rightarrow \infty} \mathcal{X}_{[0, \tau]} \Phi_{n_k} \cdot W(t) \\
 &= \lim_{k \rightarrow \infty} \Phi_{n_k} \cdot W(t \wedge \tau) \\
 &= \Phi \cdot W(t \wedge \tau).
 \end{aligned}$$

This proves the lemma.

If Φ is $\mathcal{L}_2^0$ predictable or progressively measurable and

$$E \left(\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt \right) < \infty$$

then the definition of $\Phi \cdot W(t)$ is given above. However, it is possible to extend the definition to a still more general class of processes.

53.5 The Integral For A More General Case

This considers the case where Φ is $\mathcal{L}_2(U_0, H)$ predictable or progressively measurable with $\mathcal{F}_t$ a normal filtration but instead of

$$E \left(\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 ds \right) < \infty$$

one assumes only that

$$P \left(\left[\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 ds < \infty \right] \right) = 1$$

In words, Φ is $\mathcal{L}_2(U_0, H)$ predictable or progressively measurable and $\|\Phi\|_{\mathcal{L}_2^0}^2$ is stochastically integrable.

Lemma 53.5.1 *Suppose Φ is $\mathcal{L}_2(U_0, H)$ predictable or progressively measurable and*

$$P \left(\left[\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 ds < \infty \right] \right) = 1.$$

Define

$$\tau_n(\omega) \equiv \inf \left\{ t \in [0, T] : \int_0^t \|\Phi\|_{\mathcal{L}_2^0}^2 ds \geq n \right\},$$

By convention, let $\inf \emptyset = T$. Then τ_n is a stopping time. Furthermore, τ_n has the following properties.

1. $\{\tau_n\}$ is an increasing sequence and for ω outside a set of measure zero N , for every $t \in [0, T]$ there exists n such that $\tau_n(\omega) \geq t$
2. For each n

$$\mathcal{X}_{[0, \tau_n]} \Phi \in N_W^2(0, T; \mathcal{L}_2^0)$$

that is $\mathcal{X}_{[0, \tau_n]} \Phi$ is $\mathcal{L}_2^0$ predictable or progressively measurable and

$$E \left(\int_0^T \|\mathcal{X}_{[0, \tau_n]} \Phi\|_{\mathcal{L}_2^0}^2 dt \right) < \infty$$

Proof: First of all, the integral is well defined and

$$(t, \omega) \rightarrow \int_0^t \|\Phi\|_{\mathcal{L}_2^0}^2 ds$$

is $\mathcal{F}_t$ adapted thanks to Lemma 53.2.4 or Proposition ?? and the observation that $\|\Phi\|_{\mathcal{L}_2^0}^2$ is either real predictable or progressively measurable. I need to show $[\tau_n \leq t] \in \mathcal{F}_t$. However, from the definition of τ_n ,

$$[\tau_n \leq t_0] = \left[\int_0^{t_0} \|\Phi\|_{\mathcal{L}_2^0}^2 ds \geq n \right].$$

To see this, suppose ω is in the right. Then t_0 is in the set of numbers whose infimum is defined to be $\tau_n(\omega)$. Therefore, $\tau_n(\omega) \leq t_0$. Next suppose $\tau_n(\omega) \leq t_0$. Then if $t > t_0$ it follows

$$\int_0^t \|\Phi\|_{\mathcal{L}_2^0}^2 ds \geq n$$

because such t is larger than $\tau_n(\omega)$. It follows from continuity that

$$\int_0^{t_0} \|\Phi\|_{\mathcal{L}_2^0}^2 ds \geq n$$

Now it follows from Lemma 53.2.4 and Proposition 53.2.2 or Proposition ?? that

$$[\tau_n \leq t_0] = \left[\int_0^{t_0} \|\Phi\|_{\mathcal{L}_2^0}^2 ds \geq n \right] \in \mathcal{F}_{t_0}$$

It remains to verify the two claims. There exists a set of measure 0, N such that for $\omega \notin N$

$$\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt < \infty$$

Therefore, for such ω , there exists n large enough that

$$\int_0^t \|\Phi\|_{\mathcal{L}_2^0}^2 ds < n$$

and so $\tau_n(\omega) \geq t$. Now consider the second claim.

$$E \left(\int_0^T \|\mathcal{X}_{[0, \tau_n]} \Phi\|_{\mathcal{L}_2^0}^2 dt \right) = E \left(\int_0^{\tau_n(\omega)} \|\Phi\|_{\mathcal{L}_2^0}^2 dt \right) \leq E(n) = n.$$

This proves the lemma.

With this lemma, it is possible to give the following definition.

Definition 53.5.2 Suppose Φ is $\mathcal{L}_2(U_0, H)$ predictable or progressively measurable and

$$P \left(\left[\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 ds < \infty \right] \right) = 1. \quad (53.5.26)$$

Then let τ_n denote a sequence of stopping times having the two properties of Lemma 53.5.1. Then for all ω not in the exceptional set N .

$$\Phi \cdot W(t) \equiv \mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t)$$

where $\tau_n(\omega) \geq t$. In other words, you find τ_n such that $\tau_n(\omega) \geq t$ and then apply the above definition.

Lemma 53.5.3 *The above definition is well defined and can be written as: For all ω not in a set of measure zero,*

$$\Phi \cdot W(t)(\omega) \equiv \lim_{n \rightarrow \infty} \mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t)(\omega)$$

the function on the right being constant for all n large enough. The random variable $\Phi \cdot W(t)$ is also $\mathcal{F}_t$ adapted.

Proof: Let $\{\tau_n\}$ be a sequence of stopping times as described in 1 and 2 of Lemma 53.5.1. Such a sequence exists by Lemma 53.5.1. It makes sense to define

$$\mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t)$$

Now what if both τ_m and τ_n are at least as large as t ? Does the same random variable result from applying the definition to both τ_m and τ_n ? Say $m > n$ so that $\tau_m \geq \tau_n \geq t$. By Lemma 53.4.4,

$$\begin{aligned} \mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t) &= \mathcal{X}_{[0, \tau_n]} \mathcal{X}_{[0, \tau_m]} \Phi \cdot W(t) \\ &= \mathcal{X}_{[0, \tau_m]} \Phi \cdot W(t \wedge \tau_n) \\ &= \mathcal{X}_{[0, \tau_m]} \Phi \cdot W(t). \end{aligned}$$

$\Phi \cdot W(t)$ is $\mathcal{F}_t$ adapted because for U an open set in H ,

$$\Phi \cdot W(t)^{-1}(U) = \cup_{n=1}^{\infty} \left((\mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t))^{-1}(U) \cap [\tau_n \geq t] \right) \in \mathcal{F}_t$$

This proves the lemma.

Lemma 53.5.4 *Let Φ be predictable or progressively measurable and satisfy*

$$P \left(\left[\int_0^T \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds < \infty \right] \right) = 1$$

Then if σ is a stopping time,

$$\begin{aligned} \int_0^{t \wedge \sigma} \Phi dW(s) &\equiv \Phi \cdot W^\sigma(t) \\ &= \mathcal{X}_{[0, \sigma]} \Phi \cdot W(t) = \int_0^t \mathcal{X}_{[0, \sigma]} \Phi dW(s) \end{aligned}$$

Proof: Let $\{\tau_n\}$ be the localizing sequence described above for which, when the local martingale is stopped, it results in a martingale, and suppose ω is such that $\tau_n(\omega) > t$. Then by definition, for that ω ,

$$\Phi \cdot W(t) = \mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t)$$

and so from Lemma 53.4.4, for that ω

$$\begin{aligned}
 \int_0^{t \wedge \sigma} \Phi dW(s) &\equiv \Phi \cdot W^\sigma(t) \equiv \mathcal{X}_{[0, \tau_n]} \Phi \cdot W^\sigma(t) \\
 &= \mathcal{X}_{[0, \sigma]} \mathcal{X}_{[0, \tau_n]} \Phi \cdot W(t) \\
 &= \int_0^t \mathcal{X}_{[0, \tau_n]} \mathcal{X}_{[0, \sigma]} \Phi dW(s) \\
 &\equiv \int_0^t \mathcal{X}_{[0, \sigma]} \Phi dW(s)
 \end{aligned}$$

This proves the lemma.

The lemma says that even when $\int_0^t \Phi(s) dW(s)$ is only a local martingale because Φ is only stochastically integrable, it is still the case that

$$\int_0^{t \wedge \sigma} \Phi dW(s) = \int_0^t \mathcal{X}_{[0, \sigma]} \Phi dW(s).$$

An important corollary of Lemma 53.4.4 concerns the quadratic variation of $\Phi \cdot W(t)$ for $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$.

Corollary 53.5.5 *Let $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$, Φ being $\mathcal{L}_2^0$ predictable or progressively measurable. Then the quadratic variation, $[\Phi \cdot W]$ is given by the formula*

$$[\Phi \cdot W](t) = \int_0^t \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds$$

If $\Phi \cdot W(t)$ is only a local martingale, because Φ is predictable or progressively measurable but only satisfies

$$P\left(\left[\int_0^T \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds < \infty\right]\right) = 1$$

the same conclusion holds.

Proof: Let σ be a stopping time having values in $[0, T]$. Then using Lemma 53.4.4 and the Ito isometry, Theorem 53.2.8,

$$\begin{aligned}
 &E\left(\left\|\int_0^{T \wedge \sigma} \Phi(s) dW(s)\right\|^2 - \int_0^{T \wedge \sigma} \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds\right) \\
 &= E\left(\left\|\int_0^T \mathcal{X}_{[0, \sigma]} \Phi(s) dW(s)\right\|^2 - \int_0^T \mathcal{X}_{[0, \sigma]} \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds\right) \\
 &= \int_0^T \int_\Omega \|\mathcal{X}_{[0, \sigma]} \Phi(s)\|_{\mathcal{L}_2^0}^2 dP ds - \int_\Omega \int_0^T \mathcal{X}_{[0, \sigma]} \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 ds dP
 \end{aligned}$$

$$= \int_0^T \int_{\Omega} \|\mathcal{X}_{[0,\sigma]} \Phi(s)\|_{\mathcal{L}_2^0}^2 dP ds - \int_{\Omega} \int_0^T \|\mathcal{X}_{[0,\sigma]} \Phi(s)\|_{\mathcal{L}_2(U_0,H)}^2 ds dP$$

and this equals 0 because the integrands are assumed product measurable which implies the iterated integrals are equal. Therefore, by Lemma 51.1.1, the lemma about recognizing a martingale when you see one, the stochastic process

$$\left\| \int_0^t \Phi(s) dW(s) \right\|^2 - \int_0^t \|\Phi(s)\|_{\mathcal{L}_2(U_0,H)}^2 ds$$

is a martingale because you could also take $\sigma = t$, the constant and the result would still hold. The process

$$\int_0^t \|\Phi(s)\|_{\mathcal{L}_2(U_0,H)}^2 ds$$

is adapted because the integrand is predictable. It is also increasing. Therefore by uniqueness of the quadratic variation, Theorem 51.4.4, it follows this process must be the quadratic variation.

Finally, consider the case where $\Phi \cdot W(t)$ is only a local martingale. Then letting $\{\tau_n\}$ be a localizing sequence, the above result along with Lemma 53.5.4 applies to the stopped process. Thus by Theorem 51.4.4 and letting $\{\tau_n\}$ be a localizing sequence, then from the above and the localization result,

$$\Phi \cdot W^{\tau_n}(t) = (\mathcal{X}_{[0,\tau_n]} \Phi) \cdot W(t),$$

where $\tau_n(\omega) > t$, it follows

$$\begin{aligned} \text{martingale} &= \left(\|\Phi \cdot W\|^2 - [\Phi \cdot W] \right)^{\tau_n}(t) \\ &= \|\Phi \cdot W^{\tau_n}\|^2(t) - [\Phi \cdot W^{\tau_n}](t) \\ &= \left(\|\Phi \cdot W\|^2 \right)^{\tau_n}(t) - \int_0^t \|\mathcal{X}_{[0,\tau_n]}(s) \Phi(s)\|^2 ds \\ &= \left(\|\Phi \cdot W\|^2 \right)^{\tau_n}(t) - \int_0^{t \wedge \tau_n} \|\Phi(s)\|^2 ds \end{aligned}$$

Therefore, $\|\Phi \cdot W\|^2(t) - \int_0^t \|\Phi(s)\|^2 ds$ is a local martingale and by the uniqueness part of this theorem

$$[\Phi \cdot W](t) = \int_0^t \|\Phi(s)\|^2 ds$$

This proves the corollary.

53.6 Taking Out A Linear Transformation

When is

$$L \int_0^T \Phi dW(s) = \int_0^T L \Phi dW(s)?$$

Here I am using the notation

$$\int_0^T \Phi dW(s) \equiv \Phi \cdot W(T)$$

and it is assumed $L \in \mathcal{L}(H, H_1)$ where H_1 is another separable real Hilbert space. First of all, here is a lemma which shows $L\Phi \cdot W(T)$ at least makes sense.

Proposition 53.6.1 *Let $\Phi \in N_W^2(0, T; \mathcal{L}_2(U_0, H))$. Then $L\Phi \in N_W^2(0, T; \mathcal{L}_2(U_0, H_1))$. Furthermore, for each $t \in [0, T]$*

$$\int_0^t L\Phi dW(s) = L \int_0^t \Phi dW(s)$$

Proof: First note that if $\Phi \in \mathcal{L}_2(U_0, H)$, then $L\Phi \in \mathcal{L}_2(U_0, H_1)$. This follows because if $\{g_k\}$ is an orthonormal basis for U_0 , then

$$\sum_k \|L\Phi g_k\|_{H_1}^2 \leq \|L\|^2 \sum_k \|\Phi g_k\|_H^2 < \infty$$

because $\Phi \in \mathcal{L}_2(U_0, H)$. Also this shows the map $\Phi \rightarrow L\Phi$ is continuous. It follows $L\Phi$ is $\mathcal{L}_2(U_0, H_1)$ predictable. All that remains is to check the appropriate integral.

$$\int_{\Omega} \int_0^T \|L\Phi\|_{\mathcal{L}_2(U_0, H_1)}^2 dt dP \leq \int_{\Omega} \int_0^T \|L\|^2 \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt dP < \infty$$

and so this proves $L\Phi \in N_W^2(0, T; \mathcal{L}_2(U_0, H_1))$.

If $\Phi \in N_W^2(0, T; \mathcal{L}_2(U_0, H))$ it follows one can consider

$$\int_0^T L\Phi dW(s).$$

Letting $\{\Phi_n\}$ be an approximating sequence of elementary functions satisfying

$$E \left(\int_0^T \|\Phi_n - \Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt \right) \rightarrow 0,$$

it is also the case that

$$E \left(\int_0^T \|L\Phi_n - L\Phi\|_{\mathcal{L}_2(U_0, H_1)}^2 dt \right) \rightarrow 0$$

By Theorem 53.2.8, for each t

$$\begin{aligned} \int_0^t L\Phi dW(s) &= \lim_{n \rightarrow \infty} \int_0^t L\Phi_n dW(s) = \lim_{n \rightarrow \infty} L \int_0^t \Phi_n dW(s) \\ &= L \lim_{n \rightarrow \infty} \int_0^t \Phi_n dW(s) = L \int_0^t \Phi dW(s) \end{aligned}$$

The second equality is obvious for elementary functions.

Now consider the case where $\Phi \in N_W(0, T; \mathcal{L}_2(U_0, H))$ so that all is known is that

$$P\left(\left[\int_0^T \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt < \infty\right]\right) = 1.$$

Then define τ_n as above

$$\tau_n \equiv \inf \left\{ t : \int_0^t \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt \geq n \right\}$$

This sequence of stopping times works for $L\Phi$ also. Recall there were two conditions the sequence of stopping times needed to satisfy. The first is obvious. Here is why the second holds.

$$\begin{aligned} \int_0^T \|\mathcal{X}_{[0, \tau_n]} L\Phi\|_{\mathcal{L}_2(U_0, H_1)}^2 dt &\leq \|L\|^2 \int_0^T \|\mathcal{X}_{[0, \tau_n]} \Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt \\ &= \|L\|^2 \int_0^{\tau_n} \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt \leq \|L\|^2 n \end{aligned}$$

Then let t be given and pick n such that $\tau_n \geq t$. Then from the first part

$$\begin{aligned} L \int_0^t \Phi dW(s) &\equiv L \int_0^t \mathcal{X}_{[0, \tau_n]} \Phi dW(s) \\ &= \int_0^t L \mathcal{X}_{[0, \tau_n]} \Phi dW(s) \\ &= \int_0^t \mathcal{X}_{[0, \tau_n]} L\Phi dW(s) \equiv \int_0^t L\Phi dW(s) \end{aligned}$$

This proves the proposition.

53.7 The Case Where Q Is Not Trace Class

A positive self adjoint continuous linear operator has a unique square root according to Theorem 15.10.4. Thus using Theorem 15.2.2, one can still consider the Hilbert space $U_0 \equiv Q^{1/2}U$. Is there some way to define a stochastic integral even though the covariance Q is not of trace class? This involves the concept of a cylindrical Wiener process. It is a cheap trick which causes great confusion. One generalizes the definition by using another space which does have a Q Wiener process for Q of trace class and one defines the integral in terms of the Wiener process on this space. Also note that this is an issue which is never a problem in finite dimensions.

Let $U_0 \equiv Q^{1/2}U$ using the construction of Theorem 15.2.2. Then let

$$J : U_0 \rightarrow U_1$$

be a one to one Hilbert Schmidt operator where U_1 is a separable real Hilbert space.

Lemma 53.7.1 *Let Q be a positive self adjoint bounded linear transformation defined on U a separable real Hilbert space. Let $U_0 = Q^{1/2}U$ as in Theorem 15.2.2. Then there exists a one to one Hilbert Schmidt operator $J : U_0 \rightarrow U_1$ where U_1 is a separable real Hilbert space. In fact you can take $U_1 = U$.*

Proof: Let $\alpha_k > 0$ and $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Then let $\{g_k\}$ be an orthonormal basis for $U_0 \equiv Q^{1/2}U$, the inner product and norm given in Theorem 15.2.2, and let

$$Jx \equiv \sum_{k=1}^{\infty} (x, g_k)_{U_0} \alpha_k g_k$$

Then it is clear that $J \in \mathcal{L}(U_0, U)$. This is because by the norm estimate of Theorem 15.2.2

$$\begin{aligned} \|Jx\|_U &\leq \sum_{k=1}^{\infty} |(x, g_k)_{U_0}| \alpha_k \|g_k\|_U \\ &\leq C \sum_{k=1}^{\infty} |(x, g_k)_{U_0}| \alpha_k \overbrace{\|g_k\|_{U_0}}^{=1} \\ &\leq C \left(\sum_{k=1}^{\infty} |(x, g_k)_{U_0}|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2} \\ &= C \left(\sum_{k=1}^{\infty} \alpha_k^2 \right)^{1/2} \|x\|_{U_0} \end{aligned}$$

Also, by the norm estimate of Theorem 15.2.2,

$$\sum_{j=1}^{\infty} \|Jg_j\|_U^2 = \sum_{j=1}^{\infty} \alpha_j^2 \|g_j\|_U^2 \leq C \sum_{j=1}^{\infty} \alpha_j^2 < \infty$$

and so J is also a Hilbert Schmidt operator which maps U_0 to U . It is clear that J is one to one because each $\alpha_k > 0$. If U_0 is finite dimensional, just make the above sum finite. (When you run out of g_k , quit summing.) This proves the lemma.

Now consider the operator $Q_1 \equiv JJ^*$ which is defined on U_1 . This is a positive self adjoint operator and furthermore, this operator is trace class. To see this, let $\{f_m\}$ be an orthonormal basis for U_1 . Then letting $\{e_k\}$ be a complete orthonormal basis for U_0 ,

$$\begin{aligned} \sum_{k=1}^{\infty} (JJ^* f_k, f_k)_{U_1} &= \sum_{k=1}^{\infty} (J^* f_k, J^* f_k)_U = \sum_{k=1}^{\infty} \|J^* f_k\|_U^2 \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (J^* f_k, e_j)^2 = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} (f_k, J e_j)^2 = \sum_{j=1}^{\infty} \|J e_j\|^2 < \infty \end{aligned}$$

because J is Hilbert Schmidt.

Lemma 53.7.2 $J : Q^{1/2}U = U_0 \rightarrow Q_1^{1/2}U_1 = JU_0$ is an isometry, the norms on these spaces given by Definition 15.2.1.

$$\begin{array}{c} U_1 \\ \cup \\ U \end{array} \quad \begin{array}{c} Q^{1/2} \\ \rightarrow \end{array} \quad Q^{1/2}U = U_0 \quad \xrightarrow{J} \quad JU_0 = Q_1^{1/2}U_1 \quad \xleftarrow{Q_1^{1/2}} \quad U_1$$

Proof: There are two things which need to be shown. One is that $JU_0 = Q_1^{1/2}U_1 \subseteq U_1$ and the other is that J is an isometry. These things follow from Theorem 15.2.4. For $x \in U_1$,

$$\begin{aligned} \|J^*x\|_{U_0}^2 &= (J^*x, J^*x)_{U_0} = (JJ^*x, x)_{U_1} \\ &= (Q_1x, x)_{U_1} = \|Q_1^{1/2}x\|_{U_1}^2 = \|(Q_1^{1/2})^*x\|_{U_1}^2 \end{aligned}$$

and so it follows from Theorem 15.2.4 that $Q_1^{1/2}(U_1) = J(U_0)$. In applying this theorem, the H there is the same as U_1 here and the two U_i are U_0 and U_1 . Also from this theorem

$$\|x\|_{JU_0} \equiv \|J^{-1}x\|_{U_0} = \|Q_1^{-1/2}x\|_{U_1} \equiv \|x\|_{Q_1^{1/2}U_1}$$

for all $x \in Q_1^{1/2}U_1 = JU_0$. The notation is in the sense of Definition 15.2.1 but in this case since J is one to one, J^{-1} has the usual meaning. Now let $y \in U_0$ so $Jy \in Q_1^{1/2}U_1 = JU_0$. Then by Definition 15.2.1

$$\|Jy\|_{Q_1^{1/2}U_1} = \|Jy\|_{JU_0} \equiv \|J^{-1}Jy\|_{U_0} = \|y\|_{U_0}$$

This proves the lemma.

Now let $\{g_k\}_{k=1}^L$ be any orthonormal basis for U_0 and consider $\{Jg_k\}_{k=1}^L$. Since J is an isometry this implies $\{Jg_k\}_{k=1}^L$ is also an orthonormal basis for JU_0 . This is because

$$\begin{aligned} (Jg_k, Jg_l)_{JU_0} &= \frac{1}{4} \left(\|Jg_k + Jg_l\|^2 - \|Jg_k - Jg_l\|^2 \right) \\ &= \frac{1}{4} \left(\|g_k + g_l\|^2 - \|g_k - g_l\|^2 \right) = (g_k, g_l)_{U_0} \end{aligned}$$

Consider

$$W(t) \equiv \sum_{k=1}^{\infty} \psi_k(t) Jg_k$$

It follows from Theorem 52.5.2 this is a Q_1 Wiener process in U_1 . However, it is interesting to review why this is. First of all, why does the series converge? Since the $\{Jg_k\}$ are orthonormal,

$$\int_{\Omega} \left| \sum_{k=m}^n (\psi_k(t) - \psi_k(s)) Jg_k \right|_{U_1}^2 dP = \int_{\Omega} \sum_{k=m}^n (\psi_k(t) - \psi_k(s))^2 \|Jg_k\|_{U_1}^2$$

$$= (t-s) \sum_{k=m}^n \|Jg_k\|_{U_1}^2$$

and as $m, n \rightarrow \infty$ this converges to 0 because of the assumption that J is a Hilbert Schmidt operator. Why is its covariance Q_1 ?

$$\begin{aligned} & E((a, W(t-s))_{U_1} (b, W(t-s))_{U_1}) \\ &= \lim_{n \rightarrow \infty} E \left(\left(a, \sum_{k=1}^n (\psi_k(t) - \psi_k(s)) Jg_k \right) \left(b, \sum_{j=1}^n (\psi_j(t) - \psi_j(s)) Jg_j \right) \right) \\ &= \lim_{n \rightarrow \infty} E \left(\sum_{k=1}^n (a, Jg_k) (\psi_k(t) - \psi_k(s)) \sum_{j=1}^n (b, Jg_j) (\psi_j(t) - \psi_j(s)) \right) \\ &= \lim_{n \rightarrow \infty} (t-s) \sum_{k=1}^n (a, Jg_k) (b, Jg_k) = (t-s) \sum_{k=1}^{\infty} (J^*a, g_k) (J^*b, g_k) \\ &= (t-s) (J^*a, J^*b) = (t-s) (JJ^*a, b) = (t-s) (Q_1a, b) \end{aligned}$$

This shows $W(t)$ is indeed a Q_1 Wiener process.

To get an idea how to define an integral here is a lemma.

Lemma 53.7.3

$$\Phi \in \mathcal{L}_2(Q^{1/2}U, H) = \mathcal{L}_2(U_0, H)$$

if and only if

$$\Phi \circ J^{-1} \in \mathcal{L}_2(Q_1^{1/2}U_1, H)$$

where the norms of $Q_1^{1/2}U_1$ and $Q^{1/2}U$ are in the sense of Theorem 15.2.2. Furthermore,

$$\|\Phi \circ J^{-1}\|_{\mathcal{L}_2(Q_1^{1/2}U_1, H)}^2 = \|\Phi\|_{\mathcal{L}_2(Q^{1/2}U, H)}^2 = \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2$$

Proof: Let $\{g_k\}_{k=1}^L$ be an orthonormal basis for $U_0 \equiv Q^{1/2}U$. What about $\{Jg_k\}$? Is it an orthonormal basis in $JU_0 = Q_1^{1/2}U_1$?

$$(Jg_k, Jg_l)_{JU_0} \equiv (g_k, g_l)_{U_0}$$

from the definition of the inner product in $JU_0 = Q_1^{1/2}U_1$. Thus $\{Jg_k\}$ is orthonormal. Is it complete? Yes because by Lemma 53.7.2 J is an isometry of $U_0 = Q^{1/2}U$

and $Q_1^{1/2}U_1$. It follows that if Jx is an arbitrary thing in $Q_1^{1/2}(U_1)$,

$$\begin{aligned} Jx &= J \left(\sum_{k=1}^L (x, g_k)_{U_0} g_k \right) = \sum_{k=1}^L (x, g_k)_{U_0} Jg_k \\ &= \sum_{k=1}^L (J^{-1}Jx, J^{-1}Jg_k)_{U_0} Jg_k \equiv \sum_{k=1}^L (Jx, Jg_k)_{JU_0} Jg_k \\ &= \sum_{k=1}^L (Jx, Jg_k)_{Q_1^{1/2}U_1} Jg_k \end{aligned}$$

Now

$$\begin{aligned} \|\Phi \circ J^{-1}\|_{\mathcal{L}_2(Q_1^{1/2}U_1, H)}^2 &= \sum_{k=1}^L \|\Phi \circ J^{-1}(Jg_k)\|_H^2 \\ &= \sum_{k=1}^L \|\Phi g_k\|_H^2 = \|\Phi\|_{\mathcal{L}_2(Q^{1/2}U, H)}^2 \end{aligned}$$

This proves the lemma.

With this lemma the stochastic integral can be defined in the case that Q is not of trace class.

Definition 53.7.4 Let U, H be separable real Hilbert spaces and suppose Q is a nonnegative continuous self adjoint operator defined on U . Let $U_0 = Q^{1/2}U$ as in Definition 15.2.1 on Page 427. Also let Φ be $\mathcal{L}_2(U_0, H)$ predictable and satisfy

$$\int_0^T \int_{\Omega} \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dP dt < \infty.$$

Let $J : U_0 \rightarrow U_1$ be a Hilbert Schmidt operator where U_1 is another separable real Hilbert space. Then define

$$W(t) \equiv \sum_{k=1}^L \psi_k(t) Jg_k \quad (53.7.27)$$

where g_k is an orthonormal basis for U_0 and the ψ_k are the independent real Wiener processes of Lemma 52.3.1 so that $W(t)$ is a Wiener process with values in U_1 . Then define the stochastic process $\Phi \cdot W(t)$, having values in H as follows

$$\Phi \cdot W(t) \equiv \int_0^t \Phi dW(s) \equiv \int_0^t \Phi \circ J^{-1} dW(s) \equiv \Phi \circ J^{-1} \cdot W(t)$$

where the last definition is just the stochastic integral defined earlier in terms of the Wiener process $W(t)$. In the case that Φ is $\mathcal{L}_2(U_0, H)$ predictable and it is only assumed

$$P \left(\left[\int_0^T \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt < \infty \right] \right) = 1$$

then by Corollary 53.1.7 $U_0 = Q^{1/2}U$ and by Lemma 53.7.3 $\Phi \circ J^{-1}$ is $\mathcal{L}_2(Q_1^{1/2}U_1, H)$ predictable and in fact

$$P\left(\left[\int_0^T \|\Phi \circ J^{-1}\|_{\mathcal{L}_2(Q_1^{1/2}U_1, H)}^2 dt < \infty\right]\right) = 1$$

so one can still define

$$\Phi \cdot W(t) \equiv \int_0^t \Phi dW(s) \equiv \int_0^t \Phi \circ J^{-1} dW(s) \equiv \Phi \circ J^{-1} \cdot W(t).$$

This definition is independent of J and U_1 . Here is why. Let Φ be an elementary function having values in $\mathcal{L}(U, H)$. Then $\Phi(t_k) \circ J^{-1}$ is also an elementary function with values in $\mathcal{L}(U_1, H)$. From the definition given above and 53.7.27, it follows from Theorem 52.5.2, letting N denote the subsequence there giving uniform convergence of the sum off a set of measure zero that for a.e. ω

$$\begin{aligned} \Phi \cdot W(t) &= \sum_{k=0}^{n-1} \Phi(t_k) \circ J^{-1} \left(\sum_{j=1}^{\infty} (\psi_j(t \wedge t_{k+1}) - \psi_j(t \wedge t_k)) Jg_j \right) \\ &= \sum_{k=0}^{n-1} \Phi(t_k) \circ J^{-1} \lim_{N \rightarrow \infty} \left(\sum_{j=1}^N (\psi_j(t \wedge t_{k+1}) - \psi_j(t \wedge t_k)) Jg_j \right) \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^{n-1} \Phi(t_k) \circ J^{-1} \left(\sum_{j=1}^N (\psi_j(t \wedge t_{k+1}) - \psi_j(t \wedge t_k)) Jg_j \right) \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^{n-1} \Phi(t_k) \left(\sum_{j=1}^N (\psi_j(t \wedge t_{k+1}) - \psi_j(t \wedge t_k)) g_j \right) \end{aligned}$$

which does not depend on J .

It seems the reason for this generalization is to allow the definition of a stochastic integral having values in H in the case where Q is not trace class, including the case where $Q = I$.

The Ito Formula

54.1 The Case Of A Q Wiener Process

First recall 52.5.22 where it is shown that for every α

$$E(|W(t) - W(s)|^\alpha) \leq C|t - s|^{\alpha/2},$$

and so by Kolmogorov Čentsov continuity theorem

$$|W(t) - W(s)| \leq C_\gamma |t - s|^\gamma \quad (54.1.1)$$

for every $\gamma < 1/2$.

Recall it was important that W be a Q Wiener process where Q was nuclear. That is

$$Q = \sum_k \lambda_k e_k \otimes e_k$$

where $\sum_k \lambda_k < \infty$. The Ito formula is the following.

Theorem 54.1.1 *Let Φ be an $\mathcal{L}_2^0$ valued predictable or progressively measurable process which is stochastically integrable in $[0, T]$ because*

$$P\left(\left[\int_0^T \|\Phi\|^2 dt < \infty\right]\right) = 1$$

and let $\phi : [0, T] \times \Omega \rightarrow H$ be predictable or progressively measurable process which is Bochner integrable on $[0, T]$ and let $X(0)$ be $\mathcal{F}_0$ measurable and H valued. Let

$$X(t) \equiv X(0) + \int_0^t \phi(s) ds + \Phi \cdot W(t).$$

Let $F : [0, T] \times H \times \Omega \rightarrow \mathbb{R}^1$ have continuous partial derivatives F_t, F_x, F_{xx} which are uniformly continuous on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$ and let it be progressively measurable. Then the following formula holds for a.e. ω .

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) +$$

$$\int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q)\} ds$$

The dependence of F on ω is suppressed.

To begin with, here is a lemma which has most of the important ideas.

Lemma 54.1.2 *Suppose the simpler case that*

$$X(t) = X(0) + \phi_0 t + \Phi_0 \cdot W(t)$$

where Φ_0 is constant in t and takes finitely many values in $\mathcal{L}(U, H)$. (Recall from Lemma 53.1.9 the restrictions of such operators to U_0 are dense in $\mathcal{L}_2(U_0, H)$.) Then the theorem holds.

Proof: Let $\{t_0, \dots, t_m\}$ be a partition of $[0, t]$ with $|t_i - t_{i-1}| = t/m$. Then

$$\begin{aligned} F(t, X(t)) - F(0, X(0)) &= \sum_{k=0}^{m-1} (F(t_{k+1}, X(t_{k+1})) - F(t_k, X(t_{k+1}))) \\ &\quad + \sum_{k=0}^{m-1} (F(t_k, X(t_{k+1})) - F(t_k, X(t_k))) \end{aligned}$$

which by Taylor's formula and mean value theorem equals

$$\sum_{k=0}^{m-1} F_t(t_k, X(t_{k+1})) \Delta t_k + \sum_{k=0}^{m-1} (F_t(\tilde{t}_k, X(t_{k+1})) - F_t(t_k, X(t_{k+1}))) \Delta t_k + \quad (54.1.2)$$

$$+ \sum_{k=0}^{m-1} F_x(t_k, X(t_k)) \Delta X_k + \frac{1}{2} F_{xx}(t_k, X(t_k)) \Delta X_k (\Delta X_k) + \quad (54.1.3)$$

$$\sum_{k=0}^{m-1} \left(\frac{1}{2} F_{xx}(t_k, \tilde{X}_k) - \frac{1}{2} F_{xx}(t_k, X(t_k)) \right) \Delta X_k (\Delta X_k) \quad (54.1.4)$$

Here $\tilde{X}_k$ is between $X(t_k)$ and $X(t_{k+1})$, similarly for $\tilde{t}_k$. By assumptions on F , the expressions in 54.1.4 and the last term in 54.1.2 both converge to 0 as $m \rightarrow \infty$. It remains to consider the other terms. The first term of 54.1.2 converges to

$$\int_0^t F_t(s, X(s)) ds \quad (54.1.5)$$

thanks to the continuity of X . First consider the first term of 54.1.3.

$$\begin{aligned} \Delta X_k &= \phi_0(t_{k+1} - t_k) + \Phi_0 \cdot W(t_{k+1}) - \Phi_0 \cdot W(t_k) \\ &= \phi_0(t_{k+1} - t_k) + \Phi_0 W(t_{k+1}) - \Phi_0 W(t_k) \end{aligned} \quad (54.1.6)$$

$$\begin{aligned} \sum_{k=0}^{m-1} F_x(t_k, X(t_k)) \Delta X_k &= \sum_{k=0}^{m-1} F_x(t_k, X(t_k)) \phi_0(t_{k+1} - t_k) \\ &+ \sum_{k=0}^{m-1} F_x(t_k, X(t_k)) \Phi_0 W(t_{k+1}) - F_x(t_k, X(t_k)) \Phi_0 W(t_k) \end{aligned}$$

By continuity of X and F_x , this converges to

$$\int_0^t F_x(s, X(s)) \phi(s) ds + F_x(\cdot, X) \Phi_0 \cdot W(t) \quad (54.1.7)$$

where in this case $\phi(s) = \phi_0$, the constant.

It remains to consider the remaining term in 54.1.3. From 54.1.6 this term is

$$\sum_{k=0}^{m-1} \frac{1}{2} F_{xx}(t_k, X(t_k)) \Delta X_k (\Delta X_k) =$$

$$\sum_{k=0}^{m-1} \frac{1}{2} F_{xx}(t_k, X(t_k)) (\phi_0(t_{k+1} - t_k)) (\phi_0(t_{k+1} - t_k)) \quad (54.1.8)$$

$$+ \sum_{k=0}^{m-1} \frac{1}{2} F_{xx}(t_k, X(t_k)) (\Phi_0 \Delta W(t_k)) (\Phi_0 \Delta W(t_k)) \quad (54.1.9)$$

$$+ \sum_{k=0}^{m-1} \frac{1}{2} F_{xx}(t_k, X(t_k)) (\phi_0) (\Phi_0 \Delta W(t_k)) (t_{k+1} - t_k) \quad (54.1.10)$$

$$+ \sum_{k=0}^{m-1} \frac{1}{2} F_{xx}(t_k, X(t_k)) (\Phi_0 \Delta W(t_k)) (\phi_0) (t_{k+1} - t_k) \quad (54.1.11)$$

Of course, by continuity of F_{xx} the two last terms are the same and they each converge to 0 in $L^2(\Omega)$ thanks to 54.1.1. Because of the factor $(t_{k+1} - t_k)^2$ in 54.1.8, this term also converges to 0 as $m \rightarrow \infty$.

It remains to consider the expression in 54.1.9.

$$F_{xx}(t_k, X(t_k)) \in \mathcal{L}(H, \mathcal{L}(H, \mathbb{R}))$$

so

$$F_{xx}(t_k, X(t_k)) (\Phi_0 \Delta W(t_k)) \in \mathcal{L}(H, \mathbb{R})$$

and so recalling Φ_0^* maps H to U ,

$$\begin{aligned} &F_{xx}(t_k, X(t_k)) (\Phi_0 \Delta W(t_k)) (\Phi_0 \Delta W(t_k)) \\ &= (\Phi_0^* (F_{xx}(t_k, X(t_k)) (\Phi_0 \Delta W(t_k))), \Delta W(t_k))_U \\ &= (\Phi_0^* F_{xx}(t_k, X(t_k)) \Phi_0 \Delta W(t_k), \Delta W(t_k))_U \end{aligned}$$

Thus this expression equals

$$\sum_{k=0}^{m-1} \frac{1}{2} (\Phi_0^* F_{xx}(t_k, X(t_k)) \Phi_0 \Delta W(t_k), \Delta W(t_k))_U$$

To save on notation let $\zeta_k \equiv \Phi_0^* F_{xx}(t_k, X(t_k)) \Phi_0$. Thus ζ_k is a self adjoint operator defined on U and the above equals

$$\sum_{k=0}^{m-1} \frac{1}{2} (\zeta_k \Delta W(t_k), \Delta W(t_k))_U \quad (54.1.12)$$

I need to find what this converges to. Let $\{e_j\}$ be the orthonormal basis such that $Qe_i = \lambda_i e_i$. Let $A \in \mathcal{F}_k$.

$$\begin{aligned} \int_A (E((\zeta_k \Delta W(t_k), \Delta W(t_k))_U | \mathcal{F}_k)) dP &= \int_A (\zeta_k \Delta W(t_k), \Delta W(t_k))_U dP \\ &= \int_A \left(\zeta_k \overbrace{\sum_{j=1}^{\infty} (\Delta W(t_k), e_j) e_j}^{\Delta W(t_k)}, \Delta W(t_k) \right) dP \\ &= \int_A \left(\sum_{j=1}^{\infty} (\Delta W(t_k), e_j) (\zeta_k e_j, \Delta W(t_k)) \right) dP \\ &= \sum_{j=1}^{\infty} \int_{\Omega} \mathcal{X}_A((\Delta W(t_k), e_j) (\zeta_k e_j, \Delta W(t_k))) dP \end{aligned}$$

Now by independence of $\mathcal{X}_A$ and $\Delta W(t_k)$ the above equals

$$= P(A) \sum_{j=1}^{\infty} \int_{\Omega} ((\Delta W(t_k), e_j) (\zeta_k e_j, \Delta W(t_k))) dP$$

If ζ_k were not a random variable but was just a constant operator mapping U to U , this would reduce to

$$\begin{aligned} P(A) \sum_{j=1}^{\infty} \Delta t_k (\zeta_k e_j, Qe_j) &= P(A) \Delta t_k \sum_{j=1}^{\infty} (e_j, \zeta_k Qe_j) \\ &= P(A) \Delta t_k \text{trace}(\zeta_k Q). \end{aligned}$$

I claim this is still the case even though ζ_k is a random variable. The reason for this is you can get a sequence of simple functions, $\{s_n\}$ converging uniformly to ζ_k which are of the form

$$\sum_{i=1}^{p_n} c_i^n \mathcal{X}_{A_i^n}(\omega)$$

where A_i^n is measurable with respect to $\mathcal{F}_k$. Thus by independence of the increments in the Wiener process, the functions $\mathcal{X}_{A_i^n}$ and $\Delta W(t_k)$ are independent. Therefore,

$$\begin{aligned}
 & P(A) \sum_{j=1}^{\infty} \int_{\Omega} ((\Delta W(t_k), e_j) (\zeta_k e_j, \Delta W(t_k))) dP \\
 &= \lim_{n \rightarrow \infty} P(A) \sum_{j=1}^{\infty} \int_{\Omega} \left((\Delta W(t_k), e_j) \left(\sum_{i=1}^{p_n} c_i^n \mathcal{X}_{A_i^n} e_j, \Delta W(t_k) \right) \right) dP \\
 &= \lim_{n \rightarrow \infty} P(A) \sum_{j=1}^{\infty} \int_{\Omega} \left(\sum_{i=1}^{p_n} \mathcal{X}_{A_i^n} (\Delta W(t_k), e_j) (c_i^n e_j, \Delta W(t_k)) \right) dP \\
 &= \lim_{n \rightarrow \infty} P(A) \sum_{j=1}^{\infty} \left(\sum_{i=1}^{p_n} \int_{\Omega} \mathcal{X}_{A_i^n} (\Delta W(t_k), e_j) (c_i^n e_j, \Delta W(t_k)) \right) dP \\
 &= \lim_{n \rightarrow \infty} P(A) \sum_{j=1}^{\infty} \sum_{i=1}^{p_n} P(A_i^n) \int_{\Omega} (\Delta W(t_k), e_j) (c_i^n e_j, \Delta W(t_k)) dP \\
 &= \lim_{n \rightarrow \infty} P(A) \Delta t_k \sum_{j=1}^{\infty} \sum_{i=1}^{p_n} P(A_i^n) (Q c_i^n e_j, e_j) \\
 &= \lim_{n \rightarrow \infty} P(A) \Delta t_k \sum_{j=1}^{\infty} \int_{\Omega} (Q s_n e_j, e_j) dP \\
 &= \Delta t_k P(A) \sum_{j=1}^{\infty} \int_{\Omega} (Q \zeta_j e_j, e_j) dP \\
 &= P(A) \Delta t_k \sum_{j=1}^{\infty} (e_j, \zeta_k Q e_j) \\
 &\equiv P(A) \Delta t_k \text{trace}(\zeta_k Q) = \int_A \Delta t_k \text{trace}(\zeta_k Q) dP
 \end{aligned}$$

Now return to 54.1.12,

$$\sum_{k=0}^{m-1} \frac{1}{2} (\zeta_k \Delta W(t_k), \Delta W(t_k))_U$$

I have just shown

$$E \left(\frac{1}{2} (\zeta_k \Delta W(t_k), \Delta W(t_k))_U | \mathcal{F}_{t_k} \right) = \frac{1}{2} \Delta t_k \text{trace}(\zeta_k Q). \quad (54.1.13)$$

Now here is an interesting lemma.

Lemma 54.1.3 Suppose η_j are random variables $E(\eta_j^2) < \infty$, such that η_j is measurable with respect to $\mathcal{G}_k$ for all $k > j$ where $\{\mathcal{G}_k\}$ is increasing. Then

$$\begin{aligned} & E \left(\left[\sum_{k=0}^{m-1} \eta_k - E(\eta_k | \mathcal{G}_k) \right]^2 \right) \\ &= E \left(\sum_{k=0}^{m-1} \eta_k^2 - E(\eta_k | \mathcal{G}_k)^2 \right) \end{aligned} \quad (54.1.14)$$

Proof of the lemma: First consider a mixed term $i < k$.

$$E((\eta_i - E(\eta_i | \mathcal{G}_i))(\eta_k - E(\eta_k | \mathcal{G}_k)))$$

This equals

$$\begin{aligned} & E(\eta_i \eta_k) - E(\eta_i E(\eta_k | \mathcal{G}_k)) - E(\eta_k E(\eta_i | \mathcal{G}_i)) + E(\eta_i | \mathcal{G}_i) E(\eta_k | \mathcal{G}_k) \\ &= E(\eta_i \eta_k) - E(E(\eta_i \eta_k | \mathcal{G}_k)) - E(\eta_k E(\eta_i | \mathcal{G}_i)) + E(\eta_k E(\eta_i | \mathcal{G}_i) | \mathcal{G}_k) \\ &= E(\eta_i \eta_k) - E(E(\eta_i \eta_k | \mathcal{G}_k)) - E(\eta_k E(\eta_i | \mathcal{G}_i)) + E(\eta_k E(\eta_i | \mathcal{G}_i)) = 0. \end{aligned}$$

Thus 54.1.14 equals

$$\sum_{k=0}^{m-1} E((\eta_k - E(\eta_k | \mathcal{G}_k))^2)$$

which equals

$$\begin{aligned} & \sum_{k=0}^{m-1} E(\eta_k^2) - 2E(\eta_k E(\eta_k | \mathcal{G}_k)) + E(E(\eta_k | \mathcal{G}_k)^2) \\ &= \sum_{k=0}^{m-1} E(\eta_k^2) - 2E(E(\eta_k E(\eta_k | \mathcal{G}_k)) | \mathcal{G}_k) + E(E(\eta_k | \mathcal{G}_k)^2) \\ &= \sum_{k=0}^{m-1} E(\eta_k^2) - 2E(E(\eta_k | \mathcal{G}_k) E(\eta_k | \mathcal{G}_k)) + E(E(\eta_k | \mathcal{G}_k)^2) \\ &= \sum_{k=0}^{m-1} E(\eta_k^2) - 2E(E(\eta_k | \mathcal{G}_k)^2) + E(E(\eta_k | \mathcal{G}_k)^2) \\ &= \sum_{k=0}^{m-1} E(\eta_k^2) - E(E(\eta_k | \mathcal{G}_k)^2) \end{aligned}$$

and this proves the lemma.

Now from this lemma and 54.1.13, this implies

$$\begin{aligned} & E \left(\left(\sum_{k=0}^{m-1} \frac{1}{2} (\zeta_k \Delta W(t_k), \Delta W(t_k))_U - \frac{1}{2} \Delta t_k \text{trace}(\zeta_k Q) \right)^2 \right) \\ &= \frac{1}{4} E \left(\sum_{k=0}^{m-1} (\zeta_k \Delta W(t_k), \Delta W(t_k))_U^2 - (\Delta t_k \text{trace}(\zeta_k Q))^2 \right) \\ &\leq C \sum_{k=0}^{m-1} E \left(|\Delta W(t_k)|^4 + \Delta t_k^2 \right) \end{aligned}$$

which converges to 0 as $m \rightarrow \infty$ by 54.1.1. Thus

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=0}^{m-1} \frac{1}{2} F_{xx}(t_k, X(t_k)) (\Phi_0 \Delta W(t_k)) (\Phi_0 \Delta W(t_k)) \\ &= \frac{1}{2} \int_0^t \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q) ds, \end{aligned}$$

the convergence taking place in $L^2(\Omega)$. This proves the lemma.

Now from this lemma, the following is immediate.

Lemma 54.1.4 *Let Φ, ϕ be elementary having values in $\mathcal{L}(U, H)$ and H respectively. Then the conclusion of Theorem 54.1.1 holds.*

Proof: You apply what was just shown on each subinterval.

Lemma 54.1.5 *Now assume Φ is $\mathcal{L}_2^0$ predictable or progressively measurable and ϕ is H predictable or progressively measurable, and*

$$\int_0^T |\phi(s)|_H ds < C, \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt < C \text{ a.e. } \omega \quad (54.1.15)$$

Also letting

$$X(t) \equiv X(0) + \int_0^t \phi(s) ds + \Phi \cdot W(t), \quad (54.1.16)$$

assume that for a.e. ω

$$|X(t, \omega)|_H < C \quad (54.1.17)$$

for all t . Then the conclusion of Theorem 54.1.1 holds.

Proof: By Proposition 53.2.6 there exists a sequence of predictable elementary functions $\{\Phi_n\}$ having values in $\mathcal{L}(U, H)$ and a sequence of predictable elementary functions $\{\phi_n\}$ having values in H such that

$$\int_0^T \int_{\Omega} |\phi_n - \phi| dP ds, \int_0^T \int_{\Omega} \|\Phi_n - \Phi\|_{\mathcal{L}_2^0}^2 dP ds$$

both converge to 0. One can also assume 54.1.15 holds for ϕ_n, Φ_n in place of ϕ and Φ . By taking a further subsequence if needed,

$$\phi_n(s, \omega) \rightarrow \phi(s, \omega), \quad \Phi_n(s, \omega) \rightarrow \Phi(s, \omega)$$

pointwise a.e. Also let $X_n(t)$ satisfy 54.1.16 with ϕ and Φ replaced with ϕ_n and Φ_n respectively.

I need to verify there exists a subsequence still denoted by $\{X_n(t)\}$ such that for a.e. ω this sequence converges uniformly to $X(t)$ in H on $[0, T]$. Recall from Theorem 53.2.8, both $\{\Phi_n \cdot W(t)\}$ and $\{\Phi \cdot W(t)\}$ are martingales. Therefore, from the maximal estimate of Theorem 50.3.5, and the Ito isometry of Theorem 53.2.8

$$\begin{aligned} P \left(\left[\sup_{t \in [0, T]} |\Phi_n \cdot W(t) - \Phi \cdot W(t)| > \alpha \right] \right) &\leq \frac{1}{\alpha} E(|(\Phi_n - \Phi) \cdot W(T)|) \\ &\leq \frac{1}{\alpha} E(|(\Phi_n - \Phi) \cdot W(T)|^2) = \frac{1}{\alpha} \int_0^T \int_{\Omega} \|\Phi_n - \Phi\|_{\mathcal{L}_2^0}^2 dP dt. \end{aligned} \quad (54.1.18)$$

It follows there exists a subsequence, still denoted by n such that

$$P \left(\left[\sup_{t \in [0, T]} |\Phi_n \cdot W(t) - \Phi \cdot W(t)| > 2^{-n} \right] \right) \leq 2^{-n}.$$

(For $\alpha = 2^{-k}$ in 54.1.18, you let n_k be so large that the right side is less than 2^{-k} for every $n \geq n_k$. Then denote n_k by k to save notation.) By the Borel Cantelli lemma, Lemma 47.1.2, the collection of ω which are in infinitely many of the sets

$$\left[\sup_{t \in [0, T]} |\Phi_n \cdot W(t) - \Phi \cdot W(t)| > 2^{-n} \right]$$

has measure 0. Therefore, $\{\Phi_n \cdot W(t)\}$ is uniformly Cauchy with values in H for all ω not in a set of measure 0 and it converges to $\Phi \cdot W(t)$ for these ω . Therefore, for these ω it can be assumed

$$|\Phi_n \cdot W(t)| < D \quad (54.1.19)$$

for some constant D . This is because by 54.1.15 and 54.1.17, there is a set of measure zero such that for ω not in this set,

$$|\Phi \cdot W(t)| < D$$

and now the uniform convergence implies for all n large enough, 54.1.19. Similarly,

$$\begin{aligned} &\overline{P} \left(\left[\sup_{t \in [0, T]} \left| \int_0^t (\phi(s) - \phi_n(s)) ds \right|_H > \alpha \right] \right) \\ &\leq P \left(\left[\int_0^T |\phi(s) - \phi_n(s)|_H ds > \alpha \right] \right) \\ &< \frac{1}{\alpha} \int_{\Omega} \int_0^T |\phi(s) - \phi_n(s)| ds dP. \end{aligned}$$

and so, taking a further subsequence, one can obtain

$$\begin{aligned} & \bar{P} \left(\left[\sup_{t \in [0, T]} \left| \int_0^t (\phi(s) - \phi_{n_k}(s)) ds \right|_H > 2^{-k} \right] \right) \\ & \leq P \left(\left[\int_0^T |\phi(s) - \phi_n(s)|_H ds > 2^{-k} \right] \right) < 2^{-k} \end{aligned}$$

showing that off a possibly larger set of measure 0 it follows that

$$\left\{ \int_0^t \phi_{n_k}(s) ds \right\} \quad (54.1.20)$$

converges uniformly on $[0, T]$ to $\int_0^t \phi(s) ds$ and that $\int_0^T |\phi_{n_k} - \phi|_H ds$ also converges to 0. Thus there exists a set of measure zero such that for ω not in this set, there is a constant D such that for all k large enough

$$\left| \int_0^t \phi_{n_k}(s) ds \right| < D$$

for all t . Similarly, one can obtain a set of measure 0 and a further subsequence such that off this set of measure zero,

$$\int_0^T \|\Phi_{n_k} - \Phi\|_{\mathcal{L}_2^0}^2 ds \quad (54.1.21)$$

converges to 0 and

$$\int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 ds < \infty.$$

Letting

$$X_n(t) \equiv X(0) + \int_0^t \phi_n(s) ds + \Phi_n \cdot W(t),$$

it follows from the above observations, $\{X_{n_k}(t, \omega)\}$ converges uniformly to $X(t, \omega)$ on $[0, T]$ for ω off a set of measure 0 and in fact there is some constant such that for ω not in this exceptional set,

$$|X_n(t, \omega)| < D \quad (54.1.22)$$

for all $t \in T$. For simplicity, refer to this sequence as $\{X_n(t)\}$.

From Lemma 54.1.4, the Ito formula holds for each X_n . That is

$$F(t, X_n(t)) = F(0, X(0)) + F_x(\cdot, X_n(\cdot)) \Phi_n \cdot W(t) +$$

$$\begin{aligned} & \int_0^t \{F_t(s, X_n(s)) + F_x(s, X_n(s))(\phi_n(s)) + \\ & + \frac{1}{2} \text{trace}(\Phi_n(s)^* F_{xx}(s, X_n(s)) \Phi_n(s) Q)\} ds \end{aligned} \quad (54.1.23)$$

The uniform convergence of X_n to X on $[0, T]$ and the dominated convergence theorem along with the fact F_x is uniformly continuous on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$ implies with 54.1.22 that

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \|F_x(\cdot, X_n(\cdot)) \Phi_n - F_x(\cdot, X(\cdot)) \Phi\|_{\mathcal{L}_2^0}^2 dP dt = 0$$

Therefore, the Ito isometry implies

$$F_x(\cdot, X_n(\cdot)) \Phi_n \cdot W(t) \rightarrow F_x(\cdot, X(\cdot)) \Phi \cdot W(t)$$

in $L^2(\Omega)$. It follows then from the usual use of the maximal inequality and the fact these Ito integrals are martingales there exists a set of measure zero such that for ω not in this set, the above convergence is actually uniform in $t \in [0, T]$.

It remains to consider the other terms in 54.1.23. The limit

$$\begin{aligned} & \int_0^t F_t(s, X_n(s)) + F_x(s, X_n(s))(\phi_n(s)) ds \\ & \rightarrow \int_0^t F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) ds \text{ a.e. } \omega \end{aligned}$$

is also routine from the uniform convergence of the various functions above and the boundedness of the $X_n(t, \omega)$ for ω not in a set of measure zero. It only remains to consider the term involving F_{xx} .

$$\int_0^t \frac{1}{2} \text{trace}(\Phi_n(s)^* F_{xx}(s, X_n(s)) \Phi_n(s) Q) ds \quad (54.1.24)$$

and show it converges to

$$\int_0^t \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q) ds. \quad (54.1.25)$$

From the definition of what is meant by trace,

$$\begin{aligned} & \text{trace}(\Phi_n(s)^* F_{xx}(s, X_n(s)) \Phi_n(s) Q) \\ &= \sum_k (F_{xx}(s, X_n(s)) \Phi_n(s) Q e_k, \Phi_n(s) e_k) \\ &= \sum_k \lambda_k (F_{xx}(s, X_n(s)) \Phi_n(s) e_k, \Phi_n(s) e_k) \end{aligned}$$

Similarly,

$$\begin{aligned} & \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q) \\ &= \sum_k \lambda_k (F_{xx}(s, X(s)) \Phi(s) e_k, \Phi(s) e_k) \end{aligned}$$

Since X_n is bounded there exists a constant M from F_{xx} such that

$$|\text{trace}(\Phi_n(s)^* F_{xx}(s, X_n(s)) \Phi_n(s) Q) - \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q)|$$

is dominated by an expression of the form

$$\begin{aligned} &= \sum_k \lambda_k M \|\Phi_n - \Phi\| \|\Phi_n\| \\ &\quad + \sum_k \lambda_k \|F_{xx}(s, X_n(s)) - F_{xx}(s, X(s))\| \|\Phi_n\| \|\Phi\| \end{aligned}$$

It follows the norm of the difference of the integrals of these terms is dominated by an expression of the form

$$\begin{aligned} &\sum_k \lambda_k M \int_0^t \|\Phi_n - \Phi\| \|\Phi_n\| ds \\ &+ \sum_k \lambda_k \int_0^t \|F_{xx}(s, X_n(s)) - F_{xx}(s, X(s))\| \|\Phi_n\| \|\Phi\| ds \end{aligned}$$

From Holder's inequality the first term is dominated by an expression of the form

$$\left(\sum_k \lambda_k M \right) \left(\int_0^t \|\Phi_n - \Phi\|^2 dt \right)^{1/2} \left(\int_0^t \|\Phi_n\|^2 dt \right)^{1/2}$$

and by 54.1.21 this converges to 0. Consider the second term.

$$\begin{aligned} &\int_0^t \|F_{xx}(s, X_n(s)) - F_{xx}(s, X(s))\| \|\Phi_n\| \|\Phi\| ds \\ &\leq \left(\int_0^t (\|F_{xx}(s, X_n(s)) - F_{xx}(s, X(s))\| \|\Phi\|)^2 ds \right)^{1/2} \cdot \left(\int_0^t \|\Phi_n\|^2 ds \right)^{1/2} \end{aligned}$$

The first term in the product converges to 0 by the uniform convergence of $X_n(s)$ to $X(s)$ which implies $\|F_{xx}(s, X_n(s)) - F_{xx}(s, X(s))\| \rightarrow 0$ uniformly. Then this term converges to zero by the dominated convergence theorem and the boundedness of $F_{xx}(s, X_n(s))$ which follows from the boundedness of X_n . The second term in the product is bounded by 54.1.21 which says that

$$\lim_{n \rightarrow \infty} \int_0^T \|\Phi_n - \Phi\|^2 ds = 0.$$

Therefore, one can pass to the limit in 54.1.23 and obtain the Ito formula,

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) +$$

$$\begin{aligned}
& + \int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) \\
& + \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q)\} ds
\end{aligned} \quad (54.1.26)$$

This proves the lemma.

Now with these lemmas, it is easy to prove Theorem 54.1.1, the Ito formula. By now, it is likely one has forgotten what this theorem said. Therefore, here it is stated again.

Theorem 54.1.6 *Let Φ be an $\mathcal{L}_2^0$ valued predictable or progressively measurable process which is stochastically integrable in $[0, T]$ because*

$$P\left(\left[\int_0^T \|\Phi\|^2 dt < \infty\right]\right) = 1$$

and let $\phi : [0, T] \times \Omega \rightarrow H$ be predictable or progressively measurable process which is Bochner integrable on $[0, T]$ and let $X(0)$ be $\mathcal{F}_0$ measurable and H valued. Let

$$X(t) \equiv X(0) + \int_0^t \phi(s) ds + \Phi \cdot W(t).$$

Let $F : [0, T] \times H \times \Omega \rightarrow \mathbb{R}^1$ have continuous partial derivatives F_t, F_x, F_{xx} which are uniformly continuous on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$ and let it be progressively measurable. Then the following formula holds for a.e. ω .

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) +$$

$$\begin{aligned}
& \int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \\
& + \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q)\} ds
\end{aligned}$$

The dependence of F on ω is suppressed.

Proof: Define a stopping time,

$$\tau(\omega) = \inf \left\{ t \geq 0 : |X(t, \omega)|_H > C \text{ or } \int_0^t \|\Phi\|^2 ds > C \text{ or } \int_0^t |\phi| ds > C \right\}$$

Then by the localization lemma, Lemma 53.4.4,

$$\begin{aligned}
X(t \wedge \tau) &= X(0) + \int_0^{t \wedge \tau} \phi(s) ds + \Phi \cdot W(t \wedge \tau) \\
&= X(0) + \int_0^t \mathcal{X}_{[0, \tau]}(s) \phi(s) ds + \mathcal{X}_{[0, \tau]} \Phi \cdot W(t).
\end{aligned}$$

Thus

$$\int_{\Omega} \int_0^T |\mathcal{X}_{[0,\tau]} \phi(s)| ds dP = \int_{\Omega} \int_0^{\tau} |\phi(s)| ds dP \leq CP(\Omega) = C$$

and

$$\int_{\Omega} \int_0^T \|\mathcal{X}_{[0,\tau]} \Phi\|_{\mathcal{L}_2^0}^2 ds dP = \int_{\Omega} \int_0^{\tau} \|\Phi\|_{\mathcal{L}_2^0}^2 ds dP \leq CP(\Omega) = C$$

In addition,

$$|X(t \wedge \tau)|_H \leq C$$

and so, letting

$$\begin{aligned} X_C(t) &\equiv X(t \wedge \tau), \quad \phi_C(t) \equiv \mathcal{X}_{[0,\tau]} \phi(t), \\ \Phi_C(t) &= \mathcal{X}_{[0,\tau]} \Phi(t), \end{aligned}$$

These functions satisfy the conditions of Lemma 54.1.5 and so since $X(0) = X_C(0)$,

$$F(t, X_C(t)) = F(0, X(0)) + F_x(\cdot, X_C(\cdot)) \Phi_C \cdot W(t) +$$

$$\begin{aligned} &\int_0^t \{F_t(s, X_C(s)) + F_x(s, X_C(s))(\phi_C(s)) + \\ &+ \frac{1}{2} \text{trace}(\Phi_C(s)^* F_{xx}(s, X_C(s)) \Phi_C(s) Q)\} ds \end{aligned} \quad (54.1.27)$$

Now by assumption, there is a set of measure zero such that for ω not in this set,

$$\int_0^T \|\Phi\|^2 ds < \infty \text{ and } \int_0^T |\phi| ds < \infty$$

Thus for fixed ω not in this set of measure zero, it follows that for large enough C ,

$$\int_0^T \|\Phi\|^2 ds < C \text{ and } \int_0^T |\phi| ds < C.$$

Also for these ω , $t \rightarrow X(t, \omega)$ is continuous and so eventually, for large enough C , $|X(t, \omega)|_H < C$ for all $t \in [0, T]$. Thus letting $C \rightarrow \infty$, the expression in 54.1.27 implies the desired formula. This proves the theorem.

54.2 The Case Where Q Is Not Trace Class

In this case $Q \in \mathcal{L}(U, U)$ and Q is self adjoint and nonnegative. Then let $U_0 \equiv Q^{1/2}U$ and $J : U_0 \rightarrow U_1$ is a Hilbert Schmidt operator. Then $W(t)$ is a Q_1 Wiener process converging in U_1 where $Q_1 \equiv JJ^*$ and Q_1 is trace class. Suppose as above

$$X(t) = X(0) + \int_0^t \phi(s) ds + \int_0^t \Phi(s) dW(s)$$

where.

$$P \left(\left[\int_0^T \|\Phi\|_{\mathcal{L}_2(U_0, H)}^2 dt < \infty \right] \right) = 1$$

The definition of the last integral is

$$\int_0^t \Phi(s) \circ J^{-1} dW(s)$$

What does the Ito formula say in this setting? According to Theorem 54.1.1 under the conditions stated there for F ,

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \circ J^{-1} \cdot W(t) +$$

$$\begin{aligned} & \int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \\ & + \frac{1}{2} \text{trace} \left((\Phi(s) \circ J^{-1})^* F_{xx}(s, X(s)) \Phi(s) \circ J^{-1} Q_1 \right) \} ds \end{aligned}$$

By definition of the integral, this is the same as

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) +$$

$$\begin{aligned} & \int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \\ & + \frac{1}{2} \text{trace} \left((\Phi(s) \circ J^{-1})^* F_{xx}(s, X(s)) \Phi(s) \circ J^{-1} Q_1 \right) \} ds \end{aligned}$$

Consider that messy term at the end. Letting $Q_1 f_k = \lambda_k f_k$ for $\{f_k\}$ orthonormal in U_1 , it equals

$$\begin{aligned} & \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k (F_{xx}(s, X(s)) \Phi(s) \circ J^{-1} f_k, \Phi(s) \circ J^{-1} f_k) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} (F_{xx}(s, X(s)) \Phi(s) \circ J^{-1} \sqrt{\lambda_k} f_k, \Phi(s) \circ J^{-1} \sqrt{\lambda_k} f_k) \\ &= \frac{1}{2} \sum_{k=1}^{\infty} (F_{xx}(s, X(s)) \Phi(s) \circ J^{-1} Q_1^{1/2} f_k, \Phi(s) \circ J^{-1} Q_1^{1/2} f_k) \end{aligned}$$

Consider F_{xx} . This is in $\mathcal{L}(H, \mathcal{L}(H, \mathbb{R}))$. Suppose it is just multiplication by a scalar function of s called $f(s)$. Then the above reduces right away to

$$\begin{aligned} & \frac{f(s)}{2} \sum_{k=1}^{\infty} (\Phi(s) \circ J^{-1} Q_1^{1/2} f_k, \Phi(s) \circ J^{-1} Q_1^{1/2} f_k) \\ &= \frac{f(s)}{2} \|\Phi(s) \circ J^{-1}\|_{\mathcal{L}_2(Q_1^{1/2} U_1, H)}^2 = \frac{f(s)}{2} \|\Phi(s)\|_{\mathcal{L}_2(U_0, H)}^2 \end{aligned}$$

by Lemma 53.7.3. In particular if one considers the case where $Q = I$, then this term reduces to

$$\frac{f(s)}{2} \|\Phi(s)\|_{\mathcal{L}_2(U,H)}^2$$

which is relatively pleasant to behold since it does not involve U_0 or any other noxious and intricate Banach space. This is summarized in the following theorem.

Theorem 54.2.1 *Let $F : [0, T] \times H \times \Omega \rightarrow \mathbb{R}$ be progressively measurable and have continuous partial derivatives F_t, F_x, F_{xx} which are uniformly continuous on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$. Also suppose Φ is an $\mathcal{L}_2(U_0, H)$ predictable or progressively measurable process which is stochastically integrable in $[0, T]$,*

$$P \left(\left[\int_0^T \|\Phi\| dt < \infty \right] \right)^2 dt = 1,$$

where $U_0 \equiv Q^{1/2}U$ for $Q \in \mathcal{L}(U, U)$ a continuous self adjoint operator $(Qx, x)_U \geq 0$. Also let ϕ be an H valued predictable or progressively measurable process which is Bochner integrable on $[0, T]$ and let $X(0)$ be $\mathcal{F}_0$ measurable and H valued. Let

$$X(t) \equiv X(0) + \int_0^t \phi(s) ds + \int_0^t \Phi(s) dW(s)$$

where $W(t)$ is a Q_1 Wiener process defined on U_1 where $J : U_0 \rightarrow U_1$ is Hilbert Schmidt and $Q_1 \equiv JJ^*$ is trace class. Then one has the following Ito formula.

$$F(t, X(t)) = F(0, X(0)) + \int_0^t F_x(s, X(s)) \Phi(s) dW(s) +$$

$$\begin{aligned} & \int_0^t [F_t(s, X(s)) + F_x(s, X(s)) \phi(s) + \\ & \frac{1}{2} \text{trace} \left((\Phi(s) \circ J^{-1})^* F_{xx}(s, X(s)) \Phi(s) \circ J^{-1} Q_1 \right)] ds \end{aligned}$$

The stochastic integral on the right is defined as

$$\int_0^t F_x(s, X(s)) \Phi(s) \circ J^{-1} dW(s).$$

(The most important observation is that it is a Martingale or submartingale.) In addition to this, in the case where $F_{xx}(s, x)$ reduces to multiplication by a scalar function of s , the above formula simplifies to

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t F_x(s, X(s)) \Phi(s) dW(s) + \\ & \int_0^t \left[F_t(s, X(s)) + F_x(s, X(s)) \phi(s) + \frac{f(s)}{2} \|\Phi(s)\|_{\mathcal{L}_2(U_0,H)}^2 \right] ds \end{aligned}$$

In the special case where $Q = I$ the identity, it simplifies even more to an expression of the form

$$F(t, X(t)) = F(0, X(0)) + \int_0^t F_x(s, X(s)) \Phi(s) dW(s) + \int_0^t \left[F_t(s, X(s)) + F_x(s, X(s)) \phi(s) + \frac{f(s)}{2} \|\Phi(s)\|_{\mathcal{L}_2(U, H)}^2 \right] ds.$$

Stochastic Differential Equations

55.1 Theorems Based On Contraction Mapping

55.1.1 Predictable And Stochastic Continuity

First of all, here is an important observation.

Proposition 55.1.1 *Let $X(t)$ be a stochastic process having values in E a complete metric space and let it be $\mathcal{F}_t$ adapted and left continuous where $\mathcal{F}_t$ is a normal filtration. Then it is predictable. If $t \rightarrow X(t, \omega)$ is continuous for all $\omega \notin N$, $P(N) = 0$, then $(t, \omega) \rightarrow X(t, \omega) \mathcal{X}_{N^c}(\omega)$ is predictable. Also, if $X(t)$ is stochastically continuous and adapted on $[0, T]$, then it has a predictable version. If $X \in C([0, T]; L^p(\Omega; F))$, $p \geq 1$ for F a Banach space, then X is stochastically continuous.*

Proof: First suppose X is continuous for all $\omega \in \Omega$. Define

$$I_{m,k} \equiv ((k-1)2^{-m}T, k2^{-m}T]$$

if $k \geq 1$ and $I_{m,0} = \{0\}$ if $k = 1$. Then define

$$\begin{aligned} X_m(t) &\equiv \sum_{k=1}^{2^m} X((k-1)2^{-m}T) \mathcal{X}_{((k-1)2^{-m}T, k2^{-m}T]}(t) \\ &\quad + X(0) \mathcal{X}_{[0,0]}(t) \end{aligned}$$

Here the sum means that $X_m(t)$ has value $X((k-1)2^{-m}T)$ on the interval

$$((k-1)2^{-m}T, k2^{-m}T].$$

Thus X_m is predictable because each term in the formal sum is. Thus

$$X_m^{-1}(U) = \cup_{k=1}^{2^m} (X((k-1)2^{-m}T) \mathcal{X}_{((k-1)2^{-m}T, k2^{-m}T]}^{-1}(U))$$

$$= \cup_{k=1}^{2^m} ((k-1)2^{-m}T, k2^{-m}T] \times (X((k-1)2^{-m}))^{-1}(U),$$

a finite union of predictable sets. Since X is left continuous,

$$X(t, \omega) = \lim_{m \rightarrow \infty} X_m(t, \omega)$$

and so X is predictable.

Now suppose that for $\omega \notin N$, $P(N) = 0$, $t \rightarrow X(t, \omega)$ is continuous. Then applying the above argument to $X(t) \mathcal{X}_{N^c}$ it follows $X(t) \mathcal{X}_{N^c}$ is predictable by completeness of $\mathcal{F}_t$, $X(t) \mathcal{X}_{N^c}$ is $\mathcal{F}_t$ measurable.

Next consider the other claim. Since X is stochastically continuous on $[0, T]$ it is uniformly stochastically continuous on this interval by Lemma 50.1.1. Therefore, there exists a sequence of partitions of $[0, T]$, the m^{th} being

$$0 = t_{m,0} < t_{m,1} < \cdots < t_{m,n(m)} = T$$

such that for X_m defined as above, then for each t

$$P([d(X_m(t), X(t)) \geq 2^{-m}]) \leq 2^{-m} \tag{55.1.1}$$

Then as above, X_m is predictable. Let A denote those points of $\mathcal{P}_T$ at which $X_m(t, \omega)$ converges. Thus A is a predictable set because it is just the set where $X_m(t, \omega)$ is a Cauchy sequence. Now define the predictable function Y

$$Y(t, \omega) \equiv \begin{cases} \lim_{m \rightarrow \infty} X_m(t, \omega) & \text{if } (t, \omega) \in A \\ 0 & \text{if } (t, \omega) \notin A \end{cases}$$

From 55.1.1 it follows from the Borel Cantelli lemma that for fixed t , the set of ω which are in infinitely many of the sets,

$$[d(X_m(t), X(t)) \geq 2^{-m}]$$

has measure zero. Therefore, for each t , there exists a set of measure zero, $N(t)$ such that for $\omega \notin N(t)$ and all m large enough

$$[d(X_m(t, \omega), X(t, \omega)) < 2^{-m}]$$

Hence for $\omega \notin N(t)$, $(t, \omega) \in A$ and so $X_m(t, \omega) \rightarrow Y(t, \omega)$ which shows

$$d(Y(t, \omega), X(t, \omega)) = 0 \text{ if } \omega \notin N(t).$$

The predictable version of $X(t)$ is $Y(t)$.

Finally consider the claim about the specific example where

$$X \in C([0, T]; L^p(\Omega; F)).$$

$$P(\|X(t) - X(s)\|_F \geq \varepsilon) \varepsilon^p \leq \int_{\Omega} \|X(t) - X(s)\|_F^p dP \leq \varepsilon^p \delta$$

provided $|s - t|$ sufficiently small. Thus

$$P(\|X(t) - X(s)\|_F \geq \varepsilon) < \delta$$

when $|s - t|$ is small enough. This proves the proposition.

Definition 55.1.2 Let U, H be real separable Hilbert spaces and let Q be a self adjoint nonnegative operator and

$$U_0 \equiv Q^{1/2}U$$

Let

$$\sigma : [0, T] \times H \times \Omega \rightarrow \mathcal{L}_2(U_0, H) \quad (55.1.2)$$

satisfy

$$\|\sigma(t, x_2, \omega) - \sigma(t, x_1, \omega)\|_{\mathcal{L}_2(U_0, H)} \leq K \|x_1 - x_2\|_H \quad (55.1.3)$$

$$(s, \omega) \rightarrow \|\sigma(s, 0, \omega)\|_{\mathcal{L}_2(U_0, H)} \in L^2([0, T] \times \Omega) \quad (55.1.4)$$

$$\sigma \text{ is progressively measurable} \quad (55.1.5)$$

Recall from Proposition 53.3.6 this means that the restriction of σ to $[0, t] \times H \times \Omega$ is $\mathcal{B}([0, t]) \times \mathcal{B}(H) \times \mathcal{F}_t$ measurable. Let $\mathcal{F}_t$ be a normal filtration corresponding to a Wiener process $W(t)$ defined on U_1 where $J : U_0 \rightarrow U_1$ is a Hilbert Schmidt operator. Also let $C([0, T]; L^2(\Omega, U))$ denote the space of continuous functions defined on $[0, T]$ having values in $L^2(\Omega, H)$. Then let $V_{\mathcal{F}}$ denote those functions X in $C([0, T]; L^2(\Omega, H))$ with the property that $X(t)$ is $\mathcal{F}_t$ measurable. ($X(t)$ is adapted.)

I will often suppress the dependence of σ on ω in what follows.

Lemma 55.1.3 Let σ be given in the above definition and let $X \in V_{\mathcal{F}}$. Then

$$(t, \omega) \rightarrow \sigma(t, X_1(t, \omega), \omega)$$

is progressively measurable where X_1 is a predictable version of X and

$$\int_0^T \int_{\Omega} \|\sigma(t, X_1(t, \omega), \omega)\|_{\mathcal{L}_2(U_0, H)}^2 dP dt < \infty$$

Proof: Since $X \in V_{\mathcal{F}}$ it is continuous and $\mathcal{F}_t$ adapted, it has a predictable version X_1 . Therefore, by Proposition 53.3.6 the first claim follows right away.

Next I need to verify the claim about the integral. However,

$$\begin{aligned} & \int_0^T \int_{\Omega} \|\sigma(t, X_1(t, \omega))\|_{\mathcal{L}_2(U_0, H)}^2 dP dt \\ & \leq 2 \int_0^T \int_{\Omega} \left(\|\sigma(t, X_1(t, \omega)) - \sigma(t, 0)\|_{\mathcal{L}_2(U_0, H)}^2 + \|\sigma(t, 0)\|_{\mathcal{L}_2(U_0, H)}^2 \right) dP dt \\ & \leq 2K^2 \int_0^T \int_{\Omega} \|X_1(t, \omega)\|_H^2 dP dt + 2 \int_0^T \|\sigma(t, 0)\|_{\mathcal{L}_2(U_0, H)}^2 dt < \infty. \end{aligned}$$

This proves the lemma.

It follows from this lemma that for $\sigma, V_{\mathcal{F}}$ given in Definition 55.1.2 and $X \in V_{\mathcal{F}}$, one can define the stochastic integral

$$\int_0^t \sigma(s, X_1(s)) dW(s)$$

The following will be assumed on $b(t, x, \omega)$.

$$|b(t, x_2, \omega) - b(t, x_1, \omega)|_H \leq K |x_1 - x_2|_H \quad (55.1.6)$$

$$(s, \omega) \rightarrow |b(s, 0, \omega)|_H \in L^2([0, T] \times \Omega) \quad (55.1.7)$$

$$b : [0, T] \times H \times \Omega \rightarrow H \text{ is progressively measurable.} \quad (55.1.8)$$

Now here is a useful function space described in the following lemma.

Lemma 55.1.4 *Let $C([0, T]; L^2(\Omega; H))$ denote the space of continuous functions with values in $L^2(\Omega; H)$ and for each $\lambda \geq 0$, denote for $X \in C([0, T]; L^2(\Omega; H))$*

$$\|X\|_{\lambda} \equiv \max \left\{ e^{-\lambda t} \|X(t)\|_{L^2(\Omega; H)} : t \in [0, T] \right\}$$

Then all these norms are equivalent and $(C([0, T]; L^2(\Omega; H)), \|\cdot\|_{\lambda})$ is a Banach space. If $V_{\mathcal{F}}$ denotes those functions in $C([0, T]; L^2(\Omega; H))$ which are adapted to $\mathcal{F}_t$, a normal filtration, then $V_{\mathcal{F}}$ is a closed subspace of

$$C([0, T]; L^2(\Omega; H)).$$

Proof: The assertion about the norms and the Banach space are all obvious.

It only remains to consider the last claim about $V_{\mathcal{F}}$. Suppose $\{X_n\}_{n=1}^{\infty} \subseteq V_{\mathcal{F}}$ and $\|X_n - Y\|_{\lambda} \rightarrow 0$. I need to verify Y is $\mathcal{F}_t$ adapted. However, taking a subsequence, still called n and letting $\lambda = 0$ for convenience, it can be assumed

$$\|X_n - Y\|_0 < 2^{-n}$$

and so it follows for each $t \in [0, T]$, $X_n(t)(\omega) \rightarrow Y(t)(\omega)$ a.e. ω . It follows since $(\Omega, \mathcal{F}_t, P)$ is complete, $Y(t)$ is $\mathcal{F}_t$ measurable for each t . This proves the lemma.

Lemma 55.1.5 *Let X_1 be a predictable version of $X \in V_{\mathcal{F}}$ as above and let b be as above. Then there exists a set of measure 0 N such that for $\omega \notin N$,*

$$t \rightarrow \int_0^t b(s, X_1(s, \omega), \omega) ds$$

is continuous with values in H and adapted. The above stochastic process

$$F(t)(\omega) \equiv \int_0^t b(s, X_1(s, \omega)) ds$$

is $\mathcal{F}_t$ adapted and

$$F \in C([0, T]; L^2(\Omega; H)).$$

Thus $F \in V_{\mathcal{F}}$.

Proof: First note that $(s, \omega) \rightarrow b(s, X_1(s, \omega), \omega)$ is progressively measurable because X_1 is and b is progressively measurable which implies this follows from Proposition 53.3.6. That $F(t)$ given above is adapted follows easily from Proposition ?? and Pettis theorem applied to positive and negative parts of $(b(s, X_1(s, \omega)), h)_H$. It only remains to verify $F \in C([0, T]; L^2(\Omega; H))$.

Let $s < t$.

$$\begin{aligned} \|F(t) - F(s)\|_{L^2(\Omega; H)}^2 &= \int_{\Omega} \left| \int_s^t b(r, X_1(r, \omega), \omega) dr \right|_H^2 dP \\ &\leq (t-s) \int_{\Omega} \int_s^t |b(r, X_1(r, \omega), \omega)|^2 dr dP \\ &\leq (t-s) \int_0^T \int_{\Omega} |b(r, X_1(r, \omega), \omega)|^2 dP dr \end{aligned} \quad (55.1.9)$$

and from the above assumptions on b ,

$$\begin{aligned} \int_0^T \int_{\Omega} |b(r, X_1(r, \omega), \omega)|^2 dP dr &\leq 2 \int_0^T \int_{\Omega} |b(r, X_1(r, \omega)) - b(s, 0)|^2 dP dr \\ + 2 \int_0^T \int_{\Omega} |b(r, 0, \omega)|^2 dP dr &< \infty \end{aligned}$$

Thus the last term in 55.1.9 converges to 0 as $t \rightarrow s$ or as $s \rightarrow t$ and so $F \in \mathcal{V}_{\mathcal{F}}$. This proves the lemma.

55.1.2 Existence And Uniqueness For A Stochastic Equation

With this preparation, it is time to prove an existence and uniqueness theorem for the solution to the stochastic differential equation. This is described in the following definition.

Definition 55.1.6 *The solution X of the stochastic differential equation*

$$dX = b(t, X(t), \omega) dt + \sigma(t, X(t), \omega) dW, \quad X(0) = X_0$$

where X_0 is $\mathcal{F}_0$ measurable, will be an $\mathcal{F}_t$ adapted stochastic process $X(t)$ which satisfies the following integral equation for all $\omega \notin N$ where $P(N) = 0$.

$$X(t, \omega) = X_0 + \int_0^t b(s, X(s, \omega), \omega) ds + \int_0^t \sigma(s, X(s)) dW(s)(\omega)$$

First note that if X_1 and X_2 are predictable versions of a stochastic process X , then the integral on the right with X replaced with X_i in the above integral

equation will be the same. This follows from the Ito isometry and the assumed properties of σ .

$$\begin{aligned} & E \left(\left(\int_0^t \sigma(s, X_1(s)) dW(s) - \int_0^t \sigma(s, X_2(s)) dW(s) \right)^2 \right) \\ &= \int_0^t \int_{\Omega} \|\sigma(s, X_1(s)) - \sigma(s, X_2(s))\|^2 dP ds \\ &\leq K^2 \int_0^t \int_{\Omega} |X_1(s) - X_2(s)|_H^2 dP ds = 0 \end{aligned}$$

Also note that

$$\begin{aligned} & E \left(\left(\int_0^t b(s, X_1(s)) ds - \int_0^t b(s, X_2(s)) ds \right)^2 \right) \\ &= E \left(\left(\int_0^t (b(s, X_1(s)) - b(s, X_2(s))) ds \right)^2 \right) \\ &\leq T \int_{\Omega} \int_0^t |b(s, X_1(s)) - b(s, X_2(s))|_H^2 ds dP \\ &\leq TK^2 \int_0^t \int_{\Omega} |X_1(s) - X_2(s)|_H^2 dP ds = 0, \end{aligned}$$

the interchange of integrals justified by the fact $(s, \omega) \rightarrow b(s, X_i(s, \omega), \omega)$ is product measurable and so the deterministic integral involving b is also unchanged with a change in predictable version of $X(t)$.

Lemma 55.1.7 *Let $X_0 \in L^2(\Omega; H)$ and X_0 is $\mathcal{F}_0$ measurable. Then there exists a solution $X \in V_{\mathcal{F}}$ to the integral equation*

$$X(t) = X_0 + \int_0^t b(s, X_1(s)) ds + \int_0^t \sigma(s, X_1(s)) dW(s).$$

where X_1 is a predictable version of X .

Proof: First of all, if $X \in V_{\mathcal{F}}$ there exists a predictable version of $X(t)$ because it is adapted and stochastically continuous by Proposition 55.1.1. Using Lemma 55.1.5, define a mapping $\Psi : V_{\mathcal{F}} \rightarrow V_{\mathcal{F}}$ given by

$$\Psi X(t)(\omega) \equiv X_0 + \int_0^t b(s, X_1(s, \omega)) ds + \int_0^t \sigma(s, X_1(s)) dW(s)(\omega)$$

where X_1 is a predictable version of X . On the exceptional set N where the right side is not a continuous function of t , let $\Psi X(t) \equiv 0$. By Lemma 55.1.5 and properties of the stochastic integral this does define a mapping of $V_{\mathcal{F}}$ to $V_{\mathcal{F}}$. Of course two different predictable versions for $X \in V_{\mathcal{F}}$ could yield different exceptional sets of

measure zero but from the above simple observations before the lemma, Ψ of the two different predictable versions would be the same in $V_{\mathcal{F}}$. Therefore, this is a well defined map. I will show that for λ large enough Ψ is a contraction map. Let $X, Y \in V_{\mathcal{F}}$. Then using the Ito isometry and properties of σ listed in 55.1.2 - 55.1.4,

$$\begin{aligned}
 & \left\| e^{-\lambda t} \left(\int_0^t \sigma(s, X_1(s)) dW(s) - \int_0^t \sigma(s, Y_1(s)) dW(s) \right) \right\|_{L^2(\Omega; H)}^2 \\
 &= e^{-2\lambda t} \int_0^t \int_{\Omega} \|\sigma(s, X_1(s)) - \sigma(s, Y_1(s))\|_{\mathcal{L}_2(U_0, H)}^2 dP ds \\
 &\leq K^2 e^{-2\lambda t} \int_0^t \int_{\Omega} \|X_1(s) - Y_1(s)\|_H^2 dP ds \\
 &= K^2 e^{-2\lambda t} \int_0^t e^{2\lambda s} \left(e^{-2\lambda s} \|X(s) - Y(s)\|_{L^2(\Omega; H)}^2 \right) ds \\
 &\leq K^2 \int_0^t e^{-2\lambda(t-s)} ds \|X - Y\|_{\lambda}^2 \leq \frac{K^2}{\lambda} \|X - Y\|_{\lambda}^2
 \end{aligned}$$

Therefore,

$$\left\| \left(\int_0^{(\cdot)} \sigma(s, X_1(s)) dW(s) - \int_0^{(\cdot)} \sigma(s, Y_1(s)) dW(s) \right) \right\|_{\lambda} \leq \frac{K}{\sqrt{\lambda}} \|X - Y\|_{\lambda}$$

Now consider the term involving b .

$$\begin{aligned}
 & \left\| e^{-\lambda t} \left(\int_0^t b(s, X_1(s)) ds - \int_0^t b(s, Y_1(s)) ds \right) \right\|_{L^2(\Omega; H)}^2 \\
 &= e^{-2\lambda t} \left\| \left(\int_0^t b(s, X_1(s)) ds - \int_0^t b(s, Y_1(s)) ds \right) \right\|_{L^2(\Omega; H)}^2 \\
 &\leq e^{-2\lambda t} T K^2 \int_0^t \|X_1(s) - Y_1(s)\|_{L^2(\Omega; H)}^2 ds \\
 &= e^{-2\lambda t} T K^2 \int_0^t e^{2\lambda s} e^{-2\lambda s} \|X_1(s) - Y_1(s)\|_{L^2(\Omega; H)}^2 ds \\
 &\leq T K^2 \int_0^t e^{-2\lambda(t-s)} ds \|X - Y\|_{\lambda}^2 \leq \frac{T K^2}{\lambda} \|X - Y\|_{\lambda}^2
 \end{aligned}$$

and so

$$\left\| \left(\int_0^{(\cdot)} b(s, X_1(s)) ds - \int_0^{(\cdot)} b(s, Y_1(s)) ds \right) \right\|_{\lambda} \leq \frac{\sqrt{T} K}{\sqrt{\lambda}} \|X - Y\|_{\lambda}$$

Thus

$$\|\Psi X - \Psi Y\|_\lambda \leq \left(\frac{K}{\sqrt{\lambda}} + \frac{\sqrt{TK}}{\sqrt{\lambda}} \right) \|X - Y\|_\lambda$$

which implies if λ is large enough, then Ψ is a contraction map. It follows there exists a unique fixed point for Ψ . This proves the lemma.

Note that in the above argument $X_1 = X$ in $V_{\mathcal{F}}$ because for each t these functions are equal a.e. This is why I can speak of a fixed point of Ψ even though the two sides don't look the same.

Now for X the unique solution to Lemma 55.1.7, $X_1(t) = X(t)$ a.e. for each t because X_1 is defined to be a version of X . Of course the exceptional set may depend on t . Now define for $\omega \notin N$, the exceptional set of probability 0 where the right side involving the integrals is not a continuous function of t ,

$$\begin{aligned} Y(t)(\omega) &\equiv X_0(\omega) + \int_0^t b(s, X_1(s, \omega), \omega) ds \\ &\quad + \mathcal{X}_{N^c}(\omega) \int_0^t \sigma(s, X_1(s)) dW(s)(\omega) \\ Y(t)(\omega) &\equiv 0 \text{ if } \omega \in N. \end{aligned} \tag{55.1.10}$$

Here N is such that for $\omega \notin N$, both the integral terms are continuous. Therefore, $Y(t)(\omega) = X(t)(\omega) = X_1(t)(\omega)$ a.e. ω . As pointed out above, off a set of measure 0 the nonstochastic term is continuous and so it follows from Proposition 55.1.1 and the assumption $\mathcal{F}_t$ is normal that this term is predictable. The stochastic integral term is also predictable thanks to Proposition 55.1.1 and the continuity in t of this term for ω not in a set of measure zero so it is also predictable. Therefore, Y is predictable. Hence by completeness of the $\mathcal{F}_t$, the right side of 55.1.10 is unchanged if X_1 is replaced with Y . Therefore, for $\omega \notin N$

$$Y(t)(\omega) \equiv X_0(\omega) + \int_0^t b(s, Y(s, \omega), \omega) ds + \int_0^t \sigma(s, Y(s)) dW(s)(\omega)$$

This proves the existence part of the following theorem which is the main result.

Theorem 55.1.8 *Let U, H be separable real Hilbert spaces and let Q be a nonnegative self adjoint operator defined on U . Let $U_0 \equiv Q^{1/2}U$ where $Q^{1/2}$ is defined in Theorem 15.10.4 with the inner product and norm given by Definition 15.2.1. Also let U_1 be another separable Hilbert space and $J : U_0 \rightarrow U_1$ be a Hilbert Schmidt operator. Let $W(t)$ be a Wiener process defined on U_1 as in Definition 53.7.4 with the normal filtration $\mathcal{F}_t$ defined as in Lemma 52.4.2. For $\Phi \in N_W^2(0, T; \mathcal{L}_2(U_0, H))$ define $\Phi \cdot W(t) \equiv \int_0^t \Phi dW$ also according to Definition 15.2.1. Let b satisfy 55.1.6 - 55.1.8 and let σ satisfy 55.1.2 - 55.1.5. Also let $X_0 \in L^2(\Omega; H)$ with X_0 being $\mathcal{F}_0$ measurable. Then there exists a set of probability zero, N and a stochastic process $Y(t)$ such that for $\omega \notin N, t \rightarrow Y(t, \omega)$ is continuous and*

$$Y(t)(\omega) \equiv X_0(\omega) + \int_0^t b(s, Y(s, \omega), \omega) ds + \int_0^t \sigma(s, Y(s)) dW(s)(\omega)$$

This stochastic process is unique in the sense that if $Z(t)$ is another such stochastic process, then $Z(t, \omega) = Y(t, \omega)$ for a.e. ω .

Proof: It only remains to verify the uniqueness assertion. By Proposition 55.1.1 and letting N denote the union of the two exceptional sets, both $Y\mathcal{X}_{N^C}$ and $Z\mathcal{X}_{N^C}$ are predictable. Therefore, the following manipulations involving the integrals are all justified. For $\omega \notin N$

$$\begin{aligned} |Y(t, \omega) - Z(t, \omega)|_H^2 &\leq 2 \left(T \int_0^t |b(s, Y(s, \omega), \omega) - b(s, Z(s, \omega), \omega)|_H^2 ds + \right. \\ &\quad \left. \left| \int_0^t (\sigma(s, Y(s)) - \sigma(s, Z(s))) dW(s) \right|_H^2 \right) \end{aligned}$$

and so, taking the integral of both sides over N^C and using the Ito isometry and the isometry of Lemma 53.7.3 along with the properties of b and σ this yields

$$\begin{aligned} \int_{N^C} |Y(t) - Z(t)|_H^2 dP &\leq 2T \int_{N^C} \int_0^t |b(s, Y(s, \omega)) - b(s, Z(s, \omega))|_H^2 ds dP \\ &\quad + 2 \int_{N^C} \left| \int_0^t (\sigma(s, Y(s)) - \sigma(s, Z(s))) dW(s) \right|_H^2 dP \\ &= 2T \int_{N^C} \int_0^t |b(s, Y(s, \omega)) - b(s, Z(s, \omega))|_H^2 ds dP \\ &\quad + 2 \int_0^t \int_{N^C} \|\sigma(s, Y(s)) - \sigma(s, Z(s))\|_{\mathcal{L}_2(U_0, H)}^2 dP ds \\ &\leq 2TK^2 \int_0^t \int_{N^C} |Y(s) - Z(s)|_H^2 dP ds + 2 \int_0^t \int_{N^C} |Y(s) - Z(s)|_H^2 dP ds \end{aligned}$$

By Gronwall's inequality this proves the uniqueness assertion. This proves the theorem.

55.2 Gelfand Triples

Let H be a separable real Hilbert space and let $V \subseteq H$ be a separable Banach space which is embedded continuously into H and which is also dense in H . Then identifying H and H' you can write

$$V \subseteq H = H' \subseteq V'.$$

This is called a Gelfand triple. If V is reflexive, you could conclude separability of V from the separability of H . However, if V is not reflexive, this might not happen. For example, you could take $V = L^\infty(0, 1)$ and $H = L^2(0, 1)$.

Proposition 55.2.1 *Suppose V is reflexive and a subset of H a separable Hilbert space with the inclusion map continuous. Suppose also that V is dense in H . Then identifying H and H' , it follows that H is dense in V' and V is separable.*

Proof: If H is not dense in V' , then by the Hahn Banach theorem, there exists $\phi^{**} \in V''$ such that $\phi^{**}(H) = 0$ but $\phi^{**}(\phi^*) \neq 0$ for some $\phi^* \in V' \setminus \overline{H}$. Since V is reflexive there exists $v \in V$ such that $\phi^{**} = Jv$ for J the standard mapping from V to V'' . Thus

$$\phi^{**}(h) \equiv (v, h)_H = 0$$

for all $h \in H$. Therefore, $v = 0$ and so $Jv = 0 = \phi^{**}$ which contradicts $\phi^{**}(\phi^*) \neq 0$. Therefore, H is dense in V' . Now by Theorem 23.1.13 which says separability of the dual space implies separability of the space, it follows V is separable as claimed. This proves the proposition.

From now on it is assumed V and V' are both separable and that H is dense in V' . This is summarized in the following definition.

Definition 55.2.2 *V, H, V' will be called a Gelfand triple if V, V' are separable, $V \subseteq H$ with the inclusion map continuous, $H = H'$, and $H = H'$ is dense in V' .*

What about the Borel sets on V and H ?

Proposition 55.2.3 *Denote by $\mathcal{B}(X)$ the Borel sets of X where X is any separable Banach space. Then*

$$\mathcal{B}(X) = \sigma(X').$$

Here $\sigma(X')$ is the smallest σ algebra such that each $\phi \in X'$ is measurable. Also in the context of the above definition, $\mathcal{B}(V) = \sigma(i^*H')$ because H' is dense in V' . Here i^* is the restriction to V so that $i^*h(v) \equiv h(v) \equiv (h, v)_H$ for all $v \in V$ and $\sigma(i^*H')$ denotes the smallest σ algebra such that i^*h is measurable for each $h \in H'$.

Proof: By Lemma 23.1.4 there exists a countable subset of the unit ball in X'

$$\{\phi_n\}_{n=1}^\infty = D'$$

such that

$$\|v\|_X = \sup \{|\phi(v)| : \phi \in D'\}.$$

Consider a closed ball $\overline{B}(v_0, r)$ in X . This equals

$$\left\{ v \in X : \sup_n |\phi_n(v) - \phi_n(v_0)| \leq r \right\} = \cap_{n=1}^\infty \phi_n^{-1} \left(\overline{B}(\phi_n(v_0), r) \right)$$

and this last set is in $\sigma(D')$. Therefore, every closed ball is in $\sigma(D')$ which implies every open ball is also in $\sigma(D')$ since open balls are the countable union of closed balls. Since X is separable, it follows every open set is the countable union of balls and so every open set is in $\sigma(D')$. It follows $\mathcal{B}(X) \subseteq \sigma(D') \subseteq \sigma(X')$. On the other hand, every $\phi \in X'$ is continuous and so it is Borel measurable. Hence $\sigma(X') \subseteq \mathcal{B}(X)$.

Now consider the last claim. From Lemma 23.1.4 and density of $H' = H$ in V' , it can be assumed $D' \subseteq H = H'$. Therefore, from the first part of the argument

$$\mathcal{B}(V) \subseteq \sigma(D') \subseteq \sigma(i^*H')$$

Also each i^*h is continuous on V so in fact, equality holds in the above because $\sigma(i^*H') \subseteq \mathcal{B}(V)$. This proves the proposition.

Next I want to verify that V is in $\mathcal{B}(H)$. This will be true if V is reflexive. More generally, here is an interesting result.

Proposition 55.2.4 *Let $X \subseteq Y$, X dense in Y and suppose X, Y are Banach spaces and that X is reflexive. Then $X \in \mathcal{B}(Y)$.*

Proof: Define the functional

$$\phi(x) \equiv \begin{cases} \|x\|_X & \text{if } x \in X \\ \infty & \text{if } x \in Y \setminus X \end{cases}$$

Then ϕ is lower semicontinuous on Y . Here is why. Suppose $(x, a) \notin \text{epi}(\phi)$ so that $a < \phi(x)$. I need to verify this situation persists for (x, b) near (x, a) . If this is not so, there exists $x_n \rightarrow x$ and $a_n \rightarrow a$ such that $a_n \geq \phi(x_n)$. If $\liminf_{n \rightarrow \infty} \phi(x_n) < \infty$, then there exists a subsequence still denoted by n such that $\|x_n\|_X$ is bounded. Then by the Eberlein Smulian theorem, there exists a further subsequence such that x_n converges weakly in X to some z . Now since X is dense in Y it follows Y' can be considered a subspace of X' and so for $f \in Y'$

$$f(x_n) \rightarrow f(z), \quad f(x_n) \rightarrow f(x)$$

and so $f(z - x) = 0$ for all $f \in Y'$ which requires $z = x$. Now $x \rightarrow \|x\|_X$ is convex and lower semicontinuous on X so it follows from Corollary 14.2.10

$$a = \lim_{n \rightarrow \infty} \inf a_n \geq \lim_{n \rightarrow \infty} \inf \phi(x_n) \geq \phi(x) > a$$

which is a contradiction. If $\liminf_{n \rightarrow \infty} \phi(x_n) = \infty$, then

$$\infty > a = \lim_{n \rightarrow \infty} \inf a_n = \infty$$

another contradiction. Therefore, $\text{epi}(\phi)$ is closed and so ϕ is lower semicontinuous as claimed. Therefore,

$$X = Y \setminus (\cap_{n=1}^{\infty} \phi^{-1}((n, \infty)))$$

and since ϕ is lower semicontinuous, each $\phi^{-1}((n, \infty))$ is open. Hence X is a Borel subset of Y . This proves the proposition.

55.3 An Ito Formula

This Ito formula seems to be the fundamental idea which allows one to obtain solutions to stochastic partial differential equations using a variational point of view.

I am following the treatment found in [55]. The following lemma is fundamental to the presentation. It approximates a function with a sequence of two step functions X_k^r, X_k^l where X_k^r has the value of X at the right end of each interval and X_k^l gives the value X at the left end of the interval. The lemma is very interesting for its own sake. You can obviously do this sort of thing for a continuous function but here the function is not continuous and in addition, it is a stochastic process depending on ω also.

Lemma 55.3.1 *Let $X : [0, T] \times \Omega \rightarrow V$, be $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable and suppose*

$$X \in K \equiv L^p([0, T] \times \Omega; V).$$

Then there exists a sequence of nested partitions, $\mathcal{P}_k \subseteq \mathcal{P}_{k+1}$,

$$\mathcal{P}_k \equiv \{t_0^k, \dots, t_{m_k}^k\}$$

such that the step functions given by

$$\begin{aligned} X_k^r(t) &\equiv \sum_{j=1}^{m_k} X(t_j^k) \mathcal{X}_{[t_{j-1}^k, t_j^k)}(t) \\ X_k^l(t) &\equiv \sum_{j=1}^{m_k} X(t_{j-1}^k) \mathcal{X}_{[t_{j-1}^k, t_j^k)}(t) \end{aligned}$$

both converge to X in K as $k \rightarrow \infty$ and

$$\lim_{k \rightarrow \infty} \max \{|t_j^k - t_{j+1}^k| : j \in \{0, \dots, m_k\}\} = 0$$

Proof: For $t \in \mathbb{R}$ let $\gamma_n(t) \equiv k/2^n$ and let $\delta_n(t) \equiv (k+1)/2^n$ where $t \in [k/2^n, (k+1)/2^n)$. Also suppose X is defined to equal 0 on $[0, T]^C \times \Omega$. There exists a set of measure zero N such that for $\omega \notin N, t \rightarrow X(t, \omega)$ is in $L^p(\mathbb{R})$. Therefore by continuity of translation, as $n \rightarrow \infty$ it follows that for $\omega \notin N$, and $t \in [0, T]$,

$$\int_{\mathbb{R}} \|X(\gamma_n(t) + s) - X(t + s)\|_V^p ds \rightarrow 0$$

The above is dominated by

$$\int_{\mathbb{R}} 2^{p-1} (\|X(s)\|^p + \|X(s)\|^p) \mathcal{X}_{[-2T, 2T]}(s) ds < \infty$$

Consider

$$\int_{\Omega} \int_0^T \left(\int_{\mathbb{R}} \|X(\gamma_n(t) + s) - X(t + s)\|_V^p ds \right) dt dP$$

By the dominated convergence theorem, this converges to 0 as $n \rightarrow \infty$. Now Fubini. This yields

$$\int_{\Omega} \int_0^T \left(\int_{\mathbb{R}} \|X(\gamma_n(t) + s) - X(t + s)\|_V^p ds \right) dt dP$$

Next change the variable in the integrand.

$$= \int_{\Omega} \int_0^T \left(\int_{\mathbb{R}} \|X(\gamma_n(t-s) + s) - X(t)\|_V^p ds \right) dt dP$$

which converges to 0 as $n \rightarrow \infty$. Therefore,

$$\int_{\Omega} \int_0^T \left(\int_0^T \|X(\gamma_n(t-s) + s) - X(t)\|_V^p ds \right) dt dP \rightarrow 0$$

also. Now interchange the order of integration again to obtain

$$\int_0^T \left(\int_{\Omega} \int_0^T \|X(\gamma_n(t-s) + s) - X(t)\|_V^p dt dP \right) ds \rightarrow 0$$

At this point note that $t \rightarrow X(\gamma_n(t-s) + s)$ is a step function of the form $X_n^l(t)$ for each $s \in [0, T]$ and $t \rightarrow X(\delta_n(t-s) + s)$ is a step function of the form $X_n^r(t)$. This is because $\gamma_n(t-s) + s \leq t$ and for fixed s , $\gamma_n(t-s) + s$ is constant on small intervals of the form $[a, b]$ where $b - a = 2^{-n}$ while $\delta_n(t-s) + s > t$ and is similarly constant on small intervals. Also $\gamma_n(t-s) + s \geq -2^{-n}$ for every $t \in [0, T]$. It follows the function

$$s \rightarrow \int_{\Omega} \int_0^T \|X(\gamma_n(t-s) + s) - X(t)\|_V^p dt dP \quad (55.3.11)$$

converges to 0 in $L^1([0, T])$. Similarly, the function

$$s \rightarrow \int_{\Omega} \int_0^T \|X(\delta_n(t-s) + s) - X(t)\|_V^p dt dP$$

converges to 0 in $L^1([0, T])$.

The problem here is that $\gamma_n(t-s) + s$ could be negative. Therefore, replace it with $(\gamma_n(t-s) + s)^+$. This will not affect the convergence of the integral to 0 in 55.3.11 as $n \rightarrow \infty$ because the modification takes place at most on an interval of length 2^{-n} provided $X(0) \in L^p(\Omega; V)$. If this is not the case, simply redefine it at 0 to equal 0. Therefore, there exists a subsequence n_k such that

$$\int_{\Omega} \int_0^T \left\| X((\gamma_{n_k}(t-s) + s)^+) - X(t) \right\|_V^p + \|X(\delta_{n_k}(t-s) + s) - X(t)\|_V^p dt dP$$

converges to 0 for a.e. $s \in [0, T]$ where $X(0)$ is redefined to equal 0 if it happens that $X(0) \notin L^p(\Omega; V)$. Pick such an s . Let

$$X_k^l(t) \equiv X((\gamma_{n_k}(t-s) + s)^+), \quad X_k^r(t) \equiv X(\delta_{n_k}(t-s) + s).$$

This proves the lemma.

Remark 55.3.2 The mesh points other than 0 of each of the k^{th} partitions can be assumed to not be in some given set of measure zero N . The variable s must avoid a set of measure 0 already. If you have another set of measure zero simply adjust s . When you slide s to the right or to the left, you are just moving the mesh points, except for the first one which is always 0. For each n there exists a closed interval $[a_n, b_n]$ such that for s in this interval, the mesh points for $\mathcal{P}_n$ avoid the set of measure zero n . You can also have these intervals be nested. Therefore, there exists a value of s such that the partition points of the step functions determined as above avoid the set of measure zero for all n . The interesting case is the set of measure zero where $\bar{X}$ is a particular measurable representative of X . The set to avoid would be those values of $t > 0$ where $X(t)$ is not equal to $\bar{X}(t)$ a.e. In the case where it is also the case that $X \in L^2([0, T] \times \Omega; H)$ it follows that s can be chosen such that these step functions also converge in this space in addition to K . Whenever convenient, this will be assumed.

Now consider the following situation.

Situation 55.3.3 Let X satisfy the following.

$$X(t) = X_0 + \int_0^t Y(s) ds + \int_0^t Z(s) dW(s), \quad (55.3.12)$$

$X_0 \in L^2(\Omega; H)$ and is $\mathcal{F}_0$ measurable, where Z is $\mathcal{L}_2(U, H)$ predictable and

$$\int_0^T \int_{\Omega} \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 dP dt < \infty$$

so that the stochastic integral can be defined. Also X has a measurable representative $\bar{X}$ (For a.e. t , $\bar{X}(t) = X(t)$ for P a.e. ω) such that

$$\bar{X} \in L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}, H) \cap L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}, V)$$

so for each t , $\bar{X}(t) \in L^p(\Omega; V) \cap L^2(\Omega; H)$. Also $W(t)$ is a Wiener process on another Hilbert space U_1 and $Q = I$. Suppose $Y \in K' = L^{p'}([0, T] \times \Omega; V')$ where $1/p' + 1/p = 1$ and Y is V' predictable. Thus X is continuous into V' although it has values in V and it is also V' predictable thanks to Proposition 55.1.1. The stochastic integral is defined as

$$\int_0^t Z(s) \circ J^{-1} dW(s)$$

where $J \in \mathcal{L}_2(U, U_1)$ and $W(t)$ is a Wiener process with values in U_1 . Also $Q : U \rightarrow U$ is just the identity. Recall also that by Lemma 53.7.3

$$\|\Phi\|_{\mathcal{L}_2(U, H)} = \|\Phi \circ J^{-1}\|_{\mathcal{L}_2(Q_1^{1/2} U_1, H)}$$

where $Q_1 = JJ^*$ and the last norm is the one which occurs in the usual Ito isometry. The goal is to prove the following Ito formula.

$$|X(t)|^2 = |X_0|^2 + \int_0^t \left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds$$

$$+ 2 \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* X(s) \right) dW(s) \quad (55.3.13)$$

where $\mathcal{R}$ is the Riesz map which takes U_1 to U'_1 . The main thing is that the last term above be a local martingale.

Lemma 55.3.4 *Let X be as in Situation 55.3.3 and let $\overline{X}_k^l$ be as in Lemma 55.3.1 corresponding to the measurable representative $\overline{X}$ above. Then $\overline{X}_k^l$ is predictable. In determining $\overline{X}_k^l$, the mesh points always are chosen such that for $a > 0$ one of them $X(a) = \overline{X}(a)$ a.e. **Note** $a > 0$*

Proof: This is a step function and a typical term is of the form $\overline{X}(a) \mathcal{X}_{[a,b)}(t)$. I will try and show this is predictable and then it will follow the given step function is predictable. Let a_n be an increasing sequence converging to a and let b_n be an increasing sequence converging to b . Then for a.e. ω ,

$$X(a_n) \mathcal{X}_{(a_n, b_n]}(t) \rightarrow \overline{X}(a) \mathcal{X}_{[a,b)}(t)$$

in V' due to the fact that $t \rightarrow X(t)$ is continuous into V' for a.e. ω . Therefore, letting $v \in V$ be given it follows that for a.e. ω

$$\langle X(a_n) \mathcal{X}_{(a_n, b_n]}(t), v \rangle \rightarrow \langle \overline{X}(a) \mathcal{X}_{[a,b)}(t), v \rangle$$

and since the filtration is a normal filtration in which $(P, \mathcal{F}_t)$ is complete, this shows

$$(t, \omega) \rightarrow \langle \overline{X}(a) \mathcal{X}_{[a,b)}(t), v \rangle$$

is real predictable because it is the pointwise limit of real predictable functions, those in the sequence being real predictable because of the continuity of $X(t)$ into V' and Proposition 55.1.1. Now since $H \subseteq V'$ it follows that for all $v \in V$,

$$(t, \omega) \rightarrow (\overline{X}(a) \mathcal{X}_{[a,b)}(t), v)$$

is real predictable. Now this holds for $h \in H$ replacing v in the above because V is dense in H . By the Pettis theorem, this proves the lemma.

Lemma 55.3.5 *In Situation 55.3.3 the following formula holds for $0 < s < t$ where $M(t) \equiv \int_0^t Z(u) dW(u)$. In this formula it is assumed s, t are such that each of $X(s) = \overline{X}(s)$, $X(t) = \overline{X}(t)$ so these are each in $L^p(\Omega; V) \cap L^2(\Omega; H)$. $|\cdot|$ denotes the norm in H and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V, V' .*

$$\begin{aligned} |X(t)|^2 &= |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle du + 2(X(s), M(t) - M(s)) \\ &\quad + |M(t) - M(s)|^2 - |X(t) - X(s) - (M(t) - M(s))|^2 \end{aligned} \quad (55.3.14)$$

Also for $t > 0$

$$\begin{aligned} |X(t)|^2 &= |X_0|^2 + 2 \int_0^t \langle Y(u), X(t) \rangle du + 2(X_0, M(t)) \\ &\quad + |M(t)|^2 - |X(t) - X_0 - M(t)|^2 \end{aligned} \quad (55.3.15)$$

Proof: For a.e. t , $X(t) = \overline{X}(t)$ a.e. Assume s, t are not in the exceptional set just described. The formula is a straight forward computation which holds a.e. ω .

$$\begin{aligned}
 & |M(t) - M(s)|^2 - |X(t) - X(s) - (M(t) - M(s))|^2 + 2(X(s), M(t) - M(s)) \\
 &= |M(t) - M(s)|^2 - |X(t) - X(s)|^2 - |M(t) - M(s)|^2 \\
 &\quad + 2(X(t) - X(s), M(t) - M(s)) + 2(X(s), M(t) - M(s)) \\
 &= -|X(t) - X(s)|^2 + 2(X(t), M(t) - M(s)) \\
 &= -|X(t) - X(s)|^2 + 2(X(t), X(t) - X(s)) - 2\left\langle \int_s^t Y(u) du, X(t) \right\rangle \\
 &= -|X(t)|^2 - |X(s)|^2 + 2(X(t), X(s)) + 2|X(t)|^2 - 2(X(t), X(s)) \\
 &\quad - 2\int_s^t \langle Y(u), X(t) \rangle du \\
 &= |X(t)|^2 - |X(s)|^2 - 2\int_s^t \langle Y(u), X(t) \rangle du
 \end{aligned}$$

Now it is time to prove the other assertion.

$$\begin{aligned}
 & |M(t)|^2 - |X(t) - X_0 - M(t)|^2 + 2(X_0, M(t)) \\
 &= -|X(t) - X_0|^2 + 2(X(t) - X_0, M(t)) + 2(X_0, M(t)) \\
 &= -|X(t) - X_0|^2 + 2(X(t), M(t)) \\
 &= -|X(t) - X_0|^2 + 2(X(t), X(t) - X_0) - 2\left\langle \int_0^t Y(s) ds, X(t) \right\rangle \\
 &= |X(t)|^2 - |X_0|^2 - 2\left\langle \int_0^t \langle Y(s), X(t) \rangle ds \right\rangle
 \end{aligned}$$

This proves the lemma.

Noting that $X(0) = X_0$ the first formula works in both cases.

Lemma 55.3.6 *In the Situation 55.3.3,*

$$E\left(\sup_{t \in [0, T]} |X(t)|_H\right) < \infty.$$

Thus each $X(t) \in L^2(\Omega; H)$.

Proof: Consider the formula in Lemma 55.3.5.

$$\begin{aligned} |X(t)|^2 &= |X(s)|^2 + 2 \int_s^t \langle Y(u), X(t) \rangle du + 2(X(s), M(t) - M(s)) \\ &\quad + |M(t) - M(s)|^2 - |X(t) - X(s) - (M(t) - M(s))|^2 \end{aligned} \quad (55.3.16)$$

Now let t_j denote a point of $\mathcal{P}_k$ from Lemma 55.3.1. It follows from the construction in this lemma, it can be assumed that $\mathcal{P}_k$ is such that for all $t \in \mathcal{P}_k$, $X(t) = \bar{X}(t)$ provided $t > 0$. When $t = 0$, $X(0) \equiv X_0$. At the positive partition points $X = \bar{X}$. Also $X(0) = X_0 \in L^2(\Omega; H)$.

$$|X(t_m)|^2 - |X_0|^2 = \sum_{j=0}^{m-1} |X(t_{j+1})|^2 - |X(t_j)|^2$$

Using the formula in Lemma 55.3.5 for $t = t_m$ this yields

$$\begin{aligned} |X(t_m)|^2 - |X_0|^2 &= 2 \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} \langle Y(u), X_k^r(u) \rangle du + \\ &\quad + 2 \sum_{j=0}^{m-1} \left(\int_{t_j}^{t_{j+1}} Z(u) dW(u), X(t_j) \right)_H + \sum_{j=0}^{m-1} |M(t_{j+1}) - M(t_j)|^2 \\ &\quad - \sum_{j=0}^{m-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \end{aligned} \quad (55.3.17)$$

This can be written as

$$\begin{aligned} |X(t_m)|^2 - |X_0|^2 &\leq 2 \int_0^{t_m} \langle Y(u), X_k^r(u) \rangle du \\ &\quad + 2 \int_0^{t_m} \mathcal{R} \left((Z(u) \circ J^{-1})^* \hat{X}_k^l(u) \right) dW(u) + \sum_{j=0}^{m-1} |M(t_{j+1}) - M(t_j)|^2 \\ &\quad - \sum_{j=0}^{m-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \end{aligned} \quad (55.3.18)$$

where $\mathcal{R}$ is the Riesz map from U_1 to U_1' which satisfies

$$\mathcal{R}x(y) \equiv (y, x)_{U_1}.$$

and $\hat{X}_k^l$ is a step function for which $\hat{X}_k^l = X_k^l$ at positive mesh points but at 0 $\hat{X}_k^l(0) \equiv X_0$. That stochastic integral makes sense because the integrand is predictable. Here is why. It follows from Lemma 55.3.5 that X_k^l is predictable and since X_0 is $\mathcal{F}_0$ measurable, it follows $\hat{X}_k^l$ is predictable. That the second term on

the right in 55.3.13 does reduce to what is claimed follows from noting that it is a sum of expressions of the form

$$\left(\int_s^t Z(u) \circ J^{-1} dW(u), X(s) \right)_H$$

If $Z(u)$ were an elementary function, this is a sum of things like

$$\begin{aligned} & \left(\Delta W(u), (Z(u) \circ J^{-1})^* X(s) \right)_{U_1} \\ &= \mathcal{R} \left((Z(u) \circ J^{-1})^* X(s) \right) \Delta W(u) \end{aligned}$$

where $\mathcal{R} : U_1 \rightarrow U'_1$ is the Riesz map, and so, approximating $Z(u)$ by elementary functions which have terms like the above and passing to the limit, one obtains this term in the case where the appropriate integrability condition holds. What about the question whether the function $\mathcal{R} \left((Z(u) \circ J^{-1})^* X(s) \right)$ is appropriately integrable? This is not clearly the case because $X(s)$ might not be uniformly bounded. However, this term is a local martingale. You could define stopping times

$$\tau_k(\omega) \equiv \inf \{t : |X(t, \omega)| > k\}$$

and this would be a localizing sequence. Thus the second term on the right is a local martingale of the form claimed above.

Now it follows on discarding the negative terms,

$$\begin{aligned} & \sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 \leq |X_0|^2 + 2 \int_0^T |\langle Y(u), X_k^r(u) \rangle| du \\ & + 2 \sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* \hat{X}_k^l(u) \right) dW(u) \right| + \sum_{j=0}^{m_k-1} \left| \int_{t_j}^{t_{j+1}} Z(u) dW(u) \right|^2 \end{aligned}$$

where there are $m_k + 1$ points in $\mathcal{P}_k$.

Do $\int_\Omega$ to both sides. Using the Ito isometry, this yields

$$\begin{aligned} & \int_\Omega \left(\sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 \right) dP \leq E \left(|X_0|^2 \right) + 2 \|Y\|_{K'} \|X_k^r\|_K + \sum_{j=0}^{m_k-1} \int_{t_j}^{t_{j+1}} \int_\Omega \|Z(u)\|^2 dP du \\ & + 2 \int_\Omega \left(\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* \hat{X}_k^l(u) \right) dW(u) \right| \right) dP \\ & \leq C + \int_0^T \int_\Omega \|Z(u)\|^2 dP du + \\ & + 2 \int_\Omega \left(\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* \hat{X}_k^l(u) \right) dW(u) \right| \right) dP \end{aligned}$$

$$\leq C + 2 \int_{\Omega} \left(\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* \widehat{X}_k^l(u) \right) dW(u) \right| \right) dP$$

where the result of Lemma 55.3.1 that X_k^r converges to X in K shows the term $2 \|Y\|_{K'} \|X_k^r\|_K$ is bounded. The term involving the stochastic integral is next.

Applying the Burkholder Davis Gundy inequality, Theorem 51.4.7 for $F(r) = r$ along with the description of the quadratic variation of the Ito integral found in Corollary 53.5.5

$$\begin{aligned} \int_{\Omega} \sup_{t_j \in \mathcal{P}_k} |X(t_k)|^2 dP &\leq C + C \int_{\Omega} \left(\int_0^T \left\| \mathcal{R} \left((Z(u) \circ J^{-1})^* \widehat{X}_k^l(u) \right) \right\|^2 du \right)^{1/2} dP \\ &\leq C + C \int_{\Omega} \left(\int_0^T \|Z(u)\|^2 |\widehat{X}_k^l(u)|^2 du \right)^{1/2} dP \end{aligned}$$

Now for each ω , there are only finitely many values of $\widehat{X}_k^l(u)$ and they are $X(t_j)$ for $t_j \in \mathcal{P}_k$. Therefore, the above is dominated by

$$\begin{aligned} C + C \int_{\Omega} \left(\sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 \right)^{1/2} \left(\int_0^T \|Z(u)\|^2 du \right)^{1/2} dP \\ \leq C + \frac{1}{2} \int_{\Omega} \sup_{t_j \in \mathcal{P}_k} |X(t_j)|^2 + C \int_{\Omega} \int_0^T \|Z(u)\|_{\mathcal{L}_2(U, H)}^2 dudP \end{aligned}$$

and so

$$\frac{1}{2} \int_{\Omega} \sup_{t_j \in \mathcal{P}_k} |X(t_k)|^2 dP \leq C$$

for some constant C independent of $\mathcal{P}_k$ dependent on $\int_{\Omega} \int_0^T \|Z(u)\|_{\mathcal{L}_2(U, H)}^2 dudP$. Let D denote the union of all the $\mathcal{P}_k$ except for 0. Thus D is a dense subset of $[0, T]$ and it has just been shown that

$$E \left(\sup_{t \in D} |X(t)|^2 \right) \leq C.$$

Let $\{e_j\}$ an orthonormal basis for H which is also contained in V . I claim that for $y \in V'$

$$|y|^2 = \sup_n \sum_{j=1}^n |\langle y, e_j \rangle|^2$$

This is certainly true if $y \in H$ because in this case

$$\langle y, e_j \rangle = (y, e_j)$$

If $y \notin H$, then the series must diverge. If not, you could consider the infinite sum

$$y \equiv \sum_{j=1}^{\infty} \langle y, e_j \rangle e_j \in H$$

and it would satisfy

$$|y|^2 = \sum_{j=1}^{\infty} |\langle y, e_j \rangle|^2$$

It follows

$$|X(t)|^2 = \sup_n \sum_{j=1}^n |\langle X(t), e_j \rangle|^2$$

and for a.e. ω this is just the sup of continuous functions of t . Therefore, for given ω ,

$$t \rightarrow |X(t)|^2$$

is lower semicontinuous. Hence letting $t \in [0, T]$ and $t_j \rightarrow t$ where $t_j \in D$,

$$|X(t)|^2 \leq \liminf_{n \rightarrow \infty} |X(t_j)|^2$$

so it follows for a.e. ω

$$\sup_{t \in [0, T]} |X(t)|^2 \leq \sup_{t \in D} |X(t)|^2 \leq \sup_{t \in [0, T]} |X(t)|^2$$

Hence

$$E \left(\sup_{t \in [0, T]} |X(t)|^2 \right) \leq C.$$

This proves the lemma.

Observation 55.3.7 *Note the above shows that for a.e. ω , $\sup_{t \in [0, T]} |X(t)|_H < \infty$ so that for such ω , $X(t)$ has values in H . Say*

$$|X(t, \omega)| \leq C(\omega)$$

Hence if $v \in V$, then for a.e. ω

$$\lim_{t \rightarrow s} \langle X(t), v \rangle = \lim_{t \rightarrow s} \langle X(t), v \rangle = \langle X(s), v \rangle = \langle X(s), v \rangle$$

Therefore, since for such ω , $|X(t, \omega)|$ is bounded, the above holds for all $h \in H$ as well. Therefore, for a.e. ω , $t \rightarrow X(t, \omega)$ is weakly continuous with values in H . Eventually, it is shown that in fact it is continuous with values in H .

I am working toward the Ito formula 55.3.13. In order to get this, there is a technical result which will be needed.

Lemma 55.3.8 *Let $X(s) - \widehat{X}_k^l(s) \equiv \Delta_k(s)$. Then the following limit occurs.*

$$\lim_{k \rightarrow \infty} P \left(\left[\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* \Delta_k(s) \right) dW(s) \right| \geq \varepsilon \right] \right) = 0. \quad (55.3.19)$$

That is, $\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* \left(X(s) - \widehat{X}_k^l(s) \right) \right) dW(s) \right|$ converges to 0 in probability. Also the stochastic integral makes sense because X is H predictable.

Proof: First note that from Lemma 55.3.6, for a.e. ω , $X(t)$ has values in H for $t \in [0, T]$ and so it makes sense to consider it in the stochastic integral provided it is H predictable. However, as noted in Situation 55.3.3, this function is automatically V' predictable. Therefore,

$$\langle X(t), v \rangle = (X(t), v)$$

is real predictable for every $v \in V$. Now if $h \in H$, let $v_n \rightarrow h$ in H and so for each ω ,

$$(X(t, \omega), v_n) \rightarrow (X(t, \omega), h)$$

By the Pettis theorem, X is H predictable.

Let $\{e_k\}$ be an orthonormal basis for H such that each e_k is in V . Let P_n be the orthogonal projection onto $\text{span}(e_1, \dots, e_n)$. First note that for each n

$$\begin{aligned} \lim_{k \rightarrow \infty} E \left(\int_0^T \left| P_n \left(X(s) - \widehat{X}_k^l(s) \right) \right|^2 d[M](s) \right) = \\ \lim_{k \rightarrow \infty} \int_0^T E \left(\left| P_n \left(X(s) - \widehat{X}_k^l(s) \right) \right|^2 \right) d[M](s) = 0 \end{aligned} \quad (55.3.20)$$

This follows because the e_k are in V and $s \rightarrow X(s)$ is continuous in V' for a.e. ω and so the integrand converges to 0. After all, $\widehat{X}_k^l(s)$ equals X evaluated at the mesh point of $\mathcal{P}_k$ no larger than s which is closest to s . Also, it was shown above $E \left(\sup_{t \in [0, T]} |X(t)|^2 \right)$ is bounded and so the dominated convergence theorem implies the above converges to 0. Therefore, in 55.3.19 if $X(s) - \widehat{X}_k^l(s)$ were replaced with $P_n \left(X(s) - \widehat{X}_k^l(s) \right)$, the desired conclusion would follow because convergence in $L^1(\Omega)$ implies convergence in probability. Here is why.

By the Burkholder Davis Gundy theorem, Theorem 51.4.7 and Corollary 53.5.5 which describes the quadratic variation of the stochastic integral

$$\begin{aligned} & \int_{\Omega} \left(\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* P_n \Delta_k(s) \right) dW(s) \right| \right) dP \\ & \leq C \int_{\Omega} \left(\int_0^T \left| P_n \left(X(s) - \widehat{X}_k^l(s) \right) \right|^2 \|Z(s)\|^2 ds \right)^{1/2} dP \\ & = CE \left(\left(\int_0^T \left| P_n \left(X(s) - \widehat{X}_k^l(s) \right) \right|^2 d[M](s) \right)^{1/2} \right) \\ & \leq CE \left(\int_0^T \left| P_n \left(X(s) - \widehat{X}_k^l(s) \right) \right|^2 d[M](s) \right)^{1/2} \end{aligned}$$

Therefore, in order to obtain the desired result, it suffices to verify the following two expressions are small provided n is large.

$$P \left(\left[\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* (1 - P_n) X(s) \right) dW(s) \right| \geq \varepsilon \right] \right) \quad (55.3.21)$$

$$P \left(\left[\sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* (1 - P_n) \widehat{X}_k^l(s) \right) dW(s) \right| \geq \varepsilon \right] \right) \quad (55.3.22)$$

By Corollary 51.4.8 which depends on the Burkholder Davis Gundy inequality and Corollary 53.5.5 which describes the quadratic variation of the stochastic integral, 55.3.21 is dominated by

$$\begin{aligned} & \frac{C}{\varepsilon} E \left(\left(\int_0^T \|Z(s)\|^2 |(1 - P_n) X(s)|^2 ds \right)^{1/2} \wedge \delta \right) \\ & + P \left(\left[\left(\int_0^T \|Z(s)\|^2 |(1 - P_n) X(s)|^2 ds \right)^{1/2} > \delta \right] \right) \\ & \leq \frac{C\delta}{\varepsilon} + P \left(\left[\left(\int_0^T \|Z(s)\|^2 |(1 - P_n) X(s)|^2 ds \right)^{1/2} > \delta \right] \right) \end{aligned} \quad (55.3.23)$$

Let $\eta > 0$ be given. Then let δ be small enough that the first term is less than η . Fix such a δ .

Consider the second of the above terms. By Corollary 53.5.5 the integral inside it equals

$$\int_0^T |(1 - P_n) X(s)|^2 d[M](s)$$

For a.e. ω ,

$$\lim_{n \rightarrow \infty} \int_0^T |(1 - P_n) X(s)|^2 d[M](s) = 0$$

because for a.e. $\omega, t \rightarrow X(t, \omega)$ is in H and so the integrand converges to 0 and is dominated by

$$\sup_{t \in [0, T]} |X(t, \omega)| < C(\omega).$$

by Lemma 55.3.6. It follows that as $n \rightarrow \infty$, the second term in 55.3.23 converges to 0. Let n be large enough that this term is less than η also. Similarly the expression in 55.3.22 is small for large enough n . This proves the lemma.

Now at long last, here is the first version of the Ito formula.

Lemma 55.3.9 *In Situation 55.3.3, let D be as above, the union of all the positive mesh points for all the $\mathcal{P}_k$. Also assume $X_0 \in L^2(\Omega; H) \cap L^p(\Omega; V)$. Then for every $t \in D$,*

$$|X(t)|^2 = |X_0|^2 + \int_0^t \left(2 \langle Y(s), \overline{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds$$

$$+ 2 \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* X(s) \right) dW(s) \quad (55.3.24)$$

Proof: Let $t \in D$. Then $t \in \mathcal{P}_k$ for all k large enough. Consider 55.3.18,

$$\begin{aligned} |X(t)|^2 - |X_0|^2 &= 2 \int_0^t \langle Y(u), X_k^r(u) \rangle du \\ &+ 2 \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* \widehat{X}_k^l(u) \right) dW(u) + \sum_{j=0}^{q_k-1} |M(t_{j+1}) - M(t_j)|^2 \\ &- \sum_{j=0}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \end{aligned} \quad (55.3.25)$$

where $t_{q_k} = t$. By Lemma 55.3.8 the second term on the right, the stochastic integral, converges to

$$2 \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* X(u) \right) dW(u)$$

in probability. The first term on the right converges to

$$2 \int_0^t \langle Y(u), \overline{X}(u) \rangle du$$

in $L^1(\Omega)$ because $X_k^r \rightarrow \overline{X}$ in K . Therefore, this also happens in probability. Consider the next term.

$$\begin{aligned} E \left(\sum_{j=0}^{q_k-1} |M(t_{j+1}) - M(t_j)|^2 \right) &= \sum_{j=0}^{q_k-1} E \left(\left| \int_{t_j}^{t_{j+1}} Z(s) dW(s) \right|^2 \right) \\ &= \sum_{j=0}^{q_k-1} \int_{t_j}^{t_{j+1}} E \left(\|Z(s)\|^2 \right) ds = \int_0^t E \left(\|Z(s)\|^2 \right) ds \end{aligned}$$

and so this term converges in $L^1(\Omega)$ to $\int_0^t \|Z(s)\|^2 ds$ which implies this term also converges in probability to $\int_0^t \|Z(s)\|^2 ds$. In fact, for all k large enough, it equals this.

Thus all the terms in 55.3.25 converge in probability except for the last term which also must converge in probability because it equals the sum of terms which do. It remains to find what this last term converges to. Thus

$$\begin{aligned} |X(t)|^2 - |X_0|^2 &= 2 \int_0^t \langle Y(u), \overline{X}(u) \rangle du \\ &+ 2 \int_0^t \mathcal{R} \left((Z(u) \circ J^{-1})^* X(u) \right) dW(u) + \int_0^t \|Z(s)\|^2 ds - a \end{aligned}$$

where a is the limit in probability of the term

$$\sum_{j=0}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2$$

Let P_n be the projection onto $\text{span}(e_1, \dots, e_n)$ as before where $\{e_k\}$ is an orthonormal basis for H with each $e_k \in V$. Then using

$$X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)) = \int_{t_j}^{t_{j+1}} Y(s) ds$$

the troublesome term above is of the form

$$\begin{aligned} & \sum_{j=0}^{q_k-1} \int_{t_j}^{t_{j+1}} \langle Y(s), X(t_{j+1}) - X(t_j) - P_n(M(t_{j+1}) - M(t_j)) \rangle ds \quad (55.3.26) \\ & - \sum_{j=0}^{q_k-1} \langle X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j)), (1 - P_n)(M(t_{j+1}) - M(t_j)) \rangle \end{aligned} \quad (55.3.27)$$

It is here that X_0 has values in V is used. When $j = 0$, $X(t_j) = X(0) = X_0$ and this needs to have values in V to do $Y(s)$ to it in 55.3.26. The sum in 55.3.27 is dominated by

$$\begin{aligned} & \left(\sum_{j=1}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2 \right)^{1/2} \\ & \left(\sum_{j=0}^{q_k-1} |(1 - P_n)(M(t_{j+1}) - M(t_j))|^2 \right)^{1/2} \end{aligned} \quad (55.3.28)$$

Now it is known that $\sum_{j=1}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2$ converges in probability to a . If you take the expectation of the other factor it is

$$\begin{aligned} & E \left(\sum_{j=1}^{q_k-1} \left| (1 - P_n) \int_{t_j}^{t_{j+1}} Z(s) dW(s) \right|^2 \right) \\ & = E \left(\sum_{j=1}^{q_k-1} \sum_{m=n+1}^{\infty} \left(\int_{t_j}^{t_{j+1}} Z(s) \circ J^{-1} dW(s), e_m \right)^2 \right) \\ & = E \left(\sum_{j=1}^{q_k-1} \sum_{m=n+1}^{\infty} \left| \int_{t_j}^{t_{j+1}} \mathcal{R} \left((Z(s) \circ J^{-1})^* e_m \right) dW(s) \right|^2 \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{q_k-1} \sum_{m=n+1}^{\infty} E \left(\left| \int_{t_j}^{t_{j+1}} \mathcal{R} \left((Z(s) \circ J^{-1})^* e_m \right) dW(s) \right|^2 \right) \\
&\leq \sum_{m=n+1}^{\infty} \int_0^T E \left(\left| (Z(s) \circ J^{-1})^* e_m \right|^2 \right) ds \\
&= \int_0^T \sum_{m=n+1}^{\infty} E \left(\left| (Z(s) \circ J^{-1})^* e_m \right|^2 \right) ds
\end{aligned}$$

As $n \rightarrow \infty$ the sum converges to 0 as $n \rightarrow \infty$ independent of k . In fact the sum is just the tail of the series which defines

$$\left\| (Z(s) \circ J^{-1})^* \right\|_{\mathcal{L}_2(H, Q_1^{1/2} U_1)}^2 = \|Z(s) \circ J^{-1}\|_{\mathcal{L}_2(Q_1^{1/2} U_1, H)}^2 = \|Z(s)\|_{\mathcal{L}_2(U, H)}^2,$$

and this is given to be integrable so by the dominated convergence theorem the term converges to 0. Thus the expression in 55.3.28 is of the form $f_k g_{nk}$ where f_k converges in probability to a as $k \rightarrow \infty$ and g_{nk} converges in probability to 0 as $n \rightarrow \infty$ independently of k . Now this implies $f_k g_{nk}$ converges in probability to 0. Here is why.

$$\begin{aligned}
P(|f_k g_{nk}| > \varepsilon) &\leq P(2\delta |f_k| > \varepsilon) + P(2C_\delta |g_{nk}| > \varepsilon) \\
&\leq P(2\delta |f_k - a| + 2\delta |a| > \varepsilon) + P(2C_\delta |g_{nk}| > \varepsilon)
\end{aligned}$$

where $\delta |f_k| + C_\delta |g_{kn}| > |f_k g_{nk}|$ and $\lim_{\delta \rightarrow 0} C_\delta = \infty$. Pick δ small enough that $\varepsilon - 2\delta |a| > \varepsilon/2$. Then this is dominated by

$$\leq P(2\delta |f_k - a| > \varepsilon/2) + P(2C_\delta |g_{nk}| > \varepsilon)$$

Fix n large enough that the second term is less than η . Now taking k large enough, the above is less than η . It follows the expression in 55.3.28 and consequently in 55.3.27 converges to 0 in probability.

Now consider the other term, 55.3.26 using the n just determined. This term is of the form

$$\begin{aligned}
&\sum_{j=0}^{q_k-1} \int_{t_j}^{t_{j+1}} \left\langle Y(s), X_k^r(s) - \widehat{X}_k^l(s) - P_n(M_k^r(s) - M_k^l(s)) \right\rangle ds \\
&= \int_0^t \left\langle Y(s), X_k^r(s) - \widehat{X}_k^l(s) - P_n(M_k^r(s) - M_k^l(s)) \right\rangle ds
\end{aligned}$$

The term

$$\int_0^t \left\langle Y(s), P_n(M_k^r(s) - M_k^l(s)) \right\rangle ds$$

converges to 0 in $L^1(\Omega)$ as $k \rightarrow \infty$. This is because the integrand converges to 0 thanks to the continuity of $M(t)$ and also since this is a projection onto a finite

dimensional subspace of V , $P_n(M(s))$ is uniformly bounded in V so there is no problem getting a dominating function for the dominated convergence theorem. Now consider

$$\int_0^t \langle Y(s), X_k^r(s) - \widehat{X}_k^l(s) \rangle ds$$

This converges to 0 in $L^1(\Omega)$ from the construction in Lemma 55.3.1. This is because $X_0 \in L^p(\Omega; V)$. Thus it equals

$$\int_0^t \langle Y(s), X_k^r(s) - X_k^l(s) \rangle ds + \int_0^{t_{1k}} \langle Y(s), \overline{X}(0) - X_0 \rangle ds$$

and the expectation of the absolute value of the second term converges to 0 as $k \rightarrow \infty$. While the expectation of the absolute value of the first term converges to 0 by the construction of the two step functions. Recall they both converge in K to $\overline{X}$. Therefore, the expression

$$\sum_{j=0}^{q_k-1} |X(t_{j+1}) - X(t_j) - (M(t_{j+1}) - M(t_j))|^2$$

converges to 0 in probability. This proves the lemma.

In fact, the formula 55.3.24 is valid for all $t \in [0, T]$

Theorem 55.3.10 *In Situation 55.3.3, also assume $X_0 \in L^p(\Omega; V) \cap L^2(\Omega; H)$. Then for every $t \in [0, T]$,*

$$\begin{aligned} |X(t)|^2 &= |X_0|^2 + \int_0^t \left(2 \langle Y(s), \overline{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds \\ &\quad + 2 \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* X(s) \right) dW(s) \end{aligned} \quad (55.3.29)$$

Furthermore, for $t \in [0, T]$, $t \rightarrow X(t)$ is continuous as a map into H . In the above $\overline{X}$ is the representative of X which is in

$$L^p([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}, V) \cap L^2([0, T] \times \Omega, \mathcal{B}([0, T]) \times \mathcal{F}, H).$$

In addition to this,

$$E(|X(t)|^2) = E(|X_0|^2) + E \left(\int_0^t \left(2 \langle Y(s), \overline{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds \right)$$

Proof: Let $t \notin D$. For $t > 0$, let $t(k)$ denote the largest point of $\mathcal{P}_k$ which is less than t . Suppose $t(m) < t(k)$. Then

$$X(t(m)) = X_0 + \int_0^{t(m)} Y(s) ds + \int_0^{t(m)} Z(s) dW(s),$$

a similar formula holding for $X(t(k))$. Thus for $t > t(m)$,

$$X(t) - X(t(m)) = \int_{t(m)}^t Y(s) ds + \int_{t(m)}^t Z(s) dW(s)$$

which is the same sort of thing studied so far except that it starts at $t(m)$ rather than at 0 and $X_0 = 0$. Therefore, from Lemma 55.3.9 it follows

$$\begin{aligned} |X(t(k)) - X(t(m))|^2 &= \int_{t(m)}^{t(k)} \left(2 \langle Y(s), \bar{X}(s) - X(t(m)) \rangle + \|Z(s)\|^2 \right) ds \\ &+ 2 \int_{t(m)}^{t(k)} \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - X(t(m))) \right) dW(s) \end{aligned} \quad (55.3.30)$$

Consider that last term. It equals

$$2 \int_{t(m)}^{t(k)} \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - X_m^l(s)) \right) dW(s) \quad (55.3.31)$$

This is dominated by

$$\begin{aligned} &2 \left| \int_0^{t(k)} \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - \hat{X}_m^l(s)) \right) dW(s) \right. \\ &\quad \left. - 2 \int_0^{t(m)} \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - \hat{X}_m^l(s)) \right) dW(s) \right| \\ &\leq 2 \left| \int_0^{t(k)} \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - \hat{X}_m^l(s)) \right) dW(s) \right| \\ &\quad + 2 \left| \int_0^{t(m)} \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - \hat{X}_m^l(s)) \right) dW(s) \right| \\ &\leq 4 \sup_{t \in [0, T]} \left| \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* (X(s) - \hat{X}_m^l(s)) \right) dW(s) \right| \end{aligned}$$

In Lemma 55.3.8 the above expression was shown to converge to 0 in probability. Therefore, by the usual appeal to the Borel Canteli lemma, there is a subsequence still referred to as $\{m\}$, such that it converges to 0 pointwise in ω for all ω off some set of measure 0 as $m \rightarrow \infty$. It follows there is a set of measure 0 such that for ω not in that set, 55.3.31 converges to 0 in H . Note that $t > 0$ is arbitrary. Similar reasoning shows the first term in the non stochastic integral of 55.3.30 is dominated by an expression of the form

$$4 \int_0^T |\langle Y(s), \bar{X}(s) - X_m^l(s) \rangle| ds$$

which clearly converges to 0 for ω not in some set of measure zero because X_m^l converges in K to $\overline{X}$. Finally, it is obvious that

$$\lim_{m \rightarrow \infty} \int_{t(m)}^{t(k)} \|Z(s)\|^2 ds = 0 \text{ for a.e. } \omega$$

due to the assumptions on Z .

This shows that for ω off a set of measure 0

$$\lim_{m, k \rightarrow \infty} |X(t(k)) - X(t(m))|^2 = 0$$

and so $\{X(t(k))\}_{k=1}^\infty$ is a convergent sequence in H . Does it converge to $X(t)$? Let $\xi(t) \in H$ be what it converges to. Let $v \in V$ then

$$(\xi(t), v) = \lim_{k \rightarrow \infty} (X(t(k)), v) = \lim_{k \rightarrow \infty} \langle X(t(k)), v \rangle = \langle X(t), v \rangle = (X(t), v)$$

and now, since V is dense in H , this implies $\xi(t) = X(t)$.

Now for every $t \in D$,

$$\begin{aligned} |X(t)|^2 &= |X_0|^2 + \int_0^t \left(2 \langle Y(s), \overline{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds \\ &\quad + 2 \int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* X(s) \right) dW(s) \end{aligned}$$

and so, using what was just shown along with the obvious continuity of the functions of t on the right of the equal sign, it follows the above holds for all $t \in [0, T]$.

It only remains to verify $t \rightarrow X(t)$ is continuous with values in H . However, the above shows $t \rightarrow |X(t)|^2$ is continuous and it was shown in Lemma 55.3.6 that $t \rightarrow X(t)$ is weakly continuous into H . Therefore, from the uniform convexity of the norm in H it follows $t \rightarrow X(t)$ is continuous. This is very easy to see in Hilbert space. Say $a_n \rightharpoonup a$ and $|a_n| \rightarrow |a|$. From the parallelogram identity.

$$|a_n - a|^2 + |a_n + a|^2 = 2|a_n|^2 + 2|a|^2$$

so

$$|a_n - a|^2 = 2|a_n|^2 + 2|a|^2 - \left(|a_n|^2 + 2(a_n, a) + |a|^2 \right)$$

Then taking $\limsup$ both sides,

$$0 \leq \limsup_{n \rightarrow \infty} |a_n - a|^2 \leq 2|a|^2 + 2|a|^2 - \left(|a|^2 + 2(a, a) + |a|^2 \right) = 0.$$

Of course this fact also holds in any uniformly convex Banach space.

Now consider the last claim. If the last term in 55.3.29 were a martingale, then there would be nothing to prove. This is because if $M(t)$ is a martingale which equals 0 when $t = 0$, then

$$E(M(t)) = E(E(M(t) | \mathcal{F}_0)) = E(M(0)) = 0.$$

However, that last term is unfortunately only a local martingale. One can obtain a localizing sequence as follows.

$$\tau_n(\omega) \equiv \inf \{t : |X(t, \omega)| > n\}$$

where as usual $\inf(\emptyset) \equiv T$. Then stopping both processes on the two sides of 55.3.29 with τ_n ,

$$\begin{aligned} |X(t \wedge \tau_n)|^2 &= |X_0|^2 + \int_0^{t \wedge \tau_n} \left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds \\ &\quad + 2 \int_0^{t \wedge \tau_n} \mathcal{R} \left((Z(s) \circ J^{-1})^* X(s) \right) dW(s) \end{aligned}$$

Now from Lemma 53.5.4,

$$\begin{aligned} |X(t \wedge \tau_n)|^2 &= |X_0|^2 + \int_0^t \mathcal{X}_{[0, \tau_n]}(s) \left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds \\ &\quad + 2 \int_0^t \mathcal{X}_{[0, \tau_n]}(s) \mathcal{R} \left((Z(s) \circ J^{-1})^* X(s) \right) dW(s) \end{aligned}$$

That last term is now a martingale and so you can take the expectation of both sides. This gives

$$\begin{aligned} E(|X(t \wedge \tau_n)|^2) &= E(|X_0|^2) \\ &\quad + E \left(\int_0^t \mathcal{X}_{[0, \tau_n]}(s) \left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds \right) \end{aligned}$$

Letting $n \rightarrow \infty$ and using the dominated convergence theorem yields the desired result. This proves the theorem.

Notation 55.3.11 *The stochastic integrals are unpleasant to look at.*

$$\begin{aligned} &\int_0^t \mathcal{R} \left((Z(s) \circ J^{-1})^* \bar{X}(s) \right) dW(s) \\ &\equiv \int_0^t (X(s), Z(s) dW(s)). \end{aligned}$$

55.4 Finite To Infinite Dimensions

55.4.1 Some Nonlinear Operators.

Definition 55.4.1 *For V a real Banach space, $A : V \rightarrow V'$ is a pseudomonotone map if whenever*

$$u_n \rightharpoonup u \tag{55.4.32}$$

and

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle \leq 0 \tag{55.4.33}$$

it follows that for all $v \in V$,

$$\liminf_{n \rightarrow \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle. \quad (55.4.34)$$

The half arrows denote weak convergence.

Definition 55.4.2 $A : V \rightarrow V'$ is monotone if for all $v, u \in V$,

$$\langle Au - Av, u - v \rangle \geq 0,$$

and A is Hemicontinuous if for all $v, u \in V$,

$$\lim_{t \rightarrow 0+} \langle A(u + t(v - u)), u - v \rangle = \langle Au, u - v \rangle.$$

Theorem 55.4.3 Let V be a Banach space and let $A : V \rightarrow V'$ be monotone and hemicontinuous. Then A is pseudomonotone.

Proof: Let A be monotone and Hemicontinuous. First here is a claim.

Claim: If 55.4.32 and 55.4.33 hold, then $\lim_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle = 0$.

Proof of the claim: Since A is monotone,

$$\langle Au_n - Au, u_n - u \rangle \geq 0$$

so

$$\langle Au_n, u_n - u \rangle \geq \langle Au, u_n - u \rangle.$$

Therefore,

$$0 = \liminf_{n \rightarrow \infty} \langle Au, u_n - u \rangle \leq \liminf_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle \leq \limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle \leq 0.$$

Now using that A is monotone again, then letting $t > 0$,

$$\langle Au_n - A(u + t(v - u)), u_n - u + t(u - v) \rangle \geq 0$$

and so

$$\langle Au_n, u_n - u + t(u - v) \rangle \geq \langle A(u + t(v - u)), u_n - u + t(u - v) \rangle.$$

Taking the $\liminf$ on both sides and using the claim and $t > 0$,

$$t \liminf_{n \rightarrow \infty} \langle Au_n, u - v \rangle \geq t \langle A(u + t(v - u)), (u - v) \rangle.$$

Next divide by t and use the Hemicontinuity of A to conclude that

$$\liminf_{n \rightarrow \infty} \langle Au_n, u - v \rangle \geq \langle Au, u - v \rangle.$$

From the claim,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle Au_n, u - v \rangle &= \liminf_{n \rightarrow \infty} (\langle Au_n, u_n - v \rangle + \langle Au_n, u - u_n \rangle) \\ &= \liminf_{n \rightarrow \infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle. \end{aligned}$$

This proves the theorem.

Monotonicity is very important in the above proof. The next example shows that even if the operator is linear and bounded, it is not necessarily pseudomonotone.

Example 55.4.4 Let H be any Hilbert space and let $A : H \rightarrow H'$ be given by

$$\langle Ax, y \rangle \equiv (-x, y)_H.$$

Then A fails to be pseudomonotone.

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal set of vectors in H . Then Parseval's inequality implies

$$\|x\|^2 \geq \sum_{n=1}^{\infty} |(x_n, x)|^2$$

and so for any $x \in H$, $\lim_{n \rightarrow \infty} (x_n, x) = 0$. Thus $x_n \rightarrow 0 \equiv x$. Also

$$\limsup_{n \rightarrow \infty} \langle Ax_n, x_n - x \rangle =$$

$$\limsup_{n \rightarrow \infty} \langle Ax_n, x_n - 0 \rangle = \limsup_{n \rightarrow \infty} (-\|x_n\|^2) = -1 \leq 0.$$

If A were pseudomonotone, we would need to be able to conclude that for all $y \in H$,

$$\liminf_{n \rightarrow \infty} \langle Ax_n, x_n - y \rangle \geq \langle Ax, x - y \rangle = 0.$$

However,

$$\liminf_{n \rightarrow \infty} \langle Ax_n, x_n - 0 \rangle = -1 < 0 = \langle A0, 0 - 0 \rangle.$$

Now the following proposition is useful.

Proposition 55.4.5 Suppose $A : V \rightarrow V'$ is pseudomonotone and bounded where V is separable. Then it must be demicontinuous. This means that if $u_n \rightarrow u$, then $Au_n \rightharpoonup Au$.

Proof: Since $u_n \rightarrow u$ is strong convergence and since Au_n is bounded, it follows

$$\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle = \lim_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle = 0.$$

Suppose this is not so that Au_n converges weakly to Au . Since A is bounded, there exists a subsequence, still denoted by n such that $Au_n \rightharpoonup \xi$. I need to verify $\xi = Au$. From the above, it follows that for all $v \in V$

$$\begin{aligned} \langle Au, u - v \rangle &\leq \liminf_{n \rightarrow \infty} \langle Au_n, u_n - v \rangle \\ &= \liminf_{n \rightarrow \infty} \langle Au_n, u - v \rangle = \langle \xi, u - v \rangle \end{aligned}$$

Hence $\xi = Au$. This proves the proposition.

55.4.2 The Situation

First recall that in the notion of the cylindrical Wiener process,

$$\int_0^t \Phi(s) dW(s) \equiv \int_0^t \Phi(s) \circ J^{-1} dW(s)$$

where $W(t)$ is a Wiener process having values in U_1 , a separable Hilbert space such that $J : U \rightarrow U_1$ is a Hilbert Schmidt operator and $W(t)$ is a Q_1 Wiener process where $Q_1 \equiv JJ^*$ is defined on U_1 . Also recall that

$$W(t) \equiv \sum_{k=1}^{\infty} \psi_k(t) J g_k$$

where $\{g_k\}$ is an orthonormal basis for U . Here $U = U_0 = Q^{1/2}U$ because it is assumed $Q = I$. Let

$$W_n(t) = \sum_{k=1}^n \psi_k(t) g_k$$

Let $V \subseteq H \subseteq V'$ be a Gelfand triple. Let $A : K \rightarrow K'$ where $K = L^p([0, T] \times \Omega; V)$ and so $K' = L^{p'}([0, T] \times \Omega; V')$. Also let $\sigma : \mathcal{H} \rightarrow L^2([0, T] \times \Omega; \mathcal{L}_2(U, H))$ where $\mathcal{H} \equiv L^2([0, T] \times \Omega; H)$. Let

$$A : [0, T] \times V \times \Omega \rightarrow V', \quad \sigma : [0, T] \times H \times \Omega \rightarrow \mathcal{L}_2(U; H)$$

be progressively measurable as in Proposition 53.3.6 so that by this proposition, if X is H progressively measurable and $\bar{X}$ is V progressively measurable, then

$$(t, \omega) \rightarrow A(t, \bar{X}(t, \omega), \omega), \quad (t, \omega) \rightarrow \sigma(t, X(t, \omega), \omega) \quad (55.4.35)$$

will be V' and $\mathcal{L}_2(U, H)$ progressively measurable. Also assume

$$X \rightarrow \sigma(t, X, \omega) \quad (55.4.36)$$

is continuous from H to $\mathcal{L}_2(U; H)$. As in [55] I will suppress the dependence on ω . Thus I will write $A(t, \bar{X}(t, \omega))$ for $A(t, \bar{X}(t, \omega), \omega)$.

The interest is in the stochastic differential equation

$$dX = A(t, X) dt + \sigma(t, X) dW(t), \quad X(0) = X_0 \in H.$$

Other assumptions will be on A and σ . Assume the following. [55]

1. (Hemicontinuity) For a.e. ω

$$\lambda \rightarrow \langle A(t, u + \lambda v, \omega), z \rangle \text{ is continuous.}$$

2. (Weak monotonicity) There exists $c \in \mathbb{R}$ such that for all $u, v \in V$,

$$\begin{aligned} & 2 \langle A(\cdot, u) - A(\cdot, v), u - v \rangle + |\sigma(\cdot, u) - \sigma(\cdot, v)|_{\mathcal{L}_2(U, H)}^2 \\ & \leq c |u - v|_H^2 \text{ on } [0, T] \times \Omega \end{aligned}$$

3. (Coercivity) There exist $p \in (1, \infty)$, $c_1 \in \mathbb{R}$, $c_2 \in (0, \infty)$, and $f \in L^1([0, T] \times \Omega)$ which is $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable and is also $\mathcal{F}_t$ adapted such that for all $v \in V, t \in [0, T]$,

$$2 \langle A(t, v), v \rangle + \|\sigma(t, v)\|_{\mathcal{L}_2(U, H)}^2 \leq c_1 |v|_H^2 - c_2 \|v\|_V^p + f(t)$$

4. (Boundedness) There exist $c_3 \in [0, \infty)$ and an $\mathcal{F}_t$ adapted process $g \in L^{p'}([0, T] \times \Omega)$ which is $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable such that for all $v \in V, t \in [0, T]$

$$\|A(t, v)\|_{V'} \leq g(t) + c_3 \|v\|_V^{p-1}.$$

Here $1/p + 1/p' = 1$.

5. (Measurability) Let

$$A : [0, T] \times V \times \Omega \rightarrow V', \quad \sigma : [0, T] \times H \times \Omega \rightarrow \mathcal{L}_2(U; H)$$

be progressively measurable.

Observation 55.4.6 *4 and 3 imply the following estimate holds.*

$$\begin{aligned} \|\sigma(t, v)\|_{\mathcal{L}_2(U, H)}^2 &\leq c_1 |v|_H^2 - c_2 \|v\|_V^2 + f(t) - 2 \langle A(t, v), v \rangle \\ &\leq c_1 |v|_H^2 - c_2 \|v\|_V^2 + f(t) + 2 \left(g(t) + c_3 \|v\|_V^{p-1} \right) \|v\|_V \\ &\leq c_1 |v|_H^2 + f(t) + 2g(t) \|v\|_V + 2c_3 \|v\|_V^p \end{aligned} \quad (55.4.37)$$

Another important observation is the following which holds for a.e. ω .

Proposition 55.4.7 *The above hypotheses imply that $u \rightarrow A(t, u)$ is demicontinuous. This means that if $u_n \rightarrow u$ in V then*

$$A(t, u_n) \rightharpoonup A(t, u)$$

where the arrow denotes weak convergence.

Proof: To place this in a more familiar context and write more simply, let $B_1(u) \equiv -A(t, u)$. Then from 2

$$\langle B_1 u - B_1 v, u - v \rangle \geq -c |u - v|^2$$

Therefore, if $B = cI + B_1$ it follows B is monotone and hemicontinuous as well as bounded. That B is bounded is obvious from 4. Therefore, the conclusion follows from Proposition 55.4.5.

Note that as usual, the above holds for a.e. ω .

55.4.3 The Finite Dimensional Problem, Estimates

In this section I will assume there exists a solution to a finite dimensional approximate problem and consider how to pass to a limit and obtain existence for the problem which is really of interest. Let $\{e_k\}$ be an orthonormal basis for H such that each $e_k \in V$. Let $H_n = \text{span}(e_1, \dots, e_n)$. Let P_n be the orthogonal projection onto H_n . Thus

$$P_n y \equiv \sum_{k=1}^n \langle y, e_k \rangle e_k$$

and so for $v \in H_n$ and $w \in V$,

$$\begin{aligned} (P_n A(t, w), v)_H &= \left(\sum_{k=1}^n \langle A(t, w), e_k \rangle e_k, v \right)_H \\ &= \left\langle A(t, w), \sum_{k=1}^n e_k (e_k, v) \right\rangle \\ &= \langle A(t, w), v \rangle \end{aligned} \quad (55.4.38)$$

What does $P_n \sigma$ mean for $\sigma \in \mathcal{L}_2(U, H)$? It will mean that for $u \in U$,

$$P_n \sigma(u) \equiv \sum_{k=1}^n (\sigma u, e_k) e_k$$

Thus $P_n \sigma \in \mathcal{L}_2(U, H_n)$ and letting $\{g_k\}$ be the orthonormal basis for U

$$\begin{aligned} \|P_n \sigma\|_{\mathcal{L}_2(U, H_n)}^2 &\equiv \sum_{j=1}^{\infty} |P_n \sigma(g_j)|_{H_n}^2 \\ &\equiv \sum_{j=1}^{\infty} \left| \sum_{k=1}^n (\sigma g_j, e_k) e_k \right|_{H_n}^2 = \sum_{j=1}^{\infty} \sum_{k=1}^n (\sigma g_j, e_k)^2 \\ &\leq \sum_{k=1}^{\infty} |\sigma g_k|_H^2 = \|\sigma\|_{\mathcal{L}_2(U, H)}^2 \end{aligned} \quad (55.4.39)$$

Assumption 55.4.8 Assume there exists a solution X_n to

$$\begin{aligned} dX_n &= P_n A(t, X_n(t)) + P_n \sigma(t, X_n(t)) dW_n(t) \\ X_n(0) &= P_n X_0 \end{aligned}$$

where $X_0 \in V$.

Since X_n is continuous for a.e. ω , it must be predictable by Proposition 55.1.1. Now estimates are obtained. First note that from 55.4.38, for $v \in H_n$,

$$\langle P_n A(t, v), v \rangle = \langle A(t, v), v \rangle$$

By the Ito formula derived earlier,

$$\begin{aligned} |X_n(t)|^2 &= |P_n X_0|^2 + \int_0^t 2 \langle A(s, X_n(s)), X_n(s) \rangle + \|P_n \sigma(s, X_n(s))\|_{\mathcal{L}_2(U, H)}^2 ds \\ &\quad + 2 \int_0^t (X_n(s), P_n \sigma(s, X_n(s)) dW_n(s)), \end{aligned}$$

the last term being a local martingale as discussed earlier.

Now let τ_k be a stopping time

$$\tau_k(\omega) \equiv \inf \{t : |X_n(t, \omega)| > k\}$$

By the localization theorems or versions of them,

$$\begin{aligned} |X_n(t \wedge \tau_k)|^2 &= |P_n X_0|^2 + \int_0^t \mathcal{X}_{[0, \tau_k]}(s) 2 \langle A(s, X_n(s)), X_n(s) \rangle \\ &\quad + \mathcal{X}_{[0, \tau_k]}(s) \|P_n \sigma(s, X_n(s))\|_{\mathcal{L}_2(U, H)}^2 ds \\ &\quad + 2 \int_0^t (\mathcal{X}_{[0, \tau_k]}(s) X_n(s), P_n \sigma(s, X_n(s)) dW_n(s)) \end{aligned}$$

Then take expectation of both sides. That last term is a martingale so its expectation is 0. Thus

$$\begin{aligned} E(|X_n(t \wedge \tau_k)|^2) &= E(|P_n X_0|^2) + \\ &\int_0^t E\left(\mathcal{X}_{[0, \tau_k]}(s) \left[2 \langle A(s, X_n(s)), X_n(s) \rangle + \|P_n \sigma(s, X_n(s))\|_{\mathcal{L}_2(U, H)}^2\right]\right) ds \end{aligned}$$

Then

$$\begin{aligned} E\left(e^{-c_1 t} |X_n(t \wedge \tau_k)|^2\right) - E(|P_n X_0|^2) &= \\ \int_0^t \frac{d}{ds} \left(E\left(e^{-c_1 s} |X_n(s \wedge \tau_k)|^2\right)\right) ds &= \\ \int_0^t -c_1 E\left(e^{-c_1 s} |X_n(s \wedge \tau_k)|^2\right) ds + \\ \int_0^t e^{-c_1 s} E\left(\mathcal{X}_{[0, \tau_k]}(s) \left[2 \langle A(s, X_n(s)), X_n(s) \rangle + \|P_n \sigma(s, X_n(s))\|_{\mathcal{L}_2(U, H)}^2\right]\right) ds \end{aligned}$$

Now apply 3 along with 55.4.38 and 55.4.39 to obtain

$$\begin{aligned} E\left(e^{-c_1 t} |X_n(t \wedge \tau_k)|^2\right) - E(|P_n X_0|^2) &\leq \int_0^t -c_1 E\left(e^{-c_1 s} |X_n(s \wedge \tau_k)|^2\right) ds + \\ \int_0^t e^{-c_1 s} E\left(\mathcal{X}_{[0, \tau_k]}(s) \left(c_1 |X_n(s \wedge \tau_k)|^2 - c_2 \|X_n(s \wedge \tau_k)\|_V^p + f(s)\right)\right) ds \end{aligned}$$

Two of these terms cancel when $\mathcal{X}_{[0, \tau_k]}(s) = 1$. Since $e^{-c_1 s} \leq 1$,

$$\begin{aligned} & E \left(e^{-c_1 t} |X_n(t \wedge \tau_k)|^2 \right) + c_2 \int_0^t \|X_n(s \wedge \tau_k)\|_V^p ds \\ & \leq E \left(|P_n X_0|^2 \right) + \int_0^T |f(t)| dt \end{aligned}$$

Let $k \rightarrow \infty$ and use Fatou's lemma to conclude

$$\begin{aligned} & E \left(e^{-c_1 t} |X_n(t)|^2 \right) + c_2 E \left(\int_0^t \|X_n(s)\|_V^p ds \right) \\ & \leq E \left(|P_n X_0|^2 \right) + \int_0^T |f(t)| dt \\ & \leq E \left(|X_0|^2 \right) + \int_0^T |f(t)| dt \end{aligned}$$

From 4 it follows $A(\cdot, X_n)$ is bounded in K' . This has proved the following lemma.

Lemma 55.4.9 *There exists a constant C independent of n such that*

$$\sup_{t \in [0, T]} E \left(|X_n(t)|^2 \right) + \|X_n\|_K + \|A(\cdot, X_n)\|_{K'} \leq C.$$

55.4.4 Passing To The Limit

With this estimate, it follows there exists a subsequence which will still be denoted by n such that

$$X_n \rightharpoonup \bar{X} \text{ weakly in } L^2([0, T] \times \Omega; \mathcal{P}; H), \quad (55.4.40)$$

$$X_n \rightharpoonup \bar{X} \text{ weakly in } K \equiv L^p([0, T] \times \Omega; \mathcal{P}; V), \quad (55.4.41)$$

$$P_n A(\cdot, X_n) \rightharpoonup Y \text{ weakly in } K' \equiv L^{p'}([0, T] \times \Omega; \mathcal{P}; V') \quad (55.4.42)$$

Thus the limit functions are all progressively measurable, $\mathcal{P}$ denoting the σ algebra of progressively measurable sets. Letting P'_n be the projection in U onto $\text{span}(g_1, \dots, g_n)$, 55.4.37 implies there is a further subsequence, still denoted by n such that for some Z

$$P_n \sigma(\cdot, X_n) P'_n \circ J^{-1} \rightharpoonup Z \circ J^{-1} \text{ weakly in } L^2([0, T] \times \Omega; \mathcal{P}; \mathcal{L}_2(U_1, H))$$

Now recall

$$X_n(t) = P_n X_0 + \int_0^t P_n A(s, X_n(s)) ds + \int_0^t P_n \sigma(s, X_n(s)) dW_n(s). \quad (55.4.43)$$

Consider the last term. It equals

$$\int_0^t P_n \sigma(s, X_n(s)) P'_n \circ J^{-1} dW(s)$$

Thus 55.4.43 is of the form

$$X_n(t) = P_n X_0 + \int_0^t P_n A(s, X_n(s)) ds + \int_0^t P_n \sigma(s, X_n(s)) P'_n \circ J^{-1} dW(s) \quad (55.4.44)$$

Let

$$I : L^2([0, T] \times \Omega; \mathcal{P}; \mathcal{L}_2(U_1, H)) \rightarrow C([0, T]; L^2(\Omega; H))$$

given by

$$I\sigma(t) \equiv \int_0^t \sigma(s) dW(s).$$

Then by the Ito isometry this is a linear continuous map and so I is continuous from the weak topology on $L^2([0, T] \times \Omega; \mathcal{P}; \mathcal{L}_2(U_1, H))$ to the weak $*$ topology on $L^\infty(0, T; \mathcal{P}; L^2(\Omega, H))$ since this is a dual space of the separable Banach space $L^1(0, T; \mathcal{P}; L^2(\Omega, H))$. See Corollary 13.4.6 and Theorem 23.8.6. Therefore,

$$\begin{aligned} \int_0^t P_n \sigma(s, X_n(s)) dW_n(s) &= \\ \int_0^{(\cdot)} P_n \sigma(s, X_n(s)) P'_n \circ J^{-1} dW(s) &\rightharpoonup \int_0^{(\cdot)} Z(s) \circ J^{-1} dW(s) \end{aligned}$$

weak $*$ in $L^\infty(0, T; \mathcal{P}; L^2(\Omega, H))$.

Next consider the deterministic integral in 55.4.43. The mapping $J : K' \rightarrow C([0, T]; L^2(\Omega; V'))$ given by

$$JY(t) \equiv \int_0^t Y(s) ds$$

is linear and continuous. Therefore, as in the stochastic integral, this is weak $*$ continuous as a map into $L^\infty(0, T; L^2(\Omega; V'))$ and so the deterministic integral in 55.4.43 converges weak $*$ in $L^\infty(0, T; L^2(\Omega; V'))$ to

$$\int_0^{(\cdot)} Y(s) ds.$$

It follows the right side of 55.4.43 converges weak $*$ in $L^\infty(0, T; L^2(\Omega; V'))$ to

$$X_0 + \int_0^{(\cdot)} Y(s) ds + \int_0^{(\cdot)} Z(s) dW(s).$$

Now define $X(t)$ by

$$X(t) \equiv X_0 + \int_0^t Y(s) ds + \int_0^t Z(s) dW(s)$$

Thus $X \in C([0, T]; V')$.

Is it the case that $\bar{X} \in K$ above is a measurable representative of X ? Since the right side of 55.4.43 converges weak $*$ in $L^\infty(0, T; L^2(\Omega; V'))$, it must be the case that the left side of this equation also converges weak $*$ in $L^\infty(0, T; L^2(\Omega; V'))$. Thus it is also the case it converges weakly in $L^2(0, T; L^2(\Omega; V'))$. Let

$$\eta \in L^2([0, T] \times \Omega; V).$$

Then

$$\begin{aligned} \int_0^T \langle \bar{X}(t), \eta(t) \rangle_{L^2(\Omega; V)} dt &= \int_0^T (\bar{X}(t), \eta(t))_{L^2(\Omega; H)} dt = \\ \int_0^T \int_\Omega \bar{X}(t, \omega) \eta(t, \omega) dP dt &= \lim_{n \rightarrow \infty} \int_0^T \int_\Omega X_n(t, \omega) \eta(t, \omega) dP dt \\ &= \lim_{n \rightarrow \infty} \int_0^T (X_n(t), \eta(t))_{L^2(\Omega; H)} dt = \lim_{n \rightarrow \infty} \int_0^T \langle X_n(t), \eta(t) \rangle dt \\ &= \int_0^T \langle X(t), \eta(t) \rangle_{L^2(\Omega; V)} dt \end{aligned}$$

Therefore, $\bar{X}(t) = X(t)$ a.e. t . Thus $\bar{X}$ is a measurable representative of X as hoped.

Therefore, the situation of the Ito formula is obtained. It remains to identify $Y(s)$ and $Z(s)$. First, from the Ito formula, Theorem ??, it follows

$$|X(t)|^2 = |X_0|^2 + \int_0^t \left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds + M(t)$$

where $M(t)$ is a local martingale. Also from this theorem,

$$E(|X(t)|^2) = E(|X_0|^2) + E\left(\int_0^t \left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds\right)$$

and so

$$\begin{aligned} E\left(e^{-ct} |X(t)|^2\right) - E(|X_0|^2) &= \int_0^t \left(e^{-cs} E(|X(s)|^2) \right)' ds \\ \int_0^t -ce^{-cs} E(|X(s)|^2) + e^{-cs} &\left(E\left(2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) \right) ds \end{aligned}$$

which implies

$$\begin{aligned} &E\left(e^{-ct} |X(t)|^2\right) - E(|X_0|^2) \\ &= E\left(\int_0^t e^{-cs} \left(-c |X(s)|^2 + 2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds\right) \quad (55.4.45) \end{aligned}$$

Now consider 55.4.44. Using 55.4.38, the same reasoning applied to this equation leads to

$$\begin{aligned} & E \left(e^{-ct} |X_n(t)|^2 \right) - E \left(|P_n X_0|^2 \right) \\ = & E \left(\int_0^t e^{-cs} \left(-c |X_n(s)|^2 + 2 \langle A(s, X_n(s)), X_n(s) \rangle \right. \right. \\ & \left. \left. + \|P_n \sigma(s, X_n(s)) P'_n\|_{\mathcal{L}_2(U, H)}^2 \right) ds \right) \end{aligned}$$

Deleting those projection operators only makes the norm larger and so

$$\begin{aligned} & E \left(e^{-ct} |X_n(t)|^2 \right) - E \left(|P_n X_0|^2 \right) \\ \leq & E \left(\int_0^t e^{-cs} \left(-c |X_n(s)|^2 + 2 \langle A(s, X_n(s)), X_n(s) \rangle \right. \right. \\ & \left. \left. + \|\sigma(s, X_n(s))\|_{\mathcal{L}_2(U, H)}^2 \right) ds \right) \end{aligned}$$

At this point, it is desired to exploit the monotonicity assumption on A . Let

$$\phi(s) \in K \cap L^2([0, T] \times \Omega; H)$$

where ϕ is progressively measurable. Then inserting this and patching things up by adding other terms,

$$\begin{aligned} & E \left(e^{-ct} |X_n(t)|^2 \right) - E \left(|P_n X_0|^2 \right) \leq \\ & E \left(\int_0^t e^{-cs} \left(-c |X_n(s) - \phi(s)|^2 + 2 \langle A(s, X_n(s)) - A(s, \phi(s)), X_n(s) - \phi(s) \rangle \right. \right. \\ & \quad \left. \left. + \|\sigma(s, X_n(s)) - \sigma(s, \phi(s))\|_{\mathcal{L}_2(U, H)}^2 \right) ds \right) \quad (55.4.46) \\ & + E \left(\int_0^t e^{-cs} \left(c |\phi(s)|^2 - 2c \langle X_n(s), \phi(s) \rangle + 2 \langle A(s, \phi(s)), X_n(s) - \phi(s) \rangle \right. \right. \\ & \quad \left. \left. + 2 \langle A(s, X_n(s)), \phi(s) \rangle + 2 \langle \sigma(s, X_n(s)), \sigma(s, \phi(s)) \rangle_{\mathcal{L}_2(U, H)} \right. \right. \\ & \quad \left. \left. - \|\sigma(s, \phi(s))\|^2 \right) ds \right) dP \end{aligned}$$

Now from 2 the expression in 55.4.46 is nonpositive and so the whole mess above reduces to

$$\begin{aligned} & E \left(e^{-ct} |X_n(t)|^2 \right) - E \left(|P_n X_0|^2 \right) \leq \\ & + E \left(\int_0^t e^{-cs} \left(c |\phi(s)|^2 - 2c \langle X_n(s), \phi(s) \rangle + 2 \langle A(s, \phi(s)), X_n(s) - \phi(s) \rangle \right. \right. \end{aligned}$$

$$+2 \langle A(s, X_n(s)), \phi(s) \rangle + 2 \langle \sigma(s, X_n(s)), \sigma(s, \phi(s)) \rangle_{\mathcal{L}_2(U, H)} \\ - \|\sigma(s, \phi(s))\|^2 ds \Big) dP$$

Now let $\psi(t) \in L^\infty(0, T)$, just a real valued function. Both sides of the above without the E are progressively measurable because this is true of X_n, ϕ and 55.4.35. Then pass to the limit in the above, * do $\int_0^T \psi(t) * dt$ Then use Fubini's theorem to obtain

$$E \left(\int_0^T \psi(t) \left(e^{-ct} |X(t)|^2 - |X_0|^2 \right) dt \right) \\ \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(c |\phi(s)|^2 - 2c \langle \bar{X}(s), \phi(s) \rangle \right. \right. \\ \left. \left. + 2 \langle A(s, \phi(s)), \bar{X}(s) - \phi(s) \rangle \right. \right. \\ \left. \left. + 2 \langle Y(s), \phi(s) \rangle + 2 \langle Z(s), \sigma(s, \phi(s)) \rangle_{\mathcal{L}_2(U, H)} \right. \right. \\ \left. \left. - \|\sigma(s, \phi(s))\|^2 \right) ds dt \right)$$

Now from 55.4.45, the left side of the above inequality can be replaced with the right side of 55.4.45 to obtain

$$E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(-c |X(s)|^2 + 2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds dt \right) \\ \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(c |\phi(s)|^2 - 2c \langle \bar{X}(s), \phi(s) \rangle \right. \right. \\ \left. \left. + 2 \langle A(s, \phi(s)), \bar{X}(s) - \phi(s) \rangle \right. \right. \\ \left. \left. + 2 \langle Y(s), \phi(s) \rangle + 2 \langle Z(s), \sigma(s, \phi(s)) \rangle_{\mathcal{L}_2(U, H)} \right. \right. \\ \left. \left. - \|\sigma(s, \phi(s))\|^2 \right) ds dt \right) \quad (55.4.47)$$

Recall that ϕ was arbitrary. Let it equal $\bar{X}$. Then the above reduces to

$$E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(-c |X(s)|^2 + 2 \langle Y(s), \bar{X}(s) \rangle + \|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds dt \right) \\ \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(-c |\bar{X}(s)|^2 \right. \right.$$

$$+2 \langle Y(s), \bar{X}(s) \rangle + 2 \langle Z(s), \sigma(s, \bar{X}(s)) \rangle_{\mathcal{L}_2(U, H)} \\ - \|\sigma(s, \bar{X}(s))\|^2 \Big) ds dt \Big) \\$$

Recall $\bar{X}(t) = X(t)$ a.e. and so the above simplifies to

$$E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(\|Z(s)\|_{\mathcal{L}_2(U, H)}^2 \right) ds dt \right) \leq \\ E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(2 \langle Z(s), \sigma(s, \bar{X}(s)) \rangle_{\mathcal{L}_2(U, H)} - e^{-cs} \|\sigma(s, \bar{X}(s))\|^2 \right) ds \right) dt \Big)$$

which simplifies further to

$$E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(\|Z(s) - \sigma(s, \bar{X}(s))\|_{\mathcal{L}_2(U, H)}^2 \right) ds dt \right) \leq 0$$

and so for a.e. ω , $Z(s) = \sigma(s, \bar{X}(s))$ a.e. s and so for a.e. ω , $Z(s) = \sigma(s, X(s))$ a.e. s . Now 55.4.47 simplifies to

$$E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(-c|X(s)|^2 + 2 \langle Y(s), \bar{X}(s) \rangle \right) ds dt \right) \\ \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(c|\phi(s)|^2 - 2c \langle \bar{X}(s), \phi(s) \rangle \right. \right. \\ \left. \left. + 2 \langle A(s, \phi(s)), \bar{X}(s) - \phi(s) \rangle + 2 \langle Y(s), \phi(s) \rangle \right) ds dt \right) \quad (55.4.48)$$

Next let $\phi = \bar{X} - \varepsilon \eta v$ where $\eta \in L^\infty([0, T] \times \Omega)$ and $v \in V$ in 55.4.48.

$$E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(-c|X(s)|^2 + 2 \langle Y(s), \bar{X}(s) \rangle \right) ds dt \right) \\ \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(c|\bar{X}(s) - \varepsilon \eta(s) v|^2 - 2c \langle \bar{X}(s), \bar{X}(s) - \varepsilon \eta(s) v \rangle \right. \right. \\ \left. \left. + 2 \langle A(s, \bar{X}(s) - \varepsilon \eta(s) v), \varepsilon \eta(s) v \rangle + 2 \langle Y(s), \bar{X}(s) - \varepsilon \eta(s) v \rangle \right) ds dt \right)$$

Simplifying this yields

$$0 \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(c\varepsilon^2 |\eta(s) v|^2 - 2\varepsilon \langle Y(s), \eta(s) v \rangle \right) \right. \\ \left. + 2 \langle A(s, \bar{X}(s) - \varepsilon \eta(s) v), \varepsilon \eta(s) v \rangle ds \right)$$

First divide by ε and get

$$0 \leq E \left(\int_0^T \psi(t) \int_0^t e^{-cs} \left(c\varepsilon |\eta(s)v|^2 - 2 \langle Y(s), \eta(s)v \rangle \right) \right. \\ \left. + 2 \langle A(s, \bar{X}(s) - \varepsilon \eta(s)v), \eta(s)v \rangle \right) ds$$

Next let $\varepsilon \rightarrow 0$ and use Hemicontinuity, 1 and the dominated convergence theorem to conclude

$$0 \leq 2E \left(\int_0^T \psi(t) \int_0^t e^{-cs} (\langle A(s, \bar{X}(s)), v \rangle - \langle Y(s), v \rangle) \eta(s) ds dt \right)$$

Since this is true for all $\psi \in L^\infty(0, T)$, it follows for a.e. ω ,

$$\int_0^t e^{-cs} (\langle A(s, \bar{X}(s)), v \rangle - \langle Y(s), v \rangle) \eta(s) ds = 0$$

for all t . In particular, this is true if $t = T$. Since η is arbitrary, this shows that for any $v \in V$,

$$\langle A(s, \bar{X}(s)), v \rangle - \langle Y(s), v \rangle = 0 \text{ a.e. } s$$

Since V is separable, there exists a set of measure zero such that for s not in this set, the above equals 0 for all $v \in V$. This has shown that for a.e. ω ,

$$A(s, \bar{X}(s)) = Y(s) \text{ a.e. } s.$$

This proves the existence part of the following theorem.

Theorem 55.4.10 *Suppose 1 - 5 on the operators A and σ and suppose also there exists a solution to the finite dimensional stochastic equation given in 55.4.8. Then there exists a unique solution to the stochastic differential equation*

$$\begin{aligned} dX &= A(t, X(t)) dt + \sigma(t, X(t)) dW, \\ X(0) &= X_0 \in L^p(\Omega; V) \cap L^2(\Omega; H) \end{aligned}$$

More precisely,

$$\begin{aligned} X(t) &= X_0 + \int_0^t A(s, \bar{X}(s)) ds + \int_0^t \sigma(s, X(s)) dW(s), \\ X_0 &\in L^p(\Omega; V) \cap L^2(\Omega; H), \quad X_0 \text{ is } \mathcal{F}_0 \text{ measurable} \end{aligned}$$

where $\bar{X}$ is a measurable representative of X which is in

$$L^p([0, T] \times \Omega; V) \cap L^2([0, T] \times \Omega; H)$$

and is V and H progressively measurable.

Proof: It only remains to verify uniqueness. Suppose then that $X_i, i = 1, 2$ are solutions corresponding to the initial condition X_{0i} . Then by the Ito formula and Theorem ??

$$\begin{aligned} E \left(|X_1(t) - X_2(t)|^2 \right) &= E \left(|X_{01} - X_{02}|^2 \right) + \\ E \left(\int_0^t 2 \langle A(s, \bar{X}_1(s)) - A(s, \bar{X}_2(s)), \bar{X}_1(s) - \bar{X}_2(s) \rangle + \right. \\ &\quad \left. ||\sigma(s, X_1(s)) - \sigma(s, X_2(s))||^2 ds \right) \end{aligned}$$

Now from 2 and Fubini's theorem,

$$\begin{aligned} E \left(|X_1(t) - X_2(t)|^2 \right) &\leq E \left(|X_{01} - X_{02}|^2 \right) + c \int_0^t E \left(|\bar{X}_1(s) - \bar{X}_2(s)|^2 \right) ds \\ &= E \left(|X_{01} - X_{02}|^2 \right) + c \int_0^t E \left(|X_1(s) - X_2(s)|^2 \right) ds \end{aligned}$$

When $X_{01} = X_{02}$ uniqueness is obtained from Gronwall's inequality. That is, for each t ,

$$X_1(t) = X_2(t) \text{ in } L^2(\Omega; H).$$

This proves the theorem.

Of course it should also be possible to relax the assumption that $X_0 \in V$. One should only have to assume $X_0 \in L^2(\Omega; H)$ and measurable in $\mathcal{F}_0$. That a theorem should hold in this more general case follows right away from the above formula for uniqueness. Let

$$X_{0n} \rightarrow X_0 \in L^2(\Omega; H)$$

the convergence being in $L^2(\Omega; H)$. Then an adaption of the argument given to prove existence and in particular, passing to the limit shows existence. Uniqueness is then immediate from a repeat of the uniqueness argument given in the proof of Theorem 55.4.10 above in the simpler case where $X_{01} = X_{02}$. This proves the following corollary.

Corollary 55.4.11 *In the situation of Theorem 55.4.10 it is only necessary to assume $X_0 \in L^2(\Omega; H)$.*

Suppose in addition to the above, that

$$u \rightarrow \sigma(t, u)^* h \tag{55.4.49}$$

is Lipschitz continuous for all $h \in H$. Suppose also that in addition to hemicontinuity, 1

$$v \rightarrow \langle A(t, v), w \rangle \tag{55.4.50}$$

is Lipschitz continuous for all $w \in V$. Then you can show the finite dimensional problem has a unique solution because it is in the situation of Theorem 55.1.8. Therefore, there exists a solution to the stochastic differential equation of Theorem 55.4.10.

Here is a simple example.

Example 55.4.12 Let $H = L^2(W)$ where W is a bounded open subset of $\mathbb{R}^n$ and let V be a closed subspace of $H^1(W)$ which contains $C_c^\infty(W)$. Let $A : V \rightarrow V'$ satisfy

$$\langle Au, v \rangle \equiv - \int_W \nabla u \cdot \nabla v dx$$

Let $\sigma \in \mathcal{L}_2(U, H)$ be a constant and let $W(t)$ be a Wiener process defined on U_1 where $J : U \rightarrow U_1$ is a Hilbert Schmidt operator. Then the assumptions 1 - 5 all hold if $K = L^2([0, T] \times \Omega; V)$. The only one of these assumptions which is not immediately obvious is 3, the one labeled coercivity. However, using the definition of the norm in V ,

$$- \langle Au, u \rangle + \|u\|_H^2 = \|u\|_V^2$$

and so

$$2 \langle Au, u \rangle + \|\sigma\|_{\mathcal{L}_2(U, H)}^2 = -\|u\|_V^2 + \|u\|_H^2 + \|\sigma\|_{\mathcal{L}_2(U, H)}^2$$

which is sufficient. The conditions 55.4.49 and 55.4.50 both hold and this is sufficient to use Theorem 55.1.8 to get existence of a solution to the finite dimensional problem. Therefore, by Corollary 55.4.11, there exists a unique solution to the equation

$$dX = AXdt + \sigma dW(t), \quad X(0) = X_0 \in H$$

The unique solution to this is called the Ornstein Uhlenbeck process. Maybe you might right this as

$$dX = \Delta X dt + \sigma dW(t), \quad X(0) = X_0 \in H.$$

Of course Corollary 55.4.11 and Theorem 55.1.8 include many more examples than this, including time dependent operators.

55.5 Finite Dimensional Problems

55.5.1 The Ito Formula

Before starting, it is a good idea to review the Ito formula, Theorem 54.1.1.

Theorem 55.5.1 Let Φ be an $\mathcal{L}_2(U_0, H)$ valued predictable or progressively measurable process which is stochastically integrable in $[0, T]$ because

$$P \left(\left[\int_0^T \|\Phi\|^2 dt < \infty \right] \right) = 1$$

and let $\phi : [0, T] \times \Omega \rightarrow H$ be predictable or progressively measurable process which is Bochner integrable on $[0, T]$ and let $X(0)$ be $\mathcal{F}_0$ measurable and H valued. Let

$$X(t) \equiv X(0) + \int_0^t \phi(s) ds + \Phi \cdot W(t).$$

Let $F : [0, T] \times H \times \Omega \rightarrow \mathbb{R}^1$ have continuous partial derivatives F_t, F_x, F_{xx} which are uniformly continuous on bounded subsets of $[0, T] \times H$ independent of $\omega \in \Omega$ and let it be progressively measurable. Then the following formula holds for a.e. ω .

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) + \int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s) Q)\} ds$$

The dependence of F on ω is suppressed.

In the case of interest here $H = \mathbb{R}^d$ so it is a finite dimensional problem and $Q \equiv I$ which is trace class because of the finite dimensions. Thus in the case of interest here, the formula is

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) + \int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s))\} ds$$

and also $U_0 = U = H = \mathbb{R}^d$ and the space of Hilbert Schmidt operators in the statement of the theorem reduces to the space of $d \times d_1$ matrices with the Frobenius norm.

55.5.2 An Unbelievably Technical Lemma

This is all from [55], see also [43]. Here $W(t)$ is a Wiener process on $\mathbb{R}^{d_1}$ with respect to the usual normal filtration $\mathcal{F}_t$. Thus $U \equiv \mathbb{R}^{d_1}$ and $Q \equiv I$. Let $H \equiv \mathbb{R}^d$.

$$(t, \mathbf{x}, \omega) \rightarrow \sigma(t, \mathbf{x}, \omega)$$

is progressively measurable and continuous in $\mathbf{x}$ into the space of $d \times d_1$ matrices with the usual Frobenius norm. Similarly

$$(t, \mathbf{x}, \omega) \rightarrow \mathbf{b}(t, \mathbf{x}, \omega)$$

is progressively measurable and continuous in $\mathbf{x}$ into $\mathbb{R}^d$. Assume also the following integrability condition.

$$\int_0^T \sup_{|\mathbf{x}| \leq R} \left\{ \|\sigma(t, \mathbf{x})\|^2 + |\mathbf{b}(t, \mathbf{x})| \right\} dt < \infty \text{ on } \Omega \quad (55.5.51)$$

The dependence on ω is suppressed. In addition to this, some other assumptions are made. For $|\mathbf{x}|, |\mathbf{y}| \leq R$,

$$2(\mathbf{x} - \mathbf{y}, \mathbf{b}(t, \mathbf{x}) - \mathbf{b}(t, \mathbf{y})) + \|\sigma(t, \mathbf{x}) - \sigma(t, \mathbf{y})\|^2 \leq K_t(R) |\mathbf{x} - \mathbf{y}|^2 \quad (55.5.52)$$

$$2(\mathbf{x}, \mathbf{b}(t, \mathbf{x})) + \|\sigma(t, \mathbf{x})\|^2 \leq K_t(1) (1 + |\mathbf{x}|^2) \quad (55.5.53)$$

In the above, $K_t(R)$ is an $[0, \infty)$ valued $\mathcal{F}_t$ adapted process for each $R \geq 0$. It is assumed to satisfy

$$\int_0^T K_t(R) dt < \infty \text{ a.e. } \omega \quad (55.5.54)$$

Also define for each $R \geq 0$ and $t \in [0, T]$

$$\alpha_t(R) \equiv \int_0^t K_s(R) ds. \quad (55.5.55)$$

To begin with, here is a useful lemma which is not unbelievably technical.

Lemma 55.5.2 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{f_n\}$ be a sequence of measurable functions. Then $f_n \rightarrow 0$ in probability if and only if for every subsequence, $\{f_{n_k}\}$ there exists a further subsequence $\{f_{n_{k_l}}\}$ which converges pointwise a.e. to 0. Also, if $f_n \rightarrow 0$ in probability and if g is any real valued measurable function then gf_n also converges in probability to 0.*

Proof: First of all, if $\{f_n\}$ converges in probability, then every subsequence converges in probability also. Now if $\{f_n\}$ does converge in probability, then there exists a subsequence f_{n_k} such that

$$P(|f_{n_k}| \geq 2^{-k}) \leq 2^{-k}$$

Then by the Borel Cantelli lemma, letting E_k denote the set $[|f_{n_k}| \geq 2^{-k}]$, it follows the set of ω which is in infinitely many of these sets has measure zero and so for a.e. ω , $|f_{n_k}(\omega)| < 2^{-k}$ for all k and so this subsequence $\{f_{n_k}\}$ converges to 0 a.e.

Now suppose every subsequence has an a.e. convergent subsequence which converges to 0. Why does $\{f_n\}$ converge to 0 in probability? Suppose it doesn't. Then there is an $\varepsilon > 0$ and a $\delta > 0$ and a subsequence $\{f_{n_k}\}$ such that

$$P(|f_{n_k}| \geq \varepsilon) \geq \delta.$$

But now, by hypothesis, there is a subsequence $f_{n_{k_l}}$ which converges to 0 a.e. and so for a.e. ω , $|f_{n_{k_l}}(\omega)| < \varepsilon$ for all l large enough. Consider

$$g_l(\omega) \equiv \sup \{|f_{n_{k_r}}(\omega)| : r \geq l\}$$

Then $\{g_l\}$ is a decreasing sequence of measurable functions and it converges to 0 a.e. Furthermore,

$$P(g_l \geq \varepsilon) \geq \delta$$

and the sets $[g_l \geq \varepsilon]$ decrease to a set of measure zero. Hence the above is a contradiction.

The last claim follows from this. If $f_n \rightarrow 0$ in probability, this means every subsequence has an a.e. convergent subsequence converging to 0. This last condition is unchanged if f_n is replaced by gf_n . This proves the lemma.

Note that $\lim_{n \rightarrow \infty} P(|gf_n| \geq \varepsilon) = 0$ and $\lim_{n \rightarrow \infty} P(|f_n| \geq \varepsilon) = 0$ could be replaced in the last claim with $\limsup_{n \rightarrow \infty}$ with no change in the conclusion. Some sort of convergence in probability to 0 is preserved when you multiply by a nonzero measurable function.

Here is a lemma which will be needed. This is not the unbelievably technical lemma either.

Lemma 55.5.3 *Let $Y(t) \in [0, \infty)$, $t \in [0, \infty)$ be a continuous $\mathcal{F}_t$ adapted stochastic process and let γ be a stopping time. Also let $\varepsilon > 0$. Let*

$$\beta_\varepsilon \equiv \inf \{t : Y(t) \geq \varepsilon\}$$

The usual convention that $\inf \emptyset = \infty$ is followed. Then

$$P\left(\left[\sup_{t \in [0, \gamma]} Y(t) \geq \varepsilon\right]\right) \leq \frac{1}{\varepsilon} E(Y(\beta_\varepsilon \wedge \gamma)).$$

Proof: From the continuity of $Y(t)$ and definition of β_ε ,

$$\left[\sup_{t \in [0, \gamma]} Y(t) \geq \varepsilon\right] = [\beta_\varepsilon \leq \gamma]$$

To see this show instead that

$$\left[\sup_{t \in [0, \gamma]} Y(t) < \varepsilon\right] = [\beta_\varepsilon > \gamma]$$

Why is this so? If $\omega \in$ left side, then for some $\delta > 0$, $Y(t) < \varepsilon - \delta$ for all $t \in [0, \gamma]$. By continuity, if $Y(t) \geq \varepsilon$, it follows $t > \gamma + \eta$ for some $\eta > 0$. Hence taking the inf of such t gives $\beta_\varepsilon \geq \gamma + \eta$ and so the left side is contained in the right side. Now suppose ω is in the right side. Then $\beta_\varepsilon > \gamma + \eta$ for some $\eta > 0$. Therefore, if $Y(t) \geq \varepsilon$, it follows $t \geq \gamma + \eta$ and so if $t \leq \gamma$ you can't have $Y(t) \geq \varepsilon$ so for such t , $Y(t) < \varepsilon$. Therefore, the right side is contained in the left side.

Now it follows

$$\left[\sup_{t \in [0, \gamma]} Y(t) \geq \varepsilon\right] = [\beta_\varepsilon \leq \gamma] \subseteq [Y(\beta_\varepsilon \wedge \gamma) \geq \varepsilon]$$

because if $\beta_\varepsilon \leq \gamma$, then $Y(\beta_\varepsilon \wedge \gamma) = Y(\beta_\varepsilon)$ and by continuity, $Y(\beta_\varepsilon) \geq \varepsilon$. Now if $Y(\beta_\varepsilon \wedge \gamma) \geq \varepsilon$, then you can't have $\gamma < \beta_\varepsilon$ and so in fact the inclusion in the above

is an equality. Now the desired result follows.

$$\begin{aligned} \varepsilon P \left(\left[\sup_{t \in [0, \gamma]} Y(t) \geq \varepsilon \right] \right) &= \varepsilon P([Y(\beta_\varepsilon \wedge \gamma) \geq \varepsilon]) \\ &\leq E(Y(\beta_\varepsilon \wedge \gamma)) \end{aligned}$$

and this proves the lemma.

Note the conclusion of the lemma is valid if $t \rightarrow Y(t)$ is only continuous off a set of measure zero. Next is a localization lemma. This lemma seems like it has very elaborate hypotheses but the conclusion is very attractive. In this lemma, the $p_n(t)$ is going to be of the form

$$X_n \left(t - \frac{1}{n} \right) - X_n(t)$$

because the lemma is a technical result related to obtaining existence to a finite dimensional stochastic process by using the Euler method. The litany of technical assumptions are things which end up holding for the proof of existence using the Euler method. The conclusion of this lemma is that

$$\sup_{t \in [0, T]} |X_n(t) - X_m(t)| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

in probability. This is very nice indeed. The proof of the lemma proceeds by first showing that

$$\sup_{t \in [0, \gamma_{n,m}]} |X_n(t) - X_m(t)| \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

in probability where $\gamma_{n,m}(R)$ is a suitable stopping time.

Lemma 55.5.4 *Let $n \in \mathbb{N}$ and $X_n(t)$ be a continuous $\mathbb{R}^d$ valued $\mathcal{F}_t$ adapted process such that $X(0) = X_0$ an $\mathcal{F}_0$ measurable function and*

$$dX_n(t) = \mathbf{b}(t, X_n(t) + p_n(t)) dt + \sigma(t, X_n(t) + p_n(t)) dW(t)$$

for $t \in [0, T]$ for $p_n(t)$ a progressively measurable process. For $n \in \mathbb{N}$ and $R \geq 0$, let $\tau_n(R)$ be stopping times such that

$$|X_n(t)| + |p_n(t)| \leq R \text{ if } t \in (0, \tau_n(R)] \text{ a.e. } \omega \quad (55.5.56)$$

$$\lim_{n \rightarrow \infty} E \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| ds \right) = 0 \quad (55.5.57)$$

There exists a function $r : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{R \rightarrow \infty} r(R) = \infty$ and

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left([\tau_n(R) \leq T] \cap \left[\sup_{s \in [0, \tau_n(R)]} |X_n(s)| \leq r(R) \right] \right) = 0 \quad (55.5.58)$$

for all $t \in [0, T]$. Then

$$\sup_{t \in [0, T]} |X_n(t) - X_m(t)| \rightarrow 0 \quad (55.5.59)$$

in probability as $n, m \rightarrow \infty$.

Proof: First note that from 55.5.51 you can replace $K_t(R)$ with

$$K_t(R) \vee \sup_{|\mathbf{x}| \leq R} \left(\|\sigma(t, \mathbf{x})\|^2 + |\mathbf{b}(t, \mathbf{x})| \right)$$

Since σ and $\mathbf{b}$ are continuous as functions of $\mathbf{x}$, you could replace the above sup with the supremum over rational numbers less than or equal to R , a countable set and so the above sup delivers a $\mathcal{F}_t$ measurable function. Therefore, with no loss of generality, assume

$$\sup_{|\mathbf{x}| \leq R} |\mathbf{b}(t, \mathbf{x})| \leq K_t(R). \quad (55.5.60)$$

You can also replace $K_t(R)$ with $K_t(R) \vee K_t(1)$ if needed and without loss of generality assume it is also true that for $R \geq 1$,

$$K_t(1) \leq K_t(R). \quad (55.5.61)$$

Next define stopping times

$$\tau(R, u) \equiv \inf \{t \geq 0 : \alpha_t(R) > u\}$$

Then from 55.5.54 it follows that for a.e. ω , the integral in 55.5.54 is less than some u and so $\tau(R, u) = \infty$ for such u . Thus

$$\lim_{u \rightarrow \infty} \tau(R, u) = \infty.$$

Let $u(R)$ denote the smallest positive integer such that

$$P([\tau(R, u(R)) \leq R]) \leq \frac{1}{R}$$

Thus letting $\tau(R) \equiv \tau(R, u(R))$, it follows the set of ω such that $\alpha_t(R)$ exceeds $u(R)$ for some $t \leq R$ has measure no more than R^{-1} . Thus

$$P([\tau(R) > R]) > 1 - \frac{1}{R} \quad (55.5.62)$$

and so $\tau(R) \rightarrow \infty$ in probability as $R \rightarrow \infty$. Also

$$\alpha_{t \wedge \tau(R)}(R) \leq u(R) \quad (55.5.63)$$

for all $t \in [0, T]$. This is because $\tau(R)$ is the first time α_t equals $u(R)$ so in the above, if $t < \tau(R)$, then strict inequality takes place and if $t \geq \tau(R)$ then equality takes place.

Now it is desired to show that the litany of technical conditions holds for the stopping times $\tau_n(R) \wedge \tau(R)$. This is obvious for 55.5.56 and 55.5.57 because by making the stopping time smaller, it only makes these two conditions hold better. Consider the condition 55.5.58. For convenience, this condition is

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \sup P \left([\tau_n(R) \leq T] \cap \left[\sup_{s \in [0, \tau_n(R)]} |X_n(s)| \leq r(R) \right] \right) = 0$$

I have to show it still holds if $\tau_n(R)$ is replaced with $\tau_n(R) \wedge \tau(R)$. However,

$$\begin{aligned}
 & P \left([\tau_n(R) \wedge \tau(R) \leq T] \cap \left[\sup_{s \in [0, \tau_n(R) \wedge \tau(R)]} |X_n(s)| \leq r(R) \right] \right) \\
 & \leq P \left(\overbrace{[\tau_n(R) \leq T] \cap \left[\sup_{s \in [0, \tau_n(R)]} |X_n(s)| \leq r(R) \right]}^{\text{smaller one is } \tau_n(R)} \cap [\tau_n(R) \leq \tau(R)] \right) \\
 & \quad + P \left(\overbrace{[\tau_n(R) > \tau(R)] \cap [\tau(R) \leq T]}^{\tau_n(R) > \tau(R)} \right)
 \end{aligned}$$

This last converges to 0 as $R \rightarrow \infty$ independently of n because as shown above in 55.5.62 $\tau(R) \rightarrow \infty$ in probability and it is smaller than

$$P([\tau(R) \leq T]).$$

Then it follows

$$\begin{aligned}
 & \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left([\tau_n(R) \wedge \tau(R) \leq T] \cap \left[\sup_{s \in [0, \tau_n(R) \wedge \tau(R)]} |X_n(s)| \leq r(R) \right] \right) \\
 & \leq \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left([\tau_n(R) \leq T] \cap \left[\sup_{s \in [0, \tau_n(R)]} |X_n(s)| \leq r(R) \right] \right) = 0.
 \end{aligned}$$

This shows $\tau_n(R)$ can be replaced with $\tau_n(R) \wedge \tau(R)$. Hence making this change, assume without loss of generality

$$\alpha_{t \wedge \tau_n(R)}(R) \leq u(R), \quad u(R) \rightarrow \infty. \quad (55.5.64)$$

Claim:

$$\lim_{n \rightarrow \infty} E \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| K_s(R) ds \right) = 0.$$

Proof of claim: First recall that from the technical assumption 55.5.56,

$$|p_n(t)| \leq R$$

if $t \leq \tau_n(R)$. Therefore, for m a positive integer,

$$\begin{aligned} & E \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| K_s(R) ds \right) \\ & \leq \overbrace{mE \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| ds \right)}^{K_s(R) < m} \end{aligned} \quad (55.5.65)$$

$$+ E \left(R \int_0^{T \wedge \tau_n(R)} K_s(R) \mathcal{X}_{[m, \infty)}(K_s(R)) ds \right) \quad (55.5.66)$$

Consider the second term on the right in the above. The expression inside the expectation is dominated by

$$R (\alpha_{T \wedge \tau_n(R)}(R)) \leq u(R) R$$

by 55.5.64. Now from 55.5.54 which says the integral of $K_t(R)$ is finite, the dominated convergence theorem implies

$$\lim_{m \rightarrow \infty} R \int_0^{T \wedge \tau_n(R)} K_s(R) \mathcal{X}_{[m, \infty)}(K_s(R)) ds = 0$$

and now by the dominated convergence theorem again,

$$\lim_{m \rightarrow \infty} E \left(R \int_0^{T \wedge \tau_n(R)} K_s(R) \mathcal{X}_{[m, \infty)}(K_s(R)) ds \right) = 0$$

and so the last term of 55.5.66 is less than ε for large enough m . Pick such an m . Then by technical assumption 55.5.57,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} E \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| K_s(R) ds \right) \\ & \leq \limsup_{n \rightarrow \infty} mE \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| ds \right) + \varepsilon = \varepsilon \end{aligned}$$

Since ε is arbitrary, this proves the claim.

Now it is assumed

$$X_n(t) = X_0 + \int_0^t \mathbf{b}(s, X_n(s) + p_n(s)) ds + \int_0^t \sigma(s, X_n(s) + p_n(s)) dW(s)$$

and so

$$X_n(t) - X_m(t) = \int_0^t (\mathbf{b}(s, X_n(s) + p_n(s)) - \mathbf{b}(s, X_m(s) + p_m(s))) ds$$

$$+ \int_0^t (\sigma(s, X_n(s) + p_n(s)) - \sigma(s, X_m(s) + p_m(s))) dW(s) \quad (55.5.67)$$

Let $\psi_t(R) \equiv \exp(-2\alpha_t(R) - |X_0|)$ and let $F(t, X) \equiv \psi_t(R) |X|^2$. Thus

$$\frac{d}{dt}(\psi_t(R)) = \exp(-2\alpha_t(R) - |X_0|) (-2K_t(R))$$

and

$$F_{xx} = 2\psi_t(R) \text{id}$$

The Ito formula is

$$F(t, X(t)) = F(0, X(0)) + F_x(\cdot, X(\cdot)) \Phi \cdot W(t) +$$

$$\int_0^t \{F_t(s, X(s)) + F_x(s, X(s))(\phi(s)) + \frac{1}{2} \text{trace}(\Phi(s)^* F_{xx}(s, X(s)) \Phi(s))\} ds$$

Then using the Ito formula on 55.5.67 this F ,

$$\psi_t(R) |X_n(t) - X_m(t)|^2 =$$

$$\int_0^t [\psi_s(R) (-2K_s(R)) |X_n(s) - X_m(s)|^2 + 2\psi_s(R) (X_n(s) - X_m(s)) \cdot \mathbf{b}(s, X_n(s) + p_n(s)) - \mathbf{b}(s, X_m(s) + p_m(s))]_{\mathbb{R}^d} ds \quad (55.5.68)$$

$$+ \int_0^t \psi_s(R) \|\sigma(s, X_n(s) + p_n(s)) - \sigma(s, X_m(s) + p_m(s))\|^2 ds + M(t) \quad (55.5.69)$$

where $M(t)$ is a continuous local martingale with $M(0) = 0$. It isn't a martingale because the right sort of integrability does not necessarily hold on the term

$$F_x(\cdot, X_n(\cdot) - X_m(\cdot)) (\sigma(\cdot, X_n(\cdot)) - \sigma(\cdot, X_m(\cdot))).$$

However, it can be localized by stopping times of the form

$$\tau_r \equiv \inf \{t : |X_n(t) - X_m(t)| + |X_n(t)| + |X_m(t)| \geq r\}.$$

This is where you use the monotonicity technical assumption 55.5.52.

$$\begin{aligned} X_n(s) - X_m(s) &= X_n(s) + p_n(s) - (X_m(s) + p_m(s)) \\ &\quad + p_m(s) - p_n(s) \end{aligned}$$

Then substituting this in to 55.5.68, 55.5.69 yields

$$\psi_t(R) |X_n(t) - X_m(t)|^2 =$$

$$\begin{aligned}
& \int_0^t \{\psi_s(R) (-2K_s(R)) |X_n(s) - X_m(s)|^2 \\
& + \int_0^t \psi_s(R) (p_m(s) - p_n(s), \mathbf{b}(s, X_n(s) + p_n(s)) - \mathbf{b}(s, X_m(s) + p_m(s))) ds \\
& + 2\psi_s(R) (X_n(s) + p_n(s) - (X_m(s) + p_m(s)), \\
& \mathbf{b}(s, X_n(s) + p_n(s)) - \mathbf{b}(s, X_m(s) + p_m(s)))_{\mathbb{R}^d} ds\} \\
& + \int_0^t \psi_s(R) \|\sigma(s, X_n(s) + p_n(s)) - \sigma(s, X_m(s) + p_m(s))\|^2 ds + M(t) \quad (55.5.70)
\end{aligned}$$

Of course $M(t)$ depends on m, n but I am going to take expectation and it will end up being 0 so I will suppress this dependence. Hence from 55.5.52,

$$\psi_t(R) |X_n(t) - X_m(t)|^2 \leq \quad (55.5.71)$$

$$\begin{aligned}
& \int_0^t 2\psi_s(R) (p_m(s) - p_n(s), \mathbf{b}(s, X_n(s) + p_n(s)) - \mathbf{b}(s, X_m(s) + p_m(s))) ds \\
& \quad (55.5.72)
\end{aligned}$$

$$+ \int_0^t \psi_s(R) (-2K_s(R)) |X_n(s) - X_m(s)|^2 ds$$

$$+ \int_0^t \psi_s(R) K_s(R) (|X_n(s) - X_m(s) + (p_n(s) - p_m(s))|^2) + M(t) \quad (55.5.73)$$

At this point, use 55.5.60 which says

$$\sup_{|\mathbf{x}| \leq R} |\mathbf{b}(t, \mathbf{x})| \leq K_t(R).$$

If $t \leq \tau_n(R) \wedge \tau_m(R)$, then from 55.5.56,

$$|X_n(s)| + |p_n(s)| \leq R, |X_m(s)| + |p_m(s)| \leq R$$

and so the integral in 55.5.72 is dominated by

$$\int_0^t 4\psi_s(R) K_s(R) |p_m(s) - p_n(s)| ds$$

It follows the whole mess from 55.5.71 - 55.5.73 implies for $t \leq \tau_n(R) \wedge \tau_m(R)$

$$\begin{aligned}
\psi_t(R) |X_n(t) - X_m(t)|^2 & \leq \int_0^t 4\psi_s(R) K_s(R) |p_m(s) - p_n(s)| ds \\
& + \int_0^t \psi_s(R) (-2K_s(R)) |X_n(s) - X_m(s)|^2 ds \\
& + \int_0^t \psi_s(R) K_s(R) |X_n(s) - X_m(s)|^2 +
\end{aligned}$$

$$\begin{aligned} & \int_0^t 2\psi_s(R) K_s(R) (X_n(s) - X_m(s), p_n(s) - p_m(s)) ds \\ & + \int_0^t \psi_s(R) K_s(R) |p_n(s) - p_m(s)|^2 ds + M(t) \end{aligned} \quad (55.5.74)$$

Now 55.5.74 is dominated by

$$\int_0^t \psi_s(R) K_s(R) |X_n(s) - X_m(s)|^2 ds + \int_0^t \psi_s(R) K_s(R) |p_n(s) - p_m(s)|^2 ds$$

Replace 55.5.74 with this and observe several things cancel and this implies

$$\begin{aligned} \psi_t(R) |X_n(t) - X_m(t)|^2 & \leq \int_0^t 4\psi_s(R) K_s(R) |p_m(s) - p_n(s)| ds + \\ & \int_0^t 2\psi_s(R) K_s(R) |p_n(s) - p_m(s)|^2 ds + M(t) \end{aligned} \quad (55.5.75)$$

Now note that $|p_k(s)| \leq R$ if $k = m, n$ because $t \leq \tau_n(R) \wedge \tau_m(R)$ and so

$$|p_n(s) - p_m(s)|^2 \leq 2R(|p_n(s)| + |p_m(s)|)$$

It follows from 55.5.75 and that $\psi_s(R) \leq 1$

$$\begin{aligned} & \psi_t(R) |X_n(t) - X_m(t)|^2 \\ & \leq (4 + 2R) \int_0^t K_s(R) (|p_n(s)| + |p_m(s)|) ds + M(t) \end{aligned}$$

Now let δ_k be a stopping time like

$$\delta_k \equiv \inf \{t : |X_n(t) - X_m(t)| + |X_n(t)| + |X_m(t)| \geq k\}$$

for $k \in \mathbb{N}$. Then stopping with δ_k and using the localization lemma, Lemma 53.5.4, it follows

$$\begin{aligned} & \psi_{t \wedge \delta_k}(R) |X_n(t \wedge \delta_k) - X_m(t \wedge \delta_k)|^2 \\ & \leq (4 + 2R) \int_0^t \mathcal{X}_{[0, \delta_k]}(s) K_s(R) (|p_n(s)| + |p_m(s)|) ds + M(t \wedge \delta_k) \end{aligned}$$

where the last term is now a martingale. Letting τ be any stopping time, $\tau \leq \tau_n(R) \wedge \tau_m(R) \wedge t$, Then you can replace t with τ in the above using localization on the last term and then taking the expectation of both sides

$$\begin{aligned} & E \left(\psi_{\tau \wedge \delta_k}(R) |X_n(\tau \wedge \delta_k) - X_m(\tau \wedge \delta_k)|^2 \right) \\ & \leq (4 + 2R) E \left(\int_0^\tau \mathcal{X}_{[0, \delta_k]}(s) K_s(R) (|p_n(s)| + |p_m(s)|) ds \right) \end{aligned}$$

See Theorem 50.5.8. Letting $k \rightarrow \infty$ and using the dominated convergence theorem (Recall the $p_k(s)$ are even bounded.) and Fatou's lemma on the left, it follows

$$E \left(\psi_\tau(R) |X_n(\tau) - X_m(\tau)|^2 \right) \leq (4 + 2R) E \left(\int_0^\tau K_s(R) (|p_n(s)| + |p_m(s)|) ds \right) \quad (55.5.76)$$

and it was shown in the above claim that as $m, n \rightarrow \infty$, the right side of this expression converges to 0. Let $\tau_n(R) \wedge \tau_m(R) \wedge T \equiv \gamma_{n,m}$.

Here is where Lemma 55.5.3 will be used.

$$P \left(\left[\sup_{s \in [0, \gamma_{n,m}]} \psi_s(R) |X_n(s) - X_m(s)|^2 \geq \varepsilon \right] \right) \leq \frac{1}{\varepsilon} E \left(\psi_{\gamma_{n,m} \wedge \beta_\varepsilon}(R) |X_n(\gamma_{n,m} \wedge \beta_\varepsilon) - X_m(\gamma_{n,m} \wedge \beta_\varepsilon)|^2 \right)$$

where

$$\beta_\varepsilon \equiv \inf \left\{ s : \psi_s(R) |X_n(s) - X_m(s)|^2 \geq \varepsilon \right\}.$$

For fixed ε it follows from 55.5.76 that the term on the right of the inequality in the above converges to 0 as $m, n \rightarrow \infty$. Of course for $t \in [0, T]$, the definition of $\psi_t(R)$ implies it is bounded below by a positive number $a(\omega) > 0$ and so this shows from Lemma 55.5.2 that

$$\sup_{s \in [0, \gamma_{n,m}]} |X_n(s) - X_m(s)|^2 \quad (55.5.77)$$

converges in probability to 0 as $m, n \rightarrow \infty$. You can multiply by $\sup_{s \in [0, \gamma_{n,m}]} (\psi_s(R))^{-1}$ in this lemma. Note also $\gamma_{n,m} = \gamma_{n,m}(R)$.

Now consider the following.

$$\begin{aligned} & \left[\sup_{t \in [0, T]} |X_n(t) - X_m(t)|^2 \geq \varepsilon \right] \subseteq \left[\sup_{t \in [0, \gamma_{n,m}(R)]} |X_n(t) - X_m(t)|^2 \geq \varepsilon \right] \\ & \cup \left[\sup_{t \in [\gamma_{n,m}(R), T]} |X_n(t) - X_m(t)|^2 \geq \varepsilon \right] \cap \left([\tau_n(R) \leq T] \cup [\tau_m(R) \leq T] \right) \\ & \subseteq \left[\sup_{t \in [0, \gamma_{n,m}(R)]} |X_n(t) - X_m(t)|^2 \geq \varepsilon \right] \cup \left([\tau_n(R) \leq T] \cup [\tau_m(R) \leq T] \right) \end{aligned}$$

The reason for the inclusion of the union of the two terms on the end is that if both $\tau_n(R), \tau_m(R) > T$, then $[0, \gamma_{n,m}(R)] = \emptyset$ and so

$$\left[\sup_{t \in [0, \gamma_{n,m}(R)]} |X_n(t) - X_m(t)|^2 \geq \varepsilon \right] = \emptyset.$$

Thus

$$P\left(\left[\sup_{t \in [0, T]} |X_n(t) - X_m(t)|^2 \geq \varepsilon\right]\right) \leq P\left(\left[\sup_{t \in [0, \gamma_{n,m}(R)]} |X_n(t) - X_m(t)|^2 \geq \varepsilon\right]\right) \\ + P([\tau_n(R) \leq T]) + P([\tau_m(R) \leq T]) \quad (55.5.78)$$

Suppose it is shown that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P([\tau_n(R) \leq T]) = 0. \quad (55.5.79)$$

Then there exists R_0 such that if $R > R_0$,

$$\limsup_{n \rightarrow \infty} P([\tau_n(R) \leq T]) < \eta$$

Then fixing such an R , there exists N such that for $n > N$,

$$P([\tau_n(R) \leq T]) < \eta.$$

Then for that R and for $m, n > N$, the sum of the two terms in 55.5.64 is less than 2η . This with 55.5.77 will imply the desired conclusion of the lemma in 55.5.59. Therefore, it suffices to show 55.5.79.

This is done by using the coercivity assumption 55.5.53 instead of the monotonicity assumption. As pointed out earlier, it can be assumed $K_t(R) \geq K_t(1)$ for all $R \geq 1$. Recall the equation solved by $X_n(t)$,

$$X_n(t) = X_0 + \int_0^t \mathbf{b}(s, X_n(s) + p_n(s)) ds + \int_0^t \sigma(s, X_n(s) + p_n(s)) dW(s)$$

Also recall $\psi_t(1) \equiv \exp(-2\alpha_t(1) - |X_0|)$, and this time let $F(t, X) \equiv \psi_t(1) |X|^2$. Then using the Ito formula for $F(t, X_n(t))$

$$\psi_t(1) |X_n(t)|^2 = |X_0|^2 e^{-|X_0|} + \int_0^t \psi_s(1) \left[(-2K_s(R)) |X_n(s)|^2 + \right. \\ \left. 2\psi_s(1) (X_n(s), \mathbf{b}(s, X_n(s) + p_n(s))) + \psi_s(1) \|\sigma(s, X_n(s) + p_n(s))\|^2 \right] ds \\ + M(t)$$

where as before, $M(t)$ is a local martingale, $M(0) = 0$. It depends on R and n but as before, this is suppressed because I am just going to take expectation and it is going to disappear. Using 55.5.53 the above implies

$$\psi_t(1) |X_n(t)|^2 \leq |X_0|^2 e^{-|X_0|} + \int_0^t \psi_s(1) \left((-2K_s(1)) |X_n(s)|^2 + \right. \\ \left. \psi_s(1) K_s(1) (1 + |X_n(s) + p_n(s)|^2) \right) ds$$

$$+ \int_0^t 2\psi_s(1) (-p_n(s), \mathbf{b}(s, X_n(s) + p_n(s))) ds + M(t) \quad (55.5.80)$$

Now for $t \leq T \wedge \tau^n(R)$, this implies $|X_n(t)| + |p_n(t)| \leq R$ and so $|\mathbf{b}(s, X_n(s) + p_n(s))| \leq K_s(R)$. Recall $K_s(R) \geq K_s(1)$. Now it is elementary to show

$$|X_n(s) + p_n(s)|^2 \leq 2|X_n(s)|^2 + 2|p_n(s)|^2$$

and using these observations along with $\psi_s(1) \leq 1$ in 55.5.80, this implies

$$\begin{aligned} \psi_t(1) |X_n(t)|^2 &\leq |X_0|^2 e^{-|X_0|^2} + \int_0^t 2K_s(1) |p_n(s)|^2 + \psi_s(1) K_s(1) ds \\ &\quad + \int_0^t 2K_s(1) |p_n(s)| ds + M(t) \\ &\leq |X_0|^2 e^{-|X_0|^2} + \int_0^t 2K_s(R) |p_n(s)| (1 + |p_n(s)|) + e^{-2\alpha_s(1)} K_s(1) ds + M(t) \end{aligned}$$

The reason for that term $e^{-2\alpha_s(1)}$ is that

$$\psi_s(1) \leq e^{-2\alpha_s(1)}$$

Now this is convenient because

$$\int_0^t e^{-2\alpha_s(1)} K_s(1) ds \leq \int_0^\infty e^{-2\alpha_s(1)} K_s(1) ds = \frac{1}{2}$$

Therefore, the above inequality implies

$$\psi_t(1) |X_n(t)|^2 \leq |X_0|^2 e^{-|X_0|^2} + \frac{1}{2} + \int_0^t 2K_s(R) |p_n(s)| (1 + |p_n(s)|) ds + M(t)$$

As before, let τ be an arbitrary stopping time with $\tau \leq T \wedge \tau_n(R)$. Also let δ_k be a localizing sequence such that $M^{\delta_k}(t)$ is a martingale. Then first localizing with δ_k and then with τ , it follows from Theorem 50.5.8 that

$$E(M^{\delta_k}(\tau)) = E(E(M^{\delta_k}(\tau) | \mathcal{F}_0)) = E(M^{\delta_k}(0)) = 0$$

and so letting $k \rightarrow \infty$, and using Fatou's lemma and dominated convergence theorem as before,

$$E(\psi_\tau(1) |X_n(\tau)|^2) \leq E(|X_0|^2) + \frac{1}{2} + 2(1+R) E\left(\int_0^\tau K_s(R) |p_n(s)| ds\right) \quad (55.5.81)$$

By the **claim**, the limit as $n \rightarrow \infty$ of that last term is 0.

Therefore, from Lemma 55.5.3,

$$P\left(\left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 \psi_t(1) \geq c\right]\right) \leq \frac{1}{c} E(\psi_\tau(1) |X_n(\tau)|^2)$$

where here $\tau \equiv \inf \{t : Y(t) \geq c\} \wedge T \wedge \tau_n(R)$. From 55.5.81 and **claim 1** which says the last term of 55.5.81 converges to 0,

$$\limsup_{n \rightarrow \infty} P \left(\left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 \psi_t(1) \geq c \right] \right) \leq \frac{1}{c} \left(E(|X_0|^2) + \frac{1}{2} \right).$$

Therefore,

$$\limsup_{n \rightarrow \infty} P \left(\left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 \inf_{t \in [0, T]} \psi_t(1) \geq c \right] \right) \leq \frac{1}{c} \left(E(|X_0|^2) + \frac{1}{2} \right) \quad (55.5.82)$$

claim 2:

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left(\left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 \geq r(R) \right] \right) = 0 \quad (55.5.83)$$

Proof of claim: Let $g(\omega) \equiv \inf_{t \in [0, T]} \psi_t(1)$. This is a positive function, $g(\omega) \in (0, 1)$. Let

$$f_n(\omega) \equiv \sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2$$

Then 55.5.82 is of the form

$$\limsup_{n \rightarrow \infty} P([f_n g \geq c]) \leq \frac{1}{c} \left(E(|X_0|^2) + \frac{1}{2} \right)$$

Recall $r(R) \rightarrow \infty$ so the above implies

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P([f_n g \geq r(R)]) = 0 \quad (55.5.84)$$

Is the same true for $f_n g$ replaced with f_n alone? This is the content of the claim. If not, there exists $\varepsilon > 0$ and an increasing sequence $R_k \rightarrow \infty$ such that

$$\limsup_{n \rightarrow \infty} P([f_n \geq r(R_k)]) \geq 2\varepsilon$$

This requires there a subsequence n_k such that

$$P([f_{n_k} \geq r(R_k)]) > \varepsilon$$

Also from 55.5.84,

$$\lim_{k \rightarrow \infty} P([f_{n_k} g \geq r(R_k)]) = 0$$

Thus there exists a further subsequence, n_{k_l} such that

$$P([f_{n_{k_l}} g \geq r(R_{k_l})]) < 2^{-l}$$

It follows from the Borel Cantelli lemma, the set N of ω which are in infinitely many of the sets $\left[f_{n_{k_l}} g \geq r(R_{k_l}) \right]$ has measure 0. Thus for $\omega \notin N$ it follows that for all l large enough,

$$f_{n_{k_l}}(\omega) g(\omega) < r(R_{k_l})$$

Since $g(\omega) > 0$, it follows that for all l large enough

$$f_{n_{k_l}}(\omega) < r(R_{k_l})$$

which shows the set of ω contained in infinitely many of the sets $\left[f_{n_{k_l}} \geq r(R_{k_l}) \right]$ is contained in N . Thus

$$\cap_{l=1}^{\infty} \cup_{m=l}^{\infty} \left[f_{n_{k_m}} \geq r(R_{k_m}) \right] \subseteq N$$

but this is impossible because

$$\begin{aligned} P\left(\cap_{l=1}^{\infty} \cup_{m=l}^{\infty} \left[f_{n_{k_m}} \geq r(R_{k_m}) \right]\right) &= \lim_{l \rightarrow \infty} P\left(\cup_{m=l}^{\infty} \left[f_{n_{k_m}} \geq r(R_{k_m}) \right]\right) \\ &\geq \liminf_{l \rightarrow \infty} P\left(\left[f_{n_{k_l}} \geq r(R_{k_l}) \right]\right) \geq \varepsilon \end{aligned}$$

and N has measure zero. This proves the claim.

Now from

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left(\left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 \geq r(R)\right]\right) = 0, \quad (55.5.85)$$

it follows

$$\begin{aligned} &\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P([\tau_n(R) \leq T]) \leq \\ &\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left([\tau_n(R) \leq T] \cap \left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 \leq r(R)\right]\right) \\ &+ \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} P\left([\tau_n(R) \leq T] \cap \left[\sup_{t \in [0, T \wedge \tau_n(R)]} |X_n(t)|^2 > r(R)\right]\right) \end{aligned}$$

and the second of these on the right equals 0 by 55.5.85 and the first equals 0 from 55.5.58. Therefore, 55.5.79 is obtained and this proves the Lemma.

55.5.3 An Existence Theorem

Here an existence theorem is presented for finite dimensional stochastic differential equations. To review, $W(t)$ is a Wiener process on $\mathbb{R}^{d_1}$ with respect to the usual normal filtration $\mathcal{F}_t$. Thus $U \equiv \mathbb{R}^{d_1}$ and $Q \equiv I$. Let $H \equiv \mathbb{R}^d$.

$$(t, \mathbf{x}, \omega) \rightarrow \sigma(t, \mathbf{x}, \omega)$$

is progressively measurable and continuous in $\mathbf{x}$ into the space of $d \times d_1$ matrices with the usual Frobenius norm. Similarly

$$(t, \mathbf{x}, \omega) \rightarrow \mathbf{b}(t, \mathbf{x}, \omega)$$

is progressively measurable and continuous in $\mathbf{x}$ into $\mathbb{R}^d$. Assume also the following integrability condition.

$$\int_0^T \sup_{|\mathbf{x}| \leq R} \left\{ \|\sigma(t, \mathbf{x})\|^2 + |\mathbf{b}(t, \mathbf{x})| \right\} dt < \infty \text{ on } \Omega \quad (55.5.86)$$

The dependence on ω is suppressed. In addition to this, some other assumptions are made. For $|\mathbf{x}|, |\mathbf{y}| \leq R$,

$$2(\mathbf{x} - \mathbf{y}, \mathbf{b}(t, \mathbf{x}) - \mathbf{b}(t, \mathbf{y})) + \|\sigma(t, \mathbf{x}) - \sigma(t, \mathbf{y})\|^2 \leq K_t(R) |\mathbf{x} - \mathbf{y}|^2 \quad (55.5.87)$$

$$2(\mathbf{x}, \mathbf{b}(t, \mathbf{x})) + \|\sigma(t, \mathbf{x})\|^2 \leq K_t(1) (1 + |\mathbf{x}|^2) \quad (55.5.88)$$

In the above, $K_t(R)$ is an $[0, \infty)$ valued $\mathcal{F}_t$ adapted process for each $R \geq 0$. It is assumed to satisfy

$$\int_0^T K_t(R) dt < \infty \text{ a.e. } \omega \quad (55.5.89)$$

Also define for each $R \geq 0$ and $t \in [0, T]$

$$\alpha_t(R) \equiv \int_0^t K_s(R) ds. \quad (55.5.90)$$

Here is the main theorem.

Theorem 55.5.5 *Under the above conditions there exists a solution to the stochastic differential equation*

$$dX = A(t, X) dt + \sigma(t, X) dW, \quad X(0) = X_0$$

where X_0 is $\mathcal{F}_0$ measurable and in $L^2(\Omega)$. If the monotonicity condition 55.5.52 is modified to have

$$\sup_{R \geq 0} \sup_{\omega \in \Omega} \int_0^T K_t(R) dt \equiv \ln C < \infty \quad (55.5.91)$$

then the solution is unique. In fact if $X_{0k} \rightarrow X_0$ in probability, then letting X_k denote the above solution corresponding to initial condition X_{0k} , it follows

$$\sup_{t \in [0, T]} |X_k(t) - X(t)| \rightarrow 0 \quad (55.5.92)$$

in probability.

Proof: Consider X_n the solution to the following approximate problem

$$X_n(t) = X_0 + \int_0^t \mathbf{b}\left(s, X_n\left(s - \frac{1}{n}\right)\right) ds + \int_0^t \sigma\left(s, X_n\left(s - \frac{1}{n}\right)\right) dW(s)$$

The explanation for this is that $X(-s) \equiv X_0$ for all $s \geq 0$. This last term is only a local martingale. One can use a localizing sequence of the form

$$\tau_m = \inf\left\{t : \int_0^t \left\|\sigma\left(s, X_n\left(s - \frac{1}{n}\right)\right)\right\|^2 ds > m\right\}$$

Let

$$p_n(t) \equiv X_n\left(t - \frac{1}{n}\right) - X_n(t)$$

and define

$$\tau_n(R) \equiv \inf\left\{t \geq 0 : |X_n(t)| > \frac{R}{3}\right\}$$

Then if $t \in (0, \tau_n(R)]$,

$$|p_n(t)| \leq \left|X_n\left(t - \frac{1}{n}\right)\right| + |X_n(t)| \leq \frac{2R}{3}$$

and so $\tau_n(R)$ is such that for $t \leq \tau_n(R)$

$$|X_n(t)| + |p_n(t)| \leq R$$

These stopping times then satisfy the condition 55.5.56 of Lemma 55.5.4.

Let

$$r(R) \equiv \frac{R}{4}.$$

That horribly technical condition 55.5.58 is satisfied trivially. It involved the limit as $R \rightarrow \infty$ of the lim sup of the following is equal to 0.

$$P\left(\left[\tau_n(R) \leq T\right] \cap \left[\sup_{s \in [0, \tau_n(R)]} |X_n(s)| \leq r(R)\right]\right)$$

However, the set inside is $\emptyset$. This is because if $\tau_n(R) \leq T$, then $|X_n(s)| > R/4 = r(R)$ whenever s is close enough to $\tau_n(R)$ and so the intersection of the two sets above is empty because if ω is in the first of them, then it is not in the second.

Next it is necessary to consider 55.5.57. Recall this condition is

$$\lim_{n \rightarrow \infty} E\left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| ds\right) = 0.$$

First note that $|p_n(s)|$ is bounded by R for all $s \leq \tau_n(R)$. The idea is to show something about convergence in probability. By definition,

$$-p_n(t) = \int_{t-1/n}^t \mathbf{b}\left(s, X_n\left(s - \frac{1}{n}\right)\right) ds + \int_{t-1/n}^t \sigma\left(s, X_n\left(s - \frac{1}{n}\right)\right) dW(s)$$

Therefore,

$$P\left([|p_n(t)| \mathcal{X}_{[0, \tau_n(R)]}(t) \geq 2\varepsilon]\right) = P\left([|p_n(t)| \geq 2\varepsilon] \cap [t \leq \tau_n(R)]\right)$$

Since only $t \leq \tau_n(R)$ are in the above set, $|X_n(t)|, |X_n(t - 1/n)| \leq R$

$$\begin{aligned} &\leq P\left(\left[\int_{t-1/n}^t \sup_{|\mathbf{x}| \leq R} |\mathbf{b}(s, \mathbf{x})| ds \geq \varepsilon\right]\right) + \\ &P\left(\left[\sup_{r \in [0, t]} \left|\int_0^{r \wedge \tau_n(R)} \mathcal{X}_{[t-1/n, T]}(s) \sigma(s, X_n(s - 1/n)) dW(s)\right| \geq \varepsilon\right]\right) \\ &= P\left(\left[\int_{t-1/n}^t \sup_{|\mathbf{x}| \leq R} |\mathbf{b}(s, \mathbf{x})| ds \geq \varepsilon\right]\right) + \\ &P\left(\left[\sup_{r \in [0, t]} \left|\int_0^r \mathcal{X}_{[0, \tau_n(R)]}(s) \mathcal{X}_{[t-1/n, T]}(s) \sigma(s, X_n(s - 1/n)) dW(s)\right| \geq \varepsilon\right]\right) \end{aligned}$$

Now this last term can be estimated by the exotic Corollary 51.4.8 which comes from the amazing Burkholder Davis Gundy inequality of Theorem 51.4.7. It is dominated by

$$\begin{aligned} &\frac{C}{\varepsilon} E\left([\mathcal{X}_{[0, \tau_n(R)]} \mathcal{X}_{[t-1/n, T]} \sigma(s, X_n(\cdot - 1/n))]^{1/2}(t) \wedge \delta\right) \\ &+ P\left([\mathcal{X}_{[0, \tau_n(R)]} \mathcal{X}_{[t-1/n, T]} \sigma(\cdot, X_n(\cdot - 1/n))]^{1/2}(t) > \delta\right) \end{aligned}$$

Then using the characterization of the quadratic variation found in Corollary 53.5.5 this is dominated by an expression of the form

$$\frac{C}{\varepsilon} \delta + P\left(\left[\left(\int_{t-1/n}^t \sup_{|\mathbf{x}| \leq R} \|\sigma(s, \mathbf{x})\|^2 ds\right)^{1/2} > \delta\right]\right)$$

Thus

$$\begin{aligned} P\left([|p_n(t)| \mathcal{X}_{[0, \tau_n(R)]}(t) \geq 2\varepsilon]\right) &\leq P\left(\left[\int_{t-1/n}^t \sup_{|\mathbf{x}| \leq R} |\mathbf{b}(s, \mathbf{x})| ds \geq \varepsilon\right]\right) \\ &+ \frac{C}{\varepsilon} \delta + P\left(\left[\left(\int_{t-1/n}^t \sup_{|\mathbf{x}| \leq R} \|\sigma(s, \mathbf{x})\|^2 ds\right)^{1/2} > \delta\right]\right) \end{aligned}$$

Now first let $n \rightarrow \infty$. Then by 55.5.86

$$\lim_{n \rightarrow \infty} \sup P\left([|p_n(t)| \mathcal{X}_{[0, \tau_n(R)]}(t) \geq 2\varepsilon]\right) \leq \frac{C}{\varepsilon} \delta$$

Next since δ is arbitrary, it follows

$$\lim_{n \rightarrow \infty} \sup P \left([|p_n(t)| \mathcal{X}_{[0, \tau_n(R)]}(t) \geq 2\varepsilon] \right) = 0$$

Thus $p_n(t) \mathcal{X}_{[0, \tau_n(R)]}(t) \rightarrow 0$ in probability.

This is enough to verify 55.5.57. By definition of $\tau_n(R)$,

$$|p_n(t)| \mathcal{X}_{[0, \tau_n(R)]}(t) \leq R$$

Then using the distribution function, Theorem 9.7.4,

$$\begin{aligned} E \left(\int_0^{T \wedge \tau_n(R)} |p_n(s)| ds \right) &= \int_0^T \int_{\Omega} \mathcal{X}_{[0, \tau_n(R)]}(t) |p_n(t)| dP dt \\ &= \int_0^T \int_0^R P \left([\mathcal{X}_{[0, \tau_n(R)]}(t) |p_n(t)| > \lambda] \right) d\lambda dt \end{aligned}$$

and now by the convergence in probability and the above bound along with the dominated convergence theorem, this converges to 0. This verifies 55.5.57.

It follows the conditions of Lemma 55.5.4 are satisfied and so

$$\sup_{t \in [0, T]} |X_n(t) - X_m(t)| \rightarrow 0$$

in probability. Let n_k be such that if $m, n \geq n_k$, then

$$P \left(\left[\sup_{t \in [0, T]} |X_n(t) - X_m(t)| \geq 2^{-k} \right] \right) \leq 2^{-k}$$

It follows there exists a set of measure zero N such that if $\omega \notin N$ then $\{X_{n_k}(t)\}_{k=1}^{\infty}$ is uniformly Cauchy. Thus there exists a continuous $t \rightarrow X(t)$ such that X_{n_k} converges uniformly to X . Now recall

$$X_{n_k}(t) = X_0 + \int_0^t \mathbf{b}(s, X_{n_k}(s - 1/n_k)) ds + \int_0^t \sigma(s, X_{n_k}(s - 1/n_k)) dW(s) \quad (55.5.93)$$

and it is necessary to pass to the limit. Consider the deterministic integral first. For $\omega \notin N$

$$\begin{aligned} & \left| \int_0^t \mathbf{b}(s, X_{n_k}(s - 1/n_k)) ds - \int_0^t \mathbf{b}(s, X(s)) ds \right| \\ & \leq \left| \int_0^t \mathbf{b}(s, X_{n_k}(s - 1/n_k)) ds - \int_0^t \mathbf{b}(s, X(s - 1/n_k)) ds \right| \\ & \quad + \left| \int_0^t \mathbf{b}(s, X(s - 1/n_k)) ds - \int_0^t \mathbf{b}(s, X(s)) ds \right| \quad (55.5.94) \end{aligned}$$

In order to pass to a limit, define

$$S(t) \equiv \sup_{k \in \mathbb{N}} |X_{n_k}(t - 1/n_k)|$$

Now by uniform convergence,

$$|X_{n_k}(t - 1/n_k)| \leq |X_{n_k}(t - 1/n_k) - X(t - 1/n_k)| + |X(t - 1/n_k)|$$

Hence $S(t)$ is bounded independent of $t \in [0, T]$ for each $\omega \notin N$. Now define

$$\tau(R) \equiv \inf \{t \in [0, T] : S(t) > R\}$$

Then it follows from the integrability condition 55.5.86 and the continuity of $\mathbf{b}$ in $\mathbf{x}$ and 55.5.94 that

$$\lim_{k \rightarrow \infty} \int_0^t \mathcal{X}_{[0, \tau(R)]} \mathbf{b}(s, X_{n_k}(s - 1/n_k)) ds = \int_0^t \mathcal{X}_{[0, \tau(R)]} \mathbf{b}(s, X(s)) ds$$

Next consider the stochastic term. Let another stopping time be defined.

$$\tau_N(R) \equiv \inf \left\{ t \in [0, T] : \int_0^t \sup_{|\mathbf{x}| \leq R} \|\sigma(s, \mathbf{x})\|^2 ds > N \right\} \wedge \tau(R)$$

Then by 55.5.86 again and dominated convergence theorem

$$\lim_{k \rightarrow \infty} E \left(\int_0^{\tau_N(R)} \|\sigma(s, X_{n_k}(s - 1/n_k)) - \sigma(s, X(s))\|^2 ds \right) = 0$$

Note how 55.5.86 provides the dominating function and then the convergence of the integrand comes from continuity of σ in the second argument. By the Ito isometry, it follows as $k \rightarrow \infty$,

$$\int_0^t \mathcal{X}_{[0, \tau_N(R)]} \sigma(s, X_{n_k}(s - 1/n_k)) dW(s) \rightarrow \int_0^t \mathcal{X}_{[0, \tau_N(R)]} \sigma(s, X(s)) dW(s)$$

in $L^2(\Omega)$. Hence this convergence also takes place in probability. Thus for each $N \in \mathbb{N}$ there exists a set of measure zero, S_N and a further subsequence denoted by $\{j\}$ such that for $\omega \notin S_N$

$$\int_0^t \mathcal{X}_{[0, \tau_N(R)]} \sigma(s, X_j(s - 1/j)) dW(s) \rightarrow \int_0^t \mathcal{X}_{[0, \tau_N(R)]} \sigma(s, X(s)) dW(s)$$

pointwise. It is eventually the case that for N large enough, $\tau_N(R) = \tau(R)$. This follows from the integrability condition again 55.5.86,

$$\inf \left\{ t \in [0, T] : \int_0^t \sup_{|\mathbf{x}| \leq R} \|\sigma(s, \mathbf{x})\|^2 ds > N \right\} = \infty$$

if N is large enough. Then if

$$S \equiv \cup_{N \in \mathbb{N}} S_N$$

it is a set of measure zero and using a Cantor diagonalization procedure if necessary, there exists a subsequence $\{i\}$ such that if $\omega \notin S$, then

$$\int_0^t \mathcal{X}_{[0, \tau_N(R)]} \sigma(s, X_i(s - 1/i)) dW(s) \rightarrow \int_0^t \mathcal{X}_{[0, \tau_N(R)]} \sigma(s, X(s)) dW(s)$$

pointwise for each N . Using this subsequence in 55.5.93, it follows that for a.e. ω ,

$$X(t) = X_0 + \int_0^t \mathcal{X}_{[0, \tau(R)]} \mathbf{b}(s, X(s)) ds + \int_0^t \mathcal{X}_{[0, \tau(R)]} \sigma(s, X(s)) dW(s)$$

where $\tau(R) \equiv \inf \{t \in [0, T] : S(t) > R\}$. Since

$$\sup_{t \in [0, T]} S(t) < \infty$$

for a.e. ω , it follows that for any $t \in [0, T]$, eventually for R large enough, $\tau(R) = \infty$ and so the above equation implies

$$X(t) = X_0 + \int_0^t \mathbf{b}(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW(s)$$

This proves the existence part of the theorem.

Now consider the uniqueness part, 55.5.92. From assumption

$$X_k(t) = X_{0k} + \int_0^t \mathbf{b}(s, X_k(s)) ds + \int_0^t \sigma(s, X_k(s)) dW(s) \quad (55.5.95)$$

$$X(t) = X_0 + \int_0^t \mathbf{b}(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW(s) \quad (55.5.96)$$

Since $X_{0k} \rightarrow X_0$ in probability, it follows

$$\sup_k |X_{0k}| < \infty$$

for a.e. ω . This is because X_0 has values in $\mathbb{R}^d$. Consider then the positive function

$$F(t, \mathbf{x}) \equiv e^{-\alpha_t(R) - (\sup_k |X_{0k}| + |X_0|) |\mathbf{x}|^2} \equiv \phi_t(R) |\mathbf{x}|^2$$

where

$$\alpha_t(R) \equiv \int_0^t K_s(R) ds.$$

Note

$$\phi_\tau(R) \geq \phi_T(R) \geq \frac{1}{C} \exp \left(- \left(\sup_k |X_{0k}| + |X_0| \right) \right)$$

which is positive. As before,

$$\begin{aligned} F_t(t, \mathbf{x}) &= (-K_t(R)) \phi_t(R) |\mathbf{x}|^2, \quad F_x(t, \mathbf{x}) = 2\phi_t(R) \mathbf{x}, \\ F_{xx}(t, \mathbf{x}) &= 2\phi_t(R) \text{id} \end{aligned}$$

Then by the Ito formula 54.1.1 applied to the difference of 55.5.95 and 55.5.96,

$$\begin{aligned} & \phi_t(R) |X_k(t) - X(t)|^2 \\ &= e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 \\ & \quad + \int_0^t 2\phi_s(R) (\mathbf{b}(s, X_k(s)) - \mathbf{b}(s, X(s)), X_k(s) - X(s)) ds \\ & \quad + \int_0^t \phi_s(R) \|\sigma(s, X_k(s)) - \sigma(s, X(s))\|^2 ds + M(t) \end{aligned}$$

where $M(t)$ is a local martingale equal to 0 when $t = 0$ which equals a suitable Ito integral. Define the stopping time $\tau_k(R)$

$$\tau_k(R) \equiv \inf \{t \in [0, T] : |X(t)| + |X_k(t)| > R\}$$

Then by 55.5.52 it follows that if $t \leq \tau_k(R)$, then the estimate shown there holds. Thus

$$\begin{aligned} \phi_t(R) |X_k(t) - X(t)|^2 &\leq e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 \\ & \quad + \int_0^t K_s(R) \phi_s(R) |X_k(s) - X(s)|^2 ds + M(t) \end{aligned}$$

Now by Gronwall's inequality,

$$\phi_t(R) |X_k(t) - X(t)|^2 \leq C \left(e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 + M(t) \right)$$

where C is given as in 55.5.91. Thus $CM(t)$ is also a local martingale and to save notation denote this product by $M(t)$.

Localizing with τ a stopping time $\leq \tau_k(R)$ and then taking the expectation using Theorem 50.5.8, it follows $E(M(\tau) | \mathcal{F}_0) = M(0) = 0$ and so

$$E(M(\tau)) = E(E(M(\tau) | \mathcal{F}_0)) = E(M(0)) = 0$$

so the above implies

$$E\left(\phi_\tau(R) |X_k(\tau) - X(\tau)|^2\right) \leq E\left(C e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2\right)$$

It follows from Lemma 55.5.3 that

$$P\left(\left[\sup_{t \in [0, T \wedge \tau_k(R)]} \phi_t(R) |X_k(t) - X(t)|^2 \geq \varepsilon\right]\right) =$$

$$\begin{aligned}
& P \left(\left[\sup_{t \in [0, T]} |\phi_{t \wedge \tau_k}(R) - X(t \wedge \tau_k(R))|^2 \geq \varepsilon \right] \right) \\
& \leq \frac{1}{\varepsilon} E \left(|\phi_\tau(R) - X(\tau)|^2 \right) \leq \frac{1}{\varepsilon} E \left(C e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 \right)
\end{aligned} \tag{55.5.97}$$

where

$$\tau \equiv \inf \left\{ t \in [0, T] : |X_k(t) - X(t)|^2 \geq \varepsilon \right\} \wedge \tau_k(R)$$

Now $C e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2$ converges to 0 in probability as $k \rightarrow \infty$. Also there is some constant K independent of k such that

$$e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 \leq 2 e^{-(\sup_k |X_{0k}| + |X_0|)} \left(|X_{0k}|^2 + |X_0|^2 \right) \leq K$$

Thus from the distribution function, Theorem 9.7.4

$$\begin{aligned}
& E \left(C e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 \right) \\
& = C \int_0^K P \left(\left[e^{-(\sup_k |X_{0k}| + |X_0|)} |X_{0k} - X_0|^2 > x \right] \right) dx
\end{aligned}$$

and so by the dominated convergence theorem and the assumption $X_{0k} \rightarrow X_0$ in probability, this converges to 0. Thus from 55.5.97 and the observation that $\phi_t(R) \geq \phi_T(R)$ is positive, it follows

$$\lim_{k \rightarrow \infty} \sup P \left(\left[\sup_{t \in [0, T]} |X_k(t \wedge \tau_k(R)) - X(t \wedge \tau_k(R))|^2 \geq \varepsilon \right] \right) = 0. \tag{55.5.98}$$

It only remains to get rid of the $\tau_k(R)$ in the above.

$$\begin{aligned}
& P([\tau_k(R) < T]) \leq \\
& P \left(\left[\sup_{t \in [0, T]} |X_k(t \wedge \tau_k(R))| + |X(t \wedge \tau_k(R))| \geq R \right] \right)
\end{aligned}$$

Now

$$\begin{aligned}
& |X_k(t \wedge \tau_k(R))| + |X(t \wedge \tau_k(R))| \\
& \leq |X_k(t \wedge \tau_k(R)) - X(t \wedge \tau_k(R))| + 2 |X(t \wedge \tau_k(R))|
\end{aligned}$$

and so

$$\begin{aligned}
& P([\tau_k(R) < T]) \leq \\
& P \left(\left[\sup_{t \in [0, T]} |X_k(t \wedge \tau_k(R)) - X(t \wedge \tau_k(R))| \geq 1 \right] \right) \\
& + P \left(\left[2 \sup_{t \in [0, T]} |X(t)| \geq R - 1 \right] \right)
\end{aligned}$$

By 55.5.98 $\limsup_{k \rightarrow \infty}$ of the first of these on the right is 0 and since X is continuous, $\lim_{R \rightarrow \infty}$ of the second is also 0. Therefore,

$$\lim_{R \rightarrow \infty} \limsup_{k \rightarrow \infty} P([\tau_k(R) < T]) = 0.$$

Now with this, the desired result follows.

$$\begin{aligned} & P\left(\left[\sup_{t \in [0, T]} |X_k(t) - X(t)| \geq \varepsilon\right]\right) \leq P([\tau_k(R) < T]) + \\ & P\left(\left[\sup_{t \in [0, T]} |X_k(t \wedge \tau_k(R)) - X(t \wedge \tau_k(R))| \geq \varepsilon\right] \cap [\tau_k(R) \geq T]\right) \\ & \leq P\left(\left[\sup_{t \in [0, T]} |X_k(t \wedge \tau_k(R)) - X(t \wedge \tau_k(R))| \geq \varepsilon\right]\right) + P([\tau_k(R) < T]) \end{aligned}$$

Now take $\limsup_{k \rightarrow \infty}$ and use 55.5.98 to conclude

$$\limsup_{k \rightarrow \infty} P\left(\left[\sup_{t \in [0, T]} |X_k(t) - X(t)| \geq \varepsilon\right]\right) \leq \limsup_{k \rightarrow \infty} P([\tau_k(R) < T])$$

Then take $\lim_{R \rightarrow \infty}$ of both sides to conclude

$$\limsup_{k \rightarrow \infty} P\left(\left[\sup_{t \in [0, T]} |X_k(t) - X(t)| \geq \varepsilon\right]\right) = 0$$

and this proves the uniqueness part of the theorem.

55.6 Infinite Dimensional Problems

First Gelfand triples were considered. Then a general Ito formula was presented. Next assumptions were placed on operators which would allow one to pass from finite to infinite dimensional problems assuming a solution to appropriate finite dimensional problems was known. Finally, a general existence and uniqueness theorem was presented for a finite dimensional problem. It only remains to verify that the conditions on the operators which allowed passage from finite dimensional problems to the infinite dimensional problems are sufficient to give a unique solution to the corresponding finite dimensional problem. Recall the conditions which were used to go from finite to infinite dimensional problems. These were as follows. First you have a Gelfand triple V, H, V' .

Next there is a $Q_1 \equiv JJ^*$ Wiener process defined on U_1 where $J: U \rightarrow U_1$ is a Hilbert Schmidt operator for U a separable real Hilbert space.

$$W(t) \equiv \sum_{k=1}^{\infty} \psi_k(t) Jg_k$$

$$W_n(t) \equiv \sum_{k=1}^n \psi_k(t) g_k$$

Next there are operators A, σ as follows.

$$A : [0, T] \times V \times \Omega \rightarrow V', \quad \sigma : [0, T] \times H \times \Omega \rightarrow \mathcal{L}_2(U, H)$$

1. (Hemicontinuity) For a.e. ω

$$\lambda \rightarrow \langle A(t, u + \lambda v, \omega), z \rangle \text{ is continuous.}$$

2. (Weak monotonicity) There exists $c \in \mathbb{R}$ such that for all $u, v \in V$,

$$\begin{aligned} & 2 \langle A(\cdot, u) - A(\cdot, v), u - v \rangle + |\sigma(\cdot, u) - \sigma(\cdot, v)|_{\mathcal{L}_2(U, H)}^2 \\ & \leq c \|u - v\|_H^2 \text{ on } [0, T] \times \Omega \end{aligned}$$

3. (Coercivity) There exist $p \in (1, \infty)$, $c_1 \in \mathbb{R}$, $c_2 \in (0, \infty)$, and $f \in L^1([0, T] \times \Omega)$ which is $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable and is also $\mathcal{F}_t$ adapted such that for all $v \in V, t \in [0, T]$,

$$2 \langle A(t, v), v \rangle + \|\sigma(t, v)\|_{\mathcal{L}_2(U, H)}^2 \leq c_1 \|v\|_H^2 - c_2 \|v\|_V^p + f(t)$$

4. (Boundedness) There exist $c_3 \in [0, \infty)$ and an $\mathcal{F}_t$ adapted process $g \in L^{p'}([0, T] \times \Omega)$ which is $\mathcal{B}([0, T]) \times \mathcal{F}$ measurable such that for all $v \in V, t \in [0, T]$

$$\|A(t, v)\|_{V'} \leq g(t) + c_3 \|v\|_V^{p-1}.$$

Here $1/p + 1/p' = 1$.

5. (Measurability) Let

$$A : [0, T] \times V \times \Omega \rightarrow V', \quad \sigma : [0, T] \times H \times \Omega \rightarrow \mathcal{L}_2(U; H)$$

be progressively measurable.

Next recall the finite dimensional problems. First there is the orthogonal projection P_n which maps V' onto H_n , a finite dimensional subspace of V which consists of the span of the first n vectors in $\{e_k\}_{k=1}^\infty$, an orthonormal basis for H with the property that each $e_k \in V$. This is given by

$$P_n y \equiv \sum_{k=1}^n \langle y, e_k \rangle e_k.$$

For $\sigma \in \mathcal{L}_2(U, H)$,

$$P_n \sigma(u) \equiv \sum_{k=1}^n (\sigma u, e_k) e_k$$

Then the finite dimensional problem is of the form

$$\begin{aligned} dX_n &= P_n A(t, X_n(t)) + P_n \sigma(t, X_n(t)) dW_n(t) \\ X_n(0) &= P_n X_0 \end{aligned}$$

where to begin with $X_0 \in V$. For each $X_n \in H_n$, $P_n \sigma(t, X_n(t)) \in \mathcal{L}_2(U, H_n)$. In particular, it can be considered in $\mathcal{L}_2(U_n, H_n)$ by taking its restriction to $\text{span}(g_1, \dots, g_n)$ where $\{g_k\}$ is an orthonormal basis for U . Thus identifying H_n with $\mathbb{R}^n$ and $\text{span}(g_1, \dots, g_n)$ with $\mathbb{R}^n$ also, one can regard $P_n \sigma(t, X_n(t))$ as an $n \times n$ matrix or linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. It only remains to verify the conditions for existence for finite dimensional problems.

That $P_n A(t, \mathbf{x})$ is a continuous as a function of $\mathbf{x} \in H_n$ follows from the fact proved in Proposition 55.4.5 that $X \rightarrow A(t, X)$ is demicontinuous and the observation that H_n is finite dimensional. That $P_n \sigma(t, \mathbf{x})$ is continuous as a function of $\mathbf{x} \in H_n$ follows from 55.4.36. What about the monotonicity assumptions necessary to prove the above finite dimensional problem? For $|\mathbf{x}|_{H_n}, |\mathbf{y}|_{H_n} \leq R$,

$$\begin{aligned} &2 \langle \mathbf{x} - \mathbf{y}, P_n A(t, \mathbf{x}) - P_n A(t, \mathbf{y}) \rangle_{H_n} + \|P_n \sigma(t, \mathbf{x}) - P_n \sigma(t, \mathbf{y})\|^2 \\ &\leq 2 \langle A(t, \mathbf{x}) - A(t, \mathbf{y}), \mathbf{x} - \mathbf{y} \rangle + \|\sigma(t, \mathbf{x}) - \sigma(t, \mathbf{y})\|^2 \leq c |\mathbf{x} - \mathbf{y}|_{H_n}^2 \end{aligned}$$

by the property assumed on A in 2. This certainly implies the monotonicity condition used in proving both existence and uniqueness in the finite dimensional problem 55.5.87, 55.5.91. In fact this gives a stronger condition because there was no provision for fussing with the size of $|\mathbf{x}|, |\mathbf{y}|$.

Next consider whether the coercivity condition 55.5.88,

$$\begin{aligned} &2 \langle \mathbf{x}, P_n A(t, \mathbf{x}) \rangle + \|P_n \sigma(t, \mathbf{x})\|^2 \\ &\leq 2 \langle A(t, \mathbf{x}), \mathbf{x} \rangle + \|\sigma(t, \mathbf{x})\|^2 \end{aligned}$$

and by the coercivity condition 3, this is dominated by

$$\leq c_1 |\mathbf{x}|_{H_n}^2 - c_2 \|\mathbf{x}\|_V^p + f(t) \leq c_1 |\mathbf{x}|_{H_n}^2 + f(t) \quad (55.6.99)$$

$$\begin{aligned} &\leq f(t) \left(1 + |\mathbf{x}|_{H_n}^2\right) + c_1 \left(1 + |\mathbf{x}|_{H_n}^2\right) \\ &= (c_1 + f(t)) \left(1 + |\mathbf{x}|_{H_n}^2\right) \end{aligned}$$

and so condition 55.5.53 is verified with $K_t(1) = (c_1 + f(t))$. The integrability condition 55.5.51 is satisfied because of 4 and 55.4.37. This completes the proof of the following theorem.

Theorem 55.6.1 *Suppose 1 - 5 on the operators A and σ . Then there exists a unique solution to the stochastic differential equation*

$$\begin{aligned} dX &= A(t, X(t)) dt + \sigma(t, X(t)) dW, \\ X(0) &= X_0 \in L^p(\Omega; V) \cap L^2(\Omega; H) \end{aligned}$$

More precisely,

$$\begin{aligned} X(t) &= X_0 + \int_0^t A(s, \overline{X}(s)) ds + \int_0^t \sigma(s, X(s)) dW(s), \\ X_0 &\in L^p(\Omega; V) \cap L^2(\Omega; H), \quad X_0 \text{ is } \mathcal{F}_0 \text{ measurable} \end{aligned}$$

where $\overline{X}$ is a measurable representative of X which is in

$$L^p([0, T] \times \Omega; V) \cap L^2([0, T] \times \Omega; H)$$

and is V and H progressively measurable.

Proof: Under the conditions 1 - 5 there exists a unique solution to the finite dimensional problem 55.4.8. Therefore, the conclusion of this theorem follows from Theorem 55.4.10.

Another Approach, Semigroups

56.1 Stochastic Fubini Theorem

Let $(t, \omega, x) \rightarrow \Phi(t, \omega, x)$ be $\mathcal{P}_T \times \mathcal{B}(E)$ measurable. This is product measure where $\mathcal{P}_T$ is the predictable σ algebra and E is a real separable Banach space. Let μ be a finite measure defined on $\mathcal{B}(E)$. Suppose also that

$$\int_E \left(\int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu < \infty \quad (56.1.1)$$

It follows that for μ a.e. x , $\int_{\Omega} \int_0^T \|\Phi\|^2 dt dP < \infty$ and so for these values of x it makes perfect sense to write the stochastic integral

$$\Phi(\cdot, \cdot, x) \cdot W(T).$$

Does it also make sense to write

$$\int_E \Phi(\cdot, \cdot, x) \cdot W(T) d\mu? \quad (56.1.2)$$

You would need to have a set of μ measure 0 such that for E_0 the complement of this set,

$$x \rightarrow \chi_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)(\omega)$$

is Borel measurable and also

$$\int_E |\chi_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)|_H d\mu < \infty \quad (56.1.3)$$

at least for a.e. ω .

From the Ito Isometry and 56.1.1,

$$\begin{aligned}
 \infty &> \int_E \left(\int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu \\
 &= \int_E \left(\int_{\Omega} \left(|\Phi(\cdot, \cdot, x) \cdot W(T)|_H^2 \right) dP(\omega) \right)^{1/2} d\mu(x) \\
 &= \int_E \left(\int_{\Omega} \left(|\mathcal{X}_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)|_H^2 \right) dP(\omega) \right)^{1/2} d\mu(x) \\
 &\geq \int_E \int_{\Omega} |\mathcal{X}_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)|_H dP(\omega) d\mu(x)
 \end{aligned}$$

If it is possible to interchange the integrals, it would follow 56.1.3 would hold for a.e. ω . Thus it suffices to show for some set E_0 , the complement of a set of μ measure 0,

$$(\omega, x) \rightarrow \mathcal{X}_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)(\omega)$$

is $\mathcal{F}_T \times \mathcal{B}(E)$ measurable because this will establish both 56.1.2 and 56.1.3.

Here is an approximation lemma.

Lemma 56.1.1 *Suppose $(t, \omega, x) \rightarrow \Phi(t, \omega, x)$ is $\mathcal{P}_T \times \mathcal{B}(E)$ measurable and*

$$\int_E \left(\int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu < \infty \quad (56.1.4)$$

Then there exists a sequence $\{\Phi_n\}$ of mappings which are of the form

$$\Phi_n(t, \omega, x) \equiv \sum_{j=1}^{J_n} \sum_{k=1}^{K_n} \sum_{l=1}^{L_n} \alpha_{kl}^n H_j \phi_{jk}^n(x) \mathcal{X}_{(s_{kl}^n, t_{kl}^n]}(t) \mathcal{X}_{F_{kl}^n}(\omega)$$

where the intervals $(s_{kl}^n, t_{kl}^n]$ are contained in $[0, T]$, α_{kl}^n are constants, H_j are in $\mathcal{L}_2(U, H)$, $\phi_{jk}^n(x)$ are bounded Borel measurable functions, and $F_{kl}^n \in \mathcal{F}_{s_{kl}^n}$ such that also

$$\lim_{n \rightarrow \infty} \int_E \left(\int_{\Omega} \int_0^T \|\Phi(t, \omega, x) - \Phi_n(t, \omega, x)\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu = 0. \quad (56.1.5)$$

Now assuming the approximation lemma, it is obvious from the formula that

$$(\omega, x) \rightarrow \Phi_n(\cdot, \cdot, x) \cdot W(T)(\omega)$$

is $\mathcal{F}_T \times \mathcal{B}(E)$ measurable and in any other way desired. From 56.1.5

$$\lim_{m \rightarrow \infty} \int_E \left(\int_{\Omega} \int_0^T \|\Phi_m(t, \omega, x) - \Phi(t, \omega, x)\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu = 0$$

and so by the Ito isometry,

$$\lim_{m \rightarrow \infty} \int_E \left(\int_{\Omega} |\Phi_m(\cdot, \cdot, x) \cdot W(T) - \Phi(\cdot, \cdot, x) \cdot W(T)|_H^2 dP \right)^{1/2} d\mu = 0 \quad (56.1.6)$$

and there exists a subsequence, m_k such that

$$\mu \left(\left[\left(\int_{\Omega} |\Phi_{m_k}(\cdot, \cdot, x) \cdot W(T) - \Phi(\cdot, \cdot, x) \cdot W(T)|_H^2 dP \right)^{1/2} \geq 2^{-k} \right] \right) \leq 2^{-k}$$

and so for x off a set of measure zero,

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\Phi_{m_k}(\cdot, \cdot, x) \cdot W(T) - \Phi(\cdot, \cdot, x) \cdot W(T)|_H^2 dP = 0$$

which implies

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\Phi_{m_k}(\cdot, \cdot, x) \cdot W(T) - \Phi(\cdot, \cdot, x) \cdot W(T)|_H dP = 0.$$

Letting E_0 denote the complement of this exceptional set, the following holds for all x .

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\mathcal{X}_{E_0}(x) \Phi_{m_k}(\cdot, \cdot, x) \cdot W(T) - \mathcal{X}_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)|_H dP = 0$$

showing that $(\omega, x) \rightarrow \mathcal{X}_{E_0}(x) \Phi(\cdot, \cdot, x) \cdot W(T)(\omega)$ is $\mathcal{F}_T \times \mathcal{B}(E)$ measurable. It follows it makes sense to write

$$\int_E \Phi(\cdot, \cdot, x) \cdot W(T) d\mu.$$

Now it is reasonable to ask whether this equals

$$\int_E \Phi(\cdot, \cdot, x) d\mu \cdot W(T). \quad (56.1.7)$$

First consider whether this expression makes sense. It clearly does for Φ_n in place of Φ because by construction,

$$\int_E \Phi_n(\cdot, \cdot, x) d\mu$$

is $\mathcal{L}_2^0$ predictable and satisfies

$$\begin{aligned} & \int_{\Omega} \int_0^T \left\| \int_E \Phi_n(t, \omega, x) d\mu \right\|_{\mathcal{L}_2^0}^2 dt dP \\ & \leq C \int_{\Omega} \int_0^T \left(\int_E \|\Phi_n(t, \omega, x)\|_{\mathcal{L}_2^0}^2 d\mu \right) dt dP < \infty \end{aligned}$$

Also from 56.1.5

$$\lim_{n \rightarrow \infty} \int_E \left(\int_{\Omega} \int_0^T \|\Phi(t, \omega, x) - \Phi_n(t, \omega, x)\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu = 0$$

and so by Minkowski's inequality, (Remember Φ was $\mathcal{P}_T \times \mathcal{B}(E)$ measurable.)

$$\lim_{n \rightarrow \infty} \left(\int_{\Omega} \int_0^T \left(\left\| \int_E (\Phi(t, \omega, x) - \Phi_n(t, \omega, x)) d\mu \right\|_{\mathcal{L}_2^0} \right)^2 dt dP \right)^{1/2} = 0$$

which shows

$$\int_E \Phi_n(\cdot, \cdot, x) d\mu \rightarrow \int_E \Phi(\cdot, \cdot, x) d\mu$$

in $L^2(\Omega \times [0, T]; \mathcal{P}_T)$. It follows 56.1.7 at least makes sense and also, from the Ito isometry,

$$\int_E \Phi_n(\cdot, \cdot, x) d\mu \cdot W(T) \rightarrow \int_E \Phi(\cdot, \cdot, x) d\mu \cdot W(T)$$

in $L^2(\Omega \times [0, T]; \mathcal{P}_T)$. From 56.1.6 and the measurability result above, Minkowski's inequality yields

$$\begin{aligned} 0 &= \lim_{m \rightarrow \infty} \int_E \left(\int_{\Omega} |\Phi_m(\cdot, \cdot, x) \cdot W(T) - \Phi(\cdot, \cdot, x) \cdot W(T)|_H^2 dP \right)^{1/2} d\mu \\ &\geq \lim_{m \rightarrow \infty} \left(\int_{\Omega} \left| \int_E (\Phi_m(\cdot, \cdot, x) \cdot W(T) - \Phi(\cdot, \cdot, x) \cdot W(T)) d\mu \right|_H^2 dP \right)^{1/2} \end{aligned}$$

so that

$$\int_E \Phi_m(\cdot, \cdot, x) \cdot W(T) d\mu(x) \rightarrow \int_E \Phi(\cdot, \cdot, x) \cdot W(T) d\mu(x)$$

in $L^2([0, T] \times \Omega, \mathcal{P}_T)$. Thus

$$\begin{aligned} &\int_E \Phi_n(\cdot, \cdot, x) d\mu \cdot W(T) \\ &= \int_E \Phi_n(\cdot, \cdot, x) \cdot W(T) d\mu \\ &\rightarrow \int_E \Phi(\cdot, \cdot, x) \cdot W(T) d\mu(x) \end{aligned}$$

Hence

$$\int_E \Phi(\cdot, \cdot, x) \cdot W(T) d\mu(x) = \int_E \Phi(\cdot, \cdot, x) d\mu \cdot W(T).$$

Now it is time to prove Lemma 56.1.1.

Lemma 56.1.2 Suppose $(t, \omega, x) \rightarrow \Phi(t, \omega, x)$ is $\mathcal{P}_T \times \mathcal{B}(E)$ measurable and

$$\int_E \left(\int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu < \infty \quad (56.1.8)$$

Then there exists a sequence $\{\Phi_n\}$ of mappings which are of the form

$$\Phi_n(t, \omega, x) \equiv \sum_{j=1}^{J_n} \sum_{k=1}^{K_n} \sum_{l=1}^{L_n} \alpha_{kl}^n H_j \phi_{jk}^n(x) \mathcal{X}_{(s_{kl}^n, t_{kl}^n]}(t) \mathcal{X}_{F_{kl}^n}(\omega) \quad (56.1.9)$$

where the intervals $(s_{kl}^n, t_{kl}^n]$ are contained in $[0, T]$, α_{kl}^n are constants, H_j are in $\mathcal{L}_2(U, H)$, $\phi_{jk}^n(x)$ are bounded Borel measurable functions, and $F_{kl}^n \in \mathcal{F}_{s_{kl}^n}^n$ such that also

$$\lim_{n \rightarrow \infty} \int_E \left(\int_{\Omega} \int_0^T \|\Phi(t, \omega, x) - \Phi_n(t, \omega, x)\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu = 0. \quad (56.1.10)$$

Proof: First without loss of generality, it can be assumed $\Phi(t, \omega, x)$ is bounded.

Next, there exists a complete orthonormal set $\{H_j\}$ for $\mathcal{L}_2(U_0, H)$ such that each $H_j \in \mathcal{L}_2(U, H)$. The reason for this is that $\mathcal{L}_2(U, H)$ is dense in $\mathcal{L}_2(U_0, H)$. Letting $\{G_k\}$ be an orthonormal basis for $\mathcal{L}_2(U, H)$, you can use the Gram Schmidt process with the inner product in $\mathcal{L}_2(U_0, H)$ to obtain an orthonormal basis for $\mathcal{L}_2(U_0, H)$, $\{H_k\}$ in which each H_k happens to be in $\mathcal{L}_2(U, H)$. Then one can obtain an approximation

$$\sum_{j=1}^{J_n} H_j \phi_j(t, \omega, x)$$

where

$$\phi_j(t, \omega, x) \equiv (\Phi(t, \omega, x), H_j)_{\mathcal{L}_2^0}$$

and J_n is such that the sum up to J_n is closer to Φ in the norm of 56.1.8 than $1/n$. Now since $(t, \omega) \rightarrow \Phi(t, \omega, x)$ is $\mathcal{L}_2^0$ predictable, $(t, \omega) \rightarrow \phi_j(t, \omega, x)$ is in $L^2([0, T] \times \Omega, \mathcal{P}_T)$. Also from the assumption that Φ is product measurable with respect to $\mathcal{P}_T \times \mathcal{B}(E)$, the mapping

$$x \rightarrow \phi_j(t, \omega, x)$$

is $\mathcal{B}(E)$ measurable.

Now let $\{\varepsilon_k(t, \omega)\}$ be an orthonormal basis in $L^2([0, T] \times \Omega, \mathcal{P}_T)$. Then there exists K_n such that

$$\sum_{j=1}^{J_n} \sum_{k=1}^{K_n} H_j (\phi_j(\cdot, \cdot, x), \varepsilon_k(\cdot, \cdot))_{L^2([0, T] \times \Omega)} \varepsilon_k(t, \omega)$$

is closer to Φ in the norm of 56.1.8 than $1/n$. Denote by $\phi_{j,k}^n(x)$ the inner product in the above expression to have

$$\sum_{j=1}^{J_n} \sum_{k=1}^{K_n} H_j \phi_{j,k}^n(x) \varepsilon_k(t, \omega)$$

is closer to Φ in the norm of 56.1.8 than $1/n$ and $x \rightarrow \phi_{j,k}^n(x)$ is a bounded Borel measurable function having values in $\mathbb{R}$ due to the assumption that Φ was bounded as well as Borel measurable in x . Finally use Proposition 53.2.6 to approximate the predictable functions $\varepsilon_k(t, \omega)$ with elementary functions. Doing this, there exists L_n such that

$$\sum_{j=1}^{J_n} \sum_{k=1}^{K_n} H_j \phi_{j,k}^n(x) \sum_{l=1}^{L_n} \alpha_{k,l}^n \mathcal{X}_{(s_{k,l}^n, t_{k,l}^n]}(t) \mathcal{X}_{F_{k,l}^n}$$

is closer to Φ in the norm of 56.1.8 than $1/n$. This proves the lemma.

This completes the proof of the stochastic Fubini theorem.

Theorem 56.1.3 *Let Φ be $\mathcal{P}_T \times \mathcal{B}(E)$ measurable for E a separable Banach space and $\mathcal{P}_T$ the predictable σ algebra. Also assume*

$$\int_E \left(\int_{\Omega} \int_0^T \|\Phi\|_{\mathcal{L}_2^0}^2 dt dP \right)^{1/2} d\mu < \infty$$

where μ is a finite measure on $\mathcal{B}(E)$. Let $\{W(t)\}$ be a Q Wiener process. Then the following equation holds and both sides make sense.

$$\int_E \Phi(\cdot, \cdot, x) \cdot W(T) d\mu(x) = \int_E \Phi(\cdot, \cdot, x) d\mu(x) \cdot W(T).$$

56.2 Correlation Operators

Recall that for a Q Wiener process

$$E((W(t) - W(s), h)(W(t) - W(s), k)) = (t - s)(Qh, k)$$

where Q is a trace class operator. Also the Ito integral was discussed above. The correlation operators are a little like this but involving two different Ito integrals. Recall that $\Phi \in \mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$ meant $(t, \omega) \rightarrow \Phi(t, \omega)$ is $\mathcal{L}_2^0 \equiv \mathcal{L}_2(U_0, H)$ (where $U_0 = Q^{1/2}U$) predictable and

$$\int_{\Omega} \int_0^T \|\Phi(t, \omega)\|_{\mathcal{L}_2^0}^2 dt dP < \infty.$$

Then for such Φ the Ito integral, $\Phi \cdot W(t)$ was defined. It was also defined using localization for $\Phi \in \mathcal{N}_W(0, T; \mathcal{L}_2^0)$ which meant that

$$P\left(\left[\int_0^T \|\Phi(t, \omega)\|_{\mathcal{L}_2^0}^2 dt < \infty\right]\right) = 1.$$

Definition 56.2.1 For $\Phi_i, i = 1, 2$ in $\mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$, the correlation operators are defined by

$$V(t, s) \equiv \int_{\Omega} \int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr dP \in \mathcal{L}(H, H) \quad (56.2.11)$$

Lemma 56.2.2 The above definition makes sense.

Proof: It is necessary to show

$$\omega \rightarrow \int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr \quad (56.2.12)$$

is measurable and that the above function of ω makes sense. To do this, it suffices to show

$$\int_0^{t \wedge s} \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}(H, H)} dr < \infty$$

because then 56.2.12 will exist, there being no measurability problems due to the assumption that each Φ_i is predictable. Let $f, h \in H$

$$\begin{aligned} & \left| \left(\left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* f, h \right)_H \right| \\ & \leq \left| \left(\Phi_2(r) Q^{1/2} \right)^* h \right|_U \left| \left(\Phi_1(r) Q^{1/2} \right)^* f \right|_U \\ & \leq \left\| \left(\Phi_2(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}_2(H, U)} \left\| \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}_2(H, U)} |f|_H |h|_H \end{aligned}$$

which shows

$$\begin{aligned} & \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}(H, H)} \\ & \leq \left\| \left(\Phi_2(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}_2(H, U)} \left\| \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}_2(H, U)} \\ & = \left\| \Phi_2(r) Q^{1/2} \right\|_{\mathcal{L}_2(U, H)} \left\| \Phi_1(r) Q^{1/2} \right\|_{\mathcal{L}_2(U, H)} \end{aligned}$$

Thus

$$\begin{aligned} & \int_{\Omega} \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}(H, H)} dP \\ & \leq \left(\int_{\Omega} \left\| \Phi_1(r) Q^{1/2} \right\|_{\mathcal{L}_2}^2 dP \right)^{1/2} \left(\int_{\Omega} \left\| \Phi_2(r) Q^{1/2} \right\|_{\mathcal{L}_2}^2 dP \right)^{1/2} \\ & = \left(\int_{\Omega} \left\| \Phi_1(r) \right\|_{\mathcal{L}_2^0}^2 dP \right)^{1/2} \left(\int_{\Omega} \left\| \Phi_2(r) \right\|_{\mathcal{L}_2^0}^2 dP \right)^{1/2} \quad (56.2.13) \end{aligned}$$

since

$$\left\| \Phi(r) Q^{1/2} \right\|_{\mathcal{L}_2}^2 \equiv \sum_k \left| \Phi(r) Q^{1/2} e_k \right|_H^2$$

$$\begin{aligned}
&= \sum_{k,j} \left(\Phi(r) Q^{1/2} e_k, f_j \right)^2 = \sum_{k,j} \left(\Phi(r) \sqrt{\lambda_k} e_k, f_j \right)^2 \\
&\quad \sum_{k,j} (\Phi(r) g_k, f_j)^2 = \|\Phi(r)\|_{\mathcal{L}_2^0}^2
\end{aligned}$$

where C is the largest of the λ_k . From 56.2.13

$$\begin{aligned}
&\int_0^T \int_{\Omega} \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}(H,H)} dP dr \\
&= \int_{\Omega} \int_0^T \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}(H,H)} dr dP < \infty
\end{aligned}$$

Thus for a.e. ω ,

$$\int_0^T \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\|_{\mathcal{L}(H,H)} dr < \infty$$

and so for those ω

$$\int_0^T \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr$$

makes sense. Also this makes sense for T replaced with $t \wedge s$. Also this shows

$$\int_{\Omega} \int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr dP$$

makes sense because

$$\begin{aligned}
&\int_{\Omega} \left\| \int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr \right\| dP \\
&\leq \int_{\Omega} \int_0^{t \wedge s} \left\| \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* \right\| dr dP < \infty
\end{aligned}$$

This proves the lemma.

With this lemma, here is the main result.

Theorem 56.2.3 *Let $\Phi_i, i = 1, 2$ be in $\mathcal{N}_W^2(0, T; \mathcal{L}_2^0)$. Then*

$$E((\Phi_1 \cdot W(t), a)_H (\Phi_2 \cdot W(s), b)_H) = (V(t, s) a, b)_H \quad (56.2.14)$$

where $V(t, s)$ is given in Definition 56.2.1.

Proof: First suppose Φ_i is elementary. Then by adding in points on $[0, T]$ it can be assumed

$$\begin{aligned}
\Phi_1(r, \omega) &= \sum_{i=0}^n \mathcal{X}_{(t_i, t_{i+1}]}(r) \Phi_i(\omega) \\
\Phi_2(r, \omega) &= \sum_{i=0}^n \mathcal{X}_{(t_i, t_{i+1}]}(r) \Psi_i(\omega)
\end{aligned}$$

where $\Phi_i \Psi_i$ are $\mathcal{F}_{t_i}$ measurable. Then

$$\begin{aligned}\Phi_1 \cdot W(t) &\equiv \sum_{i=0}^n \Phi_i(\omega) (W(t \wedge t_{i+1}) - W(t \wedge t_i)) \\ \Phi_2 \cdot W(t) &\equiv \sum_{i=0}^n \Psi_i(\omega) (W(s \wedge t_{i+1}) - W(s \wedge t_i))\end{aligned}$$

The left side of 56.2.14 equals

$$\begin{aligned}E \left(\left(\sum_{i=0}^n \Phi_i (W(t \wedge t_{i+1}) - W(t \wedge t_i)), a \right) \left(\sum_{i=0}^n \Psi_i (W(s \wedge t_{i+1}) - W(s \wedge t_i)), b \right) \right) \\ = \sum_{i=0}^n E((\Phi_i (W(t \wedge t_{i+1}) - W(t \wedge t_i)), a) (\Psi_i (W(s \wedge t_{i+1}) - W(s \wedge t_i)), b))\end{aligned}$$

because of independence of the increments for the Wiener process. If Φ_i and Ψ_i were constants, this would reduce to

$$\begin{aligned}&\sum_{i=0}^n ((t \wedge s \wedge t_{i+1}) - (t \wedge s \wedge t_i)) (Q\Phi_i^* a, \Psi_i^* b) \\ &= \sum_{i=0}^n ((t \wedge s \wedge t_{i+1}) - (t \wedge s \wedge t_i)) E((Q\Phi_i^* a, \Psi_i^* b)) \\ &= \int_0^{t \wedge s} \int_{\Omega} (Q\Phi_1^* a, \Phi_2^* b) dP dr = E \left(\int_0^{t \wedge s} (Q\Phi_1^* a, \Phi_2^* b) dr \right) \\ &= E \left(\int_0^{t \wedge s} \left((Q^{1/2} \Phi_2^*)^* Q^{1/2} \Phi_1^* a, b \right) dr \right) = E \left(\int_0^{t \wedge s} \left((\Phi_2 Q^{1/2}) (\Phi_1 Q^{1/2})^* a, b \right) dr \right) \\ &= \left(E \left(\int_0^{t \wedge s} (\Phi_2 Q^{1/2}) (\Phi_1 Q^{1/2})^* a, b \right) \right) \equiv (V(t, s) a, b)\end{aligned}$$

but unfortunately, these are not constants. However, the expression still reduces to

$$\begin{aligned}&\sum_{i=0}^n ((t \wedge s \wedge t_{i+1}) - (t \wedge s \wedge t_i)) E((Q\Phi_i^* a, \Psi_i^* b)) \\ &= (V(t, s) a, b)\end{aligned}$$

and this is shown now.

There exists two sequences of simple functions measurable in $\mathcal{F}_{t_i}$,

$$\{\phi_k^i\}_{k=1}^{\infty}, \{\psi_k^i\}_{k=1}^{\infty}$$

such that

$$\lim_{k \rightarrow \infty} \|\phi_k^i - \Phi_i\|_{L^2(\Omega)} = \lim_{k \rightarrow \infty} \|\psi_k^i - \Psi_i\|_{L^2(\Omega)} = 0.$$

Consider

$$\begin{aligned} & E \left(\left(\phi_k^i (W(t \wedge t_{i+1}) - W(t \wedge t_i)), a \right) \left(\psi_k^i (W(s \wedge t_{i+1}) - W(s \wedge t_i)), b \right) \right) \\ &= E \left(\left((W(t \wedge t_{i+1}) - W(t \wedge t_i)), \phi_k^{i*} a \right) \left((W(s \wedge t_{i+1}) - W(s \wedge t_i)), \psi_k^{i*} b \right) \right) \end{aligned}$$

Say $\phi_k^{i*} a = \sum_{j=1}^m a_j \mathcal{X}_{A_j}$, $\psi_k^{i*} b = \sum_{j=1}^m b_j \mathcal{X}_{B_j}$ where $A_j, B_j \in \mathcal{F}_{t_i}$. Then the above reduces to an expression of the form

$$\sum_{j,l} E \left(\left((W(t \wedge t_{i+1}) - W(t \wedge t_i)), a_j \right) \left((W(s \wedge t_{i+1}) - W(s \wedge t_i)), b_l \right) \mathcal{X}_{A_j} \mathcal{X}_{B_l} \right)$$

Now by independence, this is

$$\begin{aligned} & \sum_{j,l} E \left(\left((W(t \wedge t_{i+1}) - W(t \wedge t_i)), a_j \right) \left((W(s \wedge t_{i+1}) - W(s \wedge t_i)), b_l \right) P(A_j \cap B_l) \right) \\ &= \left((t \wedge s \wedge t_{i+1}) - (t \wedge s \wedge t_i) \right) \sum_{j,l} (Q a_j, b_l) P(A_j \cap B_l) \\ &= \left((t \wedge s \wedge t_{i+1}) - (t \wedge s \wedge t_i) \right) E \left((Q \phi_k^{i*} a, \psi_k^{i*} b) \right) \\ &= \left((t \wedge s \wedge t_{i+1}) - (t \wedge s \wedge t_i) \right) E \left(\left(\left(\psi_k^i Q^{1/2} \right) \left(\phi_k^i Q^{1/2} \right)^* a, b \right) \right) \end{aligned}$$

Now consider 56.2.13. Applying this estimate the differences of $\phi_k^i Q^{1/2}$ and $\Phi_i Q^{1/2}$ and $\psi_k^i Q^{1/2}$ and $\Psi_i Q^{1/2}$, it follows

$$\begin{aligned} & \lim_{k \rightarrow \infty} E \left(\left(\left(\psi_k^i Q^{1/2} \right) \left(\phi_k^i Q^{1/2} \right)^* a, b \right) \right) - E \left((Q \Phi_i^* a, \Psi_i^* b) \right) \\ &= \lim_{k \rightarrow \infty} E \left(\left(\left(\psi_k^i Q^{1/2} \right) \left(\phi_k^i Q^{1/2} \right)^* a, b \right) \right) - E \left(\left(\left(\Psi_i Q^{1/2} \right) \left(\Phi_i Q^{1/2} \right)^* a, b \right) \right) = 0. \end{aligned}$$

Therefore for Φ_1, Φ_2 elementary functions,

$$E \left((\Phi_1 \cdot W(t), a)_H (\Phi_2 \cdot W(s), a)_H \right) = (V(t, s) a, b)_H$$

where

$$V(t, s) \equiv E \left(\int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr \right).$$

Now by Proposition 53.2.6 there exists sequences of elementary functions $\{\Phi_1^n\}$ and $\{\Phi_2^n\}$ satisfying

$$\lim_{n \rightarrow \infty} \int_0^T \int_{\Omega} \|\Phi_i^n - \Phi_i\|_{\mathcal{L}_2^0}^2 dP dt = 0$$

and now by the estimate 56.2.13 applied again to the differences, $\Phi_i^n - \Phi_i$, it follows

$$(V_n(t, s) a, b) \equiv \left(E \left(\int_0^{t \wedge s} \left(\Phi_2^n(r) Q^{1/2} \right) \left(\Phi_1^n(r) Q^{1/2} \right)^* dr \right) a, b \right)$$

converges to

$$(V(t, s) a, b) \equiv \left(E \left(\int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr \right) a, b \right)$$

and also by the Ito isometry,

$$E((\Phi_1^n \cdot W(t), a)(\Phi_2^n \cdot W(s), b)) = (V_n(t, s) a, b)$$

converges to

$$E((\Phi_1 \cdot W(t), a)(\Phi_2 \cdot W(s), b))$$

which shows

$$\begin{aligned} (V(t, s) a, b) &\equiv \left(E \left(\int_0^{t \wedge s} \left(\Phi_2(r) Q^{1/2} \right) \left(\Phi_1(r) Q^{1/2} \right)^* dr \right) a, b \right) \\ &= E((\Phi_1 \cdot W(t), a)(\Phi_2 \cdot W(s), b)) \end{aligned}$$

and this proves the theorem.

56.3 Weak Solutions To Stochastic Differential Equations

For A the generator of S a strongly continuous semigroup, the stochastic differential equation

$$dx(t) = [Ax(t) + f(t)] dt + BdW(t), \quad x(0) = \xi$$

is said to have a strong solution for $t \in [0, T]$ if $x(t)$ has values in $D(A)$ a.e. and

$$\int_0^T |Ax(s)| ds < \infty \text{ a.e.}$$

$$x(t) = \xi + \int_0^t [Ax(s) + f(s)] ds + BW(t) \text{ a.e.}$$

It is always assumed that f is a predictable process (H predictable) and $t \rightarrow f(t, \omega)$ is Bochner integrable on finite intervals. Also ξ is $\mathcal{F}_0$ measurable. Recall the filtration was derived from the Wiener process. That is

$$\mathcal{F}_t = \sigma(W(b) - W(a) : 0 \leq a \leq b \leq t).$$

Also $B \in \mathcal{L}(U, H)$.

56.3.1 Convolution With A Semigroup

Recall the definitions of U and $U_0 \equiv Q^{1/2}U$ where $\{e_j\}$ was an orthonormal basis for U such that $Qe_k = \lambda_k e_k$ and $\{g_k\}$ given by $g_k = \lambda_k^{1/2} e_k$ was an orthonormal basis for U_0 . Also let $\{f_j\}$ be an orthonormal basis for H . Letting $S \in \mathcal{L}(H, H)$ and $B \in \mathcal{L}(U, H)$

$$\begin{aligned}
 \|SB\|_{\mathcal{L}_2^0}^2 &\equiv \sum_k |SBg_k|_H^2 = \sum_{kj} (SBg_k, f_j)^2 \\
 &= \sum_{kj} \left(SB\sqrt{\lambda_k}e_k, f_j \right)^2 = \sum_{kj} \left(\sqrt{\lambda_k}e_k, B^*S^*f_j \right)^2 \\
 &= \sum_{kj} \left(Q^{1/2}e_k, B^*S^*f_j \right)^2 = \sum_j \sum_k \left(e_k, Q^{1/2}B^*S^*f_j \right)^2 \\
 &= \sum_j \left\| Q^{1/2}B^*S^*f_j \right\|^2 = \sum_j \left(Q^{1/2}B^*S^*f_j, Q^{1/2}B^*S^*f_j \right) \\
 &= \sum_j (f_j, SBQB^*S^*f_j) = \text{trace}(SBQB^*S^*) \quad (56.3.15)
 \end{aligned}$$

In fact, the following lemma holds.

Lemma 56.3.1 *The function $t \rightarrow S(t)B$ is continuous with values in $\mathcal{L}_2(U_0, H)$.*

Proof: Letting $\{e_k\}$ be the orthonormal basis for U and $\{g_k\}$ the orthonormal basis for U_0 with $g_k = \sqrt{\lambda_k}e_k$ as described earlier and $\{f_j\}$ an orthonormal basis for H ,

$$\begin{aligned}
 \|S(t)B - S(s)B\|_{\mathcal{L}_2^0}^2 &\equiv \sum_{kj} \left((S(t)B - S(s)B)\sqrt{\lambda_k}e_k, f_j \right)^2 \\
 &= \sum_k \sum_j \lambda_k ((S(t)B - S(s)B)e_k, f_j)^2 \\
 &= \sum_k \lambda_k |(S(t) - S(s))Be_k|_H^2
 \end{aligned}$$

and as $s \rightarrow t$, this converges to 0 from the dominated convergence theorem. Recall $\sum_k \lambda_k < \infty$. This proves the lemma.

For S a strongly continuous semigroup as described above and $B \in \mathcal{L}(U, H)$,

$$W_A(t) \equiv S(t - (\cdot))B \cdot W(t) \equiv \int_0^t S(t-s)BdW(s)$$

That is, you fix t and for $s \in [0, t]$ you consider the constant process $\Phi(s) \equiv S(t-s)B$ and $W_A(t)$ is just the Ito integral $\Phi \cdot W(t)$. Because of Lemma 56.3.1

and the definition of the Ito integral, $W_A(t)$ is approximated in $L^2(\Omega)$ by sums of the form

$$\sum_{k=0}^m S(t - t_k) B(W(t \wedge t_{k+1}) - W(t \wedge t_k))$$

because the elementary functions corresponding to these sums converge in the norm

$$\|\Phi\|^2 \equiv \int_0^T \int_{\Omega} \|\Phi\|_{\mathcal{L}_2^0}^2 dP dt.$$

Also

$$\int_a^b S(c - r) B dW(r)$$

means

$$S(c - (\cdot)) B \cdot W(b)$$

where a takes the place of 0 in the definition of the Ito integral and sums of the above form can be used to approximate this random variable also.

Theorem 56.3.2 *Suppose A generates a strongly continuous semigroup as described above. Suppose also that*

$$\int_0^t \|S(r) B\|_{\mathcal{L}_2^0}^2 ds = \int_0^t \text{trace}(S(r) BQB^* S^*(r)) dr < \infty \quad (56.3.16)$$

Then the process $\{W_A(t)\}$ is Gaussian, continuous in mean square and has a predictable version. Recall Gaussian means the random variables

$$(W_A(t), a)$$

are normally distributed. Continuous in mean square means

$$\lim_{s \rightarrow t} E(|W_A(t) - W_A(s)|_H^2) = 0$$

and predictable means that $W_A^{-1}(U) \in \mathcal{P}_T$, the smallest σ algebra containing the sets of the form $(s, r] \times A_s$ where $A_s \in \mathcal{F}_s$ and $\{0\} \times A_0$, where $A_0 \in \mathcal{F}_0$. Also

$$E((W_A(t), a)(W_A(t), b)) = \left(\left(\int_0^t S(r) BQB^* S^*(r) dr \right) a, b \right) \quad (56.3.17)$$

Thus the covariance is the linear operator

$$\int_0^t S(r) BQB^* S^*(r) dr \in \mathcal{L}(H, H).$$

In addition to this, for a.e. ω , $t \rightarrow W_A(t, \omega)$ is in $L^2(0, T; H)$ with covariance operators $\mathcal{Q}$ defined by

$$\begin{aligned} (\mathcal{Q}\phi, \psi)_{\mathcal{H}} &\equiv E((W_A, \phi)_{\mathcal{H}}(W_A, \psi)_{\mathcal{H}}) \\ &= \int_0^T \int_0^T \left(\left(\int_0^{s \wedge r} S(r - t) BQB^* S^*(s - t) dt \right) \phi(s), \psi(r) \right) ds dr. \end{aligned}$$

Proof: I want to show $W_A(t)$ is mean square integrable. For $s \leq t$

$$W_A(t) - W_A(s) = \int_s^t S(t-r) B dW(r) + \int_0^s (S(t-r) - S(s-r)) B dW(r)$$

Now the random variables defined by the two integrals are independent. This is because of the independence of the increments in Wiener process and the definition of the Ito integral as limits of things which clearly are independent. Thus

$$\begin{aligned} E \left(|W_A(t) - W_A(s)|^2 \right) &= E \left(\left| \int_s^t S(t-r) B dW(r) \right|^2 \right) \\ &+ E \left(\left| \int_0^s (S(t-r) - S(s-r)) B dW(r) \right|^2 \right) \end{aligned} \quad (56.3.18)$$

Consider the first expression on the right in the above. By the Ito isometry this equals

$$\int_{\Omega} \int_s^t \|S(t-r) B\|_{\mathcal{L}_2^0}^2 dr dP$$

and by 56.3.15 this equals

$$\begin{aligned} &\int_{\Omega} \int_s^t \text{trace}(S(t-r) B Q B^* S^*(t-r)) dr dP \\ &= \int_{\Omega} \int_0^{t-s} \text{trace}(S(r) B Q B^* S^*(r)) dr dP \\ &= \int_0^{t-s} \text{trace}(S(r) B Q B^* S^*(r)) dr \end{aligned}$$

which converges to 0 as $s \rightarrow t$ due to the assumption 56.3.16 which says

$$r \rightarrow \text{trace}(S(r) B Q B^* S^*(r))$$

is $L^1([0, T])$. Next consider the second term in 56.3.18.

$$E \left(\left| \int_0^s (S(t-r) - S(s-r)) B dW(r) \right|^2 \right) \quad (56.3.19)$$

$$\begin{aligned} &= \int_{\Omega} \int_0^s \|(S(t-r) - S(s-r)) B\|_{\mathcal{L}_2^0}^2 dr dP \\ &= \int_0^s \|(S(t-r) - S(s-r)) B\|_{\mathcal{L}_2^0}^2 dr \\ &= \int_0^s \|(S(t-s+r) - S(r)) B\|_{\mathcal{L}_2^0}^2 dr \end{aligned}$$

Which converges to 0 by Lemma 56.3.1 as $s \rightarrow t$. This shows the claim about mean square continuity.

Why is $W_A(t)$ Gaussian? That is, why is $(W_A(t), a)$ normally distributed? Observe that $W_A(t)$ is just an Ito integral. Thus it is a limit in $L^2(\Omega; H)$ of expressions of the form

$$\sum_{i=0}^m \Phi_i(W(t \wedge t_{i+1}) - W(t \wedge t_i))$$

and

$$\left(\sum_{i=0}^m \Phi_i(W(t \wedge t_{i+1}) - W(t \wedge t_i)), a \right) = \sum_{i=0}^m ((W(t \wedge t_{i+1}) - W(t \wedge t_i)), \Phi_i^* a)$$

Now by independence of the increments this is a finite sum of normally distributed independent random variables and so it is Gaussian with zero mean.

Now in general if X is the limit of a sequence of H valued random variables in $L^2(\Omega; H)$ and each X_n is Gaussian with zero mean, then passing to a subsequence still denoted by n such that the convergence is also pointwise, the dominated convergence theorem implies

$$\lim_{n \rightarrow \infty} E(e^{it(a, X_n)}) = E(e^{it(a, X)})$$

so the characteristic functions must converge to the characteristic function of X . However,

$$E(e^{it(a, X_n)}) = e^{-\frac{1}{2}t^2\sigma_n^2}$$

If a subsequence $\sigma_{n_k} \rightarrow \infty$, $e^{-\frac{1}{2}t^2\sigma_n^2} \rightarrow 0$ and it is not possible that $0 = E(e^{it(a, X)})$ which must equal 1 when $t = 0$. Therefore, there must be some upper bound to the σ_n^2 and so a further subsequence has the property that $\sigma_n^2 \rightarrow \sigma^2$. Therefore,

$$e^{-\frac{1}{2}t^2\sigma^2} = \lim_{n \rightarrow \infty} e^{-\frac{1}{2}t^2\sigma_n^2} = \lim_{n \rightarrow \infty} E(e^{it(a, X_n)}) = E(e^{it(a, X)})$$

and so X is also Gaussian with mean 0. Thus $W_A(t)$ is Gaussian.

Consider the claim about a predictable version. This follows from Proposition 50.2.6 which is stated here.

Proposition 56.3.3 *Let X be $\mathcal{F}_t$ adapted and left continuous. Then it is predictable. Also, if X is stochastically continuous and adapted on $[0, T]$, then it has a predictable version.*

Recall what it means to be stochastically continuous.

Definition 56.3.4 *X is stochastically continuous at $t_0 \in I$ means: for all $\varepsilon > 0$ and $\delta > 0$ there exists $\rho > 0$ such that*

$$P([||X(t) - X(t_0)|| \geq \varepsilon]) \leq \delta \text{ whenever } |t - t_0| < \rho, t \in I.$$

X is stochastically continuous if it is stochastically continuous at every $t \in I$.

$\{W_A(t)\}$ is stochastically continuous thanks to the assertion proved above that it is mean square continuous, and adapted. To see this, let $\varepsilon, \delta > 0$ be given and pick t_0 . Then if $|t - t_0|$ is small enough,

$$\begin{aligned} \varepsilon \delta^{1/2} &\geq \left(\int_{\Omega} |W_A(t) - W_A(t_0)|^2 dP \right)^{1/2} \\ &\geq \left(\int_{\{|W_A(t) - W_A(t_0)| \geq \varepsilon\}} |W_A(t) - W_A(t_0)|^2 dP \right)^{1/2} \\ &\geq \varepsilon P(|W_A(t) - W_A(t_0)| \geq \varepsilon)^{1/2} \end{aligned}$$

which shows that W_A is stochastically continuous. Therefore, from that proposition stated above it has a predictable version.

Next consider the claim about the correlation. I need to consider

$$E((W_A(t), a)(W_A(t), b)).$$

From Theorem 56.2.3 there was a formula like the following.

$$\begin{aligned} (V(t, s), a, b) &\equiv \left(E \left(\int_0^{t \wedge s} (\Phi_2(r) Q^{1/2}) (\Phi_1(r) Q^{1/2})^* dr \right), a, b \right) \\ &= E((\Phi_1 \cdot W(t), a)(\Phi_2 \cdot W(s), b)) \end{aligned}$$

In the situation of interest here, $t = s$ and

$$\begin{aligned} V(t, t) &= E \left(\int_0^t (S(t-r) B Q^{1/2}) (S(t-r) B Q^{1/2})^* dr \right) \\ &= \int_0^t (S(r) B Q^{1/2}) (S(r) B Q^{1/2})^* dr \\ &= \int_0^t (S(r) B Q^{1/2}) Q^{1/2} B^* S^*(r) dr \\ &= \int_0^t S(r) B Q B^* S^*(r) dr \end{aligned}$$

This proves the claim about the covariance.

It only remains to prove the last claim. It was shown above there is a measurable predictable version of W_A . By Fubini's theorem applied to this predictable version,

$$E \left(\int_0^T |W_A(t)|^2 dt \right) = \int_0^T E(|W_A(t)|^2) dt \quad (56.3.20)$$

Then from what was shown above about the covariance,

$$\begin{aligned}
 &= \int_0^T E \left(\sum_k (W_A(t) \cdot f_k)^2 \right) dt \\
 &= \int_0^T \sum_k E \left((W_A(t) \cdot f_k)^2 \right) dt \\
 &= \int_0^T \sum_k \int_0^t (S(r) B Q B^* S^*(r) f_k, f_k) dr dt \\
 &= \int_0^T \int_0^t \sum_k (S(r) B Q B^* S^*(r) f_k, f_k) dr dt \\
 &= \int_0^T \int_0^t \text{trace} (S(r) B Q B^* S^*(r)) dr dt
 \end{aligned}$$

which by 56.3.15 equals

$$\int_0^T \int_0^t \|S(r) B\|_{\mathcal{L}_2^0}^2 dr dt < \infty.$$

It follows from 56.3.20 that for a.e. ω ,

$$\int_0^T |W_A(t)|^2 dt < \infty.$$

Thus for a.e. ω ,

$$\int_0^T W_A(t) dt$$

makes sense. Modifying this on a set of measure zero, it follows the above integral is a random variable having values in H . Similarly, if $g \in L^2(0, T; H) \equiv \mathcal{H}$

$$\int_0^T (W_A(t), g(t)) dt$$

is measurable and so by the Pettis theorem, W_A is strongly measurable having values in $\mathcal{H}$. I want to show that W_A is actually Gaussian. That is, if $g \in \mathcal{H}$ then $(W_A, g)_{\mathcal{H}}$ has a normal distribution. To do this, use the definition of the Ito integral and Lemma 56.3.1 to approximate $W_A(t)$ as

$$\sum_{i=0}^n S(t - t_i) B (W(t \wedge t_{i+1}) - W(t \wedge t_i))$$

where $\{t_0, t_1, \dots, t_n\}$ is a partition of $[0, T]$ having each $|t_{i+1} - t_i| = T/n$. Of course S is only a semigroup so define $S(t - t_i) \equiv I$ whenever $t < t_i$. Then letting g be in $C([0, T]; H)$, a continuous function, an approximation to

$$\int_0^T (W_A(t), g(t)) dt$$

is

$$\int_0^T \left(\sum_{i=0}^n S(t-t_i) B(W(t \wedge t_{i+1}) - W(t \wedge t_i)), g(t) \right) dt$$

I will show later that as $n \rightarrow \infty$, the limit of the second expression converges to the first. For now, consider the second expression. For fixed n let $\{\tau_k\}_{k=0}^m$ be a uniform partition of $[0, T]$ and approximate the above integral by

$$\sum_{k=1}^m \sum_{i=0}^n (S(\tau_k - t_i) B(W(\tau_k \wedge t_{i+1}) - W(\tau_k \wedge t_i)), g(\tau_k)) (\tau_k - \tau_{k-1})$$

Thus from continuity of S and g , the above double sum converges for a.e. ω as $m \rightarrow \infty$ to

$$\int_0^T \left(\sum_{i=0}^n S(t-t_i) B(W(t \wedge t_{i+1}) - W(t \wedge t_i)), g(t) \right) dt \quad (56.3.21)$$

and so it follows that if for all m the above double sum is symmetric (mean = 0) and normally distributed, the same must be true of the above integral. This follows from looking at the characteristic functions as was done earlier in the proof. The double sum is obviously normally distributed because it is of the form

$$\sum_{k=1}^m \sum_{i=0}^n ((W(\tau_k \wedge t_{i+1}) - W(\tau_k \wedge t_i)), B^* S(\tau_k - t_i)^* (\tau_k - \tau_{k-1}) g(\tau_k)),$$

the second term in every inner product being independent of ω . By modifying the sum by adding in points as required, and rearranging, one can express the above double sum as a sum of the form

$$\sum_{l=1}^p ((W(b_l) - W(a_l)), g_l)_H$$

where the intervals (a_l, b_l) are disjoint. Then it is the sum of independent normally distributed random variables having mean 0 and is therefore normally distributed with mean 0. Thus the integral in 56.3.21,

$$\int_0^T \left(\sum_{i=0}^n S(t-t_i) B(W(t \wedge t_{i+1}) - W(t \wedge t_i)), g(t) \right) dt, \quad (56.3.22)$$

is normally distributed and has mean 0 also. I need to pass to the limit and argue that

$$\int_0^T (W_A(t), g(t)) dt$$

is normally distributed and has mean 0. To save space, let

$$\Sigma_n(t) \equiv \left(\sum_{i=0}^n S(t-t_i) B(W(t \wedge t_{i+1}) - W(t \wedge t_i)) \right)$$

It follows from the definition of the Ito integral that when $n \rightarrow \infty$ the integrand in 56.3.22 converges in $L^1(\Omega)$ to

$$(W_A(t), g(t))_H$$

for each t . Next,

$$\begin{aligned} & \int_{\Omega} \left| \int_0^T (W_A(t), g(t)) dt - \int_0^T (\Sigma_n(t), g(t)) dt \right| dP \\ & \leq \int_0^T \int_{\Omega} |(W_A(t), g(t)) - (\Sigma_n(t), g(t))| dP dt \\ & \leq C_T \left(\int_0^T \int_{\Omega} |(W_A(t) - \Sigma_n(t), g(t))|^2 dP dt \right)^{1/2} \end{aligned}$$

Then the Ito isometry can be used on the inside integral to obtain

$$\begin{aligned} & \leq C_T \left(\int_0^T \int_{\Omega} |W_A(t) - \Sigma_n(t)|^2 |g(t)|^2 dP dt \right)^{1/2} \\ & = C_T \left(\int_0^T |g(t)|^2 \int_{\Omega} |W_A(t) - \Sigma_n(t)|_H^2 dP dt \right)^{1/2} \\ & = C_T \left[\int_0^T |g(t)|^2 \int_{\Omega} \int_0^t \|S(t-s)B\right. \\ & \quad \left. - \sum_{i=0}^n S(t-t_i)B\mathcal{X}_{(t_i, t_{i+1}]}(s) \right\|_{\mathcal{L}_2^0}^2 ds dP dt \right]^{1/2} \end{aligned}$$

Then by Lemma 56.3.1 about continuity of $t \rightarrow S(t)B$ with values in $\mathcal{L}_2^0$ and consequent uniform continuity on bounded intervals, the integrand converges uniformly to 0 as $n \rightarrow \infty$ and so the above expression converges to 0. It follows that for a suitable subsequence

$$\int_0^T (\Sigma_n(t), g(t)) dt \rightarrow \int_0^T (W_A(t), g(t)) dt$$

a.e. and so it follows the random variable

$$\int_0^T (W_A(t), g(t)) dt$$

is normally distributed having mean 0. I need to verify this is true for all $g \in \mathcal{H}$, not just for continuous g . However, $C([0, T]; H)$ is dense in $\mathcal{H}$ due to regularity of

Lebesgue measure and so if g is arbitrary there exists a sequence g_n converging to g in $\mathcal{H}$ such that g_n is continuous. Then

$$\begin{aligned} & \int_{\Omega} \left| \int_0^T (W_A(t), g(t)) dt - \int_0^T (W_A(t), g_n(t)) dt \right| dP \\ & \leq \int_{\Omega} \int_0^T |(W_A(t), g(t) - g_n(t))| dt dP \\ & \leq \left(\int_{\Omega} \int_0^T |W_A(t)|^2 dt dP \right)^{1/2} \left(\int_{\Omega} \int_0^T |g(t) - g_n(t)|^2 dt dP \right)^{1/2} \end{aligned}$$

and so the random variables $\int_0^T (W_A(t), g_n(t)) dt$ converges in $L^1(\Omega)$ to

$$\int_0^T (W_A(t), g(t)) dt$$

so this random variable is normal with zero mean.

It only remains to consider the covariance operators. Let $\phi, \psi \in \mathcal{H}$

$$\begin{aligned} (\mathcal{Q}\phi, \psi) & \equiv E((W_A, \phi)_{\mathcal{H}} (W_A, \psi)_{\mathcal{H}}) = \\ & E \left(\left(\int_0^T (W_A(s), \phi(s)) ds \right) \left(\int_0^T (W_A(r), \psi(r)) dr \right) \right) \\ & = \int_0^T \int_0^T E((W_A(s), \phi(s)) (W_A(r), \psi(r))) ds dr \end{aligned} \quad (56.3.23)$$

Suppose $s > r$. Then

$$\begin{aligned} W_A(s) & = \int_0^s S(s-t) BdW(t) \\ & = \int_0^r S(s-t) BdW(t) + \int_r^s S(s-t) BdW(t) \end{aligned}$$

and the second of these integrals is independent of $W_A(r)$ and by reasoning similar to the above is symmetric and Gaussian so

$$\begin{aligned} & E((W_A(s), \phi(s)) (W_A(r), \psi(r))) \\ & = E \left(\left[\left(\int_0^r S(s-t) BdW(t) + \int_r^s S(s-t) BdW(t), \phi(s) \right) \right] (W_A(r), \psi(r)) \right) \\ & = E \left(\left(\int_0^r S(s-t) BdW(t), \phi(s) \right) (W_A(r), \psi(r)) \right) \\ & = E \left(\left(S(s-r) \int_0^r S(r-t) BdW(t), \phi(s) \right) (W_A(r), \psi(r)) \right) \\ & = E((S(s-r) W_A(r), \phi(s)) (W_A(r), \psi(r))) \\ & = E((W_A(r), S^*(s-r) \phi(s)) (W_A(r), \psi(r))) \end{aligned} \quad (56.3.24)$$

Now recall from 56.3.17 shown above,

$$E((W_A(r), a)(W_A(r), b)) = \left(\left(\int_0^r S(t) BQB^* S^*(t) dt \right) a, b \right)$$

Thus 56.3.24 equals

$$\begin{aligned} & \left(\left(\int_0^r S(t) BQB^* S^*(t) dt \right) S^*(s-r) \phi(s), \psi(r) \right) \\ &= \left(\left(\int_0^r S(r-t) BQB^* S^*(r-t) dt \right) S^*(s-r) \phi(s), \psi(r) \right) \\ &= \left(\left(\int_0^r S(r-t) BQB^* S^*(s-t) dt \right) \phi(s), \psi(r) \right) \end{aligned} \quad (56.3.25)$$

which follows from changing the variables in the integral. If $s < r$ similar considerations would yield the following for the integrand of 56.3.23.

$$\left(\left(\int_0^s S(s-t) BQB^* S^*(r-t) dt \right) \psi(r), \phi(s) \right)$$

which equals

$$\begin{aligned} & \left(\int_0^s S(s-t) BQB^* S^*(r-t) \psi(r), \phi(s) dt \right) \\ &= \left(\int_0^s \psi(r), S(r-t) BQB^* S^*(s-t) \phi(s) dt \right) \\ &= \left(\psi(r), \left(\int_0^s S(r-t) BQB^* S^*(s-t) dt \right) \phi(s) \right) \\ &= \left(\left(\int_0^s S(r-t) BQB^* S^*(s-t) dt \right) \phi(s), \psi(r) \right) \end{aligned}$$

It follows the integrand of 56.3.23 is always of the form

$$\left(\left(\int_0^{s \wedge r} S(r-t) BQB^* S^*(s-t) dt \right) \phi(s), \psi(r) \right).$$

It follows

$$\begin{aligned} & (\mathcal{Q}\phi, \psi)_{\mathcal{H}} \equiv E((W_A, \phi)_{\mathcal{H}}(W_A, \psi)_{\mathcal{H}}) \\ &= \int_0^T \int_0^T \left(\left(\int_0^{s \wedge r} S(r-t) BQB^* S^*(s-t) dt \right) \phi(s), \psi(r) \right) ds dr. \end{aligned}$$

This proves the theorem.

56.3.2 Continuity Of $W_A(t)$

The convolution with a continuous semigroup is mean square continuous as shown above. However, it is also the case that for a.e. ω , $t \rightarrow W_A(t)$ is continuous with values in H provided $B = I$.

The following lemma is an essential part of the argument.

Lemma 56.3.5 *Let X be a random variable having values in H a real separable Hilbert space which is Gaussian and symmetric with trace class covariance operator Q ,*

$$Q = \sum_k \lambda_k e_k \otimes e_k$$

where $\{e_k\}$ is orthonormal and $Qe_k = \lambda_k e_k$, $\lambda_k > 0$, $\sum_k \lambda_k < \infty$. Thus $\omega \rightarrow (h, X(\omega))$ is normally distributed having mean 0 and

$$E((X, a)(X, b)) = (Qa, b)$$

Then for each $m \in \mathbb{N}$, the positive integers,

$$E(|X|^{2m}) \leq \left(\frac{2m!}{2^m m!} \right) \left(E(|X|^2) \right)^m \quad (56.3.26)$$

Proof: First consider the case where X is normally distributed with mean 0 and variance σ^2 so X has values in $\mathbb{R}$. Then

$$E(\exp(itX)) = e^{-\frac{1}{2}t^2\sigma^2} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}\sigma^{2k}}{2^k k!}$$

It follows upon taking derivatives and setting $t = 0$ to compute moments that

$$E(X^{2m+1}) = 0, \quad E(X^{2m}) = \frac{(2m)!}{2^m m!} \sigma^{2m} = \frac{(2m)!}{2^m m!} E(X^2)^m.$$

Next consider the general case. Letting $\{e_k\}$ be as above,

$$E((X, e_k)^2) \equiv (Qe_k, e_k) = \lambda_k.$$

Therefore, from what was just shown

$$E((X, e_k)^{2m}) = \frac{(2m)!}{2^m m!} E((X, e_k)^2)^m = \frac{(2m)!}{2^m m!} \lambda_k^m$$

Now using Minkowski's inequality and the above,

$$E(|X|^{2m})^{1/m} = E\left(\left(\sum_k (X, e_k)^2\right)^m\right)^{1/m}$$

$$\begin{aligned}
&\leq \sum_k E \left((X, e_k)^{2m} \right)^{1/m} = \sum_k \left(\frac{(2m)!}{2^m m!} \lambda_k^m \right)^{1/m} \\
&= \left(\frac{(2m)!}{2^m m!} \right)^{1/m} \sum_k \lambda_k = \left(\frac{(2m)!}{2^m m!} \right)^{1/m} E \left(|X|^2 \right)
\end{aligned}$$

since

$$E \left(|X|^2 \right) = \sum_k E \left((X, e_k)^2 \right) = \sum_k \lambda_k$$

and this proves the lemma.

Next I will present a simple formula which will be used to factor the integral defining $W_A(t)$.

Lemma 56.3.6 *The following holds for $\alpha \in (0, 1)$.*

$$\int_{\sigma}^t (t-s)^{\alpha-1} (s-\sigma)^{-\alpha} ds = \frac{\pi}{\sin(\pi\alpha)}$$

Proof: First change variables to get rid of the σ . Let $y = (t-\sigma)^{-1}(s-\sigma)$. Then the integral becomes

$$\begin{aligned}
&\int_0^1 (t - [(t-\sigma)y + \sigma])^{\alpha-1} (t-\sigma)^{-\alpha} y^{-\alpha} (t-\sigma) dy \\
&= \int_0^1 ((t-\sigma)(1-y))^{\alpha-1} (t-\sigma)^{-\alpha} y^{-\alpha} (t-\sigma) dy \\
&= \int_0^1 (1-y)^{\alpha-1} y^{-\alpha} dy
\end{aligned}$$

Next let $y = x^2$. The integral is

$$2 \int_0^1 (1-x^2)^{\alpha-1} x^{1-2\alpha} dx$$

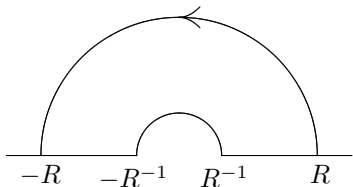
Next let $x = \sin \theta$

$$2 \int_0^{\frac{1}{2}\pi} (\cos(\theta))^{2\alpha-1} \sin^{(1-2\alpha)}(\theta) d\theta = 2 \int_0^{\frac{1}{2}\pi} \left(\frac{\cos(\theta)}{\sin(\theta)} \right)^{2\alpha-1} d\theta$$

Now change the variable again. Let $u = \cot(\theta)$. Then this yields

$$2 \int_0^{\infty} \frac{u^{2\alpha-1}}{1+u^2} du$$

This is fairly easy to evaluate using contour integrals. Consider the following contour called Γ_R for large R . As $R \rightarrow \infty$, the integral over the little circle converges to 0 and so does the integral over the big circle. There is one singularity at i .



Thus

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{e^{(\ln|z| + i \arg(z))(1-2\alpha)}}{1+z^2} dz = \\ &= (1 + \cos(1-2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1+u^2} du + i \sin((1-2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1+u^2} du \\ &= \pi \left(\cos\left(\frac{\pi}{2}(1-2\alpha)\right) + i \sin\left(\frac{\pi}{2}(1-2\alpha)\right) \right) \end{aligned}$$

Then equating the imaginary parts yields

$$\sin((1-2\alpha)\pi) \int_0^\infty \frac{u^{2\alpha-1}}{1+u^2} du = \pi \sin\left(\frac{\pi}{2}(1-2\alpha)\right)$$

and so using the trig identities for the sum of two angles,

$$\begin{aligned} \int_0^\infty \frac{u^{2\alpha-1}}{1+u^2} du &= \frac{\pi \left(\sin\left(\frac{\pi}{2}(1-2\alpha)\right) \right)}{2 \sin\left(\frac{\pi}{2}(1-2\alpha)\right) \cos\left(\frac{\pi}{2}(1-2\alpha)\right)} \\ &= \frac{\pi}{2 \cos\left(\frac{\pi}{2}(1-2\alpha)\right)} = \frac{\pi}{2 \sin(\pi\alpha)} \end{aligned}$$

This proves the lemma.

Lemma 56.3.7 *Let $W(t)$ be a Q Wiener process on H . Recall this means Q is a positive trace class operator defined on H and $H_0 \equiv Q^{1/2}H$. In terms of the usual context, $U = H$. As before, let $\mathcal{L}_2^0$ denote $\mathcal{L}_2(H_0, H)$, the Hilbert Schmidt operators mapping H_0 to H . Also let $S(t)$ be a stongly continuous semigroup on H . Then for any $\alpha \in (0, 1)$,*

$$\int_0^T t^{-\alpha} \|S(t)\|_{\mathcal{L}_2^0}^2 dt < \infty$$

Proof:

$$\begin{aligned}
 \|S(t)\|_{\mathcal{L}_2^0}^2 &\equiv \sum_k \sum_l (S(t) g_l, e_k)^2 \\
 &= \sum_l \lambda_l \sum_k (S(t) e_l, e_k)^2 \\
 &= \sum_l \lambda_l \|S(t) e_l\|_H^2 \\
 &\leq C \sum_l \lambda_l e^{\beta t}
 \end{aligned}$$

because $\|S(t)\|$ has exponential growth thanks to Theorem 15.12.3 on Page 468. This proves the lemma.

Theorem 56.3.8 *Let $W(t)$ be a Q Wiener process on H . Recall this means Q is a positive trace class operator defined on H and $H_0 \equiv Q^{1/2}H$. In terms of the usual context, $U = H$. As before, let $\mathcal{L}_2^0$ denote $\mathcal{L}_2(H_0, H)$, the Hilbert Schmidt operators mapping H_0 to H . Let $S(t)$ be a strongly continuous semigroup whose generator is A , a closed densely defined operator mapping $D(A) \subseteq H$ to H . Then*

$$t \rightarrow W_A(t) \equiv \int_0^t S(t-s) dW(s)$$

has the property that for a.e. ω , the function is continuous having values in H .¹

Proof: In the argument, m will be a positive integer such that

$$m > \frac{1}{2\alpha}$$

where $\alpha \in (0, 1/2)$. Using Lemma 56.3.6,

$$\begin{aligned}
 W_A(t) &= \int_0^t S(t-\sigma) dW(\sigma) \\
 W_A(t) &\equiv \frac{1}{\pi} \sin(\pi\alpha) \int_0^t \int_\sigma^t S(t-\sigma) (t-s)^{\alpha-1} (s-\sigma)^{-\alpha} ds dW(\sigma).
 \end{aligned}$$

Now use the Stochastic Fubini theorem, Theorem 56.1.3 to write this as

$$\begin{aligned}
 &\frac{1}{\pi} \sin(\pi\alpha) \int_0^t \int_0^s S(t-\sigma) (t-s)^{\alpha-1} (s-\sigma)^{-\alpha} dW(\sigma) ds \\
 &= \frac{1}{\pi} \sin(\pi\alpha) \int_0^t S(t-s) (t-s)^{\alpha-1} \overbrace{\int_0^s S(s-\sigma) (s-\sigma)^{-\alpha} dW(\sigma)}^{Y(s)} ds.
 \end{aligned}$$

¹Recall it was shown earlier this function was mean square continuous. This is something more.

Now consider the random variable $Y(s)$ I want to show $E(|Y(s)|_H^{2m}) < C$ for some C independent of s . From this it will follow that for a.e. ω

$$\int_0^T |Y(s)|_H^{2m} ds < \infty$$

Then when this has been shown it will follow that for these $\omega, t \rightarrow W_A(t)$ will be continuous having values in H . To show this I need to find the covariance of $Y(s)$. This random variable will end up being Gaussian as in the proof of Theorem 56.3.2. First I will show this. The integral equal to $Y(s)$ is the limit in $L^2(\Omega, H)$ of sums of the form

$$\sum_{j=0}^p S(s - \sigma_j) (W(\sigma_{j+1}) - W(\sigma_j)) (s - \sigma_j)^{-\alpha}$$

and each of these is Gaussian. The fact this limit exists follows because

$$\sigma \rightarrow S(s - \sigma) (s - \sigma)^{-\alpha}$$

is in $L^2(0, T; \mathcal{L}_2^0)$ by Lemma 56.3.7. Therefore, using the trick involving characteristic functions as in Theorem 56.3.2, it follows $Y(s)$ must also be Gaussian. I need to find and estimate the trace of the covariance.

Consider for $a, b \in H$,

$$\begin{aligned} & E((Y(s), a)(Y(s), b)) \\ & \equiv \int_{\Omega} \left(\int_0^s S(s - \sigma) (s - \sigma)^{-\alpha} dW(\sigma), a \right) \left(\int_0^s S(s - \tau) (s - \tau)^{-\alpha} dW(\tau), b \right) dP \end{aligned} \quad (56.3.28)$$

Consider one of these integrals on the inside.

$$\int_0^s S(s - \sigma) (s - \sigma)^{-\alpha} dW(\sigma)$$

is the limit in $L^2(\Omega, H)$ of sums like

$$\sum_{j=0}^p S(s - \sigma_j) (W(\sigma_{j+1}) - W(\sigma_j)) (s - \sigma_j)^{-\alpha}$$

and so $\left(\int_0^s S(s - \sigma) (s - \sigma)^{-\alpha} dW(\sigma), a \right)$ is a limit in $L^2(\Omega)$ of sums like

$$\begin{aligned} & \sum_{j=0}^p \left(S(s - \sigma_j) (W(\sigma_{j+1}) - W(\sigma_j)) (s - \sigma_j)^{-\alpha}, a \right) \\ & = \sum_{j=0}^p ((W(\sigma_{j+1}) - W(\sigma_j)), S^*(s - \sigma_j) a) (s - \sigma_j)^{-\alpha} \end{aligned}$$

Thus the expression in 56.3.28 is the limit as $p \rightarrow \infty$ of

$$E \left(\sum_{j=0}^p ((W(\sigma_{j+1}) - W(\sigma_j)), S^*(s - \sigma_j) a) (s - \sigma_j)^{-\alpha} \cdot \right. \\ \left. \sum_{i=0}^p ((W(\sigma_{i+1}) - W(\sigma_i)), S^*(s - \sigma_i) b) (s - \sigma_i)^{-\alpha} \right)$$

By independence of the increments this is of the form

$$\sum_{j=0}^p (s - \sigma_j)^{-2\alpha} E [((W(\sigma_{j+1}) - W(\sigma_j)), S^*(s - \sigma_j) a) \\ ((W(\sigma_{j+1}) - W(\sigma_j)), S^*(s - \sigma_j) b)] \\ = \sum_{j=0}^p (s - \sigma_j)^{-2\alpha} (\sigma_{j+1} - \sigma_j) (QS^*(s - \sigma_j) a, S^*(s - \sigma_j) b)$$

and this converges to

$$\int_0^s (s - \sigma)^{-2\alpha} (QS^*(s - \sigma) a, S^*(s - \sigma) b) d\sigma \\ = \int_0^s u^{-2\alpha} (QS^*(u) a, S^*(u) b) du \\ = \left(\left(\int_0^s u^{-2\alpha} S(u) QS^*(u) du \right) a, b \right)$$

which shows the covariance of $Y(s)$ is the operator

$$\left(\int_0^s u^{-2\alpha} S(u) QS^*(u) du \right). \quad (56.3.29)$$

Does it even make sense? For each u , the integrand is in $\mathcal{L}(H_0, H)$. Also

$$\begin{aligned} \|u^{-2\alpha} S(u) QS^*(u)\|_{\mathcal{L}_2^0}^2 &\equiv u^{-4\alpha} \sum_k \sum_l (e_k, S(u) QS^*(u) g_l)^2 \\ &= u^{-4\alpha} \sum_l \lambda_l \sum_k \left(Q^{1/2} S^*(u) e_k, Q^{1/2} S^*(u) e_l \right)^2 \\ &= u^{-4\alpha} \sum_l \lambda_l \left\| Q^{1/2} S^*(u) e_l \right\|_H^2 \\ &\leq u^{-4\alpha} \sum_l \lambda_l \left\| S(u) Q^{1/2} \right\|^2 = u^{-4\alpha} C \end{aligned}$$

and so

$$\|u^{-2\alpha} S(u) QS^*(u)\|_{\mathcal{L}_2^0} \leq C u^{-2\alpha}$$

which is an integrable function since $\alpha < 1/2$. It follows 56.3.29 at least makes sense and is in $\mathcal{L}_2(H_0, H)$. Furthermore, its norm in $\mathcal{L}_2(H_0, H)$ is bounded independent of s . What about the trace of this covariance operator? Let $\{e_n\}$ be the orthonormal basis for H used above. Then if $\mathcal{Q}(s)$ denotes this covariance operator,

$$\begin{aligned}
 \sum_k (\mathcal{Q}(s) e_k, e_k)_H &= \sum_k \int_0^s u^{-2\alpha} (S(u) Q S^*(u) e_k, e_k) du \\
 &= \sum_k \int_0^s u^{-2\alpha} \left| Q^{1/2} S^*(u) e_k \right|_H^2 du \\
 &= \int_0^s \sum_k u^{-2\alpha} \sum_l \left(Q^{1/2} S^*(u) e_k, e_l \right)^2 du \\
 &= \int_0^s \sum_l u^{-2\alpha} \sum_k \left(e_k, S(u) Q^{1/2} e_l \right)^2 du \\
 &= \int_0^s \sum_l u^{-2\alpha} \lambda_l \sum_k (e_k, S(u) e_l)^2 du \\
 &= \int_0^s u^{-2\alpha} \sum_l \lambda_l |S(u) e_l|_H^2 du \leq C \int_0^T u^{-2\alpha} du < \infty
 \end{aligned}$$

because of the assumption that $\alpha < 1/2$. Therefore, $\mathcal{Q}(s)$, the covariance of $Y(s)$ has trace which is bounded independent of s . Thus $E(|Y(s)|^2)$ which equals the trace of $\mathcal{Q}(s)$ is bounded independent of s . Now recall Lemma 56.3.5.

Since there exists a bound for $\text{trace}(\mathcal{Q}(s)) = E(|Y(s)|^2)$ which is independent of s , it follows there exists a constant C independent of $s \in [0, T]$ such that

$$E(|Y(s)|^{2m}) \leq C \quad (56.3.30)$$

With 56.3.30 it becomes possible to finish the proof. From 56.3.27

$$W_A(t) = \frac{1}{\pi} \sin(\pi\alpha) \int_0^t S(t-s) (t-s)^{\alpha-1} Y(s) ds$$

where thanks to 56.3.30, there is a set of measure 0 such that off this set,

$$\int_0^T |Y(s)|_H^{2m} ds < \infty.$$

Assume ω is such that the above holds.

First note that if $s \rightarrow Y(s)$ is continuous, then $t \rightarrow W_A(t)$ must also be contin-

uous. Next suppose $Y \in L^{2m}(0, T, H)$. By Holder's inequality

$$\begin{aligned} |W_A(t)| &\leq C \left(\int_0^t \|S(t-s)\|^{2m/(2m-1)} (t-s)^{(\alpha-1)2m/(2m-1)} ds \right)^{2m/(2m-1)} \\ &\quad \left(\int_0^T |Y(s)|_H^{2m} ds \right)^{1/2m} \\ &\leq C \left(\int_0^T \|S(s)\|^{2m/(2m-1)} (s)^{(\alpha-1)2m/(2m-1)} ds \right)^{2m/(2m-1)} \left(\int_0^T |Y(s)|_H^{2m} ds \right)^{1/2m} \end{aligned}$$

the first integral is finite because $\|S(t)\|$ has exponential growth due to its being a continuous semigroup and the assumption that $m > 1/(2\alpha)$ implies

$$\frac{(\alpha-1)2m}{2m-1} > \frac{((1/2m)-1)2m}{2m-1} = -1.$$

Thus

$$\sup_{t \in [0, T]} |W_A(t)| \leq C \left(\int_0^T |Y(s)|_H^{2m} ds \right)^{1/2m}.$$

Letting $W_n(t)$ be defined as $W_A(t)$ but with $Y(s)$ replaced with $Y_n(s)$ where $\{Y_n(t)\}$ is a sequence of continuous H valued functions converging in $L^{2m}(0, T; H)$ to Y , it follows

$$\sup_{t \in [0, T]} |W_A(t) - W_n(t)| \leq C \|Y_n - Y\|_{L^{2m}(0, T; H)}$$

which shows that since uniform convergence occurs, it must be the case that $t \rightarrow W_A(t)$ is continuous. This proves the theorem.

Note $\alpha \in (0, 1/2)$ was arbitrary and it was shown that if $m > 1/2\alpha$, then

$$\sup_{t \in [0, T]} E(|W_A(t)|^{2m}) < \infty$$

and so it follows for a.e. ω ,

$$\sup_{t \in [0, T]} E(|W_A(t)|^p) < K_p$$

for any $p > 2$. Note that $2m$ must be larger than 2.

Corollary 56.3.9 *In the situation of the above theorem, let $\{A_n\}$ denote a sequence of Yosida approximations such that for all $x \in H$,*

$$e^{A_n t} x \rightarrow S(t)x$$

Then for all $p > 2$

$$\lim_{n \rightarrow \infty} E \left(\sup_{t \in [0, T]} |W_A(t) - W_{A_n}(t)|^p \right) = 0.$$

Proof: It is no loss of generality to assume at the outset that $S(t)$ is a bounded semigroup since if not, you could consider $e^{-\gamma t}S(t)$ for a suitable γ which would be a bounded semigroup due to the exponential growth of any continuous semigroup. Then you could do everything for this bounded semigroup and note that the desired result would only involve a factor of $e^{\gamma t}$. More specifically, approximating by sums you can assert that for each $t \leq T$ and $S_n(t)$ the semigroup generated by A_n ,

$$\begin{aligned} & \left| \int_0^t e^{-\gamma(t-s)} (S(t-s) - S_n(t-s)) dW(s) \right| \\ & \geq e^{-\gamma T} \left| \int_0^t (S(t-s) - S_n(t-s)) dW(s) \right| \end{aligned}$$

and so, if you have shown the result for the bounded semigroups, $e^{-\gamma t}S(t)$, it will follow also for the unbounded semigroup.

Thus from the proof of the Hille Yosida theorem,

$$\|S(t)\|, \|e^{A_n t}\| \leq M.$$

This is convenient. Also let m be a large positive integer larger than p corresponding to α such that $m > 1/2\alpha$.

Now as in the proof of the above theorem,

$$W_A(t) = \frac{1}{\pi} \sin(\pi\alpha) \int_0^t S(t-s)(t-s)^{\alpha-1} Y(s) ds$$

where

$$Y(s) \equiv \int_0^s S(s-\sigma)(s-\sigma)^{-\alpha} dW(\sigma)$$

and

$$W_{A_n}(t) = \frac{1}{\pi} \sin(\pi\alpha) \int_0^t S_n(t-s)(t-s)^{\alpha-1} Y_n(s) ds$$

where

$$Y_n(s) \equiv \int_0^s S_n(s-\sigma)(s-\sigma)^{-\alpha} dW(\sigma)$$

for $S_n(t) \equiv e^{A_n t}$. Then

$$\begin{aligned} W_A(t) - W_{A_n}(t) &= \frac{\sin(\pi\alpha)}{\pi} \int_0^t (S(t-s) - S_n(t-s))(t-s)^{\alpha-1} Y(s) ds \\ &\quad + \frac{\sin(\pi\alpha)}{\pi} \int_0^t S_n(t-s)(t-s)^{\alpha-1} (Y_n(s) - Y(s)) ds \\ &\equiv I_n(t) + J_n(t) \end{aligned}$$

First consider $I_n(t)$. For fixed t ,

$$\begin{aligned}
 |I_n(t)| &\leq C \left(\int_0^t |(S(t-s) - S_n(t-s)) Y(s)|^{2m} ds \right)^{1/2m} \\
 &\quad \left(\int_0^t (t-s)^{(\alpha-1)(2m)/(2m-1)} ds \right)^{(2m-1)/2m} \\
 &\leq C \left(\int_0^T s^{(\alpha-1)(2m)/(2m-1)} ds \right)^{(2m-1)/2m} \\
 &\quad \left(\int_0^t |(S(t-s) - S_n(t-s)) Y(s)|^{2m} ds \right)^{1/2m} \\
 &= C_m \left(\int_0^T |(S(t-s) - S_n(t-s)) Y(s)|^{2m} ds \right)^{1/2m}
 \end{aligned}$$

where define both $S(r)$ and $S_n(r)$ to equal I if $r < 0$. By the choice of α , $(\alpha-1)(2m)/(2m-1) - 1$. Thus for every t ,

$$|I_n(t)|^{2m} \leq C_m 2M \int_0^T |Y(s)|^{2m} ds < \infty$$

and it follows

$$\sup_{t \in [0, T]} |I_n(t)|^{2m} < \infty.$$

Now choose t_n such that from the above,

$$\begin{aligned}
 \sup_{t \in [0, T]} |I_n(t)|^{2m} &< \frac{1}{n} + |I_n(t_n)|^{2m} \\
 &\leq \frac{1}{n} + C_m \left(\int_0^T |(S(t_n-s) - S_n(t_n-s)) Y(s)|^{2m} ds \right)
 \end{aligned}$$

The integrand converges to 0 as $n \rightarrow \infty$ and in addition is bounded by $2M |Y(s)|^{2m}$, a function in L^1 and so by dominated convergence theorem,

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \left(\sup_{t \in [0, T]} |I_n(t)|^{2m} \right) &\leq \\
 \lim_{n \rightarrow \infty} \sup_{n \rightarrow \infty} \left(\frac{1}{n} + C_m \left(\int_0^T |(S(t_n-s) - S_n(t_n-s)) Y(s)|^{2m} ds \right) \right) &= 0.
 \end{aligned}$$

Recall $S_n(t)x$ converges uniformly to $S(t)x$ on finite intervals. Thus

$$\lim_{n \rightarrow \infty} E \left(\sup_{t \in [0, T]} |I_n(t)|^{2m} \right) = 0.$$

Next consider $J_n(t)$.

$$\begin{aligned}
 |J_n(t)| &\leq C \left(\int_0^t \|S_n(t-s)(t-s)^{\alpha-1}\|^{2m/(2m-1)} ds \right)^{(2m-1)/2m} \\
 &\quad \cdot \left(\int_0^T |Y_n(s) - Y(s)|^{2m} ds \right)^{1/2m} \\
 &\leq C \left(\int_0^T \|S_n(s)(s)^{\alpha-1}\|^{2m/(2m-1)} ds \right)^{(2m-1)/2m} \\
 &\quad \cdot \left(\int_0^T |Y_n(s) - Y(s)|^{2m} ds \right)^{1/2m}
 \end{aligned}$$

and so

$$\sup_{t \in [0, T]} |J_n(t)|^{2m} \leq C_m \left(\int_0^T |Y_n(s) - Y(s)|^{2m} ds \right)^{1/2m} \quad (56.3.31)$$

Next

$$\begin{aligned}
 &E \left(|Y_n(s) - Y(s)|^2 \right) \\
 &= E \left(\left| \int_0^s (S_n(s-\sigma) - S(s-\sigma))(s-\sigma)^{-\alpha} dW(\sigma) \right|^2 \right) \quad (56.3.32)
 \end{aligned}$$

The inside stochastic integral is a limit in $L^2(\Omega, H)$ of sums of the form

$$\sum_{k=0}^p (s - \sigma_k)^{-\alpha} \Delta_n(s - \sigma_k) (W(\sigma_{k+1}) - W(\sigma_k))$$

where Δ_n represents the difference of the two semigroups. Then the following converges to 56.3.32. Using independence of the increments,

$$\begin{aligned}
 &E \left(\left| \sum_{k=0}^p (s - \sigma_k)^{-\alpha} \Delta_n(s - \sigma_k) (W(\sigma_{k+1}) - W(\sigma_k)) \right|^2 \right) \\
 &= E \left(\sum_{k=0}^p (s - \sigma_k)^{-2\alpha} |\Delta_n(s - \sigma_k) (W(\sigma_{k+1}) - W(\sigma_k))|_H^2 \right) \\
 &= \sum_{k=0}^p (s - \sigma_k)^{-2\alpha} \sum_{r=1}^{\infty} E \left(((W(\sigma_{k+1}) - W(\sigma_k)), \Delta_n(s - \sigma_k)^* e_r)^2 \right) \\
 &= \sum_{k=0}^p (s - \sigma_k)^{-2\alpha} \sum_{r=1}^{\infty} (\sigma_{k+1} - \sigma_k) (Q \Delta_n(s - \sigma_k)^* e_r, \Delta_n(s - \sigma_k)^* e_r) \\
 &= \sum_{k=0}^p (\sigma_{k+1} - \sigma_k) (s - \sigma_k)^{-2\alpha} \sum_{r=1}^{\infty} \left| Q^{1/2} \Delta_n(s - \sigma_k)^* e_r \right|_H^2
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^p (\sigma_{k+1} - \sigma_k) (s - \sigma_k)^{-2\alpha} \sum_{r=1}^{\infty} \sum_{i=1}^{\infty} \left(Q^{1/2} \Delta_n (s - \sigma_k)^* e_r, e_i \right)^2 \\
&= \sum_{k=0}^p (\sigma_{k+1} - \sigma_k) (s - \sigma_k)^{-2\alpha} \sum_{i=1}^{\infty} \sum_{r=1}^{\infty} \left(e_r, \Delta_n (s - \sigma_k) Q^{1/2} e_i \right)^2 \\
&= \sum_{k=0}^p (\sigma_{k+1} - \sigma_k) (s - \sigma_k)^{-2\alpha} \sum_{i=1}^{\infty} \lambda_i |\Delta_n (s - \sigma_k) e_i|^2
\end{aligned}$$

and so passing to a limit as $p \rightarrow \infty$, this yields

$$\begin{aligned}
&\int_0^s (s - \sigma)^{-2\alpha} \sum_{i=1}^{\infty} \lambda_i |\Delta_n (s - \sigma) e_i|^2 d\sigma \\
&= \int_0^s \sigma^{-2\alpha} \sum_{i=1}^{\infty} \lambda_i |(S(\sigma) - S_n(\sigma)) e_i|^2 d\sigma \\
&\leq \sum_{i=1}^{\infty} \lambda_i \int_0^T \sigma^{-2\alpha} |(S(\sigma) - S_n(\sigma)) e_i|^2 d\sigma
\end{aligned}$$

which converges to 0 as $n \rightarrow \infty$ independent of s . This is at least as large as the trace of the covariance of $Y_n(s) - Y(s)$ for each s and by Lemma 56.3.5, for each s ,

$$E \left(|Y_n(s) - Y(s)|^{2m} \right) \leq C_m \left(\sum_{i=1}^{\infty} \lambda_i \int_0^T \sigma^{-2\alpha} |(S(\sigma) - S_n(\sigma)) e_i|^2 d\sigma \right)^m$$

and so from 56.3.31

$$\begin{aligned}
&E \left(\sup_{t \in [0, T]} |J_n(t)|^{2m} \right) \\
&\leq C_m E \left(\int_0^T |Y_n(s) - Y(s)|^{2m} ds \right) \\
&\leq C_m \left(\sum_{i=1}^{\infty} \lambda_i \int_0^T \sigma^{-2\alpha} |(S(\sigma) - S_n(\sigma)) e_i|^2 d\sigma \right)^m
\end{aligned}$$

which converges to 0. This proves the corollary.

This corollary has another very interesting corollary. To motivate this corollary, consider the following incorrect argument which uses integration by parts.

$$\begin{aligned}
W_A(t) &\equiv \int_0^t S(t-s) dW(s) = \int_0^t S(t-s) \frac{dW}{ds} ds \\
&= W(s) S(t-s) \Big|_0^t + \int_0^t AS(t-s) W(s) ds \\
&= W(t) + A \int_0^t S(t-s) W(s) ds.
\end{aligned}$$

Even though the above argument is faulty, the formula does hold and in particular,

$$\int_0^t S(t-s) W(s) ds \in D(A).$$

Corollary 56.3.10 *In the above situation*

$$t \rightarrow \int_0^t S(t-s) W(s) ds$$

is in $C^1([0, \infty), H)$ and it has values in $D(A)$ for a.e. ω and also

$$\begin{aligned} W_A(t) &= W(t) + A \int_0^t S(t-s) W(s) ds \\ &= \frac{d}{dt} \int_0^t S(t-s) W(s) ds \end{aligned}$$

Proof: Let $\{A_n\}$ be a sequence of Yosida approximations of A and let $S_n(t)$ be the semigroup generated by A_n such that $S_n(t)x \rightarrow S(t)x$ uniformly on every finite interval. This was done in the proof of the Hille Yosida theorem in the case where $S(t)$ was bounded. In the general case this will also hold by considering the semigroup, $e^{-\alpha t} S(t)$. Then by the stochastic Fubini theorem,

$$\begin{aligned} \int_0^t A_n W_{A_n}(s) ds + W(t) &= \int_0^t A_n \int_0^s S_n(s-r) dW(r) ds + W(t) \\ &= \int_0^t \int_r^t A_n S_n(s-r) ds dW(r) + W(t) \\ &= \int_0^t (S_n(t-r) - I) dW(r) = W_{A_n}(t) - W(t) + W(t) = W_{A_n}(t). \end{aligned}$$

Thus

$$W_{A_n}(t) = W(t) + A_n \int_0^t W_{A_n}(s) ds \quad (56.3.33)$$

Recall it was shown above that $t \rightarrow W_{A_n}(t)$ is continuous a.e. This was even true for unbounded A generating a continuous semigroup. Therefore, from the above,

$$\left(e^{-A_n t} \int_0^t W_{A_n}(s) ds \right)' = e^{-A_n t} W(t)$$

Hence

$$e^{-A_n t} \int_0^t W_{A_n}(s) ds = \int_0^t e^{-A_n s} W(s) ds$$

and so

$$\int_0^t W_{A_n}(s) ds = \int_0^t e^{A_n(t-s)} W(s) ds = \int_0^t S_n(t-s) W(s) ds.$$

Thus

$$W_{A_n}(t) = W(t) + A_n \int_0^t S_n(t-s) W(s) ds \quad (56.3.34)$$

and it is clear from dominated convergence theorem that

$$\int_0^t S_n(t-s) W(s) ds \rightarrow \int_0^t S(t-s) W(s) ds \quad (56.3.35)$$

by the dominated convergence theorem.

By Corollary 56.3.9

$$\lim_{n \rightarrow \infty} E \left(\sup_{t \in [0, T]} |W_A(t) - W_{A_n}(t)|^p \right) = 0.$$

Take a subsequence still called n such that

$$E \left(\sup_{t \in [0, T]} |W_A(t) - W_{A_n}(t)|^p \right) < \frac{1}{4^n}$$

Then

$$\begin{aligned} & P \left(\left[\sup_{t \in [0, T]} |W_A(t) - W_{A_n}(t)|^p > \frac{1}{2^n} \right] \right) \frac{1}{2^n} \\ & \leq E \left(\sup_{t \in [0, T]} |W_A(t) - W_{A_n}(t)|^p \right) < \frac{1}{4^n} \end{aligned}$$

and so

$$P(B_n) \equiv P \left(\left[\sup_{t \in [0, T]} |W_A(t) - W_{A_n}(t)|^p > \frac{1}{2^n} \right] \right) \leq \frac{1}{2^n}$$

Therefore, by the Borel Cantelli lemma, there is a set of measure 0 such that for ω not in this set, ω is in only finitely many of the B_n and so for a.e. ω , $W_{A_n}(t)$ converges uniformly to $W_A(t)$ on $[0, T]$. Now taking a countable union of sets of measure 0 corresponding to $T = 1, 2, 3, \dots$, it follows there is a set of measure 0 such that $W_{A_n}(t)$ converges uniformly to $W_A(t)$ on $[0, \infty)$.

Now recall the definition of the Yosida approximations.

$$A_n = A\lambda_n(\lambda_n I - A)^{-1} \equiv AJ_n$$

and $|J_n x - x|$ converges to 0 for each $x \in D(A)$ and $\|J_n\|$ is bounded because by Proposition 15.12.5,

$$\left\| (\lambda_n I - A)^{-1} \right\| \leq \frac{M}{|\lambda_n - \alpha|}$$

for some constants, α, M . Therefore, $|J_n x - x|$ converges to 0 for each $x \in H$, not just for those in $D(A)$. By 56.3.34

$$\begin{aligned} W_{A_n}(t) &= W(t) + A_n \int_0^t S_n(t-s) W(s) ds \\ &= W(t) + AJ_n \int_0^t S_n(t-s) W(s) ds \end{aligned}$$

and it was shown above that $W_{A_n}(t) \rightarrow W_A(t)$ uniformly and so it follows from the above that

$$AJ_n \int_0^t S_n(t-s) W(s) ds \rightarrow W_A(t) - W(t)$$

Also

$$\begin{aligned} & \left| J_n \int_0^t S_n(t-s) W(s) ds - \int_0^t S(t-s) W(s) ds \right| \\ & \leq \left| J_n \int_0^t S_n(t-s) W(s) ds - J_n \int_0^t S(t-s) W(s) ds \right| \\ & \quad + \left| J_n \int_0^t S(t-s) W(s) ds - \int_0^t S(t-s) W(s) ds \right| \\ & \leq M \left| \int_0^t S_n(t-s) W(s) ds - \int_0^t S(t-s) W(s) ds \right| \\ & \quad + \left| J_n \int_0^t S(t-s) W(s) ds - \int_0^t S(t-s) W(s) ds \right| \end{aligned}$$

which converges to 0. Thus

$$\begin{aligned} J_n \int_0^t S_n(t-s) W(s) ds &\rightarrow \int_0^t S(t-s) W(s) ds, \\ J_n \int_0^t S_n(t-s) W(s) ds &\in D(A) \end{aligned}$$

and

$$AJ_n \int_0^t S_n(t-s) W(s) ds \rightarrow W_A(t) - W(t).$$

Since A is closed,

$$\int_0^t S(t-s) W(s) ds \in D(A)$$

and

$$A \int_0^t S(t-s) W(s) ds = W_A(t) - W(t).$$

Hence

$$\begin{aligned} W_A(t) &= W(t) + A \int_0^t S(t-s) W(s) ds \\ &= \frac{d}{dt} \left(\int_0^t S(t-s) W(s) ds \right). \end{aligned}$$

the reason for this last step is as follows. Taking a difference quotient for

$$\int_0^t S(t-s) W(s) ds$$

yields

$$\begin{aligned} & \frac{1}{h} \int_0^{t+h} S(h) S(t-s) W(s) ds - \frac{1}{h} \int_0^t S(t-s) W(s) ds \\ &= \frac{1}{h} \int_t^{t+h} S(t+h-s) W(s) ds + \frac{1}{h} (S(h) - I) \int_0^t S(t-s) W(s) ds \end{aligned}$$

and taking a limit,

$$\begin{aligned} & \frac{d}{dt} \left(\int_0^t S(t-s) W(s) ds \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{1}{h} \int_t^{t+h} S(t+h-s) W(s) ds + \frac{1}{h} (S(h) - I) \int_0^t S(t-s) W(s) ds \right) \\ &= W(t) + A \int_0^t S(t-s) W(s) ds = W_A(t) \end{aligned}$$

It follows from this formula that $t \rightarrow \int_0^t S(t-s) W(s) ds$ is in $C^1([0, \infty), H)$ for a.e. ω because $t \rightarrow W_A(t)$ is continuous. Also the formula shows $t \rightarrow \int_0^t S(t-s) W(s) ds$ is continuous with values in $D(A)$. This proves the corollary.

56.3.3 Temporal Regularity Analytic Semigroups

Here the semigroups will be analytic semigroups, not just continuous semigroups. This will make possible results on regularity in time and later, some other results involving fractional powers of operators which amount to spacial regularity. First the regularity in time will be considered. This presentation will make use of Corollary 30.2.7 on Page 894 which is the basic result on analytic semigroups. The theorem is very interesting because it says the convolution integral has the same sort of temporal regularity as the Wiener process.

Theorem 56.3.11 *Let $W(t)$ be a Q Wiener process where Q is of trace class and has all positive eigenvalues as described above and let A be a sectorial operator. For arbitrary $\alpha \in (0, \frac{1}{2})$, there is a set of measure zero such that for ω not in this set, the convolution integral, $W_A(t)$ has α Holder continuous trajectories on any finite interval, $[0, T]$.*

Proof: Fix $T > 0$ and let $T \geq t > s > 0$. Then

$$\begin{aligned} & E \left(|W_A(t) - W_A(s)|^2 \right) \\ &= E \left(\left| \int_0^s S(t-r) dW(r) + \int_s^t S(t-r) dW(r) - \int_0^s S(s-r) dW(r) \right|^2 \right) \end{aligned}$$

By the usual argument involving sums approximating the integrals along with the independence of the increments, (Formally, the $dW(r)$ for $r > s$ are independent of the $dW(r)$ for $r < s$.) the above expression equals

$$E \left(\left| \int_0^s (S(t-r) - S(s-r)) dW(r) \right|^2 \right) + E \left(\left| \int_s^t S(t-r) dW(r) \right|^2 \right) \quad (56.3.36)$$

Approximating by sums in the usual way, the following formal argument is justified.

$$\begin{aligned} & E \left(\left| \int_s^t S(t-r) dW(r) \right|^2 \right) = E \left(\int_s^t (S(t-r) dW(r), S(t-r) dW(r))_H \right) \\ &= \int_s^t E \left(\sum_k (S(t-r) dW(r), e_k)_H^2 \right) = \int_s^t E \left(\sum_k (dW(r), S(t-r)^* e_k)_H^2 \right) \\ &= \int_s^t \sum_k E (dW(r), S(t-r)^* e_k)_H^2 = \int_s^t \sum_k (QS(t-r)^* e_k, S(t-r)^* e_k) dr \\ &= \int_s^t \sum_k \left| Q^{1/2} S(t-s)^* e_k \right|^2 dr = \int_s^t \sum_k \sum_i \left(Q^{1/2} S(t-s)^* e_k, e_i \right)^2 dr \\ &= \int_s^t \sum_i \sum_k \left(e_k, S(t-s) Q^{1/2} e_i \right)^2 dr = \int_s^t \sum_i \lambda_i \sum_k (e_k, S(t-s) e_i)^2 dr \\ &= \int_s^t \sum_{i=1}^{\infty} \lambda_i |S(t-s) e_i|^2 dr. \end{aligned} \quad (56.3.37)$$

With exactly similar reasoning, the other term in 56.3.36 is of the form

$$\int_0^s \sum_{i=1}^{\infty} \lambda_i |(S(t-r) - S(s-r)) e_i|^2 dr. \quad (56.3.38)$$

The idea is to estimate these terms. First consider the one in 56.3.37. From Corollary 30.2.7 on Page 894, $\|S(t-s)\|$ is bounded with a bound which depends on T and so

$$\int_s^t \sum_{i=1}^{\infty} \lambda_i |S(t-s) e_i|^2 dr \leq M_T |t-s| \text{trace}(Q).$$

Now consider the term in 56.3.38. This one will lead to an estimate which, while usable, is not so good. Using Corollary 30.2.7 again,

$$\begin{aligned}
 \int_0^s \sum_{i=1}^{\infty} \lambda_i |(S(t-r) - S(s-r)) e_i|^2 dr &= \sum_i \lambda_i \int_0^s \left| \int_{s-r}^{t-r} AS(\sigma) e_i d\sigma \right|^2 dr \\
 \sum_i \lambda_i \int_0^s \left| \int_{s-r}^{t-r} |AS(\sigma) e_i| d\sigma \right|^2 dr &\leq N_T^2 \sum_i \lambda_i \int_0^s \left| \int_{s-r}^{t-r} \left(\frac{1}{\sigma} + aM_T \right) d\sigma \right|^2 dr \\
 &= N_T^2 \text{trace}(Q) \int_0^s \left| \int_{s-r}^{t-r} \frac{1}{\sigma} + aM_T d\sigma \right|^2 dr \\
 &\leq N_T^2 \text{trace}(Q) \int_0^s \left| \int_{s-r}^{t-r} \frac{1}{\sigma} d\sigma + aM_T |t-s| \right|^2 dr \\
 &\leq 2N_T^2 \text{trace}(Q) \int_0^s \left| \int_{s-r}^{t-r} \frac{1}{\sigma} d\sigma \right|^2 dr \\
 &\quad + 2T (aM_T)^2 N_T^2 \text{trace}(Q) |t-s|^2
 \end{aligned}$$

Now pick $\gamma \in (0, 1/2)$.

$$\begin{aligned}
 &= 2N_T^2 \text{trace}(Q) \int_0^s \left| \int_{s-r}^{t-r} \frac{1}{\sigma^\gamma \sigma^{1-\gamma}} d\sigma \right|^2 dr \\
 &\leq 2N_T^2 \text{trace}(Q) \int_0^s \frac{1}{(s-r)^{2\gamma}} \left| \int_{s-r}^{t-r} \sigma^{\gamma-1} d\sigma \right|^2 dr \\
 &\leq 2N_T^2 \text{trace}(Q) \int_0^s \frac{1}{(s-r)^{2\gamma}} \left| \int_{s-r}^{t-r} \sigma^{\gamma-1} d\sigma \right|^2 dr
 \end{aligned}$$

Now $s-r \geq 0$ and the exponent $\gamma-1 < 0$. Therefore,

$$\begin{aligned}
 &\leq 2N_T^2 \text{trace}(Q) \int_0^s \frac{1}{(s-r)^{2\gamma}} \left| \int_0^{t-s} \sigma^{\gamma-1} d\sigma \right|^2 dr \\
 &= 2N_T^2 \text{trace}(Q) \frac{1}{\gamma^2} (t-s)^{2\gamma} \int_0^s \frac{1}{(s-r)^{2\gamma}} dr \\
 &\leq 2N_T^2 \text{trace}(Q) \frac{1}{\gamma^2} (t-s)^{2\gamma} \int_0^T \frac{1}{(r)^{2\gamma}} dr \\
 &= 2N_T^2 \text{trace}(Q) \frac{1}{\gamma^2} \frac{T^{1-2\gamma}}{1-2\gamma} (t-s)^{2\gamma}
 \end{aligned}$$

Thus

$$\begin{aligned}
 E(|W_A(t) - W_A(s)|^2) &\leq 2N_T^2 \text{trace}(Q) \frac{1}{\gamma^2} \frac{T^{1-2\gamma}}{1-2\gamma} |t-s|^{2\gamma} \\
 M_T |t-s| \text{trace}(Q) + 2T (aM_T)^2 N_T^2 \text{trace}(Q) |t-s|^2 &\quad (56.3.39)
 \end{aligned}$$

and it was shown earlier that $W_A(t) - W_A(s)$ is a symmetric Gaussian random variable.

In general, if X is a symmetric Gaussian random variable having trace class covariance $\mathcal{Q}$, then letting $\{e_k\}$ be the eigenvectors for $\mathcal{Q}$,

$$\begin{aligned} E(|X|^2) &= E\left(\sum_k (X, e_k)^2\right) = \sum_k E((X, e_k)^2) \\ &\equiv \sum_k (\mathcal{Q}e_k, e_k) = \text{trace}(\mathcal{Q}). \end{aligned}$$

From Lemma 56.3.5, if m is a positive integer, it follows from 56.3.39 that for $t, s \in [0, T]$, since $2\gamma < 1$,

$$\begin{aligned} &E(|W_A(t) - W_A(s)|^{2m}) \\ &\leq \left(2N_T^2 \text{trace}(Q) \frac{1}{\gamma^2} \frac{T^{1-2\gamma}}{1-2\gamma} |t-s|^{2\gamma} + \right. \\ &\quad \left.M_T |t-s| \text{trace}(Q) + 2T(aM_T)^2 N_T^2 \text{trace}(Q) |t-s|^2\right)^m \\ &\leq C_{m,\gamma,T,Q} |t-s|^{2\gamma m} \end{aligned}$$

Now recall the Kolmogorov Čentsov continuity theorem, Theorem 50.1.4 on Page 1472. This theorem implies that for a.e. $\omega, t \rightarrow W_A(t)$ is Hölder continuous with exponent

$$\frac{2\gamma m - 1}{2m} = \gamma - \frac{1}{m}$$

Thus if $\alpha < 1/2$, you can choose $\gamma \in (\alpha, \frac{1}{2})$ and then choose m so large that $\gamma - 1/m > \alpha$ also. This proves the theorem on $[0, T]$. To get Hölder continuous trajectories off a set of measure zero on any finite interval, $[0, T]$, let the set of measure zero be the union of the sets of measure zero corresponding to $T = 1, 2, 3, \dots$. This proves the theorem.

56.3.4 Spacial Regularity

Next a different sort of regularity will be established. It will be shown that $W_A(t)$ is in $D((-A)^\alpha)$ for fractional powers of $(-A)$. The necessary theory of analytic semigroups is found starting on Page 902. To begin with, here is a proposition.

Proposition 56.3.12 *Let $\gamma \in (0, 1/2)$. Then $W_A(t) \in D((-A)^\gamma)$ a.e. Also $(-A)^\gamma W_A(t)$ is a Gaussian random variable and its covariance can be computed.*

Proof: First note

$$\int_0^t (-A)^\gamma S(t-s) dW(s)$$

is well defined. This follows because $s \rightarrow \|(-A)^\gamma S(t-s)\|$ is in $L^2([0, t])$ due to the estimate,

$$\|(-A)^\gamma S(t-s)\| \leq \frac{M}{|t-s|^\gamma}$$

from Theorem 30.2.18 on Page 906. This estimate shows $s \rightarrow (-A)^\gamma S(t-s)$ is in $L^2([0, T]; \mathcal{L}(H, H))$ and so the stochastic integral makes sense.

Approximating $W_A(t) \equiv \int_0^t S(t-s) dW(s)$ with sums,

$$\sum_{k=0}^m S(t-s_k) (W(s_{k+1}) - W(s_k)) \equiv S_m(t),$$

it follows since $S(t)$ is an analytic semigroup,

$$S_m(t) \in D((-A)^\gamma)$$

and

$$(-A)^\gamma S_m(t) = \sum_{k=0}^m (-A)^\gamma S(t-s_k) (W(s_{k+1}) - W(s_k))$$

which converges in $L^2(\Omega; H)$ to

$$\int_0^t (-A)^\gamma S(t-s) dW(s)$$

while $S_m(t)$ converges in $L^2(\Omega; H)$ to

$$W_A(t) \equiv \int_0^t S(t-s) dW(s)$$

Since $(-A)^\gamma$ is closed, it follows that for a.e. ω , $W_A(t) \in D((-A)^\gamma)$ and

$$(-A)^\gamma W_A(t) = \int_0^t (-A)^\gamma S(t-s) dW(s).$$

Next consider the claim that $(-A)^\gamma W_A(t)$ is Gaussian. This follows from noting that $(-A)^\gamma S_m(t)$ is Gaussian. This follows because

$$\begin{aligned} & ((-A)^\gamma S(t-s_k) (W(s_{k+1}) - W(s_k)), a) \\ &= (W(s_{k+1}) - W(s_k), ((-A)^\gamma S(t-s_k))^* a) \end{aligned}$$

and this is a normal random variable having mean 0. Then $(-A)^\gamma S_m(t)$ is a finite sum of such random variables and by the independence of the increments, it must also be a normal random variable having mean 0. Then taking a subsequence, $(-A)^\gamma S_m(t)$ converges pointwise a.e. to $(-A)^\gamma W_A(t)$ and so using the characteristic functions it follows from the dominated convergence theorem that

$$E(\exp(i\lambda((-A)^\gamma S_m(t), a)_H)) \rightarrow E(\exp(i\lambda((-A)^\gamma W_A(t), a)_H))$$

Now

$$E(\exp(i\lambda((-A)^\gamma S_m(t), a)_H)) = e^{-\frac{1}{2}\lambda^2\sigma_m^2}$$

Then σ_m^2 must be bounded since otherwise, there would exist a subsequence for which $\sigma_m^2 \rightarrow \infty$ and $0 = E(\exp(i\lambda((-A)^\gamma W_A(t), a)_H))$ which is impossible because this last must equal 1 when $\lambda = 0$. Therefore, there is a subsequence, still called σ_m^2 which converges to σ^2 and so

$$\begin{aligned} e^{-\frac{1}{2}\lambda^2\sigma^2} &= \lim_{m \rightarrow \infty} e^{-\frac{1}{2}\lambda^2\sigma_m^2} = \lim_{m \rightarrow \infty} E(\exp(i\lambda((-A)^\gamma S_m(t), a)_H)) \\ &= E(\exp(i\lambda((-A)^\gamma W_A(t), a)_H)). \end{aligned}$$

Since $a \in H$ is arbitrary, it follows $(-A)^\gamma W_A(t)$ is a Gaussian random variable.

Now it is time to compute its covariance. Let $a, b \in H$. I will compute something a little more general,

$$E(((-A)^\gamma W_A(t) - (-A)^\gamma W_A(s), a)((-A)^\gamma W_A(t) - (-A)^\gamma W_A(s), b)),$$

where $s \leq t$.

$$\begin{aligned} &E\left(\left(\int_0^t (-A)^\gamma S(t-r) dW(r) - \int_0^s (-A)^\gamma S(s-r) dW(r), a\right)\right. \\ &\quad \cdot \left.\left(\left(\int_0^t (-A)^\gamma S(t-r) dW(r) - \int_0^s (-A)^\gamma S(s-r) dW(r), b\right)\right)\right) \end{aligned}$$

Now using the independence of the increments and justifying all the manipulations by considering sums and passing to limits, the above reduces to

$$\begin{aligned} &E\left(\left(\int_0^s (-A)^\gamma (S(t-r) - S(s-r)) dW(r), a\right)\right. \\ &\quad \cdot \left.\left(\int_0^s (-A)^\gamma (S(t-r) - S(s-r)) dW(r), b\right)\right) \\ &+ E\left(\left(\int_s^t (-A)^\gamma S(t-r) dW(r), a\right)\left(\int_s^t (-A)^\gamma S(t-r) dW(r), b\right)\right) \end{aligned}$$

Consider the top first. Approximating by sums and taking the meaning suggested by the following symbols, the following formal argument yields the following computation.

$$\begin{aligned} &\int_0^s E(((-A)^\gamma (S(t-r) - S(s-r)) dW(r), a)((-A)^\gamma (S(t-r) - S(s-r)) dW(r), b)) \\ &= \int_0^s E((dW(r), ((-A)^\gamma (S(t-r) - S(s-r)))^* a)(dW(r), ((-A)^\gamma (S(t-r) - S(s-r)))) \\ &= \int_0^s (Q((-A)^\gamma (S(t-r) - S(s-r)))^* a, ((-A)^\gamma (S(t-r) - S(s-r)))^* b) dr \end{aligned}$$

$$= \left(\left[\int_0^s (-A)^\gamma (S(t-r) - S(s-r)) Q((-A)^\gamma (S(t-r) - S(s-r)))^* dr \right] a, b \right)$$

Next consider the bottom. Using a similar formal computation,

$$\begin{aligned} & E \left(\left(\int_s^t (-A)^\gamma S(t-r) dW(r), a \right) \left(\int_s^t (-A)^\gamma S(t-r) dW(r), b \right) \right) \\ &= \int_s^t E \left(((-A)^\gamma S(t-r) dW(r), a) ((-A)^\gamma S(t-r) dW(r), b) \right) \\ &= \int_s^t E \left((dW(r), ((-A)^\gamma S(t-r))^* a) (dW(r), ((-A)^\gamma S(t-r))^* b) \right) \\ &= \int_s^t (Q((-A)^\gamma S(t-r))^* a, ((-A)^\gamma S(t-r))^* b) dr \\ &= \int_0^{t-s} (Q((-A)^\gamma S(\sigma))^* a, ((-A)^\gamma S(\sigma))^* b) d\sigma \\ &= \left(\left[\int_0^{t-s} (-A)^\gamma S(\sigma) Q((-A)^\gamma S(\sigma))^* d\sigma \right] a, b \right) \end{aligned}$$

Thus the covariance is the operator taking H to H given by

$$\begin{aligned} & \left[\int_0^{t-s} (-A)^\gamma S(\sigma) Q((-A)^\gamma S(\sigma))^* d\sigma \right] \\ &+ \left[\int_0^s (-A)^\gamma (S(t-r) - S(s-r)) Q((-A)^\gamma (S(t-r) - S(s-r)))^* dr \right] \end{aligned}$$

This proves the proposition.

To reiterate, $W_A(t) \in D((-A)^\gamma)$ whenever $\gamma < 1/2$ and you can take $(-A)^\gamma$ inside the stochastic convolution integral.

For A a sectorial operator for the sector $S_{-a, \phi}$ where $-a < 0$, so that the fractional powers are defined, you can define a scale of Banach spaces, $X_\gamma \equiv D((-A)^\gamma)$ with the norm

$$\|x\|_\gamma \equiv \|(-A)^\gamma x\|$$

These are Banach spaces because $(-A)^\gamma$ is a closed operator. Recall that for $\alpha \leq \beta$,

$$D((-A)^\alpha) \subseteq D((-A)^\beta).$$

Earlier it was shown that $t \rightarrow W_A(t)$ was Holder continuous having exponent $\gamma < 1/2$ with values in H . The next theorem shows that $t \rightarrow W_A(t)$ is also Holder continuous with exponent $\alpha < 1/2 - \gamma$ with values in X_γ .

Theorem 56.3.13 *Let A be as just described. Then for all $\gamma \in (0, 1/2)$ and $\alpha \in (0, 1/2 - \gamma)$ $t \rightarrow W_A(t)$ is α Holder continuous with values in X_γ on any finite interval.*

Proof: The proof will depend on the Kolmogorov Čentsov continuity theorem, Theorem 50.1.4. Let $[0, T]$ be a finite interval. It is necessary to estimate for $0 < s < t \leq T$,

$$E \left(\left| \int_0^t (-A)^\gamma S(t-r) dW(r) - \int_0^s (-A)^\gamma S(s-r) dW(r) \right|^2 \right).$$

Using the independence of the increments and approximating the integrals with sums and then taking a limit, the above equals

$$E \left(\left| \int_s^t (-A)^\gamma S(t-r) dW(r) \right|^2 \right) + \quad (56.3.40)$$

$$E \left(\left| \int_0^s (-A)^\gamma (S(t-r) - S(s-r)) dW(r) \right|^2 \right) \quad (56.3.41)$$

First consider 56.3.40. Approximating by sums, the following formal argument can be made precise. The expression equals

$$\int_s^t E \left(|(-A)^\gamma S(t-r) dW(r)|^2 \right)$$

The integrand of this term is

$$\begin{aligned} & E \left(\sum_k (dW(r), ((-A)^\gamma S(t-r))^* e_k)^2 \right) \\ &= \sum_k E \left((dW(r), ((-A)^\gamma S(t-r))^* e_k)^2 \right) \\ &= \sum_k dr (Q(((-A)^\gamma S(t-r))^* e_k, ((-A)^\gamma S(t-r))^* e_k)) \\ &= \sum_k dr \left| Q^{1/2}(((-A)^\gamma S(t-r))^* e_k) \right|^2 \\ &= dr \sum_k \sum_j \left(Q^{1/2}(((-A)^\gamma S(t-r))^* e_k, e_j) \right)^2 \\ &= dr \sum_k \sum_j \left(((-A)^\gamma S(t-r))^* e_k, Q^{1/2} e_j \right)^2 \\ &= dr \sum_j \lambda_j \sum_k (e_k, (-A)^\gamma S(t-r) e_j)^2 \\ &= dr \sum_j \lambda_j |(-A)^\gamma S(t-r) e_j|^2 \\ &\leq dr \frac{M^2}{|t-r|^{2\gamma}} \sum_j \lambda_j = \text{trace}(Q) \frac{M^2}{|t-r|^{2\gamma}} dr \end{aligned}$$

where the estimate found in Theorem 30.2.18 is being used. Thus 56.3.40 is dominated by

$$\begin{aligned}
 & \text{trace}(Q) M^2 \int_s^t \frac{1}{(t-r)^{2\gamma}} dr \\
 = & \text{trace}(Q) M^2 \int_s^t \frac{1}{r^{2\gamma}} dr \\
 \leq & \text{trace}(Q) M^2 \int_0^{t-s} \frac{1}{r^{2\gamma}} dr \\
 = & \text{trace}(Q) M^2 \frac{1}{1-2\gamma} |t-s|^{1-2\gamma}
 \end{aligned}$$

Next consider 56.3.41. Formally, it equals

$$\int_0^s E \left(|(-A)^\gamma (S(t-r) - S(s-r)) dW(r)|^2 \right)$$

Using exactly the same computations, the integrand equals

$$\begin{aligned}
 & \sum_k E(dW(r), ((-A)^\gamma (S(t-r) - S(s-r)))^* e_k)^2 \\
 = & dr \sum_k (Q((-A)^\gamma (S(t-r) - S(s-r)))^* e_k, ((-A)^\gamma (S(t-r) - S(s-r)))^* e_k) \\
 = & dr \sum_k \left| Q^{1/2}((-A)^\gamma (S(t-r) - S(s-r)))^* e_k \right|^2 \\
 = & dr \sum_k \sum_j \left(((-A)^\gamma (S(t-r) - S(s-r)))^* e_k, Q^{1/2} e_j \right)^2 \\
 = & dr \sum_j \lambda_j \sum_k (e_k, (-A)^\gamma (S(t-r) - S(s-r)) e_j)^2 \\
 = & dr \sum_j \lambda_j |(-A)^\gamma (S(t-r) - S(s-r)) e_j|^2
 \end{aligned}$$

Thus 56.3.41 is dominated by

$$\begin{aligned}
 & \sum_j \lambda_j \int_0^s |(-A)^\gamma (S(t-r) - S(s-r)) e_j|^2 dr \\
 = & \sum_j \lambda_j \int_0^s \left\| \int_{s-r}^{t-r} (-A)^\gamma AS(\sigma) d\sigma \right\|^2 dr \\
 \leq & \text{trace}(Q) \int_0^s \left\| \int_{s-r}^{t-r} (-A)^\gamma AS(\sigma) d\sigma \right\|^2 dr
 \end{aligned}$$

Using Theorem 30.2.18 again, this is dominated by

$$\leq \operatorname{trace}(Q) \int_0^s \left(\int_{s-r}^{t-r} \frac{M}{\sigma^{1+\gamma}} d\sigma \right)^2 dr$$

Now this requires some estimating. Let $\xi \in (\gamma, 1/2)$ and write it as

$$\begin{aligned} &\leq \operatorname{trace}(Q) \int_0^s \left(\int_{s-r}^{t-r} \frac{M}{\sigma^{1+\gamma-\xi}} \frac{1}{\sigma^\xi} d\sigma \right)^2 dr \\ &\leq \operatorname{trace}(Q) \int_0^s \frac{1}{(s-r)^{2\xi}} \left(\int_{s-r}^{t-r} \frac{M}{\sigma^{1+\gamma-\xi}} d\sigma \right)^2 dr \\ &\leq \operatorname{trace}(Q) \int_0^s \frac{1}{(s-r)^{2\xi}} \left(\int_0^{t-s} \frac{M}{\sigma^{1+\gamma-\xi}} d\sigma \right)^2 dr \\ &= M^2 \operatorname{trace}(Q) \left(\frac{(t-s)^{\xi-\gamma}}{\xi-\gamma} \right)^2 \int_0^s \frac{1}{r^{2\xi}} dr \\ &= M^2 \operatorname{trace}(Q) \frac{T^{1-2\xi}}{(1-2\xi)(\xi-\gamma)^2} (t-s)^{2\xi-2\gamma} \end{aligned}$$

It follows

$$\begin{aligned} &E \left(\left| \int_0^t (-A)^\gamma S(t-r) dW(r) - \int_0^s (-A)^\gamma S(s-r) dW(r) \right|^2 \right) \\ &\leq \operatorname{trace}(Q) M^2 \frac{1}{1-2\gamma} |t-s|^{1-2\gamma} + M^2 \operatorname{trace}(Q) \frac{T^{1-2\xi}}{(1-2\xi)(\xi-\gamma)^2} (t-s)^{2\xi-2\gamma} \end{aligned}$$

Now to simplify things note $1-2\gamma \geq 2\gamma-2\xi$ and so, adjusting the constant, there exists a constant C_T such that

$$E \left(\|W_A(t) - W_A(s)\|_{X_\gamma}^2 \right) \leq C_T |t-s|^{2\xi-2\gamma}$$

Letting m be a positive integer, it follows from Lemma 56.3.5 that

$$E \left(\|W_A(t) - W_A(s)\|_{X_\gamma}^{2m} \right) \leq \left(\frac{2m!}{2^m m!} \right) C_T^m |t-s|^{2m\xi-2m\gamma}$$

Let m be large enough that $2m\xi - 2m\gamma > 1$. It follows from Kolmogorov Čentsov continuity theorem that $t \rightarrow W_A(t)$ is Hölder continuous with exponent α for any α such that

$$0 < \alpha < \frac{2m\xi - 2m\gamma - 1}{2m} = \xi - \gamma - \frac{1}{2m}.$$

Now if α is any exponent less than $1/2 - \gamma$ as described in the statement of the theorem, you can choose m large enough and ξ close enough to $1/2$ that

$$\alpha < \xi - \gamma - \frac{1}{2m}$$

and this proves the theorem for a single time interval. Taking the union of countably many sets of measure zero corresponding to the intervals $[0, n], n \in \mathbb{N}$, there exists a set of measure zero such that for ω not in this set, $W_A(t)$ has a version which is Hölder continuous with exponent $\alpha < 1/2 - \gamma$ for any finite interval. This proves the theorem.

56.3.5 Weak Solutions

Let A generate a strongly continuous semigroup as described above. Then a weak solution to the stochastic differential equation,

$$dX(t) = (AX(t) + f(t))dt + BdW(t), \quad X(0) = \xi \quad (56.3.42)$$

is a function X having values in H such that for a.e. ω , $X(\cdot, \omega)$ is Bochner integrable and for all $\zeta \in D(A^*)$ and for all $t \in [0, T]$

$$(X(t), \zeta) = (\xi, \zeta) + \int_0^t [(X(s), A^*\zeta) + (f(s), \zeta)] ds + (BW(t), \zeta) \quad \text{a.e. } \omega \quad (56.3.43)$$

Recall that in all this it is assumed f is H predictable with Bochner integrable trajectories on an arbitrary $[0, T]$. Also ξ is $\mathcal{F}_0$ measurable.

A^* is defined as follows.

$$D(A^*) \equiv \{y \in H : |(Ax, y)| \leq C|x| \text{ for all } x \in D(A)\}$$

Since $D(A)$ is assumed to be dense in H this means there exists a unique $A^*y \in H$ such that for all $x \in D(A)$,

$$(Ax, y) = (x, A^*y).$$

For $y \in D(A^*)$, it follows that for all $x \in D(A)$,

$$(x, A^*y) = (Ax, y) \equiv \lim_{h \rightarrow 0} \left(\frac{S(h)x - x}{h}, y \right) = \left(x, \lim_{h \rightarrow 0} \frac{S^*(h)y - y}{h} \right) \quad (56.3.44)$$

The solution in the following theorem is called a mild solution. .

Theorem 56.3.14 *Let ξ be $\mathcal{F}_0$ measurable and S is a strongly continuous semigroup generating A as described above. Let $B \in \mathcal{L}(U, H)$ and f be predictable with Bochner integrable trajectories. Then*

$$X(t) \equiv S(t)\xi + \int_0^t S(t-s)f(s)ds +$$

$$+ \int_0^t S(t-s) B dW(s) \quad (56.3.45)$$

is a weak solution to 56.3.42 as described in 56.3.43.

Proof: I need to show that for $X(t)$ given above, 56.3.43 holds. That is

$$(X(t), \zeta) = (\xi, \zeta) + \int_0^t [(X(s), A^* \zeta) + (f(s), \zeta)] ds + (BW(t), \zeta)$$

I will start with the right side of 56.3.43 and reduce to the left.

Let $\zeta \in D(A^*)$ and consider the term in the middle of 56.3.43 listed above. With X given in 56.3.45, this term is

$$\int_0^t [(X(s), A^* \zeta) + (f(s), \zeta)] ds \quad (56.3.46)$$

$$= \int_0^t \left[\left(\left(S(s) \xi + \int_0^s S(s-r) f(r) dr \right. \right. \right. \quad (56.3.47)$$

$$\left. \left. + \int_0^s S(s-r) B dW(r) \right), A^* \zeta \right) + (f(s), \zeta) \right] ds \quad (56.3.48)$$

The most interesting term is the one with the Stochastic integral. Consider this one first.

$$\begin{aligned} & \int_0^t \left(\int_0^s S(s-r) B dW(r), A^* \zeta \right) ds \\ &= \int_0^t \left(\int_0^s B^* S^*(s-r) A^* \zeta, dW(r) \right) ds \end{aligned} \quad (56.3.49)$$

It is important to give some discussion of the meaning of this last expression.

$$s \rightarrow B^* S^*(s-r) A^* \zeta$$

is a function having values in U and the inner product is taken in U so it is equivalent to considering this as a function having values in $U' = \mathcal{L}(U; \mathbb{R})$ which is a special case of what was done earlier in defining the stochastic integral provided this function is also in $\mathcal{L}_2(U, \mathbb{R})$. However, $\mathbb{R}$ is one dimensional and so this will be Hilbert Schmidt in the right way if

$$\sum_i (B^* S^*(s-r) A^* \zeta, e_i)^2 = |B^* S^*(s-r) A^* \zeta|_U^2 < \infty$$

which is of course the case. Thus the last integral makes sense. Approximating it with the integrals of elementary functions it is clear the two expressions in 56.3.49 are equal as claimed. Now from the Stochastic Fubini theorem, Theorem 56.1.3, applied to this second integral, it equals

$$\int_0^t \left(\int_0^s B^* S^*(s-r) A^* \zeta, dW(r) \right) ds$$

$$= \int_0^t \left(\int_r^t B^* S^* (s-r) A^* \zeta ds, dW(r) \right)$$

From Theorem 15.12.11 on Page 482 this equals

$$\begin{aligned} & \int_0^t \left(\int_r^t B^* S^* (s-r) \lim_{h \rightarrow 0} \frac{S^*(h) \zeta - \zeta}{h} ds, dW(r) \right) \\ &= \int_0^t \left(\int_r^t B^* \lim_{h \rightarrow 0} \frac{S^*(h+s-r) \zeta - S^*(s-r) \zeta}{h} ds, dW(r) \right) \\ &= \int_0^t \left(\int_r^t \frac{d}{ds} (B^* S^* (s-r) \zeta) ds, dW(r) \right) \\ &= \int_0^t (B^* S^* (t-r) \zeta - B^* \zeta, dW(r)) \\ &= \left(\zeta, \int_0^t S(t-r) B dW(r) \right) - (B^* \zeta, W(t)) \\ &= (W_A(t), \zeta) - (\zeta, BW(t)) \end{aligned}$$

Next look at the middle term in 56.3.47

$$\begin{aligned} & \int_0^t \left(\int_0^s S(s-r) f(r) dr, A^* \zeta \right) ds \\ &= \int_0^t \int_r^t (f(r), S^*(s-r) A^* \zeta) ds dr \\ &= \int_0^t \int_r^t \left(f(r), S^*(s-r) \lim_{h \rightarrow 0} \frac{S^*(h) \zeta - \zeta}{h} \right) ds dr \\ &= \int_0^t \int_r^t \left(f(r), \frac{d}{ds} (S^*(s-r) \zeta) \right) ds dr \\ &= \int_0^t (f(r), S^*(t-r) \zeta - \zeta) dr \\ &= \int_0^t (S(t-r) f(r), \zeta) dr - \int_0^t (f(r), \zeta) dr \end{aligned}$$

Collecting these terms yields 56.3.46 - 56.3.48 is of the form

$$\begin{aligned} & \int_0^t [(X(s), A^* \zeta) + (f(s), \zeta)] ds = \\ & (W_A(t), \zeta) - (\zeta, BW(t)) + \int_0^t (S(t-r) f(r), \zeta) dr - \int_0^t (f(r), \zeta) dr \\ & + \int_0^t (S(s) \xi, A^* \zeta) ds + \int_0^t (f(s), \zeta) ds \end{aligned}$$

It follows the right side of 56.3.43 which is the description of the weak solution is of the form

$$\begin{aligned}
 (W_A(t), \zeta) - (\zeta, BW(t)) + \int_0^t (S(t-r)f(r), \zeta) dr + \int_0^t (S(s)\xi, A^*\zeta) ds \\
 + (BW(t), \zeta) + (\xi, \zeta) \\
 = \int_0^t (S(t-r)f(r), \zeta) dr + \int_0^t (S(s)\xi, A^*\zeta) ds \\
 + \left(\int_0^t S(t-r)BdW(r), \zeta \right) + (\xi, \zeta)
 \end{aligned}$$

Now consider the second integral.

$$\begin{aligned}
 \int_0^t (S(s)\xi, A^*\zeta) ds &= \int_0^t \left(S(s)\xi, \lim_{h \rightarrow 0} \frac{S^*(h)\zeta - \zeta}{h} \right) ds \\
 &= \int_0^t \left(\xi, \frac{d}{ds} (S^*(s)\zeta) \right) ds \\
 &= (\xi, S^*(t)\zeta) - (\xi, \zeta) \\
 &= (S(t)\xi, \zeta) - (\xi, \zeta)
 \end{aligned}$$

and so the right side of 56.3.43 reduces to

$$\int_0^t (S(t-r)f(r), \zeta) dr + \left(\int_0^t S(t-r)BdW(r), \zeta \right) + (S(t)\xi, \zeta)$$

which is the definition of $(X(t), \zeta)$. Thus if $X(t)$ is given as the mild solution described in the statement of this theorem, this has shown

$$\begin{aligned}
 (X(t), \zeta) &= \\
 (\xi, \zeta) + \int_0^t [(X(s), A^*\zeta) + (f(s), \zeta)] ds + (BW(t), \zeta)
 \end{aligned} \tag{56.3.50}$$

This proves the theorem.

Next consider the question of uniqueness of weak solutions. Recall a weak solution, $X(t)$ is one which satisfies 56.3.50 for all $\zeta \in D(A^*)$. Recall the Ito formula

Theorem 56.3.15 *Let Φ be an $\mathcal{L}_2^0$ valued predictable process which is stochastically integrable in $[0, T]$ because*

$$P \left(\left[\int_0^T \|\Phi\|^2 dt < \infty \right] \right) = 1$$

and let ϕ be an H valued predictable process which is Bochner integrable on $[0, T]$ and let $Y(0)$ be $\mathcal{F}_0$ measurable and H valued. Let

$$Y(t) \equiv Y(0) + \int_0^t \phi(s) ds + \Phi \cdot W(t).$$

Let $G : [0, T] \times H \rightarrow \mathbb{R}^1$ have continuous partial derivatives G_t, G_x, G_{xx} which are uniformly continuous on bounded subsets of $[0, T] \times H$. Then the following formula holds for a.e. ω .

$$G(t, Y(t)) = G(0, Y(0)) + G_x(\cdot, Y(\cdot)) \Phi \cdot W(t) +$$

$$\int_0^t \{G_t(s, Y(s)) + G_x(s, Y(s))(\phi(s)) + \frac{1}{2} \text{trace}(\Phi(s)^* G_{xx}(s, Y(s)) \Phi(s) Q)\} ds$$

In proving the uniqueness the following interesting lemma and corollary will be of use.

Lemma 56.3.16 *Let $f : [0, 1] \rightarrow X$, a Banach space such that f is continuous. Define the Bernstein polynomials*

$$p_n(x) = \sum_{k=0}^n \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$$

Then $p_n(x)$ converges uniformly to $f(x)$ on $[0, 1]$ and furthermore, if $f' : [0, 1] \rightarrow X$ is also continuous, then the polynomials $p'_n(x)$ also converge to $f'(x)$ uniformly on $[0, 1]$.

Proof: By Lemma 7.1.2 on Page 163 it follows

$$\sum_{k=0}^n \binom{n}{k} (k - nx)^2 x^k (1-x)^{n-k} \leq \frac{1}{4}n \quad (56.3.51)$$

By the binomial theorem

$$\|f(x) - p_n(x)\| \leq \sum_{k=0}^n \binom{n}{k} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| x^k (1-x)^{n-k}$$

At this point you break the sum into two pieces, those values of k such that k/n is close to x and those values for k such that k/n is not so close to x . Thus

$$\begin{aligned} & \|f(x) - p_n(x)\| \\ & \leq \sum_{|x - (k/n)| < \delta} \binom{n}{k} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| x^k (1-x)^{n-k} \\ & \quad + \sum_{|x - (k/n)| \geq \delta} \binom{n}{k} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| x^k (1-x)^{n-k} \end{aligned} \quad (56.3.52)$$

where δ is a positive number chosen in an auspicious manner about to be described. Since f is continuous on $[0, 1]$, it follows f is uniformly continuous. Therefore,

letting $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - y| < \delta$, then $\|f(x) - f(y)\| < \varepsilon/2$. This is the auspicious choice for δ .

Also, $\|f(x)\|$ for $x \in [0, 1]$ is bounded by some number M . Thus 56.3.52 implies that for $x \in [0, 1]$,

$$\begin{aligned} \|f(x) - p_n(x)\| &\leq \sum_{|x - (k/n)| < \delta} \binom{n}{k} \frac{\varepsilon}{2} x^k (1-x)^{n-k} \\ &\quad + 2M \sum_{|nx - k| \geq n\delta} \binom{n}{k} x^k (1-x)^{n-k} \\ &\leq \frac{\varepsilon}{2} + 2M \sum_{|nx - k| \geq n\delta} \binom{n}{k} \frac{(k - nx)^2}{n^2 \delta^2} x^k (1-x)^{n-k} \\ &\leq \frac{\varepsilon}{2} + \frac{2M}{n^2 \delta^2} \sum_{k=0}^n \binom{n}{k} (k - nx)^2 x^k (1-x)^{n-k} \end{aligned}$$

From 56.3.51

$$\|f(x) - p_n(x)\| \leq \frac{\varepsilon}{2} + \frac{2M}{n^2 \delta^2} \frac{n}{4} = \frac{\varepsilon}{2} + \frac{M}{2n\delta^2}$$

showing the uniform convergence as claimed.

Now suppose f' is continuous. I need to verify $p'_n(x)$ converges uniformly to $f'(x)$ on $[0, 1]$. Differentiating the formula for p_n yields

$$\begin{aligned} p'_n(x) &= \sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)}{1/n} \right) x^k (1-x)^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \left(n \int_{k/n}^{(k+1)/n} f'(s) ds \right) x^k (1-x)^{n-1-k} \end{aligned}$$

while the $n-1$ Bernstein polynomial for f' is

$$q_n(x) \equiv \sum_{k=0}^{n-1} \binom{n-1}{k} f'\left(\frac{k}{n-1}\right) x^k (1-x)^{n-1-k}$$

Thus

$$\|p'_n(x) - f'(x)\| \leq \|p'_n(x) - q_n(x)\| + \|q_n(x) - f'(x)\|.$$

By the first part of the argument it follows that for all n large enough,

$$\|p'_n(x) - f'(x)\| \leq \|p'_n(x) - q_n(x)\| + \varepsilon/2$$

Consider the first term on the right. Then

$$\begin{aligned}
 & ||p'_n(x) - q_n(x)|| \\
 & \leq \sum_{k=0}^{n-1} \binom{n-1}{k} \left(n \int_{k/n}^{(k+1)/n} \left\| f'(s) - f'\left(\frac{k}{n-1}\right) \right\| ds \right) x^k (1-x)^{n-1-k} \\
 & < \sum_{k=0}^{n-1} \binom{n-1}{k} (\varepsilon/2) x^k (1-x)^{n-1-k} = \varepsilon/2
 \end{aligned}$$

whenever n is large enough due to the uniform continuity of f' on $[0, 1]$.

Corollary 56.3.17 *Suppose f, f' are continuous on $[a, b]$ having values in a Banach space, X . Then there exists a sequence of polynomials, $\{p_n\}$ such that $p_n(x)$ converges uniformly to $f(x)$ on $[a, b]$ and $p'_n(x)$ converges uniformly to $f'(x)$ on $[a, b]$. Each polynomial is a finite sum of a polynomial times a value of f .*

Proof: Let $l : [0, 1] \rightarrow [a, b]$ be one to one onto and linear. Then let $g(x) \equiv f \circ l(x)$ for all $x \in [0, 1]$. Then g is continuous with continuous derivative on $[0, 1]$ and so there exists a sequence of polynomials $\{p_n(x)\}$ converging uniformly to g on $[0, 1]$ and $p'_n(x)$ converges uniformly to $g'(x)$ on $[0, 1]$. Thus letting $q_n(y) \equiv p_n(l^{-1}(y))$, it follows $q_n(y)$ converges uniformly to $f(y)$ on $[a, b]$. Also

$$q'_n(y) = p'_n(l^{-1}(y)) (l^{-1})'(y)$$

while

$$f'(y) = g'(l^{-1}(y)) (l^{-1})'(y)$$

and so since $(l^{-1})'(y)$ is just a scalar, it follows $q'_n(y)$ converges uniformly to $f'(y)$ on $[a, b]$. This proves the corollary.

Lemma 56.3.18 *Let X be a weak solution to*

$$dX = (AX + f) dt + BdW, \quad X(0) = \xi.$$

where. That is

$$(X(t), \zeta) = (\xi, \zeta) + \int_0^t (X(s), A^* \zeta) ds + \int_0^t (f(s), \zeta) ds + (BW(t), \zeta) \quad (56.3.53)$$

Then whenever $\zeta(\cdot) \in C([0, T]; D(A^*))$, $\zeta'(\cdot) \in C([0, T]; H)$ and $t \in [0, T]$,

$$(X(t), \zeta(t)) =$$

$$\begin{aligned}
 & (\xi, \zeta(0)) + \int_0^t (\zeta(s), BdW(s)) + \int_0^t (X(s), \zeta'(s)) \\
 & + ((X(s), A^* \zeta(s)) + (f(s), \zeta(s))) ds
 \end{aligned}$$

the second term on the right being defined in the usual way as a limit of sums involving elementary functions. Also the norm in $D(A^*)$ will be the norm

$$||x|| \equiv |x|_H + |A^*(x)|_H$$

which is a norm on $D(A^*)$ making $D(A^*)$ a Banach space because A^* is closed.

Proof: First let $\phi(s)$ be in $C^1([0, T])$ and let $\zeta_0 \in D(A^*)$. then let $\zeta(t) = \phi(t)\zeta_0$. Then look at the right side of 56.3.53 with this ζ_0 in place of ζ . This is

$$\begin{aligned} F_{\zeta_0}(t) &\equiv (\xi, \zeta_0) + \int_0^t (X(s), A^*\zeta_0) ds + \int_0^t (f(s), \zeta_0) ds + (BW(t), \zeta_0) \\ &= F_{\zeta_0}(0) + \int_0^t ((X(s), A^*\zeta_0) + (f(s), \zeta_0)) ds + B^*\zeta_0 \cdot W(t) \end{aligned}$$

Now consider the function

$$G(t, x) \equiv x\phi(t)$$

Thus from the Ito formula

$$\begin{aligned} G(t, F_{\zeta_0}(t)) &= \overbrace{G(0, F_{\zeta_0}(0))}^{=(\xi, \zeta_0)\phi(0)} + \overbrace{G_x(\cdot, F_{\zeta_0}(\cdot))}^{=\phi(\cdot)} B^*\zeta_0 \cdot W(t) + \\ &\int_0^t \left\{ \overbrace{G_t(s, F_{\zeta_0}(s))}^{=\phi'(s)F_{\zeta_0}(s)} + \overbrace{G_x(s, F_{\zeta_0}(s))}^{=\phi(s)} ((X(s), A^*\zeta_0) + (f(s), \zeta_0)) + \right. \\ &\left. + \frac{1}{2} \text{trace} \left(\Phi(s)^* \overbrace{G_{xx}(s, F_{\zeta_0}(s))}^{=0} \Phi(s) Q \right) \right\} ds \end{aligned}$$

and so substituting in to the Ito formula yields

$$\begin{aligned} F_{\zeta_0}(t)\phi(t) &= (\xi, \zeta_0)\phi(0) + \phi(\cdot)B^*\zeta_0 \cdot W(t) \\ &+ \int_0^t \phi'(s)F_{\zeta_0}(s) + \phi(s)((X(s), A^*\zeta_0) + (f(s), \zeta_0)) ds \end{aligned}$$

and written in the other notation,

$$\begin{aligned} F_{\zeta_0}(t)\phi(t) &= (\xi, \zeta_0)\phi(0) + \int_0^t (\zeta(s), BdW(s)) \\ &+ \int_0^t \phi'(s)F_{\zeta_0}(s) + ((X(s), A^*\zeta(s)) + (f(s), \zeta(s))) ds \end{aligned} \quad (56.3.54)$$

Now recall from the assumption that X is a weak solution and 56.3.53

$$\begin{aligned}(X(t), \zeta_0) &= (\xi, \zeta_0) + \int_0^t ((X(s), A^* \zeta_0) + (f(s), \zeta_0)) ds + (BW(t), \zeta_0) \\ &\equiv F_{\zeta_0}(t) \text{ a.e.}\end{aligned}$$

and so 56.3.54 equals

$$\begin{aligned}F_{\zeta_0}(t) \phi(t) &= (X(t), \zeta(t)) = (\xi, \zeta_0) \phi(0) + \int_0^t (\zeta(s), BdW(s)) \\ &\quad + \int_0^t \phi'(s) (X(s), \zeta_0) + ((X(s), A^* \zeta(s)) + (f(s), \zeta(s))) ds \\ &= (\xi, \zeta(0)) + \int_0^t (\zeta(s), BdW(s)) + \int_0^t (X(s), \zeta'(s)) \\ &\quad + ((X(s), A^* \zeta(s)) + (f(s), \zeta(s))) ds\end{aligned}$$

which shows the desired result holds for these special $\zeta(s)$. Now suppose ζ is $C([0, T]; D(A^*))$ and $\zeta' \in C([0, T]; H)$. By Corollary 56.3.17 there is a sequence $\{\zeta_n\}$ such that each ζ_n is a finite sum of polynomials times a value of ζ which converges uniformly in $D(A^*)$ to $\zeta(t)$ on $[0, T]$ and $\{\zeta'_n(t)\}$ converges uniformly to $\zeta'(t)$ on $[0, T]$ in H . The formula holds for each of these ζ_n because each is a finite linear combination of functions of the form $\zeta_0 \phi(s)$ where $\phi(s)$ is $C^1([0, 1])$. Therefore, from the uniform convergence along with the Ito isometry, the formula ends up holding for all such $\zeta \in C([0, T]; D(A^*))$ with $\zeta' \in C([0, T]; H)$ because $\zeta_n \rightarrow \zeta$ uniformly in H and $A^* \zeta_n \rightarrow A^* \zeta$ uniformly in H also while $\zeta'_n \rightarrow \zeta'$ uniformly in H .

With this preparation, it is easy to give a proof of uniqueness.

Theorem 56.3.19 *There is exactly one weak solution to*

$$dX = (AX + f) dt + BdW, \quad X(0) = \xi$$

and it is the mild solution of Theorem 56.3.14.

Proof: Suppose X is a weak solution. I will show it is the mild solution of Theorem 56.3.14. Consider $\zeta(s) \equiv S^*(t-s)\zeta_0$ where $\zeta_0 \in D(A^*)$. This satisfies $\zeta \in C([0, T]; D(A^*))$, $\zeta' \in C([0, T]; H)$ and from Lemma 56.3.18 and the observation that $\zeta(0) = S^*(t)\zeta_0$ while $\zeta(t) = \zeta_0$,

$$\begin{aligned}(X(t), \zeta_0) &= (X(t), \zeta(t)) = \\ &= (\xi, S^*(t)\zeta_0) + \int_0^t (S^*(t-s)\zeta_0, BdW(s)) - \int_0^t (X(s), S^*(t-s)A^*\zeta_0) ds \\ &\quad + \int_0^t (X(s), A^*S^*(t-s)\zeta_0) ds + \int_0^t (f(s), S^*(t-s)\zeta_0) ds\end{aligned}$$

$$= (S(t)\xi, \zeta_0) + \left(\int_0^t S(t-s) B dW(s), \zeta_0 \right) + \left(\int_0^t S(t-s) f(s) ds, \zeta_0 \right)$$

Since ζ_0 is an arbitrary element of the dense set, $D(A^*)$, it follows

$$X(t) = S(t)\xi + \int_0^t S(t-s) B dW(s) + \int_0^t S(t-s) f(s) ds.$$

In other words X is a mild solution of the sort in Theorem 56.3.14. This proves uniqueness.

The Hausdorff Maximal Theorem

The purpose of this appendix is to prove the equivalence between the axiom of choice, the Hausdorff maximal theorem, and the well-ordering principle. The Hausdorff maximal theorem and the well-ordering principle are very useful but a little hard to believe; so, it may be surprising that they are equivalent to the axiom of choice. First it is shown that the axiom of choice implies the Hausdorff maximal theorem, a remarkable theorem about partially ordered sets.

A nonempty set is partially ordered if there exists a partial order, $\prec$, satisfying

$$x \prec x$$

and

$$\text{if } x \prec y \text{ and } y \prec z \text{ then } x \prec z.$$

An example of a partially ordered set is the set of all subsets of a given set and $\prec \equiv \subseteq$. Note that two elements in a partially ordered sets may not be related. In other words, just because x, y are in the partially ordered set, it does not follow that either $x \prec y$ or $y \prec x$. A subset of a partially ordered set, $\mathcal{C}$, is called a chain if $x, y \in \mathcal{C}$ implies that either $x \prec y$ or $y \prec x$. If either $x \prec y$ or $y \prec x$ then x and y are described as being comparable. A chain is also called a totally ordered set. $\mathcal{C}$ is a maximal chain if whenever $\tilde{\mathcal{C}}$ is a chain containing $\mathcal{C}$, it follows the two chains are equal. In other words $\mathcal{C}$ is a maximal chain if there is no strictly larger chain.

Lemma A.0.20 *Let $\mathcal{F}$ be a nonempty partially ordered set with partial order $\prec$. Then assuming the axiom of choice, there exists a maximal chain in $\mathcal{F}$.*

Proof: Let $\mathcal{X}$ be the set of all chains from $\mathcal{F}$. For $\mathcal{C} \in \mathcal{X}$, let

$$S_{\mathcal{C}} = \{x \in \mathcal{F} \text{ such that } \mathcal{C} \cup \{x\} \text{ is a chain strictly larger than } \mathcal{C}\}.$$

If $S_{\mathcal{C}} = \emptyset$ for any $\mathcal{C}$, then $\mathcal{C}$ is maximal. Thus, assume $S_{\mathcal{C}} \neq \emptyset$ for all $\mathcal{C} \in \mathcal{X}$. Let $f(\mathcal{C}) \in S_{\mathcal{C}}$. (This is where the axiom of choice is being used.) Let

$$g(\mathcal{C}) = \mathcal{C} \cup \{f(\mathcal{C})\}.$$

Thus $g(\mathcal{C}) \supsetneq \mathcal{C}$ and $g(\mathcal{C}) \setminus \mathcal{C} = \{f(\mathcal{C})\} = \{\text{a single element of } \mathcal{F}\}$. A subset $\mathcal{T}$ of $\mathcal{X}$ is called a tower if

$$\emptyset \in \mathcal{T},$$

$$\mathcal{C} \in \mathcal{T} \text{ implies } g(\mathcal{C}) \in \mathcal{T},$$

and if $\mathcal{S} \subseteq \mathcal{T}$ is totally ordered with respect to set inclusion, then

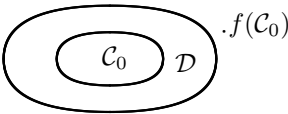
$$\cup \mathcal{S} \in \mathcal{T}.$$

Here $\mathcal{S}$ is a chain with respect to set inclusion whose elements are chains.

Note that $\mathcal{X}$ is a tower. Let $\mathcal{T}_0$ be the intersection of all towers. Thus, $\mathcal{T}_0$ is a tower, the smallest tower. Are any two sets in $\mathcal{T}_0$ comparable in the sense of set inclusion so that $\mathcal{T}_0$ is actually a chain? Let $\mathcal{C}_0$ be a set of $\mathcal{T}_0$ which is comparable to every set of $\mathcal{T}_0$. Such sets exist, $\emptyset$ being an example. Let

$$\mathcal{B} \equiv \{\mathcal{D} \in \mathcal{T}_0 : \mathcal{D} \supsetneq \mathcal{C}_0 \text{ and } f(\mathcal{C}_0) \notin \mathcal{D}\}.$$

The picture represents sets of $\mathcal{B}$. As illustrated in the picture, $\mathcal{D}$ is a set of $\mathcal{B}$ when $\mathcal{D}$ is larger than $\mathcal{C}_0$ but fails to be comparable to $g(\mathcal{C}_0)$. Thus there would be more than one chain ascending from $\mathcal{C}_0$ if $\mathcal{B} \neq \emptyset$, rather like a tree growing upward in more than one direction from a fork in the trunk. It will be shown this can't take place for any such $\mathcal{C}_0$ by showing $\mathcal{B} = \emptyset$.



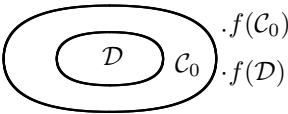
This will be accomplished by showing $\tilde{\mathcal{T}}_0 \equiv \mathcal{T}_0 \setminus \mathcal{B}$ is a tower. Since $\mathcal{T}_0$ is the smallest tower, this will require that $\tilde{\mathcal{T}}_0 = \mathcal{T}_0$ and so $\mathcal{B} = \emptyset$.

Claim: $\tilde{\mathcal{T}}_0$ is a tower and so $\mathcal{B} = \emptyset$.

Proof of the claim: It is clear that $\emptyset \in \tilde{\mathcal{T}}_0$ because for $\emptyset$ to be contained in $\mathcal{B}$ it would be required to properly contain $\mathcal{C}_0$ which is not possible. Suppose $\mathcal{D} \in \tilde{\mathcal{T}}_0$. The plan is to verify $g(\mathcal{D}) \in \tilde{\mathcal{T}}_0$.

Case 1: $f(\mathcal{D}) \in \mathcal{C}_0$. If $\mathcal{D} \subseteq \mathcal{C}_0$, then since both $\mathcal{D}$ and $\{f(\mathcal{D})\}$ are contained in $\mathcal{C}_0$, it follows $g(\mathcal{D}) \subseteq \mathcal{C}_0$ and so $g(\mathcal{D}) \notin \mathcal{B}$. On the other hand, if $\mathcal{D} \supsetneq \mathcal{C}_0$, then since $\mathcal{D} \in \tilde{\mathcal{T}}_0$, $f(\mathcal{C}_0) \in \mathcal{D}$ and so $g(\mathcal{D})$ also contains $f(\mathcal{C}_0)$ implying $g(\mathcal{D}) \notin \mathcal{B}$. These are the only two cases to consider because $\mathcal{C}_0$ is comparable to every set of $\mathcal{T}_0$.

Case 2: $f(\mathcal{D}) \notin \mathcal{C}_0$. If $\mathcal{D} \subsetneq \mathcal{C}_0$ it can't be the case that $f(\mathcal{D}) \notin \mathcal{C}_0$ because if this were so, $g(\mathcal{D})$ would not compare to $\mathcal{C}_0$.



Hence if $f(\mathcal{D}) \notin \mathcal{C}_0$, then $\mathcal{D} \supseteq \mathcal{C}_0$. If $\mathcal{D} = \mathcal{C}_0$, then $f(\mathcal{D}) = f(\mathcal{C}_0) \in g(\mathcal{D})$ so

$g(\mathcal{D}) \notin \mathcal{B}$. Therefore, assume $\mathcal{D} \supsetneq \mathcal{C}_0$. Then, since $\mathcal{D}$ is in $\widetilde{\mathcal{T}}_0$, $f(\mathcal{C}_0) \in \mathcal{D}$ and so $f(\mathcal{C}_0) \in g(\mathcal{D})$. Therefore, $g(\mathcal{D}) \in \widetilde{\mathcal{T}}_0$.

Now suppose $\mathcal{S}$ is a totally ordered subset of $\widetilde{\mathcal{T}}_0$ with respect to set inclusion. Then if every element of $\mathcal{S}$ is contained in $\mathcal{C}_0$, so is $\cup \mathcal{S}$ and so $\cup \mathcal{S} \in \widetilde{\mathcal{T}}_0$. If, on the other hand, some chain from $\mathcal{S}$, $\mathcal{C}$, contains $\mathcal{C}_0$ properly, then since $\mathcal{C} \notin \mathcal{B}$, $f(\mathcal{C}_0) \in \mathcal{C} \subseteq \cup \mathcal{S}$ showing that $\cup \mathcal{S} \notin \mathcal{B}$ also. This has proved $\widetilde{\mathcal{T}}_0$ is a tower and since $\mathcal{T}_0$ is the smallest tower, it follows $\widetilde{\mathcal{T}}_0 = \mathcal{T}_0$. This has shown roughly that no splitting into more than one ascending chain can occur at any $\mathcal{C}_0$ which is comparable to every set of $\mathcal{T}_0$. Next it is shown that every element of $\mathcal{T}_0$ has the property that it is comparable to all other elements of $\mathcal{T}_0$. This is done by showing that these elements which possess this property form a tower.

Define $\mathcal{T}_1$ to be the set of all elements of $\mathcal{T}_0$ which are comparable to every element of $\mathcal{T}_0$. (Recall the elements of $\mathcal{T}_0$ are chains from the original partial order.)

Claim: $\mathcal{T}_1$ is a tower.

Proof of the claim: It is clear that $\emptyset \in \mathcal{T}_1$ because $\emptyset$ is a subset of every set. Suppose $\mathcal{C}_0 \in \mathcal{T}_1$. It is necessary to verify that $g(\mathcal{C}_0) \in \mathcal{T}_1$. Let $\mathcal{D} \in \mathcal{T}_0$ (Thus $\mathcal{D} \subseteq \mathcal{C}_0$ or else $\mathcal{D} \supsetneq \mathcal{C}_0$.) and consider $g(\mathcal{C}_0) \equiv \mathcal{C}_0 \cup \{f(\mathcal{C}_0)\}$. If $\mathcal{D} \subseteq \mathcal{C}_0$, then $\mathcal{D} \subseteq g(\mathcal{C}_0)$ so $g(\mathcal{C}_0)$ is comparable to $\mathcal{D}$. If $\mathcal{D} \supsetneq \mathcal{C}_0$, then $\mathcal{D} \supsetneq g(\mathcal{C}_0)$ by what was just shown ($\mathcal{B} = \emptyset$). Hence $g(\mathcal{C}_0)$ is comparable to $\mathcal{D}$. Since $\mathcal{D}$ was arbitrary, it follows $g(\mathcal{C}_0)$ is comparable to every set of $\mathcal{T}_0$. Now suppose $\mathcal{S}$ is a chain of elements of $\mathcal{T}_1$ and let $\mathcal{D}$ be an element of $\mathcal{T}_0$. If every element in the chain, $\mathcal{S}$ is contained in $\mathcal{D}$, then $\cup \mathcal{S}$ is also contained in $\mathcal{D}$. On the other hand, if some set, $\mathcal{C}$, from $\mathcal{S}$ contains $\mathcal{D}$ properly, then $\cup \mathcal{S}$ also contains $\mathcal{D}$. Thus $\cup \mathcal{S} \in \mathcal{T}_1$ since it is comparable to every $\mathcal{D} \in \mathcal{T}_0$.

This shows $\mathcal{T}_1$ is a tower and proves therefore, that $\mathcal{T}_0 = \mathcal{T}_1$. Thus every set of $\mathcal{T}_0$ compares with every other set of $\mathcal{T}_0$ showing $\mathcal{T}_0$ is a chain in addition to being a tower.

Now $\cup \mathcal{T}_0, g(\cup \mathcal{T}_0) \in \mathcal{T}_0$. Hence, because $g(\cup \mathcal{T}_0)$ is an element of $\mathcal{T}_0$, and $\mathcal{T}_0$ is a chain of these, it follows $g(\cup \mathcal{T}_0) \subseteq \cup \mathcal{T}_0$. Thus

$$\cup \mathcal{T}_0 \supsetneq g(\cup \mathcal{T}_0) \supsetneq \cup \mathcal{T}_0,$$

a contradiction. Hence there must exist a maximal chain after all. This proves the lemma.

If X is a nonempty set, $\leq$ is an order on X if

$$x \leq x,$$

and if $x, y \in X$, then

$$\text{either } x \leq y \text{ or } y \leq x$$

and

$$\text{if } x \leq y \text{ and } y \leq z \text{ then } x \leq z.$$

$\leq$ is a well order and say that $(X, \leq)$ is a well-ordered set if every nonempty subset of X has a smallest element. More precisely, if $S \neq \emptyset$ and $S \subseteq X$ then there exists an $x \in S$ such that $x \leq y$ for all $y \in S$. A familiar example of a well-ordered set is the natural numbers.

Lemma A.0.21 *The Hausdorff maximal principle implies every nonempty set can be well-ordered.*

Proof: Let X be a nonempty set and let $a \in X$. Then $\{a\}$ is a well-ordered subset of X . Let

$$\mathcal{F} = \{S \subseteq X : \text{there exists a well order for } S\}.$$

Thus $\mathcal{F} \neq \emptyset$. For $S_1, S_2 \in \mathcal{F}$, define $S_1 \prec S_2$ if $S_1 \subseteq S_2$ and there exists a well order for $S_2, \leq_2$ such that

$$(S_2, \leq_2) \text{ is well-ordered}$$

and if

$$y \in S_2 \setminus S_1 \text{ then } x \leq_2 y \text{ for all } x \in S_1,$$

and if $\leq_1$ is the well order of S_1 then the two orders are consistent on S_1 . Then observe that $\prec$ is a partial order on $\mathcal{F}$. By the Hausdorff maximal principle, let $\mathcal{C}$ be a maximal chain in $\mathcal{F}$ and let

$$X_\infty \equiv \cup \mathcal{C}.$$

Define an order, $\leq$, on X_∞ as follows. If x, y are elements of X_∞ , pick $S \in \mathcal{C}$ such that x, y are both in S . Then if $\leq_S$ is the order on S , let $x \leq y$ if and only if $x \leq_S y$. This definition is well defined because of the definition of the order, $\prec$. Now let U be any nonempty subset of X_∞ . Then $S \cap U \neq \emptyset$ for some $S \in \mathcal{C}$. Because of the definition of $\leq$, if $y \in S_2 \setminus S_1, S_i \in \mathcal{C}$, then $x \leq y$ for all $x \in S_1$. Thus, if $y \in X_\infty \setminus S$ then $x \leq y$ for all $x \in S$ and so the smallest element of $S \cap U$ exists and is the smallest element in U . Therefore X_∞ is well-ordered. Now suppose there exists $z \in X \setminus X_\infty$. Define the following order, $\leq_1$, on $X_\infty \cup \{z\}$.

$$x \leq_1 y \text{ if and only if } x \leq y \text{ whenever } x, y \in X_\infty$$

$$x \leq_1 z \text{ whenever } x \in X_\infty.$$

Then let

$$\tilde{\mathcal{C}} = \{S \in \mathcal{C} \text{ or } X_\infty \cup \{z\}\}.$$

Then $\tilde{\mathcal{C}}$ is a strictly larger chain than $\mathcal{C}$ contradicting maximality of $\mathcal{C}$. Thus $X \setminus X_\infty = \emptyset$ and this shows X is well-ordered by $\leq$. This proves the lemma.

With these two lemmas the main result follows.

Theorem A.0.22 *The following are equivalent.*

The axiom of choice

The Hausdorff maximal principle

The well-ordering principle.

Proof: It only remains to prove that the well-ordering principle implies the axiom of choice. Let I be a nonempty set and let X_i be a nonempty set for each $i \in I$. Let $X = \cup\{X_i : i \in I\}$ and well order X . Let $f(i)$ be the smallest element of X_i . Then

$$f \in \prod_{i \in I} X_i.$$

A.1 The Hamel Basis

A Hamel basis is nothing more than the correct generalization of the notion of a basis for a finite dimensional vector space to vector spaces which are possibly not of finite dimension.

Definition A.1.1 Let X be a vector space. A Hamel basis is a subset of X , Λ such that every vector of X can be written as a finite linear combination of vectors of Λ and the vectors of Λ are linearly independent in the sense that if $\{x_1, \dots, x_n\} \subseteq \Lambda$ and

$$\sum_{k=1}^n c_k x_k = 0$$

then each $c_k = 0$.

The main result is the following theorem.

Theorem A.1.2 Let X be a nonzero vector space. Then it has a Hamel basis.

Proof: Let $x_1 \in X$ and $x_1 \neq 0$. Let $\mathcal{F}$ denote the collection of subsets of X , Λ containing x_1 with the property that the vectors of Λ are linearly independent as described in Definition A.1.1 partially ordered by set inclusion. By the Hausdorff maximal theorem, there exists a maximal chain, $\mathcal{C}$. Let $\Lambda = \cup \mathcal{C}$. Since $\mathcal{C}$ is a chain, it follows that if $\{x_1, \dots, x_n\} \subseteq \mathcal{C}$ then there exists a single $\Lambda' \in \mathcal{C}$ containing all these vectors. Therefore, if

$$\sum_{k=1}^n c_k x_k = 0$$

it follows each $c_k = 0$. Thus the vectors of Λ are linearly independent. Is every vector of X a finite linear combination of vectors of Λ ?

Suppose not. Then there exists z which is not equal to a finite linear combination of vectors of Λ . Consider $\Lambda \cup \{z\}$. If

$$cz + \sum_{k=1}^m c_k x_k = 0$$

where the x_k are vectors of Λ , then if $c \neq 0$ this contradicts the condition that z is not a finite linear combination of vectors of Λ . Therefore, $c = 0$ and now all the c_k must equal zero because it was just shown Λ is linearly independent. It follows $\mathcal{C} \cup \{\Lambda \cup \{z\}\}$ is a strictly larger chain than $\mathcal{C}$ and this is a contradiction. Therefore, Λ is a Hamel basis as claimed. This proves the theorem.

A.2 Exercises

1. Zorn's lemma states that in a nonempty partially ordered set, if every chain has an upper bound, there exists a maximal element, x in the partially ordered set. x is maximal, means that if $x \prec y$, it follows $y = x$. Show Zorn's lemma is equivalent to the Hausdorff maximal theorem.
2. Show that if Y, Y_1 are two Hamel bases of X , then there exists a one to one and onto map from Y to Y_1 . Thus any two Hamel bases are of the same size.
3. $\uparrow$ Using the Baire category theorem of the chapter on Banach spaces show that any Hamel basis of a Banach space is either finite or uncountable.
4. $\uparrow$ Consider the vector space of all polynomials defined on $[0, 1]$. Does there exist a norm, $\|\cdot\|$ defined on these polynomials such that with this norm, the vector space of polynomials becomes a Banach space (complete normed vector space)?

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Index

- C^1 functions, 124
- C_c^∞ , 350
- C_c^m , 350
- F_σ sets, 178
- G_δ , 363
- G_δ sets, 178
- L^1
 - weak sequential compactness, 517
- L^1_{loc} , 525
- L^p
 - compactness, 356
- L^p multipliers, 666
- $L^p(\Omega; X)$, 724
- L^∞ , 358
- L^p_{loc} , 1046
- Δ regular, 550
- π systems, 275
- σ algebra, 177
- (1,p) extension operator, 1127
- Abel's formula, 84
- Abel's theorem, 804
- absolutely continuous, 530
- adapted, 1374, 1475
- adjugate, 73
- Alexander subbasis theorem, 324
- algebra, 167
- algebra of sets, 264
- analytic continuation, 932, 1034
- Analytic functions, 791
- analytic sets, 761
- approximate identity, 351
- a priori estimates, 412
- area formula, 591, 601, 1066
- at most countable, 24
- atlas, 1089
- automorphic function, 1020
- axiom of choice, 19, 23
- axiom of extension, 19
- axiom of specification, 19
- axiom of unions, 19
- Banach Alaoglu theorem, 383
- Banach space, 341
- Banach Steinhaus theorem, 365
- basis of a topology, 153
- basis of module of periods, 1008
- Besicovitch covering theorem, 615, 630
- Bessel's inequality, 441
- Big Picard theorem, 947
- Binet Cauchy theorem, 1095
- Blaschke products, 989
- Bloch's lemma, 935
- block matrix, 80
- Bochner integrable, 706
- Bochner's theorem, 1369
- Borel Cantelli lemma, 189, 1297
- Borel measure, 225
- Borel regular, 569
- Borel sets, 177
- bounded continuous linear functions, 363
- bounded variation, 779
- box topology, 325
- branch of the logarithm, 834
- Brouwer fixed point theorem, 313, 425
- Burkholder Davis Gundy inequality, 1536
- Burkholder Gundy inequality, 1548
- Calderon Zygmund decomposition, 665
- Cantor diagonalization procedure, 150

- Cantor function, 542
- Cantor set, 542
- Caratheodory, 219
- Caratheodory extension theorem, 321
- Caratheodory's criterion, 566
- Caratheodory's procedure, 220
- Cartesian coordinates, 55
- Casorati Weierstrass theorem, 814
- Cauchy
 - general Cauchy integral formula, 820
 - integral formula for disk, 799
- Cauchy Riemann equations, 793
- Cauchy Schwarz inequality, 98, 421
- Cauchy sequence, 118
- Cauchy sequence, 110
- Cayley Hamilton theorem, 77
- central limit theorem, 1356
- chain rule, 122
- change of variables, 1084
- change of variables general case, 309
- characteristic function, 186, 1327, 1422, 1441
- characteristic polynomial, 76
- chart, 1089
- closed graph theorem, 369
- closed set, 100, 154
- closure of a set, 155
- Coarea formula, 1081
- coarea formula, 1078
- cofactor, 70
- compact, 141
- compact injection map, 1215
- compact set, 156
- complement, 100
- complete measure space, 220
- completion of measure space, 271
- conditional expectation, 1371
- conformal maps, 797, 920
- connected, 158
- connected components, 159
- continuity set, 1354
- continuous function, 99, 156
- convergence in measure, 189
- convex
 - set, 422
- convex
 - functions, 356
- convex sets, 397
- convolution, 351, 655
- convolution of measures, 1334
- Coordinates, 53
- countable, 24
- counting zeros, 844
- covariation, 1534
- Cramer's rule, 73
- cycle, 820
- cylinder sets, 1449
- cylindrical set, 1327

- Darboux, 50
- Darboux integral, 50
- derivatives, 121
- determinant, 66
 - product, 69
 - transpose, 67
- differential equations
 - global existence, 410
- differentiation
 - Radon measures, 633
- dilations, 920
- Dini derivatives, 543
- distribution, 1043, 1298
- distribution function, 247, 543, 662
- distributional derivative, 1207
- divergence theorem, 613
- DL, 1566
- dominated convergence theorem, 211, 727
- Doob Dynkin lemma, 1324
- Doob estimate, 1378
- Doob's submartingale estimate, 1480
- doubly periodic, 1006
- dual space, 375
- duality maps, 391

- Eberlein Smulian theorem, 387
- Egoroff theorem, 186, 198
- eigenvalues, 76, 849, 852
- elementary, 1623

- elementary factors, 973
- elementary function, 1550
- elementary set, 333
- elementary sets, 266
- elliptic, 1006
- entire, 809
- epsilon net, 142, 147
- equality of mixed partial derivatives, 131
- equi integrable, 519, 1566
- equivalence class, 26
- equivalence relation, 25
- Erling's lemma, 1215
- essential singularity, 815
- essentially more slowly, 559
- Euler's theorem, 1001
- events, 1301
- evolution equation
 - continuous semigroup, 477
- exchange theorem, 58
- exponential growth, 657
- extended complex plane, 777
- extension theorem, 1126

- Fatou's lemma, 204
- filtration, 1475
- finite intersection property, 146, 157
- finite measure space, 178
- first hitting time, 1492
- Fourier series
 - uniform convergence, 390
- Fourier transform L^1 , 645
- fractional linear transformations, 920, 925
 - mapping three points, 922
- fractional powers
 - sectorial operator, 902
- fractional spaces
 - reflexive, 1283
- Frechet derivative, 121
- Fredholm alternative, 458
- Fresnel integrals, 877
- Fubini theorem
 - stochastic Fubini theorem, 1756
- Fubini's theorem, 261, 270, 280

- Bochner integrable functions, 722
- function, 22
 - uniformly continuous, 27
- function element, 932, 1034
- functional equations, 1024
- fundamental theorem of algebra, 810
- fundamental theorem of calculus, 49, 527, 529
 - general Radon measures, 623

- Gamma function, 356, 578
- gamma function, 995
- gauge function, 371
- Gauss's formula, 996
- Gaussian measure, 1430, 1440
- Gerschgorin's theorem, 848
- good lambda inequality, 249
- Gram determinant, 434
- Gram matrix, 434
- Gramm Schmidt process, 86
- great Picard theorem, 946

- Hadamard three circles theorem, 839
- Hahn Banach theorem, 372
- Hamel basis, 1811
- Hardy Littlewood maximal function, 525
- Hardy's inequality, 357
- harmonic functions, 796
- Haursdorff measures, 565
- Hausdorff and Lebesgue measure, 577, 579
- Hausdorff dimension, 577
- Hausdorff maximal principle, 26, 289, 324, 371
- Hausdorff maximal theorem, 1807
- Hausdorff measure
 - translation invariant, 570
- Hausdorff measure and nonlinear maps, 587, 597
- Hausdorff measures, 565
- Hausdorff metric, 173
- Hausdorff space, 154
- Heine Borel, 27
- Heine Borel theorem, 145

- Hermitian, 89
- Hilbert Schmidt operators, 451, 1621
- Hilbert Schmidt theorem, 443, 718
- Hilbert space, 421
- Holder's inequality, 337
- homotopic to a point, 965
- Hormander condition, 666
- implicit function theorem, 132, 135, 136
- independent events, 1301
- independent random vectors, 1302
- independent sigma algebras, 1302
- indicator function, 186
- infinite products, 969
- inner product space, 421
- inner regular, 1298, 1407
- inner regular measure, 225
- interior point, 100
- inverse function theorem, 135, 136
- inverses and determinants, 72
- inversions, 920
- isodiametric inequality, 571, 575
- isogonal, 796, 919
- isolated singularity, 814
- isometric, 739
- Ito isometry, 1556, 1633
- James map, 377
- Jensen's formula, 986
- Jensens inequality, 356, 1374
- Kolmogorov Centsov theorem, 1472
- Kolmogorov extension theorem, 327, 332
- Kolmogorov zero one law, 1310
- Kolmogorov's inequality, 1312
- Lagrange multipliers, 137, 138
- Laplace expansion, 70
- Laplace transform, 658
- Laurent series, 866
- law, 1425
- Lebesgue
 - set, 529
- Lebesgue decomposition, 489
- Lebesgue measure, 285
- Lebesgue point, 527
- Levy's theorem, 1585, 1617
- limit of a function, 104
- limit point, 154
- limit points, 104
- linear combination, 57, 68
- linearly dependent, 57
- linearly independent, 57
- Liouville theorem, 809
- Lipschitz, 28, 103, 113
- Lipschitz boundary, 606, 1067
- Lipschitz manifold, 1089
- Lipschitz maps
 - extension, 1062
- little Picard theorem, 1036
- local martingale, 1542
- localization, 1645
- localizing sequence, 1542
- locally compact, 147
- locally compact , 156
- locally convex topological vector space, 394
- locally finite, 602, 1085
- locally Lipschitz, 582
- logarithm
 - branch of logarithm, 834
- lower semicontinuous, 400
- Lusin's theorem, 356
- manifold, 1089
- manifolds
 - radon measure, 1091
 - surface measure, 1091
- Marcinkiewicz interpolation, 663
- martingale, 1374
- matrix
 - left inverse, 73
 - lower triangular, 73
 - non defective, 89
 - normal, 89
 - right inverse, 73
 - upper triangular, 73
- maximal function
 - general Radon measures, 621

- maximal monotone operator, 746
- maximum modulus theorem, 835
- mean value inequality, 125
- mean value theorem
 - for integrals, 51
- measurable, 219
 - Borel, 180
- measurable function, 180
 - pointwise limits, 181
- measurable functions
 - Borel, 188
 - combinations, 186
- measurable rectangle, 266, 333
- measurable representative, 734
- measurable sets, 178, 220
- measure space, 178
 - isomorphic, 766
 - separable, 766
- Mellin transformations, 874
- meromorphic, 816
- Merten's theorem, 953
- Meyer Serrin theorem, 1100
- Mihlin's theorem, 678
- mild solution, 1797
- Minkowski functional, 390, 398
- Minkowski's inequality, 343
- minor, 70
- Mittag Leffler, 878, 980
- mixed partial derivatives, 130
- modification, 1469
- modular function, 1018, 1020
- modular group, 949, 1008
- module of periods, 1004
- mollifier, 351
- monotone class, 265
- monotone convergence theorem, 200
- Montel's theorem, 923, 945
- Morrey's inequality, 1049
- multi-index, 103, 129, 163, 637
- multipliers, 666
- Muntz's first theorem, 438
- Muntz's second theorem, 439
- natural, 1560, 1562
- Neumann series, 882
- nonlinear Fubini's theorem, 1084
- normal, 1343, 1425
- normal family of functions, 925
- normal filtration, 1475
- normal topological space, 155
- nowhere differentiable functions, 388
- nuclear operator, 448
- numerical range, 895
- one point compactification, 156, 229
- open cover, 156
- open mapping theorem, 366, 831
- open set, 100
- open sets, 154
- operator norm, 118, 363
- optional sampling theorem, 1383
- order, 995
- order of a pole, 815
- order of a zero, 807
- order of an elliptic function, 1006
- Ornstein Uhlenbeck process, 1722
- orthonormal set, 440
- outer measure, 189, 219
- outer regular, 1298, 1407
- outer regular measure, 225
- pap97, 772
- partial derivative, 124
- partial order, 26, 370
- partially ordered set, 1807
- partition, 35
- partition of unity, 233, 603, 1087
- period parallelogram, 1006
- Pettis theorem, 700
- Phragmen Lindelof theorem, 837
- pi systems, 275
- pivot space, 1211
- Plancherel theorem, 649
- point of density, 585, 595, 1055
- pointwise limits of measurable functions, 701
- polar decomposition, 501
- pole, 815
- polish space, 760
- polynomial, 163, 637

- positive, 459
- positive and negative parts of a measure, 538
- positive definite, 1422
- positive definite functions, 1364
- positive linear functional, 234
- power series
 - analytic functions, 803
- power set, 19
- precompact, 156, 173, 1215
- predictable, 1477, 1635
- primitive, 787
- principal branch of logarithm, 835
- principal ideal, 984
- probability space, 1298
- product measure, 269
- product rule, 1048
- product topology, 156
- progressively measurable, 1475, 1641, 1645
- projection in Hilbert space, 424
- Prokhorov's theorem, 1359, 1411
- properties of integral
 - properties, 47
- pseudo continuous, 1421
- pseudomonotone operator, 415

- quadratic variation, 1578
- quotient space, 1246

- Rademacher's theorem, 535, 1052, 1054
- Radon Nikodym derivative, 492
- Radon Nikodym property, 736
- Radon Nikodym Theorem
 - σ finite measures, 492
 - finite measures, 489
- Radon Nikodym theorem
 - Radon Measures, 635
- random variable, 543, 1298
- random vector, 1298
 - independent, 1320
- rank of a matrix, 74
- real Schur form, 87
- reflexive Banach Space, 378
- reflexive Banach space, 509

- region, 807
- regular measure, 225
- regular topological space, 155
- removable singularity, 814
- residue, 855
- resolvent, 471
- resolvent set, 881, 895
- Riemann criterion, 39
- Riemann integrable, 38
- Riemann integral, 38
- Riemann sphere, 777
- Riemann Stieltjes integral, 38
- Riesz map, 426
- Riesz representation theorem, 741
 - $C_0(X)$, 513
 - Hilbert space, 426
 - locally compact Hausdorff space, 234
- Riesz Representation theorem
 - $C(X)$, 512
- Riesz representation theorem L^p
 - finite measures, 503
- Riesz representation theorem L^p
 - σ finite case, 509, 745
- Riesz representation theorem for L^1
 - finite measures, 506
- right polar decomposition, 91
- Rouche's theorem, 861
- Runge's theorem, 958

- Sard's lemma, 306
- scalars, 55, 99
- scale of Banach spaces, 918
- Schauder fixed point theorem, 410
- Schottky's theorem, 943
- Schroder Bernstein theorem, 23
- Schwartz class, 1137
- Schwarz formula, 805
- Schwarz reflection principle, 829
- Schwarz's lemma, 926
- sectorial, 884
- self adjoint, 89, 459
- semigroup
 - adjoint, 482
 - contraction

- bounded, 469
 - generator, 468
 - growth estimate, 468
 - Hille Yosida theorem, 472
 - strongly continuous, 468
- seminorms, 393
- separability of $C(H)$, 1410
- separated, 158
- separation theorem, 391
- sequential compactness, 27, 517
- sequential weak* compactness, 385
- sequentially compact set, 116
- sets, 19
- Shannon sampling theorem, 660
- simple function, 193, 697
- simple functions, 182
- Skorokhod's theorem, 1415
- Sobolev Space
 - embedding theorem, 659
 - equivalent norms, 658
- Sobolev space, 1097
- Sobolev spaces, 659
- span, 57, 68
- spectral radius, 883
- Steiner symetrization, 573
- stereographic projection, 778, 944
- Stirling's formula, 997
- stochastic Fubini theorem, 1756
- stochastic integral, 1623
- stochastic process, 1469
- stong law of large numbers, 1396
- stong topology, 401
- stopped martingale, 1528
- stopped process, 1517
- stopping time, 1379, 1384, 1484, 1492
- strict convexity, 391
- strong law of large numbers, 1318, 1402
- strongly measurable, 697
- subbasis, 324
- submartingale, 1374
- submartingale convergence theorem, 1378
- subspace, 57
- sums
 - independent random variables, 1315, 1393
 - supermartingale, 1374
 - support of a function, 233
 - Suslin space, 761
- tail event, 1310
- Tietze extension theorem, 152
- tight, 1351, 1409
- topological space, 154
- topological vector space dual, 395
- total variation, 495, 531
- totally bounded set, 142
- totally ordered set, 1807
- trace, 450
- trajectories, 1469
- translation invariant, 287
- translations, 920
- trivial, 57
- Tychonoff fixed point theorem, 409
- Tychonoff theorem, 325
- uniform boundedness theorem, 365
- uniform convergence, 776
- uniform convexity, 391
- uniformly bounded, 147, 945
- uniformly continuous, 27
- uniformly equicontinuous, 147, 945
- uniformly integrable, 212, 357
- unimodular transformations, 1008
- upcrossing, 1374, 1391
- upper and lower sums, 36
- Urysohn's lemma, 227
- variational inequality, 424
- vector measures, 495, 734
- Vector valued distributions, 1207
- version, 1469
- Vitali
 - convergence theorem, 728
- Vitali convergence theorem, 213, 357
- Vitali cover, 630
- Vitali covering theorem, 290, 294, 297, 1055
- Vitali coverings, 294, 296, 1055
- Vitali theorem, 949
- volume of unit ball, 578

- weak * convergence, 1041
- weak convergence, 392
- weak convergence of measures, 1362, 1414
- weak derivative, 1043
- weak topology, 382, 406
- weak* measurable, 704
- weak* topology, 407
- weak* topology, 382
- weakly measurable, 697
- Weierstrass
 - Stone Weierstrass theorem, 168
- Weierstrass M test, 776
- Weierstrass P function, 1013
- well ordered sets, 1809
- Wiener process, 1578
- Wiener process in Hilbert Space, 1602
- winding number, 817
- Wronskian, 84

- Yankov von Neumann Aumann, 767, 769
- Young's inequality, 337, 523, 1262

- zeta function, 997